The Cognitive Awareness of Middle-Years Students in Mathematical Problem Solving

by

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THE COGNITIVE AWARENESS OF MIDDLE-YEARS STUDENTS IN MATHEMATICAL PROBLEM SOLVING

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HANHSONG B. VUONG

A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University

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Abstract

As a teacher and researcher, I studied Middle-Years students in mathematical problem solving. Over a period of two years, students solved problems related to algebraic equations and revisited their thinking processes. By considering their past work, students were able to recognize their own cognitive growth. This study investigated the students' awareness of the conceptual changes associated with progressing from being primary thinkers to complex thinkers.

The study used an action-researcher approach and a complementary accounts methodology. After nine participants were identified, I created field notes for their work collections, and conducted two sets of interviews with them.

Results indicated that students became aware of their own cognitive growth in solving mathematical problems as they moved from primary thinking (common sense and working hard) to complex thinking (making sense and working smart) processes.

Acknowledgements

I still remembered the first time stepping into Dr. Mason's office to discuss about my Master's program. What caught my attention immediately were not the books lying on the shelves or a several new Canadian one-dollar bills hanging vertically on the wall, but a wall-calendar showing a photo of an island. The picturesque island seemed peaceful, but yet made me feel anxious somehow. Suddenly, I felt that I was cast away on that island, standing there on the shore as waves gently brushed against at my feet. The island appeared to show no signs of habitation, but only conifer trees gave it green colour while outlining its shape. My stomach churned and my knees trembled, and I felt that I was alone. Many questions rushed to my mind, evolving around one concern: *Would I survive? If so, how?*

Three years later, as I sit down to write my last section for the document, past memories seem to linger on…I have survived indeed! I, however, would like to acknowledge the people who have helped me reach the final stage of my journey, and would like to express my gratitude for their continuing supports and their expert advices.

The nine participants in the study from the Winnipeg School Division No. 1

The four-committee members, and "Two thumbs up" for Ralph!

My family, and "Hats off" to Steve and Andrew!

Now, as I stand on the shore to take one final look around the island, I feel the same waves gently brush against at my feet. I, however, no longer feel anxious. Instead, my mind is at ease, knowing I have gained learning experiences, and the acquired new knowledge has enriched my personal and professional growth. These learning processes give me confidence to meet future challenges.

I, then, turn around to face the calm blue sea, and look at the horizon in the distance, still remaining deeply in thought, *My journey awaits me*, and realizing that the next adventure is yet to be determined…

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CHAPTER I

INTRODUCTION

My View on Education

Whenever I look into a mirror, the reflection of myself as a teacher shows me that I take learning to heart. Being a teacher during the day and a Master's student at night has broadened by view on education. I used to think that one of a teacher's main roles was to give a test at the end of each unit, and the test was supposed to measure the students' learning. Learning is more than that. Learning cannot be treated as an end product; it is a continuous process with no beginning or end. Each stepping stone, major or minor, is significant because the step can allow the student to have a better understanding of her/his ontological and epistemological world. Being able to understand the student's own personal learning style creates awareness of their classroom roles.

My teaching assignment includes grades six to eight mathematics and grade seven and eight English language arts. I am fortunate to be able to teach the same students for three consecutive years. Within these three years, students develop socially and mentally. I am always interested in seeing students' learning growth, either in the English language arts or in the mathematics program. Learning can be compared to a flower because the blooming process takes time. Snapshots of the flower must be taken every few minutes in order to see the growth; otherwise, the growth is too little to be noticeable.

Applying this analogy to learning growth, I believe that students go through a similar process as blooming flowers. The changes in the students' progress often seem little and insignificant. When progress is examined after a certain period of time, I am then able to notice the growth in students' learning. The nature of changes that students undergo is important to me as a teacher. I believe, however, that students should become aware of their own progress as well. To see their own learning, students must document their thinking processes on a regular basis, as they engage in problem solving. Students' work samples represent snapshots of their learning that students and I can revisit. Written documents provide feedback for the students to see changes in their thinking processes. As the students go through the process, they become aware of their capabilities in solving mathematical problems. Consequently, students are able to personalize their goals, develop their own strategic plans, execute the plans, and constantly monitor their own learning. The learning process encourages students to become active participants.

The Research Question

We seldom identify *how* we accomplish our achievements. Things take their course, and we move on to our next goal. If we could take a few moments each day to explain one skill out of the many we use in the course of a day, we would be able to notice changes, little by little. Recognizing and appreciating these little changes is significant, because when we recognize changes, we can strive for betterment, steering and accelerating our learning rather than just allowing it to take place.

There should be no difference in mathematics classrooms. During any given period, students are engaged in changing their knowledge about mathematical ideas. If students are to intentionally become better at their mathematical learning, they will need to set goals, develop plans, and execute those plans. First, however, they must notice and then articulate their own learning processes.

In *How to Solve It*, Pòlya, (1957) outlines how a teacher can help a learner notice her/his cognition, gain confidence as a result, and become more intentional as a learner as follows:

> The best is, however, to help the students naturally. The teacher should put himself [sic] in the student's place, he [sic] should see the student's case, he [sic] should try to understand what is going on in the student's mind, and ask a question or indicate a step that *could have occurred to the student himself* [sic] (p. 1, original italic text).
>
> If the student has really conceived a plan, the teacher has now a relatively peaceful time. The main danger is that the student forgets his [sic] plan. This may easily happen if the student received his [sic] plan from outside, and accepted it on the authority of the teacher; but if he [sic] worked for it himself [sic], even with some help, and conceived the final idea with satisfaction, he [sic] will not lose this idea easily. Yet the teacher must insist that the student should *check each step* (p. 13, original italic text).
>
> One of the first and foremost duties of the teacher is not to give his [sic] students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. The students will find looking back at its solution really interesting if they have made

> an honest effort, and have the consciousness of having done well. Then they are eager to see what else they could accomplish with that effort, and how they could do equally well another time...(p. 15)

Pòlya's (1957) ideas suggest that mathematics classrooms should be constructed in a manner that facilitates intentional learning. Learning should be structured by the learners themselves. Mathematics instruction needs to provide opportunities for each learner to adapt learning opportunities to her/his individual needs. Generating and establishing personal learning goals needs to be followed by reflecting on and celebrating success as those goals are attained. Success can be motivating, and success can provide learners with an opportunity to become aware of the progresses that led to their success, and this can enable them to strive consciously for similar results.

Students' roles in mathematics classrooms become more complex when students are expected to become active participants in adapting their behaviours to achieve goals. Students take charge of their own learning when they take action to fulfill their personal learning needs. Students must be encouraged to develop and execute plans, which can enable them to meet their needs, and they must be encouraged to reflect on and celebrate their accomplishments along the way. Students need to be taught to analyze their learning and ultimately internalize the learning process. This is what Pòlya (1957) intends when he describes helping students naturally to become better at learning.

When students in mathematics classrooms must construct their own meanings of mathematics, the teacher's roles become challenging to establish, but less central to the learning processes of the students. To establish a learning environment, the mathematics teacher must try to understand the learning capability of each student. Skemp (1987) explains, "So the teacher of mathematics has a triple task: to fit the mathematical material to the state of development of the learners' mathematical schemas; to also fit the manner of presentation to the modes of thinking…of which the learners are capable; and, finally, to increase gradually the learners' abilities to [be independent of the teacher]" (p. 45). As the teacher becomes more familiar with the learners, the teacher is better able to decide how to adjust instruction to help students fulfill their responsibilities. As the appropriate learning environment develops, the teacher's roles are simplified. Learning processes are directed and completed more by the students than by the teacher. The teacher facilitates by monitoring and guiding, and encourages students to apply their acquired knowledge to solving mathematical problems.

This research study will take place within a sequence of problem-solving activities for my grade six, seven and eight students. In the activities students solve a specific problem and record their solution and their solution processes. The students then use a rubric intended to lead them toward becoming more aware of and expressing their strategies and celebrating their progress. In this study, I will be focusing on the students' awareness of their progress in solving problems. The research question is:

What is the nature of the conceptual changes that students undergo as they become aware of their own growth in mathematical problem solving?

Limitations of the Study

Over the course of designing and executing the study, several limitations began to surface that *might* have influenced the results of the study, and hence, the purpose of this section is to discuss three limitations that I perceived to be relevant to the study. The first limitation was about me being a teacher/researcher in this study. The nine participants had been taking my mathematics courses for three consecutive years. Consequently, I had built a collegial relationship with the students. Even though I had explained the study with care, through the eyes of the participants, they still knew me as the teacher who had the authority governing their marks. In this situation my presence *might* have influenced the nature of their responses. Also our collegial relationships *might* have influenced the manner in which I would interpret their balance work and their responses. My perspectives on the data *might* have been biased. What if I was not their grade eight mathematics teacher, and what if I was an external researcher? How *might* the situations have influenced the data? The dilemma was unavoidable. That is why, when I proposed the study to the students, I ensured that I had defined their roles and my roles clearly and carefully so it would reduce the chance for the data to be influenced by our biases.

The second limitation was the number of students involved in the study. In fact, nine students *might* be considered by some to be an insufficient number of participants. If the intent of the study was to extend beyond my classroom to other classrooms in different schools and divisions, the subjects, then, *might* have been well represented. This *might* affect the outcome of the study. With larger number of participants, the outcome *might* have become richer in illuminating the research question, perhaps. Because the study was a qualitative research, it would not be feasible to have a large number of participants. For one factor, the workload of making detailed notes on individual participants would be overwhelming.

The third limitation was the study involved only the French Immersion students. The nine participants received mathematics instruction in French. Because the study was conducted in English, could it have hindered the participants when they wanted to express their thoughts clearly and precisely about their mathematical knowledge? The language barrier *might* have been an issue in this situation. Also, French Immersion students tended to be known as "well-disciplined" and "conscientious-learners", and subsequently, "high-achievers" in general. What if the perception were true, how *might* it have influenced the data? How about the participants' social-economic backgrounds, *might* it have influenced the data? Acknowledging this limitation, I would hesitate to generalize my findings. The outcome of the study would reflect only on this specific group of participants. What I have done is to make recommendations for teachers based on the theory and practical aspects of the study.

With careful planning and executing the study, I considered my research processes to be well articulated, and I have confidence in the findings. I would not deny claiming that my study was imperfect, but the fact is no study could be considered perfect. As Ellsworth (1997) points out, there can be no perfect fit between the targeted audience and learning. There will always be volatile space between the study and the participants. This compares to Lush's (one of the participants) perspective on perfection when she said, "I can't say the explanation was perfect, because *nothing's perfect*. But like it is, because [Betsy] explained of what she thought of like the observations and she [verified]…" [italics added] Like Lush stated, as long as we had articulated the situation thoroughly, we would accept the study as the next best thing to a perfect study. The study satiated the theorist's words and ideas that echoed in one of the participants' words.

CHAPTER II

SITUATING THE RESEARCH

Developing activities for the problem-solving component of the mathematics curriculum is a most challenging role for the teacher. The activities must engage a wide range of students in thoughtfully solving problems while constructing mathematical knowledge. Over time, the activities must offer different levels of complexity while maintaining similarity of form so that students are able to notice and explain their cognition as it develops. In the problem-solving strand of instruction, I use a rubric to guide students to write about their thinking, and to communicate my expectations to them. The rubric encourages a focus on learning, and then guides students to express in words their own learning in solving the mathematical problems. (See Appendix 1.) Students are able to reflect on their thought processes, causing them to become aware of their own learning growth in solving the problems. On this point, Barnes (2000) wrote, "Students should be encouraged to reflect, not only mathematical ideas they are developing, but also on how they come to know them and on aesthetic aspects of them…[A] focus of reflection may help students to become more aware of their own thinking processes" (p. 41).

Teaching and Learning to Solve Problems in Mathematics

Pòlya (1957) is considered the principal theorist regarding students' own awareness of their learning processes in solving mathematical problems. Even though Pòlya (1957) might not be considered a current theorist, his theories are much alive and relevant today in mathematics education. As Schoenfeld (1992) points out:

> Discussions of problem-solving strategies in mathematics…must begin with Pòlya. Simply put, *How to Solve It* (1945) planted the seeds of the problem-solving “movement” that flowered in the 1980s: Open the 1980 NCTM (National Council of Teachers of Mathematics) *Yearbook* (Krulik, 1980) to any page, and you are likely to find Pòlya invoked, either directly or by inference in the discussion of problem-solving examples. The *Yearbook* begins by reproducing the *How to Solve It* problem-solving plan on its flyleaf and continues with numerous discussions of how to implement Pòlya-like strategies in the classroom…Nonetheless, a close examination reveals that while his name is frequently invoked, his ideas are often trivialized. Little that goes in the name of Pòlya also goes in the spirit of his work… (p.352)

Pòlya’s (1957) model of problem solving describes a set of four independent phases: understanding the problem, devising a plan, carrying out the plan, and looking back. This model, moreover, lays the foundation for Manitoba mathematics curriculum. As the students are engaging in these phases, they also are encouraged to become aware of their own progress.

Pòlya (1957), furthermore, suggests how teachers can become more aware of students’ learning progress. Hence, this study will make extensive use of Pòlya’s (1957) sense of mathematical problem solving. The essence to solving a mathematical problem is finding the connections between the givens and the unknown. Thus, connecting all of the givens with the unknown essentially will lead to the solution. The process of bringing one more given into play can be perceived to be a sign of progress. Pòlya (1957) defines progress to be a “step forward” (p. 182) in the learning process, and with each step made; the student is one step closer to the solution of the mathematical problem. "Our undertaking may be important or unimportant, our problem of any kind – when we are working intensely, we watch eagerly for signs of progress[, and to come to terms with] what can be reasonably regarded as a sign of approaching the solution" (p. 176).

There are two types of sign of progress: *clearly expressible signs*, and *less clearly expressible signs* (Pòlya, 1957). Becoming more aware of signs of progress, students feel that they have made progress. Having understood this essential point, students can express with some clarity the nature of still other signs of progress.

> Thus, understanding clearly the nature of the unknown means progress. Clearly disposing the various data so that we can easily recall any one also means progress. Visualizing vividly the condition as a whole may mean an essential advance; and separating the condition into appropriate parts may be an important step forward. When we have found a figure that we can easily imagine, or a notation that we can easily retain, we can reasonably believe that we have made some progress. Recalling a *problem related to ours and solved before* may be a decisive move in the right direction. (p. 183, original italic text)

When students work intently, they can certainly feel the pace of progress. If the progress is immediate, they feel excited, and if it is slow, they may feel desperate for any distinct sign of progress. Emotions responses to the obscurity of some problem-solving situations can prevent students from progressing.

> …Since the more clearly expressible signs of progress are connected with the success or failure of certain rather definite mental operations, we may suspect that

> our less clearly expressible guiding feelings may be similarly connected with other, more obscure, mental activities – perhaps with activities whose nature is more "psychological" and less "logical" (Pòlya, 1957, p. 184).

Once students have identified and determined goals to meet their learning needs, a plan is drawn, and the signs of progress can help them determine the appropriate actions to pursue. The presence of these signs may cause students to apply efforts to the right spot, and "…if the signs become more frequent as [students] proceed, if they multiply, [students'] hesitation fades, [their] spirits rise, and [they] move with increasing confidence…" (Pòlya, 1957, p. 185). On the other hand, the absence of these signs warns students of a detour ahead. They may be discouraged, have to retrace the path, and find another means to continue. Consequently, students will gain experiences in reading signs of progress as they are approaching them, and take appropriate actions to ensure that success is at solving mathematical problems.

Problem Solving

There are all types of problems. Problems can arrange from simple (e.g., what is one plus one?), to complex (e.g., what is the square root of negative one?). Each problem, major or minor, needs to be solved. Charles and Lester (1982) define a problem as a task for which:

1. The person confronting it *wants or needs to find a solution.*
2. The person has *no readily available procedure* for finding the solution.
3. The person must *make an attempt* to find a solution (p. 5, original italic text).

In a classroom situation, when students are given a mathematical problem, they need to find a strategy, within the context of the students' mathematical knowledge, to attempt a solution. Schoenfeld (1992) states, according to Pòlya, part of that engagement can be active discovery which involves guessing. In addition, mathematics appears sometimes as a guessing game to a mathematician. She/he has to guess a mathematical theorem before proving it.

Since mathematics is a living subject that seeks to understand the pattern, then the learning of mathematics has to emphasize pattern-seeking. The instructional style and content have to move students beyond mathematical rules by focusing on:

- seeking solutions, not just memorizing procedures;
- exploring patterns, not just memorizing formulas; and
- formulating conjectures, not just doing exercises (Schoenfeld, 1992, p. 335).

To reflect these emphases, teachers have to create learning environments that will stimulate and motivate the students to explore and conceptualize mathematical ideas instead of memorizing closed body of mathematical laws. Schoenfeld (1989) describes his sense of how mathematics should be: "If you really understand it, you don't have to memorize a lot, because you can figure it out…" (p. 87). Consequently, problem-solving activities play a significant role in making students recognize that mathematics is really about patterns and not only about numbers.

Even though problem solving has been used with multiple meanings that range from "working rote exercises" to "doing mathematics as a professional" (Schoenfeld, 1992), it should promote and cause the students to learn to think mathematically. Consequently, the students should learn mathematics in much the way that mathematical ideas were discovered. This problem-solving idea also appears in *Grades 5 – 8 Mathematics Foundations* document (Manitoba Education and Training (MET), 1997).

> **Problem-Solving** – Students are exposed to a wide variety of problems in all areas of mathematics. They explore a variety of methods for solving and verifying both routine and non-routine problems. In addition, they are challenged to find multiple solutions for problems and to create their own problems… (p. A-4, original bold text)

The notion of problem solving plays a prominent role in Manitoba mathematics curriculum that demands the classroom instruction to cause learners to think mathematically.

Schroeder and Lester (1989) discuss three approaches to problem-solving instruction. The first approach is teaching *about* problem solving. This approach highlights Pòlya's (1957) model of problem solving. The teacher who teaches *about* problem solving demonstrates a number of strategies for the students, and from which they can choose or which they should use in devising and carrying out their problem-solving plans. Schroeder and Lester (1989) point out one limitation of the approach: "If teaching *about* problem solving is the focus, the danger is that 'problem solving' will be regarded as a strand to be added to the curriculum. Instead of problem solving serving as a context in which mathematics is learned and applied, it may become just another topic, taught in isolation from the content and relationships of mathematics" (p. 34, original italic text). This limitation also reflects the Manitoba Foundations document's (MET, 1997) intention of integrating problem solving into all areas of mathematics, and it should not be treated as an isolated strand.

The second approach is teaching *for* problem solving. The emphasis of this approach concentrates on ways in which the mathematical concepts being taught can be applied in the solution of both routine and non-routine problems. The teacher who teaches *for* problem solving is concerned about students' ability to transfer what they have learned from one problem context to others. Schroeder and Lester (1989) address one concern with the approach as follows:

> ...When this approach is interpreted narrowly, problem solving is viewed as an activity students engage in only *after* the introduction of a new concept or following work on a computational skill or algorithm. The purpose is to give the solution of real-world problems...Often, solutions to these problems can be obtained simply by following the pattern established in the sample, and when students encounter problems that do not follow the sample, they often feel at a loss (p. 34, original italic text).

The Manitoba Foundations document (MET,1997) also shares this concern, because teaching *for* problem solving can be interpreted narrowly: "Traditional teaching methods of show-and-tell followed by students working on either textbook pages or worksheets" (p. A-6). Students should learn mathematics in ways that develop concepts, skills, and understanding.

The third approach is teaching via problem solving. This approach is most consistent with the recommendations of NCTM's (1987) Standards Commission that:

1. mathematics concepts and skills be learned in the context of solving problems;
2. the development of higher-level thinking processes be fostered through problem-solving experiences; and

3. mathematics instruction take place in an inquiry-oriented, problem-solving atmosphere (Schroeder & Lester, 1989, p. 34).

Teaching *via* problem solving fosters students to see mathematics as a sense-making activity, and they have to internalize it as such. As Schoenfeld (1989) points out: "Every theorem is, in essence, a statement of the following type: 'Things fit together in a particular way, for the following reasons.' In short, mathematicians spend most of their time making sense of things. Doing mathematics is sense-making, and becoming a mathematician includes developing (or internalizing) the mathematician's aesthetic, a predilection to analyze and understand, to perceive structure and structural relationships, to see how things fit together" (p.87). When students engage in a problem solving activity and are making sense of the context of the problem, they come to understand and use mathematics in meaningful ways.

Teaching *via* problem solving can be compared to "Learning through problem solving" (MET, 1997, p. A-6), because students are encouraged to take risk, communicate, and interpret of mathematical ideas. "This learning involves solving a revealed problem rather than remembering as demonstrated ritual or definition. The intent is to engage students in thinking about what they are learning and why they are learning it" (p. A-6.) The teacher needs to create a learning environment that fosters students to think mathematically. Hence, "Problem solving in grades 6-8 should promote mathematical learning. Students can learn about, and deepen their understanding of, mathematical concepts by working through carefully selected problems that allow applications of mathematics to other context" (NCTM, 2000, p.256).

Mathematics Reform

In recent years, mathematics curriculum has been focusing on students' construction of their own meanings of mathematics. Instruction that reflects mathematical reform provides students with experiences that proceed from simple to complex, and from concrete to abstract.

More and more educators have come to realize and recognize the essence of teaching mathematics is to provide students with opportunities to explore and develop their understandings in authentic learning environments. Unlike traditional approaches, reform approaches involves linking learning to the situation or context in which it takes place. Students' knowledge is shaped within the context of their experiences, as they construct their sense of their experiences to enable them to interact appropriately with the environment that encompasses their existence. Learning can be considered as embedded dimensions of activity that will stimulate learners' metacognition to enrich knowledge and expand experiences in the process of learning (Brown, 1995).

Boaler (1998) provides two terms for student activity: *rule-following behaviour* and *situated learning*. In traditional education, students are conditioned to rule-following behaviour. This behaviour might stem from beliefs that mathematics is memorizing a vast number of rules, formulas, and equations. Unlike traditional education, reform education involves learning linked to the situation or context in which it takes place, known as situated learning, and activates students' mathematical thinking on what they think is expected of them rather than on the arithmetic within a question.

Learning mathematics no longer emphasizes memorizing a vast array of mathematical rules, then applying them in practice questions. It involves:

1. how students show their understanding, explain their reasoning, and assist their peers in discussions;
2. how students support each other's learning either in pairs, small groups or large groups; and
3. how students extend their acquired knowledge to other similar situations in the future (Fraivillig, Murphy & Fuson, 1999).

Discussions create an awareness of each student's understanding of the subject matter. To be able to explain to others, students must internalize their learning, thus developing a network of schematic knowledge, linking declarative (concepts) and procedural (rules) knowledge (Marshall, 1989). This network of knowledge paves the path for students to transfer their understanding from one situation to the next.

To achieve this kind of learning, the pedagogy of mathematics becomes complex. The pedagogy is rather simple if students are able to learn by finding solutions to questions in textbooks, but this is not the case. Students should learn mathematics by inquiry. In this context, "inquiry" is used where students construct their own meanings of mathematics, as they become active participants in mathematical activities. Inquiry provides them with first-hand experiences in exploring ideas, and developing and executing strategic plans to solve questions. An inquiry approach leads students to discover mathematical concepts, which stimulates them to take ownership of their learning. "Not only must students be active learners who construct their own mathematical meaning from their experiences, they must also communicate their understandings and their thought processes as they investigate, create, validate, and learn" (MET, 1997, p. A-6). To facilitate such learning, the mathematics curriculum must include authentic activities, opportunities for students to enhance their understanding of mathematics concepts. These activities allow students to view the procedures they learn as tools that they can adapt and use. The understandings and perceptions that result from these experiences lead to increased competence in learning mathematics in novel situations. The students are more likely to develop a predisposition to think about and to use mathematics in novel situations. This tendency rests on two important principles: 1) the students have the belief that mathematics involves active and flexible thought; and 2) the students have developed an ability to adapt and change methods to fit new situations (Boaler, 1998).

From Cognition to Metacognition

Beyer (1997) has recommended that cognitive skills and principles related to cognitive processing to be part of classroom learning goals. The thinking processes are necessary to develop subject-matter knowledge, and these thinking skills eventually turn students into lifelong learners.

In order for students to improve their problem-solving processes, they must have opportunities to witness their cognition through writing. Burton and Morgan (2000) state, "*Natural language* within which the symbolic or special vocabulary and structures particular to mathematics are embedded...*Natural language* serves in the construction of the identities of the author and reader and of the epistemological and ontological assumptions underlying the writing" (p. 430, italics added). The act of writing is complex, but it is significant for students to be able to articulate the learning process. The act of writing, furthermore, assists students in visualizing the cognitive process in problem solving, and the strategies used to arrive at the solution, resulting in their awareness of their own learning and progress in problem solving. Each previous piece of writing, then, becomes part of the knowledge base about the student's learning for the next piece of writing – that is, they supplement each other for building mathematical knowledge and they can navigate in the direction of their learning needs to meet their mathematical abilities in problem solving.

Manitoba Mathematics Curriculum

The importance of exposing all students to a wide variety of problems in all areas of mathematics has been recognized in the *Grades 5 to 8 Mathematics – A Foundation for Implementation* document. The document outlines seven Mathematical Processes known as Big Ideas, which are to be implemented in the structure of mathematics to permeate teaching, learning, and assessing. These seven processes become the essence of mathematics instruction. Two of the seven Big Ideas are problem solving and reasoning. Problem solving comprises three kinds of processes: learning through problem solving, solving routine problems, and solving non-routine problems. Mathematical reasoning involves systematic thinking, conjecturing, and validating. These processes help students understand that mathematics makes sense. They are encouraged to draw logical conclusions and then justify their solutions in a variety of ways. In fact, good reasoning is as important as finding correct answers (MET, 1997). The central focus for this study is on two Big Ideas: problem solving and mathematical reasoning, I consider the Balance Series can address these ideas.

In the Balance Series, each problem shows a balance scale, and according to the *Grades 5 to 8 Mathematics – A Foundation for Implementation* document (MET, 1997) "The balance scale provides a concrete way for student to understand the principle of "balance" in equality" (p. B-60). This pictorial representation is a symbolic representation of an algebraic equation. The students "develop [their] understanding of solving equations by having them move step by step from concrete materials and diagrams to symbolic representations" (p. B-82). The sequence of the balance problems contains patterns, mobiles suspended from mobiles, allowing students to relate and predict the number of balances and the number of geometric forms from one problem to the next problems. After recognizing the patterns, the students eventually can design their own balance problem.

In the Balance Series

The specific problem-solving sequence of activities is called "The Balance Series". (See Appendix 2 for an example from the series of 25 problems appropriate for grades 7 and 8.) Each problem contains a diagram of a mobile. The mobile shows a two-armed balance suspended at its midpoint. Geometric forms suspended from each arm represent masses. Students must determine numerical values for the mass of each form, so that the mobile will balance. To assist students, each problem has one or more mathematical statements. The balances allow students to use informal algebra, and trigger them to develop their intuitive knowledge of algebraic concepts.

Each diagram of a mobile is a representation of an algebraic equation. (See Appendix 2.) The fulcrum (balancing point) can represent the equals sign in the equation; each geometric form can be seen as a variable; each arm with geometric forms symbolizes terms on each side of the equation that must be balanced. When the students determine the value for each geometric form, so that the mobile will be balanced, the geometric forms are seen as specific numerical values. The series of problems offers one context for engaging with the concepts involved in an algebraic equation. The complexity of each balance problem in the series involves balances suspended from balances, more literal symbols, and more complex mathematical statements as clues.

As the students solve the balance problems, they are informally thinking about algebraic ideas. Throughout the problem-solving process, students must be conscious of maintaining the balance of both sides; this is a contextual example of the algebra concept of equality. Students learn that one geometric form can have different values in different contexts, but in each context, each geometric form can have only one value even when it is used more than once. This is how variables operate in formal algebra. The supplementary clues, in the form of mathematical statements using literal symbols, bring the forms of algebra into the problem-solving context. Students can think about algebra statements as ways of expressing their understandings about the mobiles and the problems they are solving.

According to the *Grades 5 – 8 Mathematics Foundations* document (MET, 1997), balance problems are considered as routine problem solving, because "...there are readily identifiable models (the meanings of the arithmetic operations and the patterns/templates) to apply to problem situations" (p. A-10). However, the balance problems can also be considered as non-routine problem solving. The first step in solving each mobile is to divide its total weight into separate elements. The students learn that weights suspended below the fulcrum can have a numerical value independent of the weights on the two extended arms. This offers one context for using the strategies, guess-and-check or think-backwards. Most students find solving the balance problems motivating and challenging, engaging them in a context where they can "deepen and extend understandings" (p. A-10) of algebraic concepts. The Balance Series "...can be seen as evoking an 'I tried this and I tried that, and eureka, I finally figured it out' reaction. This process involves a search for heuristics (strategies seeking to discover)" (p. A-10).

Teaching and Learning Algebra

A number of research studies have investigated students' intuitive understandings of informal uses of school algebra prior to studying formal algebra (Swafford and Langrall, 2000). Some of the topics that have been widely studied are students' uses of variables or literal symbols, students' intuitive equation-solving methods, students' abilities to intuitively apply the distributive property, and their interpretations of the equal sign and grouping signs. These studies illustrate that students have begun formal algebra instruction with these powerful sense-making skills. They also highlight some of the difficulties students encounter when they begin their study of algebra. In *Psychology of Learning Math*, for example, Skemp (1987) states,

> In the building up of the structure of successive abstractions, if a particular level is imperfectly understood, everything form then on is in peril. This dependency is probably greater in mathematics than in any other subject. ...But to understand algebra without ever having really understood arithmetic is impossibility, for much of the algebra we learn at school is generalized arithmetic (p.20).
>
> [Symbols] act as combined handles and labels for their associated concepts. ...But the function of symbols is for manipulating and communicating mathematical concepts, and these are the true operands in relational mathematics (p.169).

No amount of research about students' pre-instructional knowledge in algebra can suffice for teachers' decisions about what and how to teach (Swafford and Langrall, 2000). Many studies, however, have shown that after students have been exposed to informal algebra in the early years, instruction can be built on students' intuitive knowledge to access particular algebraic ideas (Day & Jones, 1997; Fouche, 1997). The Balance Series is designed to engage students in informal algebraic thinking, preparing them to understand formal algebra later in future studies.

Students must be encouraged to learn and understand the big ideas of algebra prior to studying formal algebra. Woodbury (2000) reminds us of the two conceptual areas that encompass the big ideas of algebra:

1. Ideas about numbers, number systems, and number theory, encompassing big ideas about properties of numbers and operations in different number systems;
2. Symbolic representation and theory of equation, including the big ideas of the use of variables and the properties of equations and operations on equations (p. 227).

These conceptual areas should be used as guides to teach a particular algebraic topic. The focus of teaching and learning experiences about real numbers as a number system is to build a relational understanding of the big ideas about real numbers. Students should learn the property of real numbers in relation to integers and rational numbers in the number systems, the distributive property that holds in those different number systems, and the importance of using these concepts in problem-solving activities. These conceptual areas can be found in each Balance problem that enhances the understanding of the big ideas about algebra.

Stacy and MacGregor (1997) state that students who are beginning algebra experience difficulties, because they have a limited understanding of number and operations, and students cannot write what they know, which causes them to make errors. They present four practical strategies to overcome these difficulties.

1. Seeing the operation, not just the answer
2. Understanding the equal sign
3. Understanding the properties of numbers
4. Being able to use all numbers, not just whole numbers (p. 253)

The Balance Series addresses these needs. Each Balance problem is designed to give students opportunities to use and practice these strategies, and as a result the Balance activities can assist students' understanding of learning formal algebra.

Postmodern View

There are multiple discourses in learning algebraic concepts. The first is the manipulative discourse of learing algebra. Students manipulate an actual scale balance weighing an object by comparing it to fixed weighing blocks. The second is the pictorial discourse of learning algebra. A simulation of a scale balance or a mobile is drawn on a sheet of paper, and students make sense of the diagram by observing, thinking creatively, and relating the diagram to the actual 3-dimensional models. The third is the abstract discourse of learning algebra. An algebraic equation is a simulacrum of a scale balance or a mobile. Which one of these discourses underlies the truth about learning algebraic concepts?

Postmodernists are constantly questioning the nature of truth. If the truth is constrained by the five senses, then the first discourse should represent a complete sense of algebra. Students can see, feel, and touch an actual scale balance. These experiences can lead students to construct a true knowledge about algebra. Then what is the true knowledge about algebra? If the "truth" about algebra lies within a linear string of numbers and letters off setting each side of an algebraic equation, then what signifies the "truth" underlying the composition of these algebraic signs? An algebraic equation can be used to explain the phenomenon of a sequence of mathematical concepts, which provides only the scope of the mathematical knowledge, and not the truth.

Pictorial discourses like the Balance Series substitute the numbers with weights, and the letters with geometric forms. A mobile diagram substitutes for a 3-dimensional scale balance. The "truth" does not lie within the Balance Series, but the series creates a simulation of algebra for students. This simulacrum stimulates students to process making sense of what algebra is, and construct a meaningful way of coming to terms with the concepts. Ellsworth (1997) states that there is no perfect fit between the targeted audience and learning. The learning experience that students gain from participating in the discourse is at the discretion of an individual student. The meaning making is based on the students' interpretations, which are always drawing on references to personal histories that are shaped by cultural, historical, social, and environmental factors. Therefore, the success of the Balance Series in a sense depends on the students. By becoming active participants, students will internalize their thinking processes, and enhance their learning experiences. The series only offers an interpretation of the algebraic truth, if only the "truth" exists.

The semiotics (a mobile diagram) lies at the intersection between algebraic ideas and algebraic signs constructing algebraic knowledge. This intersection gives the underlying authority to the author, the text and the reader. The authority of an author creates the text intentionally. The authority of a text is to communicate its ideas and signs to the reader, but its readers govern the underlying authority. Basically, readers have the authority to see what they want to see. How one reader interprets the text does not confirm with the uniformity of another reader's interpretation of the same text. Taking the Balance 17 (See Appendix 13.) for example, one student might see that the total weight of the mobile is shared equally between its two extended arms as a starting point; another might see the importance of one rectangle equalling two squares as a starting point; and another might see the importance of identifying inequality between forms as a starting point. The ultimate goal for these students is to successfully solve the problem, and by doing so, they will gain learning experiences. These learning experiences will create new consciousness, leading students to construct their own understanding of pre-algebraic concepts (Slattery, 1995).

Pedagogy would be rather simple if students were able to learn by finding solutions to questions in textbooks, but this is not the case. Students should learn mathematics by inquiry. Inquiry provides them with first-hand experiences in exploring ideas, and developing and executing strategic plans to solve questions. An inquiry approach leads students to discover mathematical concepts which stimulates them to take ownership of their learning. "Not only must students be active learners who construct their own mathematical meaning from their experiences, they must also communicate their understandings and their thought processes as they investigate, create, validate, and learn" (MET, 1997).

The Balance Series, however, does not present its information in a neutral language. Instead, each Balance problem offers multiple ways of viewing honouring multiple interpretations and values eclecticism rather than one method. Postmodern theory supports the deconstruction of "texts" to create multiple and authentic meanings to individuals (Hlynka & Yeaman, 1992). During the problem-solving process, students must deconstruct the diagram of a mobile by taking the mobile apart, and examining each part separately under a microscopic lens. Some students might start at the top of the mobile, some at the right, some at the left, and some at the bottom. Students are processing their abstract thoughts of meaning making, while deconstructing the text. After having solved each part of the mobile, students are ready to consolidate their findings into a whole. The reconstruction process gives students a different view of the mobile. This new view is different from the initial view, because the holistic perspective enriched with problem-solving skills, meaning making, and learning experiences contributes to students' mathematical knowledge. This knowledge is not neutral. Then the Balance Series can be considered as a postmodern text, because "[t]he postmodern textbook should…present the reader with a multiplicity of views of a given field of knowledge" (Spring, 1991, p.197).

The Balance Curriculum

The Balance curriculum consists of having students work through and solve a sequence of problems. A mobile diagram invites students to become active participants in the discourse. The discourse encourages and fosters students to perform critical thinking, and connect the situation at hand to their personal experience with the previous problems.

The Balance problems allow students to view the procedures they learn as tools that they can use and adapt. The understandings and perceptions that result from these experiences lead to increased competence in learning mathematics in novel situations. The students are more likely to develop a predisposition to think about and to use mathematics in novel situations. This tendency rests on two important principles: 1) the students have the belief that mathematics involves active and flexible thought; and 2) the students have developed an ability to adapt and change methods to fit new situations (Boaler, 1998).

During the activities students begin to develop their own strategic approach to problem solving, and students are also asked to do a self-assessment. The feedback from the students themselves and from the teacher increases the students' awareness of the self. This self-awareness guides students to reinterpret their beliefs, structures, and traditional concepts (Slattery, 1995). The Balance curriculum directs students to appreciate their learning, and to value learned concepts and ideas of the balances.

The Balance curriculum supports Kliebard's (1972) three metaphorical roots of curriculum design. First is the metaphor of product, known as simulacrum. Students come to work on the first Balance problem without having prior knowledge of what the problem is. The simple structure of the first mobile might become complex simulacrum for students. Kliebard (1972) refers to the students during this stage as the raw materials. The highly skilled technician (the teacher) asks students to express their thought processes in writing. The focus of writing is to promote the students' own awareness of their thinking and to create a concrete record of their strategic plans in successfully or unsuccessfully solving the Balance problem. The process of writing, then, is shared with peers in a class discourse facilitated by the teacher. "Only through participation guided by others will students develop the knowledge-in-action that will enable them to participate effectively on their own" (Applebee, 1996, p.62). As a result, students are encouraged to think critically, and to explain their thought processes.

The second metaphorical root is the metaphor of growth. The Balance curriculum does not prescribe a working time frame. This suggests that the implementation of the curriculum can be as short as one semester or a number of years, depending on the teaching circumstance. Kliebard (1972) refers to the students during this period as the plants, and the teacher as the wise and patient gardener who "treats each plant according to its needs, so that each plant comes to flower"(p. 403). Another part of the curricular domain is structured as self-monitoring so that students are actively involved in seeing their learning growth. The conversation shows its participants their own growth in planning, executing and assessing their problem-solving methods, writing and explaining their thought processes, and developing and constructing their algebraic understandings. Applebee (1996) states the four important elements of effective conversation as quality, quantity, relatedness, and manner. The quality of the students' work collections supports meaningful conversation. Over time, the quantity of their Balance work will provide students with enough substance to make their own generalizations about their learning progress. Since the Balance Series contains a sequence of problems, and each problem supports another, the continuity gives students a cumulative conversation of their learning progress. One of the curricular intentions is to consider the progress of each student as an individual; there are no comparisons between students being made. This methodology will encourage student participation. The Balance curriculum designs for individual learners so that each student becomes a critical observer of her/his progress in fulfilling her/his learning needs. "All plants are nurtured with great solicitude, but no attempt is made to divert the inherent potential of the individual plant from its own metamorphosis or development of the whims and desires of the gardener" (Kliebard, 1972, p. 404).

The third metaphorical root is the metaphor of travel. The curricular domain is to intentionally create a journey for students to experience learning growth in solving mathematical problems. It is "...a route over which students will travel under the leadership of an experienced guide and companion" (Kliebard, 1972, p. 404). The teacher, in this situation, will guide students safely to their destinations, while offering companionship when students encounter misfortunes during the journey. The role of the teacher is not to be an answer giver, but an advice giver. The final destination for each student is to arrive at the fourth level in the problem-solving assessment rubric. (See Appendix 1.) If the teacher can bring all of her/his students to the final destination without much difficulty, then what does that say about the curriculum? Looking realistically, however, students have authority over their own destiny. Some students will be unable to reach the final destination either due to their mathematical abilities, or their willingness to cooperate. Ellsworth (1997) describes the journey as being dependent on the volatile space between the addresser (curriculum) and the response (students). The gap cannot be avoided under any circumstance, because:

1. The space of difference between address and response is a social space, formed and informed by historical conjunctures of power and of social and cultural difference;
2. The space of difference between address and response is a space that bears the traces and unpredictable workings of the unconscious, and this make it able to escape surveillance and control by both teachers and student; and
3. The space of difference between address and response is available to teachers as a powerful and surprising pedagogical resource. However, and paradoxically, teachers can't control mode of address – even through pedagogical practices that are intended to regulate it. Practices like dialogue, for instance (Ellsworth, 1997, p. 38).

The best way is to treat each response as an individual case. Then the teacher is able to work on a plan with an individual student in meeting her/his needs.

"Each traveller [student] will be affected differently by the journey since its effect is at least as much a function of the predilections, intelligence, interests, and intent of the traveller as it is of the contours of the route…no effort is made to anticipate the exact nature of the effect on the traveller; but a great effort is made to plot the route so that the journey will be as rich, as fascinating, and as memorable as possible" (Kliebard, 1972, p. 404).

Research Method

I am an action researcher who undertakes the dual roles of teacher/researcher. I am using my own teaching as a site for research, with my students as research participants. Simultaneously maintaining the dual roles of teacher/researcher must be undertaken with care. Wong (1995) perceives an inherent conflict. He defines a teacher's roles as being concerned with the sphere of process or change in human characteristics, and a researcher's roles as being concerned with knowing or understanding these characteristics. Furthermore, a teacher's primary goal is to choose and take appropriate actions to cause the change. A researcher's primary goal, however, is to carefully observe, reflect and inquire to understand the situation. Wong (1995) emphasizes his struggles to come to terms with his teacher roles in the classroom, while negotiating his researcher roles. He believes that teacher roles are to act responsibly and with compassion and respect. The researcher is merely responsible for interrogating subjects, and asking probing questions until she/he knows and understands her/his subjects' perspectives. Wong (1995) explains that these two roles are distinct, and there should not be any crossovers. The teacher/researcher can merely play one role, either the teacher or the researcher, and not both simultaneously. The main difference between these two roles is the teacher is considered human, and the researcher, non-human.

> As a researcher, I was greatly interested in students' conceptions of the natural world and their ways of interpreting and explaining them. I felt that any assistance from me would adversely influence the expression of their tenuous understanding. On the other hand, as a teacher I felt obligated to instruct, to change Toni's notions or her way of thinking (p. 25).

In contrast, Wilson (1995) claims that the teacher's role and the researcher's role can be unified. The teacher/researcher's primary roles are to produce knowledge; researchers produce knowledge through writing and publishing papers and textbooks, giving presentations and generating theories, whereas teachers produce knowledge through teaching and exploring ideas. Wilson (1995) believes that if the teacher/researcher is too conscious of separating one role from another, then she/he is constrained to interact with her/his epistemology. Instead, she considers it to be the role of teachers and researchers alike to find out how to clarify any student confusion, and to examine the situation at hand closely. Wong (1995), on the other hand, claims that it is without doubt that the action of teacher/researcher can interfere and change the intention of the research to certain degree, but many teacher/researchers are incorporating such intervention as an opportunity to examine the change.

I agree with Wilson (1995):

> [The] teacher must teach. But teaching, for me, entails *everything* I must do to help my students learn, including asking questions. In his portrait of both teaching and research, Wong focuses on questioning as a tool of research and ignores the fact that it is equally a tool of teaching. Questions help teachers, as Dewey (1904/1964) reminds us, study pupils' minds. Furthermore, from Socrates on, many educators have used questions as a powerful pedagogical tool (cf., Haroutunian-Gordon, 1991). (p.20, original italic text)

The first aspect of my research is to investigate students' work samples from 1999-2001 school year. My task as action researcher is to analyze the students' work to see progress in their learning to explain their cognition through writing. Obviously, this task assists me, a researcher, in knowing and understanding how students demonstrate their learning progress, and in determining and defining what their learning progress is. At the same time, as a teacher it is valuable to see and be part of students' learning progress so I can apply what I learn in my research analysis in the planning of my instruction. Evidently, there is no need for negotiating between the two roles because essentially both roles are striving for the same agenda – that is the knowledge of students becoming aware of their problem-solving processes and their understanding of the balance concept. This example illustrates what Wilson (1995) meant by the compatibility of the two roles.

As a teacher/researcher, I have to acknowledge that the teacher and the researcher at the fundamental level are two distinct professional roles. As Wong (1995) points out, "…reducing our conceptions of teaching and research to their common denominator eliminates much of what make each a disciplined, professional practice" (p. 23). As the teacher I am to teach and explore mathematical ideas with the students, and being the researcher I am to observe and understand such actions. These two roles, however, do not have to operate independent of each other. Instead, both roles can be viewed as knowledge producers. The acquired knowledge, then, can foster two kinds of learning – 1) the researcher's learning and understanding the process or change in participants' characteristics, and 2) the teacher's learning of his own practice. In my belief, I have the need to interact with the students in my classroom in order to become aware of my pedagogy. In other words, the more I understand the situation at hand, the more I will understand my own practice, and as a result, I will be able to gain success with students, and encourage their learning growth.

My duties as teacher include ensuring that every student is treated equally and fairly. Although the students cannot opt out of instruction, it is within students' discretion, in this case along with their legal guardians, to either participate or not in research activity. I must ensure that the choice to participate or not to participate in research activity will not affect the students' course marks. My roles as researcher include defining clearly the parameters for my data collection, and being ethical in fulfilling my responsibilities to be discrete and respect the confidentiality of participants. I may need to separate research interactions from classroom interactions, and collect data equitably from every student who chooses to participate in the research. I am, however, entitled to select and use all or only parts of the collected data in the research report. These considerations will help to resolve the ethical complications of the teacher/researcher duality.

<u>The Complementary Accounts Methodology</u>

My research design will be an adaptation of the qualitative approach that Clarke (1997) named *complementary accounts methodology*. With this method, a teacher/researcher seeks to understand the learning that occurs in complex settings such as classrooms, recognizing that the learning must reflect and accommodate that complexity. The focus of the methodology is to study the interactions between teacher/students and among students in a classroom (social context). This social activity is experienced as personal meaning, and any theory of learning must accommodate this constructed self. This accommodation can occur through a data-collection process that generates an appropriately rich data set, and then the research employs analytical techniques sensitive to the multifaceted and multiply connected nature of data.

In *complementary accounts methodology*, the research team utilizes available technology to combine videotape data with participants' reconstructions of classroom events. Clarke (1997) states that *complementary accounts methodology* is distinguished from other approaches to classroom research by:

1. the nature of the data collection procedures, leading to the construction of "integrated data sets" combining videotape and interviews data;
2. the inclusion of the reflective voice of participant students in the data set; and
3. an analytical approach that utilizes a research team with complementary but diverse areas of expertise to carry out a multifaceted analysis of a common body of classroom data (p.98).

The challenge for this type of classroom research is to portray the learning process of individuals that occurs in a highly complex social context. The learning process is taken to be an integration of the obvious social events that are captured on the videotape and the constructions of individuals' construal of those events, and the memories invoked that take place during the interview.

Central to this research procedure is the use of videotaped classroom lessons and video-stimulated recall techniques within an interview protocol that seeks to obtain the following:

1. Students' perceptions of their own constructed meanings in the course of lesson and the associated memories and existing meanings employed in the constructive process;
2. Students' sources of conviction for the construction of their mathematical meanings; and
3. The individuals, experiences, arguments, or actions in which students believed mathematical (academic content) authority to reside (p.100).

Although I will not be using the same technologies, I will be using a similar approach to collecting students' perceptions, source of convictions, and views on their learning experiences with the Balance Series.

The Rationale for Complementary Accounts Research

The focus of the research is to seek an understanding of learning growth that integrates the students' learning process in problem-solving activities and their reflection on their role as problem solvers. Since the research gives attention to the students' interpretations and coming to terms with their learning growth in problem solving, the methods are qualitative. The research method employed in this study is similar to *complementary accounts methodology* (Clarke, 1997). The data-collection processes are the same, but the instruments used are different. The first set of data is obtained from the activities that occur in the mathematics classroom. The collected data, then, are to be analyzed and interpreted by the researcher. The researcher generates field notes and collates the notes to the corresponding documents in the students' collections. The researcher sets up for the interviews with the students, making direct reference to the collections. The conversations during these interviews are to be recorded on audiocassettes for analysis later.

There is one clear distinction in method between this design and Clarke (1997). Instead of videotaping classroom events and using Cvideo software to collate field notes to the corresponding events in the video recorder, this research will utilize students' collections of past written work in solving the Balance problems. I will generate field notes on the word processor and manually collate the notes to the corresponding documents in the collections. These collections of writing provide an in-depth look at students' thinking process, and their problem-solving strategies used in the Balance problems.

CHAPTER III

THE RESEARCH PROCESS

Instructional Program

One problem from The Balance Series is done each week. Each time, the students work for one period of forty-five minutes on one problem, describing their solution and their process on the problem sheet. Sometimes, more class time is provided, and usually they are able to take the problem home to complete their solution if necessary. Later in the week, the teacher leads a whole-class lesson, discussing students' solutions and their representations of their problem-solving processes. Often, this is done with transparencies of students' written work. After the discussion, students are asked to do self-assessment based on the four-level problem-solving rubric. (See Appendix 1.) On their balance sheets, they must indicate their level, along with their own explanations of why they know that they are working at that level. The teacher collects these sheets for formal assessment of their mathematical reasoning and their self-assessment. The teacher writes comments only when the student's self-assessment indicates that the student is not using the rubric appropriately in the self-assessment. Because the intention for this program is to support the progress of each student as an individual, there are no comparisons between students made.

Students store their Balance Series sheets in a duo-tang. The students' collection of their work provides opportunities for them to revisit their work and recognize that they are making progress: solving more difficult problems and expressing their thinking processes more richly.

When Middle-Years students express their mathematical thinking processes, they are writing algebraic steps informally. The students must show their understandings and the process that brings them to the solution. When they determine a numerical value for a geometric form (a single variable) and use it to determine other forms suspended on the balance, they have developed their own algebraic procedures. Each time the students solve a mobile suspended from another mobile, they are solving first-degree equations. See Appendix 3 for a list describing the use of Balance Series worksheets with the students in the study.

Research Procedure

The setting for this study is a Winnipeg School Division Number One French Immersion School, with students from nursery to grade eight. Most of the students acquired the French language as early as nursery. Almost all the subject areas are taught in French, except for English language arts, and a few special health mini-programs such as Family Life, Lions Quest, and The Real Game, which are taught in English.

The eighth-grade students who were participants in this study had been actively engaged in solving the Balance problems in my mathematics class since sixth-grade. The collections used in the research were accumulations of two years of schoolwork.

Because of qualitative research, my intention was to include a small number of participants. The invitation, however, was open to every eighth-grade student, and it was left to each student's discretion to either participate or not in the study. Because of their age, a letter of consent outlining the intentions of the study was sent to the participants' legal guardians. Nine letters were returned, and this determined the number of participants for my research study.

Prior to conducting the first sets of interviews, I took the participants' work collections home to analyze. The key elements focused on three areas: 1) problem-solving skills, 2) strategic plans, and 3) written thought processes. The first key element was to identify the students' struggles and successes in solving the Balance problems. Becoming more aware of the changes necessary to become successful problem solvers helped the students and myself see the gradual changes in adapting and fine-tuning their problem-solving skills to fit new situations. The second key element was the problem-solving strategies that the students used in the activities. By recognizing the distinction between successful and unsuccessful strategies, the students could make necessary changes to their strategic plans to meet their learning goals. The third key element was their written thought processes. Writing in mathematics is already a difficult process for most students, making writing about their thinking processes even more difficult. Becoming more aware of the changes in the way they expressed their thought processes helped my students and me identify and correct any misconceptions that might play a major role in their thinking about the Balance problems.

The students participated in an initial thirty-minute audio-taped interview, in the school library during lunch hour. Each interview involved three participants by their own preferences, and I ensured that the previous work of every participant was discussed in the interview.

To conduct the first interviews, I shared my analysis of key elements with the participants, and made direct reference to their written work in their collections. This is called a "document-stimulated" interview process (Clarke, 1997, p. 100). The students' work collections triggered their memories and helped them in constructing their own meanings and understandings of the Balance problems. The participants were asked to react to my statements. I attempted to provide a conversational quality to the interview, and thus, the participants were encouraged to lead the conversation by engaging in personal commentary within the context of their mathematical knowledge. Since each interview involved three students, the participants were invited and encouraged to provide feedback on each other's responses where applicable. During the interviews, I intentionally made comparisons among the participants' work. I wanted to introduce two terms: working hard and working smart. "Working hard" is seeing and depending on numerical values to solve a problem, while "working smart" is making generalizations about the relationships among the geometric forms on a mobile before applying numerical values to obtain solutions.

All three sets of the first interviews were transcribed. These transcripts focused on the highlighted elements. Key elements/statements from the first set of data were illuminated, and I was ready to schedule the second set of interviews with all nine participants.

In the second set of interviews, I provided the participants with highlights of the participants' previous interview. This time, the interviews were partly document-stimulated (Clark, 1997). I prepared at least one key statement for each participant, and once again I encouraged them to be active participants. For the remaining time of the interview, I asked the participants to work cooperatively on solving Puzzle 17 (previously seen) by showing only the mobile. (See Appendix 13.) The given values were removed. My intention emphasized the idea of working smart.

After transcribing the second three sets of interviews, the next step for me was to analyze and interpret the data.

Research Chronology

During my second year in the Master of Education program, I started to explore possible ideas for my thesis. I knew that I wanted to do a research study that might help my pedagogy, and at the same time, benefit my students. One of the ideas was to study the students' metacognition. The idea was not feasible for three reasons. First, the concept of making students think of their own thinking was difficult for the prescribed grade level that I was planning to study. It might be a long process involving direct teaching, and students practicing and exercising their metacognition. Second, the concept of studying a person's mind was complex. Third, I would not be able to substantiate whether what students report about their metacognition in problem solving was actually what they were thinking.

Ideas kept on generating as time went on. With a research study I would gain personal perspectives on students' thinking and learning in solving mathematical problems encompassing my own pedagogy. After arranging several meetings with Dr. Mason, my advisor, the idea for the thesis topic began to gradually take shape. The idea of studying the nature of the conceptual changes that students undergo as they become aware of their own growth in mathematical problem solving provided opportunities for students to gain perspectives on their own learning, while I would become aware of my own pedagogy.

I did not have much difficulty in finding a suitable research method that would complement the study. *The complementary accounts methodology* (Clark, 1997) happened to be in one of my literature reading materials. From there the ball started rolling, and I was able to arrange for a proposal defence in the last week of September, 2001.

Immediately after the proposal defence, I was motivated to have a research plan ready for the university's Ethics Review Committee. The final plan was ready, and it was in the ethics committee's hands by the first week of October, 2001. The committee made several major and minor changes to the existing plan before giving further consideration. The entire process took about two months before the proposal for my thesis received the committee's approval.

Since the research study involved the participants of students in the Winnipeg School Division No. 1, I needed to obtain the approval of the division's ethics committee prior to starting the first step of data collecting process. The committee would not accept a research application without first receiving the approval from the university. In December 2001, I forwarded my research application, and I waited patiently for the response. The last week of January 2002, I finally received the Winnipeg School Division's approval.

I spent the next two weeks explaining my research to the participants, and obtaining the consent of the participants and their legal guardians. At last I was ready to analyze the participants' work collections and schedule the first set of three interviews.

The data collection processes took about a month. I finished transcribing the field notes, and I worked on drawing out common themes that would assist me in illuminating my research question. The writing process for my thesis started in April 2002, and I had my thesis defence in August 2002.

CHAPTER IV

DATA DESCRIPTION

The purpose of this chapter is to describe the research process of generating and collecting data. After determining nine participants for the study, I began to examine and analyze the participants' Balance work collections that had accumulated over a two-year period.

Preliminary Stage

Prior to forming groups, the nine participants were asked to give themselves a pseudonym name and to appoint their own group members. A few participants did not specify any preference. So three groups of three participants each were formed. The groups were labelled as groups A, B, and C.

The participants in each group were: A – Jade, Mara and Kendra; B – Betsy, Lush and Amanda; and C – Kiko, Riley and Kiara. I will now describe the key elements for each participant that will illuminate the research question, starting from group A and ending with group C. The description of each participant described in the section is based on my observations and interactions the participants over two years.

Jade

Jade became more attentive in listening to mathematics lessons in grade eight than in earlier grades. The main focus for Jade's Balance work was to see the progress she made from not understanding the balance concept to making attempts at verifying her work. The first problem showed she was unable to balance the mobile. Jade was able to divide the total weight into two equal weights, because there were two arms on the mobile. In the second problem, Jade tried to explain the value that she assigned to each geometric form, but did not include her thought processes. After the observation and thinking method was introduced as a strategy, Jade used it in Puzzle 9. Puzzle 10 showed for the first time that Jade successfully solved the problem, and the problem had no other solutions. In the second reflection sheet, she wrote, "I want to have a 4 because last year I was not really good but now I understand." The "4" meant the fourth level of the problem-solving assessment rubric. (See Appendix 1.) I wondered what she meant by understanding now, but not then. Obviously, Puzzle 13 and Puzzle 16 showed that she made sense of her answers by mentioning the supplementary clues in her explanation. Puzzle 20 was the first time that Jade tried to verify her solution. Her explanation was weak, but the attempt was there.

Mara

Mara had a good understanding of mathematical concepts. Her best effort could be found only in tasks perceiving to be important and valuable to her. The main focus for Mara was to examine what parts of balance activities that she found worthwhile to learn and explain her thought. Mara was able to grasp the mobile concept in the first problem, but did not explain or show the solving strategy. Mara recognized and stated the supplementary clue accompanying the mobile in Puzzle 2's explanation, and it was her first attempt at verifying her solution. In the first reflection sheet Mara wrote, "Through September to now (April), I have not improved much. I noticed that I had difficulties with things that I would know now." I wondered what did she know now, and not then. How did she notice the change? Moreover, "I would like to see a 4 (out of 4), and work harder." Was there a distinction between working hard and working smart? Puzzle 7 showed that Mara verified each step of her work. In Puzzle 22, Mara used the observation and thinking method for the first time, and stated the relational idea when explaining her thought processes. In the third reflection sheet, she wrote, "...The reason I was doing not so well was because I didn't know *how* to work the problem to balance. (Now I know.)" What did she know that had helped her successfully solve the problem? What did she have to do to balance the mobile? What strategy did she use?

Kendra

Kendra started her grade seven program at the school, and thus, she did not start from Puzzle 1, but was integrated into the regular program. Because Kendra was a strong mathematics student, it was easy for her to conceptualize the concept of balance. The main focus for Kendra was to learn about her thought processes during problem-solving activities. Puzzle 9 (Kendra's first Balance problem) showed she understood the context. She, furthermore, acknowledged the supplementary clue, accompanying the mobile. In Puzzle 10 and Puzzle 12, Kendra used the observation and thinking method. She explained her thought processes by stating the relationships among the forms before assigning numerical values to them.

Betsy

Because of her easy-going personality Betsy could casually speak her mind over a conversation. During these occasions, I discovered that Betsy displayed a sound mathematical knowledge. Communicating her thinking formally either through writing or speaking, however, was not her strength. She needed direct instruction, and needed more time to process the information. The main focus for Betsy was to examine her struggle and success in solving Balance problems. Betsy's first problem showed she did not understand the mobile concept. Puzzle 3 showed the written explanation was slightly clearer than the previous ones. For example: forty-in-two and this make twenty and twenty. Betsy began to realize that each geometric form had a unique numerical value. The first time she showed an understanding of the mobile concept was in Puzzle 5. Betsy acknowledged two supplementary clues in the problem, but ended up using one clue. In the first reflection sheet she wrote, "I explain why I got most of the shapes right because they are all different values...I would like to see more explanation." What did she mean by more explanation? Was it because of the fact (according to the assessment rubric – Appendix 1) that more explanation meant a better mark? Betsy's first attempt at using the observation and thinking method was found in Puzzle 11. In Puzzle 17 Betsy used different coloured pens to highlight specific parts of the written explanation corresponding to the mobile diagram. She also showed her first attempt at verifying her solution. For the first time in Puzzle 19 Betsy successfully explained the possibility of having only one solution to the problem.

Lush

Lush listened attentively during mathematics lessons. She demonstrated self-discipline in learning. The main focus for Lush was to examine her problem-solving strategies, and the written explanation of her thought processes. Lush explained the mathematical steps in the first problem. The supplementary clue was mentioned, but did not appear in the written explanation. In Puzzle 2, Lush began to write about her thinking and include the supplementary clue. The explanation in Puzzle 3 clearly showed the distinction between certainty and hypothesizing. Lush also found and explained another possible solution to the problem. In Puzzle 7, Lush gave reasons for yielding many possible solutions. In the first reflection sheet she wrote, "I am still doing the work too fast..." I know that there was time constrained on her part to completely solve the problem at the end of period. I wondered what and how could she slow down. Puzzle 17 showed Lush generated the relationship between the triangle and the rectangle from using the table of values (the T-chart). She, however, did not justify the decision-making. That was the reason why she started with a certain value for the triangle.

Amanda

Amanda learned mathematics by rote. She consistently needed direct instruction and repetition. The main focus for Amanda was to examine her struggle and success in solving Balance problems. Amanda showed a limited understanding of the mobile concept in the first problem. For examples: two arms of a mobile must equal to the total weight, and each geometric form had a unique numerical value. The written explanation focused only on mathematical steps. There was no evidence of thought processes. Amanda began to notice the supplementary clue in Puzzle 5. Puzzle 8 showed Amanda did not realize that all extended arms of the mobile needed to be balanced. In the first reflection sheet, she mentioned, "...knowing by heart". This statement confirmed my earlier saying about Amanda's learning style. For the first time in Puzzle 10 Amanda was able to successfully solve the problem. In Puzzle 20 Amanda shared her thought processes by stating the relationships among the geometric forms before assigning numerical values to them. The process helped her justify her decisions. Amanda verified her solution for the first time.

Kiara

Kiara applied her best effort in learning mathematics. Kiara still benefited from direct instruction. The main focus for Kiara was to examine her problem-solving strategies and her writing of the thought process. In Puzzle 1 Kiara did not understand the mobile concept. Puzzle 3 showed Kiara began to understand the Balance problem, but still weak in her written explanation. In Puzzle 5 Kiara realized that each geometric form had a unique numerical value. In Puzzle 9 Kiara successfully solved the problem for the first time. In Puzzle 10 Kiara used the observation and thinking method, and for the first time she verified her solution. In Puzzle 20, she mentioned the supplementary clues in the explanation. The written explanation of her thought processes became more sophisticated as time went on.

Kiko

Kiko applied his best effort in learning mathematics. Unlike learning how to play sports, mathematics did not come naturally to him. Kiko needed direct instruction to ensure his success. The main focus for Kiko was to examine the strategies used to solve Balance problems. The first problem showed Kiko did not understand the mobile concept. For example: the square was eighteen and the circle was zero. Evidently, if the circle were zero, it would not even show up in the diagram. Each geometric form suspended from the mobile must weigh more than zero. For the first time Kiko started to understand that both sides (big or small) of a mobile were equal in weights. Kiko showed his thinking by drawing on the diagram, and gave some sort of explanation. He began to notice the two supplementary clues in Puzzle 7, but ended up using only one clue to solve the problem. His explanation became clearer than the previous ones. For example: the total weight was forty-eight and $24 + 24 = 48$. Kiko successfully solved Puzzle 8 for the first time. He replaced the addition by the multiplication, for example: $32 \times 2 = 64$. In the first reflection sheet Kiko wrote, "I notice that I added more information in every balance. I noticed that I worked hard on every balance...think more..." Was there a distinction between working hard and working smart? In Puzzle 11 Kiko gave the reason for creating an unbalance situation by simply increasing or decreasing the numerical value of a geometric form. The multiplication symbol changed to the parentheses. This notation was employed in class. In the second reflection sheet Kiko mentioned, "...studying more...listening more..." I assumed that he meant by receiving more opportunities to practice in solving Balance problems and to listen to others in-group discussions. In Puzzle 19 Kiko shared his thought processes as he worked through the problem. The observation and thinking method was introduced to the students as early as Puzzle 9; Kiko started to use it in Puzzle 20. This actually confirmed my interpretation I made about Kiko in the second reflection.

Riley

Riley was the first student to speculate about the association between the Balance Series and algebra. She learned mathematics by internalizing concepts and applied knowledge to the task at hand. The main focus for Riley was to examine the strategies and her writing of her thought processes. From problems one to six, Riley did not understand the mobile concept. She did not realize that both arms of a mobile needed to be balanced, and no two different geometric forms had the same value. Riley successfully solved Puzzle 7, but the supplementary clue was not included in the explanation. The observation and thinking method was employed in Puzzle 9. Puzzle 10 showed Riley's first attempt at finding another solution to the problem. Riley still depended on numerical values to solve the problem. In Puzzle 17 Riley discussed her second solution to the problem extensively. In Puzzle 18 she used a table of values (T-chart) as a strategy. Riley explained reasons for obtaining a numerical value for each geometric form.

After I analyzed each participant's work, I realized that all the elements mentioned were interesting topics to pursue. Knowing the amount of time I had for each interview, I decided to analyze each participant's work the second time. This time I wanted to have a key element for each participant so each element would help me illuminate the research question.

In the following paragraphs, I highlighted a key element for each participant, and these elements were used in the first interviews.

The key element for Jade was to find out about her knowledge of the balance concept. Because Puzzle 16 (Appendix 4) represented a solid understanding, it would be appropriate for me to ask Jade to explain whether she still believed that Puzzle 16 (Appendix 4) represented a good understanding of Balance problems.

The key element for Mara was to establish a distinction between working hard and working smart. To complement the first reflection sheet (Appendix 5-A), I would reference Puzzle 22 (Appendix 5-B). The emphasis on explaining the thought process was on finding the relationships among the geometric forms.

The key element for Kendra was to get at the importance of stating the relational idea instead of depending on numerical values. Stating the relationships among the geometric forms would help her explain the thought process. Puzzle 10 (Appendix 6-A) and Puzzle 12 (Appendix 6-B) also would complement Jade and Mara's papers, because they contained good examples of working smart.

The key element for Betsy was to explore an ideal situation where she believed in reaching the fourth level of the problem-solving assessment rubric (Appendix 1). What thought processes must include in the explanation to create and satisfy such ideal situation. Puzzle 19 (Appendix 7) would be used as example for the conversation.

The key element for Lush was to identify the importance of justifying her decision-making. Puzzle 17 (Appendix 8-A) provided a good example of incomplete thoughts. To show thought processes in the written explanation, Lush must include the rationale for taking each action toward the solution.

The key element for Amanda was to direct and encourage her to write more about the thought process. In Puzzle 20 (Appendix 9), Amanda justified her decision-making during the process of solving the problem, but did not seem like she was aware of it.

The key element for Kiara was to identify the importance of justifying the decision-making. Puzzle 10 (Appendix 10) provided a good example of incomplete thoughts. To show thought processes in the written explanation, Kiara must include the rationale for taking each action toward the solution.

The key element for Kiko was to establish a distinction between working hard and working smart. In the first reflection sheet (Appendix 11-A), Kiko discussed these two ideas briefly. Puzzle 19 (Appendix 11-B) would be used as example to reinforce the ideas.

The key element for Riley was to get at the importance of stating relational ideas instead of depending on numerical values. Stating the relationships among the geometric forms would help her explain the thought process. Puzzle 11 (Appendix 12) would address this intention.

Prior to conducting the interviews, I extracted and photocopied corresponding Balance sheets and used them to facilitate the conversation. (See Appendices 4 – 12.)

The First Interviews

All interviews were audio taped, lasted up to thirty minutes, and were conducted in the school library during lunch hours. The three participants were seated around a circular table. In this section, I used a neutral voice to describe each interview as it happened so I would not contaminate the data. My analysis and interpretation of the data will be discussed in Chapter V.

Group A

The interview started with Jade's Puzzle 16. (See Appendix 4.) All the participants looked at the problem sheet for a moment before I asked Jade to compare what she saw and what she knew presently about the balance concept. Jade suggested that she would use different methods of arriving at the solution, because "...you can receive a higher mark, and you understand the problem a lot more..." For understanding more about the problem Jade would use another strategy to verify her solution.

I continued the interview by asking Kendra to comment on extras. An extra was a supplementary component generated by the participants. This component communicated ideas or knowledge that exceeded my expectations of them at the grade level. (See Appendix 1.)

Kendra: It shows that you can understand the problem...

I placed Kendra's Puzzle 10 (Appendix 6-A) next to Jade's Puzzle 16 (Appendix 4), and asked Kendra to compare the two sheets. Kendra noticed that Jade started in the middle arm. Kendra made random guesses for the numerical value of each geometric form.

Kendra: Mostly, I just guessed and then I tried to verify. If one of them was wrong, I just change it.

In Mara's first reflection sheet (Appendix 5-A) she mentioned something about working hard, and I asked her to explain the term.

Mara: Read over your work, making sure if it's making sense.

Then I took the opportunity to ask Mara to discuss the distinction between working hard and working smart.

Mara: Working hard is doing extras and like not wasting any time, and working smart is common sense.

Kendra: ...Working hard to me is trying your best and working smart is like not wasting your time; it's just doing what you have to do.

Jade: I think working hard is like Kendra said trying your best and not wasting any time...and working smart is like knowing what you're doing...

Judging by the participants' responses, they had some notion of what working hard and working smart mean, so I decided to do direct teaching by referencing Kendra's written explanation in Puzzle 10 (Appendix 6-A).

I pointed to the statement where Kendra explained the relationship between the two forms before assigning them numerical values. I, furthermore, explained the difference was that working hard relied on using numerical values (trial and error methods), while working hard established general relationships among the forms (eliminating and reducing methods). The process required more complex thinking, which helped the participants to see that most of the forms could be expressed under one form, and the chance of guessing had been reduced if not completely avoided.

I placed Mara's Puzzle 22 (Appendix 5-B) beside the two previous Balance sheets. Mara discussed the first few steps she did. I invited all the participants to identify working smart in Mara's explanation.

> Mara: I know those four shapes – a rectangle suspended from a triangle on the left-hand side and a pentagon from a trapezoid on the right – equal 26. The other one, a rhombus, has to be the same. So it has to be equal on both sides.

I commented on the rationale for her decisions, which showed Mara was thinking smart.

I asked Kendra to look at Puzzle 12. (See Appendix 6-B.) Kendra described her first step was to divide the total weight by two. I asked all the participants to focus on the next step. Jade immediately noticed that Kendra had stated the supplementary clue.

> Jade: …She's using her smarts.

Mara and Jade agreed with me that by stating the relationships among the forms would give Kendra an idea where to start. I asked Mara if she would see herself adopting the technique in solving Balance problems.

> Mara: I don't know. I don't really explain that but I probably do. I just don't write it down…Actually, it's good for explaining…so that you know why you use that number.

I asked Kendra to explain the importance of stating the relational idea.

Kendra: ...It helps you to get a rough idea how big the value for each form is...because if I said that circle plus triangle equals to oval, then I'd look over that again and to make sure that it actually does.

Jade: I would probably try to guess which one goes where and then see they all fit together and double check something like that.

Jade also realized that by stating relational ideas before guessing the numerical values for each geometric form, "so you have more ideas of what to guess." When I asked Jade whether Kendra's strategy would be useful to her in solving Balance problems, she replied, "I'd probably try it for something new to see it if I could do better with a different strategy...I don't really try that often."

To conclude the first interview, I recapped the ideas of working hard and working smart, and the emphasis was on establishing the relationships among the geometric forms before making random guesses.

I felt that the conversation would help Mara, Kendra, and especially Jade see the importance of working smart. I believe that they understood the idea, because at the beginning of the conversation, they were giggling (especially Mara) not taking it seriously. During the conversation, however, I began to notice that all of them became more articulated when examining writing samples. The discussion contained much deeper thoughts.

Group B

I started the interview by showing Betsy's Puzzle 19 (Appendix 7) to the participants. I allowed them a few minutes to become acquainted with the page. While reading the document, they were supposed to think whether the written part was well-articulated, a fourth level work (according to the problem-solving assessment rubric - Appendix 1).

> Betsy: I guess I can have more explanation, and I can do a different way of getting the answer."

By a different way she meant the observation and thinking method that I introduced to the eighth-graders at the beginning of their seventh-grade program.

> Betsy: For math you just show what you did like 68 divided by 2 equals to whatever, and the verification you're supposed to say how you got that and why did that."

When I asked Lush and Amanda to assess Betsy's explanation whether they considered it to be at the fourth level standard, Betsy immediately asked, "You're not going to change my mark?" I quickly reassured her that the mark was part of the grade seven's mark, and it would remain as it was.

> Amanda: The explanation was perfect, because [Betsy] showed everything that she did explain why the answer is like 34...and she has the right answer...and she showed a verification of what she did."
>
> Lush: I can't say the explanation was perfect, because nothing's perfect. But like it is, because [Betsy] explained of what she thought of like the observations and she [verified]...
>
> Betsy: It shows that I understand.

I placed Lush's Puzzle 17 (Appendix 8-A) beside Betsy's document. I explained Lush's choice of word. Words such as "know" and "must be" appearing in the document distinctively showed Lush's certainty and uncertainty. I probed the participants to examine and find the differences between their explanations.

Amanda and Betsy identified three differences between the two Balance sheets. I immediately invited each participant to readily share one difference, so everyone would have something to contribute to the conversation. Betsy noticed that Lush did not have the observation column, but had only the thinking column. Amanda stated that Lush had a different writing style. Lush found that they both had different ways of organizing their thoughts and ideas.

I pointed to Lush's work, the third level from the left-hand side where two squares suspended from a triangle, and asked her to explain the reason why she started there. Lush immediately realized that she did not justify her decision-making.

I showed the participants Amanda's Puzzle 20. (See Appendix 9.) Amanda provided the reasons for her decisions.

> Amanda: Because I looked at the indice (hint) and it shows that the triangle minus the square has to equal to the triangle.

I decided to explain the differences between Lush and Amanda's documents: one emphasizing on using numerical values, while the other on finding the relationships among the geometric forms.

After I probed Lush to further explain the reason for the initial step in Puzzle 17 (Appendix 8-A).

> Lush: Probably because of the 60 here (the total weight), then you have to divide it and divide it again instead of here (the right arm where a sub-mobile suspended from the circle) because if you don't know that (circle) you can't divide it again.

I commented on the fact that Lush knew the reason for her decision, but she needed to explain her thought.

Amanda explained her thought processes by stating the reason for her decision.

Amanda: That's easy to find it if it's like 15 minus it by a number that's not pair (even) that the other end would be equal to something like pair (even).

By referring to Lush and Amanda's documents, I pointed out that Lush made a leap from the first to the second step without explaining her decisions. In the table of values (T-chart), Lush gave the triangle a value of one.

Lush: I think because if I found the value of the triangle then I can divide…the whole part has to be equal to 15. Then if you take out the triangle the answer has to be equal to 2 squares. I started with 1 because…it doesn't say why.

Lush quickly scanned through the explanation and realized that it was indeed not explained.

Lush: I would probably start with 1 because then you can work your way down to like 9 or 10 or something.

Lush turned to the third self-reflection sheet dated April 17, 2001 (Appendix 8-B), I asked the participants to look at the word "focus".

Lush: Because usually when I do my balances I look at the time and oh my gosh and I got to finish so I don't concentrate on what I'm doing. I feel like I have to finish it right away, and then I get all mixed up and then I don't know what I'm doing after.

I asked Amanda and Betsy if they felt the same pressure as Lush. If Betsy had more time to work on Puzzle 20 (Appendix 9), she said, "I'd take a very long time."

I, furthermore, asked the participants to indicate their main focus while solving Balance problems.

Betsy: ...Try to get the answer.

Lush: Probably find a strategy first to get the answer.

Amanda: ...Look at the hint, try to look at the balance and find out how you can get the answer.

Obtaining the answer seemed to be the priority for all the participants.

Betsy: The answer's not really [important]. It's more the explanation of how you got it. The answer even though it's not right, because you (referring to me as the teacher) only take out one mark if you got the answer but all of the explanations.

Lush: If I were to do Puzzle 20 (Appendix 9), I'd start on the left side because of the circle here. If I could find the valeur (value) of the circle, then I can put it here (the right-hand side where two circles suspended from a hexagon). And then I'd add those two together and take it off 15 and I have the answer there (hexagon).

Amanda and Betsy agreed with Lush. I interjected the conversation by stating that I would start on the right, and I wanted the participants to think about my reasons.

Betsy: Because maybe if you get these answers (pointing to two circles), and you'd get this (hexagon), practically the other way around.

Lush: Because of this I think...Here, I probably would guess...because I know that it can't be 5 (the triangle on the left-hand side of the mobile)...because if it's 5 then this has to be 5 (the square on the same side as the triangle); because you

> can't have two shapes of the same number if they're different shapes. And it could be 6 but then it would complicate everything, and I probably would say 7.

I commented on the fact that Lush was relying on using numerical values. I asked Lush whether there was any other way that she did not have to use numerical values. She replied, "Maybe." I returned to the part where Amanda talked about the triangle being an odd number as example of finding the relational idea.

I asked the participants to look at Puzzle 20 (Appendix 9). I would like them to determine the value for the circle on the left-hand side by ignoring the values inscribed on the page.

> Betsy: ...I don't know how to describe the circle, but I can describe that (square). It's lighter than the hexagon...because over here, the hint says that the square is lighter than the hexagon...because there's less shape on the other side (the left-hand side of the mobile), and these (circles) have to be the same...(The hexagon) has to weigh more than the square, but it doesn't have to be more than this (circle).

I probed the participants to think about which one of the forms would weigh the most.

> Lush: This (square) might look like one of the heaviest but...because this (square) weights as much as these two so it's not heavy but...it might not be too heavy but it's heavy enough to balance out the two other shapes (the circle and the rectangle)...The circle and the rectangle have to be equal to the square.

Finally, Lush could see the relationships among the geometric forms on the mobile.

I concluded the interview by defining working hard and working smart. I felt good about the conversation, and I personally think that Lush and Betsy had applied deep thinking while engaging in finding relational ideas.

Group C

The interview began with Kiara's Puzzle 10 (Appendix 10), and after a few minutes of looking over the document, I asked Kiko to comment on Kiara's initial step. Kiko would begin the solving process the same way. Kiara, however, did not give the reason for starting on the right-hand side.

Kiko: I don't think so. I don't really see it.

Kiara: ...I don't think I explained why I started on the right side, [but]...I knew at the time I had to start on the right side because it's easiest to divide...Like there wasn't a form in the way, like another object in the way, so all I have to do is divided it in 2 and I've already knew a value for one of the forms.

I explained the importance of justifying her decision-making so the reader would be able to see the thought process.

I directed the participants' attention to one part of Kiko's first self-reflection sheet (Appendix 11-A): "I noticed that I added more information..."

Kiko: Before I didn't write down much to explain like she's (Kiara) doing. I didn't do that. But I think I did more on the other ones. When you corrected it, or when someone corrected it they wrote that I didn't say...whatever is missing...so to explain more why you got the answer.

I asked each participant to define the term working hard appearing in Kiko's self-reflection sheet (Appendix 11-A) in her/his own words.

Kiko: Like trying my best...the effort...I just took my time on doing the work.

Kiara: You're putting effort into what you're doing...I'm taking the time to explain things and not just ditching out the answer like actually giving the effort.

Riley: Working hard is like showing...what you've learned.

I, then, asked the participants to think about working smart by hinting them to think about someone who did not have to work hard to work smart. I wanted them to realize that working smart would be more efficient way of solving problems.

Riley: Do you mean like thinking? ...Thinking can show by drawing, writing of what they think, and how they solve [the problem]..

I wanted Riley to justify her thinking so I directed her to Kiara's Puzzle 10 (Appendix 9).

Riley: You can tell that she's thinking of ways to separate two things just joining it together, which makes it harder to read like observation and thinking.

Riley tried to say that Kiara should use the observation and thinking method, so her ideas would be organized and present clearly to the reader.

I asked Kiara to comment on her thought processes in Puzzle 10 (Appendix 10).

Kiara: It was one of the first balances, so it was kinda new to me. So I explained more mathematically than my thoughts like observing stuff. So I did really give out the answer, but I did try to explain my observation and not as much as I should of...That's really important to add observations because it made it really hard to understand what you're doing.

Kiara was aware of the importance of explaining her thought processes.

Kiara: If I were to do it again, I'd still probably start out the way...I started and divided it into observation and thinking but then I think I would elaborate more on my thinking and how I came to that...I would add why I chose to do the right side first. So I notice now that I didn't do that. I didn't show why I divided that one first before anything else...so [the reader] would know that you are not guessing like you are not just picking a side and doing anything out of the blue.

I took the opportunity to explain the distinction between mathematical steps and thought processes. By reinforcing the idea of processes, I asked the participants to look at Riley's Puzzle 11. (See Appendix 12.) A minute later, I directed Riley's attention to the third step of her explanation. Riley stated the triangle was equal to seven on side one because the total weight on this side was fourteen. The next step was, the circle was equal to one, but no reasons were offered in the written explanation.

Riley: It doesn't work because circle needs to be where square plus rectangle plus rectangle is equal to circle, and that said on the side two...because it's 7, and you won't get much higher than if it's 2 of them. So I thought maybe it has a small number like 1.

I define working smart by referencing Riley's explanation. I also pointed out, the explanation of Riley's thought processes should go between the third and the fourth step.

The participants would be working smart by finding the relational idea. Riley indicated that she would feel more confident having numerical values to work with than without them.

Riley: ...If you didn't know the number you can use the form like algebra using the letters.

By placing Kiko's Puzzle 19 (Appendix 11-B) on the table beside the two other sheets, I asked all the participants to look at their own documents, and see if there was a place in the explanation that they could state the relationships among the forms without relying on numerical values. To clarify the task, I instructed the participants to pretend that they only had the mobile diagram on the page, and nothing else.

Kiara noticed in Puzzle 10 (Appendix 10), "...that square and diamond are equal to pentagon." By finding the relationships among the geometric forms, "...you'll know the square, and square plus diamond is equal to pentagon. Then the square has to equal to in order to balance it out so the whole thing would balance." If the squares were removed from each side of the mobile, both sides still remained equal, "Because both of them (the squares) are the same weight so it stays at 0. It wouldn't be more weight or less." Kiara, however, was unable to see that the other geometric forms could be expressed by the squrare.

> Kiko: Mine (Puzzle 19 –Appendix 11-B), I say you have to guess because there's not like Kiara's because there is a form that block the way to divide it. You have to know the total [weight]. You have to guess.

I wanted to divert Kiko's thinking from check-and-guess methods to examine relational ideas. I asked Kiko which one of the forms would weigh the most.

> Riley: Circle...You can't really tell because without using the numbers you are guessing. You got nothing. With those forms you're not sure if how much and you don't have a total [weight] so we're not sure.
>
> Kiara: No, but the number you can see that circle and triangle are basically the same weight as two diamonds. You have no number that says which one is bigger and which one is smaller.

If the triangle were compared to the diamond, however, Kiara would agree that the diamond was larger, "Because the triangle is like half of the diamond…if you put two triangles back to back would equal to the diamond."

I talked about Puzzle 19 (Appendix 11-B) being the difficult problem for them to find the relationships among the forms, because of the middle form blocking the path. The most common way to continue with the problem was to make random guesses.

I decided to move on to Riley's Puzzle 11 (Appendix 12), and see what she had noticed.

Riley: I would think that the triangle…

Kiko: These two (circle and rectangle) have to equal the triangle.

Riley: If I don't have anything else I would…

Kiko: The triangle or the circle…

Riley: And you can tell it right here that the circle is smaller than the triangle…and right here (far right-hand side with a pentagon and a triangle).

I emphasized the importance of finding the relational idea, and asked the participants to reflect on the idea of working smart.

Riley: To know what you are doing right now.

Kiara: …The numbers can change but the forms are always stayed the same.

Riley: It'll help you with other things like algebra. Just like letters are the same as numbers.

Kiko: The relationship makes more sense…Like how we explain why we thought the triangle was greater because the circle with the rectangle is equal to the triangle.

Even though Kiko did not participate much toward the end of the interview, but I was glade to hear him make that statement. He was not being passive, but was actively thinking about the conversation.

Kiko: …The relationship makes more sense.

Riley could relate the balance to algebra. I would like to find out whether she knew both of them were similar in concepts. However, I noticed that Riley was not quick to point out the relationships among the forms. I needed to find out more about this in the next interview.

Overall, I was impressed with all the interviews, because the conversation started with individuals, but toward the end, they began to engage in sharing their thinking and became less conscious of the audio recorder.

The Second Interviews

The format and structure of second interviews was similar to the first interviews. During the first half of the interviews, I referenced participants' responses from the first interviews and allowed them to react to my interpretations of their statements. Afterwards, I gave the participants a revised page of Puzzle 17 (Appendix 13) to work on together. The problem showed only the mobile diagram. The mobile's total weight and the supplementary clues were removed from the problem. My intention here was to reinforce the idea of working smart.

Group A

The conversation started with the statement made by Jade, "working hard is like knowing what you're doing, using your smart, and you have more idea of what to guess." I invited Jade and the other participants to interact with this statement.

Jade: ...I just think that working hard is basically trying your best and pushing your limits...[and working smart is] basically knowing what you're doing, but sort of different from working hard because even if you don't know what you're doing you can still be working as hard as you can.

I reference Kendra's way of stating the supplementary clue or the relationships among the forms was to have a rough idea of how big or small value for each geometric form.

Kendra: Basically the same thing if there's no hint then I'd just look at the balance and it tells me almost the same thing. I'll look for that.

Mara: ...As soon as I get the value for each shape, and then I go back and verify the hint...If a shape is on its own...and go for the values.

I asked Mara to discuss the distinction between common sense and making sense.

Mara: Common sense is just smart thinking, and then making sense is when you try to find out. It's almost like working smart and thinking smart.

Kendra: Common sense is...what you should know, and making sense is like your sense has to make sense. If it doesn't work then it doesn't make sense.

Jade agreed with Mara and Kendra.

From the first interview I summarized the difference between the numerical-value idea and the relational idea. Also I mentioned that some of the participants relied on using numerical values when solving Balance problems, while others attempted at looking for relational ideas. The participants agreed that making sense basically looked for relationships among the geometric forms. According to the participants, one has to make sense of the task at hand before applying one's common sense to solve the problem.

For the remainder of the interview time, I set the stage for the participants to work cooperatively with each other on Puzzle 17 (Appendix 13) as mentioned earlier.

Mara: You know the circle is equal to that one (two squares and a rectangle).

Jade: This (the rhombus) is obviously a high value because it's equal to -

Mara: a circle, a rectangle and two squares…These (a rhombus and a rectangle) are equal to three squares, a triangle and a circle.

Kendra: So I think…the diamond is more heavy than the rectangle because…the rectangle is equal to two squares, and this is like equal to the rest (three squares, a triangle and a circle).

I asked the participants to indicate their first step in solving the problem.

Kendra: I'd start on the right side, two squares on the bottom.

Mara: Yes, two squares and a rectangle, and all these (two squares and a rectangle) are equal to this (a circle).

Jade: If you got this (a square), basically you have the answers for over here (a rectangle, a circle and a rhombus).

My intention was to have the participants re-construct the same mobile by using only one form. Jade said immediately that she could do it by using different colours. I realized that the participants did not understand what I wanted them to do. I rephrased my question.

Mara: Squares have to equal a rectangle. So if this one (rectangle) has to be 4, then this (square) has to be 2.

Mara was using numerical values to reinforce her understanding of the relationship between the square and the rectangle. I took the opportunity to show the first process of elimination and see whether the participants were aware of the strategy. If they could not see the relationships between the two forms then they definitely would be working hard instead of working smart.

Jade: The circle…we can replace it with 4 squares.

Seeing the process of elimination Kendra said, "We can eliminate [all of the forms]."

Sometimes, the participants needed to be shown an example, and they would catch on to the idea. In this situation, the participants, in fact, were aware of the relational idea, because they could see that one geometric form could represent the entire balance. As in this case that particular form was the square.

Kendra: For the diamond you would have…the triangle would 5 –

Mara: 5 so…

The reason that Mara stuck here was because she and Kendra looked at the far left-hand side of the mobile (the rhombus and the rectangle). I did not know why they would jump to the opposite side instead of exploring around the right-hand side. If they persisted in working on the right-hand side, they probably could have solved the problem. I instigated the participants to think whether the triangle was worth five squares.

Kendra: …The circle's 4 squares.

Mara: The diamond is equal to 5, because –

Kendra: Is equal to 8.

Mara: Five (the triangle)…Four (the circle)…I was looking at this one (a diamond with a rectangle on the left-hand side), and I got twelve.

Kendra: But this one (rectangle) takes off 2.

Mara: Oh yea, it does. 1, 2…4, 5…12 minus 2 is equal to 10.

Here, Mara counted the number of squares on the left-hand side of the balance equalling one rhombus. Kendra reassured Mara by sharing her thought processes.

Kendra and Mara: 10 squares (the diamond).

Kendra: (On the right-hand side) Two squares, this (the rectangle) is 2 squares, and then this (the circle) is 4squares.

I asked the participants how many squares would equal one diamond. Both, Kendra and Mara consulted with each other. After waiting for a few seconds, I decided to redirect their attention to the right-hand side.

Mara: 2 squares.

Kendra: 2 (the rectangle), 4 (the circle), and the diamond is 8. But here is 10.

Kendra and Mara were confused at this moment, because they could not justify the difference between the two sides. I recapped what Kendra and Mara knew about the diamond being ten squares on the left and eight on the right, and hopefully, they would see the discrepancies.

Mara: This (triangle) is 3 squares, and the circle is…

Jade: 4…7

I forgot all about Jade. This meant that she had followed the conversation all along.

Mara: 5

Kendra: Wait. That's not right. Each side is 5.

Kendra was referring to the right-hand side of the medium mobile.

Mara: Oh that's the problem.

Kendra: I didn't see any of that. These 2 squares at the bottom (they are suspended from the triangle).

Mara: 7? No. It's 5.

Kendra: This is 5, right here, and you have to take the 3 squares. The triangle is 3 squares.

Jade: Yea, because the 5 takes away 2.

Kendra: 8 (a diamond).

After a long pause…

Mara: Then what's the circle again?

Kendra: 4

Mara: 4. Yea.

I urged the participants to identify the form that could reconstruct the original mobile. The participants replied simultaneously, "Squares."

I led the participants in summarizing the solving process that they went through. The process of how one geometric form could be expressed by a number of squares served not only as verifying the solution, but also as describing the relational idea.

To conclude the interview, I asked the three participants to comment on the nature of the conversation. I wanted to know what they learned from participating in the interviews.

Jade: I like the second interview, the one we just had, because it showed us a new strategy.

Jade stated that she probably would use the strategy in future Balance problems, but would still be tempted to use numerical values, if applicable, in solving problems.

Jade: The new strategy is better because you can explain better than you guess, and it's more challenging…You have to guess a couple of times, but after you guessed it and you got the right answer you can explain how you got that answer and why all work.

I reinforced the idea of establishing the relationships among the geometric forms would alleviate the chance of using guess-and-check or trial-and-error methods.

From the conversation, I noticed that all the participants had a clear distinction between the relational idea and the numerical value. If the participants seek only for the answer, then eventually, they would have to guess at some point. The participants did not have to make random guesses when exploring the relationships among the forms, because the process of elimination would reduce all the other forms on the mobile to only one form. The value of this form could be obtained quite easily by dividing the total weight by the number of this form appearing in the mobile.

Group B

If the total weight of the mobile were given only in a problem, what would Amanda do? What strategy would she use to solve the problem? And what thought processes would she undergo in this situation?

Amanda: Then I just have to like balance it on both sides like divide it by two. I have to try [an even number] and I wouldn't have a number…odd number.

Amanda tried to narrow down the ranges for her guess. Lush considered of using the same approach as Amanda's.

I asked Betsy how important marks are to her? From the last conversation, she seemed to display more concerns for her marks than her own learning.

Betsy: I'm not sure.

Me: Would you do a balance if, let say, I don't give you a mark?

Betsy: Probably not.

Me: So that means you are really aiming for a mark more than learning, is it correct?

Betsy: Yea.

Me: "More explanation means more mark" what do you mean by the explanation?

Betsy: Explain how you found out the answers.

Me: If someone who is going through step-by-step, I divided by this and I divided by that and the answer is five let say, do you think that it's enough explanation?

Betsy: You can still put a little more detail.

Me: Like what?

Betsy: Why did you divide it by this number, why did you choose to divide it, and why did you need to divide it.

Me: How about you, Lush? Do you think marks are important than your learning?

Lush: I say they're probably the same thing. Because learning is important if you don't learn, say you just write it down and you don't understand it then when you do the test or something, you are not going to get a mark anyway.

Me: So would you do a balance in class if I said I wouldn't grade you on it, would you still continue to put your effort in and put your thinking smart into your work?

Lush: I probably still do it because my mum always wants me to do stuff even though it doesn't matter to me. I probably still do it.

Me: Amanda, between marks and learning, which is more important to you?

Amanda: Both of them but if you learn more, you would get the marks easier. So learning is more important than getting a mark.

There were two types of strategies that we discussed in the first interview: one involving the numerical value, and the other one, the relational idea.

Lush: …Probably it depends on how the balance look like, because if it just kept on dividing then I just probably do numbers anyways, but if it didn't then probably shapes.

Amanda: I think the relationships among the forms, because if you find the value of one form and you found other forms so you can balance both of them.

I asked Betsy what she knew about the relational idea. She could only see the same form had the same value. This showed that Betsy still relied on numerical values when solving Balance problems.

The participants were asked to collaborate on solving Puzzle 17 (Appendix 13), and again, all the information, except the mobile diagram, was removed.

Betsy: Well, you can see 5 different forms.

Lush: The circle is I think half of the diamond.

Lush had stated an important relationship between the circle and the diamond.

Betsy: Well, you know that you have to divide the number in two so it would be on both sides. I think it would be easier if we start on this (right-hand) side.

Betsy: Because if you find out this number or one of these two numbers (circle and square), you can find out the value of the circle (on the right-hand side).

Lush: And if we did start there to make sure that it actually works on that (left-hand) side. If we figure out the value of the rectangle with the diamond here, then to make sure it really works, like on the left-hand side where there's the diamond and the rectangle…because if you just do one side, and this side is wrong, then the whole thing is wrong you got to start all over. So if we find out that one and all other stuff on this (right-hand) side first then…

Betsy realized the relationships among the geometric forms when she said, "This diamond has to be higher than the circle, the rectangle and the squares…because when you put all these together it should equal to [diamond]."

Amanda: And if you find the circle up here on the left-hand side and you divide it by two, and if the diamond is like 24 or something, then that would mean that this would be equal to 24 too.

I wanted to correct the misconception by pointing out Lush's observation about the circle being half of the diamond. Lush believed that she was wrong, but she, immediately, corrected herself.

Lush: …Yea. I think so…On the left-hand side where the diamond is then on the other side, it divided it into two, and on the side where the circle is there's only one shape and then it kinda show that it's half…2 squares and 1 rectangle is equal 1 circle. And 2 squares, a rectangle and a square equal diamond.

Betsy: The triangle has to be an [odd] number.

Betsy did not explain her reasoning behind her statement, and I did not ask her to explain.

Amanda: I know. On the right side if there's two shapes the same has to equal the same number. Otherwise, it won't work because if you put different numbers like 3…

I probed the participants to focus on the squares, and their positions on the mobile.

Lush: They are mostly like together, except for that one with the circle.

Lush did not know what I was after so I directed the participants' attention to the rectangles.

Betsy: It's the second smallest.

Betsy somehow seemed to have a way of observing the relationships among the forms. She was right on, but had difficulties in expressing her thoughts.

Betsy: The squares are the smallest because two of the squares, square plus square's equal to a rectangle. A rectangle plus a rectangle would equal to a circle, and a circle plus a circle would equal to a diamond.

Lush: There's a triangle. You don't if these two equal that one right? But then hold on. Say the square is two, then these two (squares) equal to 4…

Amanda: Then the rectangle would have to be 4?

Lush: No…because, hang on.

Betsy: I don't know where this (diamond) goes. I don't if this is the highest number.

Lush: Because the triangle, that one is not in here so you never know really, but I don't know.

I instigated the participants to replace all the geometric forms with one form. In other words, this form should reconstruct the original mobile. The participants were confused.

> Lush: They are all different then it can work. It can't work because if it's all squares. You can't have 1 square here and 1 square there, because it's uneven after because there's so many going on…unless they're different squares.

I mentioned that the squares could stack up on top of each other. With this statement, the lights turned on suddenly, and the participants continued to solve the problem.

> Lush: I…because there's 2 squares so I did 2 squares on that side and 2 squares are on that side, 2 squares on that side, 2 squares no wait!
>
> Amanda & Lush: 2 squares.
>
> Lush: And the circle is 4 squares, and the diamond is 8 squares, I think…
>
> Amanda: Yea.
>
> Lush: Yea.
>
> Amanda & Lush: [The diamond is worth] 8 squares...[The circle,] 4 squares…[The rectangle,] 2 squares…[The triangle,] 3 squares.
>
> Lush: There's 2 squares on the bottom here on the right-hand side, and then on the other side of the triangle, there's a circle and a square, and if you change the circle into 4 squares that's 5 squares. So I did 5 squares minus 2 squares equal 3 squares.
>
> Me: How many squares are on the far right-hand side?
>
> Lush: 10 squares…Rectangle is equal to two squares; diamond is 8 squares. 2 plus 8 equals to 10, and it balance on the other side.

I wanted the participants to realize the exercise that they just did was considered as working smart. I went through the solving process with the participants ensuring that they understood what working smart was.

To conclude the interview, I asked the participants to reflect on the strategy used in the exercise.

> Betsy: Not sure, because if we do that one on the test, and the question was to find the value of each symbol, but then you would divide it the symbols…because you are not used to that kind of…
>
> Lush: I might…because it kinda made everything sort of easier to look at; because you can see that a relationship between everything.
>
> Betsy: Yea. And we are used to with one of the numbers not with shapes.
>
> Lush: Because it depends on where the shape is on the balance; because for this one, the shape's (square) at the bottom, and then it's easy to divide it by two all the time (from the top). If the situation changed it might be more different.

The participants were being honest about their feelings towards the new way of thinking. I was glad to observe that Lush still valued the old strategy working with numerical values, but recognized the new strategy as an alternative method.

<u>**Group C**</u>

Kiara did not show up for the second interview. The two other participants and me waited for a few minutes, and then we decided to start the interview.

I picked up from the last conversation about the relational idea where Kiko stated that it made sense to him. Pursuing this idea, I asked the participants to distinguish between managing the numerical value and finding the relational idea.

Kiko: I think I'll do both because without the numbers you can see which form is equal as the circle or whatever, and with the numbers then you'll know how to cut in half.

Riley: I would do with numbers without numbers all you're going to have is some kind of equation like algebra, and if you don't find the circle is equal to then you don't how to divide it…

Here was a good opportunity to find out how much Riley knew about the connecting ideas between algebra and the balance.

Riley: Forms that have a number but you don't know what it is so you have to solve it by taking your highest number and divide it.

Apparently, Riley did not know more than what I already knew from the first interview.

Riley and Kiko were asked to collaborate on solving Puzzle 17 (Appendix 13).

Kiko: Here, we can see that these 2 squares are equal to a rectangle, and the rectangle and the squares are equal to the circle. And the circle and the rest (a rectangle and squares) are equal to the diamond.

Numerical values were not given in the problem; Kiko had no choice but to rely on finding the relationships among the geometric forms.

Riley: I also see the diamond as a highest total, because you can see that there's 4 forms that equal a diamond, which is 1 form, and you can also see there's a diamond and a rectangle equal about 6 other forms, no 5. That's how I see it.

Me: How many different forms are there?

Kiko: I think there's 5…yea, 5.

Me: Is it possible now to replace everything by using one form?

Riley: No, it is not possible…because you can see that there's 4 forms that equal 1 form. You can't have like the same form with 1…

Here, Riley looked at the farther left-hand side of the mobile, and this suggested that she started from the top and worked downward. This was a natural process for students, because every balance gave the total weight, and students were used to divide this value by the number of arms on the mobile.

I talked briefly about Riley and Kiko's strategies. I, then, referred to Kiko's strategy, because I wanted them to observe the relationship between the square and the rectangle.

Kiko: The 2 squares equal the rectangle.

Riley: That they balance, and that there's 2 forms that are equal 1.

Me: Now, is it possible that we can eliminate one of the forms? Which one would it be?

Riley: The square like just take out 1 square…No, the rectangle.

Me: So if I remove the rectangle, what can I replace it by?

Riley: 2 squares…So that means the other side taking out the circle and you'll have 4 squares on that side (the left-hand side). So you're just dividing it.

Kiko: One the last side (the left-hand side) of the balance there's a diamond and under it there's a rectangle. So since we picked off the rectangle, there's going to be 2 squares replacing the rectangle.

Me: Now, there are how many different forms do we have?

Kiko: 4.

Riley: Replacing [the circle] by 4 [squares]...You can make the diamond and put it to 8 squares.

Kiko: Because a diamond equals the circle, the rectangle and 2 squares.

Riley: [The rectangle is worth] 2 [squares]...[The circle is] 4 [squares].

Me: There are two sides, and therefore, the diamond is worth –

Riley: 8.

Me: How about the other side? We have a diamond and a rectangle so how many squares are they?

Riley: 8, 9, 10.

Kiko: The circle is how much?

Riley: The circle...

Kiko: How much is the circle? There's 4 squares, right?

Riley: I was counting this one...there's a diamond and then there's a triangle...so now we can solve it is circle because we already know that circle is 4 so 4 + 1 square = 5 squares –

Kiko: Then there's 2 squares –

Riley: 5 squares take away from 2 equals 3 that equals –

Kiko: The triangle.

Me: It is possible isn't it? What makes you think that you couldn't do it?

Riley: Because I thought that you can't just make it longer, and I didn't know that you can divide it. I thought you have to keep the same forms in the same position without cutting it.

Obviously, Riley and Kiko had a good reason for thinking that they could not express the original mobile by using one particular form, because throughout the Balance Series, there were no same forms stacking up on top of each other, and they thought that it was one of the rules. This also explained the same confusion those participants in groups A and B experienced. Once I succeeded in helping Riley and Kiko see that a rectangle was indeed worth two squares, everything came into perspectives for them.

I was confident that Riley and Kiko recognized and valued the new way of thinking. I did not have time to ask them to reflect on the new strategy, but throughout the solving process, they demonstrated some degree of comfort in using the strategy. More practises would be recommended.

Overall, I was impressed with all the first and second interviews. The participants engaged in thinking and discussing their thought processes. They displayed a sound knowledge of the balance concept. This knowledge was suppressed in their inner thoughts, and they did not know how to express it. After coaching the participants through the solving process, they became aware of what they knew and understood, and how their learning experiences would enable them to see beyond the numerical-value context. The idea of working smart appeared to be new for the participants. From the interviews, I realized that the term might be unfamiliar to them, but the idea was not. All the participants needed were some guidance from me to help them notice it.

CHAPTER V

DATA DISCUSSION

Research Themes

Over the course of collecting and analyzing the data, several themes began to surface, which illuminated the research question. In the first theme, the participants showed two types of thinking after they developed the foundation for understanding mobile concepts. The first type of thinking I call *primary thinking*, and the second I call *complex thinking*. The participants' work collections and the interviews show that the participants progress through the two types of thinking. The participants must start with *primary thinking*, focusing on information apparent to their senses, before progressing to *complex thinking*, focusing on relationships among what is apparent.

The second theme is the distinction between *common sense* and *making sense*. The participants already had some notion of the distinctions. *Common sense* can be considered as part of *primary thinking*, and *making sense* part of *complex thinking*. The participants applied their *common sense* to arrive at the solution. The participants, however, applied their *making sense* to ensure that the process of arriving at the solution is in fact feasible and logical.

The third theme is the distinction between *working hard* and *working smart*. If the participants relied solely on *primary thinking* in their strategies, they would have to work hard to achieve their intended goals of solving Balance problems. If the participants' strategies were to seek the relationships among the geometric forms of a mobile, then they would have to work smart, and apply *complex thinking* to substantiate their intended goals.

The purpose of this chapter is to discuss these three themes that I consider to be illuminating in regard to the research question.

The Cognitive Process

As the participants were involved in solving Balance problems, and later became engaged in the discourse of their own work, they enriched their learning experiences. Over two years of interacting with the series, their cognition gradually developed. Some participants underwent changes in their cognitive processes more rapidly than others, but in general, every participant demonstrated the changes necessary to be considered successful.

Cognition is the mental process that interprets what the eyes see, the hands touch, and the ears hear. Cognition eventually leads to metaphoric analysis and interpretation within a given context. The internal process triggers the body and mind to react accordingly and respond to the situation. Learning to solve Balance problems is no different. The initial cognitive process that the participants undergo is simple thinking, and I called this process *primary thinking*. Later, as the participants develop understandings of the concept of the mobile, they begin a more advanced cognitive process that I call *complex thinking*.

Primary Thinking

In *primary thinking*, the participants are thinking at a superficial level. On a Balance problem sheet, they see the diagram of a mobile, the geometric forms suspending from the mobile, the given total weight, and frequently supplementary hints. First, the participants immediately observe the mobile diagram, and notice the total weight. One fundamental concept of the mobile that the participants learn is obtaining and maintaining equilibrium on both sides of the mobile. The participants learn to appreciate and become dependant on the total weight, because it helps them take the initial step towards solving Balance problems.

> Betsy: Well, you know that you have to divide the number (the total weight) in two so it would be on both sides.
>
> Kiko: [Kiara] started with the total…weight, and then she divided it into 3 because there's three sides.

The total weight is divided by two or three depending on the number of arms on the mobile. The total weight in this context is given as a numerical value, and the participants feel that they have a concrete concept, such as a number, to manipulate. One half of twenty-four is twelve, and twelve can be reduced further by dividing by two giving six, then three. In situated learning (Boaler, 1998) the participants' mathematical thinking focused on what they think is expected of them. They are expected to distribute equal weights among the arms of the mobile. The participants, however, only need to apply *primary thinking* in performing the division.

> Kiara: I knew at the time I had to start on the right side because it's easiest to divide.

The mobile, in this case, is structured in a manner that allows Kiara to see that the weight on the right side is divisible by two. Kiara only needs to apply *primary thinking.*

Second, the participants learn to understand that each geometric form suspended on a mobile has a unique weight, meaning that each form is represented by a unique numerical value. The participants can literally see the distinction between a circle and a rectangle, and between the value four and five. In fact, numerical values serve as a security blanket for the participants, enabling them to solve Balance problems. When I asked the participants to imagine if the total weight was removed from one of Balance problems during the first interview, Kiko and Riley, being their first attempt, showed confusion, and did not quite know how to react appropriately to the new situation.

> Kiko: Mine, #19 I say you have to guess because there's not like Kiara's because there is a form that blocks the way to divide it. You have to know the total [weight]...
>
> Me: Which form is the largest?
>
> Riley: Circle.
>
> Me: Why's that?
>
> Riley: You can't really tell because without using the numbers...you got nothing. With those forms you're not sure if how much and you don't have a total so we're not sure.

Integers were finite and specific quantities that the participants can understand and manipulate. By taking away the numbers the participants had no signs that can guide them through the problem-solving process. With numerical values students can obtain immediate feedback telling them whether a numerical value they assigned to the geometric form was actually right or wrong. The immediate feedback helped the participants fulfill their needs for obtaining an approval to proceed with the problem at hand, and the feedback, furthermore, encouraged them to make other attempts at the problem.

Lush: And if we did start there to make sure that it actually works on that (left-hand) side. If we figure out the value of the rectangle with the diamond here, then to make sure it really works, like on the left-hand side where there's the diamond and the rectangle…because if you just do one side, and this side is wrong, then the whole thing is wrong you got to start all over. So if we find out that one and all other stuff on this (right-hand) side first then…

In this example, Lush is substituting numerical values that she selects by guessing into all rectangles and rhombus on the mobile to see whether they all fit. If they do, then Lush can continue to figure out a numerical value for another geometric form. Otherwise, Lush has to return to the rectangle and rhombus, and try to figure out other values that might fit the forms.

The use of specific numerical values is considered as *primary thinking*, because the participants can literally compare its quantity to other values. A numerical value like seven obviously has the same quantity as another seven but is greater than five, and less than nine. As Skemp (1987) describes, the importance of learning algebra is to understand arithmetic, involving numerical values. This notion helps Amanda and Lush understand that one geometric form can have one numerical value, and a numerical value can be the weight of one specific form.

Amanda: I know. On the right side if there's two shapes the same has to be equal the same number. Otherwise, it won't work because if you put different numbers like 3…

Lush: Because I know that it can't be 5 (triangle on the left-hand side of the balance); because if it's 5 then this has to be 5 (square on the same side as triangle); because you can't have two shapes of the same number if they're different shapes...

Primary thinking allows the participants to apply their general mathematical knowledge and their understanding of a mobile to what they observe and see in the mobile diagram. This type of thinking does not require the participants to analyze and interpret. Instead, *primary thinking* requires the participants to use their *common sense* to solve Balance problems.

Common Sense

In the Balance series, *common sense* is considered as general mathematical knowledge such as the four basic operations, equality and inequality of integers. For example, the total weight has to be distributed equally between two sides of a balance; *common sense* tells the participants to divide the total weight by two, so both sides will have equal shares. If the mobile has three arms, one on the left, one centre, and one right, *common sense* here is to divide the total weight by three in order to obtain the same weight for each arm.

After the participants understand that each geometric form has a unique numerical value representing weights (the sum of all the forms has to equal the total weight), these learned experiences become part of the participants' *common sense*. Kendra describes *common sense* as "...basically what you should know..." and Mara describes it as "...just smart thinking..." The participants can apply their *common sense* (what you should know or smart thinking) to solve the problems. After obtaining a value for a particular form, Kiara, for example, applies her *common sense* to determine a numerical value for another form on the mobile.

Kiara: Like there wasn't a form in the way, like another object in the way so all I have to do is divided it in 2 and I've already known a value for one of the forms.

Common sense, furthermore, helps the participants in justifying decisions during the solving process, and determining the appropriate side of the mobile to start. *Common sense* tells the participants to avoid starting on the side where a sub-mobile is suspended from a geometric form.

Lush: Probably because of the 60 here (the total weight), then you have to divide it and divide it again instead of here (the right arm where a sub-mobile is suspended from the circle) because if you don't know that (circle) you can't divide it again.

In this situation, *common sense* can only be applied when the participants begin to make guesses. The natural way to avoid being stuck is to assign a numerical value to the form that is blocking, as Kiko describes, so the participants can continue to determine the values of the rest of other forms. The participants, however, often do not establish the appropriate criteria before attempting to make guesses. Kendra, for example, does not establish the basis for guessing.

Kendra: Mostly I just guessed and then I try to verify, and if one of them was wrong, and I just change it.

The trial and error method, like Kendra described, can be time consuming. Kendra will be able to solve the problem if she happens to guess the correct value for the geometric form, but what is the probability of being successful on the first attempt? Does this mean that the participants must depend on their luck with guessing in order to solve Balance problems? So if the participants' luck holds up, they would be able to determine every numerical value for each of the forms. Jade is experiencing the "eureka" moment in solving Balance problems as described in the *Grades 5-8 Mathematics Foundations* document (MET, 1997). If the core of solving mathematical problems depends on guessing and luck, then the participants are probably *working hard.*

Working Hard

In the Balance Series, *working hard* is considered as making random guesses. For example:

> Jade: Probably try to guess which one goes where and then see they all fit together…

The strategy that Jade uses, however, is time consuming, and probably Jade is *working hard* in this case. In this case, Mara thinks that Jade is not wasting time in making random guesses.

> Mara: Working hard is doing extras and like not wasting any time…

There is merit in Mara's thought, because Jade is spending most of her class time in guessing and checking the appropriateness of each numerical value. Jade is basically working harder than she needs to in order to solve the problem. Jade's strategy is only feasible in the case where she would have all the time she needs to complete the task, but in reality, I cannot allow the participants to have more than two forty-five-minute periods to solve a problem. In a classroom situation, I am seldom able to provide the students with the luxury of excess amount of time to solve Balance problems. Instead, the participants are given a limited number of periods to spend on each problem, depending on the time of year. Lush has expressed her concerns about it in one of the self-reflection sheets: " I am still doing the work too fast…" and later during the interview Lush explains the pressure she felt during problem-solving activities.

> Lush: Because usually when I do my balances I look at the time and oh my gosh and I got to finish so I don't concentrate on what I'm doing. I feel like I have to finish it right away, and then I get all mixed and then I don't know what I'm going after.

Lush, apparently, is not alone; Amanda and Betsy share the same concern. When I ask the participants to imagine having more time to work on Balance problems,

> Betsy: I'd take a very long time.

This statement suggests that Betsy is *working hard*, because the process of making random guesses can be lengthy. Unless Betsy begins to notice the mobile's physical structure, and the arrangements of the geometric forms, she will continue to work hard on her guesses until she successfully finds the correct numerical values for each form. As in Lush's case, she does not clearly define the range for her guesses.

> Lush: I think because if I found the value of the triangle then I can divide…the whole part has to equal to 15. Then if you take out the triangle the answer has to be equal to 2 squares …I would probably start with 1 because then you can work your way down to like 9 or 10 or something.

Common sense has little use in these situations. The participants have to employ *complex thinking*, rather than *primary thinking*, to solve the problem.

Complex Thinking

Complex thinking requires the participants to readily integrate their acquired knowledge of mathematical ideas in solving Balance problems. *Complex thinking* will help the participants justify their decisions, and subsequently reduce the time spent on guessing.

Kiara: So they would know that you are not guessing like you are not just picking a side and doing anything out of the blue.

The participants, however, are not used to this kind of *complex thinking*. The strategy of looking for the relationships among the geometric forms on a mobile seems remote and foreign to the participants. This, however, does not suggest that the participants are not ready to learn and apply *complex thinking*, but instead, it suggests that more guidance through direct teaching is required. For example, in the vignette below, I ask Jade to describe the steps in Kendra's Puzzle 12.

Jade: She used the same method by dividing first.

Me: And what's the next step after dividing?

Jade: Looking at the hint.

Me: After the hint.

Jade: And she's using her smarts.

Me: By stating the relationships among the forms. Does that give [Kendra] ideas where to focus on?

Mara/Jade: Yea.

Me: Mara, do you see yourself doing that in the future?

Mara: I don't know. I don't really explain that but I probably do. I just don't write it down.

The vignette shows two aspects of *complex thinking*, suggesting that Jade and Mara are aware of what *complex thinking* might look like. The first aspect is that Jade is able to recognize that Kendra is using other than *primary thinking* to establish the relationships among the forms. Jade refers to Kendra's thinking as using "her smarts". The second aspect is that Mara is acknowledging that she might be using *complex thinking* similar to that used by Kendra. Mara believes that there is little relevancy in writing down and recording such thought processes.

Primary thinking requires the participants to readily apply their mathematical knowledge and learning experiences to the physical structure of a mobile. The nature of *primary thinking* is to focus mainly on the literal aspect of a problem. Brown (1995) states that learning activities need embedded dimensions that will stimulate the participants' metacognition to enrich knowledge and expand experiences in the process of learning. In other words, the participants need to be challenged, and encouraged to think beyond their visual capacity. Unlike *primary thinking*, *complex thinking* focuses on the metaphorical aspects of the problem. The participants develop the understandings and perceptions of the balance and algebraic concept from solving Balance problems. They understand that numerical values representing the geometric forms can change as long as the mobile remains in a state of equilibrium.

> Kiara: That the numbers can change but the forms are always stayed the same.
>
> Riley: [The relationships will] help you with other things like algebra. Just like letters are the same as numbers.

Kiara and Riley are creating the relationships between mobile and algebraic concepts. These experiences lead the participants to increase competence in learning mathematics in novel situations. The participants, then, employ their mathematical knowledge and learning experiences in analyzing and interpreting the structural design of the mobile.

Complex thinking permits the participants to deconstruct (Spring, 1991) a mobile by examining the relationships among the geometric forms, and reconstructing the mobile by verifying the relationships as a whole.

In most Balance problems, most geometric forms can be expressed in terms of other forms. As Pòlya (1957) points out, the essence to solving a mathematical problem is to find the connections between the givens and the unknown. The physical structure of Puzzle 17 (Appendix 13), for instance, enables the participants to replace all the other forms with one form, the square.

> Kiara: On the last side (the left-hand side) of the balance there's a diamond and under it there's a rectangle. So since we picked off the rectangle, there's going to be two squares replacing the rectangle.

Kiara is stating the relationship between the rectangle and the square. Riley and Kiko are relating the diamond, the rectangle and the circle to the squares.

> Kiko: Here, we can see that these two squares are equal to a rectangle, and the rectangle and the squares are equal to the circle. And the circle and the rest (a rectangle and squares) are equal to the diamond.
>
> Riley: You can make the diamond and [replace it by] 8 squares.
>
> Kiko: Because a diamond equals the circle, the rectangle and two squares.
>
> Riley: [The rectangle is worth] 2 [squares]...[the circle is worth] 4.

To recognize these relationships the participants employ *complex thinking*, because the process requires them to deconstruct the structure of the mobile, and analyze a portion at a time.

> Lush: The circle is I think half of the diamond.

Betsy: This (the triangle) has to be an [odd] number.

These two statements require that Lush and Betsy analyze the relationship within the situation, and interpret the quantity of the forms relating to each other. The relationships among the geometric forms described by Kiara, Lush and Betsy are hidden information within the problems. They have to successfully extract the information, and the cognitive process involves *complex thinking*.

Complex thinking leads the participants away from relying on the applications of numerical values and away from trial and error methods.

Riley: You can use numbers but in another word if you didn't know the number you can use the form like algebra using the letters.

Riley is able to notice and recognize the association among the numerical values, the geometric forms, and the algebraic variables. If Riley realizes that the forms and the values are similar concepts, then with due time, she will become less dependant on the values, and more focused on finding the relationships among the forms. Numerical values, as described earlier in the chapter, represent quantities, which represent concrete concepts, and by nature, numerical values are part of *primary thinking*, and the use of numerical values in guesses is part of *common sense*. Variable-like forms, on the other hand, represent abstract concepts, and by nature, they are part of *complex thinking*, and the interpretation of variable-like forms is part of *making sense*.

Making Sense

In *complex thinking*, the participants attempt to make sense of learning experiences governing problem-solving, mathematical and cognitive knowledge with Balance problems. The experiences intertwine with mathematical knowledge governing the essence of *making sense* of the situation. *Common sense* demands the participants to interact directly with the given information, while *making sense* demands that they extract pertinent information, which might not be explicitly stated. The implication falls upon the discretion of each participant to make sense of the problem. As Mara suggests, "Read over your work making sure if it's making sense."

Mara defines *making sense* as a process of analyzing her interpretation of the situation, and ensuring that the interpretation is sensible and contextual.

> Riley: … I also see the diamond as a highest total, because you can see that there's 4 forms that equal a diamond, which is one form, and you can also see there's a diamond and a rectangle equal about 6 other forms, no 5. That's how I see it.

In this case, Riley sees the diamond as the highest value on the mobile although not every participant would see it in the same way. Betsy sees the same situation slightly different than Riley.

> Betsy: Well, this diamond has to be higher than the circle, the rectangle and the squares.

Betsy and Riley essentially see the inequality between the forms. Betsy and Riley demonstrate *primary thinking* while looking at the diamond. They demonstrate *complex thinking* while interpreting the weight of the diamond being a high or the highest value. The interpretation depends on *making sense* of the situation.

Amanda, for instance, notices the supplementary clue that the difference between the triangle and the square is the rectangle. She attempts to make sense of the given information in the context of the problem.

Amanda: Because I looked at the indice (hint) and it shows that the triangle minus the square has to equal to the rectangle.

Because of the clue Amanda is able to justify her decision about which side of the mobile to commence.

Mara, Jade and Kendra have engaged in a conversation about the relationships among the geometric forms in the vignette below. They attempt to make sense of the relationships by determining which one of the forms weighs the most.

Mara: *You know* the circle is equal to that one (2 squares and a rectangle).

Jade: *This* (the rhombus) *is obviously* a high value because it's equal to –

Mara: A circle, a rectangle and 2 squares…These (a rhombus and a rectangle) are equal to 3 squares, a triangle and a circle.

Kendra: *So I think*…the diamond is more heavy than the rectangle because…the rectangle is equal to 2 squares, and this is like equal to the rest (3 squares, a triangle and a circle) [italics added].

The italicized words show that the participants are making sense of the relationships. Mara identifies the relationship between the circle and the other forms. The circle makes sense to her, and she directs the others to that particular area. Jade, immediately, makes sense of the situation, and knows where Mara is heading, and sees the rhombus as having a high value. Kendra, then, makes sense of what Jade has said, and infers that the diamond is heavier than the other forms. As Pòlya (1957) states, the process of bringing one more given into play should be perceived as a sign of progress. This vignette clearly shows that the participants are building on each other's ideas to make sense of the problem. Each participant interprets and makes sense of the same text according to her experiences with balances. The participants in this vignette see and are encouraged by the signs of progress. As these signs of progress becoming more frequent as they proceed, they alleviate the participants' hesitation to continue to solve the problem. The signs of progress, furthermore, raise their spirits, and increase their confidence (Pòlya, 1957). The dynamic of this situation is, in fact, allowing each participant to exercise her *complex thinking* by engaging in a conversation. The participants actively enter into the discourse so they will gain knowledge-of-action. As a result, the discourse encourages them to become active participants (Applebee, 1996).

Kiki, furthermore, believes that the strategy of finding the relationships among the geometric forms makes sense. When he begins to recognize the sense making of the relational idea, he is *working smart*, instead of *working hard*.

Working Smart

In the Balance Series, the participants are *working smart* when they attempt to determine the relationships among the geometric forms. *Working smart* requires the participants to access and employ *complex thinking* in solving Balance problems. A basis of teaching mathematics is permitting the participants to work smart; that is, discovering the shortest route to solving a mathematical problem. Lush and the other participants are concerned about the amount of time that is available for them to completely solve Balance problems, and explain and write out their cognitive processes. Lush spends most of her time making random guesses, which represents *working hard* rather than *working smart*. She should be *working smart*, by using the time to explore the relationships. The relationships will lead Lush to establish the range for guessing. By knowing and specifying ranges for guessing, the participants do not have to start out cold, and consequently, time becomes a minor issue. The relationship gives the participants a way to avoid making random guesses.

Jade: So you have more ideas of what to guess.

Jades defines, "…working smart is like knowing what you're doing…" The participants need to execute their plans with intentions and purposefulness so the results can be fruitful. Otherwise, they are aimlessly working through the problem.

Kiara: So they would know that you are not guessing like you are not just picking a side and doing anything out of the blue.

Complex thinking involves *making sense* of Balance problems, and *working smart* to obtain desired results. *Primary thinking* on the other hand, involves applying *common sense* to the problem situations, while *working hard* to obtain the desired results. Students' cognitive processes, however, gradually develop an association linking *primary thinking* to *complex thinking*. During the process of making the transition from primary to *complex thinking*, the participants undergo conceptual changes as they become aware of their growth in problem solving. The participants realize the significance of employing *complex thinking*, rather than *primary thinking*, to become better problem solvers.

Each participant shows her/his impression of the elimination process. From the examples below, the participants show awareness of *complex thinking*, and consequently, the conceptual changes as problem solvers.

Kendra: Well, [the relationship] helps you to get a rough idea how big the value for each form is.

Kendra is aware that *complex thinking* can help her determine approximately the weight of each form.

Kiko: [The relationship] makes more sense…like how we explain why we thought the triangle was greater, because the circle with the rectangle is equal to the triangle.

Kiko is aware that *complex thinking* makes more sense to him, because he can explain and justify the inequality among the forms.

Jade: Because you can explain better [with the relationships] than you guess, and it's more challenging.

Jade is aware that *complex thinking* can be challenging, but admits that the relationships help her justify the thought process better than random guesses.

Betsy: …And we are used to with one of the numbers not with shapes.

Betsy is aware that *complex thinking* is a different strategy than what she is accustomed to. The statement also suggests that Betsy can work with the geometric forms to arrive at the solution, instead of working solely with numerical values.

Riley: [The relationship]'ll help you with other things like algebra. Just like letters are same as numbers.

Riley is aware that *complex thinking* can help her gain more precise ideas on how the forms relate to each other.

Mara: But the relationships. we didn't really guess.

Mara is aware that employing *complex thinking* can avoid working harder.

Lush: …It depends on how the balance look like, because if it just keep on dividing then I just probably do numbers anyways, but if it didn't then probably

> [relationships among the forms]...because it kinda make everything sort of easier to look at, [and] because you can see that relationship between everything.

Lush will continue to employ *primary thinking* if the problem situation permits. Lush, however, is aware of *complex thinking*, because when the Balance problem becomes complicated, she would employ *complex thinking*, instead of guessing for the numerical values.

The participants show signs of progress in their abilities to solve Balance problems and explain their thought processes. The act of acknowledging the new way of thinking is in fact a step in the participants' learning progress from being primary thinkers to becoming complex thinkers. *Complex thinking* encourages the participants to examine and compare ideas. They show promising signs of adopting and using *complex thinking* in problem solving, but value *primary thinking*. The participants express that they will continue to employ *primary thinking*, depending on the context of the problem. I believe that the participants have undergone conceptual changes, which allow them to distinguish between primary and *complex thinking*, and employ either or both types of cognition to explain the thought process.

The conceptual changes occurred when the participants engaged in employing *complex thinking*. As they worked cooperatively on solving Puzzle 17 (Appendix 13) during the second interviews, they realized that applying *complex thinking* reduced the number of random guesses. Furthermore, the participants became less dependent on numerical values to help them solve the problem.

The participants experienced a shortage of time in completely explicating their thought processes because of making random guesses. A strategy based on establishing relationships is more efficient than guessing, because the relationships clearly show the links between the geometric forms, and one form can be expressed in terms of another form. The importance of *complex thinking* is that it provided the participants with a way of compressing the information presented in Balance problems. By compressing the information, the participants reduced the problem to a simpler form. The reduced structure enabled them to avoid making random guesses. "The information contained in that string can be compressed. Its information content can be captured in a string that is substantially shorter that the string itself, so we cannot say that it is completely random" (Tasic, 2001, p.82). Randomness can be time consuming, time the participants did not have. If the participants could express the entire mobile using one form, then the process of determining a numerical value for this form would help them obtain the values for the other forms. The participants have learned the importance of *complex thinking*, and thus, have demonstrated conceptual changes from being primary thinkers to complex thinkers. These signs of progress show that the participants also became aware of their own learning growth.

CHAPTER VI

CONCLUSION

Theme Synthesis

Because learning is a continuous process, it cannot be treated as an end product. Even though the research has come to a conclusion, this does not mean that the students who participated in the research are at the final stage of their learning at problem solving. The research study has served as a mirror for the students and for myself, as an action researcher. The reflections have illuminated first, the students' perspectives of their own learning, and second, my perspective of the students' learning. The purpose of this chapter is to discuss the nature of these reflections that provide an understanding of how the students process their thinking while engaging in problem solving, and how I can monitor my pedagogy to cause the students to become aware of their own learning growth.

The foundation of learning to solve Balance problems is to understand the key concepts of the mobiles. The key concepts include an understanding of the role of the total weight, that two arms of a mobile must be balanced, and that each geometric form on the mobile must have a unique value. The knowledge that the students acquired becomes their *common sense* as they engage in solving Balance problems. The students, then, can begin to examine the physical structure of a mobile represented by a diagram.

The mobile diagram provides a pictorial representation. Because of its concreteness, the students need only access *primary thinking* to examine the structural composition of a mobile. By applying *common sense*, the students know and recognize the intention of the total weight as valuable information. Because of the total weight, the students are able to reduce it down to its lowest form, and by doing so they are able to obtain a numerical value for one of the forms. The rest of the other forms nicely fall into place, and the students experience success. Having successful experiences encourages the students to begin explaining the mathematical steps leading them to the solution.

During the writing process, the students describe their *primary thinking*, and the *common sense* they use to arrive at the solution. The students, then, verify their solution using the simplest method - finding the sum of each side of the mobile and comparing the sum, assuring that both sides have the same weight or the same numerical value. Then the sum of these two sides must be equal to the given total weight. This is a superficial way of verifying the solution, because the students feel that they have satisfied the literal sense of the mobile. *Primary thinking* does not require the students to analyze and interpret any underlying relationships among the forms. Primary thinkers will accept the given information at face value, without trying to make any changes to the given structure. *Primary thinking* leads the students to see concretely that in fact overall weight is the sum of all the sides on a mobile.

From the research study, I discovered that *primary thinking* is a significant stage for gaining learning experiences in problem solving. These experiences provide opportunities for students to learn to observe pertinent information in a novel situation. The students, furthermore, can exercise their *common sense* during solving and verifying the solution, and apply what they learn about mobile concepts in the problems. All the nine participants have gone through this stage of learning. This stage should not and cannot be rushed. Some students will reach this stage from working on a few Balance problems, and some will take over one school year. This is not anything that is out of ordinary, because, as a teacher, I should expect to have diverse learners in the classroom.

Respecting the diversity of learners in the classroom, however, I need to create a learning environment where the students can benefit from recognizing their own learning on a regular basis. Thus far, I have asked the students to do a formai self-reflection on their learning in solving Balance problems at the end of each term, but I believe that the students should have more opportunities to do it informally. I intended to arrange the students in a group of two or three as in the interview situation, to share, compare, and explain their own strategies. By discussing their strategies, the students were engaging in explaining their thinking processes. In groups, each student would select a piece of work to share with the others, and the other students are allowed to make constructive comments. During the interviews, I found that while the students are comparing their work with their peers, they help each other in to see which strategy works and which does not, and what pertinent information about their thinking is important to report in writing. They also help each other clarify and correct any misconceptions. The most valuable learning moments for me were to witness the exchange among students. As a result, they were able to progress in their learning, because they were informed about their own thinking. The comparisons and suggestions given by their peers helped them identify their next step in problem solving. By engaging in the discourse, the students become active participants.

Establishing and maintaining a meaningful conversation can be a challenge, however. I need to coach the students to look for specific ideas in the problems so they have a particular area to focus on. At the beginning of the interviews, the participants were only ready to answer my questions. As a teacher, I need to guide the students by asking and probing them with questions. The questions must direct their attention to specific learning goals that I want the students to explore. The questions would not be targeted specifically at anyone, but would facilitate a conversation among the members of the groups. While participating in a guided discourse, the students become aware of their own learning by either listening to other members' comments, or by looking at the other members' work. The students can see the others' thought processes, and in turn, their own thought processes as well.

On the first few occasions, the students are likely to depend on probing questions to help facilitate the conversation, but as time goes on they will become more at ease with the discussion of their work, meaning depending less on my questions. During Group A's second interview, the ideas bounce off one another, and the participants build on each other's ideas. Let us re-examine the vignette again.

> Mara: You know the circle is equal to that one (2 squares and a rectangle).
>
> Jade: This (the rhombus) is obviously a high value because it's equal to –
>
> Mara: A circle, a rectangle and 2 squares...These (a rhombus and a rectangle) are equal to 3 squares, a triangle and a circle.
>
> Kendra: So I think...the diamond is more heavy than the rectangle because...the rectangle is equal to 2 squares, and this is like equal to the rest (3 squares, a triangle and a circle).

This is a learning moment for Mara, Jade and Kendra. Each participant shows that she is listening attentively to the others, and knowledge is built from a cluster of intertwined ideas. The students are working collaboratively, and constructing meaning and understanding of the situation by engaging in *complex thinking*. I believe that thinking is a non-linear process. As in the vignette, the students experience flashes of insight as fully developed ideas wash over the mind. The more the students are exposed to and are actively taking part in the discourse, the more the students will become aware of their own learning progress. As Pòlya (1957) points out, each time the students unfold new information they are making progress. The signs of progress in this situation encourage the students to continue to look for the relationships among the forms.

The research study shows that *complex thinking* is less apparent to the students than *primary thinking*. *Primary thinking* only processes the visibly apparent concepts, and if the students come to a dead end, they either make random guesses to help them get out of it or make a detour and restart. Having clearly defined the range for guessing through *complex thinking*, the entire process can be less time consuming. I believe that students need to go through and acquire the experience of making random guesses, because the experience will enable them to adopt the relationship strategy, making analyses and interpretations of the pertinent information in a mathematical problem.

Students need direct instruction in these processes. I learned that I have to move the students away from their comfort zone, and challenge them to process *complex thinking*. *Complex thinking* poses some challenges to the students, but it is necessary, because *complex thinking* requires the students to analyze and interpret given information at the metaphorical level. The students should look for implicit information in a problem. By drawing the students' attention to closely examining the relationships among the geometric forms, the students can avoid making random guesses. Using *common sense* in guessing is less efficient than *making sense* of the situation. By making sense of the situation, the students will save time in solving the problem, and they will learn to find and make connections between Balance problems and the strategies. These ideas and concepts are embedded in their strategic knowledge about how they learn mathematics. The students must internalize their learning by connecting declarative and procedural knowledge. My responsibility as a teacher is to help the students see and apply the schematic knowledge to problem solving. The cognitive skills and principles are the basis for cognitive processes. The thinking skills eventually turn the students into lifelong learners.

If my primary goal as a teacher is to encourage the students to become lifelong learners, I need to make it explicit to the students that mathematics involves connecting ideas and concepts.

When I formally introduce algebra, I refer the students' attention to the balances. By using balance concepts to explain the roles of variables, coefficients, constants, equal signs in algebraic equations or expressions, I find that the students are able to better understand algebra. With these ideas and concepts, the students can construct the mathematical knowledge that they require to become more successful learners.

To become better problem-solvers, the students have to undergo conceptual changes from being primary thinkers to complex thinkers. That means, they are not just examining and recognizing the information in a problem, but they are ready to analyze and interpret the given information intentionally. The information triggers thinking. The thinking facilitates strategies. The strategies must respect time and effort. Time and effort are needed to bring one more conception into what is already known. The new knowledge, then, leads to clearly expressible signs of progress.

To fully appreciate the signs of progress, the students must be given opportunities to "get their hands dirty" during problem-solving activities. Consequently, the students will gain from being active participants. By being active participants in a discourse, the students will internalize their thinking, and enhance their learning experiences from the opportunity to compare, share, and discuss with other students. From interacting with their peers, the students will undergo conceptual changes in their problem-solving skills. They, furthermore, will become aware of where they stand in the process of learning how to solve mathematical problems when they revisit their attempts to explain their own thinking. At this point, the students will become aware of their own learning growth.

The research question was: *What is the nature of the conceptual changes that students undergo as they become aware of their own growth in mathematical problem solving?* The response is: *The findings suggest, as students move from* primary thinking *(*common sense *and* working hard*) to* complex thinking *(*making sense *and* working smart*) processes, they become aware of their own cognitive growth in solving mathematical problems.*

Recommendations for Teachers

I believe that learning is a lifelong process. Being a graduate student and a teacher, I learn to be patient with myself by giving myself time to learn. I have to be a patient learner by accepting and learning from my mistakes, and only time can ensure me of my success as a learner.

The process of conducting the research study has supplemented my professional growth. To witness the students undergo the conceptual changes in problem solving, and become aware of their own learning growth, is to reflect on my own practise as a mathematics teacher. Students will learn, but will learn more under the influence of their teachers. I believe that I have created a problem-solving environment, stimulating the students to think, and encouraging them to learn in their own ways.

Learning experiences gained by observing, interacting, and listening to students talking about their learning, make me realize how problem-solving activities can foster students to grow and learn. In this section of the chapter, my learning experiences are framed as five recommendations for teachers of mathematics.

1. Mathematical problem-solving activities need to be structured to allow students to exercise their thinking. To ensure students' success at solving problems is to lead them from being just primary thinkers. Students need to learn how to analyze and interpret pertinent information in various situations. Teachers have to be sensitive to their needs to become good problem-solvers, by providing problem-solving activities on a regular basis. The more opportunities students have to explain their thought processes, the more they will be able to manage *complex thinking*. I believe that mathematical problem solving can be a good opportunity to help learners to become more complex in their thinking.
2. Students are comfortable in performing arithmetic with numerical values, and are less comfortable in performing similar tasks without numbers. I believe that teachers should attempt to lead students away from working with numerical values, and focus on and examine relational ideas in a

mathematical problem, because relational ideas require students to engage in *complex thinking*.

3. One important aspect of structuring mathematical problem-solving activities to be considered is to allow students to interact with each other. The interaction permits students to share and discuss mathematical ideas and strategies with each other. Consequently, students will reflect on their own thinking and learning while being active participants in a discourse.
4. Teachers need to challenge students to move toward a higher level of thinking when solving mathematical problems. Otherwise, students tend to be satisfied with problem-solving skills they have. Teachers should be actively involved in students' learning to foster and encourage students to work smart instead of *working hard*. Teachers can obtain a desired result by making suggestions, and doing direct teaching.
5. Listening to students is important to teachers. When teachers listen, they will learn things about their students that they never knew. Listening does not only imply oral discussions, referring also to reading written texts in students' work collections. Looking at students' self-reflection sheets can help teachers develop instructional plans meeting the needs of individual learners.

These recommendations will help me develop instructional plans for mathematical problem-solving activities in the classroom. Even though my study has concluded, the ideas from the study will continue to be refined and extended. Because learning is a continuous process, I still see myself continuing to grow personally and professionally.

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Appendix 1

Problem Solving: The Balances

The Assessment Rubric

4 - Expert Response:

- the explanation is very clear and complete
- each step shows logical thinking
- the correct answer

and must satisfy one of the below criteria

- show and explain a strategy when verifying the answer
- show another or more possible solutions to the problem
- relate the problem to something in daily life

3 - Practitioner Response:

- the explanation is clear
- each step shows logical thinking
- the correct answer

2 - Apprentice Response:

- show a little or no explanation
- the correct answer

OR

- the explanation shows clear/right thinking
- the answer contains a few minor errors

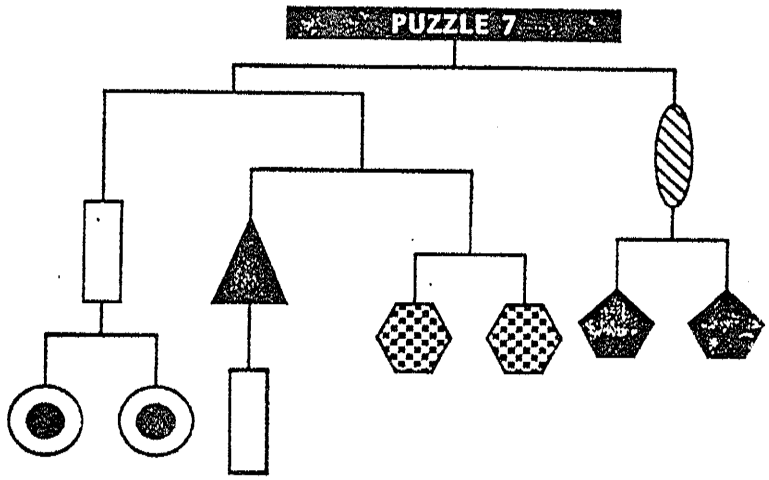
1 - Novice Response:

- try the problem
- show difficulties in understanding the problem

Note: ½ mark will be deducted from the response level for a minor computation error.

H. Vuong April 2002 - revised

Name ____________________



Discover the value of each of the shapes.
The total weight is 48. Clues:

Appendix 3

Chronology of Balance Series Worksheets

Puzzle 1	August 1999 – Grade 6
Puzzle 2	November 1999 – Grade 6
Puzzle 3	November 1999 – Grade 6
Puzzle 4	November 1999 – Grade 6
Puzzle 5	January 2000 – Grade 6
Puzzle 6	February 2000 – Grade 6
Puzzle 7	March 2000 – Grade 6
Puzzle 8	April 2000 – Grade 6
Self-reflection 1	April 2000 – Grade 6
*Puzzle 8	September 2000 – Grade 7
Puzzle 9	September 2000 – Grade 7
Puzzle 10	September 2000 – Grade 7
Puzzle 11	September 2000 – Grade 7
Puzzle 12	October 2000 – Grade 7
Puzzle 13	October 2000 – Grade 7
Puzzle 14	October 2000 – Grade 7
Puzzle 15	October 2000 – Grade 7
Puzzle 16	December 2000 – Grade 7
Puzzle 17	January 2001 – Grade 7
Puzzle 18	January 2001 – Grade 7

* This problem was used as a transition activity from grade 6 to 7.

Self-reflection 2 .... February 2001 – Grade 7

Puzzle 19 .... February 2001 – Grade 7

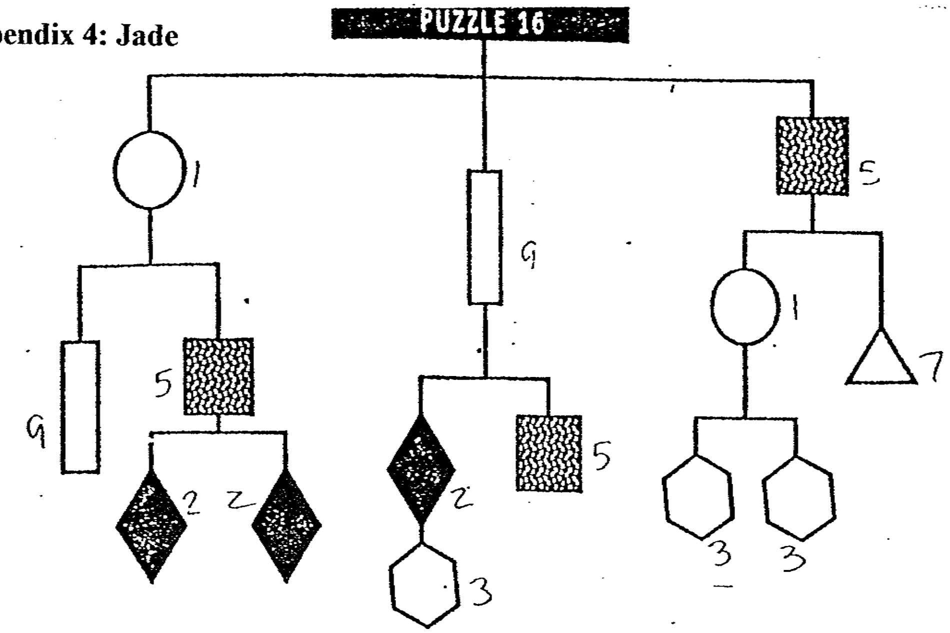
Puzzle 20 .... March 2001 – Grade 7

Puzzle 21 .... March 2001 – Grade 7

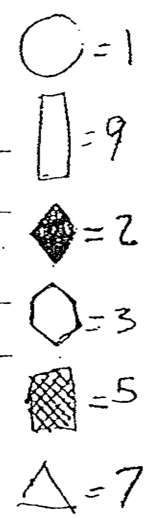
Puzzle 22 .... April 2001 – Grade 7

Self-reflection 3 .... April 2001 – Grade 7

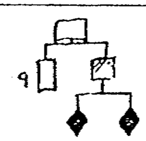
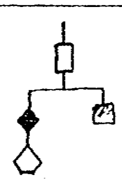
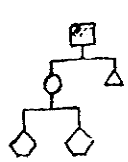
Appendix 4: Jade



Découvrir la valeur de chaque forme. Le poids total est 57. Chaque bras a le même poids.
L'indice supplémentaire: ▯ est un multiple de trois.



The explication: How do you find the value for each geometric form?

Observation	Thinking
1:	I divided 57 by 3 because there are 3 pieces. 57 ÷ 3 = 19
2. Hint: ▯ is a multiple of 3.	I tried 9 because it is a multiple of 3.
3. 	It has to balance. Then all together it must equal to 5. After I did 9 + 9 = 18. Then I assigned the O = 1 and that made 19.
4. 	▯ = 9 because the piece 1. ▤ = 5 because piece 1. ◇ = 2 because piece 1. ⌂ = 3 because 9 + 5 + 2 + 3 = 19
5. 	▤ = 5 because the piece 1 and 2. O = 1 because piece 1. ⌂ = 3 because piece 2. Δ = 7 because all have to balance, and it makes 19.

Appendix 5-A: Mara

Grade: 6 Name: Mara

Self-Reflection Sheet: Balances

After reviewing my collection of mathematics work, overall, I would like myself to notice...

I noticed that my average mark was a 2. Through September to now, I have not improved much. I noticed that I had some difficulty's with things that I would know now.

Please refer to the following sheets that are best highlighted my accomplishments in problem solving.

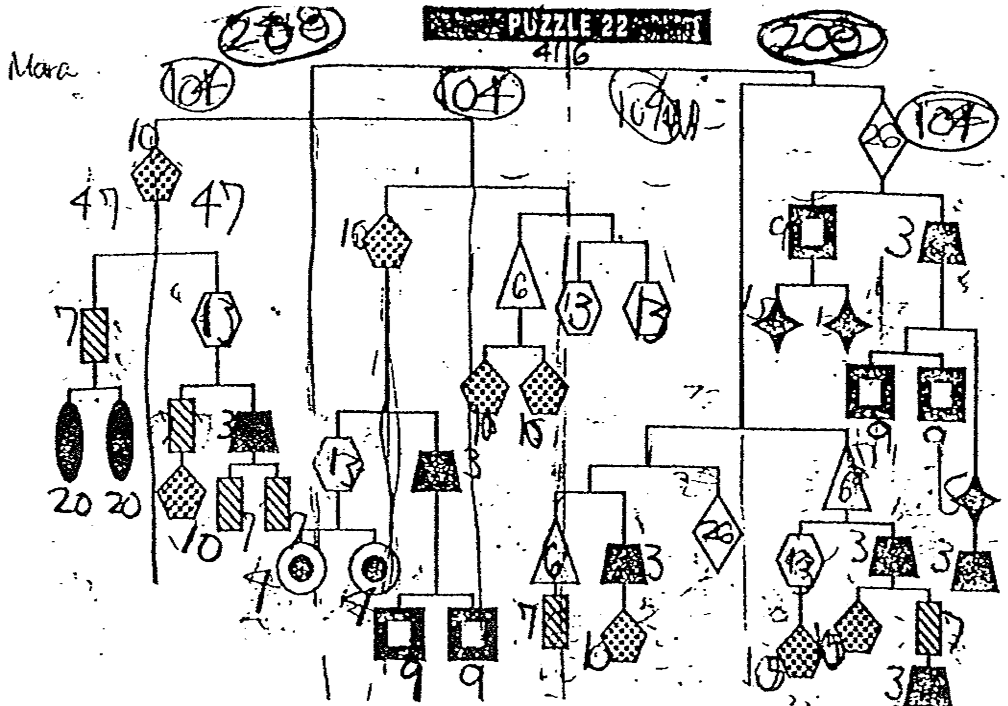
1. # 3 2½/4 It's the best mark
2. #2, 3, 4, 6, 8, 7 2/4
3. 1/4 # 5

The next step for me would be...

I would like to see a 4, and work harder. I like to get a 4, on a lot of problems.

Date: April 6th

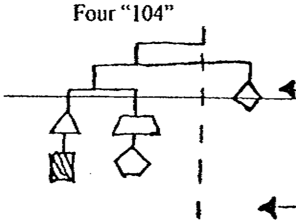
Appendix 5-B: Mara (1/2)



Discover the value of each of the shapes. The total weight is 416.

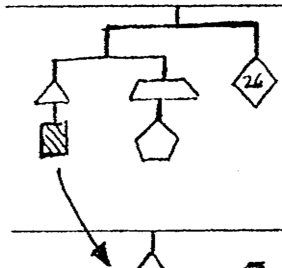
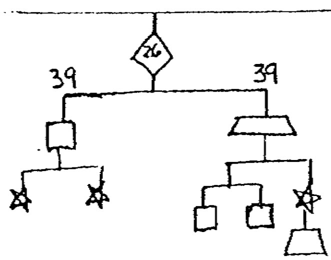
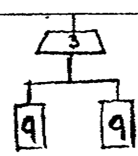
Clue: ▓ - Δ = 1

The explication: How do you find the value for each geometric form?

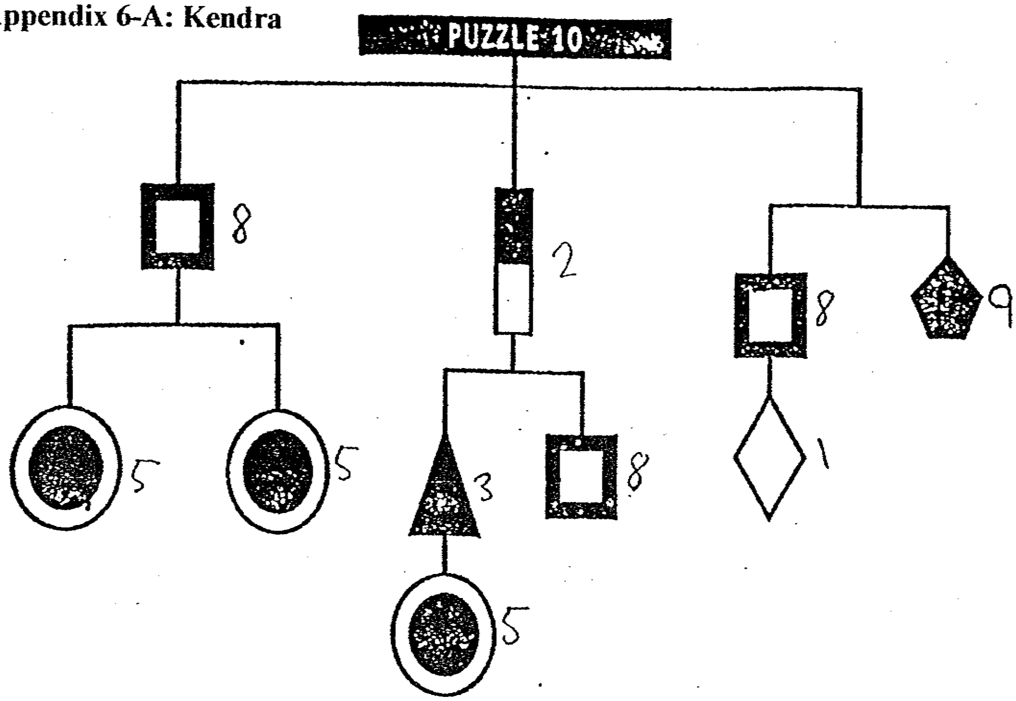
Observation	Thinking
"The total weight is 416."	416 ÷ 2 = 208 ($\frac{208\\208}{416}$) Then the balance is cut in two.
298 must be cut in two again because there are more than two sides.	208 ÷ 2 = 104
It continues…	104 ÷ 2 = 52 52 ÷ 2 = 26
There are two branches Four "104" 	where do I start? I scanned the diagram for the places where there is an only form on the branches. I found:
	I knew that the ◇ was 26 because the separated line is just left side of ◇.

Appendix 5-B: Mara (2/2)

Mara 2.

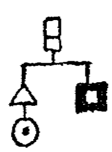
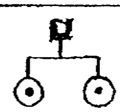
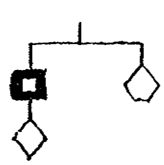
	I know that Δ, ■, ⏢, and ⌂ = 26 because it is on the other side of the ◇.
 = (■ - Δ = 1) clue	I know that side 1 is equal to second side because they are on the opposite side. 26 ÷ 2 = 13. Then each side is 13.
(same thing)	Δ → 6 ■ → 7 It must be bigger than Δ because ■ - Δ = 1 Then it can be 7 – 6 = 1. Also, 6 + 7 = 13
	You cannot find out the values for the forms because you did not see the other forms but we know that ⌂ + ⏢ = 13.
	I moved on to the far left side for finding the values of ⌂ and Δ = 26 then 104 – 26 = 78 78 ÷ 2 (2 sides) = 39. Then the total weight minus 26 is 39. 39 for each side.
	I tried the trapezoid is 3, and the ■ 9, ⏢ 3, and 15. 9 9 3 15 ——— 39 If the ⏢ were the biggest, it does not work on the other side, etc. (after) We know that Δ = 6 and ⌂ = 5. The ⏢ + ■ = ⌂ because it is on the other side and that the ⌂ + ⏢ = 13, and because 26 ÷ 2 = 13.

Appendix 6-A: Kendra

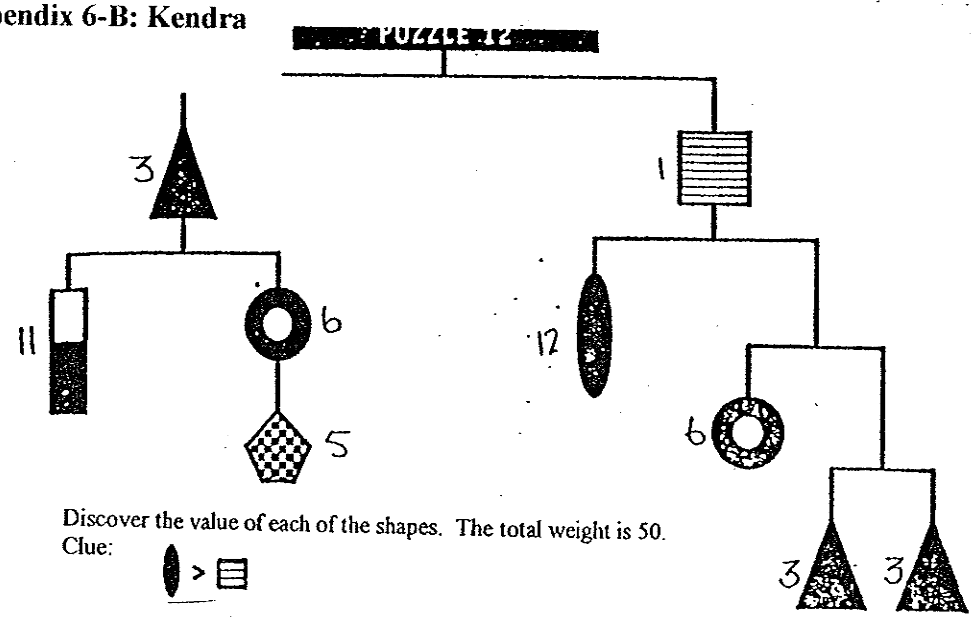


Discover the value of each of the shapes. The total weight is 54. The three arms are equal in weight.

The explication: How do you find the value for each geometric form?

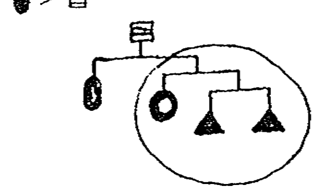
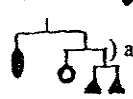
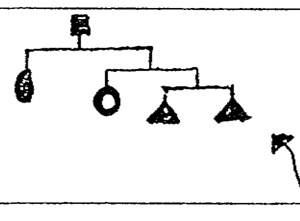
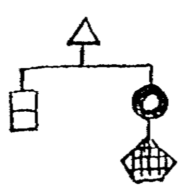
Observation	Thinking
	Each side = 18 ⊙ + Δ = ◘ ◘ can be 8 Δ = 5 or 3 (neither 6 nor 2 because ▯ = 2) I think 3 and the ⊙ = 5
	◘ = 8 ⊙ + ⊙ + ◘ = 18 ⊙ = 5
	◘ = 8 ⬠ needs to be 9 because it cannot be 8 because ◘ + ◇ = ⬠ is neither 10 nor more because it will be more than 10 then it is 9 ◇ = 1

Appendix 6-B: Kendra



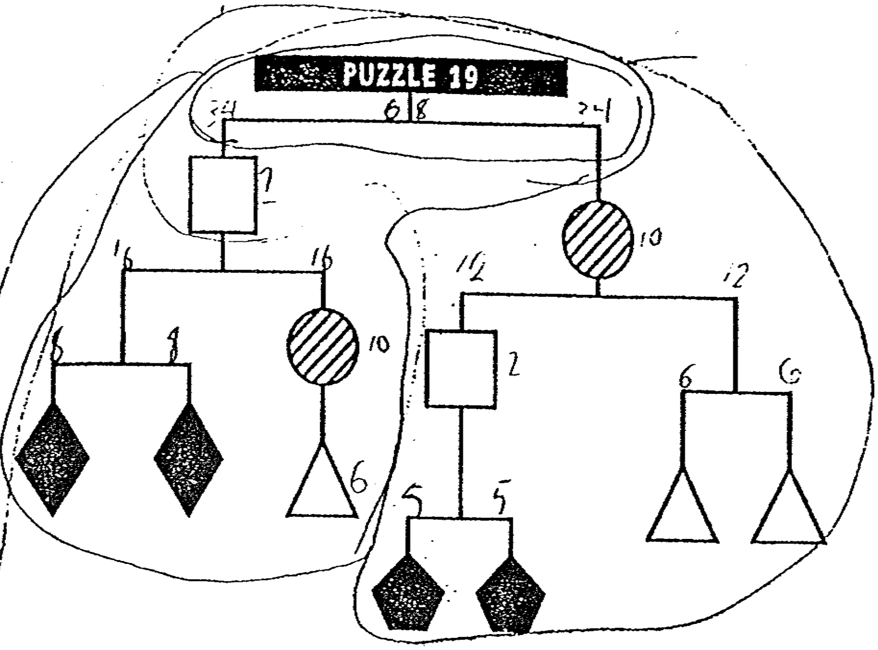
Discover the value of each of the shapes. The total weight is 50.
Clue:
⬮ > ▤

The explication: How do you find the value for each geometric form?

Observation	Thinking
It says that the total weight is 50	$50 \div 2 = 25$ Each side is worth 25
⬮ > ▤ 	⬮ is bigger than the ▤ O > ▲ because two ▲ is equal to one O ⬮ cannot be more than 12 because ▲ + ▲ + O = ⬮ () and it will be too large
	Then for the ▲ I try 3. The O must be 6 and the ⬮ must be 12 and the ▤ must be 1 $3 + 3 + 6 + 12 + 1 = 25$
	▲ = 3 O = 6 because on the other side The only way that it works: ⬟ = 5 ▯ = 11 because 6 + 5 = 11 + 11 = 22 + 3= 25

Appendix 7: Betsy (1/2)

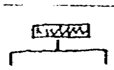
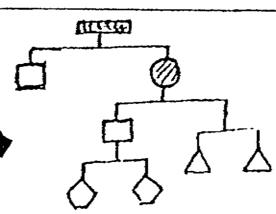
Betsy



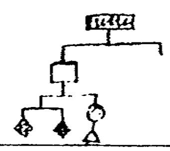
Découvrir la valeur de chaque forme. Le poids total est 68.
L'indice supplémentaire:

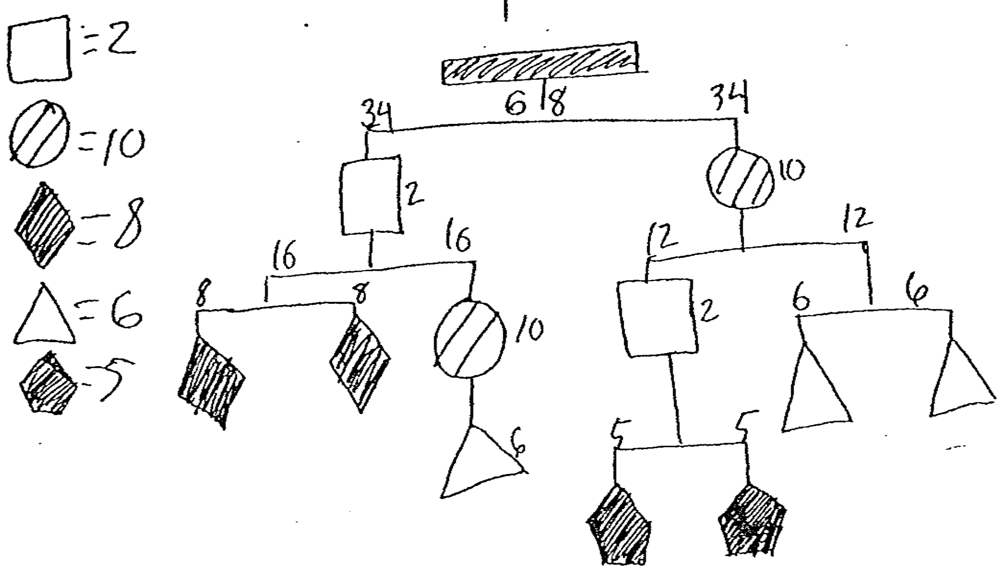
□ + ◆ = ⊘

The explication: How do you find the value for each geometric form?

Observation	Thinking
	The total weight is 68. □ + ◆ = ⊘
	68 ÷ 2 = 34 then I put 34 on each side
	First, I searched for the easiest side but the two sides are the same. Then I just started with the right-hand side. I tried to make ⊘ become one. Then 34 – 10 = 24 ÷ 2 = 12 Then I put 12 on each side. Δ = 6 because 12 divided by 2 = 6 on the other side. □ = 2 because I need to have an even number. ⌂ = 5 because 10 ÷ 2 = 5

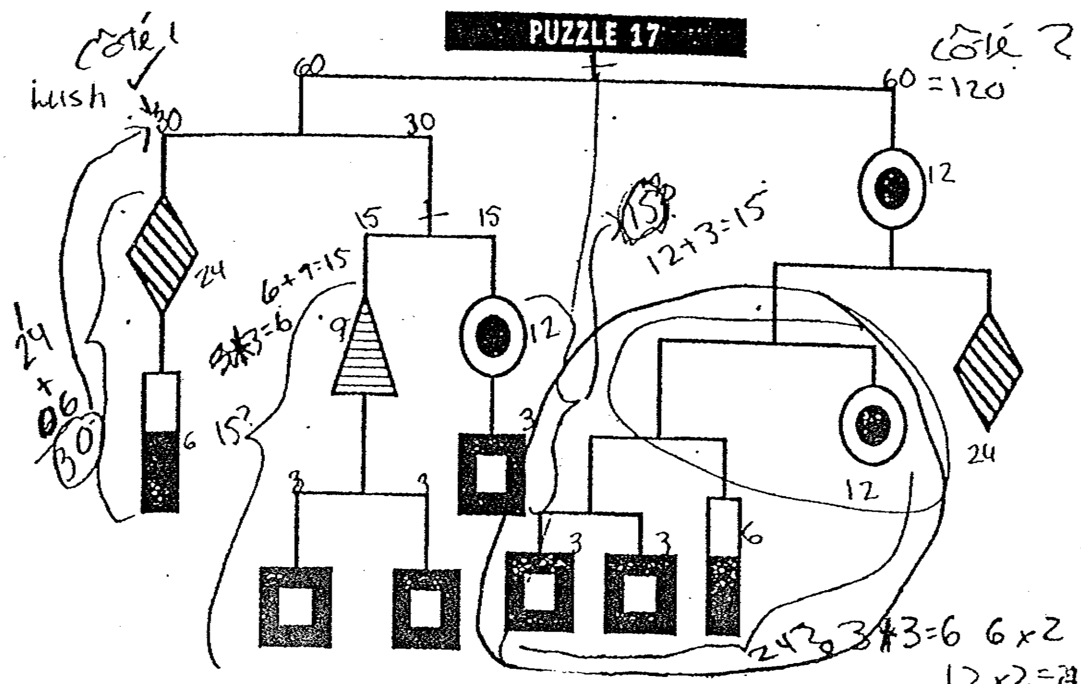
Appendix 7: Betsy (2/2)

	□ = 2, ⊘ = 10, Δ = 6 because this says that on the other side ◆ = 8 because 16 + 2 = 18.
□ + ◆ = ⊘	2 + 8 = 10



There is not any way because if you made ⊘ equal to 2, Δ = 8, □ = 6, ⌂ = 5. It does not work because on the left ⊘ + Δ = 10 and the other side is the same as □. It is supposed to be 14 and there is already 5. And if you have it more or less the thing, it will not work.

Appendix 8-A: Rush (1/2)



Discover the value of each of the shapes. The total weight is 120.

Clue: All shapes are multiples of ◘.

The explication: How do you find the value for each geometric form?

Observation	Thinking
The total weight is 120.	Each side is equal to 60 because there are two sides and you do $120 \div 2 = 60$.
Δ = 9 ◘ = 3	Δ = 9 because I did it. I started with Δ = 1. Δ: 1 | 2 | 3 | 4 | ... | 8 | 9 ◘: 7 | 6.5 | 6 | 5.5 | ... | 3.5 | 3
	does not work because 1 is not a multiple of 7. It is a factor of 7. The next one does not work because 2 is not a multiple of 6.5. The next one does not work because 3 is not a multiple of 6 but a factor of 6. Same thing with the next one, 4 is not a multiple of 5.5. I skipped to 8 because I saw a pattern, which was integer, decimal, integer, decimal, etc. for the ◘. I know that 8 is not a multiple of 3.5 so I tried 9. 9 is a multiple of 3. I made certain that it was a multiple of ◘ because the supplementary clue says that all the values are multiples of 8.
O = 12	O = 12 because it must equal to 15. $15 - 12 = 3$, and it works.

Appendix 8-A: Lush (2/2)

Lush 2

▯ = 6	▯ = 6 because on the side 2 or on the circled side it is divided by 2 again. The form on the other part es 12 so 12 ÷ 2 = 6.
◇ = 24	◇ = 24 because in the circle part when you add all the forms it is equal to 24, and it has to be equal.
Extra: O = 12, ▯ = 6, ◇ = 24, Δ = 9, ◘ = 3 are they all multiples of 3?	3 6 9 12 15 18 24…multiples of 3. ◘ ▯ Δ O ◇

Verifying the balance!

I added each part then verified each arm is equal to 60. Verified all the numbers that are multiples of 3.

Appendix 8-B: Lush

Grade: 7 Name: Lush

Self-Reflection Sheet: The balances

After reviewing my collection of mathematics work, overall, I would like myself to notice... That el didn't achive all my goals and that I still rushed. My goals were to take my time, slow down, focus and get more 3½ and 4. I focused but didn't take my time. I barely slowed down. I noticed that I had 3 fours, 3 three and a halfs and 1 three.

Please refer to the following sheets that are best highlighted my accomplishments in problem solving.

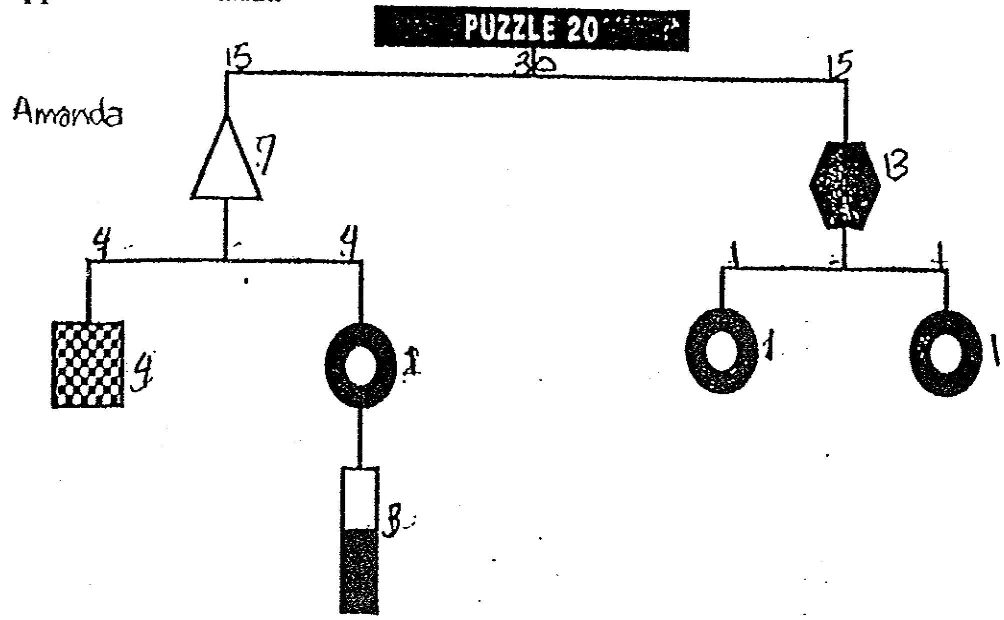
1. Balance 22 4/4 el did 9 pages of explication it really showed how I think! Included information learns in maths
2. Balance 21 3½/4 2 versions good explication
3. Balance 19 3/4 Good answer, not enough explanation but there was extra

The next step for me would be...

To try and take more time as before. To try not to keep looking at the clock, which makes me nervous and rush

Date: April 17, 2001

Appendix 9: Amanda



Discover the value of each of the shapes. The total weight is 30.

Clue: 3▩ < ⬢   △ - ▩ = ▮

The explication: How do you find the value for each geometric form?

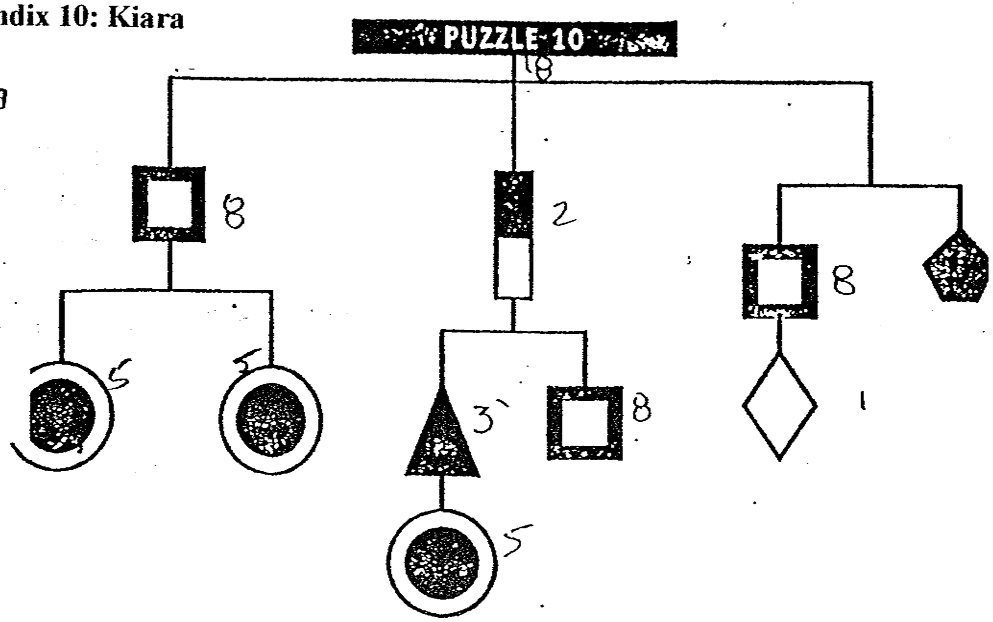
First I made the half of 30, which is 15. Now it is 15 out of 15 it is balancing. Then I did the left side because of the clue: Δ - ▩ = ▮ then I started on the left I put an odd number because the divided arms on the bottom will be equal. I put 4 on the bottom because 15 – 7 = 8 and 8 ÷ 2 = 4 then I put 4 on each side. I assigned 4 to the square because it is the only form there and the side is equal to 4 then it is 4. On the other side on the bottom of the right side I assigned 3 to the rectangle and 1 to the circle because Δ - ▩ = ▮ then 7Δ - 4▩ = 3▮ and 3 + 1 = 4. The other side on the left it is 15 then we've already known that the circle is 1, then 15 – 2 because there are two circles 1 + 1 = 2, 15 – 2 = 13 then the ⬢ form = to 13.

Verification:

1. Total weight = 30
2. $\begin{array}{r} 15 \\ 2\overline{)30} \\ \underline{30} \end{array}$
3. 15 on each side to balance.
4. Put an odd number to have an even number provided you do not use decimal numbers.
5. All satisfies the clue.

Appendix 10: Kiara

Kiara



Découvrir la valeur de chaque forme. Le poids total est 54. Les trois bras ont le même poids chacun.

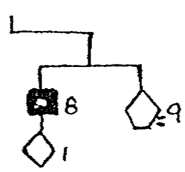
The explication: How do you find the value for each geometric form?

	Observation	Thinking
1.	I did 54 ÷ 3 for each side.	Then I assigned 18 to each side.
2.	I saw that the O must be the same.	Then I assigned each circle to be 5.
3.	I saw that rest was the ■.	Then I assigned it an 8.

I assigned the ▮ = 2 and I assigned the Δ = 3 and the ◇ = 1 and the ⬟ = 9. The reason that I assigned these numbers to these forms because all the numbers were balanced together. ⬟ = 5

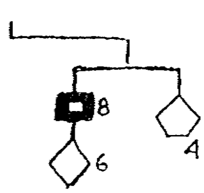
I knew that my numbers worked because if you try to change the numbers it is not going to be balanced.

E.g. 1.



All together 18 it balanced. ← good answer

E.g. 2:



All together 18 but the 6 and the 4 are not balanced.

Appendix 11-A: Kiko

Name: KIKO

Self-Reflection Sheet: The Balances

After reviewing my collection of mathematics work, overall, I would like myself to notice... I noticed that from the starting of the years I got 1 and 2 and now I got 3 and 4. I noticed that I got mostly 2 and 3 on my balances. I noticed that I added more information in every balance. I noticed that I worked hard for every balance.

Please refer to the following sheets that are best highlighted my accomplishments in problem solving.

1. The best balance I chose was balance number 8 because I worked hard and I got a good mark.
2. The second best was balance number 7 because I only got 1 forms wrong
3. My worst one was balance number 5 because I thought it was very hard but I didn't think that much.

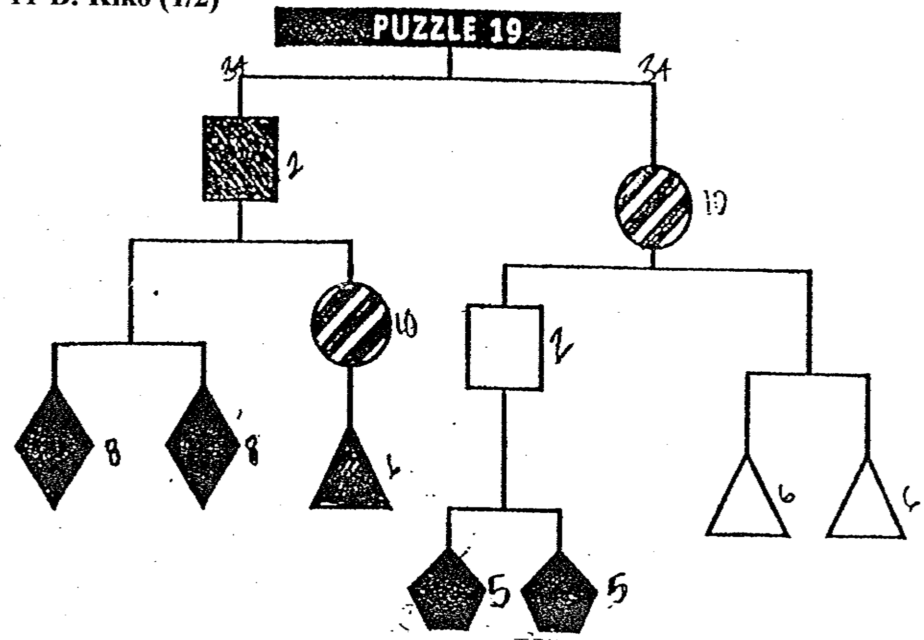
The next step for me would be...

I would like to see better marks by studying more and think more while I'm doing the balances.

Date: April 6 Signature: KIKO

Appendix 11-B: Kiko (1/2)

Kiko



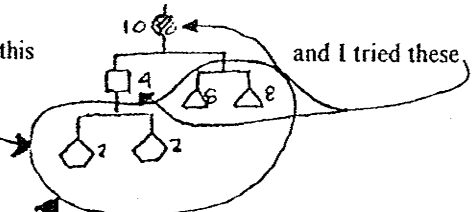
Discover the value of each of the shapes. The total weight is 68.

Clue: □ + ◇ = ⊘

The explication: How do you find the value for each geometric form?

It says that the total weight is 68, then I assigned 34 to one side and 34 to the other because there are two sides and 68 ÷ 2 = 34. I tried □ = 4, ◆ = 6, ⊘ = 10, Δ = 8, ⌂ = 5 but it did not work.

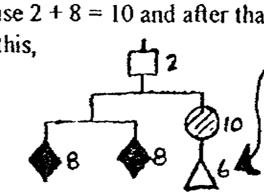
All these = 68 but on the right side it is like this numbers it did not work because



and I tried these

these did not balance. After that I tried □ = 2, ◆ = 8, ⊘ = 10, Δ = 6, ⌂ = 5 and it worked.

I tried to let ⊘ = 10, it says that the □ + the ◆ = 6 then I assigned the □ 2 and the ◆ 8. This works because 2 + 8 = 10 and after that I assigned the Δ 6 because it is like this,

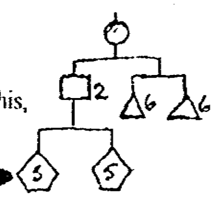


for this I assigned it a 6 because if you add all together it = 34 and you can see that it balanced because 8 + 8 = 16 and 10 + 6 = 16.

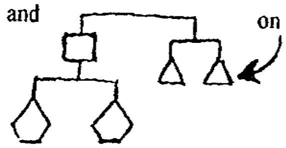
Appendix 11-B: Kiko (2/2)

Kiko 2

On the other side, all I did was I assigned the ⌂ 5 because it is like this,
we know that the □ = 2, the , ⦸ = 10 and the Δ = 6
then you know that I assigned 5 to



because if you add all these = 34 and



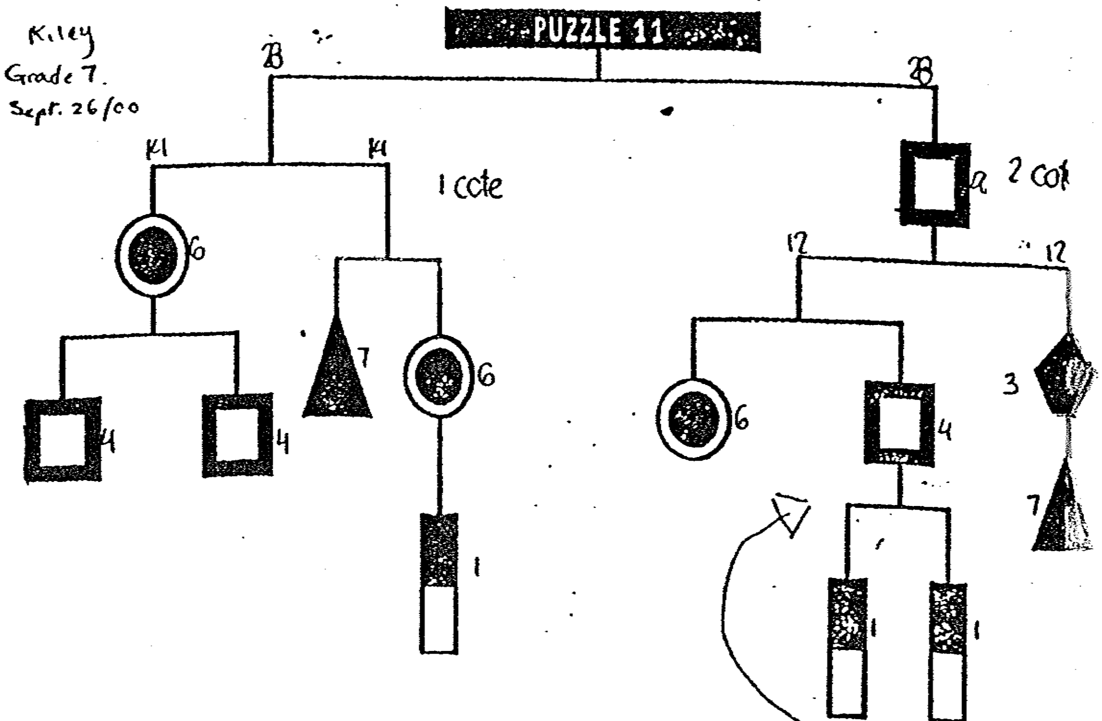
on this side it = 12

and on this side it = 12 also + the ⦸ = 10 together it is equal to 34. And if you put the two sides together (34 + 34) it = 68.

I do not think there is another way because the first time I tried □ = 4, ◆ = 6, ⦸ = 10, Δ = 8, ⌂ = 2 it did not work. After I tried □ = 2, ◆ = 8, ⦸ = 10, Δ = 6, ⌂ = 5 and it worked. After that I tried to find another answer but it did not work.

Appendix 12: Riley (1/2)

Riley
Grade 7.
Sept. 26/00



Découvrir la valeur de chaque forme. Le poids total est 56.

The explication: how do you find the value for each form?

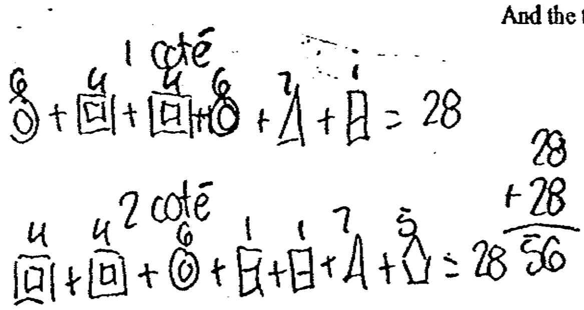
Observation	Thinking
1. The total weight is 56	56 ÷ 2 = 28
2. The total weight is 28 (side 1)	28 ÷ 2 = 14
3. The Δ = 7 (side 1)	because if you divide 14 into two that gives 7 for each side and that give 7 to the Δ.
4. The ◎ = 1 (side 1)	That does not work because ◎ needs to equal ▯+▯+▯=◎ and that says on the side 2.
5. The ◎ = 6 (side 1)	because I thought that the ◎ needed to be a large number because ▯+▯+▯ = ◎. I did not use 7 because there was another form and you cannot assign the form a 0.
6. 6 – 14 = 8 (side 1)	I did 6 – 14 = because I already knew that the ◎ = 6 and that the total weight was 14.
7. 8 ÷ 2 = 4 (side 1)	I did 8 ÷ 2 = 4 because there are two ▯ which need showing the same weight.

Appendix 12: Riley (2/2)

9. $24 \div 2 = 12$	I did $24 \div 2 = 12$ because there is side 2 that needs showing the same weight.
10. $= 4$	because I already knew it.
11. $= 6$	because I already knew it.
12. $= 1$	because I already knew it.
13. $= 7$	because I already knew it.
14. $7 - 12 = 5$	I did it because $\Delta = 7$ and the total weight is 12.
15. $= 3$	because there is just 5 left.

Verification

And the total weight equals 56.



Appendix 13: revised