INJECTIVE MODULES OVER A PRINCIPAL LEFT AND RIGHT IDEAL DOMAIN, WITH APPLICATIONS

By

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I would like to dedicate this thesis to my loving husband ***Sebastian****. Without his love, encouragement, patience, and sacrifice, this thesis would not exist. I would also like to dedicate this thesis to my daughter* ***Andrea****, who I thank for her understanding and well-timed distraction from mathematics.*

Abstract

I study the (indecomposable) injective modules over some noetherian rings. The main results of this dissertation take place in the setting of a principal left and right ideal domain R. One indecomposable injective is the injective envelope (divisible hull) of the module ${}_R R$, and is isomorphic to the classical ring of quotients $\mathcal{Q}$ (or division algebra) of R. The other indecomposable injective modules are (up to isomorphism) in a one-to-one correspondence with the prime elements of the ring (up to similarity).

Motivated by a classic treatment of O.Ore [31], I take advantage of the factorization theory in R and investigate the internal structure of an indecomposable injective $E \neq \mathcal{Q}$. I describe it as a "layered" structure in two ways: first as the union of its socle series, and secondly, as the union of its elementary socle series, a concept from model theory.

More powerful results concerning these objects are obtained via the technique of localization by considering their description over an extension $\mathfrak{R}$ of R which is also a principal left and right ideal domain, but with a unique simple module (up to isomorphism). Each socle factor $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ is a semisimple $\mathfrak{R}$-module, and I show that once a choice of basis at the level $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$ is made, there is a canonical way to extend it through all levels so that the arithmetic of E is understandable in terms of this basis.

In addition, I analyze the right module structure of E over its endomorphism ring and study the relationship with the elementary socle series of ${}_{\mathfrak{R}}E$. Also, the bicommutator of ${}_{\mathfrak{R}}E$ is shown to be the completion of the ring $\mathfrak{R}$ in the E-adic topology.

As a consequence of all these results, I am able to classify and describe the indecomposable injective modules over the first Weyl algebra (as well as over similar algebras), which—as wild modules—were thought to have an "unmanageable" structure [20].

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Contents

Introduction

Since the 1950s, there has been an increase in interest regarding non-commutative noetherian algebraic structures, mainly because of their appearance in various areas of mathematics. The skew polynomial rings are one of the main classes of non-commutative noetherian rings, and in particular, rings of differential operators are one of the most important noncommutative noetherian algebras, with an important role in the representation theory of Lie algebras and in the algebraic analysis of systems of partial differential equations.

The important role played by the injective modules over noetherian rings is obvious from the characterization of such rings via the injective modules, i.e., a ring is left noetherian if and only if every injective left module is a direct sum of indecomposable injective modules, by [27], [32].

The problem of classifying and describing the indecomposable injective modules over a left Noetherian ring has been studied extensively in Jategaonkar [20], with a dominant role being played by the prime ideals of the ring, but also by the technique of localization. There, the indecomposable (uniform) injective modules have been classified as either 'tame' or 'wild'. The tame ones are uniquely determined by the prime ideals of the ring. Jategaonkar [20] has made the remark that "wild modules are, as a rule, unmanageable", while "the tame ones are manageable". So far, there are few known descriptions of wild uniform injectives over noetherian (non-commutative) rings, so the study of such objects is indeed a very difficult task, if not almost impossible. This is especially unfortunate in the case of a non-commutative simple noetherian domain, where there are no prime ideals except the zero ideal. The only tame indecomposable injective module is the injective envelope of the ring itself, the quotient field of the ring, all the other ones being wild. That is why the study of injective modules over the first Weyl algebra, the original motivation for this thesis, seemed to be such an unchallenged ground for research, yet with great potential interest.

In the study of injective modules over a principal left and right ideal domain, we deal with two contradictory facts. The good news is that each wild indecomposable injective E is the union of its socle series—as a torsion module over an HNP ring ([20, A.1.4]). The bad news, though, is that for each wild indecomposable injective module, the socle layers do not give additional information about the module itself past their semisimple structure.

In some sense, a principal left and right ideal domain is 'almost commutative'. There is a relation of similarity between the elements of the ring allowing permutations of the order in a product by doing 'interchanges of factors'. Therefore, it is natural to expect that some of the structural properties of the indecomposable injectives over a commutative noetherian ring can be generalized to principal left and right ideal domains. However, more modern techniques seem to be unusable for dealing with wild injectives, so I will make use of some non-commutative tools developed earlier in the century, my main inspiration being an old paper of Ore [31].

Chapter 1 presents general definitions and results needed throughout my presentation. It is structured in two parts. The first one deals with notions from the theory of modules over (non-commutative) rings. On the other hand, the second part, much shorter, contains definitions and results regarding the modules over a ring, but from a totally different point of view, that of the model theory of modules.

Chapter 2 provides the background for the development of the main results of this thesis. In particular, for a principal left and right ideal domain R, we deal with divisibility and the notions associated with it, such as least common left/right multiple, greatest common right/left divisor. The research extensively employs the equivalence of similarity and its properties. I have taken Ore's paper [31] as the primary inspiration in the development of the factorization and decomposition theorems in the last part of the chapter, although Ore's results were set in an Euclidean domain, a ring of skew polynomials.

The core of this dissertation is **Chapter 3**, where we work in a principal left and right ideal domain R. First, I classify the indecomposable injective R-modules, then describe them. It is surprising how powerful the arithmetic properties of the ring are in the study of the layered structure of its indecomposable injectives.

The indecomposable injective R-modules are $E(R)$ and $E(R/Rp)$, where p ranges over a set of representatives of the similarity classes of primes of R. This classification offers an analogue of Matlis' theory for (non-commutative) principal left and right ideal domains, where the irreducible left ideals generated by $\widehat{p}$-indecomposable elements correspond to the P-primary ideals in the commutative case.

For the rest of **Chapter 3**, I investigate an indecomposable injective R-module $E = E(R/Rp)$, and a good start towards describing the layered structure of E is Theorem 3.7, where I give a sufficient condition for an element $0 \neq e \in E$ to be in $\mathrm{soc}_n(E)$, $n \geq 0$, but the condition is not necessary, as some examples show. On the other hand, the elementary socle series $(\mathrm{soc}^n(E))_{n<\omega}$ of E also exhausts the module E, as it happened with the socle series, but this time I present a complete characterization of $\mathrm{soc}^n(E)$, $n \geq 0$, in Theorem 3.24.

Also in the third chapter, I use the localization of the ring R at the set of all elements acting regularly on E, in order to obtain a principal left and right ideal domain $\mathfrak{R}$ with a unique simple module (up to isomorphism). Over this ring, I find a more explicit description of the socle layers $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ $(n \geq 1)$ by constructing bases from an arbitrary (fixed) basis of $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$. On the other hand, the elementary socle series is preserved under the process of localization.

The final section of **Chapter 3** consists of results regarding the right structure of E as an $\mathrm{End}({}_{\mathfrak{R}}E)$-module. Furthermore, the bicommutator of ${}_{\mathfrak{R}}E$ is the Hausdorff completion of $\mathfrak{R}$ in the E-adic topology.

In **Chapter 4** we discuss different applications of the theory developed in Chapters 2 and 3. In particular, we offer answers to the problem of classifying and describing the indecomposable injective modules over the first Weyl algebra, and similarly, we recognize a certain technique which can be applied to different rings, such as the first quantized Weyl algebra and some generalized Weyl algebras.

Chapter 1

Preliminaries

Rings and Modules

1.1 Terminology and Notation. Basic Definitions

The basic references for ring theory are Goodearl & Warfield [15], Lam [23] and [24], Rowen [34], Stenström [36].

All rings considered are associative with 1, and not necessarily commutative. A *domain* is a ring without zero-divisors.

"*Ideal*" will always mean two-sided ideal, and "*module*" will always mean unital left module, unless otherwise stated. If R is a ring, the class of all left R-modules will be denoted by R-Mod. The notation $N \leq M$ will be used to indicate that N is a *submodule* of the module M.

A proper ideal P of a ring R is called *prime* if, for any ideals $I, J \leq R$ with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. A ring R is *prime* if 0 is a prime ideal. Any domain is a prime ring.

A non-zero ring is said to be *simple* if it has no proper (two-sided) ideals.

The group of all units of a ring R will be denoted by $\mathrm{U}(R)$. If R is a domain and

$u \in R$, then u is left invertible if and only if u is right invertible if and only if u is a unit.

A domain R is said to be a *principal left ideal domain* if every left ideal I is principal, i.e., $I = Rr$ for some $r \in R$. If every right ideal of a domain R is principal, then R is a *principal right ideal domain.* If the domain R is both a principal left ideal domain and a principal right ideal domain, then it is said to be a principal ideal domain, or PID.

A ring R is *left noetherian* if it satisfies the ascending chain condition on left ideals, or equivalently, every left ideal is finitely generated. Thus, a principal left ideal domain is left noetherian.

If M is an R-module and $\emptyset \neq X \subseteq M$, the *(left) annihilator of X in R* is the set

$$\mathrm{ann}_R(X) = \{r \in R : \ rx = 0, \ \forall x \in X\}$$

and is a left ideal of R.

The *annihilator of $\emptyset \neq Y \subseteq R$ in M* is the set

$$\mathrm{ann}_M(Y) = \{m \in M : \ Ym = 0, \ \}$$

and is a subgroup of the group $(M, +)$. If Y is a right ideal of R, then $\mathrm{ann}_M(Y)$ is a submodule of M.

A non-zero R-module M is called *simple* if M has no non-zero proper submodules. It is well-known that M is a simple module if and only if $M = Rx \cong R/\mathrm{ann}_R(x)$, for every non-zero $x \in M$; in this case, $\mathrm{ann}_R(x)$ is a maximal left ideal of R for every non-zero $x \in M$. A non-zero R-module M is called *semisimple* if M is the sum of all its simple submodules. Equivalently, M is semisimple if and only if it is isomorphic to a direct sum of simple modules.

Definition 1.1 Let $M \in R$-Mod and $\emptyset \neq X \subseteq M \setminus \{0\}$.

(1) X is said to be *R-linearly independent* if for any distinct $x_1, \ldots, x_n \in X$ and any distinct $a_1, \ldots, a_n \in R$, the relation $\sum_{i=1}^{n} a_i x_i = 0$ implies $a_i x_i = 0$, for all $i \leq n$.

(2) If $M = \sum_{x \in X} Rx$ and X is R-linearly independent, then X is called a *basis* for M.

We make the remark that not every R-module has a basis, but if M is semisimple then there exists a basis X for M, and in this case,

$$M = \sum_{x \in X} Rx \cong \bigoplus\{Rx : \ x \in X\},$$

where every Rx is a simple submodule of M. This is the context that the term "basis" will be used in this dissertation.

If M is a left R-module and we let $T = \mathrm{End}_R(M)^{\mathrm{op}}$ then M has a right module structure over T given by the action $m \cdot f = f(m) \in M$ for all $m \in M$, $f \in T$. Then M is an R-T bimodule. If I is a left ideal of R, then $\mathrm{ann}_M(I)$ is a submodule of M_T.

A non-zero module is *indecomposable* if it does not have any proper non-zero direct summands.

Theorem 1.2 *An R-module M is indecomposable if and only if* $\mathrm{End}_R(M)$ *has no nontrivial idempotents.*

1.2 Injective Modules

Baer [2] introduced the idea of injective module over an arbitrary ring, and worked with what he called "smallest complete group" containing a given group, an object which is nowadays called the injective hull/envelope. In the same paper, we first encounter *Baer's criterion* characterizing an injective module. We should point out that the actual name "injective module" was first used in [10] by Eckmann and Schopf who proved that the injective envelope (or injective hull) of a module is a maximal essential extension.

In this section we cover the basic properties of injective modules, and explore only those special properties directly relevant to the results of this thesis.

Definition 1.3 A module E over a ring R is *injective* if for any $K, M \in R$-Mod and for any monomorphism $e : K \longrightarrow M$ and homomorphism $f : K \longrightarrow E$, there is a homomorphism $\overline{f} : M \longrightarrow E$ such that $f = \overline{f}e$.

In other words, every diagram in R-Mod with exact top row:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "e"] \arrow[d, "f"'] & M \arrow[dl, dashed, "\overline{f}"] \\ & E & \end{tikzcd}$$

can be completed commutatively, as shown.

Theorem 1.4 *For any left module E over a ring R, the following are equivalent:*

(i) *E is injective.*

(ii) *For any left ideal I of R, any homomorphism $f : I \longrightarrow E$ can be extended to a homomorphism $f : R \longrightarrow E$, that is, there exists e $\in E$ such that $f(x) = xe$ for all $x \in I$.* [Baer's Criterion]

(iii) *Every consistent set of linear equations in arbitrarily many free variables and with parameters from E has a solution in E, i.e., if the system has a solution in some $M \geq E$ then it has a solution in E.*

(iv) *Every consistent set of linear equations in one variable and with parameters from E has a solution in E.* [equivalent to Baer's Criterion]

(v) *E is a direct summand of every module that contains it.*

Corollary 1.5 *Let E be an injective left R-module, and let $a, b \in E$ such that $ann_R(a) \subseteq ann_R(b)$. Then there exists $f \in \mathrm{End}(E)$ such that $f(a) = b$.*

PROOF. Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & R/\mathrm{ann}(a) \arrow[r, "g"] \arrow[d, "h"'] & E \arrow[dl, dashed, "f"] \\ & E & \end{tikzcd}$$

where the homomorphisms g, h are defined by: $g(1+\mathrm{ann}(a)) = a$ and $h(1+\mathrm{ann}(a)) = b$. The map g is (trivially) well-defined, and injective.

If $r_1, r_2 \in R$ such that $r_1 - r_2 \in \mathrm{ann}(a) \subseteq \mathrm{ann}(b)$, then from $r_1 b = r_2 b$, it follows that $h(r_1 + \mathrm{ann}(a)) = h(r_2 + \mathrm{ann}(a))$, so the map h is also well-defined.
By the injectivity of E, the diagram can be extended commutatively to an R-homomorphism $f : E \longrightarrow E$ such that $f(a) = b$. ∎

Definition 1.6 A left R-module M is said to be *divisible* if for each non-zero divisor $r \in R$ and each $x \in M$, there exists $a \in M$ such that $x = ra$.

If E is an injective left module and $r \in R$ is not a zero-divisor, consider the homomorphism $f : Rr \longrightarrow E$ defined by $f(r) = x$. Then the definition of injectivity gives $a \in E$ such that $x = f(r) = ra$.

Lemma 1.7 *Every injective left module is divisible.*

In general, divisible modules are not injective, but this is true under some conditions. In particular, by Baer's criterion it is immediate that:

Proposition 1.8 *If R is a principal left ideal domain, then a left module is injective if and only if it is divisible.*

Definition 1.9 An injective module E is an *injective cogenerator* if $\text{Hom}_R(M, E) \neq 0$, for every non-zero module M.

Proposition 1.10 [36, (6.7)] *An injective module is a cogenerator if and only if it contains an isomorphic copy of each simple module.*

Theorem 1.11 [Baer] *For any ring R, any left R-module can be embedded in an injective left R-module.*

Definition 1.12 N is an *essential submodule* of a module M if $N \bigcap L \neq 0$ for every non-zero submodule $L \leq M$.

Equivalently, N is an essential submodule of M if for every $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq rm \in N$; that is, every non-zero element of M satisfies a non-trivial equation over N.

NOTATION: $N \leq_e M$.

In this case, we also say that M is an *essential extension* of N.

Clearly, if $N \leq_e M$ and $M \leq_e P$, then $N \leq_e P$. In other words, the essentiality is transitive.

Proposition 1.13 [Eckmann, Schopf] *A module E is injective if and only if E has no proper essential extensions.*

Theorem 1.14 *For modules $M \leq E$, the following are equivalent:*

(i) *E is a minimal injective extension of M.*

(ii) *E is a maximal essential extension of M.*

(iii) *E is injective, and an essential extension of M.*

Definition 1.15 If the left R-modules $M \leq E$ satisfy one of the equivalent properties in Theorem 1.14, we say that E is an *injective envelope* (or *injective hull*) of M.

Theorem 1.16 *Let M be a left R-module. Then:*

(i) *M has an injective envelope* [10].

(ii) *The injective envelope of M is unique up to isomorphism over M. That is, for any two injective envelopes E, E' of M, there exists an isomorphism $g : E \longrightarrow E'$ which is the identity on M.*

NOTATION: $E_R(M)$, or simply $E(M)$, denotes the injective envelope of $M \in R$-Mod. ▮

Corollary 1.17 (i) *If E is an injective left module containing M, then E contains a copy of $E(M)$ as a direct summand.*

(ii) *If $M \leq_e N$, then N can be enlarged to a copy of $E(M)$. In fact, $E(M) = E(N)$.*

In 1958, Matlis simplified for noetherian rings the problem of finding and eventually describing the injective left R-modules by reducing it to the case of indecomposable injective left R-modules.

Theorem 1.18 [27] *Any injective left module over a left noetherian ring R can be decomposed as a direct sum of a family of indecomposable injective left R-modules. Such a family of indecomposable injective submodules of E is uniquely determined by E up to permutation and R-isomorphism.*

Theorem 1.19 [27], [32] *For any ring R, the following are equivalent:*

(i) *R is left noetherian.*

(ii) *Any injective R-module E is a direct sum of indecomposable injective submodules. Such family is uniquely determined by E up to permutation and R-isomorphism.*

(iii) *The direct sum of any family of injective R-modules is injective.*

Example 1.20 1. The indecomposable injective modules over a commutative noetherian ring R are of the form $E(R/P)$, P a prime ideal of R.

2. Let R be a commutative PID with quotient field K. Let P be a complete set of non-zero prime elements of R (up to associates). Any indecomposable injective R-module is either $E(R) = K$, or has the form $E(R/Rp)$, for $p \in P$.
3. Some indecomposable injective modules over a polynomial ring in n variables are described in [30].
4. The indecomposable injective modules over the ring constructed by Jategaonkar in [19] are described by Kucera in [22].

Definition 1.21 [27] A left ideal $I \neq R$ of a ring R is said to be *meet-irreducible* (or simply, irreducible) if for two left ideals $K, L \supseteq I$, $K \cap L = I$ implies that $K = I$ or $L = I$.

Proposition 1.22 [21] *If R is left (right) noetherian, then every left (right) ideal can be written as an intersection of a finite number of irreducible left (right) ideals.*

Remark 1.23 [Irredundant decomposition] If $I = I_1 \cap \ldots \cap I_n$ is a representation of I as a finite intersection of irreducible left ideals, then either this representation is irredundant (i.e., for any i, $\bigcap\limits_{j \neq i} I_j \nsubseteq I_i$), or it can be modified to an irredundant one by eliminating the components containing the intersection of the remaining ones.

Theorem 1.24 [27] *A left module E over a ring R is an indecomposable, injective module if and only if $E \cong E_R(R/I)$, where I is an irreducible left ideal of R.*

Furthermore, for every $0 \neq e \in E$, $ann_R e = \{\, r \in R \; : \; re = 0 \,\}$ is an irreducible left ideal and $E \cong E_R(R/ann_R e)$.

Definition 1.25 A ring R is *left hereditary* if every left ideal of R is a projective module.

Proposition 1.26 [36, (9.5)] *A ring R is left hereditary if and only if every quotient of an injective left module is injective.*

In the following, the notation $\mathcal{K}.\dim({}_RM)$ is used for the Krull dimension of a left R-module M.

Proposition 1.27 [29, (6.2.9)] *If R is left hereditary noetherian domain, but not a division ring, then $\mathcal{K}.\dim(R) = 1$ and $\mathcal{K}.\dim(R/I) = 0$ for every non-zero left ideal I of R.*

In particular, each R/I is left artinian (since it has $\mathcal{K}.\dim$ 0), and contains a simple submodule.

The next corollary is a consequence of Theorem 1.24 and Proposition 1.27.

Corollary 1.28 *If R is a left hereditary noetherian domain, but not a division ring, then every indecomposable injective left R-module other than $E({}_RR)$, the quotient division hull of ${}_RR$, is isomorphic to the injective envelope of some simple R-module.*

Example 1.29 Every principal left ideal domain is (left) hereditary noetherian prime (HNP).

Theorem 1.30 [27] *For any injective left R-module E, the following are equivalent:*

(i) *E is indecomposable.*

(ii) *$E \neq 0$, and $E = E(M)$ for any nonzero submodule $M \leq E$.*

(iii) *$T = End(E)$ is a local ring (i.e., $End(E)$ has a unique maximal two-sided ideal, the Jacobson radical $\mathrm{Jac}(T)$).*

In this case, $f \in T$ is an automorphism if and only if $\ker(f) = 0$.

Lemma 1.31 [23] *For $y \in R$, the following statements are equivalent:*

(i) *$y \in \mathrm{Jac}(R)$;*

(ii) *$1 - xy$ is left invertible for any $x \in R$;*

(iii) *$yM = 0$ for every simple left R-module M.*

and symmetrically,

(ii)$'$ *$1 - yx$ is right invertible for any $x \in R$;*

(iii)$'$ *$Ny = 0$ for every simple right R-module N.*

Theorem 1.32 [Matlis] *If R is a commutative noetherian ring, then there is a one-to-one correspondence between the prime ideals of R and the isomorphism classes of indecomposable injective R-modules given by $P \leftrightarrow E(R/P)$, for a prime ideal P of R. Furthermore, if Q is an irreducible P-primary ideal, then $E(R/Q) \cong E(R/P)$.*

As attractive as this theorem is, the example of simple noetherian rings which are not division rings shows that nothing like 1.32 can be true in general for non-commutative noetherian rings. However, there are still some useful partial results regarding the classification of the indecomposable injectives.

Definition 1.33 Two irreducible left ideals I, J of R are called *related* if there exist elements $s \notin I, t \notin J$ such that

$$Is^{-1} = Jt^{-1}, \text{ where } Is^{-1} = \{ r \in R \mid rs \in I \}$$

Lemma 1.34 [26] *Let I be an irreducible left ideal of R.*
Then for every $s \notin I$, $E(R/I) \cong E(R/Is^{-1})$. Moreover, Is^{-1} is also irreducible.

PROOF. This is a consequence of 1.24, for $0 \neq e = s + I \in R/I \subset E_R(R/I)$, and of the fact that $Is^{-1} = \mathrm{ann}_R(e)$. ∎

Remark 1.35 If $\mathrm{ann}_R(a) = I$ then $\mathrm{ann}_R(sa) = Is^{-1}$.

Proposition 1.36 [26] *The irreducible left ideals I, J are related if and only if*

$$E(R/I) \cong E(R/J).$$

Corollary 1.37 [26] *Relatedness is an equivalence relation on the set of all irreducible left ideals of R.*

1.3 Localization

In the 1930s, Øystein Ore [31] presented the start of a new theory, namely that of localization of a non-commutative ring, or the birth of what was then called "non-commutative polynomials" (i.e., the ring of fractions). Since the notion of a ring of fractions (or quotient ring, or ring of quotients) plays an extremely important role in the discussion of our topic, I will start by giving a formal definition, but quickly simplify it to a more standard notation for elements of the ring of fractions. The basic references for localization are Goodearl & Warfield [15] and Stenström [36].

Let R be a ring with 1, and let Σ be a multiplicatively closed subset of R (i.e., $1 \in \Sigma$, $0 \notin \Sigma$ and $\Sigma \cdot \Sigma \subseteq \Sigma$).

Recall that an element of the ring $r \in R$ is called *regular* if it is neither a left nor a right zero-divisor.

Definition 1.38 A *left localization of R at* Σ, or *a left ring of fractions of R with respect to* Σ, is a ring denoted by $\Sigma^{-1}R$ together with a ring homomorphism $\varphi : R \to \Sigma^{-1}R$ satisfying:

(1) $\varphi(s)$ is invertible for every $s \in \Sigma$.

(2) Every element in $\Sigma^{-1}R$ has the form $\varphi(s)^{-1}\varphi(r)$, where $r \in R$ and $s \in \Sigma$.

(3) $\varphi(r) = 0$ if and only if $sr = 0$ for some $s \in \Sigma$.

If Σ contains only regular elements, then φ is clearly one-to-one. In particular, if R is a domain, φ is one-to-one.

Theorem 1.39 [Uniqueness]
If $\Sigma^{-1}R$ *exists, it is unique up to isomorphism.*

Definition 1.40 Let R be a ring with identity. A *left denominator set* is a multiplicatively closed subset Σ of R satisfying:

(1) $\Sigma r \cap Rs \neq \emptyset$ for all $s \in \Sigma$, $r \in R$ (*left Ore condition*).

(2) If $rs = 0$, for $r \in R$, $s \in \Sigma$ then $\exists s' \in \Sigma$ with $s'r = 0$ (*left reversibility*).

Note that if Σ satisfies the left Ore condition (i.e. Σ is a left Ore set) and contains only regular elements (e.g., if R is a domain) then Σ is a left denominator set.

The left Ore condition appeared in the work of Øystein Ore in the 1930s, as he was trying to extend the idea of fractions in a non-commutative setting. He determined that this is a necessary and sufficient condition under which a non-commutative domain could be embedded in a division ring Q such that Q is the left ring of fractions of R. Later, Asano proved that a ring has a left ring of fractions with respect to to a multiplicative set Σ of regular elements if and only if Σ satisfies the left Ore condition[1939, 1949]. The necessary and sufficient conditions for the existence of a left ring of fractions with respect to a multiplicatively closed subset of R were found in the 1960s by Gabriel.

Theorem 1.41 [Gabriel, 1962] *Let Σ be a multiplicatively closed subset of the ring R. There exists a left ring of fractions for R with respect to Σ if and only if Σ is a left denominator set of R.*
In this case, the left ring of fractions of R with respect to Σ has the form:

$$\Sigma^{-1}R \cong (S \times R) / \sim,$$

where $\sim$ is the equivalence relation defined by: $(s_1, r_1) \sim (s_2, r_2)$ if there exist $r, r' \in R$ such that $rr_1 = r'r_2$ and $rs_1 = r's_2 \in \Sigma$.

Remark 1.42 Suppose that Σ is a set of regular elements. Then, if the left ring of fractions $\varphi : R \to \Sigma^{-1}R$ exists, the notation $s^{-1}r$ is customarily used for an element $\varphi(s)^{-1}\varphi(r)$ of the ring of fractions. Also, the image in $\Sigma^{-1}R$ of an element $r \in R$ should be denoted $1^{-1}r = r$, being viewed as a fraction, and for an element $s \in \Sigma$, the inverse is denoted by s^{-1}, rather than $s^{-1}1$. In other words, the unique left localization of R at a left denominator set Σ is the ring $\Sigma^{-1}R$ containing R as a subring such that every $s \in \Sigma$ is invertible in $\Sigma^{-1}R$ and $\Sigma^{-1}R = \{s^{-1}r \ : \ s \in \Sigma, \ r \in R\}$.

Proposition 1.43 (Asano, 1949) *Let R be a ring and $\Sigma \subseteq R$ a right and left Ore set of regular elements. Then $R\Sigma^{-1} = \Sigma^{-1}R$, that is, any right ring of fractions with respect to Σ is also a left ring of fractions for R with respect to Σ, and vice versa.*

Definition 1.44 (1) A classical left quotient ring $\mathcal{Q}^l_{cl}(R)$ for a ring R is a left ring of fractions for R with respect to the set of all regular elements in R.

(2) A *left Ore domain* is any domain R in which the non-zero elements form a left denominator set.

In this case, its classical left quotient ring is usually called the left Ore quotient (division) ring of R, or simply the left division algebra of R. For a left and right Ore domain, the left Ore quotient ring and the right Ore quotient ring of R are equal, and is usually called the division algebra of R, $\mathcal{Q}_{cl}(R)$.

In the 1950s, Goldie's research opened new avenues for a systematic study of the theory of non-commutative noetherian rings, and although he did not publish his proof that a left noetherian domain must satisfy the left Ore condition, its simplicity made it part of the folklore.

Proposition 1.45 [Goldie] *If R is a left noetherian domain, then R is left Ore, so R has a classical left quotient ring $Q^l_{cl}(R)$.*

Corollary 1.46 *If R is a left noetherian domain, then $Q^l_{cl}(R)$ is the unique divisible hull $E({}_RR)$ of ${}_RR$.*

In his 1933 paper [31], Ore gave examples of non-commutative domains that satisfy the left (and right) Ore condition. The kinds of rings of "non-commutative polynomials" given in these examples are now called *Ore extensions*, or skew polynomial rings. Since most of the rings for which the theory in this dissertation are developed are of this type, we consider that this class of rings needs to be formally introduced.

Definition 1.47 Let R be a domain with 1, and let σ be an endomorphism of R.

(1) A *σ-derivation* δ on R is a group homomorphism of $(R, +)$ satisfying

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \text{ for all } a, b \in R.$$

(2) The set of all polynomials $\left\{ \sum_{i=1}^{n} a_i y^i \mid n \in \mathbb{N}, a_i \in R \right\}$, can be made into an associative ring, called the *skew polynomial ring*, and is denoted by $R[y; \sigma, \delta]$.

The multiplication in this ring is subject to the rule

$$ya = \sigma(a)y + \delta(a), \text{ for all } a \in R.$$

For basic results about skew polynomial rings see, for instance, Cohn [6], Goodearl & Warfield [15], or McConnell & Robson [29]. The following theorem presents some of the properties of this type of ring, under certain conditions for R, σ and δ.

Theorem 1.48 [29] *Let $S = R[y; \sigma, \delta]$.*

(i) *If σ is injective and R is a domain, then S is a domain.*

(ii) *If σ is injective and R is a division ring, then S is a principal left ideal domain.*

(iii) *If σ is an automorphism and R is a prime ring, then S is a prime ring.*

(iv) *If σ is an automorphism and R is right (or left) noetherian, then S is a right (respectively left) noetherian ring.*

Theorem 1.49 [6]*Let K be a field, σ an endomorphism and δ a σ-derivation of K.*

(i) *The left skew polynomial ring $K[y;\sigma,\delta]$ is a right Euclidean domain, hence it is a principal right ideal domain.*

(ii) *If σ is an automorphism of K, then $K[y;\sigma,\delta]$ is also a right skew polynomial ring, that is the polynomials of R can be written with coefficients on the left, or on the right, as needed. Furthermore, $K[y;\sigma,\delta]$ is also a left Euclidean domain, and consequently, a principal ideal domain (PID).*

Example 1.50 [7], [8] A simple but very important example of a skew construction is obtained by taking $R = k[x]$, k a field of $\mathrm{char}(k) = 0$, $\sigma = id_{k[x]}$ and δ the differentiation with respect to x. This ring is called **the first Weyl algebra**, and is denoted by

$$\mathbb{A} = k[x][y;\sigma = id_{k[x]}, \delta = \partial/\partial x]$$

This algebra is also described in terms of generators and relations, as the k-algebra in two generators x, y subject to

$$yx - xy = 1.$$

Equivalently, it is the quotient $k\langle x,y\rangle/(yx - xy - 1)$ of the free k-algebra $k\langle x,y\rangle$.

At the same time, the first Weyl algebra $\mathbb{A}$ can be defined as the ring of formal differential operators with polynomial function coefficients: $\mathbb{A} \cong D(k[x])$. The generators x and y of $\mathbb{A}$ are the operators of $k[x]$ defined on a polynomial $f \in k[x]$ by the formulae:

$$x(f) = x \cdot f \text{ and } y(f) = \partial f/\partial x$$

Last, but not least, we should mention another representation of the first Weyl algebra $\mathbb{A}$ as the k-subalgebra of $\mathcal{M}_\infty(k)$ generated by the following two matrices:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The mathematical structure of $\mathbb{A}$ showed up in may different contexts. It first appeared in 1925 in the work of Heisenberg, Born and Jordan, who were trying to develop the principles of mechanics that were to to explain the behaviour of the atom. The dynamical variables needed to be represented by matrices in quantum theory, so by denoting the position matrix by q and the momentum by p, the equation for a system with one degree of freedom has the form $pq - qp = \imath\hbar$; they realized that the matrices of quantum theory had to be infinite.

Another early appearance of this structure was in the work on Schrödinger on wave mechanics. At the same time, Dirac was developing an alternative interpretation of the quantum formalism. He was interested in polynomial expressions in the dynamical variables momentum p and position q which satisfied the (normalized) relation $pq - qp = 1$. He called this algebra the "quantum algebra".

The first algebraist to study this "quantum algebra" was Littlewood (1933) who constructed infinite matrices satisfying the equation $pq - qp = 1$ and established many of the basic properties of this algebra. In the 1960's, J. Dixmier published a series of papers where proved a multitude of properties. This is the first time the name "Weyl algebra" is used, after Hermann Weyl, who in 1950 wrote an impressive survey of Dirac's point of view, called "The theory of groups and quantum mechanics".

The importance of the first Weyl algebra for algebraists has grown steadily in the past 30–40 years, as it appears as a quotient of enveloping algebras of nilpotent Lie algebras by primitive ideals. Furthermore, the characterization of the first Weyl algebra as a ring of formal differential operators of $k[x]$ puts it at the heart of the theory of D-modules, with vast applications in the representation theory of Lie algebras, differential equations, mathematical physics, singularity theory and even number theory.

Example 1.51 Another example of a skew polynomial ring that will be heavily used in this dissertation is the ring $k(x)[y; \sigma = id, \delta = \partial/\partial x]$, or simply $k(x)[y, \partial/\partial x]$, which can also be described as the ring of formal differential operators on $k(x)$ with rational functions coefficients, where x acts on $k(x)$ by multiplication, while the action of y is the usual differentiation of rational functions. By Theorems 1.48 and 1.49, $k(x)[y, \partial/\partial x]$ is a left and right Euclidean domain, hence a PID.

Example 1.52 We offer next a few more examples of skew polynomial constructions:

1. $k[y]$ is a skew polynomial ring where $R = k,\ \sigma = id,\ \delta = 0$.
2. The first quantized Weyl algebra $\mathcal{A}_1^q(k)$ over a field k (char(k) $= 0$, $q \in k^\times, q \neq 1$) is the k-algebra in two generators x and y subject to $yx - qxy = 1$.
   $\mathcal{A}_1^q(k)$ is a skew polynomial ring $R[y; \sigma, \delta]$, where $R = k[x]$, $\sigma(x) = qx$, and $\delta(f) = \dfrac{\sigma(f) - f}{\sigma(x) - x}$, for all $f \in k[x]$.
3. The enveloping algebra of the 2-dimensional solvable Lie algebra over a field k with basis $\{x, y\}$ such that $yx - xy = x$ is the skew polynomial ring

$$k[x][y; \sigma = id_{k[x]}, \delta = x(\partial/\partial x)].$$

Having the necessary theory for a (left) ring of fractions, we continue by introducing the construction of modules of fractions.

Definition 1.53 Suppose that $\Sigma^{-1}R$ exists and let M be a left R-module. The *module of fractions* $\Sigma^{-1}M$ with respect to Σ can be constructed analogously to the ring of fractions $\Sigma^{-1}R$. Thus, $\Sigma^{-1}M$ is a left $\Sigma^{-1}R$-module which is canonically isomorphic to the induced module $\Sigma^{-1}M = \Sigma^{-1}R \otimes M$, and the canonical map $\mu_M : M \to \Sigma^{-1}M$ has the universal property.

Definition 1.54 Let $M \in R$-Mod and $\Sigma \subseteq R$.
Denote by $\mathrm{tor}_\Sigma(M) = \{x \in M : sx = 0 \text{ for some } s \in \Sigma\}$ the set of all Σ-torsion elements of M. If $\mathrm{tor}_\Sigma(M) = M$, then M is called Σ-torsion, and if $\mathrm{tor}_\Sigma(M) = 0$, then M is called Σ-torsionfree.

Remark 1.55 Let $M \in R$-Mod and let Σ be a left denominator set of R.

(i) The set $\mathrm{tor}_\Sigma(M)$ is a submodule of M.

(ii) If M is a simple module, then M is either Σ-torsion or Σ-torsionfree.

Lemma 1.56 *The canonical map $\mu_M : M \to \Sigma^{-1}M$ is an embedding if and only if M is Σ-torsionfree.*

Proposition 1.57 $\Sigma^{-1}M \cong \Sigma \times M/\sim$, *where $\sim$ is the equivalence relation defined by: $(s_1, x_1) \sim (s_2, x_2)$ if there exist $r, r' \in R$ such that $rx_1 = r'x_2 \in M$ and $rs_1 = r's_2 \in \Sigma$.*

NOTATION: Let M be Σ-torsionfree. Then the embedding μ_M allows us to simplify the notation. Similarly to the localization of a ring, we will denote the elements of $\Sigma^{-1}M$ by $s^{-1}x$, for some $x \in M, s \in \Sigma$, and identify an element x of the module M with $1^{-1}x \in \Sigma^{-1}M$. ∎

Using the results from 1.41 and 1.57, we can explicitly write the definitions for two equivalent fractions, as well as the definitions of the operations in $\Sigma^{-1}R$ and $\Sigma^{-1}M$.

Remark 1.58 Let Σ be a left denominator set in a ring R, and let M be a Σ-torsionfree left R-module.

1. Two fractions $s_1^{-1}r_1$ and $s_2^{-1}r_2$ are equal in $\Sigma^{-1}R$ if there exist $r, r' \in R$ such that $rr_1 = r'r_2$ and $rs_1 = r's_2 \in \Sigma$.

   The addition and multiplication in $\Sigma^{-1}R$ are defined as follows:

   $s_1^{-1}r_1 + s_2^{-1}r_2 = s^{-1}(ar_1 + rr_2)$, where $a \in R,\ r \in \Sigma$ such that $s = as_1 = rs_2 \in \Sigma$.

   $s_1^{-1}r_1 \cdot s_2^{-1}r_2 = (ss_1)^{-1}(rr_2)$, where $r \in R,\ s \in \Sigma$ such that $sr_1 = rs_2$ and $ss_1 \in \Sigma$.

2. Two fractions $s_1^{-1}x_1$ and $s_2^{-1}x_2$ are equal in $\Sigma^{-1}M$ if there exist $r, r' \in R$ such that $rx_1 = r'x_2$ and $rs_1 = r's_2 \in \Sigma$.

   The addition and multiplication by scalars in $\Sigma^{-1}M$ are defined as follows:

   $s_1^{-1}x_1 + s_2^{-1}x_2 = s^{-1}(ax_1 + rx_2)$, where $a \in R,\ r \in \Sigma$ such that $s = as_1 = rs_2 \in \Sigma$.

   $s_1^{-1}r_1 \cdot s_2^{-1}x = (ss_1)^{-1}(rx)$, where $r \in R,\ s \in \Sigma$ such that $sr_1 = rs_2$.

Proposition 1.59 [15] *Let Σ be any left denominator in a ring R, and let $Q = \Sigma^{-1}R$. Assume that Σ contains only regular elements of R, so that R is a subring of Q.*

(i) *For a left ideal L of Q, let $L^c = L \bigcap R \leq R = \{r \in R : \ 1^{-1}r \in L\}$ denote the contraction of L to R.*
Then $L = QL^c$.

(ii) *For a left ideal I of R, let $I^e = \Sigma^{-1}I = QI \leq Q$, $I^e = \{s^{-1}x : \ s \in \Sigma, x \in I\}$ denote the extension of the left ideal ${}_RI$ to the ring Q.*
Then $I^e \bigcap R = \{r \in R : \ sr \in I$ for some $s \in \Sigma\}$.

(iii) *If $I \leq R$ is a left ideal of R, then $I = I^{ec}$ if and only if R/I is Σ-torsionfree. That is, $I = R \cap (\Sigma^{-1}I)$.*

(iv) *Let M be a Σ-torsionfree R-module. Then M is a submodule of $\Sigma^{-1}M$, and we can define the contraction of a submodule P of $\Sigma^{-1}M$ to be $P^c = P \cap M$, and the extension of a submodule N of M to be $N^e = \Sigma^{-1}N$.*

Contraction and extension provide inverse lattice isomorphisms between the lattice of Q-submodules of $\Sigma^{-1}M$ and the lattice of those R-submodules N of M such that M/N is Σ-torsionfree.

In particular, the canonical map from the set of isomorphism classes of simple Σ-torsionfree R-modules to the set of isomorphism classes of simple Q-modules defined by $M \mapsto \Sigma^{-1}M$, for every simple module M, is injective (up to isomorphism) and $\mathrm{soc}_R(\Sigma^{-1}M) \neq 0$. Furthermore, if P is a simple Q-module and contains a simple R-module N, then $P \cong \Sigma^{-1}N$.

Definition 1.60 Let $\Sigma \subseteq R$. An R-module M is Σ-*divisible* if $M = sM$ for each $s \in \Sigma$. Note that a divisible module, as defined in 1.6, is Σ-divisible for Σ the set of all regular elements of R.

Theorem 1.61 [36] *Let Σ be a left denominator set for a ring R and $M \in R$-Mod. Then $M \cong \Sigma^{-1}M$ if and only if M is Σ-torsionfree and Σ-divisible.*

Proposition 1.62 [14], [36] *Let Σ be a left denominator set for a ring R, and let M be an R-module.*

(i) *If M is noetherian, then so is $\Sigma^{-1}M$ regarded as a $\Sigma^{-1}R$-module.*

(ii) *If M is simple, then $\Sigma^{-1}M$ is either zero or a simple $\Sigma^{-1}R$-module.*

(iii) *If M is a left R-module and N is a left $\Sigma^{-1}R$-module, then*

$$\mathrm{Hom}_{\Sigma^{-1}R}(\Sigma^{-1}M, N) \cong \mathrm{Hom}_R(M, N)$$

Proposition 1.63 [20] *Let Σ be a left denominator set in R.*

(i) *Any Σ-torsionfree injective left R-module E can be uniquely regarded as a left $\Sigma^{-1}R$-module and, as such, it is injective.*

(ii) *Any injective left $\Sigma^{-1}R$-module E, when viewed as a left R-module, is Σ-torsionfree and injective.*

Proposition 1.64 [20] *Let Σ be a left denominator set in R, and let M be a Σ-torsionfree R-module.*

(i) *The R-injective envelope of M is $\Sigma^{-1}R$-isomorphic to the $\Sigma^{-1}R$-injective envelope of the left quotient module $\Sigma^{-1}M$:*

$$E_R(M) \cong_{\Sigma^{-1}R} E_{\Sigma^{-1}R}(\Sigma^{-1}M) = E$$

(ii) $\mathrm{End}_R(E) = \mathrm{End}_{\Sigma^{-1}R}(E)$

Proof. (i) $E_R(M)$ is an injective left R-module, and it is enough to show that it is also a Σ-torsionfree R-module. By the previous proposition, it will then follow that $E_R(M)$ is an injective left $\Sigma^{-1}R$-module. To prove that is Σ-torsionfree, let e be a non-zero element in $E_R(M)$ such that $ae = 0$, for some $a \in \Sigma$. The injective left R-module $E_R(M)$ is essential over M, so there exist $r \in R$ and $m \in M$ such that $re = m \neq 0$. On the other hand, Σ is a left denominator set, so there are $a_1 \in \Sigma$, $r_1 \in R$ such that $a_1 r = r_1 a$. But then, from $ae = 0$, it follows that $r_1 ae = 0$, so $0 = a_1(re) = a_1 m$. Since M is Σ-torsionfree, it will follow that $m = 0$, which contradicts the choice of $m \neq 0$. Thus, $E_R(M)$ is Σ-torsionfree, and furthermore, by applying the previous proposition, $E_R(M)$ is a injective left $\Sigma^{-1}R$-module. Then $E({}_RM)$ is an essential extension of M, hence of $\Sigma^{-1}M$, and by uniqueness of the injective envelope, $E_R(M) \cong E_{\Sigma^{-1}R}(\Sigma^{-1}M)$.

(ii) By 1.62, any R-endomorphism of E is a $\Sigma^{-1}R$-endomorphism of E. ∎

1.4 Socle Series

The problem of determining the structure of a module is often solved by examining certain components or pieces of the module. One of the basic techniques is to look at its socle, a concept introduced by Krull in the 1920's. The socle of a module—as a semisimple module—has a more transparent structure, and it often plays an important role in the study of arbitrary modules. The process can be refined even further in this direction, by looking at the socle series of the module. With this motivation in mind, we give the definitions and some basic properties of these notions.

Definition 1.65 Let M be a left R-module. The sum of all simple (minimal) submodules of M is called the *socle of* M, and is denoted by $\mathrm{soc}(M)$. If M has no minimal submodules, then $\mathrm{soc}(M) = 0$.

Proposition 1.66 *The socle of any module M is a direct sum of simple submodules of M.*

It follows immediately that the socle of an indecomposable injective module is either 0 or a simple module. We list some of the properties of the socle of a module.

Proposition 1.67 [37] *Let $M \in R$-Mod and $T = \mathrm{End}(M)^{\mathrm{op}}$.*

(i) $\mathrm{soc}(M)$ *is an R-T bimodule.*

(ii) $\mathrm{soc}(\bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathrm{soc}(M_\lambda)$.

(iii) $\mathrm{Jac}(R)\mathrm{soc}(M) = 0$, *or equivalently,* $\mathrm{ann}_R(\mathrm{soc}(M)) \supseteq \mathrm{Jac}(R)$.

Note that if $E = E(M)$ and M is semisimple, then $\mathrm{soc}(E) = M$.

Definition 1.68 The *socle series* of a left module M over a ring R is the ascending chain

$$\mathrm{soc}_0(M) \le \mathrm{soc}_1(M) \le \mathrm{soc}_2(M) \le \dots$$

of submodules of M defined inductively by setting

$$\mathrm{soc}_0(M) = 0 \text{ and } \mathrm{soc}_{n+1}(M)/\mathrm{soc}_n(M) = \mathrm{soc}(M/\mathrm{soc}_n(M))$$

for all integers $n \ge 0$.
In other words, $\mathrm{soc}_{n+1}(M)$ is the full inverse image of $\mathrm{soc}(M/\mathrm{soc}_n(M))$ in M.

In the commutative case, Matlis [27] described an indecomposable injective module over a noetherian ring as the union of its socle series; this suggests that the socle series might be of great help in the task of describing the injectives as structures built up in layers. In this case, $\mathrm{soc}_{n+1}(M)/\mathrm{soc}_n(M)$ has the structure of a finite dimensional vector space over a field. The following are two well-known examples from the commutative case.

Example 1.69 The only indecomposable $\mathbb{Z}$-injectives are $\mathbb{Q}$, and the Prüfer groups $\mathbb{Z}_{p^\infty}$, where p is a prime number and the latter are given by:

$$\mathbb{Z}_{p^\infty} = \bigcup_{n=0}^{\infty} \mathrm{soc}_n(\mathbb{Z}_{p^\infty}), \text{ and } \mathrm{soc}_{n+1}(\mathbb{Z}_{p^\infty})/\mathrm{soc}_n(\mathbb{Z}_{p^\infty}) \cong \mathbb{Z}_p,$$

for all p prime and all $n \geq 0$. Furthermore, $\mathrm{soc}_n(\mathbb{Z}_{p^\infty}) \cong \mathbb{Z}_{p^n}$, for all $n \geq 1$.

Example 1.70 [30]. Consider the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field k. The simple left R-module $R/\left(\sum_{i=1}^{n} Rx_i\right)$ has an indecomposable injective envelope whose description was presented by Northcott in [30].

Let E be the k-vector space with basis $\{x_1^{-m_1} x_2^{-m_2} \ldots x_n^{-m_n} \mid m_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. The vector space E can be endowed with an R-module structure by defining the action of an arbitrary monomial from R on the elements of the basis of E as follows:

$$\left(x_1^{j_1} \ldots x_n^{j_n}\right)\left(x_1^{-m_1} \ldots x_n^{-m_n}\right) = \begin{cases} x_1^{j_1 - m_1} \ldots x_n^{j_n - m_n}, & \text{if } j_i \leq m_i \text{ for all } 1 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

By [30], the module E of "inverse polynomials" is the injective envelope of $R/\left(\sum_{i=1}^{n} Rx_i\right)$, and can be described further as the union of its socle layers: $E = \bigcup_{j=1}^{\infty} \mathrm{soc}_j(E)$.

The previous examples are clear clues that it is natural to study the concept of socle series when trying to understand the structure of an indecomposable injective. There are interesting examples of this kind of construction in a noncommutative setting as well. Such constructions may be found in [20] and [22].

1.5 E-adic Completion. Bicommutator

Consider a ring R with 1, and let E be an injective left module over R.

The class $\mathcal{E}$ of all left R-modules isomorphic to submodules of E^n, for some natural number n, is closed under isomorphic images, submodules and finite direct products, and, associated with this injective module E, we can define the E-adic topology on each R-module M (see [1], [25]).

Proposition 1.71 [E-adic topology and Hausdorff completion, [25]]

*If E is an injective left R-module, then we can introduce a topology on every left R-module M, the E-*adic topology. *A fundamental system of neighborhoods of* 0 *is:*

$$\mathcal{B}_E(M) = \{\ker(f) : f \in \mathrm{Hom}(M, E^n),\ n \in \mathbb{N}\}$$

Let $M_0 = \bigcap\{\ker f : f \in \mathrm{Hom}(M, E^n), n \in \mathbb{N}\}$ *and*

$\widehat{M} = \varprojlim\{\mathrm{Im}(f) : f \in \mathrm{Hom}(M, E^n), n \in \mathbb{N}\} = \varprojlim\{M/M' : M' \in \mathcal{B}_E(M)\}$.

Then:

(i) *R endowed with the E-adic topology described above is a topological ring.*

(ii) *M is a topological R-module.*

(iii) *Every R-homomorphism is continuous in this topology.*

(iv) *M/M_0 is Hausdorff.*

(v) *$\widehat{M}$ is the completion of M/M_0.*

(vi) *$R \to \widehat{R}$ is a continuous ring homomorphism.*

(vii) *$\widehat{M}$ is a topological $\widehat{R}$-module.*

The composite mapping $M \to M/M_0 \to \widehat{M}$ is called the **Hausdorff completion of** M.

In particular, the E-adic topology on R is defined by considering as a basis of open neighborhoods of 0 in R the left ideals of R which are annihilators of finite subsets of E:

$$\mathcal{B}_E(R) = \{\mathrm{ann}_R(F) : F \text{ finite subset of } E\}$$

A left ideal I is open in this topology if and only if it contains an open neighborhood of 0.

Proposition 1.72 [25, (1.4)] *If N is a submodule of M and M has the E-adic topology, then the induced topology on N is also the E-adic topology.*

Proposition 1.73 [25] *If N is a submodule of M and M/N is E-torsionfree (that is,* $\mathrm{Hom}(M/N, E) \neq 0$*), then N is closed in the E-adic topology. In particular, every E-torsionfree module is Hausdorff in the E-adic topology.*

In some cases, the E-adic completion can be explicitly described. In order to present these results, we need to introduce the concepts of dual and double dual of a module [36].

Let R, T be rings, and fix a bimodule ${}_RM_T$. For every left R-module N, the homomorphism group $\mathrm{Hom}_R(N, M)$ can be endowed with a structure of a right T-module as follows: $(f \cdot t)(x) = f(x)t$, for all $f \in \mathrm{Hom}_R(N, M), t \in T, x \in N$. Similarly, for every right T-module L, the homomorphism group $\mathrm{Hom}_T(L, M)$ can be endowed with a structure of a left R-module as follows: $(r \cdot g)(x) = rg(x)$, for all $g \in \mathrm{Hom}_R(L, M), r \in R, x \in L$.

Definition 1.74 1. The right T-module $\mathrm{Hom}_R(N, M) = N^*$ is called the *M-dual of N.*

2. The left R-module $\mathrm{Hom}_T(N^*, M) = N^{**}$ is called the *M-double dual of N.*

Usually special kinds of modules M (such as injective modules, cogenerators) are chosen for duality.

Definition 1.75 The *bicommutator B of M* is defined to be the M-double dual of R:

$$B = R^{**} = \mathrm{Hom}_T(\mathrm{Hom}_R(R, M),\ M_T) \cong \mathrm{Hom}_T(M_T, M_T) = \mathrm{End}_T(M)$$

Proposition 1.76 [1, (2.1)] *The E-adic completion of a left R-module M can be explicitly described as*

$$\widehat{M} = \{h \in M^{**} :\ \forall n \in \mathbb{N},\ \forall f_1, f_2, \ldots, f_n \in M^*,\ \exists a \in M,\ h(f_i) = f_i(a),\ 1 \le i \le n\},$$

where M^* is the E-dual of M. In particular, the E-adic completion of the ring R is the subring $\{b \in B = R^{**} :\ \forall n \in \mathbb{N},\ \forall e_1, e_2, \ldots, e_n \in E,\ \exists r \in R,\ be_i = re_i,\ 1 \le i \le n\}$.

Proposition 1.77 [1, (2.7)] *Let $E, M \in R$-Mod.*
Then $\widehat{M} = M^{**}$ (with respect to E) if and only if M is E-dense.
(M is said to be E-dense if for any $h \in M^{**} = \mathrm{Hom}(\mathrm{Hom}(M, E), E)$ and finitely many $f_1, \ldots, f_n \in M^* = \mathrm{Hom}_R(M, E)$, there exists an element $a \in M$ such that $h(f_i) = f_i(a) \in E$, for all *$i \le n$.)*

Proposition 1.78 *Let E be an injective cogenerator, $M \in R$-Mod, and let $T = \mathrm{End}_R(E)^{\mathrm{op}}$. Then*

(i) $M_0 = \bigcap\{\ker f\ :\ f \in \mathrm{Hom}(M, E^n),\ n \in \mathbb{N}\} = 0$

(ii) *The R-module M is E-dense.* [37](47.6(4))

(iii) *The Hausdorff completion $\widehat{M}$ of M in the E-adic topology is isomorphic to the E-double dual M^{**} of M:*

$$\widehat{M} \cong M^{**} = \mathrm{Hom}_T(\mathrm{Hom}_R(M, E), E).$$

(iv) *The Hausdorff completion $\widehat{R}$ of R in the E-adic topology is isomorphic to the bicommutator B of ${}_RE$, and*

$$\widehat{R} = \varprojlim \{R/I : I \in \mathcal{B}_E(R)\} \cong \mathrm{End}(E_T)$$

Model Theory

The main source for the basic notions from model theory is Prest [33].

1.6 Positive Primitive Formulas. pp-definable Subgroups

Let R be an associative ring with unity and consider the language $\mathcal{L}(R)$ used for the model theory of left R-modules with signature $\mathcal{L} = (+, -, 0, r)_{r\in R}$, where $+$ is a binary symbol, $-$ is a unary function symbol, 0 is a constant and r is a unary function associated with an arbitrary element r of the ring R interpreting the left scalar action of r on a left R-module.

A *positive primitive formula* (pp-formula) $\varphi = \varphi(\mathbf{v})$ is an existentially quantified finite system of linear equations, having the general form: $\exists \mathbf{w}\,(\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_n)$, where all γ_i $(i = 1 \ldots n)$ are equations of the form

$$\sum_{j=1}^{m} r_{ij}v_j + \sum_{k=1}^{l} s_{ik}w_k = 0$$

with r_{ij}, $s_{ik} \in R$ and $\mathbf{w}$ a finite sequence of variables. Equivalently, there are matrices $A = (r_{ij})_{i,j} \in \mathcal{M}_{n\times m}(R)$ and $B = (s_{ik})_{i,k} \in \mathcal{M}_{n\times l}(R)$ such that φ has the form $\exists \mathbf{w}\,(\,A\mathbf{v} + B\mathbf{w} = 0)$. More generally, we call φ positive primitive if it is logically equivalent to a formula of the form given.

If M is a left R-module, then a vector $\mathbf{a} \in M^m$ satisfies $\varphi(\mathbf{v}) = \exists \mathbf{w}(A\mathbf{v} + B\mathbf{w} = 0)$, and we write $M \models \varphi[\mathbf{a}]$ if and only if there exists $\mathbf{b} \in M^l$ such that $A\mathbf{a} + B\mathbf{b} = 0$.

A pp-formula $\varphi(\mathbf{v}) = \exists \mathbf{w}(A\mathbf{v} + B\mathbf{w} = 0)$ defines a subgroup $\varphi[M]$ of M^m:

$$\varphi[M] = \{\, \mathbf{a} \in M^m \mid M \models \varphi[\mathbf{a}]\,\}$$

and all such subgroups will be called *pp-definable subgroups* of M. (We follow the usual model-theoretic convention of not distinguishing between elements and tuples from M.)

Lemma 1.79 *Let $\varphi(\mathbf{v})$ be a pp-formula in one variable, and let $M, N \in R$-Mod.*
If $f \in \mathrm{Hom}(M, N)$, then $f\,[\varphi(M)] \subseteq \varphi[N]$, implying that $f^{-1}\,(\varphi[N)] \supseteq \varphi[M]$.
In particular, the pp-definable subgroups of M are preserved under endomorphisms of M.

Definition 1.80 1. Let $\varphi(\mathbf{v})$ and $\psi(\mathbf{v})$ be two formulae in the same tuple of variables. A partial order on the set of pp-formulas can be defined as follows: $\varphi \leq \psi$ if and only if for every left R-module M there is an inclusion $\varphi[M] \subseteq \psi[M]$ of the corresponding pp-definable subgroups.

Note that this definition makes sense only when both formulae have the same free variables.

2. The pp-definable subgroups of M^n form a modular lattice under the operations $\cap$ and $+$. If $\varphi[M]$ and $\psi[M]$ are two pp-definable subgroups of M^n, then:

$$\varphi[M] \cap \psi[M] = (\varphi \wedge \psi)[M]$$

$$\varphi[M] + \psi[M] = (\varphi + \psi)[M],$$

where $(\varphi + \psi)(\mathbf{v}) := \exists \mathbf{u}\, \exists \mathbf{w}\, (\varphi(\mathbf{u}) \wedge \psi(\mathbf{w}) \wedge (\mathbf{v} = \mathbf{u} + \mathbf{w}))$.

Lemma 1.81 (i) [11] *If R is a left noetherian ring and E is an indecomposable injective left R-module, then all of its pp-definable subgroups (in one variable) have the form*

$$\mathcal{A}_I = \{x \in E \,:\, Ix = 0\},$$

where I is a left ideal of R. That is, a pp-definable subgroup of E is the solution set of a finite system of homogeneous linear equations.

(ii) *If, in addition, R is a principal left ideal domain, then a pp-definable subgroup of ${}_RE$ is the solution set of one homogeneous linear equation $r \cdot v = 0$, i.e., all pp-definable subgroups have the form* $\operatorname{ann}_E(r)$ *for some $r \in R$.*

Definition 1.82 A left R-module M is totally transcendental (tt) if and only if it satisfies the descending chain condition on pp-definable subgroups.

Lemma 1.83 *Every injective module over a left noetherian ring is totally transcendental.*

1.7 Elementary Socle Series

An important role in the description of an injective module will be played by the concept of elementary socle series of a totally transcendental module which was introduced by

Herzog in [16] as the elementary analogue of the socle series of a module. The term 'elementary' is used as an addition to the name of the well-known concept of 'socle series' to indicate that we are using a 'first-order definable' analogue.

In order to present the definition of the elementary socle series of a module, the following definition is needed:

Definition 1.84 A pp-definable subgroup G of a module M is *minimal over a subgroup* X of M if $G \nsubseteq X$ and for any pp-definable subgroup H of M, $H \subsetneq G$ implies that $H \subseteq X$.

Note that the above definition can be given in terms of pp-formulas, interpreting 1.80.

Definition 1.85 [16] Let M be a left module over the ring R. The *elementary socle series* of M is defined by recursion on ordinals α as follows:

$$\begin{aligned}
\mathrm{soc}^0(M) &= 0 \\
\mathrm{soc}^{\alpha+1}(M) &= \mathrm{soc}^\alpha(M) + \sum\{G : G \text{ pp-definable subgroup of } M, \\
&\qquad\qquad G \text{ minimal over } \mathrm{soc}^\alpha(M)\} \\
\mathrm{soc}^\lambda(M) &= \bigcup\{\, \mathrm{soc}^\alpha(M) : \alpha < \lambda\,\} \text{ if } \lambda \text{ is a limit ordinal.}
\end{aligned}$$

Note that if M is a totally transcendental module over a left noetherian ring, then M has the descending chain condition on pp-definable subgroups, so the elementary socle series exhausts M; that is, $\mathrm{soc}^\alpha(M) = M$ for some ordinal α. Furthermore, for any β, $M \neq \mathrm{soc}^\beta(M)$ implies that $\mathrm{soc}^\beta(M) \neq \mathrm{soc}^{\beta+1}(M)$.

Remark 1.86 Note the parallel between the two notions, that of socle series and the elementary socle series, in the sense that while the first one uses the minimality condition on the submodules, for the latter the minimality condition applies to pp-definable subgroups (pp-definable formulas).

Although Herzog [16] did not give a proof of the fact that each term of the elementary socle series of M was a submodule, the result was certainly known to him. Later Kucera and Prest showed that in fact each term of the series was a definably closed submodule, but this result was not published either. The proof yielded a simple characterization of the elementary socle series, which will be needed here. The following proposition is given for the case of indecomposable injective modules over a left noetherian ring, and the proof is based on notes of Kucera.

Let R be a left noetherian ring, ${}_RE$ indecomposable injective, and $T = \mathrm{End}_R(E)^{\mathrm{op}}$. Recall that a typical pp-definable subgroup of E has the form

$$\mathcal{A}_I = \mathrm{ann}_E(I) = \{ e \in E : Ie = 0 \},$$

for some left ideal $I \leq R$.

Remark 1.87 1. If $f \in T$, then $f(\mathcal{A}_I) \subseteq \mathcal{A}_I$, for any left ideal I, and consequently $f(\mathrm{soc}^\alpha(E)) \subseteq \mathrm{soc}^\alpha(E)$, for any α. Thus $\mathcal{A}_I$ and $\mathrm{soc}^\alpha(E)$ are right T-modules.

2. E is indecomposable injective, so the ring T is local with unique maximal ideal $\mathrm{Jac}(T)$, and $\Delta = T/\mathrm{Jac}(T)$ is a division ring. If $\mathcal{A}_I$ is a pp-definable subgroup of E minimal over $\mathrm{soc}^\alpha(E) \neq E$, then $(\mathcal{A}_I + \mathrm{soc}^\alpha(E))/\mathrm{soc}^\alpha(E)$ is a simple T-submodule of $\mathrm{soc}^{\alpha+1}(E)/\mathrm{soc}^\alpha(E)$, and it follows from Lemma 1.31(iii′) that every $f \in \mathrm{Jac}(T)$ will annihilate it. Hence each $\mathrm{soc}^{\alpha+1}(E)/\mathrm{soc}^\alpha(E)$ is a $T/\mathrm{Jac}(T)$-vector space.

Lemma 1.88 *If $f \in T$ is such that* $\mathrm{ann}_R(e) = \mathrm{ann}_R(f(e))$ *for some $0 \neq e \in E$, then f is an automorphism of E, and* $\mathrm{ann}_R(e') = \mathrm{ann}_R(f(e'))$ *for all $e' \in E$.*

Proof. If $\ker(f) \neq 0$, then $\ker(f)$ is essential in E, so there exists $r \in R$ such that $0 \neq re \in \ker(f)$. Then $0 = f(re) = rf(e)$ implies that $r \in \mathrm{ann}_R(f(e)) = \mathrm{ann}_R(e)$, so $re = 0$, which contradicts the choice of r. Thus $\ker(f) = 0$, and by Theorem 1.30, f is an automorphism.

If $e' \in E$, then we have that

$$\mathrm{ann}_R(e') \subseteq \mathrm{ann}_R(f(e')) \subseteq \mathrm{ann}_R(f^{-1}f(e')) = \mathrm{ann}_R(e'),$$

and it follows that $\mathrm{ann}_R(e') = \mathrm{ann}_R(f(e'))$. ∎

Proposition 1.89 *Let R be a left noetherian ring and let ${}_RE$ be indecomposable injective.*

(i) *For each α,* $\mathrm{soc}^\alpha(E)$ *is a submodule of E. Furthermore, if $s \in R$ acts regularly on E then s acts regularly on* $\mathrm{soc}^\alpha(E)$*; that is, for every $e \in \mathrm{soc}^\alpha(E)$ there is a unique $e' \in \mathrm{soc}^\alpha(E)$ such that $se' = e$.*

(ii) *For each α such that* $\mathrm{soc}^{\alpha}(E) \neq E$,

$$\mathrm{soc}^{\alpha+1}(E) = \mathrm{soc}^{\alpha}(E) + \bigcup\{\mathcal{A}_I : \mathcal{A}_I \text{ minimal over } \mathrm{soc}^{\alpha}(E)\}$$

PROOF. The proof of (ii) is a consequence of the proof of (i), which will be proved by induction on α. Clearly (i) holds for $\mathrm{soc}^0(E) = \{0\}$, and if (i) holds for all $\alpha < \lambda$, λ a limit ordinal, then (i) holds for $\mathrm{soc}^{\lambda}(E)$.

Suppose that (i) holds for $\mathrm{soc}^{\alpha}(E)$. By definition, $\mathrm{soc}^{\alpha+1}(E)$ is closed under addition, so for both parts of (i) it suffices to show that if $r, s \in R$ are such that s acts regularly on E and $e \in \mathrm{soc}^{\alpha+1}(E)$ then $se' = re$ implies that $e' \in \mathrm{soc}^{\alpha+1}(E)$.

Since $e \in \mathrm{soc}^{\alpha+1}(E)$, there are $e_0 \in \mathrm{soc}^{\alpha}(E)$ and $e_1, \ldots, e_n \in E$ such that $e = e_0 + e_1 + \ldots + e_n$, where for every $i = 1, \ldots, n$, there is $I_i \leq R$ such that $e_i \in \mathcal{A}_{I_i} \backslash \mathrm{soc}^{\alpha}(E)$, $\mathcal{A}_{I_i}$ minimal over $\mathrm{soc}^{\alpha}(E)$. W.l.o.g., we can assume that $\mathrm{ann}_R(e_i) = I_i$. [If $\mathrm{ann}_R(e_i) = J_i$, then from $e_i \in \mathcal{A}_{I_i}$ it follows that $I_i \leq J_i$, so $\mathcal{A}_{J_i} \subseteq \mathcal{A}_{I_i}$. Since $e_i \in \mathcal{A}_{J_i} \setminus \mathrm{soc}^{\alpha}(E)$, then $\mathcal{A}_{J_i} \not\subseteq \mathrm{soc}^{\alpha}(E)$ and since $\mathcal{A}_i$ is minimal over $\mathrm{soc}^{\alpha}(E)$ it follows that $\mathcal{A}_{J_i} = \mathcal{A}_{I_i}$, where $J = \mathrm{ann}_R(e_i)$.]

If $e' \in \mathrm{soc}^{\alpha}(E)$ then there is nothing to prove, so assume that $e' \notin \mathrm{soc}^{\alpha}(E)$. If we let $I = \mathrm{ann}_R(e')$, then $e' \in \mathcal{A}_I \not\subseteq \mathrm{soc}^{\alpha}(E)$; we will show that $\mathcal{A}_I$ is minimal over $\mathrm{soc}^{\alpha}(E)$.

If $\mathcal{A}_I$ is not minimal over $\mathrm{soc}^{\alpha}(E)$ there exists a left ideal $J \gneq I$ such that $\mathcal{A}_J \subsetneq \mathcal{A}_I$ and $\mathcal{A}_J$ is minimal over $\mathrm{soc}^{\alpha}(E)$. W.l.o.g., we can choose $d \in \mathcal{A}_J \setminus \mathrm{soc}^{\alpha}(E)$ such that $\mathrm{ann}_R(d) = J$. Since $\mathrm{ann}_R(e') = I \lneq J = \mathrm{ann}_R(d)$, by Lemma 1.5, there is $f \in \mathrm{End}(E)$ such that $f(e') = d$. Furthermore, $I_i = \mathrm{ann}_R(e_i) \lneq \mathrm{ann}_R(f(e_i)) = I_i'$ for all $i = 1, \ldots, n$ (otherwise f would be an automorphism, and by 1.88, $\mathrm{ann}_R(e') = \mathrm{ann}_R(d)$, contradicting the choice of d). It follows that $\mathcal{A}_{I_i'} \subsetneq \mathcal{A}_{I_i}$, and by the minimality of $\mathcal{A}_{I_i}$, $\mathcal{A}_{I_i'} \subseteq \mathrm{soc}^{\alpha}(E)$, and in particular, $f(e_i) \in \mathrm{soc}^{\alpha}(E)$, for all $i = 1, \ldots, n$.

Now $se' = re = r(e_0 + e_1 + \ldots + e_n)$, so $sf(e') = r\,(f(e_0) + f(e_1) + \ldots + f(e_n))$. But $f(e_0) \in \mathrm{soc}^{\alpha}(E)$ since $e_0 \in \mathrm{soc}^{\alpha}(E)$; thus by the induction hypothesis $f(e') \in \mathrm{soc}^{\alpha}(E)$. But $f(e') = d \notin \mathrm{soc}^{\alpha}(E)$, a contradiction. Thus $\mathcal{A}_I$ is minimal over $\mathrm{soc}^{\alpha}(E)$, so it follows that $e' \in \mathcal{A}_I \subseteq \mathrm{soc}^{\alpha+1}(E)$. The induction is complete.

The fact that $\mathrm{soc}^{\alpha}(E)$ is a submodule of E is immediate if we take $s = 1$.

If $s \in R$ acts regularly on E and $e \in \mathrm{soc}^{\alpha}(E)$, then by the injectivity of E, there exists a unique $e' \in E$ such that $se = e'$. The proof above showed that $e' \in \mathrm{soc}^{\alpha}(E)$.

(ii) In the the proof for (i), if we consider $s = r = 1$ then $e' = e_0 + e_1 + \ldots + e_n$, and it was shown that if $e' \notin \mathrm{soc}^\alpha(E)$ and $\mathrm{ann}_R(e') = I$, then $\mathcal{A}_I$ is minimal over $\mathrm{soc}^\alpha(E)$. ∎

Example 1.90 Consider a commutative noetherian ring R, and P a prime ideal of R. The indecomposable injective left module $E = E(R/P)$ is tt, and thus the elementary socle series of E exists, and can be described as follows: $\mathrm{soc}^n(E) = \{e \in E \mid P^n e = 0\}$. Note that when P is not a maximal ideal of R, then $\mathrm{soc}_n(E) = 0$, for all $n \geq 0$.

On the other hand, over the localization R_P of R, the socle series and the elementary socle series coincide as R_P-modules, and $E = \mathrm{soc}^\alpha(E)$, for some $\alpha \leq \omega$.

It is worth emphasizing that while the socle series of a module does not always exhaust the module, the elementary socle series of a tt-module always "covers" the module entirely [see Example 1.90]. This property becomes even more powerful in the context of a left noetherian ring when the module considered is indecomposable injective. This is why the use of the elementary socle series of an indecomposable injective over a left noetherian ring is another alternative with a good chance of success at describing the injective module as a union of a chain of submodules.

Chapter 2

Principal Left and Right Ideal Domains

Throughout this chapter, unless otherwise stated, the ring R is a left and right principal ideal domain (PID). Here we study the arithmetic properties of such a ring. The following are examples of PID's (commutative and non-commutative):

(1) The ring of integers.

(2) Any division ring.

(3) The ring of differential operators with rational function coefficients: $k(x)[y, \partial/\partial x]$, where k is a field.

(4) More generally, the (left) skew polynomial ring $K[y; \sigma, \delta]$, where K is a field, $\sigma \in \mathrm{Aut}(K)$, δ a σ-derivation of K.

Recall from Chapter 1 that

$$K[y; \sigma, \delta] = \left\{ \sum_{i=1}^{n} a_i y^i \ : \ n \in \mathbb{N}, a_i \in K \right\},$$

where the multiplication is subject to the rule:

$$ya = \sigma(a)y + \delta(a), \text{ for all } a \in K.$$

Since the coefficients are elements of a field K and σ is a surjection, the elements of $K[y; \sigma, \delta]$ can also be considered as polynomials in y with coefficients written on the right. The ring $K[y; \sigma, \delta]$ is left and right Euclidean, hence it is a PID.

Note that $k(x)[y, \partial/\partial x]$ is a member of this class of skew polynomial rings, where $K = k(x), \sigma = \mathrm{id}_{k(x)}, \delta = \partial/\partial x$.

(5) The skew Laurent polynomial ring $K[y, y^{-1}, \sigma]$, where K is a field and $\sigma \in \mathrm{Aut}(K)$, is left and right Euclidean, hence a PID.

(6) Hurwitz's ring of integral quaternions, a subring of Hamilton's real quaternions $\mathbb{H} = \mathbb{R}1 \bigoplus \mathbb{R}i \bigoplus \mathbb{R}j \bigoplus \mathbb{R}k$. The Hurwitz numbers are of the form

$$(a + bi + cj + dk)/2 ,$$

where $a, b, c, d \in \mathbb{Z}$ are either all even, or all odd. This ring is a left and right Euclidean domain with respect to the norm inherited from $\mathbb{H}$, hence it is a PID.

Additional examples of PIDs can be obtained by diverse algebraic constructions, such as localization.

The work on this chapter and the following one was initially done with the ring $k(x)[y, \partial/\partial x]$ in mind, but later the more general results became obvious. This ring, and more generally, $K[y; \sigma, \delta]$ ($\sigma \in \mathrm{Aut}(K)$, δ a σ-derivation of K), has had a lot of attention during the first half of the last century, Ore being the one that initiated much of the work that was done on non-commutative polynomials.

In the introduction of his paper [31], Ore mentions that "one could have deduced this theory using the theory of moduli studied by Noether and Schmeidler". Instead, Ore preferred to "build up the theory directly, that is, to use only the properties of the polynomials themselves". While most of Ore's research on this subject consists of "brute force" methods, it provided me with inspiration and great insight into the structure of such rings and how they "work". Furthermore, I have taken full advantage of the fact that a PID is left (and right) noetherian, since every left (respectively right) ideal is 1-generated.

The investigation of the factorization theory in a PID is not new (see [18] and [31]), but with the help of a new exposition, it will be heavily used in Chapter 3, where the injective modules over such rings will be studied.

2.1 Divisibility. Similarity

The classical notions introduced here in relation to (left or right) divisibility could be presented in the more general setting of a principal left ideal domain (principal right ideal domain, respectively), or even in an associative ring (domain) with unit, but we will restrict our work to a ring R which is a PID. Most of the results related to divisibility are part of folklore, but when necessary for further development of the work, some of the proofs will be presented.

Definition 2.1 Let R be a domain and $a, d \in R \setminus \{0\}$.
d is called a *right divisor of* a if there exists $u \in R$ such that $a = ud$.
In this case, we also say that a *is right divisible by* d, or that a *is a left multiple of* d.

Remark 2.2 d is a right divisor of a if and only if $Ra \leq Rd$.

Remark 2.3 If a is right divisible by b and b is right divisible by a, then a and b are equal up to left multiplication by units. That is, $a = ub$ for some $u \in \mathrm{U}(R)$.

PROOF. There are $u, v \in R$ such that $a = ub$ and $b = va$; then $a = ub = u(va) = (uv)a$, so $(uv - 1)a = 0$, and hence $uv = 1$. Thus u is right invertible, and since R is a domain, it follows that u is a unit. ∎

In view of the previous remark, the following definition arises naturally:

Definition 2.4 Let $0 \neq a, b \in R$.
An element a is called a *left associate of* b if there exists $u \in \mathrm{U}(R)$ such that $a = ub$.
Define the relation $\doteq$ on R by : $a \doteq b$ if and only if a is a left associate of b.
It is obvious that $a \doteq b$ if and only if $Ra = Rb$, so $\doteq$ is an equivalence relation on R.

The concepts of greatest common right divisor and least common left multiple are introduced by generalizing the corresponding concepts from the commutative case.

Definition 2.5 [18] Let a, b be two non-zero elements of the ring R.

1. We say that $0 \neq d \in R$ is *a greatest common right divisor of a and b* if d is a right divisor of both a and b, and if any common right divisor of a and b is also a right divisor of d.
2. We say that $0 \neq m \in R$ is *a least common left multiple of a and b* if m is a left multiple of both a and b, and if any non-zero common left multiple of a and b is also a left multiple of m.

NOTATION: The set of all greatest common right divisors of a and b is denoted by $(a, b)_*$, and the set of all least common left multiples of a and b is denoted by $[a, b]_*$. The $(*)$ on the right-hand side indicates that right divisibility is considered. ∎

Lemma 2.6 *Let R be a PID, and $a, b \in R$, $ab \neq 0$. Then*

(i) $(a, b)_* \neq \emptyset$. *Furthermore, $d \in (a, b)_*$ if and only if $Rd = Ra + Rb$.*

(ii) $[a, b]_* \neq \emptyset$. *Furthermore, $m \in [a, b]_*$ if and only if $Rm = Ra \cap Rb$.*

PROOF. (i) The left ideal $Ra + Rb$ is principal, so there exists $d \in R$ such that $Rd = Ra + Rb$. Then $a \in Ra \subseteq Rd$ and $b \in Rb \subseteq Rd$, so d is a common right divisor of a and b. At the same time, if d' is another common right divisor of a and b, that is $a, b \in Rd'$, then $d \in Rd = Ra + Rb \subseteq Rd'$, so d' is a right divisor of d. Consequently, $d \in (a, b)_*$.

For the second part of (i), we only need to prove the direct implication. Assume that $Ra + Rb = Rd'$; then $d' \in (a, b)_*$, implying that d' and d are right divisors of each other; thus $Rd = Rd' = Ra + Rb$.

(ii) The left ideal $Ra \cap Rb$ is principal, so there exists $m \in R$ such that $Rm = Ra \cap Rb$. Since $m \in Ra \cap Rb \subseteq Ra, Rb$, then m is a common left multiple of a and b. At the same time, if m' is another common left multiple of a and b, then $m' \in Ra \cap Rb = Rm$, and it follows that m' is a left multiple of m; thus $m \in [a, b]_*$.

In order to prove the direct implication, assume that $Ra \cap Rb = Rm'$, or equivalently, $m' \in [a, b]_*$. Since $m \in [a, b]_*$, it follows that m and m' are left multiples of each other. Consequently, $Rm = Rm' = Ra \cap Rb$. ∎

The following remarks are immediate consequences of the previous results.

Remark 2.7 Let R be a PID, and let $a, b \in R$, $ab \neq 0$.

1. If $d \in (a, b)_*$, then $d' \in (a, b)_*$ if and only if $d \doteq d'$.
   Thus $(a, b)_*$ is the equivalence class of all left associates of d.
2. If $m \in [a, b]_*$, then $m' \in [a, b]_*$ if and only if $m \doteq m'$.
   Thus $[a, b]_*$ is the equivalence class of all left associates of m.

In other words, up to left multiplication by units, a and b have a unique greatest common right divisor and a unique least common left multiple.

Remark 2.8
1. Obviously, $(a, b)_* = (b, a)_*$ and $[a, b]_* = [b, a]_*$, for all $a, b \in R$.
2. If $a \doteq a'$ and $b \doteq b'$, then $(a, b)_* = (a', b')_*$, and $[a, b]_* = [a', b']_*$.
3. If $a \in \mathrm{U}(R)$, and $b \in R$, then $(a, b)_* = \mathrm{U}(R)$, and $b \in [a, b]_*$.
4. If $a = a_1 d$ and $b = b_1 d$, then $(a, b)_* = (a_1, b_1)_* \, d$ and $[a, b]_* = [a_1, b_1]_* \, d$.
5. If $a = a_1 b$, then $b \in (a, b)_*$ and $a \in [a, b]_*$.

Definition 2.9 If $(a, b)_* = \mathrm{U}(R)$, or equivalently $Ra + Rb = R$, then we say that *a and b are right relatively prime.*

Definition 2.10 Let R be a PID, and $a, b \in R$, $ab \neq 0$.
Symmetrically, we define *a greatest common left divisor* and *a least common right multiple* of a and b by considering divisibility from the left. Also, the equivalence relation defined by the *right associates* can be introduced by symmetry.

The set of all greatest common left divisors of a and b is nonempty and will be denoted by ${}_*(a, b)$, while for the set of all least common right multiples of a and b, which is also nonempty, we will use ${}_*[a, b]$.

Example 2.11 Let $R = k(x)[y, \partial/\partial x]$. The elements y and $\frac{1}{x}y$ are left associates, but are not right associates. If $y = \frac{1}{x}y \cdot f$ ($f \in k(x)$), then $xy = yf = fy + f'$, implying that $f = x$ and $f' = 0$: contradiction. Thus y and $\frac{1}{x}y$ are not right associates.

Lemma 2.12 *Let R be a PID, and $a, b \in R$, $ab \neq 0$.*

(i) ${}_*(a, b) \neq \emptyset$. *Furthermore, $d \in {}_*(a, b)$ if and only if $aR + bR = dR$.*

(ii) ${}_*[\,a\,,b\,] \neq \emptyset$. *Furthermore,* $m \in {}_*[\,a\,,b\,]$ *if and only if* $aR \cap bR = mR$.

Thus, up to right multiplication by units (or up to right associates), a *and* b *have a unique greatest common left divisor and a unique least common right multiple.*

All of the properties that were deduced and the ones that will be deduced for notions introduced by considering divisibility from the right are true for divisibility from the left, by the symmetry of the ring R, which is a PID.

Lemma 2.13 *Let* R *be a PID, and* $a, b \in R$, $ab \neq 0$. *Then*

(i) $d \in (\,a\,,b\,)_*$ *if and only if there exist (unique)* $u, v \in R$ *such that* $a = ud$ *and* $b = vd$, *and* u, v *are right relatively prime, i.e.,* $(\,u\,,v\,)_* = \mathrm{U}(R)$.

(ii) $m \in [\,a\,,b\,]_*$ *if and only if there exist (unique)* $a_1, b_1 \in R$ *such that* $m = a_1a = b_1b$, *and* a_1 *and* b_1 *are left relatively prime, i.e.,* ${}_*(\,a_1\,,b_1\,) = \mathrm{U}(R)$.

Proof. (i) Assume $d \in (\,a\,,b\,)_*$. Since $Ra, Rb \subseteq Ra + Rb = Rd$, there are $u, v \in R$ such that $a = ud$ and $b = vd$. If $c \in (\,u\,,v\,)_*$, then there exist $u', v' \in R$ such that $u = u'c$, and $v = v'c$. It follows that $a = u'cd$ and $b = v'cd$, and so

$$Ra + Rb \subseteq Rcd \subseteq Rd = Ra + Rb.$$

Thus $Rcd = Rd$, and consequently, $c \in \mathrm{U}(R)$.

On the other hand, assume that $a = ud$ and $b = vd$, for some $u, v \in R$ such that $(\,u\,,v\,)_* = \mathrm{U}(R)$, or equivalently, $Ru + Rv = R$. Let $c \in (\,a\,,b\,)_*$; then d is a common right divisor of a and b, so by the definition of a greatest common right divisor, $c \in Rd$. Since $R = Ru + Rv$, there exist $p, q \in R$ such that $1 = pu + qv$. Then

$$d = pud + qvd = pa + qb \in Ra + Rb = Rc,$$

and since $c \in Rd$, it follows that $Rd = Rc = Ra + Rb$, so $d \in (\,a\,,b\,)_*$.

(ii) Assume $m \in [\,a\,,b\,]_*$. From $Rm = Ra \cap Rb \subseteq Ra, Rb$ there exist $a_1, b_1 \in R$ such that $m = a_1a = b_1b$. If $d \in {}_*(\,a_1\,,b_1\,)$, then $a_1 = da_2$ and $b_1 = db_2$ for some $a_2, b_2 \in R$; from $m = da_2a = db_2b = dm'$ it follows that $a_2a = b_2b = m' \in Ra \cap Rb$. Then $Rm \subseteq Rm' \subseteq Ra \cap Rb = Rm$, so $m' \doteq m = dm'$, implying that $d \in \mathrm{U}(R)$.

On the other hand, assume that $m = a_1 b = b_1 a$, with ${}_*(a_1, b_1) = \mathrm{U}(R)$, and let $m' \in [a, b]_*$. Since m is a common left multiple of a and b, m is also a left multiple of m', and there is $u \in R$ such that $m = um'$.

Also, by the direct implication, since $m' \in [a, b]_*$, there exist $a_2, b_2 \in R$ such that $m' = a_2 b = b_2 a$ and ${}_*(a_2, b_2) = \mathrm{U}(R)$. We have the following implications:

$$m = um' \Rightarrow \begin{cases} a_1 b = u a_2 b \\ b_1 a = u b_2 a \end{cases} \Rightarrow \begin{cases} a_1 = u a_2 \\ b_1 = u b_2 \end{cases} \Rightarrow \begin{cases} a_1 R \subseteq uR \\ b_1 R \subseteq uR \end{cases}$$

It follows that $R = a_1 R + b_1 R \subseteq uR \subseteq R$, so $u \in U(R)$. Thus $m \doteq m' \in [a, b]_*$.

For both parts, the uniqueness follows from the fact that R is a domain. ∎

The following theorem summarizes all the equivalent definitions for the greatest common right/left divisor and least common left/right multiple of two elements of a PID.

Theorem 2.14 *Let R be a PID, and let $a, b \in R$, $ab \neq 0$.*

I. For $0 \neq d \in R$, the following are equivalent:

1. $d \in (a, b)_*$
2. $Rd = Ra + Rb$
3. *There are unique $u, v \in R$ such that $a = ud$, $b = vd$, and $(u, v)_* = \mathrm{U}(R)$.*

II. For $0 \neq m \in R$, the following are equivalent:

1. $m \in [a, b]_*$
2. $Rm = Ra \cap Rb$
3. *There are unique $a_1, b_1 \in R$ such that $m = b_1 a = a_1 b$, and ${}_*(a_1, b_1) = \mathrm{U}(R)$.*

I'. For $0 \neq d' \in R$, the following are equivalent:

1. $d' \in {}_*(a, b)$
2. $d'R = aR + bR$
3. *There are unique $u', v' \in R$ such that $a = d'u'$, $b = d'v'$, and ${}_*(u', v') = \mathrm{U}(R)$.*

II'. For $0 \neq m' \in R$, the following are equivalent:

1. $m' \in {}_*[a, b]$
2. $m'R = aR \cap bR$
3. *There are unique $a_2, b_2 \in R$ such that $m' = ab_2 = ba_2$, and $(a_2, b_2)_* = \mathrm{U}(R)$.*

Example 2.15 The ring of differential operators with rational function coefficients over a field k, $k(x)[y, \partial/\partial x]$, is a principal ideal domain. Here the computations may be made on fairly familiar grounds, since this is an algebra of formal differential operators on $k(x)$, and a great amount of research related to these objects has been done in the field of differential equations.

In particular, least common left multiples are an important tool in my analysis of injective modules, and general examples such as shown here are useful in understanding what is true—and not true—about the types of computations needed to be carried out.

Let $R = k(x)[y, \partial/\partial x]$, and let f, g, h be non-zero elements of $k(x)$.

1. $(\, y + f\, , h\,)_* = \mathrm{U}(R)$ and $h(y+f) = \left(y + f - \frac{h'}{h}\right) h \in [\, y + f\, , h\,]_*$.
   Trivially, $y + f \in [\, y + f\, , h\,]_*$.
2. $(\, y^2 + fy + g\, , h\,)_* = \mathrm{U}(R)$ and
   $h(y^2+fy+g) = \left[y^2 + (f - 2\frac{h'}{h})y + g - fh' + h'' - \frac{(h')^2}{h}\right] h \in [\, y^2 + fy + g\, , h\,]_*$.
   Trivially, $y^2 + fy + g \in [\, y^2 + fy + g\, , h\,]_*$.
3. If $f \neq g$, then $(\, y + f\, , y + g\,)_* = \mathrm{U}(R)$ and
   $\left(y + g - \frac{f'-g'}{f-g}\right)(y + f) = \left(y + f - \frac{g'-f'}{g-f}\right)(y + g) \in [\, y + f\, , y + g\,]_*$.
4. If $(\, y^2 + fy + g\, , y + h\,)_* = \mathrm{U}(R)$, then
   $\left(y - \frac{a'}{a} + h\right)(y^2 + fy + g) = \left((y - \frac{a'}{a} + h)(y + f - h) + a\right)(y + h)$ is a least common left multiple of $y^2 + fy + g$ and $y + h$, where $0 \neq a = g - h' - fh + h^2$.
5. $\left(y - \frac{n}{x}\right)(y^2 + x^n) = \left(y^2 - \frac{n}{x}y + x^n\right) y \in [\, y^2 + x^n\, , y\,]_*$, for $n \in \mathbb{N}$.
   $\left(y - \frac{n}{x}\right)(y^2 + x^n) = \left(y^2 - \frac{n}{x}y + x^n\right) y \in {}_*[\, y - \frac{n}{x}\, , y^2 - \frac{n}{x}y + x^n\,]$
6. $y - 1/x \in (\, y^2\, , xy - 1\,)_*$
7. $(\, y\, , y^2 + x\,)_* = {}_*(\, y\, , y^2 + x\,) = \mathrm{U}(R)$
8. $y \in \left(\, (y^2 + x)\, y\, , \left(y + \frac{1}{x}\right) y\, \right)_* \neq \mathrm{U}(R)$, but ${}_*\left(\, (y^2 + x)\, y\, , \left(y + \frac{1}{x}\right) y\, \right) = \mathrm{U}(R)$.

In a commutative domain, when we study the multiplicative structure of the domain, for all practical purposes two elements are taken to be equal if they are unit multiples of each other. In a non-commutative domain, the situation is much more awkward, and the relation "$a \sim b$ if and only if $au = vb$ for some units u, v" does not capture the full character of the multiplicative theory in a way that allows a good formulation of unique factorization theorems.

The proper relation is a little more complicated to formulate, but it was recognized very early that a more relaxed condition on the multipliers u and v was essential for a good theory. We introduce next the relation of *similarity* between two elements of a PID. In my presentation this concept will be used mainly in the arithmetic sense, but since in most of the newer references it is presented by using cyclic modules, I will formulate this definition by considering equivalent conditions (see [31], [18], [5]).

Definition 2.16 Let R be a PID, and $a, b \in R \setminus \{0\}$. Then $b \in R$ is said to be *right similar to* a if one of the following equivalent conditions are satisfied:

(i) There exist $0 \neq u, v \in R$ with $(u, b)_* = \mathrm{U}(R)$ such that $au = vb \in [u, b]_*$.

(ii) $R/Ra \cong R/Rb$

(iii) $R/aR \cong R/bR$

In this case, from the characterization of $[\ ,\ ]_*$ we also have $_*(a, v) = \mathrm{U}(R)$.

We will show next that the conditions in Definition 2.16 are equivalent.

PROOF. If a is a unit, then any of the conditions (ii) or (iii) will imply that $b \in \mathrm{U}(R)$. If $a \in \mathrm{U}(R)$ and (i) is satisfied, then $b \in Rb \leq Rb + Ra^{-1}vb = Rb + Ru = R$, since $(u, b)_* = \mathrm{U}(R)$. Thus (i) also implies that $b \in \mathrm{U}(R)$ whenever $a \in \mathrm{U}(R)$. So in this case, a is a unit if and only if b is a unit, and the equivalences are trivial.

We will prove the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) for $a, b \in R \setminus \mathrm{U}(R)$.

(i) $\Rightarrow$ (ii): Assume that there exist $u, v \in R$ such that

$$au = vb \in [u, b]_*\,, \text{ where } (u, b)_* = \mathrm{U}(R).$$

Let $\Psi : R \to R/Rb$ be the R-homomorphism defined by $\Psi(1) = u + Rb$. Then $\Psi \neq 0$ because otherwise $u \in Rb$ would imply that $b \in (u, b)_* = \mathrm{U}(R)$, contradicting the assumption $b \in R \setminus \mathrm{U}(R)$.

Let $\ker(\Psi) = Ra_1$, for some $a_1 \in R$; then $\Psi(a) = a\Psi(1) = au + Rb = Rb$, so $a \in \ker(\Psi)$. At the same time, $Rb = \Psi(a_1) = a_1\Psi(1) = a_1u + Rb$, hence $a_1u \in Rb$. From $au \in [u, b]_*$ we have the following implications:

$$a_1u \in Ru \cap Rb = Rau \Rightarrow Ra_1 \subseteq Ra \subseteq \ker(\Psi) = Ra_1 \Rightarrow Ra = Ra_1 = \ker(\Psi)\,.$$

To prove that Ψ is onto, consider $c + Rb \in R/Rb$. Since $(u, b)_* \doteq 1$, or equivalently $Ru + Rb = R$, there exist $p, q \in R$ such that $1 = pu + qb$. Then $c = cpu + cqb$ and it follows that $c + Rb = cpu + Rb$, so Ψ is onto, since $\Psi(cp) = cp\Psi(1) = cpu + Rb = c + Rb$. Consequently, the induced morphism $\Psi : R/Ra \to R/Rb$ is an isomorphism.

(ii) $\Rightarrow$ (i): Let $\Psi : R/Ra \to R/Rb$ be an isomorphism and denote by $u + Rb$ the image under Ψ of $1 + Ra$, i.e. $\Psi(1 + Ra) = u + Rb \neq 0$. It follows that

$$0 = \Psi(a + Ra) = a\Psi(1 + Ra) = au + Rb,$$

so $au = vb$ for some $v \in R$. At the same time, the coset $1 + Rb$ is the image of some non-zero element in R/Ra, $\Psi(c + Ra) = 1 + Rb$. Thus

$$\Psi(c + Ra) = c\Psi(1 + Ra) = cu + Rb,$$

and therefore there exists $q \in R$ such that $cu + qb = 1$, which indicates that $Ru + Rb = R$, or equivalently, $(u, b)_* = \mathrm{U}(R)$. Since $au = vb$ is right divisible by both u and b, it is right divisible by any of their least common left multiples $m = a_1 u = v_1 b$, where ${}_*(a_1, v_1) = \mathrm{U}(R)$. Then $au = vb = \alpha m = \alpha a_1 u$ for some $\alpha \in R$, so $a = \alpha a_1 \in Ra_1$.

On the other hand,

$$\Psi(a_1 + Ra) = a_1\Psi(1 + Ra) = a_1 u + Rb = v_1 b + Rb = 0,$$

which means that $a_1 \in Ra$. Since $a \in Ra_1$, it follows that $a \doteq a_1$, and so $\alpha \in \mathrm{U}(R)$. Thus $vb = au = \alpha m \in [u, b]_*$, with $(u, b)_* = \mathrm{U}(R)$, so (i) is satisfied.

(ii) $\Rightarrow$ (iii): From (i) $\Leftrightarrow$ (ii), there exist $u, v \in R$ such that $au = vb \in [u, b]_*$, with $(u, b)_* = \mathrm{U}(R)$. In addition, by 2.14(II.3), we also have that ${}_*(a, v) = \mathrm{U}(R)$.

Let $\Phi : R \to R/aR$ be the R-homomorphism defined by $\Phi(1) = v + aR$, which is non-zero since ${}_*(a, v) = \mathrm{U}(R)$ and a is assumed to be a non-unit. Let $\ker(\Phi) = b_1 R$, for some $b_1 \in R$; then $b \in \ker(\Phi) = b_1 R$, since

$$\Phi(b) = \Phi(1)b = vb + aR = au + aR = aR.$$

Also, $aR = \Phi(b_1) = \Phi(1)b_1 = vb_1 + aR$, so $vb_1 \in aR$, and it follows that $vb_1 \in aR \cap vR$. By Theorem 2.14, $au = vb \in {}_*[a, v]$. Thus $vb_1 \in aR \cap vR = vbR$, so $b_1 \in bR$, proving that $bR = b_1 R = \ker(\Phi)$.

To prove that Φ is onto, consider $d + aR \in R/aR$. Since $_*(a, v) = \mathrm{U}(R)$, or equivalently $aR + vR = R$, there exist $p, q \in R$ such that $1 = ap + vq$; then $d = apd + vqd$, implying that $d + aR = vqd + aR$. Thus $\Phi(qd) = \Phi(1)qd = vqd + aR = d + aR$, so Φ is onto, and the induced homomorphism $\Phi : R/aR \to R/bR$ is an isomorphism.

The remaining implication (iii) $\Rightarrow$ (ii) follows by symmetry. ∎

I would like to remark that the equivalence (ii) $\Leftrightarrow$ (iii) is true for any domain R, a result that can be found in [12].

As a consequence of the previous result, the notion of similarity is left-right symmetric, and we will omit the reference to left or right, and simply refer to similarity.

Corollary 2.17 *Let a, b be two distinct non-zero elements of R. Then*

(i) *a is right similar to b if and only if a is left similar to b.*

(ii) *Similarity is an equivalence relation on $R \setminus \{0\}$, and will be denoted by $\sim$. The similarity class of an element $0 \neq a \in R$ will be denoted by $\widehat{a}$.*

Remark 2.18 Let $0 \neq a \in R$.

1. Any non-zero left/right associate of a is similar to a. Thus all left and right associates of a are in $\widehat{a}$.
2. If a is a unit in R, then $\widehat{a} = \mathrm{U}(R)$.

The next example shows that the similarity relation defined in a PID has additional properties in the case when the ring is Euclidean relative to a degree function.

Example 2.19 Let $R = K[y; \sigma, \delta]$, where K a field, $\sigma \in \mathrm{Aut}(K)$ and δ a σ-derivation. As usual, the degree of $r = \sum_{i=0}^{n} a_i y^i$ is defined to be the highest power of y that occurs with a non-zero coefficient: $\deg(r) = \max\{i \mid a_i \neq 0\}$. Furthermore, from [31], for any $0 \neq a, b \in K$,

$$\deg([a, b]_*) + \deg((a, b)_*) = \deg(a) + \deg(b) = \deg(_*[a, b]) + \deg(_*(a, b))$$

Then for all $0 \neq r_1, r_2 \in R$, if $r_1 \sim r_2$, then $\deg(r_1) = \deg(r_2)$.

PROOF. If $r_1 \sim r_2$, there are $s_1, s_2 \in R$, $(r_1, s_1)_* \doteq 1$ such that $s_2 r_1 = r_2 s_1 \doteq [r_1, s_1]_*$. We have the following equalities:

$$\deg([r_1, s_1]_*) = \deg(r_2 s_1) = \deg(r_2) + \deg(s_1)$$

and

$$\deg([r_1, s_1]_*) + \deg((r_1, s_1)_*) = \deg(r_1) + \deg(s_1).$$

But $(r_1, s_1)_* \doteq 1$, so $\deg((r_1, s_1)_*) = 0$, and it follows easily from the above equalities that $\deg(r_1) = \deg(r_2)$. ∎

Example 2.20 Let $R = k(x)[y, \partial/\partial x]$.

1. The ring R is in the class of rings described in 2.19, so the similarity class of $y \in R$ will contain only non-zero elements of degree 1 in y. Furthermore, by Proposition 4.4 in [28], if $f, g \in k(x)$, then $R/R(y+f) \cong R/R(y+g)$ if and only if there is $0 \neq h \in k(x)$ such that $f - g = \frac{h'}{h}$.
   Thus, the similarity class of y is $\left\{g_1 \left(y - \frac{f'}{f}\right) g_2 : \ 0 \neq f, g_1, g_2 \in k(x)\right\}$.
   Note that the similarity class of y contains all right, left, and right and left associates of y, since $fy\frac{1}{f} = y - \frac{f'}{f}$ for $f \neq 0$, so $g_1 \left(y - \frac{f'}{f}\right) g_2 = g_1 f y \frac{g_2}{f}$.
   Also, note that although $y - \frac{f'}{f}$ and y are left-and-right associates, $y - \frac{f'}{f}$ is neither a left multiple of y, nor a right multiple of y.
2. The similarity class of $(y+h)$ is $\left\{g_1 \left(y + h - \frac{f'}{f}\right) g_2 : 0 \neq f, g_1, g_2 \in k(x)\right\}$.
3. Since $\left(y - \frac{1}{x}\right)(y^2 + x) = \left(y^2 - \frac{1}{x}y + x\right) y \in [y^2 + x, y]_*$ and $(y^2 + x, y)_* = \mathrm{U}(R)$, then $y^2 + x \sim y^2 - \frac{1}{x}y + x$.
4. More generally, for all $f, g \in k(x)$ and $0 \neq h \in k(x)$,
   $$h\left[y^2 + (f - \tfrac{2h'}{h})y + \left(g - fh' + h'' - \tfrac{(h')^2}{h}\right)\right] = (y^2+fy+g)h \in [y^2 + fy + g, h]_*,$$
   so it follows that $\left[y^2 + (f - \frac{2h'}{h})y + \left(g - fh' + h'' - \frac{(h')^2}{h}\right)\right] \sim (y^2 + fy + g)$.

We present next the necessary conditions for a least common left multiple to be a least common right multiple.

Consider $0 \neq a, b \in R$ such that $(a, b)_* = \mathrm{U}(R)$. If $m \in [a, b]_*$, there exist $a_1, b_1 \in R$ such that $m = b_1 a = a_1 b$, and ${}_*(a_1, b_1) = \mathrm{U}(R)$. Then $m \in {}_*[a_1, b_1]$, by Theorem 2.14(II$'$.3).

Lemma 2.21 *Let $a, b \in R$, $ab \neq 0$.*

(i) *If $(a,b)_* = \mathrm{U}(R)$ and $m = b_1 a = a_1 b \in [a,b]_*$, then $m \in {}_*[a_1, b_1]$. Furthermore, a_1 is right similar to a and b_1 is right similar to b.*

(ii) *If ${}_*(a,b) = \mathrm{U}(R)$ and $m' = ab_2 = ba_2 \in {}_*[a,b]$, then $m' \in [a_2, b_2]_*$. Furthermore, a_2 is left similar to a and b_2 is left similar to b.*

PROOF. The proofs are easy consequences of Theorem 2.14. ∎

Example 2.22 If $R = k(x)[y, \partial/\partial x]$, then $y^2 = yy = (y + 1/x)(y - 1/x)$. Thus,

$$y^2 \in [y, y - 1/x]_* \text{ and } y^2 \in {}_*[y, y + 1/x]$$

Remark 2.23 Recall that $\doteq$ means equality up to left associates:

$$a \doteq b \text{ if and only if } a = ub \text{ for some unit } u.$$

We have seen that $[a,b]_*$ is defined only up to left associates, that is, if $m \in [a,b]_*$, then $m' \in [a,b]_*$ iff $m' \doteq m$, and similarly for $(a,b)_*$. Rather than dealing with $[a,b]_*$ and $(a,b)_*$ as sets of elements, it is more practical to write equations in terms of the equivalence relation $\doteq$:

$$m \doteq [a,b]_* \text{ and } d \doteq (a,b)_* \text{ instead of } m \in [a,b]_* \text{ and } d \in (a,b)_*,$$

and similarly for more complicated equations.

The following concept is introduced for computational purposes, and will provide great help in dealing with products of similar elements.

Definition 2.24 [31] Let $a, b \in R$, $ab \neq 0$.
An element a_1 *is called a (right) transform of a by b* if $[a,b]_* \doteq a_1 b$.

If a_2 is a left associate of the transform a_1 of a by b, then $a_2 b \doteq a_1 b \doteq [a,b]_*$, so the following lemma is obvious:

Lemma 2.25 *Every left associate of a transform of a by b is also a transform of a by b. That is, the right transform of a by b is unique up to left multiplication by units.*

NOTATION: In view of the previous lemma, denote by ${}^b a$ any right transform of a by b. In this case, we write: $[a,b]_* \doteq {}^b a\, b \doteq {}^a b\, a$. ∎

Lemma 2.26 (i) *The transform of any $a \in \mathrm{U}(R)$ by any $b \in R$ is also a unit, and furthermore ${}^{b}a \doteq a$ and ${}^{a}b \doteq ba^{-1}$, where a^{-1} is the inverse of $a \in \mathrm{U}(R)$.*

(ii) *If $a \in \mathrm{U}(R)$ and $b \in R$, then ${}^{a}(ba^{-1}) \doteq b$ and ${}^{(a^{-1})}b \doteq ba$.*

PROOF. By Remark 2.8, $[\,a\,,b\,]_* \doteq b \doteq (ba^{-1})a \doteq ab$, so a is a transform of a by b. ∎

Example 2.27 Let $R = k(x)[y, \partial/\partial x]$, and let $f, g, h \in k(x)$.

1. ${}^{g}(y+f) = y + f - \frac{g'}{g}$, where $g \neq 0$.
2. ${}^{y}(y^2 + 1) = y^2 + 1$ and ${}^{(y^2+1)}y = y$.
3. ${}^{y}(y^2 + x) = y^2 - \frac{1}{x}y + x$

Proposition 2.28 *Let $0 \neq a, a_1 \in R$.*
$a \sim a_1$ if and only if there exists $b \in R$ such that $a_1 \doteq {}^{b}a$ and $(\,a\,,b\,)_* \doteq 1$.

PROOF. The result is a direct consequence of Definition 2.16 and Lemma 2.21. ∎

Example 2.29 Let $R = k(x)[y, \partial/\partial x]$, and let $f, g, h \in k(x)$.

1. $y \sim y - \frac{f'}{f} = {}^{f}y$
2. $y^2 + x \sim y^2 - \frac{1}{x}y + x = {}^{y}(y^2 + x)$, where $(\,y^2 + x\,, y\,)_* = \mathrm{U}(R)$.
   Furthermore, $y^2 - \frac{1}{x}y + x \neq {}^{h}(y^2 + x)$, for all $0 \neq h \in k(x)$. This is an example of two similar elements which are not right associates, left associates, or right and left associates, indicating that the set inclusion mentioned in 2.18(i) is in general strict.

Proposition 2.30 [31]

(i) *If $a + Rc = b + Rc$, then ${}^{a}c \doteq {}^{b}c$.*

(ii) *If $a = a_1 d$ and $b = b_1 d$, for some a_1 and b_1, then ${}^{a}b \doteq {}^{a_1}b_1$.*

(iii) *If the product ab is right divisible by c, then a is right divisible by ${}^{b}c$.*

(iv) *For any a, b, $c \in R$, ${}^{cb}a \doteq {}^{c}({}^{b}a)$.*

(v) *The transform of the least common left multiple is equal to the least common left multiple of the transforms of the components:*

$$ {}^{c}[a, b]_* \doteq [\,{}^{c}a, {}^{c}b\,]_* $$

(vi) *If* $a, b, c \in R \setminus \{0\}$, *then*

$$^{c}(ab) \doteq {}^{(^{b}c)}a \cdot {}^{c}b$$

Furthermore, if $(ab, c)_* \doteq 1$, *then the factors of* $^{c}(ab)$ *are similar to the corresponding factors of ab; that is,* $^{(^{b}c)}a \sim a$ *and* $^{c}b \sim b$.

Moreover, for any $n \geq 1$, *if* $(a_1 \dots a_n, c)_* \doteq 1$, *then the factors of the transform of the product* $a_1 \dots a_n$ *by c are similar to the corresponding factors of the original product, that is:*

$$^{c}(a_1 a_2 \dots a_n) \doteq b_1 b_2 \dots b_n \text{ and } a_i \sim b_i \text{ , for all } i = 1, 2, \dots n.$$

PROOF. (i) If $a + Rc = b + Rc$, then $a = b + uc$, for some $u \in R$. From $[a, c]_* \doteq {}^{a}c\, a \doteq {}^{a}c(b + uc) \doteq {}^{a}c\, b + {}^{a}c\, uc$ it follows that $^{a}c\, b$ is a common left multiple of c and b, and therefore it is a left multiple of $[b, c]_* \doteq {}^{b}c\, b$, thus $^{a}c \in R\,{}^{b}c$. Similarly, $^{b}c \in R\,{}^{a}c$, and the conclusion $^{a}c \doteq {}^{b}c$ follows.

(ii) This property is a direct consequence of Remark 2.8(4).

(iii) The product ab is right-hand divisible by both b and c, and therefore ab is a left multiple of $[b, c]_*$, so $ab \doteq u\,[b, c]_* \doteq u\,{}^{b}c\, b$, which yields $a \doteq u\,{}^{b}c$.

(iv) Since $[a, bc]_* \doteq [a, c, bc]_* \doteq [[a, c]_*, bc]_* \doteq [{}^{c}a\, c, bc]_* \doteq [{}^{c}a, b]_* c$, it follows that $^{(bc)}a(bc) \doteq {}^{b}({}^{c}a)\, b\, c$, implying that $^{(bc)}a \doteq {}^{b}({}^{c}a)$.

(v) $^{c}[a, b]_* c \doteq [[a, b]_*, c]_* \doteq [[a, c]_*, [b, c]_*]_* \doteq [{}^{c}a\ c, {}^{c}b\ c]_* \doteq [{}^{c}a, {}^{c}b]_* c$, and after cancellation on the right by c the required equality is proved.

(vi) $^{c}(ab)\ c \doteq [ab, c]_* \doteq [[ab, b]_*, c]_* \doteq [ab, [b, c]_*]_* \doteq [ab, {}^{b}c\, b]_*$
$\doteq [a, {}^{b}c]_*\ b \doteq {}^{(^{b}c)}a\ {}^{b}c\ b \doteq {}^{(^{b}c)}a\ [b, c]_* \doteq {}^{(^{b}c)}a\ {}^{c}b\ c$

It follows that $^{c}(ab) \doteq {}^{(^{b}c)}a\ {}^{c}b$.

When $(ab, c)_* \doteq 1$, then $(b, c)_* \doteq 1$ and we have that $^{c}b \sim b$.

Let $(a, {}^{b}c)_* \doteq d$, or equivalently, $Rd = Ra + R\,{}^{b}c$. Since the lattice of left ideals of R is modular, it follows from $Rab \leq Rb$ that $Rab + (Rb \cap Rc) = Rb \cap (Rab + Rc)$, and we have the following equalities:

$$Rdb = Rab + R\,{}^{b}c\, b = Rab + R\,[b, c]_* = Rab + (Rb \cap Rc) = Rb \cap (Rab + Rc)$$

From $(ab, c)_* \doteq 1$ it follows that $Rab + Rc = R$, so $Rdb = Rb + R = Rb$, implying that $b = udb$ for some $u \in R$, and it follows that $ud = 1$, hence $d \in \mathrm{U}(R)$. Thus $(a, {}^{b}c)_* \doteq 1$, and by Proposition 2.28, ${}^{({}^{b}c)}a \sim a$.

The proof of the second part of (vi) is an easy induction on n. ∎

Example 2.31 Let $R = K[y; \sigma, \delta]$, where K a field, $\sigma \in \mathrm{Aut}(K)$ and δ a σ-derivation. For all $0 \neq r_1, r_2 \in R$, if $r_1 \sim r_2$, then there exists $a \in R$ with $\deg(a) < \deg(r_1) = \deg(r_2)$ such that $r_2 = {}^{a}r_1$, and $(a, r_1)_* = 1$.

Proof. If $r_1 \sim r_2$, there is $s \in R$, $(r_1, s)_* = 1$ such that $r_2 = {}^{s}r_1$. Furthermore, if we divide s on the left by r_1, there is $0 \neq a \in R$ with $\deg(a) < \deg(r_1)$ such that $s + Rr_1 = a + Rr_1$, and by 2.30(i), ${}^{s}r_1 = {}^{a}r_1 = r_2$. ∎

2.2 Factorization and Decomposition Theorems

In this section we study various arithmetic properties of the elements of a PID, results which will be used later in systematically describing the socle series of an indecomposable injective. The results in this section can be found in [18], or they are generalizations for a PID from [31]. Some of these results also appear in [5], where the factorization theory was developed for unique factorization domains.

Definition 2.32 A non-unit $0 \neq a \in R$ is said to be *prime* if whenever $a = bc$ then either $b \in \mathrm{U}(R)$ or $c \in \mathrm{U}(R)$.

The standard terminology would be to call the elements satisfying the conditions of Definition 2.32 "irreducible" rather the "prime". Since in a PID the two concepts are equivalent, the term "prime" element is used in this thesis to avoid confusion when the "irreducible" left ideals will come into picture, since such an ideal is not necessarily generated by an element satisfying 2.32.

Remark 2.33 (i) If p is a prime element in R and $0 \neq c \in R$, then $(c, p)_* \doteq 1$ or $(c, p)_* \doteq p$.

(ii) An element $a \in R$ is prime if and only if the left ideal Ra is maximal, if and only if the right ideal aR is maximal.

Thus, every simple left module of the ring R is isomorphic to a left quotient module R/Ra, for some prime element $a \in R$.

Lemma 2.34 (i) *A left (right) associate of a prime element is also prime.*

(ii) *A nonzero element similar to a prime element is also prime.*

(iii) *A transform of a prime element is a prime element or a unit.*

PROOF. (i) This is a special case of (ii).

(ii) This is immediate from Definition 2.16 and the characterization of prime elements via maximal left ideals.

(iii) Let p be prime and $0 \neq c \in R$.
If $(c, p)_* \doteq 1$, then ${}^{c}p$ is prime by (ii).
If $(c, p)_* \neq 1$, then $(c, p)_* \doteq p$, so $c \doteq [c, p]_* \doteq {}^{c}p\, c$, and consequently ${}^{c}p \doteq 1$. ∎

Lemma 2.35 *Let $a = p_1 \dots p_n$ be a product of a finite number of prime factors $p_1, \dots, p_n$. If a is right divisible by a prime element q, there exists $i \in \{1, \dots, n\}$ such that $p_i \sim q$. Furthermore, the index i can be chosen such that:*
$p_i \doteq {}^{(p_{i+1} \cdots p_n)}q$ *and* $a \doteq p_1 \dots p_{i-1}\, \overline{p}_{i+1} \cdots \overline{p}_n\, q$, *where* $\overline{p}_j \sim p_j$ $(j \geq i+1)$.

Similar results hold for left divisibility.

PROOF. Let $i \in \{1, \dots, n\}$ be maximal such that the prime q is a right divisor of $p_i \dots p_n$.

If $i = n$, then $p_n \doteq q$, and the rest of the conclusion follows trivially.

Assume now that $i < n$. Then, by the maximality of i, $(p_{i+1} \cdots p_n, q)_* \doteq 1$.
The product $p_i p_{i+1} \dots p_n$ is a common left multiple of q (by the choice of i), and of $p_{i+1} \cdots p_n$, hence it is a multiple of their least common left multiple; thus there is $0 \neq u \in R$ such that $p_i p_{i+1} \dots p_n \doteq u \cdot [p_{i+1} \dots p_n, q]_* \doteq u \cdot \left({}^{(p_{i+1} \cdots p_n)}q\right) p_{i+1} \dots p_n$ and so $p_i \doteq u \cdot \left({}^{(p_{i+1} \cdots p_n)}q\right)$. Then $u \in \mathrm{U}(R)$, since both p_i and ${}^{(p_{i+1} \cdots p_n)}q$ are prime, so $p_i \doteq {}^{(p_{i+1} \cdots p_n)}q$, and it follows that

$$p_i \cdots p_n \doteq \left({}^{(p_{i+1} \cdots p_n)}q\right) p_{i+1} \dots p_n \doteq [(p_{i+1} \dots p_n), q]_* \doteq {}^{q}(p_{i+1} \dots p_n)\, q \doteq \overline{p}_{i+1} \cdots \overline{p}_n\, q,$$

where $\overline{p}_j \sim p_j$ for all $j = i+1, \ldots, n$, by 2.30(vi). Then

$$\begin{aligned} a &= p_1 \ldots p_{i-1} p_i p_{i+1} \ldots p_n \\ &\doteq p_1 \ldots p_{i-1} \left({}^{(p_{i+1} \ldots p_n)} q \right) p_{i+1} \ldots p_n \\ &\doteq p_1 \ldots p_{i-1} \overline{p}_{i+1} \ldots \overline{p}_n \, q. \end{aligned}$$

∎

Lemma 2.36 *Every non-zero element $a \in R \backslash \mathrm{U}(R)$ has a prime right divisor and a prime left divisor.*

Proof. The ring R is left noetherian, and since $Ra \neq R$, Ra is contained in a maximal left ideal Rp. Then p is prime, and a right divisor of a.

Symmetrically, a has a prime left divisor. ∎

Theorem 2.37 [Prime Factorization Theorem]

Every non-zero element $a \in R \setminus \mathrm{U}(R)$ has a representation as a finite product of prime elements: $a = b_1 b_2 \ldots b_n$.

Furthermore, two different prime factorizations of a have the same number of factors and the factors are similar in pairs. That is, if $a = b_1 b_2 \ldots b_n \doteq c_1 c_2 \ldots c_m$, where all b_i, c_j are prime, then $n = m$ and there exists a permutation π of $\{1, \ldots, n\}$ such that $b_i \sim c_{\pi(i)}$.

Proof. By Lemma 2.36, $a = b_1 u_1$ for some b_1 prime. If $u_1 \in \mathrm{U}(R)$, then we are done; if $u_1 \notin \mathrm{U}(R)$, then we apply the same Lemma 2.36 to u_1: $u_1 = b_2 u_2$, where b_2 is prime. This process generates an increasing chain of right ideals of R, $Ra \leq Ru_1 \leq Ru_2 \leq \ldots \leq R$, and since R is left noetherian, there is an integer $n \geq 1$ such that $Ru_n = Ru_{n+1}$, and together with $u_n = u_{n+1} b_{n+1}$ it follows that $Ru_{n+1} = Ru_n = Rb_{n+1} u_{n+1}$, implying that $b_{n+1} \in \mathrm{U}(R)$, which is not possible since b_{n+1} was chosen to be a prime right divisor of u_n; this indicates that u_n does not have a prime right divisor, so u_n must be a unit. The element $b_n u_n$ is a right associate of a prime, hence it is a prime element.

We have obtained a prime factorization of $a = b_1 b_2 \ldots b_n$, where the prime $b_n u_n$ has been "replaced" by b_n.

The proof for the second part of the theorem will use induction on n: if a product of n prime factors is equal to a product of a finite number of prime factors, then it will be shown that the two prime factorizations have the same number of prime factors, and they are similar in pairs. The proof by induction on n we are giving here is similar to the one proving the corresponding factorization theorem in [31, Theorem 1].

Consider two different prime factorizations of a, $a = b_1 b_2 \dots b_n \doteq c_1 c_2 \dots c_m$, where all the factors are prime.

If $n = 1$, then the equality $b_1 = c_1 \dots c_m$ holds if only if $n = 1$ and $b_1 \doteq c_1$.

Assume now that the conclusion holds for any product of $n-1$ or less prime factors. If $b_1 b_2 \dots b_n \doteq c_1 c_2 \dots c_m$, then b_n is a right-hand divisor of $c_1 c_2 \dots c_m$. By Lemma 2.35, there is $k \leq m$ such that $c_k \doteq {}^{(c_{k+1} \dots c_m)} b_n \sim b_n$ so

$a = c_1 \dots c_k \dots c_m \doteq c_1 \dots c_{k-1} \left({}^{(c_{k+1} \dots c_m)} b_n \right) c_{k+1} \dots c_m \doteq c_1 \dots c_{k-1} \, {}^{b_n}(c_{k+1} \dots c_m)\, b_n$.

Then $a = b_1 \dots b_{n-1} b_n \doteq c_1 \dots c_{k-1} \, {}^{b_n}(c_{k+1} \dots c_m)\, b_n$, so it follows that

$b_1 \dots b_{n-1} \doteq c_1 \dots c_{k-1} \, {}^{b_n}(c_{k+1} \dots c_m)$, and by Proposition 2.30(vi),

$b_1 \dots b_{n-1} \doteq c_1 \dots c_{k-1} \, \overline{c}_{k+1} \dots \overline{c}_m$, where $\overline{c}_l \sim c_l$, for all $l \geq k+1$.

We now apply now the induction hypothesis on the prime factorization $b_1 \dots b_{n-1}$ with $n-1$ prime factors. It follows that $n-1 = m-1$, so $n = m$.

Furthermore, if we let $d_j = c_j$ ($j \leq k-1$) and $d_j = \overline{c}_{j+1}$ ($k \leq j \leq n-1$), there exists a permutation π of $\{1, \dots, n-1\}$ such that $b_i \sim d_{\pi(i)}$. We can extend π to a permutation π' of $\{1, \dots, n\}$ in the following way:

$$\pi'(i) = \begin{cases} \pi(i), & \text{for } i = 1, \dots, k-1 \\ \pi(i+1), & \text{for } i = k, \dots, n-1 \\ \pi'(i) = k, & \text{for } i = n \end{cases}$$

It follows now that $b_i \sim c_{\pi'(i)}$ for all $i \leq n$. The induction is complete, and the second part of the theorem is now proved. ∎

Remark 2.38 Although these rings are not commutative, sometimes the order of a product can be permuted, up to similarity.

Let $0 \neq a, b \in R \setminus \mathrm{U}(R)$, $a \neq b$. Ore [31] defined a to be (right) interchangeable with b if there exist $a_1 \sim a$, $b_1 \sim b$ such that $ab = b_1 a_1 \doteq [\, b \,, a_1 \,]_*$.

In this case, $b_1 \doteq {}^{a_1} b$ and $a \doteq {}^{b} a_1$. Thus:

$$\begin{array}{ccccccc} [\, a_1 \,, b \,]_* & \doteq & a \cdot b & \doteq & {}^{a_1} b & \cdot & a_1 \\ & & & & \uparrow & & \uparrow \\ & & & & b_1 \sim b & & a_1 \sim a \end{array}$$

Then we say that b_1a_1 is obtained from ab through *(right) interchange of factors* if $a_1 \sim a \doteq {}^{b}a_1$, $b \sim b_1 \doteq {}^{a_1}b$, and consequently, $[\,a_1\,,b\,]_* \doteq ab = b_1a_1$. In this case, ab is obtained from b_1a_1 through *(right) interchange of factors* as well. Remark that according to this definition, a non-unit $0 \neq a \in R$ is not interchangeable with itself.

It follows by an easy computation from Proposition 2.30 that if $(\,a\,,bc\,)_* \doteq 1$ and a is interchangeable with b, $ab = b_1a_1$, and a_1 is interchangeable with c, then a is interchangeable with bc.

Example 2.39 Let $R = k(x)[y, \partial/\partial x]$, and let $u = y^4 - \frac{1}{x}y^3 + xy^2$. Then the following are some different forms of the prime factorization of u:

$$u \;=\; (y - \tfrac{1}{x})(y^2 + x)y \;=\; (y^2 - \tfrac{1}{x}y + x)yy = (y^2 - \tfrac{1}{x}y + x)(y + 1/x)(y - 1/x)$$

Definition 2.40 Let $0 \neq a \in R$.

1. The *length* of a is defined to be the number $l(a)$ of prime factors in a factorization of a if $a \notin \mathrm{U}(R)$; if $a \in \mathrm{U}(R)$, then $l(a) = 0$.
2. The number $l_p(a)$ of prime factors in a prime factorization of $a \notin \mathrm{U}(R)$ which are similar to a prime element p is defined to be the *p-length* of a; if $a \in \mathrm{U}(R)$, then $l_p(a) = 0$.

Note that $l_p(a) \leq l(a)$, for any prime $p \in R$.

Lemma 2.41 *Let $0 \neq a, b \in R$, $p \in R$, where p is a prime element. Then*

(i) $l(a) = 0$ *if and only if* $a \in \mathrm{U}(R)$*, and* $l_p(a) = 0$ *if and only if a has no prime factor similar to p.*

(ii) $l(ab) = l(a) + l(b)$ *and* $l_p(ab) = l_p(a) + l_p(b)$.

(iii) $l([\,a\,,b\,]_*) + l((\,a\,,b\,)_*) = l(a) + l(b)$ *and* $l_p([\,a\,,b\,]_*) + l_p((\,a\,,b\,)_*) = l_p(a) + l_p(b)$.

(iv) *If $a \sim b$ then* $l(a) = l(b)$ *and* $l_p(a) = l_p(b)$.

(v) $l({}^{b}a) \leq l(a)$*, with equality if and only if* $(\,a\,,b\,)_* \doteq 1$*, and* $l_p({}^{b}a) \leq l_p(a)$*, with equality if and only if* $l_p((\,a\,,b\,)_*) = 0$.

Proof. (iv) and (v) are consequences of 2.37 and 2.30. The other results are obvious from the definition of the length and p-length of an element. ∎

Definition 2.42 ('decomposable' appears as 'distributive' in [31])
An element $0 \neq u \in R$ is said to be *right decomposable* if it can be written as a least common left multiple of two of its proper right factors (i.e. not left associates of u, nor units):

$$u \in [\, a\,,b\,]_* \text{ (or simply, } u \doteq [\, a\,,b\,]_*)$$

If u is not right decomposable it is called *right indecomposable.*

Symmetrically, an element v is *left decomposable* if it is a least common right multiple of proper left divisors:

$$v \in {}_*[\, a\,,b\,]$$

If v is not left decomposable, then it is called *left indecomposable.*

Remark 2.43 Obviously, a prime element is right (and left) indecomposable.

Example 2.44 Let $R = k(x)[y, \partial/\partial x]$.
It is easy to generate examples of elements which are right decomposable, by considering least common left multiples of non-units that are not left multiples of each other:
$u = y^2 = (y + 1/x)(y - 1/x) \doteq [\, y\,, y - 1/x\,]_*$.
The following elements are right indecomposable:
$v_1 = (y^2 + x)y$, $v_2 = (y^2 + x^3)y$, $v_3 = (y + \dfrac{1}{x})y$, $v_4 = (y + 1/x)(y^2 + x)$.

The following proposition indicates the relationship between right and left decomposability. That is, it gives a condition that guarantees that a right decomposable element will also be left decomposable.

Proposition 2.45 *Let* $0 \neq u \in R$.

(i) *If u is right decomposable and can be written as the least common left multiple of two relatively right prime proper right divisors, $u \doteq [\, a\,,b\,]_*$, $(\, a\,,b\,)_* \doteq 1$, then u is also left decomposable.*

(ii) *If $l(u) = 2$, then u is right decomposable (indecomposable) if and only if u is left decomposable (indecomposable).*

Proof. (i) Since $u = b_1 a = a_1 b \doteq [\, a\,,b\,]_*$, $(\, a\,,b\,)_* = 1$, it follows from Lemma 2.21 that $u \in {}_*[\, a_1\,,b_1\,]$, where a_1, b_1 are proper left divisors of u, since a, b are proper right divisors.

(ii) This is a consequence of the first part. ∎

Example 2.46 $R = k(x)[y, \partial/\partial x]$.

1. The elements $(y^2+x)y$, $(y+\frac{1}{x})y$, $y(y-\frac{1}{x}), y(y-\frac{1}{x+1})$ are left and right indecomposable.
2. The element $(y-\frac{1}{x})(y^2+x)$ is right and left decomposable, while $(y+\frac{1}{x})(y^2+x)$ is right and left indecomposable.
3. $(y+f+\frac{1}{x-\alpha})(y+f)$ is left and right indecomposable for all $f \in k(x), \alpha \in k$.
4. For all $f, g, h \in k(x)$ $(gh \neq 0)$, $(y+f-\frac{g'}{g})(y+f-\frac{h'}{h})$ is right and left decomposable if and only if $\int \frac{g}{h}\,dx \in k(x)$.

Theorem 2.47 [31] *Let* $0 \neq u \in R \setminus \mathrm{U}(R)$.
Then u *is right indecomposable if and only if it has a unique prime left divisor.*
[Note: The uniqueness is up to right multiplication by units.]

Proof. Assume that u has two distinct (up to right multiplication by units) prime left divisors p_1, p_2; obviously, ${}_*(p_1, p_2) \doteq 1$. By the definition of ${}_*[\ ,\ ]$, u is left divisible by any of the least common right multiples of p_1, p_2. Thus, if $p_1q_2 = p_2q_1 \in {}_*[p_1, p_2]$, with $(q_1, q_2)_* \doteq 1$, then $u = p_1q_2v = p_2q_1v$ for some $v \in R$. From the equivalent definition for the least common left multiple (see Theorem 2.14(ii)), we have $u \in [q_1v, q_2v]_*$, since ${}_*(p_1, p_2) \doteq 1$. The components q_1v, q_2v are proper right divisors of u, so u is right decomposable.

On the other hand, assume that u is right decomposable. Then $u \doteq [u_1, u_2]_*$ for some proper right divisors u_1, u_2; so it follows that $u = w_2u_1 = w_1u_2$, for some non-units $w_1, w_2 \in R$, ${}_*(w_1, w_2) \doteq 1$. By Lemma 2.36, each of w_1 and w_2 has at least one prime left divisor, p_1 and p_2, respectively, and the primes p_1, p_2 are distinct since ${}_*(w_1, w_2) \doteq 1$.

Since $u = w_2u_1 = w_1u_2$ is left divisible by w_1 and w_2, it is also left divisible by p_1, p_2. Therefore u has two distinct prime left divisors, p_1 and p_2. ∎

The notion of decomposability and indecomposability are not usually symmetric, since there are, for instance, elements which are right indecomposable and left decomposable.

Example 2.48 Let $R = k(x)[y, \partial/\partial x]$.

1. The element $a = (y+1/x)y$ is right indecomposable, having the unique monic prime left divisor $p = (y + 1/x)$. Obviously, every right associate pf of p $(0 \neq f \in k(x))$ is also a prime left divisor of a.
2. The element $y\ {}_*\big[\,y - \frac{1}{x}\,,y - \frac{1}{x+1}\,\big] \;=\; {}_*\big[\,y(y - \frac{1}{x})\,,y(y - \frac{1}{x+1})\,\big]$ is left decomposable, but right indecomposable since $y(y - \frac{1}{x})$, $y(y - \frac{1}{x+1})$ are right indecomposable with the same unique prime left divisor y.
3. The element $\big[\,(y^2 + x)y\,,(y + \frac{1}{x})y\,\big]_* \;\doteq\; \big[\,y^2 + x\,,y + \frac{1}{x}\,\big]_*\,y$ is right decomposable, but left indecomposable since $(y^2 + x)y$, $(y + \frac{1}{x})y$ are left indecomposable with the same unique prime right divisor y.
4. $(y + \frac{1}{x})(y^2 + x)y$ is left and right indecomposable.

Remark 2.49 In view of Theorem 2.47 and Proposition 2.45, an element u with $l(u) = 2$ is right (and left) indecomposable if and only if its prime factorization is unique up to associates. That is, if $u = p_1q_1 = p_2q_2$, then p_1 and p_2 are left associates, and q_1 and q_2 are right associates.

For the remaining of this thesis, for brevity, the term "indecomposable" will be used for "right indecomposable".

Definition 2.50
1. If $0 \neq u \in R\setminus \mathrm{U}(R)$ is right indecomposable with unique prime left divisor p, then u is called *p-indecomposable.*
2. If u is q-indecomposable for some $q \in \widehat{p}$, then we say that u is $\widehat{p}$-indecomposable. (Recall that $\widehat{p}$ denotes the similarity class of the prime element p.)
3. Denote by $\widehat{\mathcal{P}}$ the set of all $\widehat{p}$-indecomposable elements.

Remark 2.51 We will see later that in fact, there is an equivalence relation on the set of all indecomposable elements, an equivalence class consisting of indecomposable elements with unique prime left divisors in the same similarity class $\widehat{p}$ of some prime p, thus of the form $\widehat{\mathcal{P}}$.

Corollary 2.52 *Let $0 \neq u \in R \setminus \mathrm{U}(R)$. The following are equivalent:*

(i) *Ru is an irreducible left ideal of R.*

(ii) *u is right indecomposable.*

(iii) *uR is contained in a unique maximal right ideal.*

In the case (ii) $\Leftrightarrow$ (iii), *u is p-indecomposable if and only if pR is the unique maximal right ideal containing uR.*

PROOF. The equivalences are easy consequences of Definition 1.21, Lemma 2.6(ii), and Theorem 2.47. ∎

The next proposition provides some tools for generating indecomposable elements.

Proposition 2.53 [31]

(i) *Every left associate of a $\widehat{p}$-indecomposable element is $\widehat{p}$-indecomposable.*

(ii) *Every proper left divisor of a p-indecomposable element is also p-indecomposable.*

(iii) *A transform of a $\widehat{p}$-indecomposable element is $\widehat{p}$-indecomposable or a unit. In particular, if $u \in \widehat{\mathcal{P}}$ and $a \in R$, $(a\,,u\,)_* \neq u$, then ${}^{a}u$ is also $\widehat{p}$-indecomposable.*

(iv) *If u is $\widehat{p}$-indecomposable and $v \sim u$, then v is $\widehat{p}$-indecomposable.*

PROOF. Let u be a $\widehat{p}$-indecomposable element, and w.l.o.g., we assume that $u = pu_1$.

(i) and (ii) are immediate by Theorem 2.47.

(iii) Let $v \doteq {}^{a}u$. If $(a\,,u\,)_* \doteq u$, then $a \in Ru$ and it follows that $v \in \mathrm{U}(R)$. Assume that $(a\,,u\,)_* \doteq d \neq u$. Then by 2.30(ii), $v \doteq {}^{b}w$, where $u = wd$, $a = bd$, and $(w\,,b\,)_* \doteq 1$. Since w is a proper left divisor of u, it is also $\widehat{p}$-indecomposable of the form $w = pw_1$. Then $v \doteq {}^{b}w \doteq {}^{b}(pw_1) \doteq {}^{(w_1 b)}p\,{}^{b}w_1$, and p is similar to ${}^{(w_1 b)}p$, by 2.30(vi). So $q =^{(w_1 b)} p$ is a prime left divisor of v, similar to p.

If v is decomposable, then $v = [\,v_1\,,v_2\,]_*$ for some proper right divisors v_1, v_2. Then $w = {}^{c}v = {}^{c}[\,v_1\,,v_2\,]_* = [\,{}^{c}v_1\,,\,{}^{c}v_2\,]_*$. From $(c\,,v\,)_* \doteq 1$ it follows that $(c\,,v_1\,)_* \doteq 1$, $(c\,,v_2\,)_* \doteq 1$, and since v_1, v_2 are proper right divisors of v, it follows that ${}^{c}v_1$, ${}^{c}v_2$ are proper divisors of w, a contradiction. Thus v is q-indecomposable, i.e. $\widehat{p}$-indecomposable.

(iv) This part is a consequence of the definition of similar elements and of part (iii). ∎

Lemma 2.54 *Let $0 \neq a, b \in R$, where $a \in \widehat{\mathcal{P}}$ and $l_p(a) = l_p(b)$.*

Then a is right divisible by b if and only if $a \doteq b$.

In this case, b is also $\widehat{\mathfrak{p}}$-indecomposable.

Proof. If $a = cb$ for some $c \in R$, then $l_p(b) = l_p(a) = l_p(cb) = l_p(c) + l_p(b)$, so $l_p(c) = 0$. Thus $c \in \mathrm{U}(R)$ or c has only prime factors not similar to p, but that would mean that a has a prime left divisor not similar to p, contradicting that a is $\widehat{\mathfrak{p}}$-indecomposable. Thus c is a unit, and $a \doteq b$.

The converse is trivially true. ∎

Proposition 2.55 *Every non-zero element $u \in R \setminus \mathrm{U}(R)$ is a least common left multiple of a finite family of indecomposable elements: $u \doteq [u_1, \ldots, u_n]_*$, where every u_i is indecomposable. Furthermore, such a decomposition can be found so that it is irredundant, in the sense that no u_i is a right divisor of the least common left multiple of the rest of the components.*

Proof. By Proposition 1.22, the left ideal Ru can be written as a finite intersection of irreducible left ideals, $Ru = I_1 \cap \ldots \cap I_n$, and by 2.52, every I_j is generated by an indecomposable element u_j, $I_j = Ru_j$. Then $Ru = Ru_1 \cap \ldots \cap Ru_n$, or equivalently, $u \doteq [u_1, \ldots, u_n]_*$.

The second part is a direct consequence of the fact that any representation of a left ideal as a finite intersection of irreducible ideals is either irredundant or can be modified into an irredundant one. ∎

From now on, every time we consider such a decomposition of an element as a least common left multiple of indecomposables, we will assume that it is irredundant.

Lemma 2.56 *Let $0 \neq u = pv \in R$, with p a prime element and $v \notin \mathrm{U}(R)$.*

(i) *If u is decomposable, then p is interchangeable with some prime left divisor of v.*

(ii) *u is p-indecomposable if and only if p is not interchangeable with any of the prime left divisors of v.*

PROOF. (i) If u is decomposable, then it has at least another prime left divisor $q \not\sim p$. Then u is left divisible by any least common right multiple $pq_1 = qp_1 \in {}_*[p,q]$ of p and q; thus $u = pq_1\overline{u} = qp_1\overline{u} = pv$, and so $v = q_1\overline{u}$. Since $p \sim p_1$, $q \sim q_1$ and $pq_1 = qp_1 \in [q_1, p_1]_*$, it follows that p is interchangeable with q_1, the prime left divisor of v.

(ii) This is an immediate consequence of part(i) and Theorem 2.47. ∎

Proposition 2.57 *Let $0 \neq u = pv \in R$, with p prime, $v \notin \mathrm{U}(R)$.*
There exist $u_1, u_2 \in R$ such that $u = u_1u_2$ and $u_2 \in \widehat{\mathcal{P}}$.

PROOF. We use induction on $l(v)$.

Assume that v is a prime element. If u is indecomposable, then take $u_1 = 1$ and $u_2 = u$. If u is decomposable, then by Lemma 2.56(i), p is interchangeable with v, $u = pv = v_1q$, where $q \sim p$. The conclusion follows by taking $u_1 = v_1$ and $u_2 = q$.

Assume now that $l(v) = n$ and the result is true for any element $u' = p'v'$, where p' is prime and $l(v') \leq n-1$.

If u is indecomposable, then take $u_1 = 1$ and $u_2 = u$. If u is decomposable, then by Lemma 2.56(i), p is interchangeable with some prime left divisor q of $v = qw$, so we can write: $u = pv = pqw = \overline{q}\,\overline{p}\,\overline{w}$, with $\overline{p} \sim p$. Then $l(\overline{w}) = n-1$, and the induction hypothesis can be applied to $\overline{p}\,\overline{w}$. Thus there exist w_1, w_2 such that $\overline{p}\,\overline{w} = w_1w_2$, and w_2 is $\overline{p}$-indecomposable. Then we write $u = u_1u_2$, with $u_1 = \overline{q}w_1$ and $u_2 = w_2 \in \widehat{\mathcal{P}}$.

The induction is now complete. ∎

Theorem 2.58 *Let $0 \neq u \in R \setminus \mathrm{U}(R)$ and let $s = l_p(u)$. Then there exist elements $v, v_1, v_2, \ldots, v_s \in R$ such that:*

1. $u = vv_1v_2\ldots v_s$;
2. $l_p(v) = 0$;
3. $v_i \in \widehat{\mathcal{P}}$, *with* $l_p(v_i) = 1$, *for all* $i \leq s$.

PROOF. (Induction on s) Let $u = p_1p_2\ldots p_n$ be a prime factorization of u.

If $s = 1$ then there is a (unique) $i \leq n$ such that $p_i \sim p$. We can apply proposition 2.57 to $p_ip_{i+1}\ldots p_n$, and there exist u_1, u_2 such that u_2 is indecomposable with unique prime left divisor similar to p and $p_i p_{i+1} \dots p_n = u_1 u_2$, from which it follows that $u = p_1 \dots p_{i-1} u_1 u_2$. Let $v = p_1 \dots p_{i-1} u_1$ and $v_1 = u_2$. Then $l_p(v) = 0$, since u has only one factor similar to p which appears as the unique prime left divisor of v_1.

Assume now that $l_p(u) = s > 1$, and the result is true for any u' with $l_p(u') \leq s - 1$. Let $k = \min\{\, i \;:\; 1 \leq i \leq n \,,\; p_i \sim p \,\}$. The element $a = p_{k+1} \dots p_n$ has $l_p(a) = s - 1$, and by induction hypothesis, there are $b, v_2, v_3, \dots, v_s$ such that $a = bv_2 \dots v_s$, $l_p(b) = 0$ and all $v_2, \dots v_s \in \widehat{\mathcal{P}}$ with $l_p(v_j) = 1$, $2 \leq j \leq s$. But $p_1 \dots p_k b$ has one prime factor similar to p, so by induction hypothesis it can be written as $p_1 \dots p_k b = v v_1$, $l_p(v) = 0$, $v_1 \in \widehat{\mathcal{P}}$ and $l_p(v_1) = 1$. Then $u = v v_1 \dots v_s$, where $v, v_1 \dots v_s$ satisfy all conditions in the theorem. ∎

Remark 2.59 The factorization mentioned in Theorem 2.58 is not unique. This is obvious for instance, when u is decomposable with $l_p(u) = l(u) \geq 2$. In this case, a factorization of the type presented in Theorem 2.58 corresponds to a prime factorization as described in 2.37, and it is not unique since u has more that one prime left divisor. We should mention that two different factorizations of the type presented in Theorem 2.58 can be obtained from each other by 2.30(vi) and 2.53(iii).

Lemma 2.60 *Let $u, u_1, \dots, u_n \in \widehat{\mathcal{P}}$ such that $l_p(u) = l_p(u_i) = 1$ for all i. If $[u_1, u_2, \dots, u_n]_* \in Ru$ then $[u_1, u_2, \dots, u_n]_* \doteq [u_1, \dots, u_{i-1}, u, u_{i+1}, \dots, u_n]_*$ for some $i \leq n$.*

Proof. First we note that if $a, b \in \widehat{\mathcal{P}}$, $l_p(a) = l_p(b) = 1$, and $a \not\doteq b$, then $(\,a\,,b\,)_* \neq b$ (otherwise $a = cb$ for some $c \in R$; but both a and b have unique prime left divisors in $\widehat{p}$, so necessarily $c \in \mathrm{U}(R)$, contradicting $a \not\doteq b$). Then by Proposition 2.53(iii), ${}^{a}b \in \widehat{\mathcal{P}}$, and similarly, ${}^{b}a \in \widehat{\mathcal{P}}$, and since $l_p(a) = l_p(b) = 1$, it follows trivially that $l_p({}^{a}b) = l_p({}^{b}a) = 1$.

The result will be proved by induction on n. Let $r \doteq [u_1, u_2, \dots, u_n]_*$ and w.l.o.g. assume for all $i \neq j$ that $u_i \not\doteq u_j$.

For $n = 1$, if $u_1 \in Ru$ then $u_1 \doteq u$, by Lemma 2.54.

Assume that $n > 1$, and that the result is true for any least common left multiple of any $n - 1$ or fewer distinct elements from $\widehat{\mathcal{P}}$, each with one prime factor from $\widehat{p}$. Let $u \in \widehat{\mathcal{P}}$ with $l_p(u) = 1$, a right divisor of r.

If $u \doteq u_1$, then we are done.

If $u \not\doteq u_1$ then ${}^{u_1}u \in \widehat{\mathcal{P}}$ with $l_p({}^{u_1}u) = 1$, by the remark at the beginning of this proof. Let $v \doteq [u_2, \ldots, u_n]_*$. Then $r \doteq [u_1, v]_* \doteq {}^{u_1}v\, u_1$ is a left multiple of u and it follows from 2.30(iii) that ${}^{u_1}v$ is right-hand divisible by ${}^{u_1}u$. Furthermore, $u_1 \not\doteq u_i$ for all $i \geq 2$, so ${}^{u_1}u,\ {}^{u_1}u_i \in \widehat{\mathcal{P}}$ and $l_p({}^{u_1}u_i) = 1$, and therefore the induction hypothesis applies to

$$ {}^{u_1}v \doteq [{}^{u_1}u_2, \ldots, {}^{u_1}u_n]_* \in R\,{}^{u_1}u \ . $$

Thus there is $2 \leq i \leq n$ such that

$$ [{}^{u_1}u_2, \ldots, {}^{u_1}u_{i\text{-}1}, {}^{u_1}u, {}^{u_1}u_{i+1}, \ldots, {}^{u_1}u_n]_* \doteq [{}^{u_1}u_2, \ldots, {}^{u_1}u_n]_*, $$

or equivalently, ${}^{u_1}[u_2, \ldots, u_{i\text{-}1}, u, u_{i+1}, \ldots, u_n]_* \doteq {}^{u_1}[u_2, \ldots, u_n]_*$. By multiplying on the right by u_1 we obtain:

$$ [u_1, u_2, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n]_* \doteq [u_1, u_2, \ldots, u_n]_* $$

∎

Chapter 3

Injective Modules over a Principal Ideal Domain

In this chapter, I examine the injective modules over a principal (left and right) ideal domain R, and in particular, I describe the layered structure of an indecomposable injective module by investigating its socle series. Later I bring into play the concept of elementary socle series, a "first-order definable" analogue of the notion of socle series also modeled on the idea of layers, and we will look into the relationship between these socle notions.

With the use of the technique of localization, I am able to inductively construct bases for each quotient of the socle series starting with an arbitrary choice of basis for the $\mathfrak{R}$-module $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$. In this extension $\mathfrak{R}$ of R, we will notice some similarities with the layered structure of an indecomposable injective module over a commutative noetherian ring (see [27]).

Finally, I will study some aspects of the endomorphism ring and the bicommutator of an indecomposable injective module.

3.1 Injective Modules over a Principal Ideal Domain

In this section, I will identify all of the indecomposable injective modules over a PID. This will generalize, in some sense, the classification given by Matlis in 1958 [27] for a commutative noetherian ring. He showed that by associating with each prime ideal P of R the injective envelope $E = E(R/P)$ of R/P there is a one-to-one correspondence between the prime ideals of R and the indecomposable injective R-modules.

In the same paper, a classification of the injective modules over a noetherian ring R was also given: every injective left R-module has a decomposition as a direct sum of indecomposable, injective submodules. This result reduces the problem of finding and eventually describing the injective left R-modules to the problem of finding all the indecomposable injective left R-modules. Matlis proved that a left R-module E is an indecomposable injective module if and only if $E \cong E_R(R/J)$, for some irreducible left ideal J of R.

In 1973 Lambek and Michler [26] established that over a left noetherian ring, there is a one-to-one correspondence between the isomorphism classes of indecomposable injective modules and related critical prime left ideals, where a critical prime left ideal is characterized as the irreducible left ideal maximal in its equivalence class of related ideals. This correspondence can be made more explicit in the particular case when R is a PID. I will deduce this classification directly from the algebraic properties of the ring R, which is a PID, without appealing to the above mentioned result.

We remarked in Corollary 2.52 that over a PID, a left ideal $0 \neq Ra$ is irreducible if and only if a is right indecomposable. This is a key observation used to prove the next theorem.

Theorem 3.1 *Let R be a PID and let p be a prime element in R.*
For every $\widehat{p}$-indecomposable element $a \in R$, $E_R(R/Ra) \cong E_R(R/Rp)$.

PROOF. If a is $\widehat{p}$-indecomposable, then $a = qs$ has a unique prime left divisor $q \sim p$, and by 2.52, the left ideal Ra is irreducible, so by Theorem 1.24, the injective envelope $E = E(R/Ra)$ of R/Ra is indecomposable. If e is the image of $0 \neq s + Ra \in R/Ra$ in $E(R/Ra)$, with $\mathrm{ann}_R(e) = \mathrm{ann}_R(s + Ra)$, then by the same Theorem 1.24 it follows that $E \cong E(R/\mathrm{ann}_R(e))$. But $q \in \mathrm{ann}_R(s + Ra)$, from $a = qs$, so $Rq \subseteq \mathrm{ann}_R(s + Ra)$; since the left ideal Rq is maximal and $\mathrm{ann}_R(s + Ra) \neq R$ (from $s + Ra \neq 0$), it follows that $\mathrm{ann}_R(e) = \mathrm{ann}_R(s + Ra) = Rq$, and by applying Matlis' Theorem 1.24,

$$E_R(R/Ra) \cong E_R(R/Rq).$$

But $q \sim p$, so $R/Rq \cong R/Rp$, and consequently, $E_R(R/Ra) \cong E_R(R/Rp)$. ∎

Corollary 3.2 *Let R be a PID and let $a, b \in R$. If a is p-indecomposable and b is q-indecomposable (p, q prime), then*

$$E_R(R/Ra) \cong E_R(R/Rb) \text{ if and only if } p \sim q$$

PROOF. Immediate by Theorem 3.1 and Definition 2.16. ∎

Remark 3.3 In the commutative case, by Theorem 1.32, $E(R/P) \cong E(R/I)$ if and only if I is an irreducible P-primary ideal. Thus Theorem 3.1 is an analogue of Matlis's theory for (non-commutative) PID's, where the irreducible left ideals which are generated by $\widehat{p}$-indecomposable elements correspond to the P-primary ideals in the commutative case.

The following theorem presents the classification of the indecomposable injective modules over a PID.

Theorem 3.4 *Let R be a PID.*
There is a one-to-one correspondence between the isomorphism classes of indecomposable injective left R-modules (corresponding to non-zero irreducible ideals) and the similarity classes of all prime elements in R:

$$\widehat{p} \leftrightarrow E_R(R/Rp)$$

Thus the indecomposable injective R-modules are: $E(R)$ and $E(R/Rp)$, where p ranges over a set of representatives of the similarity classes of primes of R.

A well-known result states that every left noetherian domain is left Ore, so in particular, a PID is left Ore and the classical left ring of quotients $Q^l_{cl}(R)$ of R can be constructed. In this extension of R every non-zero element of R becomes invertible, and thus $Q^l_{cl}(R)$ is the (unique) divisible hull of R, that is, $E(R) = Q^l_{cl}(R)$.

In the next section we will restrict our attention to the more interesting indecomposable injectives of the form $E(R/Rp)$, where $p \in R$ is a prime element.

3.2 The Socle Series of $E_R(R/Rp)$, p prime

Throughout this section, R is a PID, and E is the indecomposable injective R-module $E(R/Rp)$ for some fixed prime p.

Similarly to Matlis' approach, I will investigate the annihilators in E with the intention of finding an increasing chain of submodules that will cover E entirely. Recall that all left annihilators in R of non-zero elements of E are irreducible left ideals generated by right $\widehat{p}$-indecomposable elements. We are thus concerned with characterizing the $\widehat{p}$-indecomposable elements which annihilate the non-zero elements of E.

Lemma 3.5 (i) $\mathrm{ann}_R(a + Rp) = R(\,^a p)$, *for every* $a \notin Rp$.
(ii) *Let* $e \in E$ *with* $\mathrm{ann}_R(e) = Ru$, $u \in \widehat{\mathcal{P}}$, $l_p(u) = 1$, *and let* $r \in R$.
If $re \neq 0$, *then* $\mathrm{ann}_R(re) = R(\,^r u)$.

PROOF. (i) If $a \notin Rp$, then $\mathrm{ann}_R(a + Rp) \neq R$. At the same time,

$$[p\,,a]_* \doteq \,^a p \cdot a \doteq \,^p a \cdot p\,,$$

and it follows that $^a p\, a \in Rp$, or equivalently, $^a p \in \mathrm{ann}_R(a + Rp)$. But $^a p \in \widehat{p}$ is prime, so it generates a maximal left ideal, and it follows from $R(\,^a p) \subseteq \mathrm{ann}_R(a + Rp) \neq R$ that $\mathrm{ann}_R(a + Rp) = R(\,^a p)$.

(ii) Since $ue = 0$, it follows that $^u r\, ue = \,^r u(re) = 0$, so $^r u \in \mathrm{ann}_R(re) \neq R$, and by 2.53, $^r u \in \widehat{\mathcal{P}}$. At the same time, $0 < l_p(\,^r u) \leq l_p(u) = 1$, so $l_p(\,^r u) = 1$, implying that $\mathrm{ann}_R(re) = R(\,^r u)$. ∎

Lemma 3.6 *Let* $0 \neq e \in E$ *with* $\mathrm{ann}_R(e) = Ru$. *If* $u = vw$, *then* $\mathrm{ann}_R(we) = Rv$.

PROOF. Let $\mathrm{ann}_R(we) = Ra$. From $0 = ue = v(we)$ it follows that $v \in Ra$. At the same time, from $0 = a(we) = (aw)e$, we have that $aw \in \mathrm{ann}_R(e) = Ru = Rvw$, so $a \in Rv$. Then $Rv = Ra = \mathrm{ann}_R(we)$. ∎

The next theorem gives in (i) and (ii) a partial description of the layers of the socle series of E, while part (iii) states that the indecomposable injective module E is the union of its socle layers, a result which is true for any torsion module over a hereditary noetherian prime ring ([20, A.1.4]).

Theorem 3.7 *Let R be a PID and let $E = E(R/Rp)$, $p \in R$ prime.*

(i) $\mathrm{soc}_1(E) = R/Rp = \{e : e \in E,\ ue = 0 \text{ for some } u \in \widehat{p}\}$

(ii) *Let $e \in E$ and $\mathrm{ann}_R(e) = Ru$. If $l(u) \leq n$ then $e \in \mathrm{soc}_n(E)$.*

(iii) $E = \bigcup_{n=0}^{\infty} \mathrm{soc}_n(E)$.

PROOF. (i) The socle of an indecomposable injective module is either 0 or simple, but since R/Rp is simple, $\mathrm{soc}(E) \neq 0$; hence $\mathrm{soc}_1(E) = R/Rp$. The rest of the description of $\mathrm{soc}_1(E)$ follows easily from Lemma 3.5.

(ii) The result will be proved by induction on n; it is true for $n = 0$.

If $l(u) = 1$, then u is a prime element. If $u \not\sim p$, $e \in E$ and $ue = 0$, then $e = 0$. If $u \sim p$ and $ue = 0$, then $e \in \mathrm{soc}_1(E)$ by part (i). Hence (ii) holds for $n = 0, 1$.

Assume that $n \geq 2$ and that (ii) is true for $0 \leq m < n$. Let $0 \neq e \in E$ with $\mathrm{ann}_R(e) = Ru$, $l(u) \geq 1$.

If $l(u) < n$, then by induction hypothesis, $e \in \mathrm{soc}_{n-1}(E) \subseteq \mathrm{soc}_n(E)$.

If $l(u) = n$, there are $v, w \in R$ such that $u = vw$ and $l(v) = n - 1$, $l(w) = 1$. Then by 3.6, $\mathrm{ann}_R(we) = Rv$, and since $l(v) = n - 1 < n$ it follows by induction hypothesis that $we \in \mathrm{soc}_{n-1}(E)$, hence $Rw \leq \mathrm{ann}_R(e + \mathrm{soc}_{n-1}(E))$. Since Rw is maximal, we have $\mathrm{ann}_R(e + \mathrm{soc}_{n-1}(E)) = R$, hence $e \in \mathrm{soc}_{n-1}(E) \subseteq \mathrm{soc}_n(E)$, or $\mathrm{ann}_R(e + \mathrm{soc}_{n-1}(E)) = Rw$ maximal, hence $R(e + \mathrm{soc}_{n-1}(E))$ is a simple submodule of $E/\mathrm{soc}_{n-1}(E)$. Thus $e \in \mathrm{soc}_n(E)$ and the induction is complete.

(iii) The inclusion "$\subseteq$" is an immediate consequence of part (ii). ∎

Corollary 3.8 *Let $E = E(R/Rp)$.*

(i) $\mathrm{soc}_2(E) = \sum \left\{Re : e \in E,\ ue = 0 \text{ for some } u \in \widehat{\mathcal{P}},\ l(u) = 2\right\}$

(ii) *Let $e \in E$ and $\mathrm{ann}_R(e) = Ru$. If $e \notin \mathrm{soc}_{n-1}(E)$ then $l(u) \geq n$.*

PROOF. (i) Let $e \in E \setminus \mathrm{soc}_1(E)$ be such that $R(e + \mathrm{soc}_1(E))$ is simple. It follows that $\mathrm{ann}_R(e + \mathrm{soc}_1(E)) = Ru_1$, for some prime u_1, so $u_1 e \in \mathrm{soc}_1(E)$, and by 3.7(i), there is a prime $u_2 \in \widehat{p}$ such that $u_2(u_1 e) = 0$, where obviously $l(u_2 u_1) = 2$.

On the other hand, if $e \in E$ and $ue = 0$ for some $u = q_1 q_2 \in R$ with q_1, q_2 prime, then by 3.7(i), $q_2 e \in \mathrm{soc}_1(E)$. If $q_2 e = 0$, then $e \in \mathrm{soc}_1(E) \subseteq \mathrm{soc}_2(E)$. If $q_2 e \neq 0$, then

$\mathrm{ann}_R(e + \mathrm{soc}_1(E))$ contains Rq_2, which is a maximal left ideal of R, and it follows that $R(e + \mathrm{soc}_1(E))$ is zero or a simple submodule of $E/\mathrm{soc}_1(E)$. Thus $e \in \mathrm{soc}_2(E)$.

(ii) Immediate from part (ii) of Theorem 3.7. ∎

Example 3.9 Let $R = k(x)[y, \partial/\partial x]$. The most natural example of a simple module over R is $k(x)$, where the left R-action of an element $f \in k(x) \subseteq R$ is defined by the usual multiplication in $k(x)$, while y acts on $k(x)$ as a derivative; that is $y \cdot f = \partial f/\partial x$, for all $f \in k(x)$. Furthermore, $k(x) \cong R/Ry$ as left R-modules.

Let $E = E_R(R/Ry) \cong E_R(k(x))$, so there is an embedding of $k(x)$ in E, and w.l.o.g. we consider $k(x) \leq E$.

1. By Theorem 3.7, $\mathrm{soc}_1(E) = R/Ry \cong k(x)$, and w.l.o.g. we can identify $k(x)$ with $\mathrm{soc}^1(E)$. Then for every $0 \neq f \in k(x)$, $\mathrm{ann}_R(f) = R\left({}^f y\right) = R(y - \frac{f'}{f})$.
2. The elements $(y + \frac{1}{x})(y^2 + x)$ and $(y + \frac{1}{x})y$ are both $\widehat{y}$-indecomposable of length 2, so there exist $e_1, e_2 \in E$ such that

$$\mathrm{ann}_R(e_1) = R(y + \frac{1}{x})(y^2 + x) \text{ and } \mathrm{ann}_R(e_2) = R(y + \frac{1}{x})y.$$

Then $R(e_1 + \mathrm{soc}_1(E)) \cong R/R(y^2 + x)$ and $R(e_2 + \mathrm{soc}_1(E)) \cong R/Ry$ are simple (non-isomorphic) submodules of $E/\mathrm{soc}_1(E)$, so $e_1, e_2 \in \mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$.

Let $e = e_1 + e_2 \in \mathrm{soc}_2(E)$; then $[\,y^2 + x\,, y\,]_* \in \mathrm{ann}_R(e + \mathrm{soc}_1(E)) = Ru$, for some $0 \neq u \in R$. It is immediate that $u \notin \mathrm{U}(R)$ since otherwise $e \in \mathrm{soc}_1(E)$, and it would follow that $ye = ye_1 + ye_2 \in \mathrm{soc}_1(E)$, and we obtain the following contradiction:

$$y \in \mathrm{ann}_R(e_1 + \mathrm{soc}_1(E)) = R(y^2 + x).$$

Also, if we assume that u is prime, then by 2.60, $u \doteq y^2 + x$ or $u \doteq y$. But from $ue = ue_1 + ue_2 \in \mathrm{soc}_1(E)$, it would follow that $u \doteq y^2 + x \doteq y$, contradicting the fact that $y^2 + x$ and y are non-similar. Thus u is not prime either.

We have the following implications:

$ue \in \mathrm{soc}_1(E) \Rightarrow {}^u(y^2 + x)\, ue = {}^{(y^2+x)}u(y^2 + x)e_1 + {}^u(y^2 + x)\, ue_2 \in \mathrm{soc}_1(E) \Rightarrow$ ${}^u(y^2 + x)\, ue_2 \in \mathrm{soc}_1(E) \Rightarrow {}^u(y^2 + x)\, u \in \mathrm{ann}_R(e_2 + \mathrm{soc}_1(E)) = Ry.$

Since ${}^{u}(y^2+x) \sim y^2+x \not\sim y$, it follows that $u \in Ry$. Similarly, $u \in R(y^2+x)$, and consequently, $u \in R(y^2+x) \cap Ry = R\,[\,y^2+x\,,y\,]_* \leq \mathrm{ann}_R(e+\mathrm{soc}_1(E)) = Ru$, implying that $Ru = R\,[\,y^2+x\,,y\,]_*$.

This is an example of an element e of $\mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$ for which $R(e+\mathrm{soc}_1(E))$ is not a simple module.

Furthermore, $\mathrm{ann}_R(e)$ is an irreducible left ideal with generator of length 3, indicating that the description of $\mathrm{soc}_n(E)$ $(n \geq 2)$ from Theorem 3.7 can not be refined further in terms of length of the generators of the annihilators.

3. Other interesting examples of elements in $\mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$ are obtained by considering elements in E as formal solutions to differential equations. For example, the equation $y \cdot v = \frac{1}{x-\alpha}$ $(\alpha \in k)$ has non-zero solutions; fix such a solution and denote it by $\ln(x-\alpha)$. Then $\ln(x-\alpha)$ is an element of $\mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$, with $\mathrm{ann}_R(\ln(x-\alpha)) = R(y+\frac{1}{x-\alpha})y$.
   Similarly, if $\alpha \in k$, $(x-\alpha)\ln(x-\alpha) \in \mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$ satisfies the equation $(y-\frac{1}{x-\alpha}) \cdot v = 1 \in k(x)$, with $\mathrm{ann}_R((x-\alpha)\ln(x-\alpha)) = Ry(y-\frac{1}{x-\alpha})$.

We proved in Theorem 3.7 that the socle series of E exhausts the module itself. Unfortunately, as the previous examples indicate, for a non-zero element $e \in E$, the length of a generator of its annihilator in R does not determine the height of e in the socle series. The second example reveals something very interesting: the two simple submodules $R(e_1+\mathrm{soc}_1(E)) \cong R/R(y^2+1)$ and $R(e_2+\mathrm{soc}_1(E)) \cong R/Ry$ of $E/\mathrm{soc}_1(E)$ are non-isomorphic, showing that it is in fact in this way how all the (wild) indecomposable injectives enter into the structure of E already by level 2.

The results regarding the socle layers will improve once we perform a localization of the ring R, and our new objective to find a basis for each quotient $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ will be a much easier task to handle.

Before doing the localization of R, we would like to tackle the same problem of describing the indecomposable injective module $E(R/Rp)$ from a different perspective, by looking at the model theoretical analogue of the socle series, the elementary socle series of E.

3.3 The Elementary Socle Series of $E(R/Rp)$, p prime

The introduction of the elementary socle series will allow me to make additional progress in the quest for unfolding the structure of the injective envelope $E = E(R/Rp)$ of the simple R-module R/Rp. The approach will be based on certain internal conditions in the ring R. One of the advantages of working with the elementary version of the socle series is the fact that for any change of rings that is made by standard localization techniques, the elementary socle series of a module does not change.

Since the ring R is a principal left ideal domain, each pp-definable subgroup of the injective R-module E will be the solution set of one homogeneous equation, that is, the annihilator in E of some element r of R.

From now on, the notation $\mathcal{A}_r$ will be used for the solution set in E of the equation $r \cdot v = 0$. That is, $\mathcal{A}_r = \{ e \in E : r \cdot e = 0 \} = \operatorname{ann}_E(r)$, for $r \in R$.

Thus the elementary socle series of an indecomposable injective module $E = {}_RE$ (see Definition 1.85) is characterized by transfinite recursion on ordinals by:

$$\begin{aligned}
\operatorname{soc}^0(E) &= 0\\
\operatorname{soc}^1(E) &= \textstyle\sum\{ \mathcal{A}_r : \mathcal{A}_r \text{ minimal over } \operatorname{soc}^0(E) \}\\
&= \textstyle\sum\{ \mathcal{A}_r : \mathcal{A}_r \text{ minimal non-zero pp-definable subgroup}\}\\
\operatorname{soc}^{\alpha+1}(E) &= \operatorname{soc}^\alpha(E) + \textstyle\sum\{ \mathcal{A}_r : \mathcal{A}_r \text{ minimal over } \operatorname{soc}^\alpha(E) \}, \text{ for all } \alpha\\
\operatorname{soc}^\lambda(E) &= \textstyle\bigcup\{ \operatorname{soc}^\alpha(E) \mid \alpha < \lambda \} \text{ if } \lambda \text{ is a limit ordinal.}
\end{aligned}$$

In order to prove some of the main results about the elementary socle series of E, we need some obvious intermediate results regarding the pp-definable subgroups of E.

Lemma 3.10 (i) *If* $u \in \mathrm{U}(R)$, *then* $\mathcal{A}_u = 0$.
(ii) $\mathcal{A}_r \subseteq \mathcal{A}_{sr}$ *for all* $r, s \in R$.
(iii) *If* $\mathcal{A}_s = 0$, *then* $\mathcal{A}_{sr} = \mathcal{A}_r$.

Lemma 3.11 *Suppose that* $q \in R$ *is prime. Then* $\mathcal{A}_q \neq 0$ *if and only if* $q \sim p$.

Proof. If $0 \neq e \in \mathcal{A}_q$ then $\operatorname{ann}_R(e) = Rq$, since Rq is a maximal left ideal. Therefore, $E = E(R/Rp) \cong E(R/Rq)$, and by Corollary 3.2, $q \sim p$.

Conversely, if $q \sim p$ then $R/Rq \cong R/Rp \leq E$, so there exists $e \in E$ such that $\mathrm{ann}_R(e) = Rq$. Then $0 \neq e \in \mathcal{A}_q$, so $\mathcal{A}_q \neq 0$. ∎

Corollary 3.12 *Let $0 \neq a \in R$ with $l_p(a) = 0$. Then $\mathcal{A}_a = 0$, and consequently every equation $a \cdot v = e$, $e \in E$ has a unique solution in E.*

Proof. Since $l_p(a) = 0$, a is a unit or a product of primes not similar to p. Then by Lemma 3.10(i), or by 3.10(iii) and a simple induction on the length of a, it follows that $\mathcal{A}_a = 0$.

If the equation $a \cdot v = e$ has two solutions $e_1, e_2 \in E$, then $0 = a \cdot e_1 - a \cdot e_2 = a \cdot (e_1 - e_2)$, so $e_1 - e_2 \in \mathcal{A}_a = 0$, and the conclusion follows. ∎

Proposition 3.13 *Let $0 \neq a \in R$. Then $\mathcal{A}_a \neq 0$ if and only if $l_p(a) \neq 0$.*

Proof. By Corollary 3.12, it is sufficient to prove the converse direction. So suppose that $l_p(a) \neq 0$; thus $a = q_1 \ldots q_n$, where some of the primes q_i are similar to p. Let j be the least index such that $q_j \sim p$. By 3.10(iii) and Lemma 3.11, $\mathcal{A}_a = \mathcal{A}_{q_j \ldots q_n}$. Since $q_j \sim p$, it follows that $\mathcal{A}_{q_j} \neq 0$, so there is $0 \neq e \in \mathcal{A}_{q_j}$; the module E is divisible, so the equation $q_{j+1} \cdots q_n \cdot v = e$ has a (non-zero) solution in E. Thus $\mathcal{A}_a \neq 0$. ∎

Corollary 3.14 *Let $0 \neq r, s \in R$. Then $\mathcal{A}_{sr} = \mathcal{A}_r$ if and only if $s \in \mathrm{U}(R)$ or $l_p(s) = 0$.*

Proof. Suppose that $\mathcal{A}_{sr} = \mathcal{A}_r$, and assume that $s \notin \mathrm{U}(R)$ and $l_p(s) \neq 0$, so there exists $0 \neq e \in \mathcal{A}_s$. Since E is divisible, there is $e' \in E$ such that $re' = e \neq 0 \Rightarrow e' \notin \mathcal{A}_r$. Then $sre' = se = 0$, so $e' \in \mathcal{A}_{sr} = \mathcal{A}_r$, which is a contradiction.

For the reverse statement note that $\mathcal{A}_r \subseteq \mathcal{A}_{sr}$, by Lemma 3.10(ii). If $s \in \mathrm{U}(R)$ then trivially $\mathcal{A}_{sr} = \mathcal{A}_r$. If $s \notin \mathrm{U}(R)$ and $l_p(s) = 0$, and we assume that $\mathcal{A}_r \subsetneq \mathcal{A}_{sr}$, then there exists $e \in \mathcal{A}_{sr}$ such that $re \neq 0$. We have the following implications:

$$s(re) = sre = 0 \Rightarrow re \in \mathcal{A}_s \Rightarrow \mathcal{A}_s \neq 0 \Rightarrow l_p(s) \neq 0 : \text{ contradiction.}$$

∎

Remark 3.15 1. In view of Corollary 3.14, it is obvious that in the definition of the layers $\mathrm{soc}^n(E)$ of the elementary socle series of E it suffices to consider only the pp-definable subgroups $\mathcal{A}_r$ for elements r with all prime left divisors similar to p.

2. If $0 \neq r \in R$ then, according to Proposition 2.55, r can be written as the least common left multiple of a finite number of indecomposable elements: $r = [r_1, \ldots, r_n]_*$. If, in addition, we assume that r has all prime left divisors similar to p, then all the indecomposable components in this decomposition of r are $\widehat{p}$-indecomposable. [Assume there is a component, say r_1, which is q-indecomposable, with $q \not\sim p$, and that the decomposition is irredundant. Since $(r_1, [r_2, \ldots, r_n]_*)_* \neq r_1$, it follows by 2.53(iii) that ${}^{[r_2,\ldots,r_n]_*}r_1$ is $\widehat{q}$-indecomposable. Therefore $r = {}^{[r_2,\ldots,r_n]_*}r_1 [r_2, \ldots, r_n]_*$ has a prime left divisor not similar to p.]

Lemma 3.16 (i) *If a is $\widehat{p}$-indecomposable, then there exists $0 \neq e \in \mathcal{A}_a \subset E$ such that $\mathrm{ann}_R(e) = Ra$.*

(ii) *If $a, b \in R$ and a is $\widehat{p}$-indecomposable then $\mathcal{A}_a \subseteq \mathcal{A}_b$ if and only if $b \in Ra$.*

(iii) *If $a \in \widehat{\mathcal{P}}$ with $l_p(a) = 1$, then $\mathrm{ann}_R(e) = Ra$, for all $0 \neq e \in \mathcal{A}_a$.*

(iv) *If $r_1, r_2 \in \widehat{\mathcal{P}}$ with $l_p(r_1) = l_p(r_2) = 1$, then $\mathcal{A}_{r_1} \bigcap \mathcal{A}_{r_2} \neq 0$ if and only if $r_1 \doteq r_2$.*

PROOF. (i) Let $a = qr$, for some prime $q \sim p$, and $r \in R$. Since $\mathcal{A}_q \neq 0$, there is $0 \neq e_1 \in E$ such that $qe_1 = 0$, and there exists $e \in E$ such that $re = e_1$, since the module E is divisible. Thus $ae = 0$.

We claim that in fact $\mathrm{ann}_R(e) = Ra$. The left ideal $\mathrm{ann}_R(e) \neq R$ is irreducible, and thus is generated by a $\widehat{p}$-indecomposable element c. Since $a \in \mathrm{ann}_R(e)$, we have that $a = qr = dc$, for some $d \in R$. If d is not a unit, then by Proposition 2.53(ii), d is also indecomposable with unique prime left divisor q, $d = qd_1$, for some $d_1 \in R$. Then $a = qd_1c = qr$, so $r = d_1c$, and it follows that $0 \neq e_1 = re = d_1ce = 0$, a contradiction. Thus $d \in \mathrm{U}(R)$, so $a \doteq c$, and consequently, $\mathrm{ann}_R(e) = Ra$.

(ii) By (i), there exists $0 \neq e \in \mathcal{A}_a$ such that $\mathrm{ann}_R(e) = Ra$, and from $e \in \mathcal{A}_b$ it follows that $b \in \mathrm{ann}_R(e) = Ra$. The converse is immediate by Lemma 3.10(ii).

(iii) Let $0 \neq e \in \mathcal{A}_a$. Then $a \in \mathrm{ann}_R(e) = Rb \neq R$, for some $b \in \widehat{\mathcal{P}}$, and from $l_p(a) = 1$ it follows that $1 \leq l_p(b) \leq 1$, so $l_p(b) = 1$. Then according to Lemma 2.54, $Ra = Rb = \mathrm{ann}_R(e)$.

(iv) If $0 \neq e \in \mathcal{A}_{r_1} \bigcap \mathcal{A}_{r_2}$, then $r_1, r_2 \in \mathrm{ann}_R(e) = Rr$, for some $r \in \widehat{\mathcal{P}}$. But then $l_p(r_1) = l_p(r_2) = 1 \geq l_p(r)$, so $l_p(r) = 1$; since $r_i = u_ir$ for some $u_i \in R$ and

$l_p(r) = l_p(r_i) = 1$ $(i = 1, 2)$, it follows that $l_p(u_1) = l_p(u_2) = 0$. But $r \in \widehat{\mathcal{P}}$, so $u_1, u_2 \in \mathrm{U}(R)$. Then $r \doteq r_1 \doteq r_2$. ∎

Lemma 3.17 (i) *Let $0 \neq r \in R \setminus \mathrm{U}(R)$, and let $\mathcal{F}_r$ be the set of all distinct (up to left associates) $\widehat{\mathfrak{p}}$-indecomposable right divisors of r. Then $\mathcal{A}_r = \bigcup_{u \in \mathcal{F}_r} \mathcal{A}_u = \sum_{u \in \mathcal{F}_r} \mathcal{A}_u$.*

(ii) *If $r = [r_1, \dots, r_n]_*$ for some $r_1, \dots, r_n \in R$, then $\mathcal{A}_r = \sum_{i=1}^{n} \mathcal{A}_{r_i}$.*

Proof. (i) If $0 \neq e \in \mathcal{A}_r \subseteq E$, then $r \in \mathrm{ann}_R(e) = Rs$, for some $s \in \widehat{\mathcal{P}}$, and it follows that $s \in \mathcal{F}_r$ and $e \in \mathcal{A}_s \subseteq \bigcup_{u \in \mathcal{F}_r} \mathcal{A}_u$. On the other hand, $\bigcup_{u \in \mathcal{F}_r} \mathcal{A}_u \subseteq \mathcal{A}_r$ since $\mathcal{A}_u \subseteq \mathcal{A}_r$ for all $u \in \mathcal{F}_r$, by 3.10.

Part (ii) is proved by induction on the number n of components in a decomposition of r of the type mentioned. The result is trivially true for $n = 1$. Assume now that $n \geq 1$ and that the result is true for any $1 \leq n' < n$.

Let $r = [r_1, r_2, \dots, r_n]_*$. It is sufficient to prove that $\mathcal{A}_r \subseteq \sum_{i=1}^{n} \mathcal{A}_{r_i}$. If we denote $r' = [r_1, \dots, r_{n-1}]_*$ and $(r', r_n)_* = d$, then $r_n = wd$ and $r' = w'd$ for some $w, w' \in R$, and there exist $a, b \in R$ such that $ar' + br_n = d$.

Let $e \in \mathcal{A}_r$. The module E is divisible, so there exist $e_n, e' \in E$ such that $de_n = ar'e$ and $de' = br_n e$. By multiplying $ar' + br_n = d$ on the left by w and w', respectively, we write $war' + wbr_n = wd = r_n$ and $w'ar' + w'br_n = w'd = r'$. By rearranging the terms we obtain $war' = (1 - wb)r_n \in Rr' \cap Rr_n = Rr$ and $w'br_n = (1 - w'a)r' \in Rr' \cap Rr_n = Rr$. Since $e \in \mathcal{A}_r$, it follows that $0 = war'e = wde_n = r_n e_n$ and $0 = w'br_n e = w'de' = r'e'$, so $e_n \in \mathcal{A}_{r_n}$ and $e' \in \mathcal{A}_{r'}$.

From $ar' + br_n = d$, $de_n = ar'e$ and $de' = br_n e$ it follows that $de = de' + de_n$, so $e - (e' + e_n) \in \mathcal{A}_d \subseteq \mathcal{A}_{r'}, \mathcal{A}_{r_n} \subseteq \mathcal{A}_{r'} + \mathcal{A}_{r_n}$. Then $e \in \mathcal{A}_{r'} + \mathcal{A}_{r_n}$, since $e' \in \mathcal{A}_{r'}$ and $e_n \in \mathcal{A}_{r_n}$. Recall that $r' = [r_1, \dots, r_{n-1}]_*$, so by the induction hypothesis, $\mathcal{A}_{r'} = \sum_{i=1}^{n-1} \mathcal{A}_{r_i}$.

Thus $e \in \mathcal{A}_{r'} + \mathcal{A}_{r_n} = \sum_{i=1}^{n} \mathcal{A}_{r_i}$. ∎

Remark 3.18 If r is $\widehat{p}$-indecomposable, then there exists $e \in E$ such that $\mathrm{ann}_R(e) = Rr$, and it follows that $e \in \mathcal{A}_r \setminus \mathcal{A}_u$ for all $r \not\asymp u \in \mathcal{F}_r$. Therefore $\bigcup_{r \not\asymp u \in \mathcal{F}_r} \mathcal{A}_u \subsetneqq \mathcal{A}_r$.

My next goal is to describe the summands in $\mathrm{soc}^1(E)$; that is, the non-zero minimal pp-definable subgroups $\mathcal{A}_r$. Note that $\mathrm{soc}_1(E)$ is a simple module and E is uniform, and since $\mathrm{soc}^1(E)$ is a submodule of E, it follows that $\mathrm{soc}_1(E) \leq \mathrm{soc}^1(E)$. This inclusion is in general proper (see later Example 3.26(i)).

Lemma 3.19 *Let $E = E_R(R/Rp)$, p prime, and let $r \in R$.*
$\mathcal{A}_r$ is minimal such that $\mathcal{A}_r \neq 0$ if and only if $l_p(r) = 1$.

Proof. $\mathcal{A}_r \neq 0$ if and only if $l_p(r) \geq 1$, by Proposition 3.13.

If $l_p(r) \geq 2$, then by Theorem 2.58 we can factor $r = r'ab$ such that $l_p(r') = 0$, $l_p(a) = 1$, a $\widehat{p}$-indecomposable, and $l_p(b) \geq 1$. It follows that $\mathcal{A}_a \neq 0$ and $\mathcal{A}_b \neq 0$, and by Lemma 3.10(ii), $\mathcal{A}_b \subseteq \mathcal{A}_{r'ab} = \mathcal{A}_r$. We claim now that $\mathcal{A}_b \subsetneqq \mathcal{A}_r$. Let $0 \neq e \in \mathcal{A}_a$. The module E is divisible, so there exists $0 \neq e' \in E$ such that $be' = e$, so $e' \notin \mathcal{A}_b$. But $re' = r'abe' = r'a(be') = r'ae = 0$, so $0 \neq e' \in \mathcal{A}_r$.

Conversely, if $l_p(r) = 1$ and $0 \neq \mathcal{A}_s \subseteq \mathcal{A}_r$, then according to Lemma 3.16(iii), $s \in \mathrm{ann}_R(e) = Rr$ for every $0 \neq e \in \mathcal{A}_s \subseteq \mathcal{A}_r$, so $\mathcal{A}_r \subseteq \mathcal{A}_s$, and it follows that $\mathcal{A}_s = \mathcal{A}_r$. ∎

Lemma 3.20 *Let $0 \neq r \in R$ and $n \geq 0$.*

(i) *If $\mathcal{A}_r \not\subset \mathrm{soc}^n(E)$, then there is $s \in \widehat{\mathcal{P}}$ such that $\mathcal{A}_s \subseteq \mathcal{A}_r$ and $\mathcal{A}_s \not\subset \mathrm{soc}^n(E)$.*

(ii) *If $\mathcal{A}_r$ is minimal over $\mathrm{soc}^n(E)$, then there exists $s \in \widehat{\mathcal{P}}$ such that $\mathcal{A}_s = \mathcal{A}_r$; furthermore, if $e \in \mathcal{A}_r \setminus \mathrm{soc}^n(E)$, then $\mathrm{ann}_R(e) = Rs$.*

(iii) *If $r \in \widehat{\mathcal{P}}$ and $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$, $\mathcal{A}_r \not\subseteq \mathrm{soc}^{n-1}(E)$, then $\mathcal{A}_r$ is minimal over $\mathrm{soc}^{n-1}(E)$.*

Proof. (i) From $\mathcal{A}_r \not\subset \mathrm{soc}^n(E)$, it follows that there exists $e \in \mathcal{A}_r$, but $e \notin \mathrm{soc}^n(E)$. The left ideal $\mathrm{ann}_R(e) = Rs$ is irreducible, hence $s \in \widehat{\mathcal{P}}$. Furthermore $r \in \mathrm{ann}_R(e) = Rs$, so $\mathcal{A}_s \subseteq \mathcal{A}_r$, and also $\mathcal{A}_s \not\subset \mathrm{soc}^n(E)$, since $e \notin \mathrm{soc}^n(E)$.

(ii) The first part is immediate by (i) and the minimality of $\mathcal{A}_r$.
Let $e \in \mathcal{A}_r \setminus \mathrm{soc}^n(E) = \mathcal{A}_s \setminus \mathrm{soc}^n(E)$ with $\mathrm{ann}_R(e) = Ru$. By the argument of part (i),

$u \in \widehat{\mathcal{P}}$, and since $\mathcal{A}_r = \mathcal{A}_s$ is minimal over $\mathrm{soc}^n(E)$, $\mathcal{A}_u = \mathcal{A}_s$. Since both $u, s \in \widehat{\mathcal{P}}$, by 3.16(ii), $Ru = Rs$, that is, $\mathrm{ann}_R(e) = Rs$.

(iii) If $r \in \widehat{\mathcal{P}}$, then $r = qs$ for unique (up to right/left associates) $q \sim p$, $s \in R$, so every proper pp-definable subgroup of $\mathcal{A}_r$ is contained in $\mathcal{A}_s$.

For every $0 \neq e \in \mathcal{A}_r \subseteq \mathrm{soc}^n(E)$, there exists an integer k ($1 \leq k \leq n$) such that $e \in \mathrm{soc}^k(E) \setminus \mathrm{soc}^{k-1}(E)$. By 1.89(ii), there exists a pp-definable subgroup $\mathcal{A}_{r'}$ minimal over $\mathrm{soc}^{k-1}(E)$ containing e, and from part (ii) it follows that $\mathrm{ann}_R(e) = Rr'$. Then $r \in Rr'$ since $re = 0$, so $\mathcal{A}_{r'} \subseteq \mathcal{A}_r$. Therefore we can write:

$$\mathcal{A}_r = \bigcup \{\mathcal{A}_{r'} : \mathcal{A}_{r'} \text{ minimal over } \mathrm{soc}^{k-1}(E) \text{ for some } 1 \leq k \leq n \text{ and } \mathcal{A}_{r'} \subseteq \mathcal{A}_r\}.$$

If $\mathcal{A}_r$ is not minimal over $\mathrm{soc}^{n-1}(E)$, then all the components in this union are proper pp-definable subgroups of $\mathcal{A}_r$, and hence in $\mathcal{A}_s$. But then we obtain a contradiction:

$$\mathcal{A}_s \subsetneq \mathcal{A}_r = \bigcup \{\mathcal{A}_{r'} : \mathcal{A}_{r'} \text{ minimal over } \mathrm{soc}^{n-1}(E) \text{ and } \mathcal{A}_{r'} \subseteq \mathcal{A}_r\} \subseteq \mathcal{A}_s.$$

Thus $\mathcal{A}_r$ is necessarily minimal over $\mathrm{soc}^{n-1}(E)$. ∎

Remark 3.21 Let $r \in \widehat{\mathcal{P}}$, with $r = qs$, $q \in \widehat{p}$. The proof of part (iii) of the previous lemma shows that if $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$ and $\mathcal{A}_r \not\subseteq \mathrm{soc}^{n-1}(E)$, then $\mathcal{A}_r$ is minimal over $\mathrm{soc}^{n-1}(E)$ if and only if $\mathcal{A}_s \subseteq \mathrm{soc}^{n-1}(E)$.

Definition 3.22 Let $0 \neq r \in R$. The p-height $\mathrm{ht}_p(r)$ of r is defined inductively:

- If $l_p(r) = 0$, then $\mathrm{ht}_p(r) = 0$;
- If $r \in \widehat{\mathcal{P}}$ with $r = qs$ where $q \sim p$ and $s \in R$, then $\mathrm{ht}_p(r) = 1 + \mathrm{ht}_p(s)$;
- If $l_p(r) \geq 1$, then $\mathrm{ht}_p(r) = \max\{\mathrm{ht}_p(u) : u \in \mathcal{F}_r\}$, where $\mathcal{F}_r$ is the set of all $\widehat{p}$-indecomposable right divisors of r.

Remark 3.23
1. The p-height $\mathrm{ht}_p(r)$ of r is well-defined since $\mathrm{ht}_p(u) \leq l(u) \leq l(r)$ for all $u \in \widehat{\mathcal{P}}$ which are right divisors of r, and also, $\mathrm{ht}_p(r) \leq l_p(r)$.
2. Note that if $q \sim p$, then $\mathrm{ht}_p(q) = 1$, and if $r \in \widehat{\mathcal{P}}$ with $l_p(r) = 1$, then $\mathrm{ht}_p(r) = 1$.

Theorem 3.24 *Let $E = E(R/Rp)$, $p \in R$ prime.*

(i) $\mathrm{soc}^1(E) \;=\; \sum\left\{\mathcal{A}_r : r \in \widehat{\mathcal{P}},\, l_p(r) = 1\right\} \;=\; \bigcup\left\{\mathcal{A}_r : r \in \widehat{\mathcal{P}},\, l_p(r) = 1\right\}$

(ii) *For all* $0 \neq r \in R \setminus \mathrm{U}(R)$*, if* $l_p(r) \leq n$ *then* $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$*.*

(iii) *For all* $0 \neq r \in R$*,* $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$ *if and only if* $\mathrm{ht}_p(r) \leq n$*.*

(iv) $E = \bigcup\limits_{n<\omega} \mathrm{soc}^n(E)$*.*

Proof. (i) By 3.19, the minimal pp-definable subgroups are precisely those $\mathcal{A}_r$, with $r \in \widehat{\mathcal{P}}$, $l_p(r) = 1$. The result follows by the definition of $\mathrm{soc}^1(E)$ and by 1.89(ii).

(ii) We prove the result by induction on $n = l_p(r)$. The result is true for $n = 1$ by (i).

Suppose $n \geq 1$, and for all non-zero $r' \in R \setminus \mathrm{U}(R)$, if $l_p(r') \leq n$, then $r' \in \mathrm{soc}^n(E)$, and consider $r \in R$ such that $l_p(r) \leq n+1$. Clearly we only need to consider the case $l_p(r) = n+1$, and we can assume that $\mathcal{A}_r \not\subseteq \mathrm{soc}^n(E)$. There exists $e \in \mathcal{A}_r \setminus \mathrm{soc}^n(E)$, so by 1.89(ii) and 3.20 we can find a pp-definable subgroup $\mathcal{A}_s$ minimal over $\mathrm{soc}^n(E)$ such that $e \in \mathcal{A}_s$ and $\mathrm{ann}_R(e) = Rs$. Then $r \in Rs$ since $re = 0$, and it follows that $\mathcal{A}_s \subseteq \mathcal{A}_r$.

If $\mathcal{A}_r$ is not minimal over $\mathrm{soc}^n(E)$, then $\mathcal{A}_s \subsetneq \mathcal{A}_r$, and since $r = ts$ for some t, it follows that $l_p(s) \leq n+1$. But $\mathcal{A}_s \not\subseteq \mathrm{soc}^n(E)$, so $l_p(s) > n$, by the induction hypothesis. Thus $l_p(s) = n+1$, so $l_p(t) = 0$, implying from 3.14 that $\mathcal{A}_r = \mathcal{A}_s \subsetneq \mathcal{A}_r$, a contradiction.

(iii) By 3.17, $\mathcal{A}_r = \sum\limits_{u \in \mathcal{F}_r} \mathcal{A}_u$, where $\mathcal{F}_r$ is the set of all right divisors of r from $\widehat{\mathcal{P}}$. The proof will be done by induction on n. The result is true for $n = 0$. Assume $n \geq 1$.

Assume $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$ and let $u \in \mathcal{F}_r \subset \widehat{\mathcal{P}}$.

If $\mathcal{A}_u \subseteq \mathrm{soc}^{n-1}(E)$, then $\mathrm{ht}_p(u) \leq n-1 < n$.

If $\mathcal{A}_u \not\subseteq \mathrm{soc}^{n-1}(E)$, then by Lemma 3.20(iii), $\mathcal{A}_u$ is minimal over $\mathrm{soc}^{n-1}(E)$. Since $u \in \widehat{\mathcal{P}}$, there are $q \sim p$, $s \in R$ such that $u = qs$; then $\mathcal{A}_s \subset \mathrm{soc}^{n-1}(E)$ since $\mathcal{A}_s \subsetneq \mathcal{A}_u$ and $\mathcal{A}_u$ is minimal over $\mathrm{soc}^{n-1}(E)$. By the induction hypothesis, $\mathrm{ht}_p(s) \leq n-1$, so $\mathrm{ht}_p(u) = 1 + \mathrm{ht}_p(s) \leq n$.

Thus $\mathrm{ht}_p(r) = \max\{\,\mathrm{ht}_p(u) : \; u \in \mathcal{F}_r\,\} \leq n$, since $\mathrm{ht}_p(u) \leq n$ for all $u \in \mathcal{F}_r$.

Conversely, if $\mathrm{ht}_p(r) \leq n$, then $\mathrm{ht}_p(u) \leq n$, for all $u \in \mathcal{F}_r$.

If $\mathrm{ht}_p(u) \leq n-1$, then by the induction hypothesis, $\mathcal{A}_u \subseteq \mathrm{soc}^{n-1}(E) \subseteq \mathrm{soc}^n(E)$.

If $\mathrm{ht}_p(u) = n$, then $\mathrm{ht}_p(u) = \mathrm{ht}_p(r) = n$ and there exist $q \sim p$, $s \in R$ such that $u = qs$. Thus $\mathrm{ht}_p(s) = n - 1$, and by the induction hypothesis it follows that $\mathcal{A}_s \subseteq \mathrm{soc}^{n-1}(E)$. Since every proper pp-definable subgroup of $\mathcal{A}_u$ is contained in $\mathcal{A}_s$, it follows that $\mathcal{A}_u$ is minimal over $\mathrm{soc}^{n-1}(E)$, hence in $\mathrm{soc}^n(E)$. Since $\mathcal{A}_r = \sum_{u \in \mathcal{F}_r} \mathcal{A}_u$ and $\mathcal{A}_u \subset \mathrm{soc}^n(E)$ for all $u \in \mathcal{F}_r$, it follows that $\mathcal{A}_r \subseteq \mathrm{soc}^n(E)$.

(iv) Trivially, $E \supseteq \bigcup_{n<\omega} \mathrm{soc}^n(E)$. On the other hand, if $0 \neq e \in E$ and $\mathrm{ann}_R(e) = Rr$, where $n = l_p(r)$, then by (ii), $e \in \mathrm{soc}^n(E)$. ∎

Corollary 3.25 *Let $E = E(R/Rp)$, p prime, and let $e \in E$ with $\mathrm{ann}_R(e) = Ru$. If $e \notin \mathrm{soc}^{n-1}(E)$, then $l_p(u) \geq n$. Furthermore, if $\mathcal{A}_u \not\subseteq \mathrm{soc}^{n-1}(E)$ then $l_p(u) \geq n$.*

Example 3.26 Let $R = k(x)[y, \partial/\partial x]$ and $E = E(R/Ry)$, $p = y$.

1. Let $r = (y + \frac{1}{x})(y^2 + x)$, $\widehat{y}$-indecomposable with $l(r) = 2$ and $l_p(r) = 1$. Thus $\mathcal{A}_r \subseteq \mathrm{soc}^1(E)$. By contrast, we saw in Example 3.9(2) that any solution e with $\mathrm{ann}_R(e) = Rr$ is in $\mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$.
2. The solution $\ln(x)$ of $(y+\frac{1}{x})y \cdot v = 0$ is in $\mathrm{soc}^2(E) \setminus \mathrm{soc}^1(E)$, and in $\mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$. (Recall that $y + \frac{1}{x} \sim y$.)
3. Let $e \in E$ with $\mathrm{ann}_R(e) = Ru$, where $u = y \,{}_*\big[y - \frac{1}{x}, y - \frac{1}{x+1}\big] = y\,[u_1, u_2]_*$ (see 2.21), where $u_1 = y - \frac{1}{x} - \frac{2x+1}{x(x+1)}$ and $u_2 = y - \frac{1}{x+1} - \frac{2x+1}{(x+1)x}$. Then $e \in \mathrm{soc}^2(E) \setminus \mathrm{soc}^1(E)$.

   Note that every proper subgroup of $\mathcal{A}_u$ is in $\mathcal{A}_{[u_1, u_2]_*} = \mathcal{A}_{u_1} + \mathcal{A}_{u_2} \subseteq \mathrm{soc}^1(E)$ (by 3.17). Thus $\mathcal{A}_u$ is a minimal subgroup over $\mathrm{soc}^1(E)$, hence a summand in $\mathrm{soc}^2(E)$, but $l_y(u) = 3 > \mathrm{ht}_y(u) = 2$.

   This example can easily be generalized to give elements of arbitrary length, but of p-height 2, so parts (ii) and (iii) of Theorem 3.24 cannot be made sharper.

The next step in describing $E = E(R/Rp)$ would be to find canonical descriptions for either $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ or $\mathrm{soc}^n(E)/\mathrm{soc}^{n-1}(E)$. But this is difficult, if not impossible, because R may have many similarity classes of primes, all of which may enter into the structure of E (as we have seen, for instance, in Example 3.26(1) where the prime $y^2 + 1$, which is not similar to y, forces $\mathrm{soc}^1(E)$ to be different from $\mathrm{soc}_1(E)$). But if q is a prime not similar to p, there is unique division by q in E, so this suggests that we pass to a larger ring by localization.

3.4 Localization

Over a commutative ring R, Matlis described an indecomposable injective module $E = E(R/P)$, for a prime ideal P, and a multitude of results regarding E and $\mathrm{End}(E)$ were obtained by localizing the ring at P to get R_P, then by considering the P-adic completion $\overline{R}_P$ of R_P. In a natural way, over $\overline{R}_P$, E is the indecomposable, injective $\overline{R}_P$-module corresponding to the unique maximal ideal of $\overline{R}_P$. Moreover, ${}_{\overline{R}_P}E$ is the union of its socle series, and $\overline{R}_P$ coincides with the R-endomorphism ring of E.

These generalize the fact that if R is the ring of rational integers, then an indecomposable injective module is either $\mathbb{Q}$ or $E = E(\mathbb{Z}/\mathbb{Z}p) \cong \mathbb{Z}(p^\infty)$, where p is a prime number. Furthermore, when $E \neq \mathbb{Q}$, $E = \bigcup_{n\geq 1} \mathrm{soc}_n(E) = \varinjlim \mathbb{Z}/\mathbb{Z}p^n$ and the endomorphism ring of E is the ring $\overline{\mathbb{Z}}_p$ of p-adic integers.

It seems that if we hope for a more complete description of an indecomposable injective module $E = E(R/Rp)$ over a principal ideal domain R, it might be advantageous to construct some ring of fractions of R. The study of the socle series and of the elementary socle series of ${}_RE$ has given us some insight on how the elements of the ring act on the elements of E, and the previous section gives evidence in favor of the localization of the ring R at the set of all elements that act regularly on E. Actually, Goodearl has shown in [13] that when R is a PLID and ${}_RE$ is injective, the set of all elements of the ring that act regularly on E is a left denominator set in R, and the corresponding localization of R has a unique (up to isomorphism) simple left module. This kind of localization gives us a potential candidate for an extension of R that will allow us to develop some useful machinery for describing the layers of the module E.

Recall from Proposition 3.13 that the elements of R acting regularly on E are either units or have no prime factors similar to p. Hence by Goodearl's result [13] this set is left Ore. However, in the present setting we can give a straightforward proof by the arithmetic of the ring.

Theorem 3.27 *Let R be a PID, and consider p an arbitrary prime element in R. The set $\Sigma = \{\, r \in R \setminus \{0\} \,:\, l_p(r) = 0\}$ is a left denominator set for R, so according to Theorem 1.41, the ring R has a left ring of fractions $\mathfrak{R} = \Sigma^{-1}R$ with respect to Σ.*

Furthermore, Σ is also a right denominator set for R, and by 1.43,

$$\mathfrak{R} = \Sigma^{-1}R = R\Sigma^{-1}.$$

PROOF. First, note that Σ is the multiplicative monoid generated by 1 and all the prime elements not similar to p. Thus every divisor of an element of Σ is also an element of Σ. By 2.30(vi), the transform of an element of Σ is also an element of Σ. Since the ring is a domain, we will only check the left Ore condition (see Definition 1.40).

For all $s_1 \in \Sigma$, $q_1 \in R$, there exists a greatest common right divisor of these two elements: $d = (\, s_1 \,, q_1 \,)_*$ and $s_1 = \overline{s}_1 d$, $q_1 = \overline{q}_1 d$, where $\overline{s}_1$, $d \in \Sigma$, $\overline{q}_1 \in R$. Evidently $[\, \overline{s}_1 \,, \overline{q}_1 \,]_* \, d = {}^{\overline{s}_1}\overline{q}_1 \cdot \overline{s}_1 \cdot d = {}^{\overline{q}_1}\overline{s}_1 \cdot \overline{q}_1 \cdot d$, so ${}^{\overline{s}_1}\overline{q}_1 \cdot \overline{s}_1 = {}^{\overline{q}_1}\overline{s}_1 \cdot \overline{q}_1$, and by denoting $q_2 = {}^{\overline{s}_1}\overline{q}_1$ and $s_2 = {}^{\overline{q}_1}\overline{s}_1$, it follows that $q_2 s_1 = s_2 q_1$. By Lemma 2.41, it follows that $l_p(s_2) \leq l_p(\overline{s}_1) \leq l_p(s_1) = 0$, so $s_2 \in \Sigma$.

The right Ore condition for Σ follows by symmetry. ∎

Remark 3.28
1. It was shown in the proof of the previous theorem that the transform of an element of Σ is also an element of Σ.
2. Since R is a PID, the ring $\mathfrak{R} = \Sigma^{-1}R = R\Sigma^{-1}$ is also a principal left and right ideal domain.

Example 3.29 If $R = K[y; \sigma, \delta]$, where K is a field, $\sigma \in \mathrm{Aut}(K)$ and δ is a σ-derivation, then it follows from 2.19 that every $0 \neq s \in R$ with $\deg(s) < \deg(p)$ is in Σ.

We indicate next some computational properties of the ring $\mathfrak{R} = \Sigma^{-1}R$.

Lemma 3.30 *If $s^{-1}a = t^{-1}b$ in $\mathfrak{R}$ then ${}^{\overline{s}}\overline{t}\, a \doteq {}^{\overline{t}}\overline{s}\, b$ in R, where $\overline{s}, \overline{t} \in \Sigma$ are such that $(\, s \,, t \,)_* \doteq d$, $s = \overline{s}d$, $t = \overline{t}d$.*

PROOF. From the definition of equivalent fractions in $\Sigma^{-1}R$, $s^{-1}a = t^{-1}b \in \mathfrak{R}$ if and only if there exist u, $w \in R$ such that $ua = wb$ and $us = wt \in \Sigma$.

If $s^{-1}a = t^{-1}b \in \mathfrak{R}$, we can assume w.l.o.g. that there exist u, v as stated with ${}_*(u, w) = 1$ [if ${}_*(u, w) \neq 1$ we can cancel the above equations on the left by ${}_*(u, w)$]. Also notice that both u, w are elements of Σ. Then $us = wt \doteq [s, t]_* \in \Sigma$.

At the same time, if we let $d \doteq (s, t)_*$, then there exist $\overline{s}, \overline{t}$ such that $s = \overline{s}d, t = \overline{t}d$, so $[s, t]_* \doteq {}^{\overline{s}}\overline{t}\,\overline{s}d \doteq {}^{\overline{t}}\overline{s}\,\overline{t}d$ in R. Then $[s, t]_* \doteq {}^{\overline{s}}\overline{t}\, s \doteq {}^{\overline{t}}\overline{s}\, t$, and $w \doteq {}^{\overline{t}}\overline{s}$ and $u \doteq {}^{\overline{s}}\overline{t}$; from $ua = wb$ it follows that ${}^{\overline{s}}\overline{t}\, a \doteq {}^{\overline{t}}\overline{s}\, b$. ∎

Proposition 2.30(ii) helps simplifying the condition for equivalent fractions in $\mathfrak{R}$, and consequently the operations in $\mathfrak{R}$ can now be explicitly described.

Proposition 3.31 *According to 2.23 and 2.25, the least common left multiple and the transform are defined up to left multiplication by units. Then one can always choose representatives from these equivalence classes, so that the arithmetic of the ring $\mathfrak{R}$ can be described as follows:*

- $s^{-1}a = t^{-1}b$ *in* $\mathfrak{R}$ *implies that* ${}^{s}t\, a = {}^{t}s\, b$ *in* R.
- $s^{-1}a + t^{-1}b = [s, t]_*^{-1}\,({}^{s}t\, a + {}^{t}s\, b)$
- $(s^{-1}a)(t^{-1}b) = ({}^{a}t\, s)^{-1}({}^{t}a\, b)$

Furthermore, if M is a Σ-torsionfree left R-module then, according to 1.58, the addition and multiplication by elements of $\mathfrak{R}$ are defined in the corresponding module of fractions $\Sigma^{-1}M$ as follows :

- $s^{-1}x = t^{-1}y$ *in* $\Sigma^{-1}M$ *implies that* ${}^{s}t\, x = {}^{t}s\, y$ *in* M.
- $s^{-1}x + t^{-1}y = [s, t]_*^{-1}\,({}^{s}t\, x + {}^{t}s\, y)$
- $(s^{-1}a)(t^{-1}x) = ({}^{a}t\, s)^{-1}({}^{t}a\, x)$

PROOF. These rules can easily be deduced from the definition of $+$ and $\cdot$ in a left ring of fractions (module of fractions, respectively), and from the properties of $[\ ,\ ]_*$ and of the right transform of an element (Proposition 2.30), as well as from how the left Ore condition is satisfied as was checked in the proof of Proposition 3.27. ∎

Lemma 3.32 $\mathrm{U}(\mathfrak{R}) = \{s^{-1}t \in \mathfrak{R} : \ s, t \in \Sigma\}$.

PROOF. Clearly, if $s, t \in \Sigma$ then $s^{-1}t$ is a unit in $\mathfrak{R}$.
Suppose now that $u = s^{-1}a \in \mathrm{U}(\mathfrak{R})$, where $s \in \Sigma, a \in R$; thus there are $t \in \Sigma, b \in R$ such that $(s^{-1}a)(t^{-1}b) = 1$. According to 3.31, $({}^{a}t\, s)^{-1}({}^{t}a\, b) = 1$, so ${}^{t}a\, b = {}^{a}t\, s \in \Sigma$. If a had a prime factor similar to p, then so would ${}^{t}a$, thus $a \in \Sigma$. ∎

Lemma 3.33 *Let $0 \neq r \in R \setminus \mathrm{U}(R)$.*
Then r is a prime element of $\mathfrak{R}$ if and only if $l_p(r) = 1$.

PROOF. This is a direct consequence of the fact that all elements $r \in R$ with $l_p(r) = 0$ are units in $\mathfrak{R}$. ∎

Proposition 3.34 *The ring $\mathfrak{R}$ has only one similarity class $\widehat{\mathfrak{p}}$ of prime elements, and $\widehat{p} \subseteq \widehat{\mathfrak{p}}$.*

PROOF. Let $q \in \mathfrak{R}$ be prime. Then there are $a \in R$, $s \in \Sigma$ such that $q = s^{-1}a$, so $sq = a \in R$ is also prime in $\mathfrak{R}$, so $l_p(a) = 1$. Then $a = bp_1c$, for some $b, c \in \Sigma$ and $p_1 \sim p$ in R, and it follows from $sq = bp_1c$ that q is a left and right associate in $\mathfrak{R}$ of p_1, and by 2.18, q is similar to $p_1 \sim p$. We conclude that all primes in $\mathfrak{R}$ are similar to p, hence $\mathfrak{R}$ has only one similarity class of prime elements.

The last statement is a direct consequence of Lemma 3.33. ∎

Corollary 3.35 *$\mathfrak{R}$ has only one simple left module (up to isomorphism), and the indecomposable injective modules are $E = E(\mathfrak{R}/\mathfrak{R}p)$ and $E(\mathfrak{R}) \cong E(R)$.*

Remark 3.36

1. If $r \in R \subset \mathfrak{R}$, then the p-length $l_p(r)$ of r in R equals the length $l_{\mathfrak{R}}(r)$ of r in $\mathfrak{R}$: $l_p(r) = l_{\mathfrak{R}}(r)$.

   Furthermore, if $0 \neq r \in R \setminus \mathrm{U}(\mathfrak{R})$, then a factorization in R of r as presented in Theorem 2.58 corresponds to a prime factorization in $\mathfrak{R}$ of r. In other words, if $r = vv_1v_2 \dots v_n$, where $l_p(v) = 0$ and $v_i \in \widehat{\mathcal{P}}$ with $l_p(v_i) = 1$, then $r = (vv_1)v_2 \dots v_n$ is a prime factorization of r in $\mathfrak{R}$, since $(vv_1), v_2, \dots, v_n \in \widehat{\mathfrak{p}}$.
2. If Ru is irreducible in R, with $l_p(u) = 1$, then $\mathfrak{R}u$ is a maximal left ideal of $\mathfrak{R}$.
3. For every left ideal $I \leq \mathfrak{R}$, there exists $u \in R$ such that $I = \mathfrak{R}u$.

Proposition 3.37 $\Sigma^{-1}(R/Rp)\,{}_{\mathfrak{R}}\!\cong \mathfrak{R}/\mathfrak{R}p$

PROOF. First of all, we show that $M = R/Rp$ is Σ-torsionfree.
Assume that $0 \neq r + Rp \in M$ and $a \in \operatorname{ann}_R(r + Rp)$. Then $r + Rp \in \mathcal{A}_a$, and by Lemma 3.13, $l_p(a) \neq 0$. Hence $a \notin \Sigma$. Thus R/Rp is Σ-torsionfree.

By Corollary 1.62, the $\mathfrak{R}$-module $\Sigma^{-1}M$ is simple. Also, since $\widehat{p} \subseteq \widehat{\mathfrak{p}}$, $\mathfrak{R}/\mathfrak{R}p$ is also a simple module. Define $\phi : R/Rp \longrightarrow \mathfrak{R}/\mathfrak{R}p$ by $\phi(r + Rp) = r + \mathfrak{R}p$. Clearly, $0 \neq \phi$. By Proposition 1.62(iii), $\operatorname{Hom}_{\mathfrak{R}}(\Sigma^{-1}M, \mathfrak{R}/\mathfrak{R}p) \cong \operatorname{Hom}_R(M, \mathfrak{R}/\mathfrak{R}p) \neq 0$, and since both $\Sigma^{-1}M$ and $\mathfrak{R}/\mathfrak{R}p$ are simple, the conclusion follows. ∎

Proposition 3.38 $E_R(R/Rp)\,{}_R\!\cong E_{\mathfrak{R}}(\Sigma^{-1}(R/Rp))\,{}_{\mathfrak{R}}\!\cong E_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{R}p)$

PROOF. By Corollary 1.64. ∎

Theorem 3.39 *Let ${}_{\mathfrak{R}}E = E_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{R}p)$, p prime.*

(i) *The first two layers of the socle series of E are described as follows:*

$$\begin{aligned}
\operatorname{soc}_1({}_{\mathfrak{R}}E) &= \mathfrak{R}/\mathfrak{R}p = \left\{e \in E : ue = 0 \text{ for some } u \in \widehat{\mathfrak{p}}\right\} \\
&\qquad\qquad = \left\{e \in E : ue = 0 \text{ for some } u \in \widehat{\mathcal{P}} \subset R,\ l_p(u) = 1\right\} \\
\operatorname{soc}_2({}_{\mathfrak{R}}E) &= \sum \left\{\mathfrak{R}e : e \in E,\ ue = 0 \text{ for some } u \in \mathfrak{R},\ l_{\mathfrak{R}}(u) = 2\right\} \\
&= \sum \left\{\mathfrak{R}e : e \in E,\ ue = 0 \text{ for some } u \in \widehat{\mathcal{P}} \subset R,\ l_p(u) = 2\right\}
\end{aligned}$$

(ii) *Let $e \in E$ with $\operatorname{ann}_{\mathfrak{R}}(e) = \mathfrak{R}u$, $u \in \widehat{\mathcal{P}}$. If $l_p(u) \leq n$ then $e \in \operatorname{soc}_n({}_{\mathfrak{R}}E)$.*

(iii) ${}_{\mathfrak{R}}E = \displaystyle\bigcup_{n=0}^{\infty} \operatorname{soc}_n({}_{\mathfrak{R}}E)$.

PROOF. The results are direct consequences of Theorem 3.7, Corollary 3.8, and of the arithmetic in the ring $\mathfrak{R}$ that was inherited from R (see 3.36). ∎

Remark 3.40 1. $\operatorname{soc}_1({}_{\mathfrak{R}}E)$ can contain parts of $\operatorname{soc}_n({}_RE)$, for all $n \geq 1$. If $e \in E$ has $\operatorname{ann}_R(e) = Rr$ with $l(r) = n > 1$ and $l_p(r) = 1 = l_{\mathfrak{R}}(r)$, then $e \in \operatorname{soc}_n({}_RE)$ and $e \in \operatorname{soc}_1({}_{\mathfrak{R}}E)$.

2. Over $\mathfrak{R}$, $\operatorname{soc}_{n+1}(E)/\operatorname{soc}_n(E) \cong (\mathfrak{R}/\mathfrak{R}p)^{(I)}$, for some (possibly infinite) index set I, since all simple left $\mathfrak{R}$-modules are isomorphic to $\mathfrak{R}/\mathfrak{R}p$.

3. Note that ${}_R E$ and ${}_{\mathfrak{R}} E$ have exactly the same pp-definable subgroups, that is, if $s \in \Sigma$ then $\mathcal{A}_{s^{-1}r} = \mathcal{A}_r$, $r \in R$. Thus $\mathrm{soc}^n({}_{\mathfrak{R}}E) = \mathrm{soc}^n({}_R E)$, for all $n \geq 0$.
   Furthermore, from 3.24(iii) it follows that the p-height in R of an element $r \in R$ is the same as the p-height in $\mathfrak{R}$ of r.

Corollary 3.41 *Let $E = E(\mathfrak{R}/\mathfrak{R}p)$, p prime.*

(i) $\mathrm{soc}_1({}_{\mathfrak{R}}E) = \mathrm{soc}^1({}_{\mathfrak{R}}E)$

(ii) $\mathrm{soc}_2({}_{\mathfrak{R}}E) \subseteq \mathrm{soc}^2({}_{\mathfrak{R}}E)$.

PROOF. (i) Immediate from Theorem 3.24 and Theorem 3.39.

(ii) If $e \in \mathrm{soc}_2(E)$ is such that $ue = 0$ for some $u \in \widehat{\mathcal{P}}$ with $l_p(u) = 2$, then by Theorem 3.24, $e \in \mathrm{soc}^2(E)$. ∎

We should point out that the result in 3.41(ii) cannot be improved, as Example 3.26 shows that there exists $e \in \mathrm{soc}^2(E) \subseteq E$ with annihilator of p-length 3, which is not in $\mathrm{soc}_2({}_{\mathfrak{R}}E)$.

We also note that although the elementary socle series of the module E is independent of the choice of the ring at least up to Ore localizations, the classical socle series really does depend on the ring.

3.5 Bases for $\mathrm{soc}_n({}_{\mathfrak{R}}E)/\mathrm{soc}_{n-1}({}_{\mathfrak{R}}E)$, $n \geq 1$

In some sense, since there is only one simple module for $\mathfrak{R}$, the structure of the socle layers of $E = E(\mathfrak{R}/\mathfrak{R}p)$ is transparent: $\mathrm{soc}_1(E) \cong \mathfrak{R}/\mathfrak{R}p$, $\mathrm{soc}_{n+1}(E)/\mathrm{soc}_n(E) \cong (\mathfrak{R}/\mathfrak{R}p)^{(I_n)}$, for some set I_n, with basis $\langle (1+\mathfrak{R}p)_i \rangle_{i \in I_n}$. Unfortunately, this does not relate the structure of the socle factors to the structure of E in any way. The choice of basis (see Definition 1.1) at each level is completely arbitrary, with no relationship to the structure found at the previous level.

We will show that once we make a choice of basis at the level of $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$, there is a canonical way to extend it through all levels so that the arithmetic of E is understandable in terms of the basis. Furthermore, this basis for $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$ can be chosen such that $\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_1(E)) = \mathfrak{R}p$, for each basis element, corresponding to the abstract setting $\mathrm{soc}_2(E)/\mathrm{soc}_1(E) \cong (\mathfrak{R}/\mathfrak{R}p)^{(I_2)}$.

In the end, we will see that each element $e + \mathrm{soc}_{n-1}(E)$ of the basis of the $\mathfrak{R}$-module $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ has $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}u$ with $l_{\mathfrak{R}}(u) = n$ (see 3.46). This reflects the generalization of Matlis' theory in the commutative case; for $E = E(R/P)$, R commutative, P prime ideal, the elements at the n^{th} level are annihilated by P^n.

Proposition 3.42 *Let $E = E(\mathfrak{R}/\mathfrak{R}p)$, $p \in R \subset \mathfrak{R}$ prime. Then for every $n \geq 1$,*

$$\mathcal{G}_n = \{\, e + \mathrm{soc}_{n-1}(E) \in E/\mathrm{soc}_{n-1}(E) : \mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-1}(E)) = \mathfrak{R}p \,\}$$

is a generating set for $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$.

Proof. The module $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E) = \mathrm{soc}(E/\mathrm{soc}_{n-1}(E))$ is a sum of simple $\mathfrak{R}$-modules, hence each isomorphic to $\mathfrak{R}/\mathfrak{R}p$. Then for every non-zero element $e + \mathrm{soc}_{n-1}(E) \in \mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ there is a set $\{e_1, \ldots, e_m\} \subseteq E \setminus \mathrm{soc}_{n-1}(E)$ satisfying $\mathrm{ann}_{\mathfrak{R}}(e_i + \mathrm{soc}_{n-1}(E)) = \mathfrak{R}p$ such that $e + \mathrm{soc}_{n-1}(E) = \sum_{i=1}^{m} r_i(e_i + \mathrm{soc}_{n-1}(E))$, for some $r_i \in \mathfrak{R}$. Then $e_i + \mathrm{soc}_{n-1}(E) \in \mathcal{G}_n$. ∎

Notation:

- By the previous proposition, $\mathcal{G}_2$ is a generating set for $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$, that is $\mathrm{soc}_2(E)/\mathrm{soc}_1(E) = \sum_{\hat{e}\in\mathcal{G}_2} \mathfrak{R}\hat{e}$, and furthermore, there exists an $\mathfrak{R}$-linearly independent set $\mathfrak{B}_2 \subseteq \mathcal{G}_2$ such that $\mathrm{soc}_2(E)/\mathrm{soc}_1(E) \cong \bigoplus\{\mathfrak{R}\hat{e} : \hat{e} \in \mathfrak{B}_2\}$.
- Fix $B_2 \subset \mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$, a set of representatives from the cosets in $\mathfrak{B}_2$. Then there is a one-to-one correspondence between B_2 and $\mathfrak{B}_2$.

  Then $0 \neq pe \in \mathrm{soc}_1(E) = \mathfrak{R}/\mathfrak{R}p$, for every $e \in B_2$, and there exists $u_e \in \mathfrak{R}$ such that $pe = u_e + \mathfrak{R}p = u_e(1 + \mathfrak{R}p)$.
- For each $e \in B_2$ fix one such element u_e of $\mathfrak{R}$ such that $0 \neq pe = u_e(1 + \mathfrak{R}p)$, and let $\mathcal{F}$ be the set of all such elements.

Note that $(u, p)_* = 1$ for all $u \in \mathcal{F}$; that is $\mathcal{F} \cap \mathfrak{R}p = \emptyset$, and there is a one-to-one correspondence between B_2 and $\mathcal{F}$. ∎

Lemma 3.43 *For every $u \in \mathcal{F}$, the element ${}^{u}p\,p$ is left and right indecomposable. Thus ${}^{u}p\,p$ has a unique (up to left multiplication by units) prime left divisor ${}^{u}p$ and a unique (up to right multiplication by units) prime right divisor p.*

PROOF. Let $u \in \mathcal{F}$. Since $(u\,,p)_* = 1$, from 2.34(iii) it follows that ${}^{u}p$ is a prime element in $\widehat{p}$, and by 2.45, it is sufficient to prove that ${}^{u}p\,p$ is right indecomposable.

Let $e \in B_2$ be such that $pe = u + \mathfrak{R}p$. Then ${}^{u}p\,pe = {}^{u}p\,u(1+\mathfrak{R}p) = {}^{p}u\,p(1+\mathfrak{R}p) = 0$, so ${}^{u}p\,p \in \operatorname{ann}_{\mathfrak{R}}(e) = \mathfrak{R}a$, a indecomposable. It follows that $l_{\mathfrak{R}}(a) \leq l_{\mathfrak{R}}({}^{u}p\,p) = 2$, and since $e \notin \operatorname{soc}_1(E)$, according to Theorem 3.7, $l_{\mathfrak{R}}(a) \geq 2$. Thus $l_{\mathfrak{R}}(a) = 2$, and by 2.54 it follows that ${}^{u}p\,p \doteq a$, hence right and left indecomposable. ∎

Theorem 3.44 *Let $E = E(\mathfrak{R}/\mathfrak{R}p)$, $p \in R \subset \mathfrak{R}$, p prime. Then:*

$\mathfrak{B}_1 = \{1 + \mathfrak{R}p\}$ *is a basis for* $\operatorname{soc}_1(E)/\operatorname{soc}_0(E) = \mathfrak{R}/\mathfrak{R}p$.

and

$\mathfrak{B}_n = \{e + \operatorname{soc}_{n-1}(E) \in E/\operatorname{soc}_{n-1}(E) : \; pe = re',\; r \in \mathcal{F},\; e' + \operatorname{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}\}$
is a basis for $\operatorname{soc}_n(E)/\operatorname{soc}_{n-1}(E)$ *for all* $n \geq 2$.

The following lemma is needed for the proof of the theorem.

Lemma 3.45 *For every $0 \neq r \in \mathfrak{R}$, there exist $a, b, c_i \in \mathfrak{R}$, $u_i \in \mathcal{F}$ $(1 \leq i \leq m)$ such that*

$$r = ap + pb + \sum_{i=1}^{m} c_i u_i\,, \text{ and } c_i p \in p\mathfrak{R}$$

PROOF. If $r \in \mathfrak{R}p$, then $r = ap$ for some $a \in \mathfrak{R}$ and the desired relation holds with $b = c_i = 0$.

If $r \notin \mathfrak{R}p$, then $(r\,,p)_* = 1$. The module E is divisible, so there is $e \in E$ such that

$$pe = r(1 + \mathfrak{R}p) \in \mathfrak{R}/\mathfrak{R}p = \operatorname{soc}_1(E), \tag{1}$$

and such a solution is necessarily an element of $\operatorname{soc}_2(E)$. If $e \in \operatorname{soc}_1(E) = \mathfrak{R}/\mathfrak{R}p$, then $e = b + \mathfrak{R}p$ for some $b \in R$, and it follows that $p(b + \mathfrak{R}p) = r + \mathfrak{R}p$. Then $r = ap + pb$ for some $a \in \mathfrak{R}$, and the result is satisfied in this case for a, b chosen above and all $c_i = 0$.

If $e \in \operatorname{soc}_2(E) \setminus \operatorname{soc}_1(E)$, then $e + \operatorname{soc}_1(E) \neq 0$ is generated by the elements of $\mathfrak{B}_2$:

$$e + \operatorname{soc}_1(E) = \sum_{i=1}^{m} w_i(e_i + \operatorname{soc}_1(E)), \text{ where } e_i \in B_2, w_i \in \mathfrak{R}, i \leq m. \tag{2}$$

From (1) and (2), it follows that $\sum_{i=1}^{m}(pw_i)e_i \in \mathrm{soc}_1(E)$, and since B_2 is $\mathfrak{R}$-linearly independent over $\mathrm{soc}_1(E)$, $(pw_i)e_i \in \mathrm{soc}_1(E)$ for each i, so $pw_i \in \mathrm{ann}_{\mathfrak{R}}(e_i + \mathrm{soc}_1(E)) = \mathfrak{R}p$, and thus:

$$pw_i = c_i p,\ c_i \in \mathfrak{R}. \tag{3}$$

Also, from (2) we have $e = \sum_{i=1}^{m} w_i e_i + (b + \mathfrak{R}p)$, with $e_i \in B_2$, $(b + \mathfrak{R}p) \in \mathrm{soc}_1(E)$, and from (1), it follows that

$$r \cdot (1 + \mathfrak{R}p) = pe = \sum_{i=1}^{m} pw_i e_i + pb \cdot (1 + \mathfrak{R}p) = \sum_{i=1}^{m} c_i p e_i + pb \cdot (1 + \mathfrak{R}p). \tag{4}$$

Then we can write $pe_i = u_i \cdot (1 + \mathfrak{R}p)$ for some $u_i \in \mathcal{F}$, since $e_i \in B_2$, and it follows that $r \cdot (1 + \mathfrak{R}p) = \sum_{i=1}^{m} c_i u_i \cdot (1 + \mathfrak{R}p) + pb \cdot (1 + \mathfrak{R}p)$. Then $r - \sum_{i=1}^{m} c_i u_i - pb \in \mathfrak{R}p$, or equivalently, $r = ap + pb + \sum_{i=1}^{m} c_i u_i$ for some $a \in \mathfrak{R}$, where $c_i p \in p\mathfrak{R}$, by (3). ∎

PROOF OF THEOREM 3.44: The proof that $\mathfrak{B}_n$ is a basis for $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$ is by induction on n.

The statement is true for $n = 1$ and $n = 2$.

Fix $n \geq 3$, and assume that for all $n' < n$, $\mathfrak{B}_{n'}$ is a basis of $\mathrm{soc}_{n'}(E)/\mathrm{soc}_{n'-1}(E)$.

Claim 1: $0 \notin \mathfrak{B}_n$ and for any $e + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$, $\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-1}(E)) = \mathfrak{R}p$.

The module E is divisible, so for any $u \in \mathcal{F}$ and any $e' \in \mathrm{soc}_{n-1}(E) \setminus \mathrm{soc}_{n-2}(E)$ such that $e' + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$ there exists $e \in E$ such that $pe = ue'$. Then $e \in \mathrm{soc}_n(E)$, since it is annihilated by an element of length at most n.

Assume that $e \in \mathrm{soc}_{n-1}(E)$. Since $e' + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$, by the induction hypothesis, $\mathrm{ann}_{\mathfrak{R}}(e' + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}p$, and from $u \notin \mathfrak{R}p$ it follows that $pe = ue' \notin \mathrm{soc}_{n-2}(E)$, so necessarily $e \notin \mathrm{soc}_{n-2}(E)$. Then we can find $e_1, \ldots, e_m \in \mathrm{soc}_{n-1}(E) \setminus \mathrm{soc}_{n-2}(E)$ such that $e + \mathrm{soc}_{n-2}(E) = e_1 + \ldots + e_m + \mathrm{soc}_{n-2}(E)$, where all $\mathfrak{R}(e_i + \mathrm{soc}_{n-2}(E))$ are distinct simple modules in $E/\mathrm{soc}_{n-2}(E)$, and $\mathrm{ann}_{\mathfrak{R}}(e_i + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}q_i$ are all maximal left ideals. Then

$$\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}[q_1, \ldots, q_m]_*.$$

At the same time, from $pe = ue' \in \mathrm{soc}_{n-1}(E)$ and $\mathrm{ann}_{\mathfrak{R}}(e' + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}p$, it follows that ${}^{u}p\,pe = {}^{u}p\,ue' = {}^{p}u\,pe' \in \mathrm{soc}_{n-2}(E)$. Thus

$$ {}^{u}p\,p \in \mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}[q_1, \ldots, q_m]_*. $$

Since $l({}^{u}p\,p) = 2$, it follows that $l([q_1, \ldots, q_m]_*) = 2$ or 1. If $l([q_1, \ldots, q_m]_*) = 2$, then $[q_1, \ldots, q_m]_* \doteq {}^{u}p\,p$, contradicting the fact that ${}^{u}p\,p$ is indecomposable. Thus $l([q_1, \ldots, q_m]_*) = 1$, and it follows that $\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}q$ for some prime q. Then $p \doteq q$, since ${}^{u}p\,p$ is indecomposable and ${}^{u}p\,p \in \mathfrak{R}q$, so $pe = ue' \in \mathrm{soc}_{n-2}(E)$ implying that $u \in \mathrm{ann}_{\mathfrak{R}}(e' + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}p$. We obtain that $u \in \mathfrak{R}p \cap \mathcal{F} = \emptyset$, which is an obvious contradiction, so $e \notin \mathrm{soc}_{n-1}(E)$, and consequently, $0 \notin \mathfrak{B}_n$.

Claim 2: For all $e'_1, e'_2 \in \mathrm{soc}_{n-1}(E) \setminus \mathrm{soc}_{n-2}(E)$ such that

$$ e'_1 + \mathrm{soc}_{n-2}(E) = e'_2 + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}, $$

and for all $u \in \mathcal{F}$, if $e_1, e_2 \in E$ are such that $pe_1 = ue'_1$ and $pe_2 = ue'_2$, then

$$ e_1 + \mathrm{soc}_{n-1}(E) = e_2 + \mathrm{soc}_{n-1}(E). $$

Since $e'_1 - e'_2 \in \mathrm{soc}_{n-2}(E)$ it follows that $u(e'_1 - e'_2) = p(e_1 - e_2) \in \mathrm{soc}_{n-2}(E)$. Then $p \in \mathrm{ann}_{\mathfrak{R}}((e_1 - e_2) + \mathrm{soc}_{n-2}(E))$. Then either $\mathrm{ann}_{\mathfrak{R}}((e_1 - e_2) + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}p$, implying that $e_1 - e_2 \in \mathrm{soc}_{n-1}(E)$, or $\mathrm{ann}_{\mathfrak{R}}((e_1 - e_2) + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}$, in which case $e_1 - e_2 \in \mathrm{soc}_{n-2}(E) \subseteq \mathrm{soc}_{n-1}(E)$.

Claim 3: $\mathfrak{B}_n$ is a generating set for $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$.

By Proposition 3.42, it is sufficient to prove that $\mathfrak{B}_n$ generates all $e + \mathrm{soc}_{n-1}(E)$, with $\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_{n-1}(E)) = \mathfrak{R}p$.

Let $e \in \mathrm{soc}_n(E) \setminus \mathrm{soc}_{n-1}(E)$ such that $pe \in \mathrm{soc}_{n-1}(E)$. Then $pe = \sum_{j=1}^{m} r_j e_j + \bar{e}$, since $\mathfrak{B}_{n-1}$ is a basis for $\mathrm{soc}_{n-1}(E)/\mathrm{soc}_{n-2}(E)$, where $e_j + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$, $r_j \in \mathfrak{R}$, and $\bar{e} \in \mathrm{soc}_{n-2}(E)$. By applying Lemma 3.45 to each r_j, it follows that

$$ r_j = a_j p + p b_j + \sum_{i=1}^{m_j} c_{ij} u_{ij}, $$

for some $a_j, b_j \in \mathfrak{R}$, $u_{ij} \in \mathcal{F}$, and $c_{ij} \in \mathfrak{R}$ such that $c_{ij} p = p w_{ij}$, for some $w_{ij} \in \mathfrak{R}$. Then:

$$\begin{aligned} pe &= \sum_{j=1}^{m} r_j e_j + \overline{e} \\ &= \sum_{j=1}^{m} \left(a_j p + p b_j + \sum_{i=1}^{m_j} c_{ij} u_{ij} \right) e_j + \overline{e} \\ &= \sum_{j=1}^{m} a_j p e_j + p \sum_{j=1}^{m} b_j e_j + \sum_{j=1}^{m} \sum_{i=1}^{m_j} c_{ij} u_{ij} e_j + \overline{e} \end{aligned}$$

Using that $pe_j = t_j e'_j$ for some $t_j \in \mathcal{F}$, $e'_j + \mathrm{soc}_{n-3}(E) \in \mathfrak{B}_{n-2}$, and $u_{ij} e_j = pe''_j$, for some $e''_j + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$, it follows that

$$\begin{aligned} pe &= \sum_{j=1}^{m} a_j t_j e'_j + p \sum_{j=1}^{m} b_j e_j + \sum_{j=1}^{m} \sum_{i=1}^{m_j} c_{ij} p e''_{ij} + \overline{e} \\ &= \sum_{j=1}^{m} a_j t_j e'_j + p \sum_{j=1}^{m} b_j e_j + \sum_{j=1}^{m} \sum_{i=1}^{m_j} p w_{ij} e''_{ij} + \overline{e} \end{aligned}$$

By rearranging the above equation, it follows that

$$p\,[\ \underbrace{e}_{\mathrm{soc}_n(E)} - \underbrace{\sum_{j=1}^{m} b_j e_j}_{\mathrm{soc}_{n-1}(E)} - \underbrace{\sum_{j=1}^{m} \sum_{i=1}^{m_j} w_{ij} e''_{ij}}_{\mathrm{soc}_n(E)}\ ] = \underbrace{\sum_{j=1}^{m} a_j t_j e'_j}_{\mathrm{soc}_{n-2}(E)} + \underbrace{\overline{e}}_{\mathrm{soc}_{n-2}(E)}$$

Since $e - \sum_{j=1}^{m} b_j e_j - \sum_{j=1}^{m} \sum_{i=1}^{m_j} w_{ij} e''_{ij} \in \mathrm{soc}_n(E)$ and is annihilated by p to an element in $\mathrm{soc}_{n-2}(E)$, it follows that $e - \sum_{j=1}^{m} b_j e_j - \sum_{j=1}^{m} \sum_{i=1}^{m_j} w_{ij} e''_{ij}$ is necessarily in $\mathrm{soc}_{n-1}(E)$ implying that $e + \mathrm{soc}_{n-1}(E) = \sum_{j=1}^{m} \sum_{i=1}^{m_j} w_{ij}(e''_{ij} + \mathrm{soc}_{n-1}(E))$, where $e''_{ij} + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$.

Claim 4: $\mathfrak{B}_n$ is $\mathfrak{R}$-linearly independent in $\mathrm{soc}_n(E)/\mathrm{soc}_{n-1}(E)$. That is, if $r_i \in \mathfrak{R}$, and $e_i + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$ ($i \leq m$), then $\sum_{i=1}^{m} r_i e_i \in \mathrm{soc}_{n-1}(E)$ implies that $r_i e_i \in \mathrm{soc}_{n-1}(E)$ for all $i \leq m$, or equivalently, $r_i \in \mathfrak{R}p$, by Claim 1.

The proof is done by induction on m. If $m = 1$, the statement is trivially true. Assume now that the statement is true for any $m' < m$.

If there exists l such that $r_l \notin \mathfrak{R}p$, then w.l.o.g. $l = 1$. Also, since $(r_1, p)_* = 1$, there exists $a \in \mathfrak{R}$ such that $ar_1 - 1 \in \mathfrak{R}p$; then $ar_1 e_1 + \mathrm{soc}_{n-1}(E) = e_1 + \mathrm{soc}_{n-1}(E)$, since

$pe_1 \in \mathrm{soc}_{n-1}(E)$. It follows from $a \cdot \sum_{i=1}^{m} r_i e_i \in \mathrm{soc}_{n-1}(E)$ that $e_1 + \sum_{i=2}^{m} ar_i e_i \in \mathrm{soc}_{n-1}(E)$, so $pe_1 + \sum_{i=2}^{m} par_i e_i \in \mathrm{soc}_{n-1}(E)$. But $pe_1 \in \mathrm{soc}_{n-1}(E)$, so $\sum_{i=2}^{m} par_i e_i \in \mathrm{soc}_{n-1}(E)$ is a linear combination of $(m-1)$ or less terms, and by the induction hypothesis, each $par_i \in \mathfrak{R}p$, for all $2 \leq i \leq m$. Thus:

$$par_i = v_i p, \text{ for some } v_i \in \mathfrak{R}, i = 2, \ldots, m \tag{1}$$

At the same time, since $\mathfrak{B}_{n-1}$ is a basis for $\mathrm{soc}_{n-1}(E)/\mathrm{soc}_{n-2}(E)$, there are $c_j \in \mathfrak{R} \setminus \mathfrak{R}p$, $f_j + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$, $w \in \mathrm{soc}_{n-2}(E)$ such that

$$e_1 + ar_2e_2 + \ldots + ar_m e_m = \sum_j c_j f_j + w \tag{2}$$

Since $e_i + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$, there are $u_i \in \mathcal{F}$ and $e_i' + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$ such that $pe_i = u_i e_i'$ $(1 \leq i \leq m)$, so we have the following equalities:

$$\sum_j pc_j f_j + pw \;=\; pe_1 + par_2e_2 + \ldots par_m e_m \;=\; u_1e_1' + v_2u_2e_2' + \ldots + v_m u_m e_m'$$

and it follows that $u_1e_1' + v_2u_2e_2' + \ldots + v_m u_m e_m' - \sum_j pc_j f_j = pw \in \mathrm{soc}_{n-2}(E)$, and since $u_1 e_1' \notin \mathrm{soc}_{n-2}(E)$, and $\mathfrak{B}_{n-1}$ is $\mathfrak{R}$-linearly independent, there exists a j such that $e_1' = f_j$ and $(u_1 - pc_j)e_1' \in \mathrm{soc}_{n-2}(E))$, so $u_1 - pc_j \in \mathrm{ann}_{\mathfrak{R}}(e_1' + \mathrm{soc}_{n-2}(E)) = \mathfrak{R}p$, or equivalently, $pc_j + \mathfrak{R}p = u_1 + \mathfrak{R}p$.

Then ${}^{u_1}p\, p(c_j + \mathfrak{R}p) = {}^{u_1}p(pc_j + \mathfrak{R}p) = {}^{u_1}p(u_1 + \mathfrak{R}p) = {}^{p}u_1\, p + \mathfrak{R}p = 0$, and so ${}^{u_1}p\, p \in \mathrm{ann}_{\mathfrak{R}}(c_j + \mathfrak{R}p) = \mathfrak{R}q$, for some prime $q \in \widehat{\mathfrak{p}}$. But ${}^{u_1}p\, p$ is left indecomposable and it has a unique (up to right multiplication by units) prime right divisor p. Thus $q \doteq p$, and $pc_j \in \mathfrak{R}p$. At the same time, $u_1 - pc_j \in \mathfrak{R}p$, so $u_1 \in \mathfrak{R}p \cap \mathcal{F}$, contradicting $\mathcal{F} \cap \mathfrak{R}p = \emptyset$.

Thus all $r_i \in \mathfrak{R}p$ $(1 \leq i \leq m)$, and the proof of Claim 4 is complete. ∎

Corollary 3.46 *Let* $e + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$ $(n \geq 1)$, *with* $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}r$, *where* $r \in \widehat{\mathcal{P}} \subseteq R$. *Then* $l_{\mathfrak{R}}(r) = n$, *or equivalently,* $l_p(r) = n$ *in* R.

PROOF. The proof will be done by induction on n. We notice that if $e + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$, then by Theorem 3.39, $l_{\mathfrak{R}}(r) \geq n$.

If $n = 1$, it is trivially true that $l_{\mathfrak{R}}(r) = 1$.

Assume now that the result is true for every $1 \leq n' < n$. Since $e + \mathrm{soc}_{n-1}(E) \in \mathfrak{B}_n$, there are $e' + \mathrm{soc}_{n-2}(E) \in \mathfrak{B}_{n-1}$ and $v \in \mathcal{F}$ such that $pe = ve'$. If $\mathrm{ann}_{\mathfrak{R}}(e') = \mathfrak{R}s$, $l_{\mathfrak{R}}(s) = n-1$, then ${}^{v}s\,pe = {}^{v}s\,ve' = {}^{s}v\,se' = 0$, and so ${}^{v}s\,p \in \mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}r$. It follows that $n \leq l_{\mathfrak{R}}(r) \leq l_{\mathfrak{R}}({}^{v}s\,p) \leq l_{\mathfrak{R}}(s) + l_{\mathfrak{R}}(p) = n - 1 + 1 = n$. Thus $l_{\mathfrak{R}}(r) = n$. ∎

3.6 The Endomorphism Ring and the Bicommutator of E

Throughout this section, we will continue to work with the ring $\mathfrak{R} = \Sigma^{-1}R$ that was introduced in 3.27. Being a localization of a PID, the ring $\mathfrak{R}$ is also a PID, and consequently left (and right) noetherian. The ring $\mathfrak{R}$ has all prime elements similar to the fixed prime p, and obviously $\mathfrak{R}$ has, up to isomorphism, only two indecomposable injective left modules: $E_{\mathfrak{R}}(\mathfrak{R})$ and $E_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{R}p)$.

Many investigations of the structure of an indecomposable injective module involve the study of its endomorphism ring, as well as that of its bicommutator, and this section will present some results regarding these two objects related to $E = E(\mathfrak{R}/\mathfrak{R}p)$.

Denote by $T = \mathrm{End}({}_{\mathfrak{R}}E)^{\mathrm{op}}$ and $B = \mathrm{End}(E_T)$. Since ${}_{\mathfrak{R}}E$ is indecomposable injective, its endomorphism ring is local, hence T is a local ring with unique maximal ideal $\mathrm{Jac}(T)$ (its Jacobson radical). Denote by $\Delta = T/\mathrm{Jac}(T)$ the residue division ring of T.

Recall that the left $\mathfrak{R}$-module E is a right T-module, where the right-hand multiplication of e by f is defined as usual, that is, for any $e \in E$ and any $f \in T$, $ef := f(e)$. Note that if $I = Rr$, then $\mathrm{ann}_E(I)$ is a submodule of E_T. Recall the notation $\mathcal{A}_r = \mathrm{ann}_E(r)$.

Lemma 3.47 *Let $0 \neq e \in E$ and $r \in \mathfrak{R}$ with $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}r$. Then $eT = \mathcal{A}_r$.*

Proof. Trivially, $eT \subseteq \mathcal{A}_r$. If $e' \in \mathcal{A}_r$, then $r \in \mathrm{ann}_{\mathfrak{R}}(e') = \mathfrak{R}r'$ for some $r' \in \mathfrak{R}$, and it follows that $\mathfrak{R}r = \mathrm{ann}_{\mathfrak{R}}(e) \subseteq \mathrm{ann}_{\mathfrak{R}}(e') = \mathfrak{R}r'$; by Corollary 1.5, there exists $f \in T$ such that $f(e) = e'$, hence $e' = e \cdot f \in eT$. ∎

The following results, adapted from [17], will present the relationship between the left ideals of the ring $\mathfrak{R}$ and the finitely generated T-submodules of E_T. Recall from 1.5 that $M^* = \mathrm{Hom}_{\mathfrak{R}}(M, E) \in \text{Mod-}T$ is the E-dual of a left $\mathfrak{R}$-module M, while $M^{**} = \mathrm{Hom}_T({}_TM^*, {}_TE) \in \mathfrak{R}\text{-Mod}$ is the E-double dual of M.

Proposition 3.48 (i) *For every $q \in \widehat{\mathfrak{p}}$, the pp-definable subgroup $\mathcal{A}_q$ is a simple submodule of E_T. Furthermore, all simple submodules of E_T have this form.*

(ii) *For every $0 \neq r \in \mathfrak{R}$, $\mathcal{A}_r$ is a finitely generated submodule of E_T. Furthermore, all finitely generated submodules of E_T have this form.*

(iii) *For any left ideal $I = \mathfrak{R}r \leq \mathfrak{R}$: $(\mathfrak{R}/I)^* \cong \mathrm{ann}_E(r) = \mathcal{A}_r$.*

(iv) *Thus, there is a 1-1 correspondence between the non-zero left ideals of $\mathfrak{R}$ and the finitely generated submodules of E_T:*

$$I = \mathfrak{R}r \overset{\Phi}{\longmapsto} \mathcal{A}_r = \mathrm{ann}_E(r), \text{ with inverse } M_T = \sum_{i=1}^{n} e_i T \overset{\Psi}{\longmapsto} \mathrm{ann}_{\mathfrak{R}}(M) = \bigcap_{i=1}^{n} \mathrm{ann}_{\mathfrak{R}}(e_i)$$

Proof. (i) First, we show that every non-zero element of $\mathcal{A}_q$ generates the T-module $\mathcal{A}_q$. Let $0 \neq e \in \mathcal{A}_q$. By Lemma 3.16(iii), $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}q$, and if $0 \neq e' \in \mathcal{A}_q$, then $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}q = \mathrm{ann}_{\mathfrak{R}}(e')$. Using Corollary 1.5, there exists $f \in T$ such that $f(e) = e'$, or $e' = e \cdot f \in eT$. This proves that $\mathcal{A}_q$ is a simple T-module and $eT = \mathcal{A}_q$, for all $0 \neq e \in \mathcal{A}_q$.

By Lemma 3.47, all simple submodules of E_T are minimal pp-definable subgroups, that is of the form $\mathcal{A}_r$, for some $r \in \widehat{\mathfrak{p}}$.

(ii) By Proposition 2.55, r can be written as a least common left multiple of a finite number of indecomposable elements: $r \doteq [r_1, \ldots, r_n]_*$, where $r_i \in \mathfrak{R}$ are indecomposable, and by 3.17, $\mathcal{A}_r = \sum_{i=1}^{n} \mathcal{A}_{r_i}$. Since all the r_i are $\widehat{\mathfrak{p}}$-indecomposable, there are $0 \neq e_i \in E$ such that $\mathrm{ann}_{\mathfrak{R}}(e_i) = \mathfrak{R}r_i$. The conclusion follows now from Lemma 3.47.

(iii) The required T-isomorphism $(\mathfrak{R}/I)^* \to \mathcal{A}_r$ is defined by

$$\zeta \mapsto \zeta(1 + I), \text{ where } \zeta \in (\mathfrak{R}/I)^* = \mathrm{Hom}(\mathfrak{R}/I, E).$$

(iv) Define $\Phi(\mathfrak{R}r) = \mathrm{ann}_E(r)$ and $\Psi(M_T) = \mathrm{ann}_{\mathfrak{R}}(M)$, where $0 \neq r \in \mathfrak{R}$ and M_T is finitely generated.

If $0 \neq I = \mathfrak{R}r \leq R$, then $\Phi(I) = \mathcal{A}_r = \sum_{i=1}^{n} \mathcal{A}_{r_i}$, where $r = [r_1, \ldots, r_n]_*$ is any decomposition of r as a least common left multiple of indecomposable elements. Then there exist $e_1, \ldots, e_n \in E$ such that $\mathrm{ann}_{\mathfrak{R}}(e_i) = \mathfrak{R}r_i$, and by 3.47, $\mathcal{A}_{r_i} = e_i T$. Then $\Psi(\Phi(I)) = \Psi(\mathcal{A}_r) = \Psi(\sum_{i=1}^{n} e_i T) = \bigcap_{i=1}^{n} \mathrm{ann}_{\mathfrak{R}}(e_i) = \bigcap_{i=1}^{n} \mathfrak{R}r_i = \mathfrak{R}[r_1, \ldots, r_n]_* = \mathfrak{R}r = I.$

On the other hand, let $M_T = \sum_{i=1}^{n} e_i T$, with $\operatorname{ann}_{\mathfrak{R}}(e_i) = \mathfrak{R}r_i$ for some $r_i \in \mathfrak{R}$, and denote $r = [r_1, \ldots, r_n]_*$. Then

$$\Phi(\Psi(M)) = \Phi(\bigcap_{i=1}^{n} \operatorname{ann}_{\mathfrak{R}}(e_i)) = \Phi(\bigcap_{i=1}^{n} \mathfrak{R}r_i) = \Phi(\mathfrak{R}r) = \mathcal{A}_r = \sum_{i=1}^{n} \mathcal{A}_{r_i} = \sum_{i=1}^{n} e_i T = M$$

Thus Φ and Ψ are inverses of each other. ∎

Corollary 3.49 (i) $\operatorname{soc}(E_T) = \operatorname{soc}({}_{\mathfrak{R}}E) = \operatorname{soc}^1({}_{\mathfrak{R}}E) = \mathfrak{R}/\mathfrak{R}p$.

(ii) $\operatorname{soc}(E_T)$ *is essential in* E_T.

Furthermore, $\operatorname{soc}({}_{\mathfrak{R}}E) = \operatorname{soc}(E_T)$ *is a sub-bimodule of* ${}_BE_T$.

Proof. (i) Recall that $\operatorname{soc}({}_{\mathfrak{R}}E) = \mathfrak{R}/\mathfrak{R}p = \operatorname{soc}^1({}_{\mathfrak{R}}E) = \sum \{ \mathcal{A}_r \,,\ r \in \widehat{\mathfrak{p}} \}$, and by Proposition 3.48, $\operatorname{soc}(E_T) = \sum \{ \mathcal{A}_r \,,\ r \in \widehat{\mathfrak{p}} \}$. Thus $\operatorname{soc}(E_T) = \operatorname{soc}({}_{\mathfrak{R}}E) = \operatorname{soc}^1({}_{\mathfrak{R}}E)$.

(ii) By 1.12, we need to show that for every $0 \neq e \in E_T$, there exists $f \in T$ such that $0 \neq ef = f(e) \in \operatorname{soc}(E)$. Let $\operatorname{ann}_{\mathfrak{R}}(e) = \mathfrak{R}r$, and since $e \neq 0$, $l_{\mathfrak{R}}(r) \geq 1$. Then there exists $q \in \widehat{\mathfrak{p}}$ such that $r \in \mathfrak{R}q$, and since $\mathcal{A}_q \neq 0$, there is $0 \neq e' \in \mathcal{A}_q \leq \operatorname{soc}(E_T)$ with $\operatorname{ann}_{\mathfrak{R}}(e') = \mathfrak{R}q$, by 3.16(i). From $\mathfrak{R}r = \operatorname{ann}_{\mathfrak{R}}(e) \leq \operatorname{ann}_{\mathfrak{R}}(e') = \mathfrak{R}q$, it follows by 1.5 that there is $f \in T$ such that $0 \neq e' = f(e) = ef \in \operatorname{soc}(E_T)$. ∎

Following the established terminology, let us indicate that the E-adic topology on $\mathfrak{R}$ is the left (linear) topology whose basic open neighborhoods of 0 are the annihilators in $\mathfrak{R}$ of finite subsets of E. A left ideal I is open in this topology if it contains a basic open left ideal.

Proposition 3.50 *Let* $E = E(\mathfrak{R}/\mathfrak{R}p)$, *p prime. Every non-zero left ideal of* $\mathfrak{R}$ *is an open neighborhood of* 0 *in the* E*-adic topology on* $\mathfrak{R}$.

Proof. If $0 \neq I \leq \mathfrak{R}$, then by Proposition 1.22, there exists a decomposition of I as a finite intersection of irreducible left ideals of $\mathfrak{R}$: $I = I_1 \cap \ldots \cap I_n$. Since each $I_j = \mathfrak{R}u_j$ is the annihilator of some element e_j in E (by 3.20), the ideal I will be the annihilator of $\{e_1, \ldots, e_n\}$, hence an open neighborhood of 0 in the E-adic topology on $\mathfrak{R}$. ∎

Using Proposition 3.50, Lemma 3.48 can be reformulated in the context of the E-adic topology on $\mathfrak{R}$.

Corollary 3.51 *There is a 1-1 correspondence between the open neighborhoods I of 0 in the E-adic topology on $\mathfrak{R}$ and the finitely generated submodules V_T of E_T given by $I \mapsto \mathrm{ann}_E(I)$, and with inverse given by $V_T \mapsto \mathrm{ann}_{\mathfrak{R}}(V)$.*

Remark 3.52 $E = E(\mathfrak{R}/\mathfrak{R}p)$ is an injective cogenerator for $\mathfrak{R}$, since $E(M) \cong E$ for every simple left $\mathfrak{R}$-module M.

By [37, (47.6(4))], every left $\mathfrak{R}$-module is E-dense, and in particular, ${}_{\mathfrak{R}}\mathfrak{R}$ is E-dense. Then the following result is a direct consequence of Propositions 1.77 and 1.78.

Corollary 3.53 *Let $M \in \mathfrak{R}$–Mod.*

(i) *The E-double dual of M equals the Hausdorff completion of M in the E-adic topology:*

$$M^{**} = \mathrm{Hom}_T(\mathrm{Hom}_{\mathfrak{R}}(M, E), E) = \widehat{M}$$

(ii) *The bicommutator B of ${}_{\mathfrak{R}}E$ equals the Hausdorff completion $\widehat{\mathfrak{R}}$ of ${}_{\mathfrak{R}}\mathfrak{R}$ in the E-adic topology:*

$$B = \mathrm{End}_T(E) \cong \widehat{\mathfrak{R}} = \varprojlim\{\mathfrak{R}/I \;:\; 0 \neq_{\mathfrak{R}} I \leq \mathfrak{R}\}$$

Remark 3.54 Note that the fundamental system of open neighborhoods of 0 consists of finitely generated left ideals, so the E-topology on ${}_{\mathfrak{R}}\mathfrak{R}$ is of finite type and the results in [17] can be applied to our case. In particular, it follows from [17] that $B_{\mathfrak{R}}$ is the pure-injective envelope of $\mathfrak{R}_{\mathfrak{R}}$. Note the analogy with the ring of integers where, given $\mathbb{Z}(p^\infty)$, we pass from $\mathbb{Z}$ to $\mathbb{Z}_{(p)}$, and then to the ring of p-adic integers, $\overline{\mathbb{Z}_{(p)}}$, which is the biendomorphism ring of $\mathbb{Z}(p^\infty)$ and, as an abelian group, the pure-injective envelope of $\mathbb{Z}_{(p)}$.

Remark 3.55 From Lemma 3.20 we have that if $0 \neq r \in \mathfrak{R}$, and $\mathcal{A}_r$ is minimal over $\mathrm{soc}^n(E)$, then $\mathcal{A}_r = eT$ for every $e \in \mathcal{A}_r \backslash \mathrm{soc}^{n-1}(E)$. That is, $(\mathcal{A}_r + \mathrm{soc}^{n-1}(E))/\mathrm{soc}^{n-1}(E)$ is a simple T-module, and consequently,

$$(\mathcal{A}_r + \mathrm{soc}^{n-1}(E))/\mathrm{soc}^{n-1}(E) \leq \mathrm{soc}_T(E/\mathrm{soc}^{n-1}(E)).$$

Theorem 3.56 *Let $E = E(\mathfrak{R}/\mathfrak{R}p)$, and recall that $T = \mathrm{End}({}_{\mathfrak{R}}E)^{\mathrm{op}}$, $\Delta = T/\mathrm{Jac}(T)$.*

(i) *If* $E \neq \mathrm{soc}^n(E)$ *(*$n \geq 0$*), then* $\mathrm{ann}_T(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)) = \mathrm{Jac}(T)$.
Furthermore, $\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)$ *is a right vector space over* Δ.

(ii) *If* $E \neq \mathrm{soc}^n(E)$ *(*$n \geq 0$*), then*

$$\Delta \hookrightarrow \mathrm{End}_{\mathfrak{R}}\left(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)\right)^{\mathrm{op}}$$

Furthermore, $\Delta \cong \mathrm{End}_{\mathfrak{R}}(\mathrm{soc}^1(E))^{\mathrm{op}} \cong \mathrm{End}_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{R}p)^{\mathrm{op}}$.

(iii) $\mathrm{End}_{\mathfrak{R}}(E) \cong \varprojlim \mathrm{End}_{\mathfrak{R}}(\mathrm{soc}^n({}_{\mathfrak{R}}E))$.

Proof. (i) By 1.31, $f \in \mathrm{Jac}(T)$ if and only if f annihilates every simple T-module. If $\mathrm{soc}^n(E) \neq E$, then $\mathrm{soc}^n(E) \lneq \mathrm{soc}^{n+1}(E)$, so $\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)$ is a (non-empty) sum of pp-definable subgroups of E minimal over $\mathrm{soc}^n(E)$ which, by the above remark, are simple T-modules, hence annihilated by $\mathrm{Jac}(T)$. It follows that

$$\mathrm{Jac}(T) \subseteq \mathrm{ann}_T(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)) \neq T.$$

But $\mathrm{Jac}(T)$ is a maximal ideal in the ring T, so $\mathrm{Jac}(T) = \mathrm{ann}_T(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E))$.

(ii) If $E \neq \mathrm{soc}^n(E)$, then $\mathrm{soc}^n(E) \lneq \mathrm{soc}^{n+1}(E)$. Consider the $\mathfrak{R}$-homomorphism

$$\Psi_n : T \longrightarrow \mathrm{End}_{\mathfrak{R}}\left(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)\right)^{\mathrm{op}}$$

which associates to each $f \in T$ the $\mathfrak{R}$-morphism $\overline{f} \in \mathrm{End}_{\mathfrak{R}}\left(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)\right)^{\mathrm{op}}$ that acts according to the rule $\overline{f}(e + \mathrm{soc}^n(E)) = f(e) + \mathrm{soc}^n(E)$. Then by part (i), the homomorphism Ψ_n has kernel $\mathrm{Jac}(T)$, and we have the required embedding

$$\widehat{\Psi}_n : T/\mathrm{Jac}(T) \hookrightarrow \mathrm{End}_{\mathfrak{R}}\left(\mathrm{soc}^{n+1}(E)/\mathrm{soc}^n(E)\right)^{\mathrm{op}}.$$

The second part follows easily from the fact that E is injective and consequently every homomorphism $\overline{f} : \mathrm{soc}^1(E) \to \mathrm{soc}^1(E)$ can be extended to a homomorphism $f : E \to E$ such that the restriction $f \restriction \mathrm{soc}^1(E) = \overline{f}$.

Thus the monomorphism $\widehat{\Psi}_1 : T/\mathrm{Jac}(T) \longrightarrow \mathrm{End}_{\mathfrak{R}}\left(\mathrm{soc}^1(E)\right)^{\mathrm{op}}$ is an isomorphism.

(iii) Recall that $E = \bigcup_{n \in \omega} \mathrm{soc}^n(E)$. Then:

$$\begin{aligned} \mathrm{End}_{\mathfrak{R}}(E) &= \mathrm{Hom}_{\mathfrak{R}}(\varinjlim \mathrm{soc}^n(E), E) \\ &\cong \varprojlim \mathrm{Hom}_{\mathfrak{R}}(\mathrm{soc}^n(E), E) \\ &\cong \varprojlim \mathrm{End}_{\mathfrak{R}}(\mathrm{soc}^n(E)) \end{aligned}$$

The latter holds since $\mathrm{Hom}_{\mathfrak{R}}(\mathrm{soc}^n(E), E) = \mathrm{End}_{\mathfrak{R}}(\mathrm{soc}^n(E))$, by Proposition 1.89(i). ∎

Chapter 4

Applications

In this chapter I will present several examples which showcase the use of the theory developed in the previous chapter. The first section contains some additional results for the ring $k(x)[y, \partial/\partial x]$, results that might be of importance for the research in the field of differential equations, as the elements of this algebra are nothing else but differential operators, and the injective modules over this ring can be viewed as structures in which we can solve systems of linear equations. Similar results are possible for the ring $K[y; \sigma, \delta]$ in general.

In the second part of this chapter, I will illustrate the applicability of the theory developed in Chapter 3 for describing the injective modules over the first Weyl algebra, this being my main focus when I started working on this thesis. Back then, my motivation for pursuing this topic was the lack of results related to the (indecomposable) injectives over the first Weyl algebra, and the general belief that these objects were almost impossible to describe, their structure—as wild modules—being "unmanageable" [20].

The last section of this chapter outlines additional applications of results from Chapter 3, but they are not presented in detail, since most of the work can be done in a similar manner as for the first Weyl algebra.

4.1 The Indecomposable Injective Modules over the ring $k(x)[y, \partial/\partial x]$

The main goal of this section is to use the results from Chapter 2 to find additional properties of the injective modules over the skew polynomial ring $R = k(x)[y, \partial/\partial x]$, which is left and right Euclidean, hence a PID.

As usual, the degree of an element $r \in R$ is defined to be the highest power of y that occurs in r with a non-zero coefficient, and the leading coefficient of r is defined to be the non-zero coefficient of the highest power of y. An element whose leading coefficient is 1 is said to be monic. We should point out that some of the factorization/decomposition results obtained in Chapter 2 could be deduced a lot more easily by making use of the Euclidean properties of the ring. In this ring, for all $0 \neq a, b \in R$:

$\deg([\,a\,,b\,]_*) + \deg((\,a\,,b\,)_*) = \deg(a) + \deg(b) = \deg(\,_*[\,a\,,b\,]) + \deg(\,_*(\,a\,,b\,))$ (2.19)

and this fact can be used to simplify the theory developed for a PID in general, by using monic divisors, monic least common left/right multiples, monic greatest common right/left divisors, and so on.

Notice that the similarity relation defined in a PID has additional properties in the case when the ring is Euclidean relative to a degree function (2.19), such as: if $r_1 \sim r_2$, then $\deg(r_1) = \deg(r_2)$. Furthermore, there exists $a \in R$ with $\deg(a) < \deg(r_1) = \deg(r_2)$ such that $r_2 = {}^a r_1$, and $(\,a\,, r_1\,)_* = 1$ (2.31).

Obviously, when working with a right indecomposable element, it can be assumed that it is monic and that it has a unique monic prime left divisor.

Recall from Theorem 3.4 that the indecomposable injective modules over a PID, and hence over $R = k(x)[y, \partial/\partial x]$ are: $E({}_RR)$ and $E(R/Rp)$, where p is a prime element in R. Furthermore, if p, q are prime, then $E(R/Rp) \cong E(R/Rq)$ if and only if $p \sim q$.

For the rest of this section, fix the prime $p = y$. Then $R/Ry \,{}_R\!\cong k(x)$, where the R-module structure of $k(x)$ is given by $y \cdot f(x) = \partial f/\partial f(x)$ and $x \cdot f(x) = xf(x)$, for all $f(x) \in k(x)$.

Let $E = E(R/Ry) \cong E(k(x))$. By Theorems 3.7 and 3.24,

$$E = \bigcup_{n<\omega} \mathrm{soc}_n(E) = \bigcup_{n<\omega} \mathrm{soc}^n(E)$$

Theorem 3.24 presented a good description of the elementary socle series $(\mathrm{soc}^n({}_RE))_{n<\omega}$ of E by using the length and the p-height of an element of the ring. The description for the socle series $(\mathrm{soc}_n({}_RE))_{n<\omega}$ of E was not as sharp, but by localizing the ring R to $\mathfrak{R} = \Sigma^{-1}R$ at the set Σ of all elements that act regularly on E, we were able to construct from an arbitrary basis $\mathfrak{B}_2$ of $\mathrm{soc}_2({}_{\mathfrak{R}}E)/\mathrm{soc}_1({}_{\mathfrak{R}}E)$ bases for each term of the socle series. In this construction, the basis $\mathfrak{B}_2$ was chosen arbitrarily from the generating set

$$\mathcal{G}_2 = \{\, e + \mathrm{soc}_1(E) : \mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_1(E)) = \mathfrak{R}y \,\}.$$

In the case of $k(x)[y, \partial/\partial x]$, as it is customary in linear algebra when bases are constructed, we will also find a large $\mathfrak{R}$-linearly independent set Ln in $\mathcal{G}_2$, so the basis $\mathfrak{B}_2$ of $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$ will contain a "prescribed" independent set and will be contained in the generating set $\mathcal{G}_2$.

NOTATION: For each equation $y \cdot v = \frac{1}{x-\alpha}$, $\alpha \in k$, fix a solution $e_\alpha := \ln(x - \alpha) \in E$. Note that if e' is another solution of the same equation, then $ye_\alpha = ye'$ implies that $y(e_\alpha - e') = 0$, and by 3.39(i) it follows that $e_\alpha - e' \in \mathrm{soc}_1({}_{\mathfrak{R}}E)$. ∎

Theorem 4.1 *Let $E = E(\mathfrak{R}/\mathfrak{R}y)$.*
The set $Ln = \{e_\alpha + \mathrm{soc}_1(E) \in E \,:\, \alpha \in k\}$ is $\mathfrak{R}$-linearly independent in $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$.

PROOF. (See also Example 3.9(3).)

First we show that $0 \notin Ln$. For every $e + \mathrm{soc}_1(E) \in Ln$ there exists $\alpha \in k$ such that $ye = \frac{1}{x-\alpha}$, implying that $(y + \frac{1}{x-\alpha})ye = 0$, so $(y + \frac{1}{x-\alpha})y = {}^{\frac{1}{x-\alpha}}y\, y \in \mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}u$, where u is $\widehat{y}$-indecomposable. Since $(y + \frac{1}{x-\alpha})y$ is right and left $\widehat{y}$-indecomposable (by 2.46(3)), hence with a unique monic prime right divisor y, then either

$$\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}(y + \frac{1}{x-\alpha})y$$

implying that $e \in \mathrm{soc}_2(E) \setminus \mathrm{soc}_1(E)$, or

$$\mathrm{ann}_R(e) = Ry,$$

but the latter would yield $ye = 0$, contradicting $ye = \frac{1}{x-\alpha} \neq 0$. Thus $e + \mathrm{soc}_1(E) \neq 0$, and furthermore $e + \mathrm{soc}_1(E) \in \mathcal{G}_2$ for every $e + \mathrm{soc}_1(E) \in Ln$. It was also proved that for each $e + \mathrm{soc}_1(E) \in Ln$, $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}(y + \frac{1}{x-\alpha})y$ and $\mathrm{ann}_{\mathfrak{R}}(e + \mathrm{soc}_1(E)) = \mathfrak{R}y$.

Next we prove by induction on n that if $r_i \in \mathfrak{R}$, and $e_i + \mathrm{soc}_1(E) \in Ln$ ($i \leq n$) are distinct, then $\sum_{i=1}^{n} r_i e_i \in \mathrm{soc}_1(E)$ implies that $r_i e_i \in \mathrm{soc}_1(E)$ for all $i \leq n$.

If $n = 1$, then trivially $r_1 e_1 \in \mathrm{soc}_1(E)$.

Assume that all sets $n-1$ or fewer elements of Ln are linearly independent. Let $e_i = \ln(x - \alpha_i)$, $\alpha_i \neq \alpha_j$ for $i \neq j$, and $r_i \in \mathfrak{R}$ ($i \leq n$) be such that $\sum_{i=1}^{n} r_i e_i \in \mathrm{soc}_1(E)$. Then there exists $s \in \Sigma \subset \mathrm{U}(\mathfrak{R})$ such that $\sum_{i=1}^{n} s r_i e_i \in \mathrm{soc}_1(E)$ and $s r_i \in R \subset \mathfrak{R}$ for every $i \leq n$. For every $i \leq n$, there are $u_i \in R, g_i \in k(x)$ such that $s r_i = u_i y + g_i$ (division by y). Since $u_i y e_i = u_i \frac{1}{x-\alpha_i} \in \mathrm{soc}_1(E)$, it follows that $\sum_{i=1}^{n} g_i e_i \in \mathrm{soc}_1(E)$.

If $g_j \neq 0$ for some j, then by multiplying by $1/g_j$ and assuming w.l.o.g. that $j = 1$, it follows that $e_1 + g_2 e_2 + \ldots + g_n e_n = e \in \mathrm{soc}_1(E)$, hence $y e_1 + \sum_{i=2}^{n} y g_i e_i = ye \in \mathrm{soc}_1(E)$, and so $\frac{1}{x-\alpha_1} + (y g_2) e_2 + \ldots + (y g_n) e_n = ye \in \mathrm{soc}_1(E)$. Thus

$$\frac{1}{x-\alpha_1} + (g_2 y + g_2') e_2 + \ldots + (g_n y + g_n') e_n = ye \in \mathrm{soc}_1(E),$$

and it follows that $\frac{1}{x-\alpha_1} + (g_2 \frac{1}{x-\alpha_2} + g_2' e_2) + \ldots + (g_n \frac{1}{x-\alpha_n} + g_n' e_n) = ye \in \mathrm{soc}_1(E)$, implying that $g_2' e_2 + \ldots + g_n' e_n \in \mathrm{soc}_1(E)$. By the induction hypothesis, it follows that for all $i \geq 2$ we have $g_i' e_i \in \mathrm{soc}_1(E)$, so $g_i' \in \mathrm{ann}_{\mathfrak{R}}(e_i + \mathrm{soc}_1(E)) = \mathfrak{R} y$; but $g_i' \in k(x)$, so necessarily $g_i' = 0$, thus $g_i \in k$.

Then $ye = \frac{1}{x-\alpha_1} + g_2 \frac{1}{x-\alpha_2} + \ldots + g_n \frac{1}{x-\alpha_n} = f \in k(x)$ and we have that

$$0 = {}^f y\, f = {}^f y\, ye$$

implying that ${}^f y\, y \in \mathrm{ann}_{\mathfrak{R}}(e)$. Since ${}^f y\, y$ is $\widehat{y}$-indecomposable (by 2.46(4)), and since $e \in \mathrm{soc}_1(E)$, it follows that $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R} y$ and consequently, $ye = 0$. But then it follows that $\frac{1}{x-\alpha_1} + g_2 \frac{1}{x-\alpha_2} + \ldots + g_n \frac{1}{x-\alpha_n} = 0$, contradicting the fact that the set $\{\frac{1}{x-\alpha} : \alpha \in k\}$ is k-linearly independent in the k-vector space $k(x)$. Consequently, $g_i = 0$ for all i, so

$$s r_i e_i = u_i y e_i = u_i \frac{1}{x - \alpha_i} \in \mathrm{soc}_1(E).$$

Thus it follows that $r_i e_i = s^{-1}(s r_i) e_i \in \mathrm{soc}_1(E)$. ∎

Remark 4.2 Proposition 3.42 provides a generating set $\mathcal{G}_2$ for $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$, while Theorem 4.1 states that $Ln \subset \mathcal{G}_2$. Thus we can choose a basis $\mathfrak{B}_2$ for $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$ that is contained in $\mathcal{G}_2$ and contains Ln. The independent set Ln does not generate $\mathrm{soc}_2(E)/\mathrm{soc}_1(E)$; for instance, an element $e \in E$ with $\mathrm{ann}_{\mathfrak{R}}(e) = \mathfrak{R}(y + \frac{1}{x})(y^2 + x)y$ is in $\mathrm{soc}_2(E)$, but is not generated by Ln over $\mathrm{soc}_1(E)$.

4.2 Injective modules over the first Weyl Algebra

Let k be an algebraically closed field of characteristic zero.
Recall from Example 1.50 that the first Weyl algebra $\mathbb{A}$ over k is the associative k-algebra on generators x and y subject to the relation:

$$yx - xy = 1$$

At the same time, the algebra $\mathbb{A}$ can be described as the ring of skew polynomials $k[x][y; \partial/\partial x]$, which may also be interpreted as the ring of formal differential operators with polynomial coefficients, where the elements of $\mathbb{A}$ are subject to the relationship

$$yf = fy + \frac{\partial f}{\partial x}, \text{ for all } f \in k[x].$$

The proofs of the basic properties of $\mathbb{A}$ can be found in [7], [15], or [29], and most of them are consequences of the fact that $\mathbb{A}$ is a skew polynomial ring. The ring is a simple (left and right) hereditary noetherian prime domain, but not left nor right artinian, and not a division ring. Also, $\mathbb{A}$ is not a principal left ideal domain, however every left (right) ideal is 2-generated [35].

The aim of this section is to classify the injective left $\mathbb{A}$-modules, and describe their structure. $\mathbb{A}$ is left noetherian, so it is sufficient to consider the indecomposables. One indecomposable injective can be easily pointed out: it is the injective envelope of the left module $\mathbb{A}$, and is isomorphic to the Weyl division algebra; all the other indecomposable injective modules are the injective envelopes of simple modules (by 1.28). We give next a few examples of simple left $\mathbb{A}$-modules.

Example 4.3 The following left ideals are maximal: $\mathbb{A}(x-\alpha)$, $\mathbb{A}(y-\alpha)$, for $\alpha \in k$; $\mathbb{A}(yx-\lambda)$, for $\lambda \in k \backslash \mathbb{Z}$; $\mathbb{A}(y^2+x^3)$ [9, (4.8)]; $\mathbb{A}(y^n+x)$, for $n \in \mathbb{N}$; $\mathbb{A}\left(y^2-\frac{1}{4}x^2+\frac{1}{2}+\lambda\right)$, for $\lambda \in k \setminus \mathbb{Z}$ [9, (3.3)]

Thus $\mathbb{A}/\mathbb{A}x$, $\mathbb{A}/\mathbb{A}y$, $\mathbb{A}/\mathbb{A}\left(y^2-\frac{1}{4}x^2+\frac{1}{2}+\lambda\right)$, $\mathbb{A}/\mathbb{A}(yx-\lambda)$ ($\lambda \in k\backslash\mathbb{Z}$), $\mathbb{A}/\mathbb{A}(y^2+x^3)$ are examples of simple left modules over $\mathbb{A}$.

Example 4.4 The left ideals $I_1 = \mathbb{A}y^2+\mathbb{A}(xy-1)$ and $I_2 = \mathbb{A}x^2+\mathbb{A}(yx+1)$ are maximal and non-principal, so $\mathbb{A}/I_1$ and $\mathbb{A}/I_2$ are also simple modules.
Furthermore, $\mathbb{A}/I_1 \cong \mathbb{A}/\mathbb{A}y \cong k[x]$ and $\mathbb{A}/I_2 \cong k[y]$.

Proof. The proofs will be done only for I_1, the results for I_2 will follow from the fact that I_2 is the image of I_1 under the automorphism of $\mathbb{A}$ that sends $x \mapsto -y$ and $y \mapsto x$.

To show that I_1 is non-principal, note first the following:
$y(xy-1) = yxy - y = (xy+1)y - y = xy^2 \in \mathbb{A}y^2$

Thus $\mathbb{A}(xy-1) \leq \mathbb{A}y^2 + k[x](xy-1)$, implying that $I_1 = k[x](xy-1) + \mathbb{A}y^2$, with $I_1 \bigcap k[x] = 0$.

Assume that $I_1 = \mathbb{A}a$. Then $\deg_y(a) \geq 1$, and it follows that $xy - 1 = ba$, for some $b \in \mathbb{A}$. But then $\deg_y(a) = 1, b \in k[x]$, i.e. $a = a_0 + a_1 y,\ a_0, a_1 \in k[x]$. Since $xy - 1 = ba = b(a_0 + a_1 y) = ba_0 + ba_1 y$, it follows that $ba_1 = x$ and $ba_0 = -1$, so $b \in k$. Thus $I_1 = \mathbb{A}a = \mathbb{A}(xy-1)$. Then $y^2 \in \mathbb{A}(xy-1)$, i.e. $y^2 = (\sum_{i=0}^{n} c_i y^i)(xy-1)$, for some $\sum_{i=0}^{n} c_i y^i \in \mathbb{A}$, $c_i \in k[x]$. By using the formula $y^i x = xy^i + iy^i$ $(i \geq 0)$, it follows that $y^2 = \sum_{i=1}^{n}(c_i xy + i - 1)y^i$, so $c_0 xy - 1 = 0$, impossible. Thus I_1 is non-principal.

We will show now that I_1 is maximal. Let $I_1 \subsetneq J \leq \mathbb{A}$, and $r \in J \setminus I_1$. Since $y^2 \in I_1$, we can assume w.l.o.g. that $r = u(x)y + v(x)$ for some $u(x), v(x) \in k[x]$, and if we write $u(x) = w(x)x + \alpha$, where $w(x) \in k[x], \alpha \in k$, then

$$r = w(x)xy + \alpha y + v(x) = w(x)(xy-1) + \alpha y + v(x) + w(x) \in J \setminus I_1$$

But $w(x)(xy-1) \in I_1 \lneq J$, so $\alpha y + (v(x)+w(x)) \in J$. Note that $\alpha y + v(x) + w(x) \notin I_1$, since $r \notin I_1$. Let $a(x) = v(x) + w(x)$.

If $a(x) = 0$, then $\alpha \neq 0$ and $\alpha y \in J$, and by multiplying by x we obtain that $\alpha xy = \alpha(xy-1) + \alpha \in J$. It follows that $\alpha \in J$, so $J = \mathbb{A}$.

If $a(x) \neq 0$, then let $\deg(a(x)) = n \geq 0$. Since $0 \neq \alpha y + a(x) \in J$, it follows that

$$y^{n+1}\left[\alpha y + a(x)\right] \in J$$

In $\mathbb{A}$, we have the following formula: $y^m f = \sum_{i=0}^{m} \binom{m}{i} f^{(i)} y^{m-i}$, for all $m \geq 1$ and $0 \neq f \in k[x]$, where $f^{(i)}$ denotes the i^{th} derivative of f. Then:

$$\alpha y^{n+2} + \sum_{i=0}^{n+1} \binom{n+1}{i} a^{(i)} y^{n+1-i} \in J$$

Since $\alpha y^{n+2} = \alpha y^n y^2 \in J$ and $\binom{n+1}{i} a^{(i)} y^{n+1-i} \in J$ for all $i = 0, \ldots, n-1$, it follows that $\binom{n+1}{n} a^{(n)} y + a^{(n+1)} \in J$. But $a^{(n+1)} = 0$ and $0 \neq a^{(n)} \in K$, so if we denote $\beta = \binom{n+1}{n} a^{(n)}$ we obtain that $\beta y \in J$. Then $x(\beta y) = \beta(xy-1) + \beta \in J$, and it follows that $\beta \in J$. Thus $J = \mathbb{A}$.

The isomorphism $\mathbb{A}/\mathbb{A}y \cong \mathbb{A}/I_1$ is defined by $r + \mathbb{A}y \mapsto ry + I_1$. ∎

Now having presented a few examples of simple modules, we will return to the search for a good description of the injective $\mathbb{A}$-modules. Most of the time, the analysis of the structure of an indecomposable injective module over a left noetherian ring makes use of the prime ideals of the ring, but in the case of $\mathbb{A}$, this tactic cannot be employed, since the ring is simple, and 0 is the only prime ideal. Goodearl [13] has studied "the influence of injective module structure on localization questions for non-commutative noetherian rings", and in this paper he proved the following result for $\mathbb{A}$:

Proposition 4.5 [13, (3.2)] *Let M be any simple left $\mathbb{A}$-module and let $E = E_{\mathbb{A}}(M)$. Then the set $\mathcal{N} = \mathcal{N}_{\mathbb{A}}(E)$ of all elements of $\mathbb{A}$ acting regularly on E is a left and right Ore set in $\mathbb{A}$ and $\mathcal{N}^{-1}M$ is a simple left $\mathcal{N}^{-1}\mathbb{A}$-module. Up to isomorphism, $\mathcal{N}^{-1}\mathbb{A}$ has exactly one simple left module and exactly one simple right module.*

This is a very important result, but nothing has been done with regard to the injective module E over $\mathcal{N}^{-1}\mathbb{A}$, the above mentioned paper focusing mostly on the process of localization at an injective. Although the article does contain some results on the set $\mathcal{N}(E)$ when $E = E(\mathbb{A}/\mathbb{A}y)$, it does not provide much insight into the structure of an arbitrary indecomposable injective module.

In order to say more about the indecomposable injective modules over $\mathbb{A}$, we need to look in more detail at the simple $\mathbb{A}$-modules. With this motivation in mind, a classification of the simple left $\mathbb{A}$-modules is required first, before taking on the task of describing the indecomposable injectives. McConnell & Robson [28, (5.1)] proved that a simple left $\mathbb{A}$-module is either $k[x]$-torsionfree or $k[y]$-torsionfree, and Block [4] improved this classification of the simple $\mathbb{A}$-modules.

Proposition 4.6 [28], [4] *The simple $\mathbb{A}$-modules are classified as follows:*

- *$k[x]$-torsion, and $k[y]$-torsionfree modules having the form $M = \mathbb{A}/\mathbb{A}(x-\alpha)$, $\alpha \in k$.*
- *$k[y]$-torsion, and $k[x]$-torsionfree modules having the form $M = \mathbb{A}/\mathbb{A}(y-\alpha)$, $\alpha \in k$.*
- *$k[x]$- and $k[y]$-torsionfree modules (for example, $\mathbb{A}/\mathbb{A}(yx - \lambda)$, $\lambda \in k \setminus \mathbb{Z}$)*

Both $k[x] \setminus \{0\}$ and $k[y] \setminus \{0\}$ are left denominator sets for $\mathbb{A}$, so we can use either of them to perform left localization, obtaining either $k(x)[y, \partial/\partial x]$ or $k(y)[x, -\partial/\partial y]$, which—as PID's—were studied in Chapters 2 and 3. Furthermore, since a simple $\mathbb{A}$-module is either $k[x]$- or $k[y]$-torsionfree, the analysis of its injective envelope can be lifted to the study of an injective envelope of a corresponding simple module of fractions over one of the extensions $R = k(x)[y, \partial/\partial x]$ or $k(y)[x, -\partial/\partial y]$. W.l.o.g. we will only investigate the injective envelope of an arbitrary $k[x]$-torsionfree simple module, since the case of a $k[y]$-torsionfree simple module can be easily be adapted by symmetry, or by twisting.

For the remainder of this chapter, let $S = k[x] \setminus \{0\}$ and $R = S^{-1}\mathbb{A} = k(x)[y, \partial/\partial x]$. By a result in [4] (application of 1.59(iii)), there is a one-to-one correspondence between the isomorphism classes of $k[x]$-torsionfree simple $\mathbb{A}$-modules and the simple R-modules, given by:

$$M \mapsto S^{-1}M\,, \text{ with inverse } N \mapsto \mathrm{soc}_{\mathbb{A}}(N)\,,$$

for a $k[x]$-torsionfree simple $\mathbb{A}$-module M and a simple R-module N.

Block [4] obtained an effective description of the map $N \mapsto \mathrm{soc}_{\mathbb{A}}(N)$, where N is a simple R-module (hence of the form R/Rp, for some prime element $p \in R$). The next couple of lemmas will present results concerning the map $M \mapsto S^{-1}M$.

Lemma 4.7 *If $I = \mathbb{A}a + \mathbb{A}b$ is a left ideal of $\mathbb{A}$ such that $\mathbb{A}/I$ is a $k[x]$-torsionfree module, and such that $(a, b)_* = 1$ in R, then $I = \mathbb{A}$.*

Proof. Notice that $S^{-1}I = R(\mathbb{A}a + \mathbb{A}b) = R$, since $(a, b)_* = 1 = a_1 a + b_1 b$, for some $a_1, b_1 \in R$. The conclusion follows now as an immediate consequence of 1.59: $I = I^{ec} = R^c = \mathbb{A}$. ∎

The next lemma gives a precise "recipe" for constructing the module of fractions of a simple $k[x]$-torsionfree module.

Lemma 4.8 *Let $I = \mathbb{A}a + \mathbb{A}b$ be a maximal left ideal of $\mathbb{A}$, and assume that $\mathbb{A}/I$ is $k[x]$-torsionfree. Then the corresponding simple module over the localization $R = k(x)[y, \partial/\partial x]$ is R/Rd, where $d = (a, b)_*$ in R. In other words,*

$$S^{-1}(\mathbb{A}/\mathbb{A}a + \mathbb{A}b) \cong R/R(a, b)_*.$$

Proof. The proof requires two steps. First we will prove that $(a, b)_* = d$ is a prime element of R, so Proposition 1.62(iii) can be used to show the required isomorphism.

Since I is a maximal left ideal of $\mathbb{A}$, we have that $1 \neq (a, b)_* = d = d_1 d_2$, where $d_2 \in R$ is a prime right-hand divisor of d. Then $Rd \leq Rd_2$, and it follows that $I = \mathbb{A}a + \mathbb{A}b \leq (Ra + Rb) \cap \mathbb{A} = Rd \cap \mathbb{A} \leq Rd_2 \cap \mathbb{A} \neq \mathbb{A}$. Since the left ideal I is maximal, the chain consists of equal left ideals; in particular, $Rd \cap \mathbb{A} = Rd_2 \cap \mathbb{A}$. There exists $0 \neq f \in k[x]$ such that $fd_2 \in Rd_2 \cap \mathbb{A} = Rd \cap \mathbb{A}$. But then $fd_2 = ud = ud_1 d_2$ for some $u \in R$, so it follows that $f = ud_1 \in k[x]$, indicating that d_1 is necessarily a unit in R, and that d is a prime element of R; thus R/Rd is simple.

By 1.62(iii), $\mathrm{Hom}_R(S^{-1}(\mathbb{A}/I), R/Rd) \cong \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}/I, R/Rd)$. Since both $S^{-1}(\mathbb{A}/I)$ and R/Rd are simple R-modules, it suffices to show $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}/I, R/Rd) \neq 0$.

The $\mathbb{A}$-homomorphism $\psi : \mathbb{A} \longrightarrow R/Rd$ defined by $\psi(1) = 1 + Rd$ has kernel I, since $\psi(a) = a\psi(1) = a + Rd = Rd$, $\psi(b) = b\psi(1) = b + Rd = Rd$, and I is maximal. It follows that $\mathrm{Hom}_{\mathbb{A}}(\mathbb{A}/I, R/Rd) \neq 0$, and consequently $\mathrm{Hom}_R(S^{-1}(\mathbb{A}/I), R/Rd) \neq 0$, so $S^{-1}(\mathbb{A}/I)$ and R/Rd are isomorphic. ∎

Example 4.9 The left ideal $I = \mathbb{A}y^2 + \mathbb{A}(xy - 1)$ is a maximal left ideal and $\mathbb{A}/I$ is a $k[x]$-torsionfree module. Note that in R, $y^2 = (y + \frac{1}{x})(y - \frac{1}{x})$, so $(y^2, xy - 1)_* = y - \frac{1}{x}$, and by Lemma 4.8 it follows that $S^{-1}(\mathbb{A}/I) \cong R/R(y - \frac{1}{x}) \cong R/Ry$.

Question: If M is a simple $\mathbb{A}$-module, is there a principal maximal left ideal $I = \mathbb{A}a$ such that $M \cong \mathbb{A}/I$? If not, under what conditions would this result hold? (This is an open question.)

At this point, we completely understand how the simple $\mathbb{A}$-modules extend to simple R-modules, so we can now look at the structure of the injective envelope $E = E(M)$ of a simple $k[x]$-torsionfree $\mathbb{A}$-module M.

Then by Proposition 1.64, $E = E_{\mathbb{A}}(M) \cong S^{-1}(E_{\mathbb{A}}(M)) \cong E_R(S^{-1}M)$, where $S = k[x] \setminus \{0\}$ and $R = k(x)[y, \partial/\partial x]$, and there exists a prime $p \in R$ such that $S^{-1}M \cong R/Rp$. Furthermore, $\mathrm{End}_{\mathbb{A}}(E) \cong \mathrm{End}_R(E)$. The study of the indecomposable injective $E = E(M)$ has been lifted to the ring $R = k(x)[y, \partial/\partial x]$ whose indecomposable injective $E(R/Rp)$ was analyzed in Chapter 3 and Section 4.1.

$E = \bigcup_{n<\omega} \mathrm{soc}_n(E)$, and partial results on the socle series of ${}_RE$ have been obtained in Theorem 3.7. The description of E in terms of the socle series was improved considerably after R was localized at the set Σ of all elements of the ring that act regularly on E, obtaining $\mathfrak{R} = \Sigma^{-1}R$ (Sections 3.4, 3.5). Up to isomorphism, the ring $\mathfrak{R}$ has a unique simple left module $\mathfrak{R}/\mathfrak{R}p$, and $E = E_{\mathbb{A}}(M)) \cong E_R(R/Rp) \cong E_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{R}p)$. The semisimple structure of each factor $\mathrm{soc}_n({}_{\mathfrak{R}}E)/\mathrm{soc}_{n-1}({}_{\mathfrak{R}}E)$ has been described by constructing its basis $\mathfrak{B}_n$ (Theorem 3.44).

At the same time, $E = \bigcup_{n<\omega} \mathrm{soc}^n(E)$, and the elementary socle layers $\mathrm{soc}^n(E)$ $(n < \omega)$ have been completely described in Theorem 3.24 in terms of the p-length and p-height of the elements of the ring R.

Section 3.6 contains results about ${}_{\mathfrak{R}}E_T$ and the bicommutator $B_{\mathfrak{R}}(E)$ of E, where $T = \mathrm{End}_{\mathfrak{R}}(E)^{\mathrm{op}}$. In particular, Corollary 3.53 states that the Hausdorff completion $\widehat{\mathfrak{R}}$ of $\mathfrak{R}$ in the E-adic topology is isomorphic to the bicommutator of E:

$$B = \mathrm{End}_T(E) \cong \widehat{\mathfrak{R}} = \varprojlim\{\mathfrak{R}/I \,:\, {}_{\mathfrak{R}}I \leq \mathfrak{R}\}$$

It is worth mentioning that the double localization of $\mathbb{A}$ (first at $S = k[x] \setminus \{0\}$, then at Σ) that was used to analyze an indecomposable injective $E = E_{\mathbb{A}}(M)$ yields the same extension of $\mathbb{A}$ as the localization of $\mathbb{A}$ at $\mathcal{N}(E)$ (see 4.5), $\mathfrak{R} = \Sigma^{-1}(S^{-1}\mathbb{A}) = \mathcal{N}^{-1}(E)$. However, the intermediate steps that were done in the double localization allowed a much deeper investigation of the module E.

We summarize these results in the following theorem.

Theorem 4.10 *Let M be a simple S-torsionfree $\mathbb{A}$-module, where $S = k[x] \setminus \{0\}$ or $k[y] \setminus \{0\}$. Let $E = E(M)$ denote its (indecomposable) injective envelope over $\mathbb{A}$ and $\mathcal{N}(E)$ the set of all elements of $\mathbb{A}$ acting regularly on E. Then:*

(i) *The ring $\mathfrak{R} = \mathcal{N}(E)^{-1}\mathbb{A}$ is a PID with a unique similarity class $\widehat{\mathfrak{p}}$ of prime elements, hence with a unique simple module $\mathfrak{R}/\mathfrak{R}p$ (up to isomorphism).*

(ii) *The indecomposable injective modules over $\mathfrak{R}$ are:*

$$E(\mathbb{A}) \text{ and } E_{\mathbb{A}}(M) \cong E_R(S^{-1}M) \cong E_{\mathfrak{R}}(\Sigma^{-1}(S^{-1}M)) = E.$$

(iii) $\mathrm{End}_{\mathbb{A}}(E) \cong \mathrm{End}_R(E) \cong \mathrm{End}_{\mathfrak{R}}(E)$.

(iv) *The module E is indecomposable over $\mathbb{A}$, R, and $\mathfrak{R}$ (by (ii) and 1.2).*

(v) *The bicommutators of E as an $\mathbb{A}$-module, R-module and as an $\mathfrak{R}$-module are all equal.*

4.3 Other Applications

This final section of this thesis lists a selection of examples of rings over which (some) indecomposable injective modules can be described using the theory developed in Chapters 3. Although there are a multitude of instances where such results can be used, I will focus on applications which can be worked in a similar manner as the first Weyl algebra, so the details are therefore omitted.

The common approach for the examples in this section is the following:

- Find a multiplicatively closed set S for the ring R so that $S^{-1}R$ is a PID.
- For a simple S-torsionfree R-module M, the module of fractions $S^{-1}M$ is also a simple $S^{-1}R$-module.

Then the injective envelope of ${}_RM$ is naturally the injective envelope of the simple $S^{-1}R$-module $S^{-1}M$, and since $S^{-1}R$ is a PID it can be described by using the results developed in Chapter 3.

Throughout these examples, k is an algebraically closed field with $\mathrm{char}(k) = 0$.

(1) The first quantized Weyl algebra $\mathbb{A}_1^q(k)$ over k ($q \in k, q \neq 0, q \neq 1$) is the k-algebra in two generators x and y subject to $yx - qxy = 1$.
Note that $\mathbb{A}_1^q(k) \cong k[x][y; \sigma, \delta]$, where $\sigma(x) = qx$, $\delta(f) = \frac{\sigma(f)-f}{\sigma(x)-x}$, for all $f \in k[x]$.
$S = k[x] \setminus \{0\}$ a left Ore set for $\mathcal{A}_1^q(k)$, and the localization of $\mathbb{A}_1^q(k)$ at S is the skew polynomial ring $k(x)[y; \sigma, \delta]$, which is left and right Euclidean, hence a PID. Thus the analysis of an indecomposable injective $E(M)$, where M is a simple $k[x]$-torsionfree module, can be lifted to $k(x)[y; \sigma, \delta]$.

(2) Let $\mathfrak{g}$ be the 2-dimensional solvable Lie algebra $\mathfrak{g}$ over k with basis $\{x, y\}$ such that $yx - xy = x$. Then the universal enveloping algebra $U = U(\mathfrak{g})$ is isomorphic to the skew polynomial ring $k[x][y; \sigma = id, \delta = x(\partial/\partial x)]$.

Block [4] classified the simple modules over U, with a key role being played by the localization $S^{-1}U \cong k(x)[y, x(\partial/\partial x)]$ of U at the multiplicatively closed set $S = k[x] \setminus \{0\}$. The (indecomposable) injective envelope of a simple S-torsionfree U-module M can be easily analyzed in $k(x)[y, x(\partial/\partial x)]$, as seen in Chapter 3 and Section 4.1.

In the same paper we find a classification of the simple modules over the enveloping algebra $U(\mathfrak{b})$ of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}_2(k)$; in particular, an S-torsionfree simple $U(\mathfrak{b})$-module localizes to a simple $k(x)[y, x(\partial/\partial x)]$-module.

(3) The class of generalized Weyl algebras (GWA), introduced by Bavula in [3], provides fertile ground for applications of the theory developed in this dissertation.

Let $D = k[H]$, and let $0 \neq a \in D$ and $\sigma \in \mathrm{Aut}(D)$ be such that $\sigma(H) = H - 1$. Denote by $A = D(\sigma, a)$ the generalized Weyl algebra (of degree 1), obtained from D

by adding two generators X and Y such that the following conditions are satisfied: $YX = a$, $XY = \sigma(a)$, and $X\alpha = \sigma(\alpha)X$, $Y\alpha = \sigma^{-1}(\alpha)Y$ for all $\alpha \in D$.

The localization $S^{-1}A$ of A relative to the multiplicatively closed set $S = D \setminus \{0\}$ is isomorphic to the ring of skew Laurent polynomials $K(H)[X, X^{-1}; \sigma]$ (see [3, (1.4)]), which is left and right Euclidean, hence a PID.

Once again, the study of an indecomposable injective $E(M)$, where M is a simple S-torsionfree module, is realized in a PID, the ring $K(H)[X, X^{-1}; \sigma]$.

Among the GWA's of the type presented above are the first Weyl algebra $\mathbb{A}_1(k)$, and the quantum Weyl algebra $\mathbb{A}_1^q(k)$; for more examples see [3].

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