VIBRATION OF SIMPLY SUPPORTED ORTHOTROPIC RECTANGULAR LAMINATED COMPOSITE CROSS-PLY PLATES

BY

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Abstract

An analytical solution using the propagator matrix method is developed to obtain the natural frequencies of vibration and the corresponding displacement and stress eigenvectors for simply supported, orthotropic, rectangular, laminated composite, cross-ply plates.

Three approximate theories are investigated. They consist of two equivalent, single layer theories, (namely the classical laminated plate theory and first order shear deformation theory), along with a Raleigh-Ritz type approximation that uses quadratic finite elements through the thickness are investigated. Numerical results are presented for the exact propagator matrix approach as well as for the three approximate theories. Results from the novel exact solution approach agree with published results for the natural frequencies as well as the displacement and stress eigenvectors. Results for equivalent single layer theories over predict the natural frequencies. The extent of the over prediction is affected by a plate's thickness, the magnitude of the difference in the relative stiffness of the lamina, and the relative differences of the stiffness in one direction with respect to another direction. The novel solution approach that uses the propagator matrix method is computationally efficient. A new case of a zero fundamental frequency and corresponding rigid body displacement eigenvector is uncovered for an orthotropic material whose material constants, D_{44} and D_{55}, are negligible.

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Nomenclature

Chapter 2 – Analytical Method

a, b	length of plate in x and y direction, respectively
N	number of layers or sublayers
i	subscript denoting i th layer
H	total thickness of plate
u, v, w	displacements in the x, y and z directions, respectively
v	displacement in the y direction
w	displacement in the z direction
z_i	value of z coordinate at i th interface
$[L_1]$	matrix of derivatives defined in equation (2.2)
σ_{ij}	stress components
ρ	mass density of layer
^{T}	vector or matrix transpose
D_{ij}	elastic constants of an orthotropic material
ε_{ii}, γ_{ij}	normal and shear strain components, respectively
$\phi(z)$, $\psi(z)$ and $\chi(z)$	functions of z components of displacements u, v and w, respectively
m, n	integer values corresponding to wavenumbers in x and y directions, respectively
$\hat{M}$, $\hat{N}$	normalized wavenumbers $m\pi / a$ and $n\pi / b$ in x and y directions, respectively
ω	natural frequency of free vibration
i	$\sqrt{-1}$
L	derivative with respect to z
c	variable for the roots of the characteristic equation

n	subscript denoting $n\ th$ root of characteristic equation and corresponding $n\ th$ eigenvector
c_n	$n\ th$ root of the characteristic equation
ϕ_n, ψ_n, χ_n	eigenvectors corresponding to the $n\ th$ root of c
d_1 through d_9	constants defined in equations (2.11a) through (2.11i)
$\{B_i\}$	vector of stresses and displacements at $i\ th$ interface
$[P_i]$	propagator matrix for $i\ th$ sublayer
$[G]$	matrix linking vector of stresses and displacements with functions of z
$[H]$	matrix of eigenvectors ϕ_n, ψ_n, χ_n for each root of c_n
$[E(z)]$	diagonal matrix of exponential functions $\exp(c_n z)$
$[P^G]$	global propagator matrix

2.5.2 Propagator matrix for isotropic material

ν	Poisson ratio for isotropic material
μ	isotropic shear modulus
$[P^I]$	propagator matrix for isotropic material

2.5.3 Propagator matrix for orthotropic material neglecting D_{44} and D_{55}

R_1, R_2	constants relating $\phi(z)$ to $\psi(z)$ and $\sigma_{zz}(z)$, respectively
$\{b_i\}$	vector of stresses and displacements at $i\ th$ interface for 2 x 2 propagator matrix
$[p_i]$	propagator matrix for $i\ th$ sublayer for 2 x 2 propagator matrix for orthotropic material neglecting D_{44} and D_{55}
$\{b_i^*\}$	vector of stresses and displacements at $i\ th$ interface for 4 x 4 propagator matrix

$[p_i^*]$ propagator matrix for $i\ th$ sublayer for 4 x 4 propagator matrix for orthotropic material neglecting D_{44} and D_{55}

$[p_i^{*G}]$ global 4 x 4 propagator matrix for orthotropic material neglecting D_{44} and D_{55}

Chapter 3 – Numerical methods

3.2 Equivalent single layer theory

$\tilde{u}(x,y,z,t)$, $\tilde{v}(x,y,z,t)$ and $\tilde{w}(x,y,z,t)$ displacements in the x, y and z directions respectively as functions of x, y, z and t

N number of layers through the plate's thickness

i subscript denotes $i\ th$ layer

$\tilde{U}(x,y,z,t)$ general vector of displacements as functions of x, y, z and t

$U_k(x,y,t)$ $k\ th$ partial derivative with respect to z of $\tilde{U}(x,y,z,t)$ evaluated at $z=0$

3.3 Classical laminated plate theory

Q_{ij} stress-strain material constants for an orthotropic material in a state of plane stress

δV , δT , δW virtual strain energy, virtual kinetic energy and virtual work, respectively

$\{\sigma_i\}$ vector of stresses in the $i\ th$ layer

$\{\varepsilon\}$, $\{\delta\varepsilon\}$ strain and virtual strain vector, respectively

$\{\varepsilon_0\}$, $\{\kappa\}$ first and second derivative components of strain vector $\{\varepsilon\}$, respectively

$\{\delta\varepsilon_0\}$, $\{\delta\kappa\}$ first and second derivatives components of virtual strain vector $\{\delta\varepsilon\}$, respectively

$\{N\}$, $\{M\}$ vector of force and moment components, respectively

$[A]$, $[B]$, $[C]$ first, second and third moments, respectively, of material properties $[Q]$ about midplane of plate

I_0 total weight

I_1 and I_2 first and second moments of weight about the midplane of plate, respectively

A_{ij}, B_{ij} and C_{ij} first, second and third moments, respectively, of material property Q_{ij} about midplane of plate

$\hat{U}$, $\hat{V}$ and $\hat{W}$ amplitude coefficients of u_0, v_0 and w_0, respectively

u_0, v_0 and w_0 displacement components of $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$

$[K]$ global stiffness matrix

$[M]$ global mass matrix

3.4 First order shear deformation theory

ψ_x rotation in xz plane about y axis, $\dfrac{\partial \tilde{u}}{\partial z}$

ψ_y rotation in yz plane about x axis, $\dfrac{\partial \tilde{v}}{\partial z}$

$\{N\}$ and $\{M\}$ vector of forces and moments, respectively

N_i axial force in i th direction from stress σ_i

N_{ij} shear force in ij plane from shear stress σ_{ij}

M_i moment from stress σ_i

M_{ij} moment from shear stress σ_{ij}

$\{N^*\}$ vector of forces that include the shear forces multiplied by a shear correction coefficient

$\tilde{Q}_{ij}$ shear force product of N_{ij} and shear correction coefficient

K_s shear correction coefficient

$\tilde{Q}$ shear force in beam

b width of beam

H total thickness of beam

$\sigma^{c}{}_{xz}$ and $\sigma^{f}{}_{xz}$ shear stresses in beam from 3-D elasticity theory and first order shear deformation theory, respectively

V strain energy

V^{c} and V^{f} strain energy in beam from 3-D elasticity theory and first order shear deformation theory, respectively

A cross sectional area of beam

t time

^{o} first partial derivative with respect to time, $\frac{\partial}{\partial t}$

$\widehat{U}$, $\widehat{V}$, $\widehat{W}$, $\widehat{\Psi}_x$ and $\widehat{\Psi}_y$ amplitude coefficients for displacement functions u_0, v_0, w_0, ψ_x and ψ_y, respectively

$\{\bar{d}\}$ vector of amplitude coefficients

$[K]$ global stiffness matrix

$[M]$ global mass matrix

ω natural frequency of vibration

ω_n $n\ th$ root of frequency equation

$\widehat{U}_n$, $\widehat{V}_n$, $\widehat{W}_n$, $\widehat{\Psi}_{xn}$ and $\widehat{\Psi}_{y_n}$ amplitude coefficients of ω_n

u_{0n}, v_{0n}, w_{0n}, ψ_{xn} and ψ_{y_n} displacement functions for ω_n

$\tilde{u}_n$, $\tilde{v}_n$, and $\tilde{w}_n$ displacement solutions as functions of x, y, and z for ω_n

σ_{xn}, σ_{y_n}, σ_{xy_n}, σ_{xzn} and σ_{yz_n} stress solutions as functions of x, y, and z for ω_n

Q_{ij_i} material properties Q_{ij} in $i\ th$ layer

3.5 Raleigh-Ritz approximation

x, y and z coordinates

u, v and w displacements in x, y and z directions, respectively, as functions of

x, y, z and t

h thickness of an element

N total number of elements

N_i, N_j and N_k weighting functions for top, middle and bottom nodes of an element, respectively

ξ normalized element coordinate system which has the value (-1) at the i th node, (0) at the j th node, and (+1) at the k th node

z_i value of z coordinate at the i th node of an element

u_i, u_j, u_k; v_i, v_j, v_k; w_i, w_j, w_k displacements u, v and w at the i th, j th and k th nodes, respectively

$\{U(x,y,z,t)\}$ vector of displacements $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$

$[n]$, $[b]$ and $[c]$ matrices of weighting functions defined in Appendix D

$\{q\}$ vector of displacements at i th, j th and k th nodes

$[a]$ matrix of first derivatives with respect to z weighting functions as defined in Appendix D

$\{\varepsilon\}$ vector of strains

^{o} first partial derivative with respect to time, $\frac{\partial}{\partial t}$

$,$ $\frac{\partial}{\partial x}$

$\cdot$ $\frac{\partial}{\partial y}$

$[m]$, $[k_1]$, $[k_2]$, $[k_3]$, $[k_4]$, $[k_5]$, $[k_6]$ matrices as defined in Appendix D in terms of matrices $[a]$, $[b]$ and $[c]$

L_i Lagrangian of strain energy for i th element layer

$\{q_1\}$, $\{q_2\}$ and $\{q_3\}$ vectors of displacements u, v and w, respectively, at the i th, j th and k th nodes

$\{\bar{u}\}$ vector of displacements u_i, u_j, u_k

$\{\bar{v}\}$ vector of displacements v_i, v_j, v_k

$\{\bar{w}\}$ vector of displacements w_i, w_j, w_k

$[A]$, $[B]$ and $[C]$ matrices defined in appendix D that contain products and derivatives of the weighting functions

$\{Q\}$ and $\{q\}$ global and element displacement vector, respectively

$*$ superscript which denotes matrices and vectors that are regrouped for the ease of assembly into a global system of equations

$\{q^*\}$ regrouped element displacement vector

$[K]$ and $[M]$ 9 x 9 element stiffness and mass matrices, respectively

KIJ 3 by 3 matrix components of 9 x 9 element stiffness matrix $[K]$

$[K^*]$ and $[M^*]$ regrouped element stiffness and mass matrices, respectively

$[W]$ matrix of the weighting functions evaluated at each node

σ_{zx}, σ_{zy} and σ_{zz} element stresses

$\{\bar{\sigma}_{zx}\}$, $\{\bar{\sigma}_{zy}\}$ and $\{\bar{\sigma}_{zz}\}$ vectors of element stresses at the i th, j th and k th nodes

Chapter 4 – Numerical results

a and b length of plate in x and y direction

A_j j th undetermined coefficient of exponential displacement solution

$(A_j)_k$ j th undetermined coefficient of exponential displacement solution for k th layer

ν Poisson ratio for isotropic material

h total thickness of plate

ρ mass density

ω natural frequency

Ω normalized frequency

μ shear modulus for isotropic material

m and n wavenumber in x and y directions, respectively

D_{ij} elastic constants of an orthotropic material

ν_{ij} Poisson ratio of transverse strain in the j th direction to axial strain in the i th direction for applied stress in the i th direction

E_1, E_2 and E_3 Young's modulus in the x, y and z directions, respectively

G_{12}, G_{13} and G_{23} shear modulus for the xy, xz and yz planes, respectively

t_c and t_f thickness of core and flange materials in sandwich plate example, respectively

Chapter 1

Introduction

1.1 Composite materials

There has been a dramatic increase in the use of composite materials in all types of engineering structures as well as medical prosthetic devices, electronic circuit boards, and sport equipment. Many journals and research papers attest to the fact that there has been a major effort to develop composite material systems, and analyze and design structural components made from composite materials, [1]. Composite materials are used in infrastructure as I-beams, channel sections, and joint plates. The use of composite materials is extensive in aerospace for commercial aircraft, military aircraft, and space craft. They are used commonly in the fuselage and wing structures, [2].

Composite materials consist of two or more materials combined at the macroscopic level to compose a structural unit. They can exhibit the best qualities of their constituents and, often, qualities that neither constituent possesses, [3]. The main advantages of composite materials are their very high strength to weight ratio or stiffness to weight ratio, corrosion resistance, and thermal insulation, [2]. Other properties that can be improved are attractiveness, fatigue life, temperature dependent behavior, thermal conductivity and increased acoustical insulation.

Fiber reinforced composite materials consist of high strength and high modulus fibers in a matrix material. Reinforced steel bars embedded in concrete provide an example of a fiber reinforced composite. In these composites, fibers are the principal load carrying members while the matrix material keeps the fibers together, acts as a load transfer medium between fibers and protects fibers from being exposed to the environment. When fibers are aligned in one direction, the strength and stiffness achieved in one direction may be achieved at the expense of the corresponding property in the other direction. Fiber reinforced composite materials for structural applications are often from thick layers called lamina. A laminate is a stack of lamina with various orientations of the principal material directions.

The layers of the laminate are usually bound together by the same matrix material that is used in the laminate. Structural elements such as bars, beams or plates are then formed by stacking the layers to achieve the desired strength and stiffness, [1]. The contents of this thesis are directed mainly towards oriented laminated, fiber reinforced composites. In a fiber reinforced composite the role of the fibers is to impart the stiffness and strength characteristics of the fibers by carrying a major part of the load in the direction of the fibers, [4]. The most common fiber reinforced composite consists of reinforcing fibers embedded in a binder or a matrix material. Fibers and matrix materials can be obtained commercially in a variety of forms from individual fibres or as lamina.

Fiber reinforcement is very effective because many materials in their fibrous form are much stronger than in their bulk form due to fewer imperfections present in fibers. The fibers may be organic or metallic. Commonly used fibers are glass, boron, graphite and aramid (Kevlar). Fiber diameters range from 0.008 to 0.15 millimeters. Fibers are generally 10 to 100 times stronger then the matrix, [4]. Glass is the most widely used fiber and it is relatively inexpensive but its low modulus limits its application. Graphite or carbon fibers are the most widely used advanced fibers, [2]. Matrix materials include polymers such as epoxy and polyester, ceramic, or metals such as aluminum, copper, iron, nickel, steel, and titanium . The role of the matrix is to maintain the alignment of the fibers, increase the structural stiffness, provide adequate transverse properties perpendicular to the fibers, act as a load transfer medium for discontinuous or broken fibers and protect fibers from damage caused by mutual abrasion and environmental degradation. Fillers are added to reduce the weight and protect the fibers from ultraviolet radiation.

In a continuous fiber reinforced composite laminate, individual lamina with fibers running parallel to each other are laminated together at given angles to form angle-ply laminates or cross-ply laminates where the angles are at 90 degrees. The major purpose of lamination is to tailor the directional dependence of the strength and stiffness of a material to match the loading environment of the structural element. For example, more lamina could be oriented in one direction as opposed to another to give more stiffness in one direction, [3]. Another common composite configuration is the sandwich structure where stiff face sheets are laminated together with a soft core. This configuration has extremely high flexural stiffness-to-weight ratios and it is used extensively in aerospace structures, [2].

Two principal steps involved in the manufacture of laminated fiber reinforced composite materials are layup and curing, [2]. Common material defects are interlaminar voids due to air entrapment, delamination, or lack of resin. Other possible defects include incomplete curing, excess resin between layers, excess material voids or porosity, incorrect orientation of lamina principal material directions, damaged fibers, wrinkles or ridges cause by improper compaction, winding or layer alignment; inclusion of foreign matter, unacceptable joints in layers and a variation in thickness, [3].

1.2 Nondestructive evaluation

Natural frequencies of free vibration of plates depend on the material properties, support conditions, and the mass density of the material. Testing the natural frequencies has been a method investigated for determining if there are either flaws in a material or a change in stiffness. Additionally, it is important to know the natural frequencies of vibration in the design of composite structures. Vibrations can be minimized by selecting a material or a plate that will not be excited in one of its natural frequencies.

Vibration can be used for quality control. Since the natural frequencies of vibration are sensitive to material properties, composite structural components can be tested for their natural frequencies which can be compared to either those of a quality component. If the natural frequencies differ from that of a quality structural component, the component is likely to contain a flaw.

Additionally, it has been proposed that the natural frequencies can be used to test for deteriorations of a component after some time in service. If the natural frequencies of vibration of the structural component are lower after some time in service, and knowing that generally, the natural frequencies of vibration of a material are directly proportional to the square root of the stiffness, the material has undergone a loss in stiffness or other form of degradations. Composite materials are generally quite expensive. Therefore there is a need for this type of technology. To interpret test results, it is necessary to have theoretical predictions of the natural frequencies of vibration. Many approximate models and exact models are available to solve for natural frequencies of vibration, each yielding different accuracies and requiring different degrees of computational effort.

1.3 Literature review

Several equivalent single layer (ESL) theories have been proposed to derive the dispersion relations in the form of a generalized eigenvalue problem. The most common ones are plate theories. The classical laminated plate theory (CLPT), in which it is assumed that normals to the mid-plane before deformation remain straight and normal to the plane after deformation, under predicts the deflections and over predicts the natural frequencies. These results are due to the neglect of transverse shear strains in the classical laminated plate theory. Early attempts to consider a transverse shear deformation plate theory were made by Reissner [5] and Mindlin [6] for the cases of bending and free vibration of plates, respectively. However, first order shear deformation theory (FSDT) releases merely one assumption, the neglect of the transverse shear strains, to allow for constant transverse shear strains, through the thickness. Therefore, first order shear deformation theory requires the use of a transverse shear correction factor either implicitly (Reissner) or explicitly (Mindlin). Ambartsumyan [7] is believed to have been the first person to account for a more realistic distribution of the transverse shear strains with all assumptions of the CLPT released. Whitney and Pagano [8] presented solutions for bending and for flexural vibration frequencies of symmetric and non-symmetric laminates by considering the shear deformation and rotary inertia in the same manner as Mindlin's theory for isotropic homogeneous plates. Reissner [9] introduced a higher order plate theory with in-plane and normal higher order displacements, for the special problem of plate bending.

A list of references for numerous refined plate theories for homogeneous or laminated media, consisting of isotropic or anisotropic materials, can be found in Kapania and Raciti [10], and Librescu and Reddy [11]. However, plate theories are cumbersome to use and they do not provide very accurate eigenvalues. Subsequent to these investigations, higher order theories have been developed by many researchers [12-25] to predict more realistic displacements, stress and frequency behavior of thick-laminated plates. An analytical solution by higher order, layer wise, mixed theory in which displacement and stress continuity was maintained at layer interfaces was used by K. Rao [26]. Moreover, he calculated the natural frequencies as well as the relative stress and deflection distributions through the thickness of the plate.

More recently, finite element methods (such as a three-dimensional, eight-node hybrid stress finite element method) have been developed by Sun and Liou [27] to analyze the free vibrations of laminated composite plates. They have proved that the hybrid stress finite element model can very accurately predict the free vibration behavior of an orthotropic laminate. Although higher order, displacement based, finite element formulations give sufficiently accurate results, they have inherent difficulties such as shear locking due to the over estimation of the transverse shear stiffness in the case of thin sections. Also, the displacement models are known to yield inaccurate stress components at prescribed stress boundaries. On the other hand, hybrid elements can be used to exactly satisfy stress boundary conditions, thereby improving the prediction of the stress components. Also, hybrid models provide better convergence for problems involving a stress singularity. The motivation of the original formulation of the hybrid stress model developed by Pian [28] was to circumvent the difficulty of constructing interpolation functions for the displacements in an element to fulfill the inter-element compatibility conditions of an arbitrary geometry. An approximate solution has been accomplished by assuming a number of stress modes, which satisfy equilibrium conditions only within an element. Pian and Chen [29] presented a hybrid/mixed finite element method accomplished by using the Hellinger- Reissner principle for which the stress equilibrium conditions were not introduced initially but were incorporated through the use of additional internal displacement parameters. The finite element formulation based on the assumed stress hybrid approach has been presented by Spilker [30]. Higher-order distributions were assumed through the thickness for the stresses and displacements within each layer, and a two-dimensional plain strain element was developed by restricting attention to cylindrical bending of cross ply laminates. Later, Spilker [31, 32] developed two more hybrid-stress elements based on individual-layer, non-normal, cross-section rotations (which satisfy appropriate interface continuity) and independent layer stress fields (which satisfy interface traction continuity and upper/lower surface traction free conditions). Application of the hybrid stress model to both thin and moderately thick single and multi-layer plates (including transverse shear effects) have been presented, for example, in references [33-36]. The book by Reddy [37] gives a comprehensive summary of equivalent single layer theories and higher order theories, Raleigh-Ritz approximations and various finite element formulations. The paper by Reddy and Khedir [1] provides an excellent review of the equivalent single layer theories and higher order theories, as well as finite element formulations. The paper by Kapania and Raciti [10] provides a comprehensive review of higher order theories and Raleigh-Ritz formulations.

Theories that yield accurate eigenvalues and are computationally very convenient to use are the theories derived through the stiffness methods of analysis. Dong and his colleagues (Dong and Nelson, [38]; Dong and Pauley, [39]; Dong and Huang, [40]) presented a stiffness method of analysis to study wave propagation and vibration in laminated anisotropic plates. They discretized the plate in the thickness direction with subdivision into mathematical sub-layers and used quadratic interpolation polynomials that involve only the displacements at the interfaces between sub-layers and at the middle of the sub-layers as the generalized coordinates. These methods maintain only displacement continuity at the layer interfaces. Later on, Datta et. al. [41] presented an approximate stiffness method applicable to a layered anisotropic plate with an arbitrary number of layers. In this method, both displacement and stress continuity is maintained at layer interfaces. The accuracy of the stiffness method is demonstrated by comparing the results with analytical results for homogeneous and layered fiber-reinforced plates. The effect of varying the number of sub-layers is also investigated.

Three dimensional elasticity solutions for free vibrations, bending and stability problems of simply supported, orthotropic plates were presented in the late 1960's and early 1970's. The three-dimensional solutions obtained were used as the basis for assessing the accuracy and range of validity of several two dimensional theories, [42]. A three dimensional elasticity solution, using a finite difference scheme, was given by Noor, [43]. Srinivas and his colleagues [44, 45] analyzed the bending vibration and buckling behavior of simply supported, thick homogeneous and laminated plates by formulating and solving the problem analytically using three-dimensional elasticity theory. Also Pagano [46, 47] used the same approach for the bending of cylinders and thick laminated plates. All these solutions showed considerable non-linearity in the distributions of in plane strains through the thickness, rather than the linear distributions predicted by first order shear deformation theory. An efficient method to obtain the exact frequency equation of a plate having an arbitrary number of orthotropic layers has not yet been reported. Although it is possible to obtain, using a propagator matrix approach, (see Mal [48]), the exact frequency equation governing guided waves or vibrations in a layered orthotropic plates, finding the roots of these transcendental equations is quite cumbersome and computationally very expensive. As the number of layers increases, the exact frequency equation becomes extremely complicated and requires robust search techniques to locate the roots. To circumvent this difficultly, an analytical method that combines an efficient root-locating scheme is proposed here. In this method, the exact frequency equation of the layered orthotropic plate is constructed using the propagator matrices. Muller's method, as given in Conte and Boor [49], is then used in conjunction with initial guesses (obtained from an approximate theory) to obtain the roots of the exact frequency equation. Obtaining the exact frequency equation and solving for the exact frequencies of vibration in this manner, to the author's knowledge, has not been reported.

1.4 Thesis objectives

The objective of this thesis is (1) to investigate analytically the methods involved in obtaining natural frequencies of vibrations and the corresponding eigenvectors of displacements and stresses for an orthotropic, laminated composite, cross-ply plate and (2) to develop an exact solution that is computationally efficient. Three approximate theories and one exact theory is discussed. A novel solution approach is used and the results will be compared to each another and contrasted in terms of accuracy and computational efficiency.

1.5 Organization of thesis

An exact method using the propagator matrix approach is presented in Chapter 2 for an orthotropic material incorporating all nine (9) constants of orthotrophy. Propagator matrices for an isotropic material and an orthotropic material for which the transverse shear moduli are negligible are also presented. The formulations of the three approximate numerical methods are presented in Chapter 3. First, two equivalent single layer theories, namely the classical laminated plate theory (CLPT) and a first order shear deformation theory (FSDT) are formulated. Finally, a Raleigh-Ritz type approximation is formulated using quadratic finite elements through the thickness of the plate. Numerical results in terms of the natural frequencies of vibrations, and the corresponding displacement and stress eigenvectors obtained from programs developed are presented and discussed in Chapter 4. A comparison of the computational effort is mentioned briefly. The numerical results are validated and compared to other published results. Finally, conclusions and recommendations for future work are presented in Chapter 5.

Chapter 2

Analytical Method

2.1 Introduction

An analytical model of obtaining the natural frequencies of vibration for a simply supported, laminated composite cross-ply plate is developed in this chapter. The exact formulation maintains both the stress and displacement continuity at the laminae interfaces. Roots of the frequency equation can be calculated, along with the corresponding stress and displacement eigenvectors, at the layer or sub-layer interfaces through the thickness of the plate.

2.2 Description of problem

Consider the simply supported laminated composite plate shown in Figure 2.1. The plate has length a in the x direction and length b in the y direction. The plate is simply supported at the four edges, ($x = 0,\ a$) and ($y = 0,\ b$). The plate is assumed to be composed of perfectly bonded orthotropic laminae, each one of which may have distinct mechanical properties and thicknesses. All nine unique elastic constants of orthotrophy will be taken into account. For simplicity, it will be assumed that the plane of each laminae runs parallel to the plane of the x and y axis, while the z axis runs perpendicular to the plane of the laminae. Each layer can be divided into several sub-layers in order to increase the number of interfaces or nodal points where the stress and displacement eigenvectors can be calculated using the analytical approach. However, the division of the layers into sub-layers does not increase the accuracy of the eigenvalues and eigenvectors calculated using an analytical approach because the frequency solutions are already exact. The composite plate is has N number of sub-layers with ($N+1$) sub-layer interfaces. Each lamina is assumed to have a thickness h_i, where i corresponds to the $i\ th$ sub-layer. The total thickness of the plate is H. The interfaces at interface number 1 and $(N+1)$ are traction free.

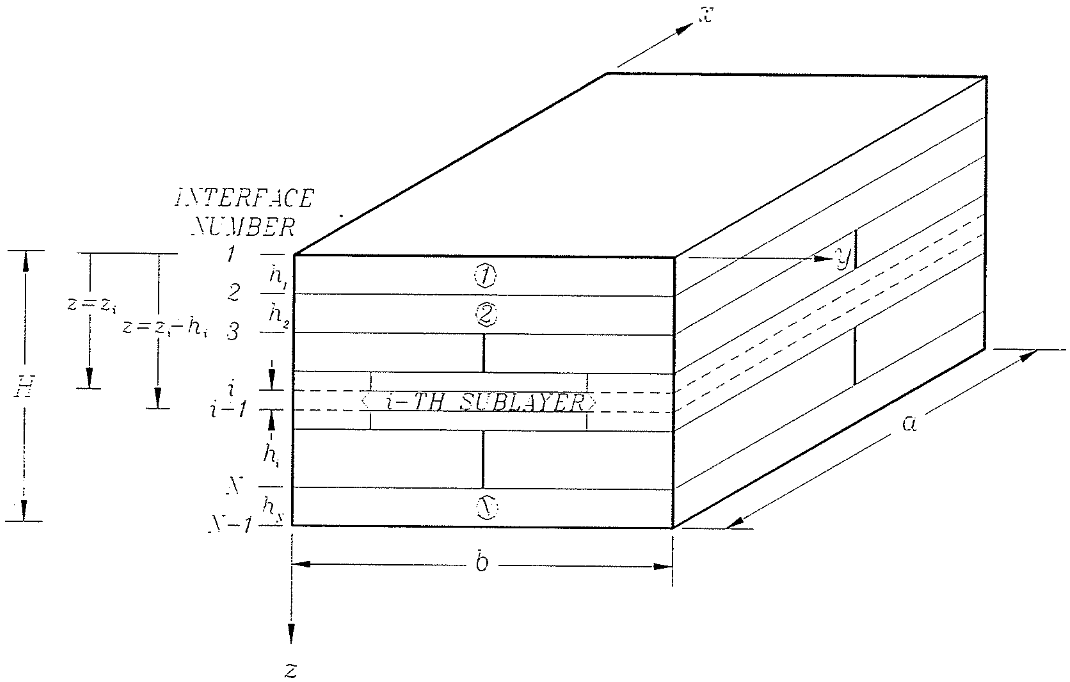

Figure 2.1: Coordinate system of a laminated composite plate used for the analytical method

2.3 Governing equations

The concern here is a plate having a large and variable number of laminates. Each laminate is assumed to have its own thickness, density, and set of nine orthotropic material elastic constants. Let t denote time and $u\ (x,y,z,t)$, $v\ (x,y,z,t)$ and $w\ (x,y,z,t)$ denote the particle displacements in the x, y and z directions, respectively. Consider the $i\ th$ sub-layer bounded by $z = z_i$ and $z = z_i + h_i$. The equations that govern the behavior of the material are the three dimensional equilibrium equations without body forces. They are given by

$$[L_1]^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \rho \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{2.1}$$

where σ_{ij} represents the stress components, ρ represents the density of the layer, and superscript T represents the matrix or vector transpose. Matrix $[L_1]^T$ is given by

$$[L_1]^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}. \tag{2.2}$$

The stress strain relations for an orthotropic material are given by McClintock [50] as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ D_{12} & D_{22} & D_{23} & & & \\ D_{13} & D_{23} & D_{33} & & & \\ & & & D_{44} & & \\ & & & & D_{55} & \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} \tag{2.3}$$

where ε_{ij} represents the strain components, and D_{ij} represents the material elastic constants. The material constants, D_{ij}, are given in terms of the Young's moduli and Poisson ratios in Appendix A. In this appendix D_{ij}, as well as its transformation in orthogonal coordinates, are also presented.

The strain-displacement relations are given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = [L_1] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} . \tag{2.4}$$

Substituting of equations (2.2), (2.3), and (2.4) into equation (2.1) yields a system of differential equations. (See equation (B.1) given in Appendix B). The equations are given by

$$\begin{bmatrix} D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\frac{\partial^2}{\partial y^2} + D_{55}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2} & (D_{12}+D_{66})\frac{\partial^2}{\partial x\partial y} & (D_{13}+D_{55})\frac{\partial^2}{\partial x\partial z} \\ (D_{12}+D_{66})\frac{\partial^2}{\partial x\partial y} & D_{66}\frac{\partial^2}{\partial x^2} + D_{22}\frac{\partial^2}{\partial y^2} + D_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2} & (D_{23}+D_{44})\frac{\partial^2}{\partial y\partial z} \\ (D_{13}+D_{55})\frac{\partial^2}{\partial x\partial z} & (D_{23}+D_{44})\frac{\partial^2}{\partial y\partial z} & D_{55}\frac{\partial^2}{\partial x^2} + D_{44}\frac{\partial^2}{\partial y^2} + D_{33}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2.5}$$

2.4 Boundary conditions and displacement solutions

The boundary conditions governing the displacement functions of x and y for a simply supported rectangular plate may be specified as, [45]:

$$\text{at } x = 0 \text{ and } a; \quad \sigma_{xx} = 0, \quad w = 0 \quad \text{and} \quad v = 0,$$
$$\text{at } y = 0 \text{ and } b; \quad \sigma_{yy} = 0, \quad w = 0 \quad \text{and} \quad u = 0. \tag{2.6}$$

Solutions for the displacements that satisfy the prescribed boundary conditions in equation (2.6) for a simply supported plate take the following form

$$\begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} \phi(z)\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \\ \psi(z)\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega t} \\ \chi(z)\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \end{Bmatrix} \tag{2.7}$$

where

$$\hat{M} = \frac{m\pi}{a}; \quad \hat{N} = \frac{n\pi}{b}; \quad \text{and} \quad m = 1,2,3,...; \quad n = 1,2,3,... \quad . \tag{2.8}$$

The $\phi(z)$, $\psi(z)$, and $\chi(z)$ are undetermined exponential functions of z which have to be determined. Substituting of equation (2.7) into equation (2.5) and canceling the trigonometric terms yields the following system of differential equations in z.

$$\begin{bmatrix} \rho\omega^2 - D_{11}\hat{M}^2 - D_{66}\hat{N}^2 + D_{55}L^2 & -\hat{M}\hat{N}(D_{12}+D_{66}) & \hat{M}L(D_{13}+D_{55}) \\ -\hat{M}\hat{N}(D_{12}+D_{66}) & \rho\omega^2 - D_{66}\hat{M}^2 - D_{22}\hat{N}^2 + D_{44}L^2 & \hat{N}L(D_{23}+D_{44}) \\ -\hat{M}L(D_{13}+D_{55}) & -\hat{N}L(D_{23}+D_{44}) & \rho\omega^2 - D_{55}\hat{M}^2 - D_{44}\hat{N}^2 + D_{33}L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(2.9)

where $L = \dfrac{d}{dz}$. Define, next, the constants

$$d_1 = \rho\omega^2 - D_{11}\hat{M}^2 - D_{66}\hat{N}^2 \tag{2.10a}$$

$$d_2 = D_{55} \tag{2.10b}$$

$$d_3 = -\hat{M}\hat{N}(D_{12} + D_{66}) \tag{2.10c}$$

$$d_4 = \hat{M}(D_{13} + D_{55}) \tag{2.10d}$$

$$d_5 = \rho\omega^2 - D_{66}\hat{M}^2 - D_{22}\hat{N}^2 \tag{2.10e}$$

$$d_6 = D_{44} \tag{2.10f}$$

$$d_7 = \hat{N}(D_{23} + D_{44}) \tag{2.10g}$$

$$d_8 = \rho\omega^2 - D_{55}\hat{M}^2 - D_{44}\hat{N}^2 \tag{2.10h}$$

and

$$d_9 = D_{33}\ . \tag{2.10i}$$

Substituting constants d_1 through d_9 into equation (2.9) yields the following equation (2.11).

$$\begin{bmatrix} d_1 + d_2L^2 & d_3 & d_4L \\ d_3 & d_5 + d_6L^2 & d_7L \\ -d_4L & -d_7L & d_8 + d_9L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2.11}$$

For a non-trivial solution of this homogeneous set of differential equations, the determinant of the matrix on the left hand side of equation (2.11) must equal zero. This requirement yields the following, sixth order polynomial auxiliary equation.

$$d_2 d_6 d_9 \left(L^2\right)^3 + \left(d_1 d_6 d_9 + d_6 d_4^{\,2} + d_2 d_5 d_9 + d_2 d_6 d_8 + d_2 d_7^{\,2}\right)\left(L^2\right)^2$$
$$+ \left(d_1 d_5 d_9 + d_1 d_6 d_8 + d_1 d_7^{\,2} + d_2 d_5 d_8 - d_3^{\,2} d_9 - 2 d_3 d_4 d_7 + d_4^{\,2} d_5\right) L^2$$
$$+ d_8\left(d_1 d_5 - d_3^{\,2}\right) = 0 \ . \qquad (2.12)$$

The six roots c_1 through c_6 of this polynomial can be solved numerically. The solutions that give the displacements as functions of z are

$$\begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \sum_{n=1}^{6} \begin{Bmatrix} \phi_n \\ \psi_n \\ \chi_n \end{Bmatrix} A_n e^{c_n z} \qquad (2.13)$$

where c_n denotes the $n\ th$ root of the six roots of equation (2.12). Substituting equation (2.13) into equation (2.11) gives the solutions for each set of eigenvector constants ϕ_n, ψ_n, and χ_n, where subscript n denotes the $n\ th$ eigenvector for the corresponding $n\ th$ root of c. The resulting eigenvector system is given as

$$\begin{bmatrix} d_1 + d_2 c_n^{\,2} & d_3 & d_4 c_n \\ d_3 & d_5 + d_6 c_n^{\,2} & d_7 c_n \\ -d_4 c_n & -d_7 c_n & d_8 + d_9 c_n^{\,2} \end{bmatrix} \begin{Bmatrix} \phi_n \\ \psi_n \\ \chi_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} . \qquad (2.14)$$

The $n\ th$ eigenvector, (see equations (B.2a) through (B.2c) located in Appendix B), is given as

$$\begin{Bmatrix} \phi_n \\ \psi_n \\ \chi_n \end{Bmatrix} = \begin{Bmatrix} d_4 c_n \left(d_5 + d_6 c_n^{\,2}\right) - c_n d_3 d_7 \\ d_7 c_n \left(d_1 + d_2 c_n^{\,2}\right) - c_n d_3 d_4 \\ -\left(d_1 + d_2 c_n^{\,2}\right)\left(d_5 + d_6 c_n^{\,2}\right) + d_3^{\,2} \end{Bmatrix} . \qquad (2.15)$$

2.5 Propagator matrix derivations

Propagator matrices for an orthotropic material, isotropic material, and an orthotropic material with negligable material coefficients D_{44} and D_{55} are presented in this section.

2.5.1 Propagator matrix for an orthotropic material

By evaluating the stresses and displacements at $z = z_i$ and $z = z_i + h_i$ (for a particular sub-layer) and by eliminating the vector of the six unknown constants A_1 through A_6, the following relation can be obtained:

$$\{B_{i+1}\} = [P_i]\{B_i\} \tag{2.16}$$

where

$$\{B_i\}^T = \langle u_i \quad v_i \quad \sigma_{zzi} \quad \sigma_{zxi} \quad \sigma_{zyi} \quad w_i \rangle . \tag{2.17}$$

The vector quantity $\{B_i\}$, yet unknown, represents the stresses and displacements at $z = z_i$ while $\{B_{i+1}\}$ represents the stresses and displacements at $z = z_i + h_i$. The matrix $[P_i]$ is the propagator matrix for the $i\ th$ sub-layer.

By substituting equations (2.3), (2.4), and (2.7) into vector $\{B_i\}$, (according to steps (B.3) through (B.7) in Appendix B), carrying out derivatives, and ignoring transcendental functions of x and y, the following relation is obtained:

$$\{B_i\} = [G]\{f(z)\} \tag{2.18}$$

where

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -D_{13}\hat{M} & -D_{23}\hat{N} & 0 & 0 & 0 & D_{33} \\ 0 & 0 & D_{55}\hat{M} & D_{55} & 0 & 0 \\ 0 & 0 & D_{44}\hat{N} & 0 & D_{44} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.19}$$

and

$$\{f(z)\} = \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \\ \dfrac{\partial\phi(z)}{\partial z} \\ \dfrac{\partial\psi(z)}{\partial z} \\ \dfrac{\partial\chi(z)}{\partial z} \end{Bmatrix} = \sum_{n=1}^{6} \begin{Bmatrix} \phi_n \\ \psi_n \\ \chi_n \\ c_n\phi_n \\ c_n\psi_n \\ c_n\chi_n \end{Bmatrix} A_n e^{c_n z} = [H][E(z)]\{A\} \tag{2.20}$$

where

$$[H] = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 \\ c_1\phi_1 & c_2\phi_2 & c_3\phi_3 & c_4\phi_4 & c_5\phi_5 & c_6\phi_6 \\ c_1\psi_1 & c_2\psi_2 & c_3\psi_3 & c_4\psi_4 & c_5\psi_5 & c_6\psi_6 \\ c_1\chi_1 & c_2\chi_2 & c_3\chi_3 & c_4\chi_4 & c_5\chi_5 & c_6\chi_6 \end{bmatrix}, \tag{2.21}$$

$$[E(z)] = \begin{bmatrix} e^{c_1 z} & & & & & \\ & e^{c_2 z} & & & & \\ & & e^{c_3 z} & & & \\ & & & e^{c_4 z} & & \\ & & & & e^{c_5 z} & \\ & & & & & e^{c_6 z} \end{bmatrix} \tag{2.22}$$

and

$$\{A\}^T = \langle A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \rangle \ . \tag{2.23}$$

Substituting equation (2.20) into (2.18) yields the following expression for the stress and displacement vector, $\{B_i\}$, at any particular interface.

$$\{B_i\}\big|_{z=z_i} = [G][H][E(z)]\big|_{z=z_i}\{A\} \ . \tag{2.24}$$

The arbitrary constants in vector $\{A\}$ are the only unknowns in the above equation. Thus an expression for stresses and displacements for the i th interface is obtained. Since it is desirable to relate the stresses and displacements at the i th interface to stresses and displacements at the $(i+1)$ th interface with a propagator matrix for the i th sub-layer, a local coordinate $z = z_i$ for $\{B_i\}$ and $z = z_i + h_i$ for $\{B_{i+1}\}$ is assumed.

Solving the matrix equation (2.24) for $\{A\}$ yields the equation

$$\{A\} = [E(z)]^{-1}\Big|_{z=z_i} [[G][H]]^{-1}\{B_i\} \ . \tag{2.25}$$

Substituting $z = z_i + h_i$ into matrix $[E(z)]$ and equation (2.25) into the stress and displacement vector $\{\mathrm{B}_{i+1}\}$ yields

$$\{\mathrm{B}_{i+1}\} = [P_i]\{B_i\} \tag{2.26}$$

where the $i\ th$ propagator matrix is given by

$$[P_i] = [G][H][E(z)]\Big|_{z=z_i+h_i} [E(z)]^{-1}\Big|_{z=z_i} [[G][H]]^{-1} = [G][H][E(z)]\Big|_{z=h_i} [[G][H]]^{-1} . \tag{2.27}$$

The global propagator matrix $[P^G]$ is obtained by repeated application of equation (2.26). This results in

$$\{B_{N+1}\} = [P^G]\{B_1\} \tag{2.28}$$

where

$$[P^G] = [P_N][P_{N-1}]...[P_i]...[P_2][P_1] \ . \tag{2.29}$$

The repeated application of equation (2.29) ensures that the continuity of the displacements and stresses are maintained at the sub-layer interfaces.

2.5.2 Propagator matrix for an isotropic material

The full derivation for the propagator matrix for an isotropic material is given in Appendix C. Isotropic material constants are given in Appendix A as

$$D_{11} = D_{22} = D_{33} = \frac{2\mu(1-\nu)}{(1-2\nu)} \tag{2.30a}$$

$$D_{12} = D_{13} = D_{23} = \frac{2\mu\nu}{(1-2\nu)} \tag{2.30b}$$

$$D_{44} = D_{55} = D_{66} = \mu \tag{2.30c}$$

where ν is Poisson's ratio, and μ is the shear modulus. The propagator matrix for an isotropic material for the $i\ th$ sub-layer, denoted by $[P_i^I]$, is given as

$$[P_i^I] = [G^I][H^I][E^I(z)]\Big|_{z=h_i} [[G^I][H^I]]^{-1} \tag{2.31}$$

where matrices $\left[G^{I}\right]$, $\left[H^{I}\right]$ and $\left[E^{I}(z)\right]$ are defined as

$$\left[G^{I}\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{2\nu}{(1-2\nu)}\hat{M} & -\frac{2\nu}{(1-2\nu)}\hat{N} & 0 & 0 & 0 & \frac{2(1-\nu)}{(1-2\nu)} \\ 0 & 0 & \hat{M} & 1 & 0 & 0 \\ 0 & 0 & \hat{N} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{2.32}$$

$$\left[H^{I}\right]=\begin{bmatrix} r\hat{M} & -r\hat{M} & \hat{N} & \hat{N} & \hat{M} & \hat{M} \\ r\hat{N} & -r\hat{N} & -\hat{M} & -\hat{M} & \hat{N} & \hat{N} \\ g^{2} & g^{2} & 0 & 0 & s & -s \\ r^{2}\hat{M} & r^{2}\hat{M} & r\hat{N} & -r\hat{N} & s\hat{M} & -s\hat{M} \\ r^{2}\hat{N} & r^{2}\hat{N} & -r\hat{M} & r\hat{M} & s\hat{N} & -s\hat{N} \\ rg^{2} & -rg^{2} & 0 & 0 & s^{2} & s^{2} \end{bmatrix} \tag{2.33}$$

and

$$\left[E^{I}(z)\right]=\begin{bmatrix} e^{rz} & & & & & \\ & e^{-rz} & & & & \\ & & e^{rz} & & & \\ & & & e^{-rz} & & \\ & & & & e^{sz} & \\ & & & & & e^{-sz} \end{bmatrix} \tag{2.34}$$

where

$$r=\sqrt{g^{2}-\lambda^{2}} \tag{2.35}$$

$$s=\sqrt{g^{2}-\lambda^{2}(1-2\nu)/(2-2\nu)} \tag{2.36}$$

$$g=\sqrt{\left(\hat{M}^{2}+\hat{N}^{2}\right)} \tag{2.37}$$

$$\lambda=\omega\sqrt{\frac{\rho}{\mu}}\ . \tag{2.38}$$

2.5.3 Propagator matrix for an orthotropic material neglecting D_{44} and D_{55}

The full derivation of the propagator matrix for an orthotropic material neglecting D_{44} and D_{55} is given in Appendix D. The governing set of differential equations is given in equation (2.9). Taking D_{44} and D_{55} as zero, the following constants in equations (2.10) are redefined as

$$d_1 = \rho\omega^2 - D_{11}\hat{M}^2 - D_{66}\hat{N}^2 \tag{2.39a}$$

$$d_3 = -\hat{M}\hat{N}(D_{12} + D_{66}) \tag{2.39b}$$

$$d_4 = \hat{M}D_{13} \tag{2.39c}$$

$$d_5 = \rho\omega^2 - D_{66}\hat{M}^2 - D_{22}\hat{N}^2 \tag{2.39d}$$

$$d_7 = \hat{N}D_{23} \tag{2.39e}$$

$$d_8 = \rho\omega^2 \tag{2.39f}$$

and

$$d_9 = D_{33} \ . \tag{2.39g}$$

The resulting three by three governing system of differential equations can be reduced to a two by two system as follows

$$\begin{bmatrix} d_3 + R_1 d_5 & d_7 L \\ -(d_4 + R_1 d_7)L & d_8 + d_9 L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.40}$$

where

$$R_1 = \frac{(d_1 d_7 - d_3 d_4)}{(d_4 d_5 - d_3 d_7)} \ . \tag{2.41}$$

The displacement function $\psi(z)$ can be eliminated from the system since it is dependent on displacement function $\phi(z)$ as

$$\psi(z) = R_1 \phi(z) \ . \tag{2.42}$$

Taking the determinant of the matrix in the equation (2.40) yields the following characteristic equation

$$\left(\mathrm{d}_3 d_9 + R_1 d_5 d_9 + d_7 d_4 + R_1 d_7{}^2\right)L^2 + \left(d_3 d_8 + R_1 d_5 d_8\right) = 0 \tag{2.43}$$

which has two roots $\pm s$ where s is given as

$$s = \frac{\sqrt{d_8}\sqrt{-(d_3 + R_1 d_5)}}{\sqrt{\left(\mathrm{d}_3 d_9 + R_1 d_5 d_9 + d_7 d_4 + R_1 d_7^{\,2}\right)}} \,. \tag{2.44}$$

The following displacement solutions are proposed through the thickness.

$$\begin{Bmatrix} \phi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \chi_1 \end{Bmatrix} C_1 e^{sz} + \begin{Bmatrix} \phi_2 \\ \chi_2 \end{Bmatrix} C_2 e^{-sz} \,. \tag{2.45}$$

Stress $\sigma_{zz}(z)$ is also dependent on $\phi(z)$ and it can be written as

$$\sigma_{zz}(z) = R_2 \phi(z) \tag{2.46}$$

where

$$R_2 = -\left(D_{13}\hat{M} + D_{23}\hat{N}R_1\right) - D_{33}\frac{(d_3 + R_1 d_5)}{d_7} \,. \tag{2.47}$$

Therefore the stress $\sigma_{zz}(z)$ and displacements $\phi(z)$ and $\chi(z)$ are dependent on one another and any one of these functions can be expressed in terms of another function multiplied by a constant. Since stress free boundary conditions are required, the stress $\sigma_{zz}(z)$ is considered the main function. Then the displacements $\phi(z)$ and $\chi(z)$ can be expressed as

$$\phi(z) = \frac{\sigma_{zz}(z)}{R_2} \tag{2.48a}$$

and

$$\psi(z) = \frac{R_1}{R_2}\sigma_{zz}(z) \,. \tag{2.48b}$$

Continuity in the stress $\sigma_{zz}(z)$ automatically ensures continuity in the displacements $\phi(z)$ and $\chi(z)$.

By evaluating the stress, $\sigma_{zz}(z)$, and displacement, $\chi(z)$, at $z = z_i$ and $z = z_i + h_i$ for a particular sub-layer and by eliminating the vector of the two undetermined constants C_1 and C_2, the following relation can be obtained:

$$\{\mathrm{b}_{i+1}\} = [p_i]\{b_i\} \tag{2.49}$$

where

$$\{b_i\}^T = \langle w_i \quad \sigma_{zzi} \rangle \,. \tag{2.50}$$

After some manipulation, the two by two propagator matrix can be expressed as

$$[p_i]=\begin{bmatrix} \frac{1}{2}\left(e^{sh_i}+e^{-sh_i}\right) & -\frac{(d_3+R_1d_5)}{2R_2sd_7}\left(e^{sh_i}-e^{-sh_i}\right) \\ -\frac{R_2sd_7}{2(d_3+R_1d_5)}\left(e^{sh_i}-e^{-sh_i}\right) & \frac{1}{2}\left(e^{sh_i}+e^{-sh_i}\right) \end{bmatrix}=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \quad (2.51)$$

Matrix $[p_i]$ and equations (2.48a) and (2.48b) can be used to construct components for a 4 by 4 propagator matrix in which the displacements u_i and v_i are included to maintain continuity of all the displacements. The stress and displacement vector is now proposed as

$$\{b^*_i\}^T=\langle u_i \quad v_i \quad w_i \quad \sigma_{zzi}\rangle \quad (2.52)$$

where the 4 by 4 propagator matrix becomes

$$[p^*_i]=\begin{bmatrix} p_{22} & 0 & \frac{p_{21}}{R_2} & 0 \\ 0 & p_{22} & \frac{R_1}{R_2}p_{21} & 0 \\ 0 & 0 & p_{11} & p_{12} \\ 0 & 0 & p_{21} & p_{22} \end{bmatrix}. \quad (2.53)$$

The global propagator matrix $\left[p^{*G}\right]$ is obtained by repeated application of equation (2.49). This results in

$$\{b^*_{N+1}\}=\left[p^{*G}\right]\{b^*_1\} \quad (2.54)$$

where

$$\left[p^{*G}\right]=\left[p^*_N\right]\left[p^*_{N-1}\right]\ldots\left[p^*_i\right]\ldots\left[p^*_2\right]\left[p^*_1\right]. \quad (2.55)$$

The repeated application of the above equation ensures that continuity of displacements and stresses are maintained at the interfaces between the sub-layers.

2.6 Frequency equations and calculation of stresses

The frequency equations and calculation of stresses and displacements using the propagator matrix method is presented here for an orthotropic, an isotropic, and an orthotropic material neglecting material constants D_{44} and D_{55}.

2.6.1 Orthotropic and isotropic materials

Denoting the elements of the 6 by 6 global propagator matrix, $\left[P^G\right]$, by P_{ij}^G and invoking zero traction conditions at interfaces 1 and ($N+1$), the following relationship can be obtained from equation (2.28).

$$\begin{bmatrix} P_{31}^G & P_{32}^G & P_{36}^G \\ P_{41}^G & P_{42}^G & P_{46}^G \\ P_{51}^G & P_{52}^G & P_{56}^G \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} . \tag{2.56}$$

The exact dispersion equation (frequency equation) for the plate is obtained by setting the determinant of the coefficient matrix to zero. Thus

$$f(\omega, \hat{M}, \hat{N}) = \begin{vmatrix} P_{31}^G & P_{32}^G & P_{36}^G \\ P_{41}^G & P_{42}^G & P_{46}^G \\ P_{51}^G & P_{52}^G & P_{56}^G \end{vmatrix} = 0 . \tag{2.57}$$

The last equation can be solved for ω given $\hat{M}$ and $\hat{N}$.

Once the roots of the dispersion equation are determined that satisfy equation (2.57), the corresponding stress and displacement eigenvectors can be obtained by letting u_1 equal unity in equation (2.56) and solving for v_1 and w_1 as outlined in equations (B.8) and (B.9) of Appendix B. The eigenvector of the stresses and displacements at the first interface is

$$\{B_1\}^T = \left\langle 1 \quad \frac{P_{46}^G P_{51}^G - P_{41}^G P_{56}^G}{P_{42}^G P_{56}^G - P_{46}^G P_{52}^G} \quad 0 \quad 0 \quad 0 \quad \frac{P_{41}^G P_{52}^G - P_{42}^G P_{51}^G}{P_{42}^G P_{56}^G - P_{46}^G P_{52}^G} \right\rangle . \tag{2.58}$$

The eigenvector of the stresses and displacements at the i th interface can then be calculated by using

$$\{\mathrm{B}_i\} = [[P_{i-1}][P_{i-2}]\ldots[P_2][P_1]]\{B_1\} . \tag{2.59}$$

2.6.2 Orthotropic material neglecting D_{44} and D_{55}

Denoting the elements of the 4 by 4 global propagator matrix, $\left[p^{*G}\right]$, by p^{*G}_{ij} and invoking zero traction conditions in terms of σ_{zz} at interfaces 1 and ($N+1$), the following relationship can be obtained from equation (2.54).

$$p^{*G}{}_{43}\{w_1\}=0\ . \tag{2.60}$$

The exact frequency equation for the plate is obtained by setting the global propagator matrix component $p^{*G}{}_{43}$ to zero. Hence

$$f(\omega,\hat{M},\hat{N})=p^{*G}{}_{43}=0\ . \tag{2.61}$$

The above equation can be solved for ω given $\hat{M}$ and $\hat{N}$.

Once the roots of the dispersion equation are determined that satisfy equation (2.61), the corresponding stress and displacement eigenvectors can be obtained by letting w_1 equal unity in equation (2.60) and letting the stress σ_{zz1} equal zero. Since the displacements u and v are dependent on the stress σ_{zz}, displacements u and v must also be zero. The eigenvector of stresses and displacements at the first interface becomes

$$\{b^*{}_1\}^T=\langle 0 \quad 0 \quad 1 \quad 0\rangle\ . \tag{2.62}$$

The eigenvector of the stresses and displacements at the $i\,th$ interface can then be calculated by using

$$\{b^*{}_i\}=\left[[p^*{}_{i-1}][p^*{}_{i-2}]\ldots[p^*{}_2][p^*{}_1]\right]\{b^*{}_1\}\ . \tag{2.63}$$

Chapter 3

Numerical Methods

3.1 Introduction

Three approximate numerical models of vibration for a simply supported, laminated composite, cross-ply plate are developed in this chapter. The first two models are derived using an Equivalent Single Layer Theory (ESL), namely classical plate bending theory (CLPT) and a first-order shear deformation theory (FSDT). Additionally, a third model is presented using a Raleigh-Ritz type approximation with quadratic finite elements through the thickness.

3.2 Equivalent single layer theory (ESL)

Classical plate bending theory and first order shear deformation theory are presented in this section, both of which are equivalent single layer theories.

3.2.1 Overview of equivalent single layer theory

An equivalent single layer theory can be used to analyze laminated composite plates where each laminate has distinct material properties and thicknesses. The laminates are assumed to be orthotropic and perfectly bonded. For the laminates to be perfectly bonded, no slipping at the laminae interfaces must occur, [3]. The material properties of each of the individual laminates are used to calculate the material properties of the entire laminated plate through the thickness, as if the plate were homogeneous. The proposed displacement solutions are continuous throughout the thickness of the plate, which ensures that displacement continuity is maintained. However, stress continuity is not maintained. The ESL theories are derived from 3-D elasticity theory by making suitable assumptions regarding the kinematics of deformation through the thickness of the plate, [37]. Since the plate is assumed to be thin, a state of plane stress is assumed for each case, [37].

3.2.2 Description of problem

Consider the simply supported laminated composite plate shown in Figure 3.1. The plate has length a in the x direction and length b in the y direction. The plate is simply supported at the four edges, ($x = 0,\ a$) and ($y = 0,\ b$). The plate is assumed to be composed of perfectly bonded orthotropic laminae, each one having distinct mechanical properties and thicknesses. For simplicity, it will be assumed that the plane of each laminae runs parallel to the plane of the x and y axis, while the z axis runs perpendicular to the plane of the laminae. The displacements in the x, y, and z directions are denoted by $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ respectively. The composite plate is composed of N number of sub-layers with ($N+1$) sub-layer interfaces. Each lamina is assumed to have thickness h_i, where i corresponds to the i th sub-layer. The total thickness through the plate is H. The interfaces at interface number 1 and $(N+1)$ are traction free.

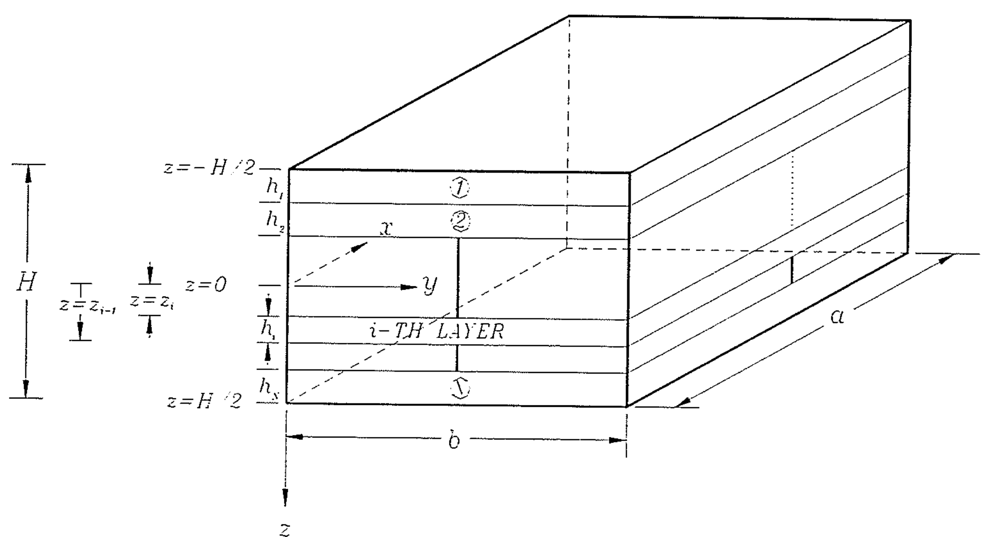

Figure 3.1: Coordinate system of a laminated composite plate used for the numerical methods

3.2.3 Displacement solutions

ESL theories are developed by assuming displacement solutions that are a power series expansion in z and continuous displacement solutions in x and y. Only a global system of coordinates is used. The displacement solutions take the following form.

$$\widetilde{U}(x,y,z,t)=\sum_{k=0}^{\infty} z^k U_k(x,y,t)=U_0(x,y,t)+zU_1(x,y,t)+z^2U_2(x,y,t)+\ldots \quad (3.1)$$

where $\widetilde{U}(x,y,z,t)$ is a displacement vector and $U_k(x,y,t)$ is the $k\ th$ partial derivative

$$U_k(x,y,t)=\left.\frac{\partial^k(\widetilde{U}(x,y,z,t))}{\partial z^k}\right|_{z=0}\times\frac{1}{k!} \quad (3.2)$$

of $\widetilde{U}(x,y,z,t)$ with respect to z evaluated at $z=0$, which contains continuous functions in x and y that are chosen with respect to the boundary conditions for the displacements, in addition to an undetermined coefficient. Thus $U_0(x,y,t)$ is the displacement at the z origin or midplane of the plate, $U_1(x,y,t)$ is the slope of the displacement at the midplane and $U_2(x,y,t)$ is the curvature of the displacement at midplane. It is difficult to assign meaning to higher order coefficients. The number of terms used determines the degree of the various possible modes of vibration through the thickness. In terms of symmetric and antisymmetric modes with respect to the displacement in the x direction denoted by u, each $k\ th$ term contributes to the following possible displacements, either symmetric or antisymmetric about the midplane of the plate, as shown in Figures 3.2a and 3.2b.

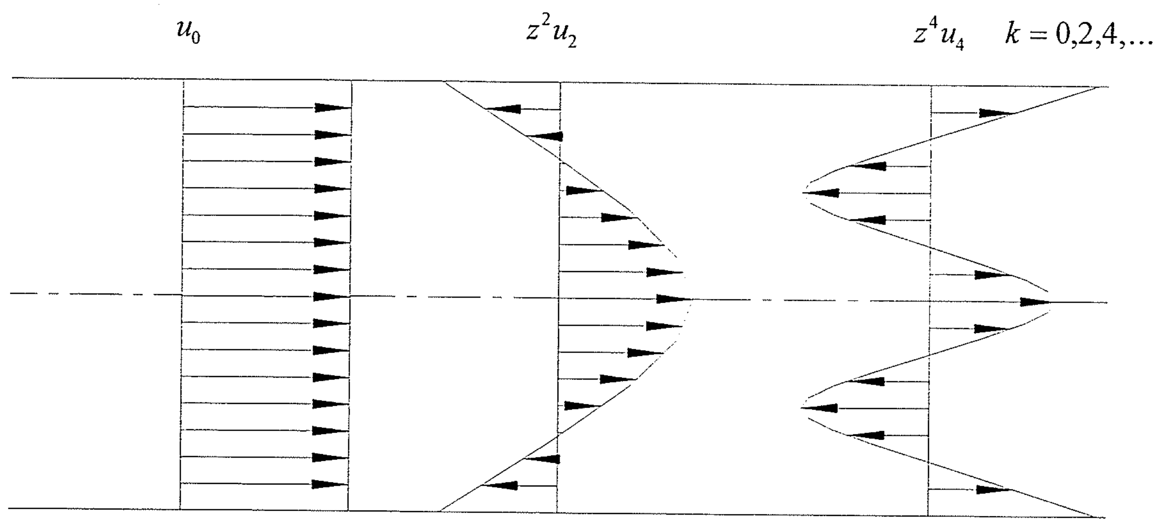

Figure 3.2a: Symmetric or extensional modes of vibration

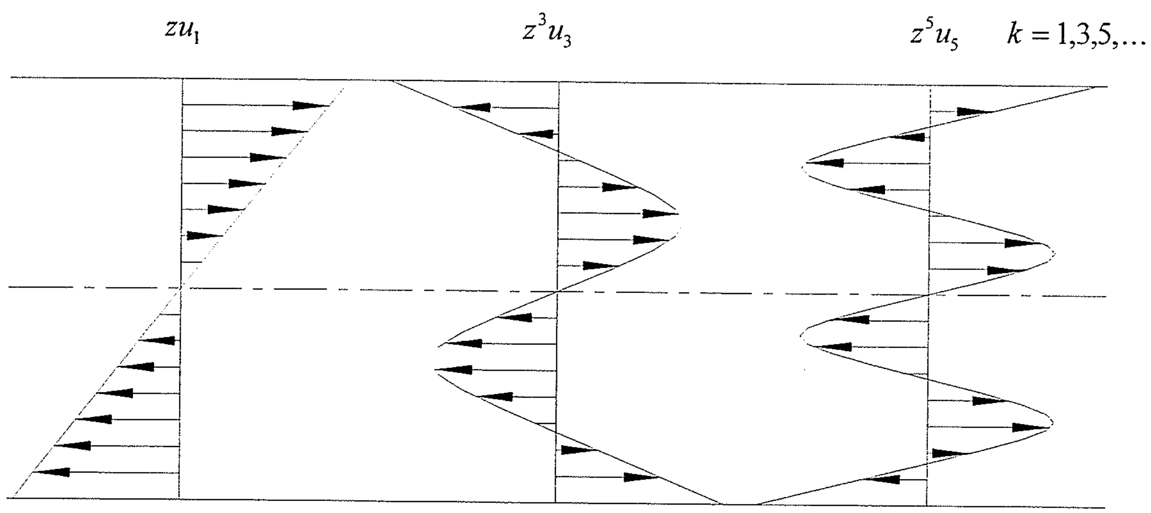

Figure 3.2b: Antisymmetric or flexural modes of vibration

The two ESL theories presented here are first order theories. The series solutions for the displacement are truncated after the first order term, except for displacement in the z direction which is truncated after the initial term. The displacement solutions are proposed in the following form.

$$\tilde{u}(x,y,z,t) = u_0(x,y,t) + zu_1(x,y,t) \tag{3.3a}$$

$$\tilde{v}(x,y,z,t) = v_0(x,y,t) + zv_1(x,y,t) \tag{3.3b}$$

$$\tilde{w}(x,y,z,t) = w_0(x,y,t) \ . \tag{3.3c}$$

These equations of displacements are able to account for the first mode of each symmetric and antisymmetric mode of displacement for $\tilde{u}$ and $\tilde{v}$, accounting for both extensional and flexural displacement. The solution for $\tilde{w}$ accounts for only the first antisymmetric mode of displacement, which enables the plate to move up or down uniformly through the thickness of the plate, with no variation of the $\tilde{w}$ displacement through the thickness. Since plane stress is assumed, and $\tilde{w}$ is not a function of z, thus no strain occurs in the z direction.

Continuity of displacements is maintained because displacement functions are continuous throughout the thickness of the plate. However, stress continuity is maintained only if the plate is homogeneous. Otherwise, solutions for the stresses will have discontinuities at layer interfaces separating layers with different material properties.

3.3 Classical laminated plate theory (CLPT)

The following section developes the classical laminated plate theory for a simply supported laminated composite plate with orthotropic laminates.

3.3.1 Kinematic assumptions

A number of kinematic assumptions are made in the formulation of this theory. The strains and displacements are assumed to be small [37], and the material is assumed to be linear elastic. Plane stress is assumed and the transverse shear stresses are assumed to be zero at the top and bottom surfaces of the plate. The displacement solutions are formulated using the Kirchhoff hypothesis. The Kirchhoff hypothesis assumes that, if the plate is very thin, a line originally straight and perpendicular to the middle surface of the laminate is assumed to remain straight and perpendicular to the middle surface when the laminate is extended or bent, [3]. The requirement that the normal to the middle surface remains straight and normal under deformation is equivalent to ignoring the transverse shear strains in planes perpendicular to the middle surface, [3]. Therefore, the transverse shear stresses σ_{xz} and σ_{yz} are zero. In addition, the length of the transverse normal is assumed to maintain a constant length through the thickness of the plate. This assumption assures that the transverse normal strain ε_z is zero, [3]. A diagram of a small element of the plate, before and after deformation, is shown in Figure 3.3.

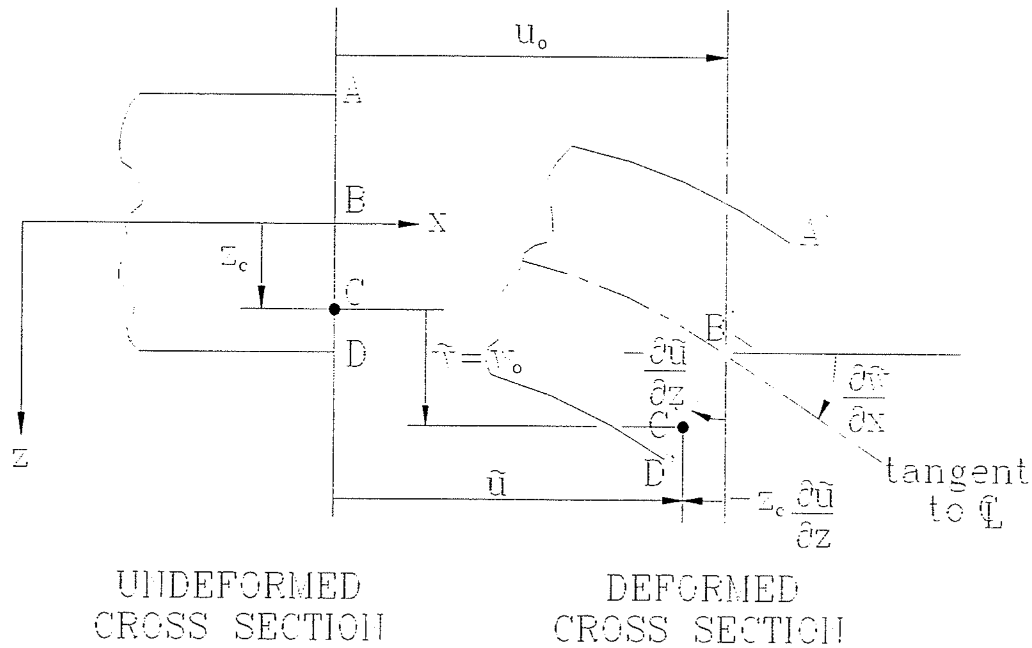

Figure 3.3: Geometry of deformation of plate element in xz plane for CLPT

From the above figure, the displacement $\tilde{u}$ can be seen to take the form

$$\tilde{u}(x,y,z,t) = u_0(x,y,t) + zu_1(x,y,t) \tag{3.4}$$

where u_0 is the distance from point B at the undeformed midplane of the plate to point B at the deformed midplane of the plate. From the above figure, it can also be seen that

$$-\frac{\partial \tilde{u}}{\partial z} = -u_1 = \frac{\partial \tilde{w}}{\partial x} = \frac{\partial w_0}{\partial x} . \tag{3.5}$$

Substituting equation (3.5) into equation (3.4) results in

$$\tilde{u} = u_0 + z\frac{\partial \tilde{u}}{\partial z} = u_0 - z\frac{\partial w_0}{\partial x} . \tag{3.6a}$$

Similarly, in the yz plane, the displacement solution is

$$\tilde{v} = v_0 - z\frac{\partial w_0}{\partial y} \; . \tag{3.6b}$$

Displacement in the z direction is independent of z, so that

$$\tilde{w} = w_o \; . \tag{3.6c}$$

3.3.2 Strain-displacement relations

The strain-displacement relations for 3-D elasticity are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{Bmatrix} . \tag{3.7}$$

Since the $\tilde{w}$ displacement is independent of z, the strain in the z direction, ε_z, is zero. In addition, substituting equation (3.5) in the above strain-displacement relations yields

$$\gamma_{xz} = \frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{w}}{\partial x} = 0 \; . \tag{3.8}$$

Similarly,

$$\gamma_{yz} = \frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{w}}{\partial y} = 0 \; . \tag{3.9}$$

Substituting the displacement solutions into remaining strain-displacement relations yields

$$\{\varepsilon\} = \{\varepsilon_0\} + z\{\kappa\} \; . \tag{3.10}$$

where

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3.11}$$

$$\{\varepsilon_0\} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \tag{3.12}$$

and

$$\{\kappa\} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}. \tag{3.13}$$

The stress-strain relations for an orthotropic material in a state of plane stress are given by, [37]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & \\ Q_{12} & Q_{22} & \\ & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{3.14}$$

The material constants, Q_{ij}, are defined in terms of D_{ij}, Young's moduli and Poisson's ratios in Appendix A. The above constitutive law for the $i\ th$ lamina can be written in the form

$$\{\sigma_i\} = [Q_i]\{\varepsilon\} = [Q_i]\{\varepsilon_0\} + z[Q_i]\{\kappa\} \tag{3.15}$$

where $\{\sigma_i\}$ stands for the stresses in the $i-th$ lamina and $[Q_i]$ stands for the material properties of the $i\ th$ lamina. The strain components $\{\varepsilon_0\}$ and $\{\kappa\}$ are valid for the entire plate.

3.3.3 Governing equations

The governing equations are derived using the principle of virtual displacements. The dynamic version of the principle of virtual displacements is given as, [37]

$$0 = \int_t (\delta V - \delta T + \delta W) dt \tag{3.16}$$

where δW is the virtual work done by the applied forces. The term, δW, is left out of the equation to solve for the solution to the free vibration problem. The term, δV, is the total virtual strain energy and δT is the total kinetic energy; both of which are integrated through the thickness of the laminated plate. Now δV is given by

$$\delta V = \int_t \int_A \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \{\sigma_i\}^T \{\delta\varepsilon\} dz dA dt \tag{3.17}$$

where the stress vector, $\{\sigma_i\}$, is given in equation (3.15) and the virtual strain vector, $\{\delta\varepsilon\}$, is

$$\{\delta\varepsilon\} = \{\delta\varepsilon_0\} + z\{\delta\kappa\} \tag{3.18}$$

The $\{\delta\varepsilon_0\}$ and $\{\delta\kappa\}$ are given below

$$\{\delta\varepsilon_0\} = \begin{Bmatrix} \dfrac{\partial \delta u_0}{\partial x} \\ \dfrac{\partial \delta v_0}{\partial y} \\ \dfrac{\partial \delta u_0}{\partial y} + \dfrac{\partial \delta v_0}{\partial x} \end{Bmatrix} \tag{3.19}$$

$$\{\delta\kappa\} = \begin{Bmatrix} -\dfrac{\partial^2 \delta w_0}{\partial x^2} \\ -\dfrac{\partial^2 \delta w_0}{\partial y^2} \\ -2\dfrac{\partial^2 \delta w_0}{\partial x \partial y} \end{Bmatrix} . \tag{3.20}$$

Substituting equation (3.18) into equation (3.17) yields

$$\delta V = \int_t \int_A \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \left(\{\sigma_i\}^T \{\delta\varepsilon_0\} + z\{\sigma_i\}^T \{\delta\kappa\} \right) dz dA dt . \tag{3.21}$$

Defining the terms, $\{N\}$, for the forces from stresses through the thickness, and, $\{M\}$, for the moments that are produced by stresses through the thickness, and using equation (3.15) for the definition of the stresses yields

$$\{N\}=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\{\sigma_i\}dz=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}([Q_i]\{\varepsilon_0\}+z[Q_i]\{\kappa\})dz=[A]\{\varepsilon_0\}+[B]\{\kappa\} \tag{3.22}$$

$$\{M\}=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}z\{\sigma_i\}dz=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}(z[Q_i]\{\varepsilon_0\}+z^2[Q_i]\{\kappa\})dz=[B]\{\varepsilon_0\}+[C]\{\kappa\} \tag{3.23}$$

where

$$\{N\}^T=\langle N_x \quad N_y \quad N_{xy}\rangle \tag{3.24a}$$

$$\{M\}^T=\langle M_x \quad M_y \quad M_{xy}\rangle \tag{3.24b}$$

and

$$[A]=\sum_{i=1}^{N}(z_{i+1}-z_i)[Q_i] \tag{3.25a}$$

$$[B]=\frac{1}{2}\sum_{i=1}^{N}(z_{i+1}{}^2-z_i{}^2)[Q_i] \tag{3.25b}$$

$$[C]=\frac{1}{3}\sum_{i=1}^{N}(z_{i+1}{}^3-z_i{}^3)[Q_i]\ . \tag{3.25c}$$

Substituting vectors $\{N\}$ and $\{M\}$ into the virtual strain energy equation (3.21), yields

$$\delta V=\int_t\int_A(\{N\}^T\{\delta\varepsilon_0\}+\{M\}^T\{\delta\kappa\})dAdt\ . \tag{3.26}$$

Expanding the above equation into its individual terms

$$\delta V=\int_t\int_A\begin{pmatrix} N_x\dfrac{\partial\delta u_0}{\partial x}+N_y\dfrac{\partial\delta v_0}{\partial y}+N_{xy}\dfrac{\partial\delta u_0}{\partial y}+N_{xy}\dfrac{\partial\delta v_0}{\partial x} \\ -M_x\dfrac{\partial^2\delta w_0}{\partial x^2}-M_y\dfrac{\partial^2\delta w_0}{\partial y^2}-2M_{xy}\dfrac{\partial^2\delta w_0}{\partial x\partial y}\end{pmatrix}dAdt\ . \tag{3.27}$$

Collecting the δ terms yields

$$\delta V=\int_t\int_A\left(\left[N_x\frac{\partial\delta u_0}{\partial x}+N_{xy}\frac{\partial\delta u_0}{\partial y}\right]+\left[N_y\frac{\partial\delta v_0}{\partial y}+N_{xy}\frac{\partial\delta v_0}{\partial x}\right]\right.$$
$$\left.-\left[M_x\frac{\partial^2\delta w_0}{\partial x^2}+M_y\frac{\partial^2\delta w_0}{\partial y^2}+2M_{xy}\frac{\partial^2\delta w_0}{\partial x\partial y}\right]\right)dAdt\ . \tag{3.28}$$

Integrating by parts can be applied in order to factor out the δ terms from the integrals in equation (3.28).

$$a\frac{\partial b}{\partial x}=\frac{\partial(ab)}{\partial x}-b\frac{\partial a}{\partial x} \tag{3.29a}$$

$$a\frac{\partial^2 b}{\partial x^2}=\frac{\partial^2(ab)}{\partial x^2}-2\frac{\partial a}{\partial x}\frac{\partial b}{\partial x}-b\frac{\partial^2 a}{\partial x^2} \tag{3.29b}$$

$$a\frac{\partial^2 b}{\partial x\partial y}=\frac{\partial^2(ab)}{\partial x\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}-\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-b\frac{\partial^2 a}{\partial x\partial y}\ . \tag{3.29c}$$

Applying integration by parts and collecting the coefficients of each of the virtual displacements together yields

$$\delta V=\int_t\int_A\left(\left[\frac{\partial(N_x\delta u_0)}{\partial x}+\frac{\partial(N_{xy}\delta u_0)}{\partial y}\right]+\left[\frac{\partial(N_y\delta v_0)}{\partial y}+N_{xy}\frac{\partial(N_{xy}\delta v_0)}{\partial x}\right]\right.$$
$$-\left[\frac{\partial^2(M_x\delta w_0)}{\partial x^2}-2\frac{\partial M_x}{\partial x}\frac{\partial\delta w_0}{\partial x}+\frac{\partial^2(M_y\delta w_0)}{\partial y^2}-2\frac{\partial M_y}{\partial y}\frac{\partial\delta w_0}{\partial y}\right.$$
$$\left.+2\frac{\partial^2(M_{xy}\delta w_0)}{\partial x\partial y}-2\frac{\partial M_{xy}}{\partial x}\frac{\partial\delta w_0}{\partial y}-2\frac{\partial M_{xy}}{\partial y}\frac{\partial\delta w_0}{\partial x}\right]$$
$$-\delta u_0\left[\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right]-\delta v_0\left[\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}\right]$$
$$\left.+\delta w_0\left[\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}\right]\right)dAdt\ . \tag{3.30}$$

The term, δT, in equation (3.16) is given as

$$\delta T=\int_t\int_A\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i\left(\overset{o}{\tilde{u}}\,\delta\overset{o}{\tilde{u}}+\overset{o}{\tilde{v}}\,\delta\overset{o}{\tilde{v}}+\overset{o}{\tilde{w}}\,\delta\overset{o}{\tilde{w}}\right)dzdAdt \tag{3.31}$$

where the superscript ^{o} symbol stands for $\frac{\partial}{\partial t}$. Substituting the displacement and virtual displacement expressions in equations (3.12), (3.13) and (3.19) as well as equation (3.20) into equation (3.31) yields

$$\delta T = \int\int_{t\,A}\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i\left(\left[\overset{o}{u}_0\,\delta\overset{o}{u}_0+\overset{o}{v}_0\,\delta\overset{o}{v}_0+\overset{o}{w}_0\,\delta\overset{o}{w}_0\right]\right.$$

$$-z\left[\overset{o}{u}_0\frac{\partial\delta\overset{o}{w}_0}{\partial x}+\overset{o}{v}_0\frac{\partial\delta\overset{o}{w}_0}{\partial y}+\frac{\partial\overset{o}{w}_0}{\partial x}\delta\overset{o}{u}_0+\frac{\partial\overset{o}{w}_0}{\partial y}\delta\overset{o}{v}_0\right]$$

$$\left.+z^2\left[\frac{\partial\overset{o}{w}_0}{\partial x}\frac{\partial\delta\overset{o}{w}_0}{\partial x}+\frac{\partial\overset{o}{w}_0}{\partial y}\frac{\partial\delta\overset{o}{w}_0}{\partial y}\right]\right)dzdAdt\ . \tag{3.32}$$

Defining terms pertinent to the inertial constants for the total weight, the first moment of the weight, and the second moment of the weight as I_0, I_1, and I_2 respectively, produces

$$I_0=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i dz=\sum_{i=1}^{N}\rho_i(z_{i+1}-z_i) \tag{3.33a}$$

$$I_1=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i z dz=\frac{1}{2}\sum_{i=1}^{N}\rho_i({z_{i+1}}^2-{z_i}^2) \tag{3.33b}$$

and

$$I_2=\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i z^2 dz=\frac{1}{3}\sum_{i=1}^{N}\rho_i({z_{i+1}}^3-{z_i}^3)\ . \tag{3.33c}$$

Substituting the constants defined in equations (3.33a) through (3.33c) into equation (3.32) and integrating by parts according to equations (3.29a) through (3.29c) (in order to factor out the δ terms) yields

$$\delta T=\int\int_{t\,A}\left(I_0\left[\left(\frac{\partial(\overset{o}{u}_0\,\delta u_0)}{\partial t}+\frac{\partial(\overset{o}{v}_0\,\delta v_0)}{\partial t}+\frac{\partial(\overset{o}{w}_0\,\delta w_0)}{\partial t}\right)-\left(\overset{oo}{u}_0\,\delta u_0+\overset{oo}{v}_0\,\delta v_0+\overset{oo}{w}_0\,\delta w_0\right)\right]\right.$$

$$-I_1\left[\left(\frac{\partial^2(\overset{o}{u}_0\,\delta w_0)}{\partial x\partial t}-\frac{\partial\delta w_0}{\partial x}\overset{oo}{u}_0-\delta\overset{o}{w}_0\frac{\partial\overset{o}{u}_0}{\partial x}+\frac{\partial^2(\overset{o}{v}_0\,\delta w_0)}{\partial y\partial t}+\ldots\right.\right.$$

$$\cdots - \frac{\partial \delta w_0}{\partial y} \overset{oo}{v_0} - \delta \overset{o}{w_0} \frac{\partial \overset{o}{v_0}}{\partial y} + \frac{\partial (\frac{\partial w_0}{\partial x} \delta \overset{o}{u_0})}{\partial t} + \frac{\partial (\frac{\partial w_0}{\partial y} \delta \overset{o}{v_0})}{\partial t} \Bigg)$$

$$- \delta w_0 \left(\frac{\partial \overset{oo}{u_0}}{\partial x} + \frac{\partial \overset{oo}{v_0}}{\partial y} \right) - \delta u_0 \left(\frac{\partial \overset{oo}{w_0}}{\partial x} \right) - \delta v_0 \left(\frac{\partial \overset{oo}{w_0}}{\partial y} \right) \Bigg]$$

$$+ I_2 \Bigg[\Bigg(\frac{\partial (\frac{\partial \overset{o}{w_0}}{\partial x} \delta w_0)}{\partial x \partial t} - \frac{\partial^2 \overset{o}{w_0}}{\partial x^2} \delta \overset{o}{w_0} - \frac{\partial \overset{oo}{w_0}}{\partial x} \frac{\partial \delta w_0}{\partial x}$$

$$+ \frac{\partial (\frac{\partial \overset{o}{w_0}}{\partial y} \delta w_0)}{\partial y \partial t} - \frac{\partial^2 \overset{o}{w_0}}{\partial y^2} \delta \overset{o}{w_0} - \frac{\partial \overset{oo}{w_0}}{\partial y} \frac{\partial \delta w_0}{\partial y} \Bigg)$$

$$- \delta w_0 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) \Bigg] \Bigg) dA dt \ . \tag{3.34}$$

The equations of motion are obtained by setting each coefficient of δu_0, δv_0, and δw_0 to zero in equations (3.30) and (3.34), where the remaining terms in both equations are boundary and initial conditions. The following equations of motion are obtained from each coefficient.

δu_0:
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right) \tag{3.35a}$$

δv_0:
$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial y} \right) \tag{3.35b}$$

δw_0:
$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} =$$

$$I_0 \frac{\partial^2 w_0}{\partial t^2} + I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) - I_2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) . \tag{3.35c}$$

Expanding the expressions for N_x, N_{xy}, N_y, M_x, M_{xy}, and M_y given in equations (3.22) and (3.23) and taking derivatives of these expressions yields the three differential equations of motion in terms of u_0, v_0, and w_0

$$A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right)+A_{12}\left(\frac{\partial^2 v_0}{\partial x\partial y}\right)+B_{11}\left(-\frac{\partial^3 w_0}{\partial x^3}\right)+B_{12}\left(-\frac{\partial^3 w_0}{\partial x\partial y^2}\right)$$

$$+A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}\right)+B_{66}\left(-2\frac{\partial^3 w_0}{\partial x\partial y^2}\right)=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \tag{3.36a}$$

$$A_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)+B_{66}\left(-2\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+A_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}\right)$$

$$+B_{12}\left(-\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+B_{22}\left(-\frac{\partial^3 w_0}{\partial y^3}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right) \tag{3.36b}$$

$$B_{11}\left(\frac{\partial^3 u_0}{\partial x^3}\right)+B_{12}\left(\frac{\partial^3 v_0}{\partial x^2\partial y}\right)+C_{11}\left(-\frac{\partial^4 w_0}{\partial x^4}\right)+C_{12}\left(-\frac{\partial^4 w_0}{\partial x^2\partial y^2}\right)$$

$$+2B_{66}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\partial y}\right)+4C_{66}\left(-\frac{\partial^4 w_0}{\partial x^2\partial y^2}\right)$$

$$+B_{12}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}\right)+B_{22}\left(\frac{\partial^3 v_0}{\partial y^3}\right)+C_{12}\left(-\frac{\partial^4 w_0}{\partial x^2\partial y^2}\right)+C_{22}\left(-\frac{\partial^4 w_0}{\partial y^4}\right)$$

$$=I_0\frac{\partial^2 w_0}{\partial t^2}+I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)-I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right). \tag{3.36c}$$

3.3.4 Boundary conditions and displacement solutions

The boundary conditions governing the displacement functions of x and y for a simply supported rectangular plate may be specified as, [45]:

$$\text{at } x=0 \text{ and } a;\quad \sigma_x=0,\quad \tilde{w}=0 \text{ and } \tilde{v}=0,$$

$$\text{at } y=0 \text{ and } b;\quad \sigma_y=0,\quad \tilde{w}=0 \text{ and } \tilde{u}=0. \tag{3.37}$$

Solutions for the displacement expression components u_0, v_0, w_0 which satisfy the above boundary conditions have the form

$$u_0=\hat{U}\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \tag{3.38a}$$

$$v_0 = \hat{V}\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega t} \tag{3.38b}$$

$$w_0 = \hat{W}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \tag{3.38c}$$

where $\hat{M}$ and $\hat{N}$ are defined as follows

$$\hat{M} = \frac{m\pi}{a}; \quad \hat{N} = \frac{n\pi}{b}; \quad \text{and} \quad m = 1,2,3,\ldots; \qquad n = 1,2,3,\ldots \tag{3.39}$$

The $\hat{U}$, $\hat{V}$, and $\hat{W}$ are amplitude coefficients and are the eigenvectors which have to be solved. Substituting the displacement solutions in equations (3.38a) through (3.38c) into the differential equations (3.36a) through (3.36c) yields the following three equations.

$$\begin{aligned}&-\cos(\hat{M}x)\sin(\hat{N}y)\Big[A_{11}\left(\hat{M}^2\hat{U}\right)+A_{12}\left(\hat{M}\hat{N}\hat{V}\right)+B_{11}\left(-\hat{M}^3\hat{W}\right)+B_{12}\left(-\hat{M}\hat{N}^2\hat{W}\right)\\&\qquad +A_{66}\left(\hat{N}^2\hat{U}+\hat{M}\hat{N}\hat{V}\right)+B_{66}\left(-2\hat{M}\hat{N}^2\hat{W}\right)=I_0\left(\omega^2\hat{U}\right)-I_1\left(\omega^2\hat{M}\hat{W}\right)\Big]\end{aligned} \tag{3.40a}$$

$$\begin{aligned}&-\sin(\hat{M}x)\cos(\hat{N}y)\Big[A_{66}\left(\hat{M}\hat{N}\hat{U}+\hat{M}^2\hat{V}\right)+B_{66}\left(-2\hat{M}^2\hat{N}\hat{W}\right)+A_{12}\left(\hat{M}\hat{N}\hat{U}\right)+A_{22}\left(\hat{N}^2\hat{V}\right)\\&\qquad +B_{12}\left(-\hat{M}^2\hat{N}\hat{W}\right)+B_{22}\left(-\hat{N}^3\hat{W}\right)=I_0\left(\omega^2\hat{V}\right)-I_1\left(\omega^2\hat{N}\hat{W}\right)\Big]\end{aligned} \tag{3.40b}$$

$$\begin{aligned}&-\sin(\hat{M}x)\sin(\hat{N}y)\Big[B_{11}\left(-\hat{M}^3\hat{U}\right)+B_{12}\left(-\hat{M}^2\hat{N}\hat{V}\right)+C_{11}\left(\hat{M}^4\hat{W}\right)+C_{12}\left(\hat{M}^2\hat{N}^2\hat{W}\right)\\&\qquad +2B_{66}\left(-\hat{M}\hat{N}^2\hat{U}-\hat{M}^2\hat{N}\hat{V}\right)+4C_{66}\left(\hat{M}^2\hat{N}^2\hat{W}\right)+B_{12}\left(-\hat{M}\hat{N}^2\hat{U}\right)+B_{22}\left(-\hat{N}^3\hat{V}\right)\\&\quad +C_{12}\left(\hat{M}^2\hat{N}^2\hat{W}\right)+C_{22}\left(\hat{N}^4\hat{W}\right)=I_0\left(\omega^2\hat{W}\right)+I_1\omega^2\left(-\hat{M}\hat{U}-\hat{N}\hat{V}\right)+I_2\omega^2\left(\hat{M}^2\hat{W}+\hat{N}^2\hat{W}\right)\Big].\end{aligned} \tag{3.40c}$$

Cancelling trigonometric terms and writing the above equations in matrix form produces the generalized eigenvalue problem that yields the solutions for the natural frequencies, ω,

$$[K]\{\vec{d}\} = \omega^2[M]\{\vec{d}\} \tag{3.41}$$

Vector, $\{\vec{d}\}$, and matrices, $[K]$ and $[M]$, are given as

$$\{\vec{d}\} = \begin{Bmatrix}\hat{U}\\ \hat{V}\\ \hat{W}\end{Bmatrix} \tag{3.42}$$

$$[K]=\begin{bmatrix} A_{11}\hat{M}^2+A_{66}\hat{N}^2 & \hat{M}\hat{N}(A_{12}+A_{66}) & -B_{11}\hat{M}^3-\hat{M}\hat{N}^2(B_{12}+2B_{66}) \\ \hat{M}\hat{N}(A_{12}+A_{66}) & A_{66}\hat{M}^2+\hat{N}^2A_{22} & -B_{22}\hat{N}^3-\hat{M}^2\hat{N}(B_{12}+2B_{66}) \\ -B_{11}\hat{M}^3-\hat{M}\hat{N}^2(B_{12}+2B_{66}) & -B_{22}\hat{N}^3-\hat{M}^2\hat{N}(B_{12}+2B_{66}) & C_{11}\hat{M}^4+\hat{M}^2\hat{N}^2(2C_{12}+4C_{66})+C_{22}\hat{N}^4 \end{bmatrix} \tag{3.43}$$

and

$$[M]=\begin{bmatrix} I_0 & 0 & -I_1\hat{M} \\ 0 & I_0 & -I_1\hat{N} \\ -I_1\hat{M} & -I_1\hat{N} & I_0+I_2\left(\hat{M}^2+\hat{N}^2\right) \end{bmatrix}. \tag{3.44}$$

The three by three eigenvalue system yields three different natural frequencies, ω. Denoting ω_n as the $n\ th$ root of the three roots of ω, there is a unique eigenvector for the amplitude coefficients $\widehat{U}_n$, $\widehat{V}_n$, and $\widehat{W}_n$ and displacements $\widetilde{u}_n$, $\widetilde{v}_n$, and $\widetilde{w}_n$ for each $n\ th$ frequency ω_n. The solutions for the functions u_o, v_o and w_o, which are contained in equations (3.6a) through (3.6c) for each natural frequency ω_n, are

$$u_{0n}=\left(\widehat{U}_n\right)\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.45a}$$

$$v_{0n}=\left(\widehat{V}_n\right)\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t} \tag{3.45b}$$

$$w_{0n}=\left(\widehat{W}_n\right)\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t}\ . \tag{3.45c}$$

Substituting the above relations into equations (3.6a) through (3.6c) yields the displacement solutions for each natural frequency, ω_n, as

$$\widetilde{u}_n=\left(\widehat{U}_n-z\hat{M}\widehat{W}_n\right)\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.46a}$$

$$\widetilde{v}_n=\left(\widehat{V}_n-z\hat{N}\widehat{W}_n\right)\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t} \tag{3.46b}$$

$$\widetilde{w}_n=\left(\widehat{W}_n\right)\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t}\ . \tag{3.46c}$$

3.3.5 Calculation of stresses

Substituting the strain-displacement equations (3.7) into the stress-strain relations of equation (3.14) yields the following equations for the stresses

$$\sigma_{xn}=Q_{11i}\frac{\partial u_0}{\partial x}+Q_{12i}\frac{\partial v_0}{\partial y}-z\left(Q_{11i}\frac{\partial^2 w_0}{\partial x^2}+Q_{12i}\frac{\partial^2 w_0}{\partial y^2}\right) \tag{3.47a}$$

$$\sigma_{y_n} = Q_{12i}\frac{\partial u_0}{\partial x} + Q_{22i}\frac{\partial v_0}{\partial y} - z\left(Q_{12i}\frac{\partial^2 w_0}{\partial x^2} + Q_{22i}\frac{\partial^2 w_0}{\partial y^2}\right) \tag{3.47b}$$

$$\sigma_{xy_n} = Q_{66i}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) - 2zQ_{66i}\frac{\partial^2 w_0}{\partial x \partial y}. \tag{3.47c}$$

The subscript n denotes the stress for the $n\,th$ eigenvector and subscript i denotes the $i\,th$ layer or lamina. The stresses are not continuous due to the difference in the material properties, Q_{ij}, at the layer or lamina interfaces. Substituting equation (3.45) into the above stress equations yields the following expressions for the stresses at the $n\,th$ frequency, ω_n, in the $i\,th$ layer or lamina.

$$\sigma_{xn} = -\left\{Q_{11i}\hat{M}\hat{U}_n + Q_{12i}\hat{N}\hat{V}_n - z\left(Q_{11i}\hat{M}^2 + Q_{12i}\hat{N}^2\right)\hat{W}_n\right\}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.48a}$$

$$\sigma_{xn} = -\left\{Q_{12i}\hat{M}\hat{U}_n + Q_{22i}\hat{N}\hat{V}_n - z\left(Q_{12i}\hat{M}^2 + Q_{22i}\hat{N}^2\right)\hat{W}_n\right\}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.48b}$$

$$\sigma_{xn} = Q_{66i}\left\{\hat{N}\hat{U}_n + \hat{M}\hat{V}_n - 2z\hat{M}\hat{N}\hat{W}_n\right\}\cos(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t}\ . \tag{3.48c}$$

3.4 First order shear deformation theory (FSDT)

The following section develops the first order shear deformation theory for a simply supported, laminated composite plate with orthotropic laminates.

3.4.1 Kinematic assumptions

The only difference between the first order shear deformation theory and the classical laminated plate theory is that the former does not use the Kirchhoff hypothesis. Normals to the midsurface after deformation do not remain perpendicular to the midsurface of the plate. Therefore, the transverse shear strains, γ_{xz} and γ_{yz}, are not zero but constant throughout the cross-section. However, the length of the transverse normal is still assumed to be constant through the thickness of the plate so that the transverse normal strain, ε_z, is zero, [3]. Other assumptions (strains and displacements are assumed to be small, [37], and the material is assumed to be linear elastic) still hold. Plane stress is assumed and the transverse shear stresses are assumed to be zero at the top and bottom surfaces of the plate. A diagram of a small element of the plate, before and after deformation, is shown in Figure 3.4.

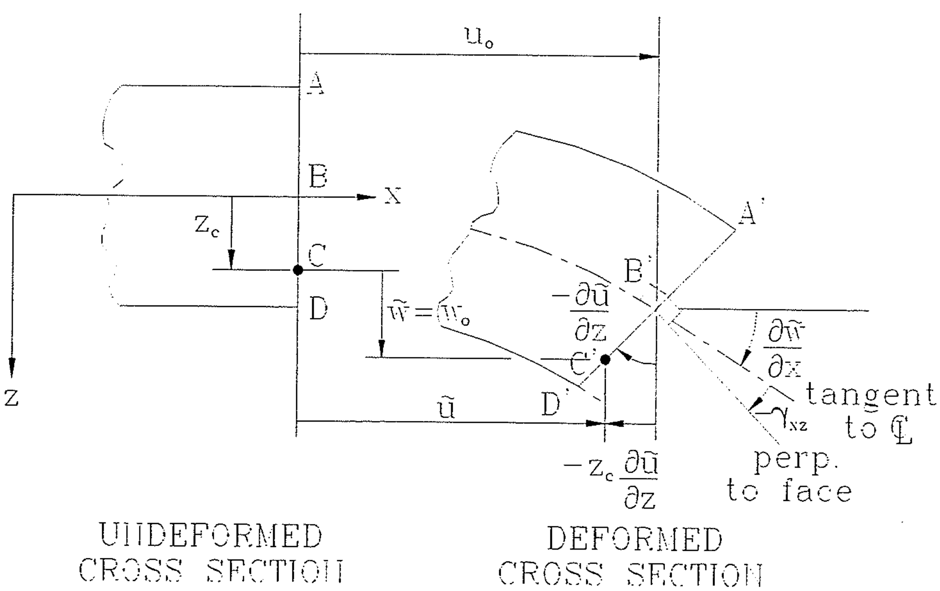

Figure 3.4: Geometry of deformation of plate element in xz plane for FSDT

From Figure 3.4, the displacement $\tilde{u}$ can be seen the form

$$\tilde{u}(x,y,z,t) = u_0(x,y,t) + zu_1(x,y,t) \tag{3.49}$$

where u_0 is the distance from point B at the undeformed midplane of the plate to point B' at the deformed midplane of the plate. The displacement solution in the xz plane is

$$\tilde{u} = u_0 + z\frac{\partial \tilde{u}}{\partial z} = u_0 + z\psi_x . \tag{3.50a}$$

Similarly, in the yz plane, the displacement solution is

$$\tilde{v} = v_0 + z\psi_y . \tag{3.50b}$$

Displacement in the z direction is independent of z. Thus

$$\tilde{w} = w_o . \tag{3.50c}$$

Slopes ψ_x and ψ_y are defined as

$$\psi_x = u_1 = \frac{\partial \tilde{u}}{\partial z} \tag{3.51a}$$

and

$$\psi_y = v_1 = \frac{\partial \tilde{v}}{\partial z} . \tag{3.51b}$$

3.4.2 Strain-displacement relations

The strain-displacement relations for 3-D elasticity are provided by equation (3.7). Since displacement $\tilde{w}$ is independent of z, the strain in the z direction, ε_z, is zero. However, the strains γ_{xz} and γ_{yz} are not zero because, from the above diagram, the derivatives $\frac{\partial \tilde{u}}{\partial z}$ and $\frac{\partial \tilde{w}}{\partial x}$ are not equal in magnitude. Substituting the displacement solutions (3.50a) through (3.50c) into the strain-displacement relations yields

$$\{\varepsilon\} = \{\varepsilon_0\} + z\{\kappa\} \tag{3.52}$$

where

$$\{\varepsilon\}^T = \langle \varepsilon_x \quad \varepsilon_y \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy} \rangle \tag{3.53}$$

$$\{\varepsilon_0\}^T = \left\langle \frac{\partial u_0}{\partial x}, \ \frac{\partial v_0}{\partial y}, \ \psi_y + \frac{\partial w_0}{\partial y}, \ \psi_x + \frac{\partial w_0}{\partial x}, \ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right\rangle \tag{3.54}$$

$$\{\kappa\}^T = \left\langle \frac{\partial\psi_x}{\partial x},\ \frac{\partial\psi_y}{\partial y},\ 0,\ 0,\ \frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x} \right\rangle. \tag{3.55}$$

Stress-strain relations for an orthotropic material in a state of plane stress are defined more conveniently in Appendix A. They are given as, [37]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & & & \\ Q_{12} & Q_{22} & & & \\ & & Q_{44} & & \\ & & & Q_{55} & \\ & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}. \tag{3.56}$$

The above constitutive law for the $i\ th$ lamina can be written in the form

$$\{\sigma_i\} = [Q_i]\{\varepsilon\} = [Q_i]\{\varepsilon_0\} + z[Q_i]\{\kappa\} \tag{3.57}$$

where $\{\sigma_i\}$ stands for the stresses in the $i\ th$ lamina and $[Q_i]$ stands for the material properties of the $i\ th$ lamina. The strain components, $\{\varepsilon_0\}$ and $\{\kappa\}$, are valid for the entire laminated plate.

3.4.3 Governing equations

The governing equations are derived using the principle of virtual displacements. The dynamic version of the principle of virtual displacements is given as, [37]

$$0 = \int_t (\delta V - \delta T + \delta W)dt \tag{3.58}$$

where δV is given by

$$\delta V = \int_t \int_A \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \{\sigma_i\}^T \{\delta\varepsilon\} dz dA dt\,. \tag{3.59}$$

The stress vector $\{\sigma_i\}$ is given in equation (3.57) and the virtual strain vector $\{\delta\varepsilon\}$ is given as

$$\{\delta\varepsilon\} = \{\delta\varepsilon_0\} + z\{\delta\kappa\} \tag{3.60}$$

where $\{\delta\varepsilon_0\}$ and $\{\delta\kappa\}$ are given as the following:

$$\{\delta\varepsilon_0\}^T = \left\langle \frac{\partial\delta u_0}{\partial x},\ \frac{\partial\delta v_0}{\partial y},\ \delta\psi_y + \frac{\partial\delta w_0}{\partial y},\ \delta\psi_x + \frac{\partial\delta w_0}{\partial x},\ \frac{\partial\delta u_0}{\partial y} + \frac{\partial\delta v_0}{\partial x} \right\rangle \quad (3.61)$$

$$\{\delta\kappa\} = \left\langle \frac{\partial\delta\psi_x}{\partial x},\ \frac{\partial\delta\psi_y}{\partial y},\ 0,\ 0,\ \frac{\partial\delta\psi_x}{\partial y} + \frac{\partial\delta\psi_y}{\partial x} \right\rangle. \quad (3.62)$$

Substituting the delta strain equation (3.60) into equation (3.59) yields

$$\delta V = \int_t \int_A \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \left(\{\sigma_i\}^T \{\delta\varepsilon_0\} + z\{\sigma_i\}^T \{\delta\kappa\} \right) dz dA dt\,. \quad (3.63)$$

Defining terms $\{N\}$ for the forces arising from the stresses through the thickness, and $\{M\}$ for the moments arising from stresses through the thickness, and using equation (3.57) for the definition of the stresses yields

$$\{N\} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \{\sigma_i\} dz = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \left([Q_i]\{\varepsilon_0\} + z[Q_i]\{\kappa\} \right) dz = [A]\{\varepsilon_0\} + [B]\{\kappa\} \quad (3.64a)$$

$$\{M\} = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} z\{\sigma_i\} dz = \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \left(z[Q_i]\{\varepsilon_0\} + z^2[Q_i]\{\kappa\} \right) dz = [B]\{\varepsilon_0\} + [C]\{\kappa\} \quad (3.64b)$$

where

$$\{N\}^T = \langle N_x \quad N_y \quad N_{yz} \quad N_{xz} \quad N_{xy} \rangle \quad (3.65a)$$

$$\{M\}^T = \langle M_x \quad M_y \quad M_{yz} \quad M_{xz} \quad M_{xy} \rangle \quad (3.65b)$$

and

$$[A] = \sum_{i=1}^{N} (z_{i+1} - z_i)[Q_i] \quad (3.66a)$$

$$[B] = \frac{1}{2}\sum_{i=1}^{N} \left(z_{i+1}{}^2 - z_i{}^2 \right)[Q_i] \quad (3.66b)$$

$$[C] = \frac{1}{3}\sum_{i=1}^{N} \left(z_{i+1}{}^3 - z_i{}^3 \right)[Q_i]. \quad (3.66c)$$

3.4.4 Shear correction coefficient

The transverse shear strains, γ_{xz} and γ_{yz}, are constant so that, from the stress-strain relations in equation (3.56), the transverse stresses, σ_{xz} and σ_{yz}, will be constant. However, from the elementary theory of homogeneous beams, the transverse shear stress varies quadratically through the thickness. The discrepancy between the actual shear stress predicted by 3-D elasticity and the constant shear stress predicted by the first order shear deformation theory is commonly corrected by using a shear correction coefficient, K_s. The shear correction coefficient is multiplied to shear force resultants N_{xz} and N_{yz}, [37]. Defining a new force vector $\{N^*\}$, we have

$$\{N^*\}^T = \langle N_x \quad N_y \quad \tilde{Q}_{yz} \quad \tilde{Q}_{xz} \quad N_{xy} \rangle \tag{3.67}$$

where the shear force components $\tilde{Q}_{xz}$ and $\tilde{Q}_{yz}$ are defined as

$$\begin{Bmatrix} \tilde{Q}_{xz} \\ \tilde{Q}_{yz} \end{Bmatrix} = K_s \begin{Bmatrix} N_{xz} \\ N_{yz} \end{Bmatrix} = K_s \sum_{i=1}^{N} \int_{z_i}^{z_{i+1}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz . \tag{3.68}$$

The shear correction factor is given by Reddy [37] as

$$K_s = \frac{5}{6} . \tag{3.69}$$

Substituting the vectors $\{N^*\}$ and $\{M\}$ into the virtual strain energy equation (3.63), gives

$$\delta V = \int_t \int_A \left(\{N^*\}^T \{\delta\varepsilon_0\} + \{M\}^T \{\delta\kappa\} \right) dA dt . \tag{3.70}$$

Expanding equation (3.70) into its individual terms, applying integration by parts, and collecting virtual terms yields the following equation (3.71)

$$\delta V = \int_t \int_A \left(\left[\frac{\partial (N_x \delta u_0)}{\partial x} + \frac{\partial (N_{xy} \delta u_0)}{\partial y} \right] - \delta u_0 \left[\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right] \right.$$

$$+ \left[\frac{\partial (N_y \delta v_0)}{\partial y} + \frac{\partial (N_{xy} \delta v_0)}{\partial x} \right] - \delta v_0 \left[\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right] + \ldots$$

$$\ldots+\left[\frac{\partial(\tilde{Q}_{yz}\delta w_0)}{\partial y}+\frac{\partial(\tilde{Q}_{xz}\delta w_0)}{\partial x}\right]-\delta w_0\left[\frac{\partial\tilde{Q}_{yz}}{\partial y}+\frac{\partial\tilde{Q}_{xz}}{\partial x}\right]$$

$$+\left[\frac{\partial(M_x\delta\psi_x)}{\partial x}+\frac{\partial(M_{xy}\delta\psi_x)}{\partial y}\right]-\delta\psi_x\left[-\tilde{Q}_{xz}+\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right]$$

$$+\left[\frac{\partial(M_y\delta\psi_y)}{\partial y}+\frac{\partial(M_{xy}\delta\psi_y)}{\partial x}\right]-\delta\psi_y\left[-\tilde{Q}_{yz}+\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right]\Bigg)dAdt\,. \qquad (3.71)$$

Using displacement and virtual displacement expressions contained in equations (3.52) and (3.60), the term δT in equation (3.58) is given as

$$\delta T=\int_t\int_A\sum_{i=1}^{N}\int_{z_i}^{z_{i+1}}\rho_i\left(\left[\dot{u}_0\,\delta\dot{u}_0+\dot{v}_0\,\delta\dot{v}_0+\dot{w}_0\,\delta\dot{w}_0\right]\right.$$

$$+z\left[\dot{u}_0\,\delta\dot{\psi}_x+\dot{\psi}_x\,\delta\dot{u}_0+\dot{v}_0\,\delta\dot{\psi}_y+\dot{\psi}_y\,\delta\dot{v}_0\right]$$

$$\left.+z^2\left[\dot{\psi}_x\,\delta\dot{\psi}_x+\dot{\psi}_y\,\delta\dot{\psi}_y\right]\right)dzdAdt\,. \qquad (3.72)$$

The inertial constants for the total weight, first moment of weight, and second moment of weight (given by I_0, I_1 and I_2, respectively) have been defined in equations (3.33a) through (3.33c). Substituting these constants and applying integration by parts to the above equation yields

$$\delta T=\int_t\int_A\left(I_0\left[\left(\frac{\partial(\dot{u}_0\,\delta u_0)}{\partial t}+\frac{\partial(\dot{v}_0\,\delta v_0)}{\partial t}+\frac{\partial(\dot{w}_0\,\delta w_0)}{\partial t}\right)-\left(\ddot{u}_0\,\delta u_0+\ddot{v}_0\,\delta v_0+\ddot{w}_0\,\delta w_0\right)\right]\right.$$

$$+I_1\left[\left(\frac{\partial(\dot{u}_0\,\delta\psi_x)}{\partial t}+\frac{\partial(\dot{\psi}_x\,\delta u_0)}{\partial t}+\frac{\partial(\dot{v}_0\,\delta\psi_y)}{\partial t}+\frac{\partial(\dot{\psi}_y\,\delta v_0)}{\partial t}\right)\right.$$

$$\left.-\left(\ddot{u}_0\,\delta\psi_x+\ddot{\psi}_x\,\delta u_0+\ddot{v}_0\,\delta\psi_y+\ddot{\psi}_y\,\delta v_0\right)\right]$$

$$\left.+I_2\left[\left(\frac{\partial(\dot{\psi}_x\,\delta\psi_x)}{\partial t}+\frac{\partial(\dot{\psi}_y\,\delta\psi_y)}{\partial t}\right)-\left(\ddot{\psi}_x\,\delta\psi_x+\ddot{\psi}_y\,\delta\psi_y\right)\right]\right)dAdt\,. \qquad (3.73)$$

The equations of motion are obtained by setting the coefficients of δu_0, δv_0, and δw_0 to zero in equations (3.71) and (3.73), thus

$$\delta u_0: \qquad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_x}{\partial t^2} \tag{3.74a}$$

$$\delta v_0: \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_y}{\partial t^2} \tag{3.74b}$$

$$\delta w_0: \qquad \frac{\partial \tilde{Q}_{xz}}{\partial x} + \frac{\partial \tilde{Q}_{yz}}{\partial y} = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{3.74c}$$

$$\delta \psi_x: \qquad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - \tilde{Q}_{xz} = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \psi_x}{\partial t^2} \tag{3.74d}$$

$$\delta \psi_y: \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - \tilde{Q}_{yz} = I_1 \frac{\partial^2 v_0}{\partial t^2} + I_2 \frac{\partial^2 \psi_y}{\partial t^2} \tag{3.74e}$$

Expanding the expressions for terms N, $\tilde{Q}$, and M given in equations (3.64a), (3.64b) and (3.67), and applying the required derivatives in the above equations, yields the five differential equations of motion in terms of u_0, v_0, and w_0 as

$$A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + A_{12}\left(\frac{\partial^2 v_0}{\partial x \partial y}\right) + B_{11}\left(\frac{\partial^2 \psi_x}{\partial x^2}\right) + B_{12}\left(\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + A_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right)$$

$$+ B_{66}\left(\frac{\partial^2 \psi_x}{\partial y^2}\right) + B_{66}\left(\frac{\partial^2 \psi_y}{\partial x \partial y}\right) = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_x}{\partial t^2} \tag{3.75a}$$

$$A_{66}\left(\frac{\partial^2 u_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial x^2}\right) + B_{66}\left(\frac{\partial^2 \psi_x}{\partial x \partial y}\right) + B_{66}\left(\frac{\partial^2 \psi_y}{\partial x^2}\right) + A_{12}\left(\frac{\partial^2 u_0}{\partial x \partial y}\right) + A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}\right)$$

$$+ B_{12}\left(\frac{\partial^2 \psi_x}{\partial x \partial y}\right) + B_{22}\left(\frac{\partial^2 \psi_y}{\partial y^2}\right) = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_y}{\partial t^2} \tag{3.75b}$$

$$K_s A_{55} \frac{\partial \psi_x}{\partial x} + K_s A_{55} \frac{\partial^2 w_0}{\partial x^2} + K_s A_{44} \frac{\partial \psi_x}{\partial y} + K_s A_{44} \frac{\partial^2 w_0}{\partial y^2} = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{3.75c}$$

$$B_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + B_{12}\left(\frac{\partial^2 v_0}{\partial x \partial y}\right) + C_{11}\left(\frac{\partial^2 \psi_x}{\partial x^2}\right) + C_{12}\left(\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + B_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right)$$

$$+ C_{66}\left(\frac{\partial^2 \psi_x}{\partial y^2}\right) + C_{66}\left(\frac{\partial^2 \psi_y}{\partial x \partial y}\right) - K_s A_{55} \psi_x - K_s A_{55} \frac{\partial w_0}{\partial x} = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \psi_x}{\partial t^2} \tag{3.75d}$$

$$B_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\right)+C_{66}\left(\frac{\partial^2 \psi_x}{\partial x\partial y}\right)+C_{66}\left(\frac{\partial^2 \psi_y}{\partial x^2}\right)+B_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}\right)+B_{22}\left(\frac{\partial^2 v_0}{\partial y^2}\right)$$

$$+C_{12}\left(\frac{\partial^2 \psi_x}{\partial x\partial y}\right)+C_{22}\left(\frac{\partial^2 \psi_y}{\partial y^2}\right)-K_sA_{44}\psi_y-K_sA_{44}\frac{\partial w_0}{\partial y}=I_1\frac{\partial^2 v_0}{\partial t^2}+I_2\frac{\partial^2 \psi_y}{\partial t^2}. \quad (3.75e)$$

3.4.5 Boundary conditions and displacement solutions

Proposing solutions for the displacement and slope expression components u_0, v_0, w_0, ψ_x, and ψ_y which satisfy the boundary conditions in equation (3.37) yields

$$u_0=\hat{U}\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \quad (3.76a)$$

$$v_0=\hat{V}\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega t} \quad (3.76b)$$

$$w_0=\hat{W}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \quad (3.76c)$$

$$\psi_x=\hat{\Psi}_x\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \quad (3.76d)$$

$$\psi_y=\hat{\Psi}_y\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega t}. \quad (3.76e)$$

The $\hat{U}$, $\hat{V}$, $\hat{W}$, $\hat{\Psi}_x$, and $\hat{\Psi}_y$ are the amplitude coefficients and they are the eigenvectors that have to be determined. Substituting the displacement solutions in equations (3.76a) through (3.76e) into the differential equations (3.75a) through (3.75e) yields the following:

$$-\cos(\hat{M}x)\sin(\hat{N}y)\Big[A_{11}\left(\hat{M}^2\hat{U}\right)+A_{12}\left(\hat{M}\hat{N}\hat{V}\right)+B_{11}\left(\hat{M}^2\hat{\Psi}_x\right)+B_{12}\left(\hat{M}\hat{N}\hat{\Psi}_y\right)$$

$$+A_{66}\left(\hat{N}^2\hat{U}+\hat{M}\hat{N}\hat{V}\right)+B_{66}\left(\hat{N}^2\hat{\Psi}_x+\hat{M}\hat{N}\hat{\Psi}_y\right)=I_0\left(\omega^2\hat{U}\right)+I_1\left(\omega^2\hat{\Psi}_x\right)\Big] \quad (3.77a)$$

$$-\sin(\hat{M}x)\cos(\hat{N}y)\Big[A_{66}\left(\hat{M}\hat{N}\hat{U}+\hat{M}^2\hat{V}\right)+B_{66}\left(\hat{M}\hat{N}\hat{\Psi}_x+\hat{M}^2\hat{\Psi}_y\right)+A_{12}\left(\hat{M}\hat{N}\hat{U}\right)+A_{22}\left(\hat{N}^2\hat{V}\right)$$

$$+B_{12}\left(\hat{M}\hat{N}\hat{\Psi}_x\right)+B_{22}\left(\hat{N}^2\hat{\Psi}_y\right)=I_0\left(\omega^2\hat{V}\right)+I_1\left(\omega^2\hat{\Psi}_y\right)\Big] \quad (3.77b)$$

$$-\sin(\hat{M}x)\sin(\hat{N}y)\Big[K_sA_{55}\left(\hat{M}\hat{\Psi}_x\right)+K_sA_{55}\left(\hat{M}^2\hat{W}\right)$$

$$+K_sA_{44}\left(\hat{N}\hat{\Psi}_y\right)+K_sA_{44}\left(\hat{N}^2\hat{W}\right)=I_0\left(\omega^2\hat{W}\right)\Big] \quad (3.77c)$$

$$-\cos(\hat{M}x)\sin(\hat{N}y)\Big[B_{11}\left(\hat{M}^2\hat{U}\right)+B_{12}\left(\hat{M}\hat{N}\hat{V}\right)+C_{11}\left(\hat{M}^2\hat{\Psi}_x\right)+C_{12}\left(\hat{M}\hat{N}\hat{\Psi}_y\right)$$

$$+B_{66}\left(\hat{N}^2\hat{U}+\hat{M}\hat{N}\hat{V}\right)+C_{66}\left(\hat{N}^2\hat{\Psi}_x+\hat{M}\hat{N}\hat{\Psi}_y\right)+K_sA_{55}\left(\hat{\Psi}_x+\hat{M}\hat{W}\right)$$

$$=I_1\left(\omega^2\hat{U}\right)+I_2\left(\omega^2\hat{\Psi}_x\right)\Big] \tag{3.77d}$$

$$-\sin(\hat{M}x)\cos(\hat{N}y)\Big[B_{66}\left(\hat{M}\hat{N}\hat{U}+\hat{M}^2\hat{V}\right)+C_{66}\left(\hat{M}\hat{N}\hat{\Psi}_x+\hat{M}^2\hat{\Psi}_y\right)+B_{12}\left(\hat{M}\hat{N}\hat{U}\right)+B_{22}\left(\hat{N}^2\hat{V}\right)$$

$$+C_{12}\left(\hat{M}\hat{N}\hat{\Psi}_x\right)+C_{22}\left(\hat{N}^2\hat{\Psi}_y\right)+K_sA_{44}\left(\hat{\Psi}_y+\hat{N}\hat{W}\right)=I_1\left(\omega^2\hat{V}\right)+I_2\left(\omega^2\hat{\Psi}_y\right)\Big]. \tag{3.77e}$$

Cancelling trigonometric terms and writting the resulting equations in matrix form, the generalized eigenvalue problem is given as

$$[K]\{\vec{d}\}=\omega^2[M]\{\vec{d}\} \tag{3.78}$$

where the vector $\{\vec{d}\}$ and matrices $[K]$ and $[M]$ are given below as

$$\{\vec{d}\}=\begin{Bmatrix}\hat{U}\\ \hat{V}\\ \hat{W}\\ \hat{\Psi}_x\\ \hat{\Psi}_y\end{Bmatrix} \tag{3.79}$$

$$[K]=\begin{bmatrix}A_{11}\hat{M}^2+A_{66}\hat{N}^2 & \hat{M}\hat{N}(A_{12}+A_{66}) & 0 & B_{11}\hat{M}^2+B_{66}\hat{N}^2 & \hat{M}\hat{N}(B_{12}+B_{66})\\ \hat{M}\hat{N}(A_{12}+A_{66}) & A_{66}\hat{M}^2+A_{22}\hat{N}^2 & 0 & \hat{M}\hat{N}(B_{12}+B_{66}) & B_{66}\hat{M}^2+B_{22}\hat{N}^2\\ 0 & 0 & K_s\hat{M}^2A_{55}+K_s\hat{N}^2A_{44} & K_s\hat{M}A_{55} & K_s\hat{N}A_{44}\\ B_{11}\hat{M}^2+B_{66}\hat{N}^2 & \hat{M}\hat{N}(B_{12}+B_{66}) & K_s\hat{M}A_{55} & C_{11}\hat{M}^2+C_{66}\hat{N}^2+K_sA_{55} & \hat{M}\hat{N}(C_{12}+C_{66})\\ \hat{M}\hat{N}(B_{12}+B_{66}) & B_{66}\hat{M}^2+B_{22}\hat{N}^2 & K_s\hat{N}A_{44} & \hat{M}\hat{N}(C_{12}+C_{66}) & C_{66}\hat{M}^2+C_{22}\hat{N}^2+K_sA_{44}\end{bmatrix} \tag{3.80}$$

$$[M]=\begin{bmatrix}I_0 & 0 & 0 & I_1 & 0\\ 0 & I_0 & 0 & 0 & I_1\\ 0 & 0 & I_0 & 0 & 0\\ I_1 & 0 & 0 & I_2 & 0\\ 0 & I_1 & 0 & 0 & I_2\end{bmatrix}. \tag{3.81}$$

The five by five eigenvalue system yields five different natural frequencies, ω. Denoting ω_n as the $n\,th$ root of the five roots of ω, there is a unique eigenvector of values for amplitude coefficients $\hat{U}_n$, $\hat{V}_n$, $\hat{W}_n$, $\hat{\Psi}_{xn}$, and $\hat{\Psi}_{yn}$ and displacements $\tilde{u}_n$, $\tilde{v}_n$, and $\widetilde{w}_n$ for each frequency, ω_n. Substituting the solutions for the functions u_o, v_o, w_o, ψ_x, and ψ_y contained in equations (3.76a) through (3.76e) into equations (3.50a) through (3.50c) yields the displacement solutions for each natural frequency, ω_n, given as

$$\widetilde{u}_n = \left(\widehat{U}_n + z\widehat{\Psi}_{x_n}\right)\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.82a}$$

$$\widetilde{v}_n = \left(\widehat{V}_n + z\widehat{\Psi}_{y_n}\right)\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t} \tag{3.82b}$$

$$\widetilde{w}_n = \left(\widehat{W}_n\right)\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t}. \tag{3.82c}$$

3.4.6 Calculation of stresses

Substituting the strain-displacement equations (3.7) into the stress-strain equations (3.56) and displacement solutions in equations (3.82a) through (3.82c) yields the following equations for the stresses

$$\sigma_{xn} = -\left\{Q_{11i}\hat{M}\widehat{U}_n + Q_{12i}\hat{N}\widehat{V}_n + z\left(Q_{11i}\hat{M}\widehat{\Psi}_{xn} + Q_{12i}\hat{N}\widehat{\Psi}_{y_n}\right)\right\}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.83a}$$

$$\sigma_{y_n} = -\left\{Q_{12i}\hat{M}\widehat{U}_n + Q_{22i}\hat{N}\widehat{V}_n + z\left(Q_{12i}\hat{M}\widehat{\Psi}_{xn} + Q_{22i}\hat{N}\widehat{\Psi}_{y_n}\right)\right\}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.83b}$$

$$\sigma_{yz_n} = Q_{44i}\left\{\hat{N}\widehat{W}_n + \widehat{\Psi}_{y_n}\right\}\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t} \tag{3.83c}$$

$$\sigma_{xzn} = Q_{55i}\left\{\hat{M}\widehat{W}_n + \widehat{\Psi}_{xn}\right\}\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega_n t} \tag{3.83d}$$

$$\sigma_{xy_n} = Q_{66i}\left\{\hat{N}\widehat{U}_n + \hat{M}\widehat{V}_n + z\left(\hat{N}\widehat{\Psi}_{xn} + \hat{M}\widehat{\Psi}_{y_n}\right)\right\}\cos(\hat{M}x)\cos(\hat{N}y)e^{-i\omega_n t}. \tag{3.83e}$$

Subscript n denotes the stress for the n *th* eigenvalue and subscript i denotes the i *th* layer or lamina. The stresses are not continuous due to the difference in the material properties, Q_{ij}, at the layer or lamina interfaces.

3.5 Raleigh-Ritz approximation

A Raleigh-Ritz approximate method is presented here which incorporates three node finite elements, through the thickness of the plate, using quadratic interpolation functions.

3.5.1 Introduction

Dong and Nelson [38], presented a numerical technique that is applicable for wave propagation in a laminated anisotropic plate. In their technique, thickness variations of the displacements are approximated using three node finite elements having quadratic interpolation functions of a thickness variable. The generalized coordinates in this representation are the displacements at the top, middle, and bottom of each mathematical sub-layer. Herein, this stiffness method is extended to orthotropic material proprieties in which all nine elastic constants of orthotrophy are incorporated into the formulation. This method uses continuous transcendental functions in the x and y directions and finite elements in the z direction through the thickness of the plate. Quadratic interpolation polynomials with three node finite elements will be used. Only displacement continuity is maintained. The thickness of the $i\ th$ element is h_i, while the total number of elements through the thickness of the plate is N. The total number of nodes used for the three node elements through the thickness will be $(2N+1)$.

3.5.2 Governing equations

The governing equation is derived using Hamilton's principle. The stress-strain relations for an orthotropic material are given in equation (2.3). Strain displacement relations are given in equation (3.84) as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \frac{\partial}{\partial x}\begin{Bmatrix} u \\ 0 \\ 0 \\ 0 \\ v \\ w \end{Bmatrix} + \frac{\partial}{\partial y}\begin{Bmatrix} 0 \\ v \\ 0 \\ w \\ 0 \\ u \end{Bmatrix} + \frac{\partial}{\partial z}\begin{Bmatrix} 0 \\ 0 \\ w \\ v \\ u \\ 0 \end{Bmatrix}. \tag{3.84}$$

Using quadratic finite elements numbered from the top downward in the z direction and continuous functions in the x and y directions, the displacements for each element are given as

$$\tilde{u}(x,y,z,t) = u_i(x,y,t)N_i(z) + u_j(x,y,t)N_j(z) + u_k(x,y,t)N_k(z) \tag{3.85a}$$

$$\tilde{v}(x,y,z,t) = v_i(x,y,t)N_i(z) + v_j(x,y,t)N_j(z) + v_k(x,y,t)N_k(z) \tag{3.85b}$$

$$\tilde{w}(x,y,z,t) = w_i(x,y,t)N_i(z) + w_j(x,y,t)N_j(z) + w_k(x,y,t)N_k(z) \tag{3.85c}$$

where the subscripts i, j, and k are used to denote the displacements and weighting functions for the top, middle, and bottom node, respectively. The displacements $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ denote the displacements in the x, y, and z directions of the local coordinate axis. The interpolation polynomials in the local coordinate, z, are given by

$$N_i = \frac{\xi}{2}(\xi - 1) \tag{3.86a}$$

$$N_j = 1 - \xi^2 \tag{3.86b}$$

$$N_k = \frac{\xi}{2}(\xi + 1). \tag{3.86c}$$

The Greek letter ξ is a normalized coordinate used for each element and has the value of (–1) at the i *th* node, (0) at the j *th* node, and (+1) at the k *th* node. The ξ is related to the local thickness coordinate, z, as follows

$$\xi(z) = \frac{2(z - z_i)}{h_i} - 1 \tag{3.87a}$$

$$\frac{\partial \xi}{\partial z} = \frac{2}{h_i} \tag{3.87b}$$

where h_i is the thickness of the element. Equations (3.85a), (3.85b), and (3.85c) can be written in matrix form as

$$\{U(x,y,z,t)\} = \begin{Bmatrix} \tilde{u}(x,y,z,t) \\ \tilde{v}(x,y,z,t) \\ \tilde{w}(x,y,z,t) \end{Bmatrix} = [n]\{q(x,y,t)\} \tag{3.88}$$

where matrix $[n]$ is defined in equation (E.5) of Appendix E. The displacement vector for the element is given as

$$\{q(x,y,t)\}^T = \langle u_i \quad u_j \quad u_k \quad v_i \quad v_j \quad v_k \quad w_i \quad w_j \quad w_k \rangle \tag{3.89}$$

where the superscript T denotes the matrix or vector transpose. These displacements contain continuous functions in x, y, and t which must fit the constraints of the boundary conditions for a simply supported plate.

Substituting equations (3.85) and (3.88) into equations (3.84) and (E.1) yields

$$\{\varepsilon\} = [b]\{q'\} + [c]\{\dot{q}\} + [a]\{q\} \tag{3.90}$$

where q' denotes $\frac{\partial q}{\partial x}$, and $\dot{q}$ denotes $\frac{\partial q}{\partial y}$. The matrices $[b]$,$[c]$, and $[a]$ are given in equations (E.2) to (E.4) of Appendix E.

The Lagrangian, L_i, for the i th element is written as

$$L_i = \frac{1}{2}\int_t\int_y\int_x\left[\int_{z_i}^{z_i+h_i}\left(\rho\left\{\overset{o}{U}\right\}^T\left\{\overset{o}{U}\right\} - \{\varepsilon\}^T[D]\{\varepsilon\}\right)dz\right]dxdydt \tag{3.91}$$

where matrix $[D]$ is the constitutive matrix for an orthotropic material defined in equation (2.3), ρ is the mass density of the material, and $\{\varepsilon\}$ is the strain vector defined in equation (3.90). Vector $\left\{\overset{o}{U}\right\}$ is the derivative of the displacement vector $\{U(x,y,z,t)\}$ in equation (3.88) with respect to time. It is defined as

$$\left\{\overset{o}{U}\right\} = [n]\left\{\overset{o}{q}\right\}. \tag{3.92}$$

Substituting the equations (3.88) and (3.92) into the total energy expression given in equation (3.91) and following steps of equations (E.6) to (E.9) of Appendix E yields the following equation:

$$L_i = \frac{1}{2}\int_t\int_y\int_x \begin{pmatrix} \left\{\overset{o}{q}\right\}^T [m]\left\{\overset{o}{q}\right\} - \{q'\}^T[k_1]\{q'\} - \{q'\}^T[k_4]^T\{\dot{q}\} \\ -\{q'\}^T[k_2]\{q\} - \{\dot{q}\}^T[k_4]\{q'\} - \{\dot{q}\}^T[k_5]\{\dot{q}\} \\ -\{\dot{q}\}^T[k_6]\{q\} - \{q\}^T[k_2]^T\{q'\} - \{q\}^T[k_6]^T\{\dot{q}\} \\ -\{q\}^T[k_3]\{q\} \end{pmatrix} dxdydt. \tag{3.93}$$

Matrices $[m]$ through $[k_6]^T$ are defined in equation (E.9) of Appendix E. Applying integration by parts (see equations (E.10) and (E.11) of Appendix E) to factor out $\{q\}^T$ and separate out the initial and boundary conditions yields the following expression

$$\begin{aligned} L_i = \frac{1}{2}\int_t\int_y\int_x \{q\}^T & \begin{pmatrix} -[m]\left\{\overset{oo}{q}\right\} + [k_1]\{q''\} + \left[[k_2]-[k_2]^T\right]\{q'\} - [k_3]\{q\} \\ +\left[[k_4]+[k_4]^T\right]\{\dot{q}'\} + [k_5]\{\ddot{q}\} + \left[[k_6]-[k_6]^T\right]\{\dot{q}\} \end{pmatrix} dxdydt \\ & + \frac{1}{2}\{q\}^T \int_y\int_x [m]\left\{\overset{o}{q}\right\}\Big|_t \, dxdy \\ & - \frac{1}{2}\{q\}^T \int_t\int_y \left([k_1]\{q'\} + [k_2]\{q\} + [k_4]^T\{\dot{q}\}\right)\Big|_x \, dydt \\ & - \frac{1}{2}\{q\}^T \int_t\int_x \left([k_4]\{q'\} + [k_5]\{q'\} + [k_6]\{q\}\right)\Big|_y \, dxdt. \end{aligned} \tag{3.94}$$

The last three lines in the above equation are the initial and boundary conditions. Taking the first variation of the Lagrangian and setting it equal to zero, the governing differential equation is

$$\begin{aligned} & -[m]\left\{\overset{oo}{q}\right\} + [k_1]\{q''\} + \left[[k_2]-[k_2]^T\right]\{q'\} - [k_3]\{q\} \\ & \qquad +\left[[k_4]+[k_4]^T\right]\{\dot{q}'\} + [k_5]\{\ddot{q}\} + \left[[k_6]-[k_6]^T\right]\{\dot{q}\} = \{0\}. \end{aligned} \tag{3.95}$$

3.5.3 Boundary conditions and displacement solutions

The boundary conditions governing the displacement functions of x and y for a simply supported rectangular plate may be specified as, [45]:

at $x = 0$ and a; $\sigma_{xx} = 0$, $\tilde{w} = 0$ and $\tilde{v} = 0$,

$$\text{at } y = 0 \text{ and } b; \quad \sigma_{yy} = 0, \quad \tilde{w} = 0 \quad \text{and} \quad \tilde{u} = 0. \tag{3.96}$$

A solution for the displacement vector $\{q(x,y,t)\}$ that satisfies the above boundary conditions is

$$\{q(x,y,t)\} = \{q_1\}\cos(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} + \{q_2\}\sin(\hat{M}x)\cos(\hat{N}y)e^{-i\omega t} + \{q_3\}\sin(\hat{M}x)\sin(\hat{N}y)e^{-i\omega t} \tag{3.97}$$

where $\hat{M}$ and $\hat{N}$ and vectors $\{q_1\}$ through $\{q_3\}$ are defined as follows

$$\hat{M} = \frac{m\pi}{a}; \quad \hat{N} = \frac{n\pi}{b}; \quad \text{and} \quad m = 1,2,3,\ldots; \quad n = 1,2,3,\ldots \tag{3.98}$$

$$\{q_1\}^T = \langle u_i \quad u_j \quad u_k \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rangle = \langle \langle \bar{u} \rangle \quad \langle 0 \rangle \quad \langle 0 \rangle \rangle \tag{3.99a}$$

$$\{q_2\}^T = \langle 0 \quad 0 \quad 0 \quad v_i \quad v_j \quad v_k \quad 0 \quad 0 \quad 0 \rangle = \langle \langle 0 \rangle \quad \langle \bar{v} \rangle \quad \langle 0 \rangle \rangle \tag{3.99b}$$

$$\{q_3\}^T = \langle 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad w_i \quad w_j \quad w_k \rangle = \langle \langle 0 \rangle \quad \langle 0 \rangle \quad \langle \bar{w} \rangle \rangle. \tag{3.99c}$$

Vectors $\{\bar{u}\}$, $\{\bar{v}\}$, and $\{\bar{w}\}$ are defined as

$$\{\bar{u}\}^T = \langle u_i \quad u_j \quad u_k \rangle \tag{3.100a}$$

$$\{\bar{v}\}^T = \langle v_i \quad v_j \quad v_k \rangle \tag{3.100b}$$

$$\{\bar{w}\}^T = \langle w_i \quad w_j \quad w_k \rangle. \tag{3.100c}$$

Substituting equation (3.97) into equation (3.95), performing derivatives and matrix multiplications, using the definitions of matrices $[m]$ through $[k_6]^T$ defined in equations (E.12) through (E.15) of Appendix E, yields the following three equations containing 3 by 3 matrices.

$$-\cos(\hat{M}x)\sin(\hat{N}y)\Big[\big(-\omega^2\rho[A] + (D_{11}\hat{M}^2 + D_{66}\hat{N}^2)[A] + D_{55}[B]\big)\{\bar{u}\} \tag{3.101a}$$

$$+\big(\hat{M}\hat{N}(D_{21} + D_{66})[A]\big)\{\bar{v}\} + \big(\hat{M}(D_{55}[E]^T - D_{13}[E])\big)\{\bar{w}\}\Big] = \{0\}$$

$$-\sin(\hat{M}x)\cos(\hat{N}y)\Big[(\hat{M}\hat{N}(D_{21} + D_{66})[A])\{\bar{u}\} \tag{3.101b}$$

$$+\big(-\omega^2\rho[A] + (\hat{M}^2 D_{66} + \hat{N}^2 D_{22})[A] + D_{44}[B]\big)\{\bar{v}\} + \big(\hat{N}(D_{44}[E]^T - D_{23}[E])\big)\{\bar{w}\}\Big] = \{0\}$$

$$-\sin(\hat{M}x)\sin(\hat{N}y)\Big[\big(\hat{M}(D_{55}[E] - D_{13}[E]^T)\big)\{\bar{u}\} \tag{3.101c}$$

$$+\left(\hat{N}(D_{44}[E]-D_{23}[E]^T)\right)\{\bar{v}\}+\left(-\omega^2\rho[A]+(\hat{M}^2D_{55}+\hat{N}^2D_{44})[A]+D_{33}[B]\right)\{\bar{w}\}\Big]=\{0\}.$$

Cancelling trigonometric functions and restating the above equations in matrix form, the generalized elemental eigenvalue equation can be found to be

$$[K]\{q\}=\omega^2[M]\{q\}. \tag{3.102}$$

The element stiffness matrix, $[K]$, element mass matrix, $[M]$, and vector, $\{q\}$, are given below as

$$[K]=\begin{bmatrix} \left[(D_{11}\hat{M}^2+D_{66}\hat{N}^2)[A]+D_{55}[B]\right] & \left[\hat{M}\hat{N}(D_{21}+D_{66})[A]\right] & \left[\hat{M}(D_{55}[E]^T-D_{13}[E])\right] \\ \left[\hat{M}\hat{N}(D_{21}+D_{66})[A]\right] & \left[(\hat{M}^2D_{66}+\hat{N}^2D_{22})[A]+D_{44}[B]\right] & \left[\hat{N}(D_{44}[E]^T-D_{23}[E])\right] \\ \left[\hat{M}(D_{55}[E]-D_{13}[E]^T)\right] & \left[\hat{N}(D_{44}[E]-D_{23}[E]^T)\right] & \left[(\hat{M}^2D_{55}+\hat{N}^2D_{44})[A]+D_{33}[B]\right] \end{bmatrix} \tag{3.103}$$

$$[M]=\begin{bmatrix} \rho[A] & & 0 \\ & \rho[A] & \\ 0 & & \rho[A] \end{bmatrix} \tag{3.104}$$

$$\{q\}=\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}. \tag{3.105}$$

Matrices $[A]$, $[B]$, and $[E]$ are defined in equations (E.13a) to (E.13c) of Appendix E. For later convenience, define each 3 by 3 matrix inside the element stiffness matrix, $[K]$, as

$$[K]=\begin{bmatrix} [K11] & [K12] & [K13] \\ [K21] & [K22] & [K23] \\ [K31] & [K32] & [K33] \end{bmatrix} \tag{3.106}$$

where each of the 3 by 3 matrices $[K11]$ through $[K33]$ matches its respective location in equation (3.103).

3.5.4 Global frequency eigenvalue equation

In order to conveniently impose these matrices into a global system where the global displacement vector $\{Q\}$ is in the form

$$\{Q\}^T = \langle u_1 \quad v_1 \quad w_1 \quad u_2 \quad v_2 \quad w_2 \quad . \quad . \quad . \quad u_{2N_e+1} \quad v_{2N_e+1} \quad w_{2N_e+1} \rangle \tag{3.107}$$

where N_e is the number of elements used so that $(2N_e + 1)$ is the total number of nodes, define a new element displacement vector, $\{q^*\}$, as

$$\{q^*\}^T = \langle u_i \quad v_i \quad w_i \quad u_j \quad v_j \quad w_j \quad u_k \quad v_k \quad w_k \rangle. \tag{3.108}$$

In doing so, the element stiffness matrix, $[K]$, and element mass matrix, $[M]$, must be rearranged to the $[K^*]$ and $[M^*]$ given in Appendix E according to the operations stated in equation (E.16) of Appendix E. Additionally, by applying the referred operations to the rows and columns of the element stiffness matrix, $[K]$, and element mass matrix, $[M]$, symmetric $[K^*]$ and $[M^*]$ matrices will be obtained, which aid in the computation of the eigenvalues. The elemental eigenvalue equation is now given as

$$[K^*]\{q^*\} = \omega^2[M^*]\{q^*\} \tag{3.109}$$

where the element stiffness matrix, $[K^*]$, and element mass matrix, $[M^*]$, are given in equations (E.17) and (E.18) of Appendix E, respectively. Note that it can be shown that, for each value in the stiffness matrix $[K^*]$, KIJ_{mn} is equal to KJI_{nm}, so that $[K^*]$ is symmetric. Since matrix $[A]$ contained in $[M^*]$ is symmetric, $[M^*]$ is also symmetric. Imposing element stiffness matrices and mass matrices into the global displacement vector $\{Q\}$ assures that the continuity of the displacement is maintained.

3.5.5 Calculation of stresses

For each frequency eigenvalue of ω^2, a unique eigenvector, $\{Q\}$, containing global displacements can be calculated. Once the displacements are known, the stresses can be evaluated in each element for the $i\,th$, $j\,th$, and $k\,th$ nodes. The solutions for $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ displacements are

$$\begin{Bmatrix} u_i(x,y,z) \\ u_j(x,y,z) \\ u_k(x,y,z) \end{Bmatrix} = [W]\{\bar{u}\} \cos(\hat{M}x)\sin(\hat{N}y) \tag{3.110a}$$

$$\begin{Bmatrix} v_i(x,y,z) \\ v_j(x,y,z) \\ v_k(x,y,z) \end{Bmatrix} = [W]\{\bar{v}\} \sin(\hat{M}x)\cos(\hat{N}y) \tag{3.110b}$$

$$\begin{Bmatrix} w_i(x,y,z) \\ w_j(x,y,z) \\ w_k(x,y,z) \end{Bmatrix} = [W]\{\bar{w}\} \sin(\hat{M}x)\sin(\hat{N}y) \tag{3.110c}$$

where vectors $\{\bar{u}\}$, $\{\bar{v}\}$, and $\{\bar{w}\}$ were defined in equations (3.100a), (3.100b), and (3.100c), respectively. Matrix $[W]$ is defined as

$$[W] = \begin{bmatrix} N_i|_{\xi=-1} & N_j|_{\xi=-1} & N_k|_{\xi=-1} \\ N_i|_{\xi=0} & N_i|_{\xi=0} & N_i|_{\xi=0} \\ N_i|_{\xi=+1} & N_i|_{\xi=+1} & N_i|_{\xi=+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I] \tag{3.111}$$

where matrix $[I]$ is the identity matrix. Note that equations for the weighting functions were given in equations (3.86a) through (3.86c). Their derivatives are

$$\frac{\partial N_i}{\partial \xi} = \xi - \frac{1}{2} \tag{3.112a}$$

$$\frac{\partial N_j}{\partial \xi} = -2\xi \tag{3.112b}$$

$$\frac{\partial N_k}{\partial \xi} = \xi + \frac{1}{2} \tag{3.112c}$$

where the normalized coordinate, which is denoted by the Greek letter ξ, is defined in equations (3.87a) and (3.87b). Using the stress-strain and strain-displacement relations given in equations (2.3) and (3.84), and substituting the expressions for the displacement given in (3.110a) through (3.110c), the stresses at the $i\,th$, $j\,th$, and $k\,th$ node of a particular element can be written as

$$\begin{Bmatrix} \sigma_{zxi} \\ \sigma_{zxj} \\ \sigma_{zxk} \end{Bmatrix} = D_{55}\left(\hat{M}[I]\{\bar{w}\} + \left(\frac{2}{h_i}\right)\left[\frac{\partial W}{\partial \xi}\right]\{\bar{u}\} \right)\cos(\hat{M}x)\sin(\hat{N}y) \tag{3.113a}$$

$$\begin{Bmatrix} \sigma_{zyi} \\ \sigma_{zyj} \\ \sigma_{zyk} \end{Bmatrix} = D_{44}\left(\hat{N}[I]\{\bar{w}\} + \left(\frac{2}{h_i}\right)\left[\frac{\partial W}{\partial \xi}\right]\{\bar{v}\} \right)\sin(\hat{M}x)\cos(\hat{N}y) \tag{3.113b}$$

$$\begin{Bmatrix} \sigma_{zzi} \\ \sigma_{zzj} \\ \sigma_{zzk} \end{Bmatrix} = \left(-\hat{M}D_{13}[I]\{\bar{u}\} - \hat{N}D_{23}[I]\{\bar{v}\} + D_{33}\left(\frac{2}{h_i}\right)\left[\frac{\partial W}{\partial \xi}\right]\{\bar{w}\} \right)\sin(\hat{M}x)\sin(\hat{N}y)$$

. (3.113c)

where $\left[\frac{\partial W}{\partial \xi}\right]$ is defined as

$$\left[\frac{\partial W}{\partial \xi}\right] = \begin{bmatrix} \left.\frac{\partial N_i}{\partial \xi}\right|_{\xi=-1} & \left.\frac{\partial N_j}{\partial \xi}\right|_{\xi=-1} & \left.\frac{\partial N_k}{\partial \xi}\right|_{\xi=-1} \\ \left.\frac{\partial N_i}{\partial \xi}\right|_{\xi=0} & \left.\frac{\partial N_j}{\partial \xi}\right|_{\xi=0} & \left.\frac{\partial N_k}{\partial \xi}\right|_{\xi=0} \\ \left.\frac{\partial N_i}{\partial \xi}\right|_{\xi=+1} & \left.\frac{\partial N_j}{\partial \xi}\right|_{\xi=+1} & \left.\frac{\partial N_k}{\partial \xi}\right|_{\xi=+1} \end{bmatrix} = \begin{bmatrix} -1.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -2 & 1.5 \end{bmatrix}. \tag{3.114}$$

Chapter 4

Numerical results and discussion

4.1 Introduction

Four different computer programs have been developed to compute the natural frequencies of vibration for simply supported laminated composite cross-ply plates as well as the corresponding eigenvectors of the stresses and displacements. First, an exact program named "Refine" is based on the formulation covered in Chapter 2. Additionally, a program named "CLPT" is based on the classical laminated plate theory, a program named "FSDT" is based on the first order shear deformation theory, and a program named "EVP2" is based on a Raleigh-Ritz procedure using quadratic finite elements through the thickness. These additional programs are based on the derivations covered in Chapter 3. A discussion comparing the computational effort of these theories will be presented. The results of these plate theories will be compared to results quoted from other references.

4.2 Computational effort versus accuracy

Results from the four numerical programs vary in accuracy, with "CLPT" being the least accurate and "Refine" being exact. However, the computational effort required to obtain results from each of the programs must be weighed against the accuracy of the results obtained.

The two equivalent single layer theories presented here, namely the classical laminated plate theory and the first order shear deformation theory, require the least computational effort. They are quite accurate for the first couple of natural frequencies for a homogeneous plate or a plate with many laminates. The greater the number of laminates in a plate, the closer the plate becomes to acting like a homogeneous plate. One advantage that these equivalent single layer programs have is that the size of the matrices used to solve for the frequency eigenvalues do not change or grow with the number of laminates in the plate. Thus, the computational effort does not increase to a great degree with an increasing number of lamina. Using the program "EVP2", based on Raleigh-Ritz quadratic finite elements through the thickness, requires the assembly of individual stiffness and mass matrices into global stiffness and mass matrices. The size of the global eigenvalue system varies directly with the number of laminates or elements used through the thickness. It can get very large when many elements are used to obtain accurate frequency eigenvalues. The computational effort to solve for the eigenvalues generally increases in proportion to the square of the number of equations. Thus, the computational effort required to solve for eigenvalues using 20 elements through the thickness is approximately four times more than using 10 elements through the thickness.

The problem can be solved analytically using the traditional method of assembling a global system of equations in terms of the undetermined coefficients, A_1 though A_6, for each layer or sublayer. The equations would consist of the boundary conditions (B. C. s) where the three stresses, σ_{zx}, σ_{zy} and σ_{zz} are zero at both of the free surfaces, forming six equations. Additionally, for each layer or sublayer interface between sub-layers, there are six continuity equations (C. C. s) of displacements, u, v and w, and stresses, σ_{zx}, σ_{zy} and σ_{zz} where the displacements and stresses at the top of one layer are equal to the displacements and stresses at the bottom of the next layer. For a plate with two layers, the system would be in the form shown in Figure 4.1.

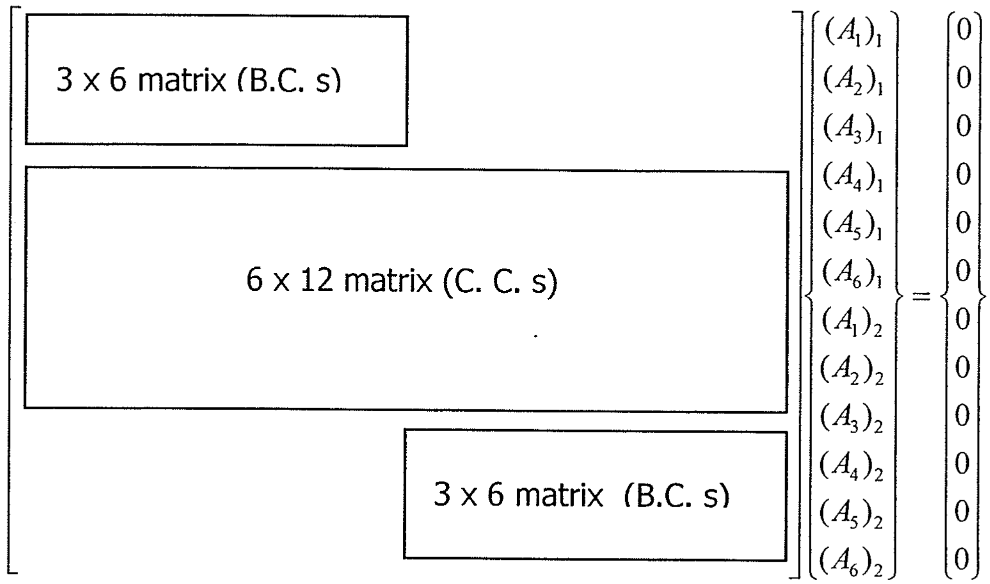

Figure 4.1: Traditional assembly of equations for two layer plate

Assembling the system of analytical equations this way results in an increase in computational effort that is proportional to the square of the number of layers or sublayers used.

The propagator matrix method used in the program "Refine", which is formulated according to Chapter 2 of this thesis, results in an increase in the computational effort that is proportional to the number of layers used. Thus, the computational effort does not increase to such a great degree when the number of layers in a composite laminated plate are greater. However, the program "Refine" uses a two step process in which the approximate output frequencies from "EVP2", run with a minimal number of elements, are used as starting "guesses" to solve for the roots of the exact frequency equation (2.33) calculated in the program "Refine". Only a few elements through the thickness are required for this process and the use of accurate starting guesses reduces the effort required by the solver in locating the roots of the frequency equation.

4.3 Numerical examples

The following six different plate examples are presented in this section. They are an:

- Homogeneous isotropic plate
- Homogeneous orthotropic plate
- ($0^o/90^o$) and ($0^o/90^o/0^o/90^o$) antisymmetric orthotropic cross ply plate
- Multi-laminated orthotropic cross-ply plates with equal layer thicknesses
- ($0^o/90^o$/isotropic/$0^o/90^o$) antisymmetric plate
- orthotropic homogeneous plate neglecting material constants D_{44} and D_{55}

Each example will follow the same format. First, a description of the plate and its material properties will be given, then a description of the numerical results is presented with respect to tables and figures, and finally, the results are discussed. Comparisons of the results will be made with the three dimensional, exact solutions of Srinivas et. al. [44,45], and the approximate solutions of Rao [26], who used a layerwise theory in which continuity of the displacement and traction was maintained between sub-layers.

4.3.1 Example 1 – Homogeneous isotropic plate

A single layer, thick, square isotropic plate has been used to validate the formulation for the isotropic propagator matrix whose formulation is presented in Appendix C. The Poisson ratio and dimensions used are $\nu = 0.3$ and size $a \times a$. The natural frequencies have been normalized by using the relation $\Omega = \omega h(\rho / \mu)^{1/2}$. The lowest five natural frequencies produced by the four different programs are compared to those given by Srinivas et al. [44] and Rao [26]. The results are shown in Table 4.1. The variations of the stresses and displacements are presented in Table 4.2 for the first and third antisymmetric and second symmetric modes of vibration for the frequencies corresponding to the wavenumbers $mh/a = nh/b = 0.3$.

The results in Table 4.1 reveal that the results from "Refine" agree completely with those published by Srinivas et. al. [44] as "Exact". The frequencies obtained from the program "EVP2" converge to the exact values for the first five frequencies when 8 elements are used. Results obtained from first order shear deformation theory entitled "FSDT" are in good agreement for the first two frequencies. The results from classical laminated plate theory are in poor agreement for the first frequency and in good agreement for the next two frequencies. Results for the variation of the stresses and displacements through the thickness show that eigenvectors obtained from "Refine" agree completely with the exact results published by Srinivas et. al. [44]. Results from Rao [26] agree closely with the exact values in both Tables 4.1 and 4.2.

4.3.2 Example 2 – Homogeneous orthotropic plate

A single layer, square orthotropic plate has been used to validate the formulation for the orthotropic propagator matrix presented in Chapter 2. The following material properties have been used which correspond to aragonite crystals: $D_{12}/D_{11} = 0.23319$, $D_{13}/D_{11} = 0.010776$, $D_{22}/D_{11} = 0.543103$, $D_{23}/D_{11} = 0.098276$, $D_{33}/D_{11} = 0.530172$, $D_{44}/D_{11} = 0.26681$, $D_{55}/D_{11} = 0.159914$, $D_{66}/D_{11} = 0.262931$. The frequencies have been normalized by using the relation $\Omega = \omega a^2 / h(\rho / D_{11})^{1/2}$. The lowest five natural frequencies resulting from the four different programs are compared to those given by Srinivas et al. [45] and Rao [26]. The results are shown in Table 4.3. The variations of the stresses and displacements are presented in Table 4.4 for the first and third antisymmetric modes, as well as the first and second symmetric modes of vibration for the frequencies corresponding to the wavenumbers $mh/a = nh/b = 0.3$. Results for the variations of the stresses and displacements are compared to the exact ones from Srinivas et. al. [45], and approximate ones from Rao [26].

Table 4.3 shows that the results obtained from the proposed theory given by "Refine" are in exact agreement with the natural frequencies predicted by the exact theory of Srinivas et. al. [45]. Results from Rao [26] also agree closely. Results from "EVP2" agree closely with as few as four elements and they converge to the exact solution for the first five frequencies when 16 elements are used. Results from the first order shear deformation theory agree closely for the first three natural frequencies.

Results from the classical laminated plate theory overestimate the first natural frequency which is an antisymmetric or bending mode, but yield good predictions for the second and third natural frequencies corresponding to symmetric or extensional modes. The latter match those yielded by "FSDT". Table 4.4 shows exact agreement between variations of stresses and displacements through the thickness predicted by the exact theory of Srinivas et. al. [45] and the proposed theory quoted as "Refine". Results from Rao [26] are also in close agreement.

Plots of the displacements and stresses through the thickness are given in Figures 4.2a and 4.2b for the first bending mode (or the first frequency for wavenumbers $mh/a = nh/b = 0.1$). Figures 4.3a and 4.4b show the variations of the displacements and stresses through the thickness for the second frequency or first extensional mode. Figure 4.2a shows both u and v cross the midplane of the plate, which show that the plate is in bending. The displacement w is constant. The stresses given in Figure 4.2b are zero at the free surface at locations $z/h = -0.5$ and $z/h = 0.5$. Figure 4.3a shows both u and v being symmetric along the vertical axis, so that the plate is in an extensional mode. Now the displacement w varies through the thickness. Table 4.3 shows that the classical laminated plate theory is more accurate for the extensional modes than the bending modes. This is because the displacements u and v are constant and displacement w is approximated as a constant value through the thickness, which approximates closely the actual displacements shown in Figure 4.3a. Since the classical laminated plate theory does not incorporate shear strain, it is not a good approximation for a plate in bending, especially for a thicker plate. Thus, the classical laminated plate theory is not as accurate as the first order shear deformation theory for the first bending mode since the shear strain is neglected. However, it is just as good an approximation as the shear deformation theory for a plate in an extensional mode because the displacement solutions yield the same displacement eigenvectors since the second terms in the displacement functions, or first derivatives, are zero.

4.3.3 Example 3 - (0°/90°/90°/0°) and (0°/90°) antisymmetric orthotropic cross-ply plate

A four layer, equal thickness, symmetric cross-ply laminated composite square plate having a lamination scheme of (0°/90°/90°/0°) and a two layer antisymmetric cross ply (0°/90°) has been analyzed. The material properties of each layer are: $E_1/E_2 = 40$, $E_1/E_3 = 40$, $G_{23}/E_2 = 0.5$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$ and $\nu_{12} = \nu_{13} = 0.25$. The non-dimensional fundamental frequencies $\Omega = \omega a^2 / h(\rho / E_{22})^{1/2}$ are shown in Table 4.5 for various length-to-thickness ratios. Variation of the stress and displacement eigenvectors of the symmetric cross ply (0°/90°/90°/0°) square plate with $a/h = 10$ and $m/a = n/b = 0.1$ have been tabulated in Table 4.6 for the first, second, and third antisymmetric bending modes and first symmetric extensional mode. The results have been compared to Rao [26].

From Table 4.5, it can be seen that classical laminated plate theory greatly overestimates the first frequencies. However, as the plate gets thinner, or as a/h increases, its accuracy improves because of the thin plate assumption used. The neglect of shear strain causes much more error for a thick plate than for a thick plate. For the first order shear deformation theory, predictions of the first frequency are relatively good, but the frequencies predicted increase in accuracy as the plate gets thinner. The same is true for the program "EVP2". "EVP2" with the use of 8 elements provides an excellent prediction of the first frequency. It provides an even closer match to the exact values predicted by "Refine" as the plate gets thinner. Tables 4.5 and 4.6 show that the results from Rao [26] are invariably in good agreement with the results from "Refine".

4.3.4 Example 4 – Multi-laminated, orthotropic cross-ply plates with equal layer thicknesses

The fundamental frequencies of the symmetric (three, five, and nine layer) and antisymmetric (two, four, six, and ten layer) cross-ply square laminated plates with equal thicknesses are presented in Tables 4.7 and 4.8, respectively, for various values of E_1/E_2. The orthotropic material properties of the individual layers in all the above laminates are E_1/E_2 = varies, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. Natural frequencies have been normalized by using the relation $\Omega = \omega b^2(\rho / E_2)^{1/2} / h$. Three-dimensional elasticity solutions using a numerical finite difference technique have been given by Noor [43] and approximate results have been presented by Rao [26]. They are considered for comparison. Results from the exact program "Refine" along with Raleigh-Ritz quadratic finite elements through the thickness "EVP2", first order shear deformation theory "FSDT" and classical laminated plate theory "CLPT" have also been considered for comparison.

Tables 4.7 and 4.8 show that the classical laminated plate theory overestimates the first natural frequency which is antisymmetric. Although the number of laminations change in Table 4.7, the total thickness of the layers between each scheme is identical. Therefore the classical laminated plate theory predicts the same frequencies for each lamination scheme. However, the classical laminated plate theory yields different values in Table 4.8, for different lamination schemes. This is due the plate being antisymmetric. The 3-D elasticity finite difference solutions presented by Noor [43] overestimate the natural frequencies. This is due to the inaccuracy of using the numerical finite difference method. The equivalent single layer theories, the classical laminated plate theory and the first order shear deformation theory, become less accurate as E_1 / E_2 gets very large or as the stiffness in one direction gets very large. Although "EVP2" is in excellent agreement, it becomes less accurate as the stiffness in one direction becomes very large. The results of Rao [26] are in good agreement with "Refine" but become less accurate as the stiffness in one direction gets very large. The accuracy of the equivalent single layer theories increase, relative to the exact values given by "Refine", as the number of laminations increases. This is due to the fact that, as the number of laminations increases, the plate acts more like a homogeneous plate.

4.3.5 Example 5 - (0°/90°/core/0°/90°) antisymmetric sandwich plate

A five layer, thin as well as thick sandwich plate of size $a \times b$ with antisymmetric cross ply faces has been analyzed in this example. The material properties of the layers are:

face sheets: E_1 = 19 x 10^6 psi (131 GPa), $E_2 = E_3$ = 1.5 x 10^6 psi (10.34 GPa), $G_{12} = G_{13}$ = 1 x 10^6 psi (6.895 GPa), G_{13} = 0.9 x 10^6 psi (6.205 GPa), $\nu_{12} = \nu_{13} = 0.22$, $\nu_{23} = 0.49$, thickness t_f and ρ = 0.3403 x 10^{-2} lb/in^3 (1627 kg/m^3).

Isotropic Core: $E_1 = E_2 = E_3 = 2G$ = 1000 psi (6.89 x 10^{-3} GPa), $G_{12} = G_{13} = G_{23}$ = 500 psi (3.45 x 10^{-3} GPa), $\nu_{12} = \nu_{13} = \nu_{23} = 0$, thickness t_c and ρ = 0.3403 x 10^{-2} lb/in^3 (97 kg/m^3).

The plate has the configuration shown in Figure 4.4.

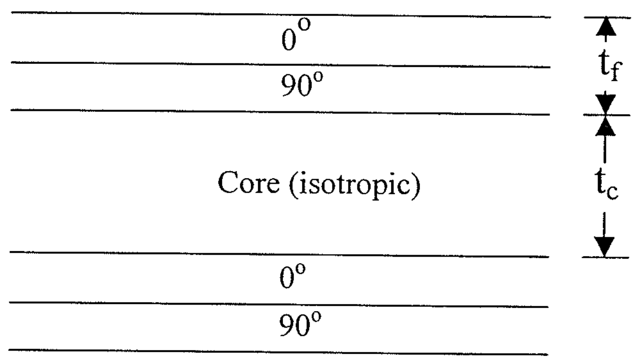

Figure 4.4: Configuration of sandwich plate in Example 5

A comparison of the results from the present study with published numerical results for a thin plate ($a/h = 100$) and a moderately thick plate ($a/h = 10$) are shown in Tables 4.9 and 4.10, respectively. The natural frequencies have been normalized using the relation $\Omega = \omega b^2 (\rho / E_2)_f^{1/2} / h$. A comparison of the results from "Refine", "EVP2", "FSDT", "CLPT" as well as the results presented by Rao [26], have been given in Table 4.11 for varying values of the thickness of the plate (a/h). Table 4.12 compares the corresponding results obtained by varying the thickness of the core to the thickness of the flange (t_c / t_f). Table 4.13 compares the corresponding results for various aspect ratios (a / b) of the plate. Table 4.9, which corresponds to a thin plate, shows Rao [26] is in good agreement with the exact results from "Refine". The equivalent single layer theories, the classical laminated plate theory and the first order shear deformation theory grossly overestimate the first natural frequency of vibration for this sandwich plate. This discrepancy can be attributed to the large difference in the stiffness between the flange material and the core material. The equivalent single layer theories overestimate the stiffness of the plate when the equivalent material properties for an equivalent single layer are obtained. An excellent estimate is obtained through "EVP2" using only 5 elements and the results match almost exactly to the exact results of "Refine". There is less error for a thin plate compared to the results for the thick plate presented in Table 4.10. The error in using equivalent single layer theories is even more pronounced when compared to the exact results from "Refine". "EVP2" matches almost exactly the exact results from "Refine" when using 14 elements. Again, more elements are needed in analyzing a thicker plate with this method as opposed to a thinner plate. (See Table 4.9). The results of Rao [26] agree closely in both Tables 4.9 and 4.10 with the exact results obtained from "Refine".

The first non-dimensional natural frequencies for various length to thickness ratios a / h have been presented in Table 4.11. Again, the equivalent single layer theories grossly overestimate the first natural frequencies. However, their accuracy increases when compared to the exact results from "Refine", for a thinner plate. The results from "EVP2" converge to the exact solutions for the program "Refine" when 14 elements are used.

In Table 4.12, the accuracy of the equivalent single layer theories increases compared to the exact values quoted under "Refine" as the ratio of the thickness of the core to the thickness of the flange increases. However, the results of the equivalent single layer theories still greatly overestimate the first frequency. "EVP2" provides an excellent estimate when 14 elements used through the thickness. The solutions from "EVP2" converge to the exact values obtained from the program "Refine" when 28 elements are used.

Table 4.13 presents a comparison of the natural frequencies obtained when the aspect ratio a/b is varied. The variation of a/b the aspect ratio does not seem to affect the accuracy of the equivalent single layer theories compared to the exact frequencies provided by "Refine". The equivalent single layer theories greatly overestimate the natural frequencies. "EVP2" provides an excellent estimate that is almost exact when 14 elements are used. Variations of the displacements and the stresses for the first frequency, which corresponds to the first bending mode of vibration for wavenumbers $mh/a = nh/b = 0.1$ are given in Figures 4.5a and 4.5b, respectively. Displacements u and v cross the vertical axis at the midplane of the plate and displacement w is constant. Figure 4.5b shows the large variations of the shear stresses, σ_{zx} and σ_{zy}, through the flange material. However, there is very little variation of the shear stresses through the core material. This shows the large difference in the shear stiffness of the flange material relative to that of the core material.

4.3.6 Example 6 – Homogeneous orthotropic plate neglecting material constants G_{13} and G_{23}

A single layer, square orthotropic plate having negligable material constants G_{13} and G_{23} (which correspond to D_{44} and D_{55}) has been analyzed in this example using the propagator matrix formulation presented in Appendix D. The following material properties of the face sheets presented in Example 5 are used: $E_1 = 19 \times 10^6$ psi (131 GPa), $E_2 = E_3 = 1.5 \times 10^6$ psi (10.34 GPa), $G_{12} = 1 \times 10^6$ psi (6.895 GPa), $G_{13} = G_{23} = 0$, $\nu_{12} = \nu_{13} = 0.22$, $\nu_{23} = 0.49$, thickness t_f and $\rho = 0.3403 \times 10^{-2}$ lb/in^3 (1627 kg/m^3). However, the natural frequencies have been normalized by using the relation $\Omega = \omega b^2 (\rho / E_2)^{1/2} / h$. A comparison of the results from "Refine", "EVP2", "FSDT", and "CLPT" have been given in Table 4.14 for various values of the wavenumbers m and n. Variations of the displacements and stresses through the thickness of the plate for the first zero frequency, first non-zero frequency, and second non-zero frequency for wavenumbers $mh/a = nh/b = 0.1$ have been shown in Figures 4.6 through 4.8, respectively.

Figure 4.6a shows that the displacements u and v are zero throughout the thickness while displacement w is constant. Figure 4.6b shows all the stresses are zero throughout the thickness. This behavior corresponds to a rigid body displacement for this zero frequency, as predicted by the derivation contained in Appendix C. Figure 4.7a shows the displacements, u and v, are zero at the free surface but vary through the thickness while displacement, w, is not zero at the free surface and varies through the thickness. Figure 4.7b shows that the shear stresses, σ_{zx} and σ_{zy}, are zero throughout the thickness as a result of material constants D_{44} and D_{55} equalling zero. Stress, σ_{zz}, is zero at the free surface and varies through the thickness.

Figure 4.8a shows u and v to be zero at the free surface and zero at the midplane while displacement w is not zero. Figure 4.8b shows the stress σ_{zz} is zero at the free surface and at the midplane and varies through the thickness. The difference between the first and second, non-zero frequency modes of vibration is the addition of a node point at the midplane of the plate where u, v and the stress, σ_{zz}, are zero. Subsequent modes of vibration are similar, with the third non-zero frequency mode having two equally spaced nodes within the thickness, and the fourth frequency non-zero mode having three equally spaced nodes within the thickness. At each node within the thickness, as well as at both the free surfaces, displacements u and v are zero and the stress σ_{zz} is zero. The shear stresses, σ_{zx} and σ_{zy}, are always zero throughout the thickness of the plate.

Table 4.1 Comparison of non-dimensional frequencies, Ω, of a homogeneous isotropic (ν = 0.3), square plate

mh/a	nh/a		I-A*	I-S	II-S	II-A	III-A
0.1	0.1	Exact [44]	0.0931	0.4443	0.7498	3.1729	3.2465
		Refine	0.0931	0.4443	0.7498	3.1729	3.2465
		Rao [26]	0.0931	0.4442	0.7498	3.1737	3.2485
		EVP2(16)	0.0931	0.4443	0.7498	3.1729	3.2465
		EVP2(8)	0.0932	0.4443	0.7498	3.1729	3.2465
		EVP2(4)	0.0932	0.4443	0.7498	3.1737	3.2473
		FSDT(5/6)	0.0930	0.4443	0.7510	3.1933	3.2741
		FSDT(π^2/12)	0.0930	0.4443	0.7510	3.1729	3.2538
		CLPT	0.0955	0.4443	0.7510	-	-
0.1	0.2	Exact [44]	0.2226	0.7025	1.1827	3.2192	3.3933
		Refine	0.2226	0.7025	1.1827	3.2192	3.3933
		Rao [26]	0.2226	0.7024	1.1827	3.2200	3.3971
		EVP2(16)	0.2226	0.7025	1.1827	3.2192	3.3933
		EVP2(8)	0.2226	0.7025	1.1827	3.2192	3.3933
		EVP2(4)	0.2226	0.7025	1.1827	3.2200	3.3942
		FSDT(5/6)	0.2219	0.7025	1.1874	3.2394	3.4310
		FSDT(π^2/12)	0.2218	0.7025	1.1874	3.2192	3.4112
		CLPT	0.2360	0.7025	1.1874	-	-
0.2	0.2	Exact [44]	0.3421	0.8886	1.4943	3.2648	3.5298
		Refine	0.3421	0.8886	1.4923	3.2648	3.5298
		Rao [26]	0.3421	0.8885	1.4922	3.2656	3.5355
		EVP2(16)	0.3421	0.8886	1.4923	3.2648	3.5298
		EVP2(8)	0.3421	0.8886	1.4923	3.2649	3.5298
		EVP2(4)	0.3421	0.8886	1.4923	3.2656	3.5308
		FSDT(5/6)	0.3406	0.8886	1.5020	3.2847	3.5774
		FSDT(π^2/12)	0.3402	0.8886	1.5020	3.2648	3.5580
		CLPT	0.3732	0.8886	1.5020	-	-
0.1	0.3	Exact [44]	0.4171	0.9935	1.6654	3.2949	3.6160
		Refine	0.4171	0.9935	1.6654	3.2949	3.6160
		Rao [26]	0.4172	0.9934	1.6654	3.2957	3.6231
		EVP2(16)	0.4171	0.9935	1.6654	3.2949	3.6160
		EVP2(8)	0.4171	0.9935	1.6654	3.2950	3.6161
		EVP2(4)	0.4172	0.9935	1.6654	3.2957	3.6170
		FSDT(5/6)	0.4149	0.9935	1.6793	3.3147	3.6702
		FSDT(π^2/12)	0.4144	0.9935	1.6793	3.2949	3.6510
		CLPT	0.4629	0.9935	1.6793	-	-
0.2	0.3	Exact [44]	0.5239	1.1327	1.8936	3.3396	3.7393
		Refine	0.5239	1.1327	1.8936	3.3396	3.7393
		Rao [26]	0.5240	1.1327	1.8936	3.3403	3.7484

		EVP2(16)	0.5239	1.1327	1.8936	3.3396	3.7393
		EVP2(8)	0.5239	1.1327	1.8936	3.3396	3.7393
		EVP2(4)	0.5239	1.1327	1.8936	3.3403	3.7404
		FSDT(5/6)	0.5206	1.1327	1.9146	3.3590	3.8032
		FSDT(π^2/12)	0.5197	1.1327	1.9146	3.3396	3.7843
		CLPT	0.5951	1.1327	1.9146	-	-
0.3	0.3	Exact [44]	0.6889	1.3329	2.2171	3.4126	3.9310
		Refine	0.6889	1.3329	2.2171	3.4126	3.9310
		Rao [26]	0.6892	1.3328	2.2171	3.4134	3.9438
		EVP2(16)	0.6889	1.3329	2.2171	3.4126	3.9310
		EVP2(8)	0.6889	1.3329	2.2171	3.4127	3.9311
		EVP2(4)	0.6890	1.3329	2.2171	3.4134	3.9323
		FSDT(5/6)	0.6834	1.3329	2.2530	3.4317	4.0111
		FSDT(π^2/12)	0.6821	1.3329	2.2530	3.4126	3.9927
		CLPT	0.8090	1.3329	2.2530	-	-

Notes:

* In headings I–A through III-A, roman numerals I through III denote the first through third modes, while "A" or "S" denotes antisymmetric or symmetric modes of vibration.
"Refine" denotes analytical method using propagator matrix
"EVP2" denotes quadratic three node finite elements through the thickness. Number in brackets is number of elements used through the entire thickness.
"FSDT" denotes first order shear deformation theory. Number in brackets is shear correction coefficient.
"CLPT" denotes classical laminated plate theory.

Table 4.2. Variations of stresses and displacements across thickness of a homogeneous isotropic plate in free vibration ($\nu = 0.3$, and mh/a = nh/b = 0.3)

z/h	u/u(0.5)			w/w(0.5)			$\tau_{xz}/\tau_{xz}(0)$			$\sigma_{zz}/\sigma_{zz}(0.1)$		
	Exact [44]	Refine	Rao	Exact [44]	Refine	Rao	Exact [44]	Refine	Rao	Exact [44]	Refine	Rao
First antisymmetric thickness mode (I-A)												
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.7561	0.7561	0.7560	1.0249	1.0249	1.0250	0.3750	0.3750	0.3750	1.5336	1.5336	1.5340
0.3	0.5420	0.5420	0.5420	1.0426	1.0426	1.0430	0.6549	0.6549	0.6550	2.0238	2.0238	2.0240
0.2	0.3496	0.3496	0.3500	1.0543	1.0543	1.0540	0.8426	0.8486	0.8480	1.7578	1.7578	1.7580
0.1	0.1713	0.1713	0.1710	1.0610	1.0610	1.0610	0.9625	0.9625	0.9620	1.0000	1.0000	1.0000
0	0.0000	0.0000	0.0000	1.0631	1.0631	1.0630	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
Third antisymmetric thickness mode (III-A)												
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9892	0.9892	0.9890	0.8148	0.8148	0.8150	0.2925	0.2925	0.2930	1.3202	1.3202	1.3200
0.3	0.8655	0.8655	0.8650	0.6166	0.6166	0.6170	0.5703	0.5703	0.5700	1.8691	1.8691	1.8690
0.2	0.6409	0.6409	0.6410	0.4426	0.4426	0.4430	0.7984	0.7983	0.7980	1.7063	1.7063	1.7060
0.1	0.3407	0.3407	0.3410	0.3243	0.3243	0.3240	0.9479	0.9479	0.9480	1.0000	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.2825	0.2825	0.2830	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
Second symmetric thickness mode (II-S)							$\tau_{xz}/\tau_{xz}(0.3)$					
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	1.0438	1.0438	1.0440	0.8280	0.8280	0.8280	0.7425	0.7425	0.7430	0.3611	0.3611	0.3610
0.3	1.0797	1.0797	1.0800	0.6377	0.6377	0.6380	1.0000	1.0000	1.0000	0.6534	0.6534	0.6530
0.2	1.1065	1.1065	1.1070	0.4333	0.4333	0.4330	0.8813	0.8813	0.8810	0.8684	0.8684	0.8680
0.1	1.1230	1.1230	1.1230	0.2191	0.2191	0.2190	0.5057	0.5058	0.5060	1.0000	1.0000	1.0000
0	1.1286	1.1286	1.1290	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0443	1.0443	1.0440

Table 4.3 Comparison of non-dimensional frequencies, Ω, of orthotropic square plate

mh/a	nh/a		I-A*	I-S	II-S	II-A	III-A
0.1	0.1	Exact [45]	0.0474	0.2170	0.3941	1.3077	1.6530
		Refine	0.0474	0.2170	0.3941	1.3077	1.6530
		Rao [26]	0.0474	0.2168	0.3940	1.3080	1.6534
		EVP2(16)	0.0474	0.2170	0.3941	1.3077	1.6530
		EVP2(8)	0.0474	0.2170	0.3941	1.3077	1.6530
		EVP2(4)	0.0474	0.2170	0.3941	1.3080	1.6534
		FSDT(5/6)	0.0474	0.2170	0.3941	1.3159	1.6646
		FSDT(π^2/12)	0.0474	0.2170	0.3941	1.3079	1.6540
		CLPT	0.0493	0.2170	0.3941	-	-
0.1	0.2	Exact [45]	0.1033	0.3450	0.5624	1.3332	1.7160
		Refine	0.1033	0.3450	0.5624	1.3331	1.7160
		Rao [26]	0.1032	0.3449	0.5623	1.3334	1.7170
		EVP2(16)	0.1033	0.3450	0.5624	1.3332	1.7160
		EVP2(8)	0.1033	0.3450	0.5624	1.3332	1.7161
		EVP2(4)	0.1033	0.3450	0.5624	1.3335	1.7165
		FSDT(5/6)	0.1032	0.3451	0.5626	1.3411	1.7305
		FSDT(π^2/12)	0.1031	0.3451	0.5626	1.3332	1.7201
		CLPT	0.1098	0.3451	0.5626	-	-
0.2	0.1	Exact [45]	0.1188	0.3515	0.6728	1.4205	1.6805
		Refine	0.1188	0.3515	0.6728	1.4205	1.6805
		Rao [26]	0.1188	0.3514	0.6727	1.4208	1.6810
		EVP2(16)	0.1188	0.3515	0.6728	1.4205	1.6805
		EVP2(8)	0.1188	0.3515	0.6728	1.4205	1.6805
		EVP2(4)	0.1188	0.3515	0.6728	1.4208	1.6809
		FSDT(5/6)	0.1187	0.3515	0.6728	1.4285	1.6921
		FSDT(π^2/12)	0.1186	0.3515	0.6728	1.4209	1.6818
		CLPT	0.1327	0.3515	0.6728		
0.2	0.2	Exact [45]	0.1694	0.4338	0.7880	1.4316	1.7509
		Refine	0.1694	0.4338	0.7880	1.4316	1.7509
		Rao [26]	0.1694	0.4336	0.7879	1.4319	1.7520
		EVP2(16)	0.1694	0.4338	0.7880	1.4316	1.7509
		EVP2(8)	0.1694	0.4338	0.7880	1.4316	1.7509
		EVP2(4)	0.1694	0.4338	0.7880	1.4319	1.7513
		FSDT(5/6)	0.1692	0.4340	0.7882	1.4393	1.7655
		FSDT(π^2/12)	0.1689	0.4340	0.7882	1.4316	1.7554
		CLPT	0.1924	0.4340	0.7882	-	-

0.1	0.3	Exact [45]	0.1888	0.4953	0.7600	1.3765	1.8115
		Refine	0.1888	0.4953	0.7600	1.3765	1.8115
		Rao [26]	0.1887	0.4951	0.7598	1.3768	1.8133
		EVP2(16)	0.1888	0.4953	0.7600	1.3765	1.8115
		EVP2(8)	0.1888	0.4953	0.7600	1.3765	1.8115
		EVP2(4)	0.1888	0.4953	0.7600	1.3768	1.8119
		FSDT(5/6)	0.1884	0.4954	0.7609	1.3841	1.8306
		FSDT(π^2/12)	0.1882	0.4954	0.7609	1.3765	1.8204
		CLPT	0.2070	0.4954	0.7609	-	-
0.3	0.1	Exact [45]	0.2180	0.5029	0.9728	1.5778	1.7334
		Refine	0.2180	0.5029	0.9728	1.5777	1.7334
		Rao [26]	0.2180	0.5028	0.9727	1.5781	1.7340
		EVP2(16)	0.2180	0.5029	0.9728	1.5777	1.7334
		EVP2(8)	0.2180	0.5029	0.9728	1.5778	1.7334
		EVP2(4)	0.2181	0.5029	0.9728	1.5780	1.7338
		FSDT(5/6)	0.2178	0.5030	0.9728	1.5857	1.7450
		FSDT(π^2/12)	0.2173	0.5030	0.9728	1.5783	1.7353
		CLPT	0.2671	0.5030	0.9728	-	-
0.3	0.3	Exact [45]	0.3320	0.6504	1.1814	1.5737	1.9289
		Refine	0.3320	0.6504	1.1814	1.5737	1.9289
		Rao [26]	0.3320	0.6502	1.1813	1.5740	1.9311
		EVP2(16)	0.3320	0.6504	1.1814	1.5737	1.9289
		EVP2(8)	0.3320	0.6504	1.1814	1.5738	1.9290
		EVP2(4)	0.3321	0.6504	1.1814	1.5740	1.9294
		FSDT(5/6)	0.3310	0.6510	1.1822	1.5812	1.9480
		FSDT(π^2/12)	0.3302	0.6510	1.1822	1.5737	1.9389
		CLPT	0.4172	0.6510	1.1822	-	-

Notes:

* In headings I–A through III-A, roman numerals I through III denote the first through third modes, while "A" or "S" denotes antisymmetric or symmetric modes of vibration.
"Refine" denotes analytical method using propagator matrix
"EVP2" denotes quadratic three node finite elements through the thickness. Number in brackets is number of elements used through the entire thickness.
"FSDT" denotes first order shear deformation theory. Number in brackets is shear correction coefficient.
"CLPT" denotes classical laminated plate theory.

Table 4.4 Variations of stresses and displacements across thickness of a homogeneous orthotropic plate in free vibration (mh/a = nh/b = 0.3)

z/h	u/u(0.5)			v/v(0.5)			w/w(0.5)			$\tau_{xz}/\tau_{xz}(0)$			$\tau_{yz}/\tau_{yz}(0)$			$\sigma_{zz}/\sigma_{zz}(0.1)$		
	Exact [45]	Refine	Rao	Exact [45]	Refine	Rao	Exact [45]	Refine	Rao	Exact [45]	Refine	Rao	Exact [45]	Refine	Rao	Exact [45]	Refine	Rao
First antisymmetric thickness mode (I-A)																		
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.6963	0.6963	0.6963	0.7644	0.7644	0.7644	1.0054	1.0054	1.0054	0.3943	0.3943	0.3942	0.3835	0.3835	0.3835	1.5685	1.5685	1.5685
0.3	0.4650	0.4650	0.4649	0.5532	0.5532	0.5532	1.0083	1.0083	1.0082	0.6738	0.6738	0.6737	0.6632	0.6632	0.6632	2.0482	2.0482	2.0482
0.2	0.2838	0.2838	0.2838	0.3595	0.3595	0.3594	1.0096	1.0096	1.0096	0.8595	0.8595	0.8594	0.8534	0.8534	0.8534	1.7657	1.7657	1.7657
0.1	0.1343	0.1343	0.1342	0.1770	0.1770	0.1769	1.0102	1.0102	1.0102	0.9655	0.9655	0.9655	0.9638	0.9638	0.9638	1.0000	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0103	1.0103	1.0103	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
Third antisymmetric thickness mode (III-A)																		
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9922	0.9922	0.9921	0.9721	0.9721	0.9720	0.9023	0.9023	0.9022	0.2942	0.2942	0.2942	0.3021	0.3021	0.3020	1.3098	1.3098	1.3097
0.3	0.8698	0.8698	0.8698	0.8399	0.8399	0.8398	0.7710	0.7710	0.7708	0.5719	0.5719	0.5718	0.5804	0.5804	0.5804	1.8614	1.8614	1.8614
0.2	0.6450	0.6450	0.6449	0.6167	0.6167	0.6166	0.6439	0.6439	0.6437	0.7992	0.7992	0.7992	0.8045	0.8045	0.8045	1.7037	1.7038	1.7037
0.1	0.3431	0.3431	0.3431	0.3262	0.3262	0.3261	0.5533	0.5533	0.5530	0.9482	0.9482	0.9481	0.9497	0.9497	0.9497	1.0000	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.5206	0.5206	0.5203	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
First symmetric thickness mode (I-S)										$\tau_{xz}/\tau_{xz}(0.3)$			$\tau_{yz}/\tau_{yz}(0.3)$					
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9890	0.9890	0.9889	1.0074	1.0074	1.0074	0.8112	0.8112	0.8112	0.7609	0.7609	0.7608	0.7614	0.7614	0.7613	0.3868	0.3868	0.3867
0.3	0.9811	0.9811	0.9810	1.0131	1.0131	1.0130	0.6148	0.6148	0.6148	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6777	0.6777	0.6778
0.2	0.9757	0.9757	0.9757	1.0170	1.0170	1.0170	0.4129	0.4129	0.4128	0.8661	0.8661	0.8660	0.8656	0.8656	0.8656	0.8804	0.8804	0.8803
0.1	0.9727	0.9727	0.9726	1.0194	1.0194	1.0194	0.2073	0.2073	0.2073	0.4919	0.4919	0.4918	0.4915	0.4915	0.4914	1.0000	1.0000	1.0000
0	0.9717	0.9717	0.9716	1.0202	1.0202	1.0202	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0395	1.0395	1.0395
Second symmetric thickness mode (II-S)																		
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	1.0073	1.0073	1.0072	1.0120	1.0120	1.0120	0.8362	0.8362	0.8362	0.7241	0.7240	0.7240	0.7336	0.7335	0.7334	0.3602	0.3602	0.3602
0.3	1.0131	1.0131	1.0131	1.0221	1.0221	1.0220	0.6489	0.6489	0.6489	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6526	0.6526	0.6525
0.2	1.0173	1.0173	1.0173	1.0296	1.0296	1.0296	0.4431	0.4431	0.4431	0.8968	0.8970	0.8970	0.8887	0.8890	0.8889	0.8680	0.8680	0.8680
0.1	1.0199	1.0199	1.0199	1.0342	1.0342	1.0342	0.2248	0.2248	0.2247	0.5197	0.5202	0.5202	0.5122	0.5128	0.5128	1.0000	1.0000	1.0000
0	1.0208	1.0208	1.0207	1.0358	1.0358	1.0358	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0444	1.0445	1.0444

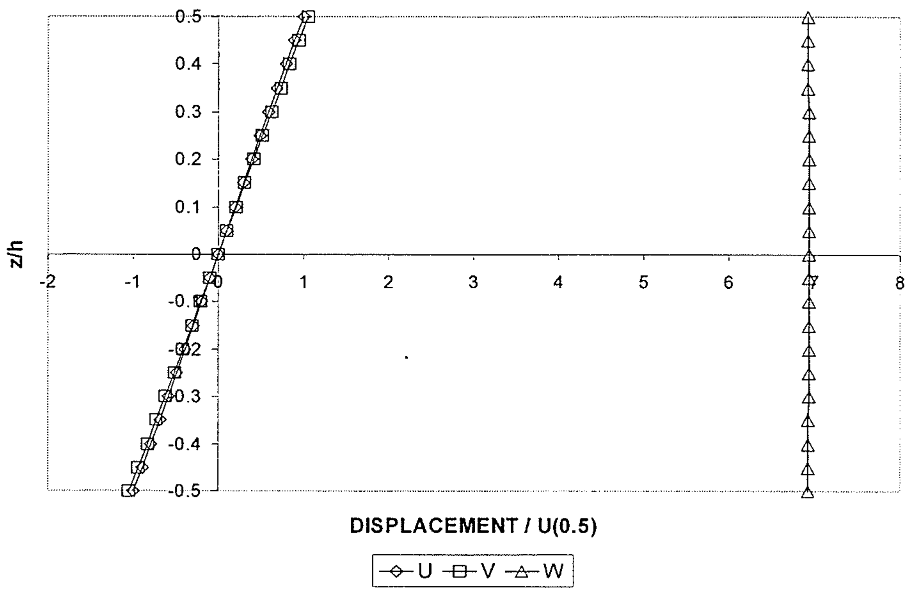

Figure 4.2a: Variations of displacements through the thickness for first bending mode of vibration of the orthotropic plate in Example 1

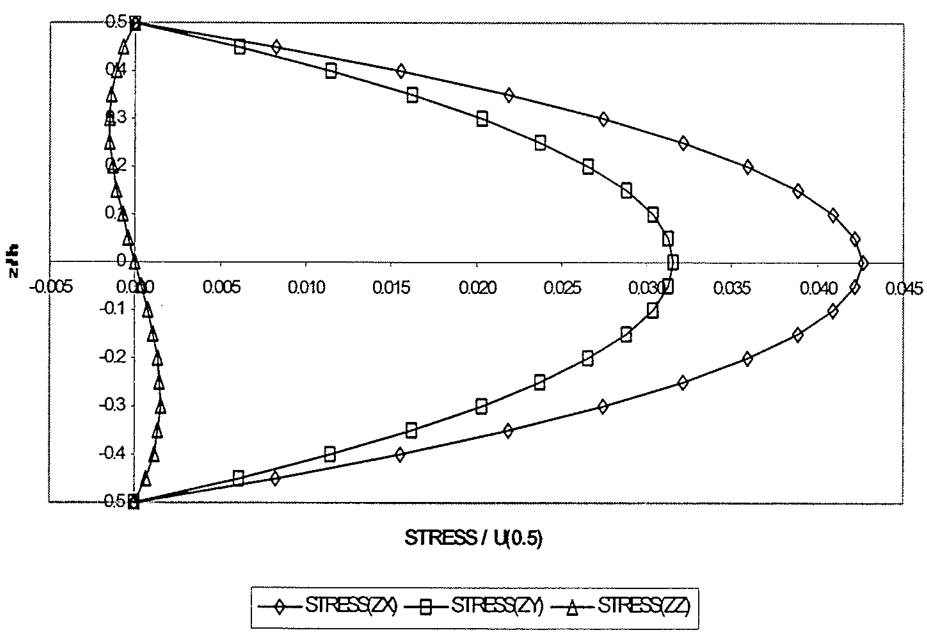

Figure 4.2b: Variations of stresses through thickness for first bending mode of vibration of the orthotropic plate in Example 1

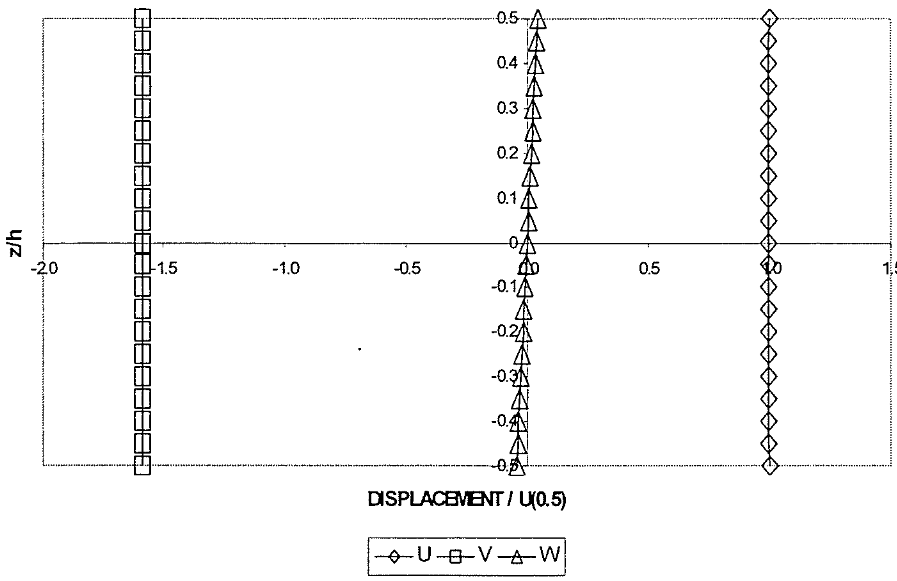

Figure 4.3a: Variations of displacements through the thickness for first extensional mode of vibration of the orthotropic plate in Example 1

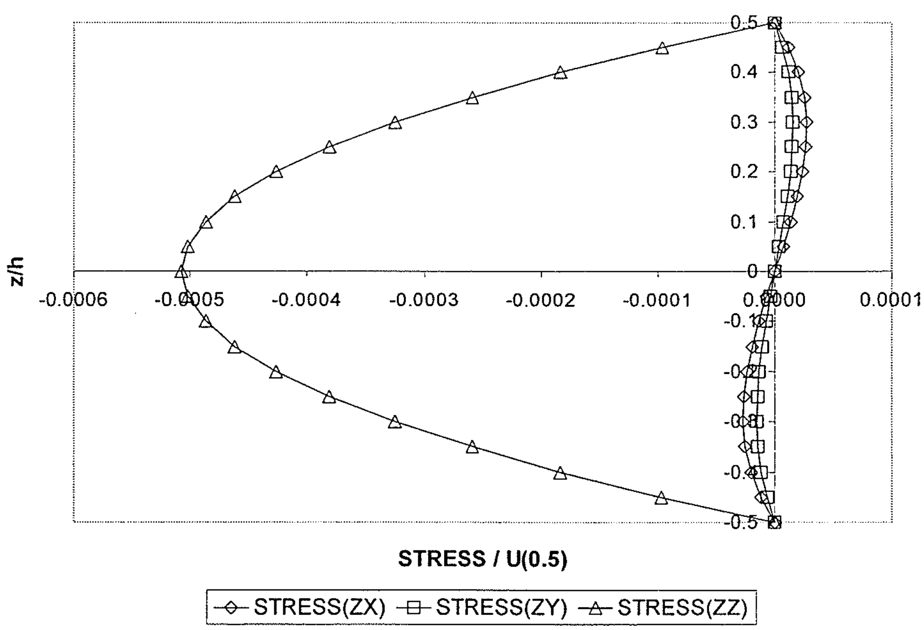

Figure 4.3b: Variations of stresses through thickness for first extensional mode of vibration of the orthotropic plate in Example 1

Table 4.5 Comparison of non-dimensional fundamental frequencies, Ω, of cross-ply square plates

a/h	[0°/90°/90°/0°]						[0°/90°]					
	CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(8)	Rao [26]	Refine	CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(8)	Rao [26]	Refine
2	15.902	5.501	5.471	5.317	5.306	5.315	8.607	5.212	5.187	4.955	4.952	4.953
5	18.299	10.862	10.820	10.683	10.675	10.682	10.721	8.840	8.821	8.528	8.520	8.527
10	18.738	15.160	15.127	15.069	15.066	15.069	11.154	10.484	10.476	10.337	10.336	10.336
20	18.853	17.684	17.670	17.636	17.634	17.636	11.269	11.091	11.089	11.037	11.040	11.037
25	18.867	18.096	18.086	18.055	18.054	18.055	11.283	11.173	11.171	11.132	11.135	11.132
50	18.885	18.701	18.699	18.670	18.669	18.670	11.302	11.285	11.284	11.264	11.268	11.264
100	18.890	18.864	18.863	18.835	18.834	18.835	11.307	11.313	11.313	11.297	11.301	11.297

Notes:

"Refine" denotes analytical method using propagator matrix
"EVP2" denotes quadratic three node finite elements through the thickness. Number in brackets is number of elements used through the entire thickness.
"FSDT" denotes first order shear deformation theory. Number in brackets is shear correction coefficient.
"CLPT" denotes classical laminated plate theory.

Table 4.6 Variations of stresses and displacements across the thickness of a cross-ply [0°/90°/90°/0°] square plate in free vibration for which a / h = 10 and m/a = n/b = 0.1

z/h	$u/u(0.5)$		$v/v(0.5)$		$w/w(0.5)$		$\tau_{xz}/\tau_{xz}(0)$		$\tau_{yz}/\tau_{yz}(0)$		$\sigma_{zz}/\sigma_{zz}(0.1)$	
	Refine	Rao	Refine	Rao	Refine	Rao	Refine	Rao	Refine	Rao	Refine	Rao
First antisymmetric thickness mode (I-A)												
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.7274	0.7270	0.7846	0.7843	1.0018	1.0006	0.4951	0.4954	0.0657	0.0655	1.3759	1.3767
0.3	0.5045	0.5040	0.5719	0.5715	1.0030	1.0009	0.8491	0.8494	0.1149	0.1143	1.8675	1.8682
0.2	0.3249	0.3245	0.3640	0.3636	1.0038	1.0011	0.9878	0.9879	0.4523	0.4518	1.7128	1.7130
0.1	0.1621	0.1618	0.1760	0.1758	1.0042	1.0013	0.9970	0.9970	0.8653	0.8652	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.0000	1.0044	1.0013	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000
Second antisymmetric thickness mode (II-A)												
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9296	0.9363	0.9393	0.9389	0.8784	0.9764	0.2398	0.2364	0.3908	0.3909	1.6266	1.5700
0.3	0.8047	0.8141	0.7601	0.7595	0.7389	0.9198	0.4504	0.4459	0.7334	0.7334	2.1603	2.1054
0.2	0.6132	0.6216	0.5142	0.5139	0.6124	0.8218	0.6985	0.6953	0.9179	0.9178	1.7970	1.7802
0.1	0.3295	0.3344	0.2627	0.2625	0.5244	0.7269	0.9222	0.9213	0.9792	0.9792	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.0000	0.4939	0.6939	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000
Third antisymmetric thickness mode (III-A)												
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	0.9664	0.9662	0.9691	0.9636	0.8312	0.8485	0.2264	0.2266	0.3817	0.3861	1.1507	1.1933
0.3	0.8634	0.8629	0.8002	0.7913	0.6254	0.6563	0.4373	0.4376	0.7253	0.7285	1.7039	1.7483
0.2	0.6728	0.6723	0.5485	0.5398	0.4345	0.5064	0.6910	0.6913	0.9133	0.9144	1.6527	1.6666
0.1	0.3660	0.3657	0.2825	0.2771	0.3017	0.4248	0.9195	0.9195	0.9782	0.9786	1.0000	1.0000
0	0.0000	0.0000	0.0000	0.0000	0.2532	0.3944	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000
Second symmetric thickness mode (II-S)							$\tau_{xz}/\tau_{xz}(0.3)$		$\tau_{yz}/\tau_{yz}(0.3)$			
0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4	1.0210	1.0189	0.9847	0.9831	0.8233	0.8306	0.4918	0.4920	0.5090	0.5095	0.3729	0.3677
0.3	1.0771	1.0732	0.9334	0.9308	0.6314	0.6400	1.0000	1.0000	1.0000	1.0000	0.6593	0.6470
0.2	1.1689	1.1635	0.8632	0.8600	0.4278	0.4286	1.0275	1.0270	0.9748	0.9743	0.8668	0.8569
0.1	1.2370	1.2306	0.8215	0.8179	0.2160	0.2141	0.5241	0.5237	0.4800	0.4796	1.0000	1.0000
0	1.2600	1.2533	0.8078	0.8040	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0461	1.0508

Table 4.7 Nondimensionalized fundamental frequencies, Ω, for simply supported, symmetric cross-ply square laminated plates where a / h = 5

Lamination and Number of Layers	Source	E_1/E_2 3	10	20	30	40
$[0^\circ/90^\circ]_s$	3 D Elasticity [43]	6.6185	8.2103	9.5603	10.2723	10.7515
	Refine	6.5767	8.2770	9.4990	10.2062	10.6822
	Rao [26]	6.5712	8.1698	9.2515	9.8595	10.2687
	EVP2(8)	6.5767	8.2771	9.4992	10.2066	10.6828
	FSDT(5/6)	6.5695	8.2982	9.5671	10.3258	10.8540
	FSDT($\pi^2/12$)	6.5612	8.2794	9.5375	10.2889	10.8117
	CLPT	7.2994	10.3159	13.5108	16.0840	18.2989
$[0^\circ/90^\circ/0^\circ]_s$	3 D Elasticity [43]	6.6468	8.5223	9.9480	10.7850	11.3435
	Refine	6.6002	8.4617	9.8769	10.7075	11.2618
	Rao [26]	6.6032	8.4383	9.8248	10.6436	11.1956
	EVP2(10)	6.6002	8.4618	9.8770	10.7078	11.2621
	FSDT(5/6)	6.5897	8.4854	9.9689	10.8683	11.4836
	FSDT($\pi^2/12$)	6.5816	8.4674	9.9394	10.8302	11.4389
	CLPT	7.2994	10.3159	13.5108	16.0840	18.2989
$[0^\circ/90^\circ/0^\circ/90^\circ/0^\circ]_s$	3 D Elasticity [43]	6.6600	8.6080	10.1368	10.0525	11.6698
	Refine	6.6095	8.5421	10.0584	10.9665	11.5789
	Rao [26]	6.6140	8.5423	10.0545	10.9624	11.5808
	EVP2(9)	6.6095	8.5422	10.0586	10.9668	11.5792
	FSDT(5/6)	6.5960	8.5423	10.0837	11.0146	11.6451
	FSDT($\pi^2/12$)	6.5880	8.5245	10.0541	10.9759	11.5994
	CLPT	7.2994	10.3159	13.5108	16.0840	18.2989

Notes:
Total sum of the thicknesses of the 0° and 90° layers are equal.

"Refine" denotes analytical method using propagator matrix
"EVP2" denotes quadratic three node finite elements through the thickness. Number in brackets is number of elements used through the entire thickness.
"FSDT" denotes first order shear deformation theory. Number in brackets is shear correction coefficient.
"CLPT" denotes classical laminated plate theory.

Table 4.8 Non-dimensionalized fundamental frequencies, Ω, for simply supported, antisymmetric cross-ply square laminated plates where a / h = 5

Lamination and Number of Layers	Source	E_1/E_2 3	10	20	30	40
$[0^o/90^o]_1$	3 D Elasticity [43]	6.2578	6.9845	7.6745	8.1763	8.5625
	Refine	6.2317	6.9555	7.6427	8.1423	8.5269
	Rao [26]	6.2336	6.9740	7.7138	8.2770	8.7266
	EVP2(8)	6.2317	6.9556	7.6430	8.1428	8.5277
	FSDT(5/6)	6.2085	6.9392	7.7060	8.3211	8.8333
	FSDT(π^2/12)	6.2020	6.9301	7.6934	8.3052	8.8142
	CLPT	6.7706	7.7419	8.8554	9.8337	10.7209
$[0^o/90^o]_2$	3 D Elasticity [43]	6.5455	8.1445	9.4055	10.1650	10.6798
	Refine	6.5044	8.0928	9.3453	10.0998	10.6111
	Rao [26]	6.5145	8.1483	9.4675	10.2731	10.8219
	EVP2(8)	6.5044	8.0928	9.3455	10.1002	10.6117
	FSDT(5/6)	6.5043	8.2246	9.6885	10.6198	11.2708
	FSDT(π^2/12)	6.4966	8.2090	9.6627	10.5856	11.2298
	CLPT	7.1690	9.7193	12.4767	14.7250	16.6726
$[0^o/90^o]_3$	3 D Elasticity [43]	6.6100	8.4143	9.8398	10.6958	11.2728
	Refine	6.5638	8.3549	9.7695	10.6190	11.1918
	Rao [26]	6.5712	8.3858	9.8353	10.7117	11.3052
	EVP2(12)	6.5638	8.3549	9.7696	10.6191	11.1919
	FSDT(5/6)	6.5569	8.4183	9.9427	11.0146	11.5264
	FSDT(π^2/12)	6.5491	8.4015	9.9148	10.9759	11.4824
	CLPT	7.2416	10.0537	13.0584	16.0839	17.5899
$[0^o/90^o]_5$	3 D Elasticity [43]	6.6458	8.5625	10.0843	11.0027	11.6245
	Refine	6.5955	8.4970	10.0064	10.9173	11.5341
	Rao [26]	6.6019	8.5164	10.0440	10.9698	11.5992
	EVP2(10)	6.5955	8.4970	10.0065	10.9175	11.5343
	FSDT(5/6)	6.5837	8.5132	10.0638	11.0058	11.6444
	FSDT(π^2/12)	6.5757	8.4958	10.0348	10.9676	11.5990
	CLPT	7.2786	10.2220	13.3492	15.8722	18.0460

Table 4.9 Comparison of non-dimensional frequencies, Ω, of an antisymmetric ($0^{o}/90^{o}$/core/$0^{o}/90^{o}$) sandwich plate where a / h = 100, a / b = 1 and t_c / t_f = 10

m	n	Considering G_{13} and G_{23} of stiff layers						
		CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(5)	EVP2(14)	Rao [26]	Refine
1	1	16.2953	16.2685	16.2681	11.9405	11.9405	11.9592	11.9405
1	2	45.1106	44.8707	44.8675	23.4015	23.4015	23.4676	23.4015
1	3	96.4483	95.3040	95.2891	36.1425	36.1425	36.2659	36.1425
2	1	45.1106	44.8707	44.8675	23.4015	23.4015	23.4673	23.4015
2	2	65.1473	64.7224	64.7169	30.9429	30.9429	31.0375	30.9429
2	3	110.6399	109.3181	109.3009	41.4466	41.4466	41.5888	41.4466
3	1	96.4483	95.3040	95.2891	36.1425	36.1425	36.2661	36.1425
3	2	110.6399	109.3181	109.3009	41.4466	41.4466	41.5884	41.4466
3	3	146.4546	144.3384	144.3111	49.7609	49.7609	49.9378	49.7609

Table 4.10 Comparison of non-dimensional frequencies, Ω, of an antisymmetric ($0^{o}/90^{o}$/core/$0^{o}/90^{o}$) sandwich plate where a / h = 10, a / b = 1 and t_c / t_f = 10

m	n	Considering G_{13} and G_{23} of stiff layers					
		CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(14)	Rao [26]	Refine
1	1	16.0224	13.9936	13.9716	1.8480	1.8548	1.8480
1	2	43.2902	31.1052	31.0057	3.2199	3.2301	3.2199
1	3	89.0968	51.9371	51.7074	5.2245	5.2376	5.2243
2	1	43.2902	31.1052	31.0057	3.2199	3.2301	3.2199
2	2	61.0874	42.2336	42.0838	4.2903	4.3024	4.2902
2	3	100.0424	59.4927	59.2323	6.0961	6.1101	6.0958
3	1	89.0968	51.9371	51.7074	5.2245	5.2376	5.2243
3	2	100.0424	59.4927	59.2323	6.0961	6.1101	6.0958
3	3	128.0226	72.8386	72.4960	7.6801	7.6951	7.6796

Table 4.11 Comparison of non-dimensional fundamental frequencies, Ω, of an antisymmetric ($0^o/90^o$/core/$0^o/90^o$) sandwich plate where $a / b = 1$ and $t_c / t_f = 10$

a/h	CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(14)	EVP2(28)	Refine
2	11.9127	5.3279	5.2975	0.7142	0.7141	0.7141
4	14.7731	9.2003	9.1615	0.9363	0.9363	0.9363
10	16.0224	13.9936	13.9716	1.8480	1.8480	1.8480
20	16.2279	15.6067	15.5989	3.4790	3.4791	3.4791
30	16.2668	15.9785	15.9748	5.0371	5.0371	5.0371
40	16.2805	16.1157	16.1136	6.4634	6.4634	6.4634
50	16.2868	16.1806	16.1792	7.7355	7.7355	7.7355
60	16.2903	16.2162	16.2153	8.8492	8.8492	8.8492
70	16.2924	16.2378	16.2371	9.8118	9.8118	9.8118
80	16.2937	16.2519	16.2513	10.6368	10.6368	10.6368
90	16.2946	16.2616	16.2612	11.3408	11.3408	11.3408
100	16.2953	16.2685	16.2681	11.9400	11.9400	11.9400

Table 4.12 Comparison of non-dimensional fundamental frequencies, Ω, of an antisymmetric ($0^o/90^o$/core/$0^o/90^o$) sandwich plate where $a / b = 1$ and $a / h = 10$

t_c/t_f	CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(14)	EVP2(28)	Refine
4	15.7514	14.0339	14.0147	1.9085	1.9084	1.9084
10	16.0224	13.9936	13.9716	1.8480	1.8480	1.8480
20	15.0489	13.0147	12.9929	2.1307	2.1307	2.1307
30	14.0326	12.0887	12.0680	2.3321	2.3321	2.3321
40	13.1555	11.3097	11.2902	2.4690	2.4690	2.4690
50	12.4104	10.6561	10.6375	2.5658	2.5658	2.5658
100	9.9357	8.5148	8.4998	2.7875	2.7875	2.7875

Table 4.13 Comparison of non-dimensional fundamental frequencies, Ω, of an antisymmetric (0°/90°/core/0°/90°) sandwich plate where $t_c / t_f = 10$ and $a / h = 10$

a/b	CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(14)	EVP2(28)	Refine
0.5	44.6484	39.8138	39.7601	5.7326	5.7326	5.7326
1.0	16.0224	13.9936	13.9716	1.8480	1.8480	1.8464
1.5	11.9740	9.6044	9.5817	1.0905	1.0905	1.0900
2.0	10.8226	7.7763	7.7514	0.8050	0.8050	0.8048
2.5	10.2754	6.6207	6.5949	0.6628	0.6627	0.6627
3.0	9.8996	5.7708	5.7453	0.5805	0.5805	0.5804
5.0	5.3388	3.7714	3.7505	0.4496	0.4495	0.4494

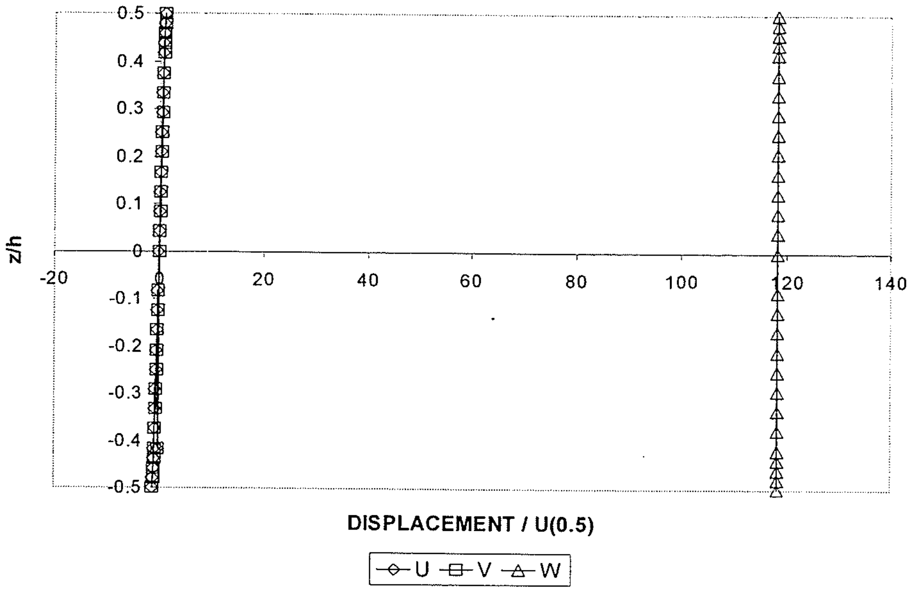

Figure 4.5a: Variations of displacements through thickness from first bending mode of vibration of (0°/90°/core/0°/90°) sandwich plate in Example 5

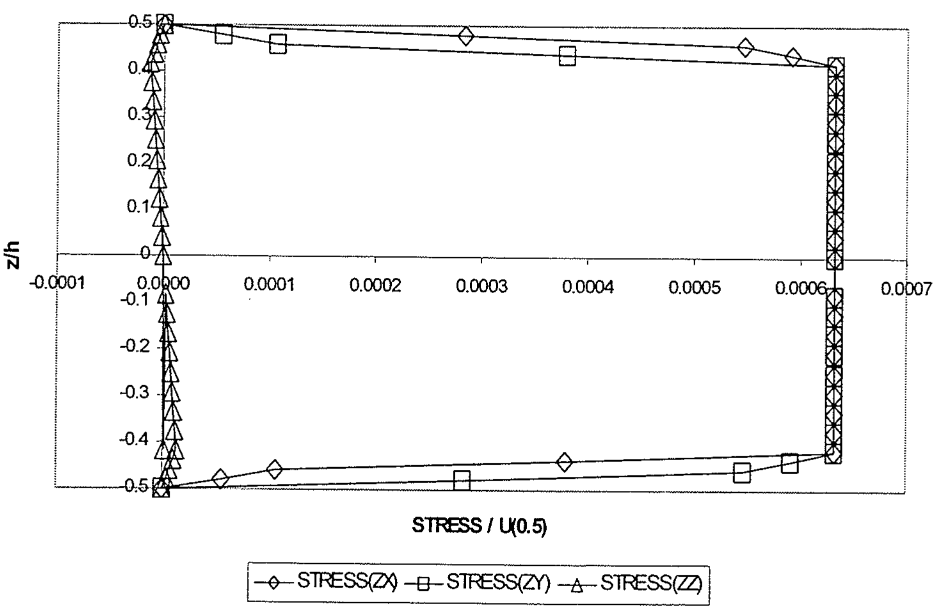

Figure 4.5b: Variations of stresses through thickness from first bending mode of vibration of (0°/90°/core/0°/90°) sandwich plate in Example 5

Table 4.14 Comparison of non-dimensional frequencies, Ω, of an orthotropic homogeneous plate
where $a / h = 100$, $a / b = 1$, and $D_{44} = D_{55} = 0$

m	n	Neglecting G_{13} and G_{23}				
		CLPT	FSDT(5/6)	FSDT(π^2/12)	EVP2(5)	Refine
1	1	397.834	397.834	397.834	397.829	397.829
1	2	658.629	658.629	658.629	658.601	658.601
1	3	941.089	941.089	941.089	941.003	941.003
2	1	596.469	596.469	596.469	596.462	596.462
2	2	795.667	795.667	795.667	795.632	795.634
2	3	1045.810	1045.810	1045.810	1045.710	1045.710
3	1	837.516	827.516	827.516	827.507	827.509
3	2	981.022	981.022	981.022	980.979	980.981
3	3	1193.500	1193.500	1193.500	1193.380	1193.390

Notes:

First frequency is zero, these are results for lowest non-zero frequency
"Refine" denotes analytical method using propagator matrix
"EVP2" denotes quadratic three node finite elements through the thickness. Number in brackets is
number of elements used through the entire thickness.
"FSDT" denotes first order shear deformation theory. Number in brackets is shear correction coefficient.
"CLPT" denotes classical laminated plate theory.

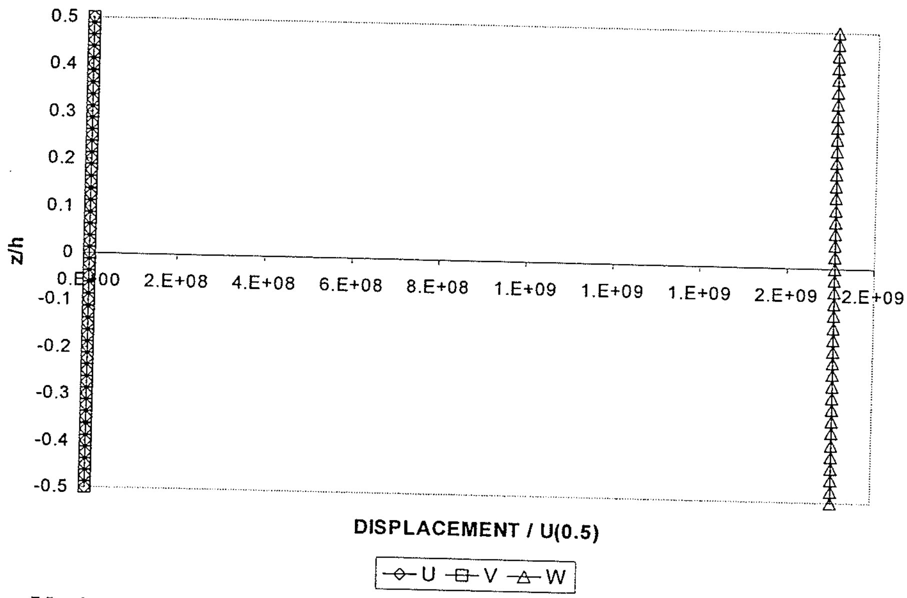

Figure 4.6a: Variations of displacements from first mode zero frequency of vibration of an orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

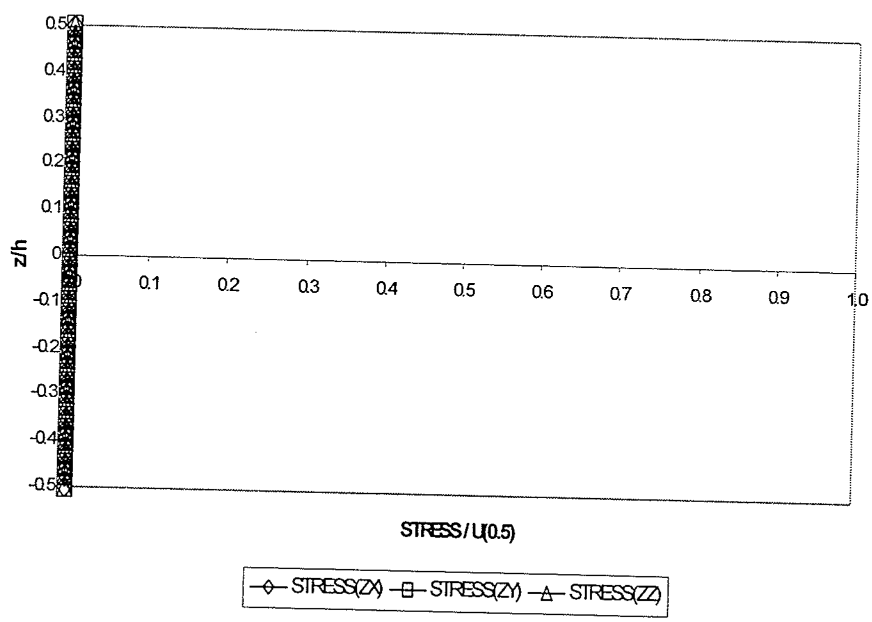

Figure 4.6b: Variations of stresses from first mode zero frequency of vibration of an orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

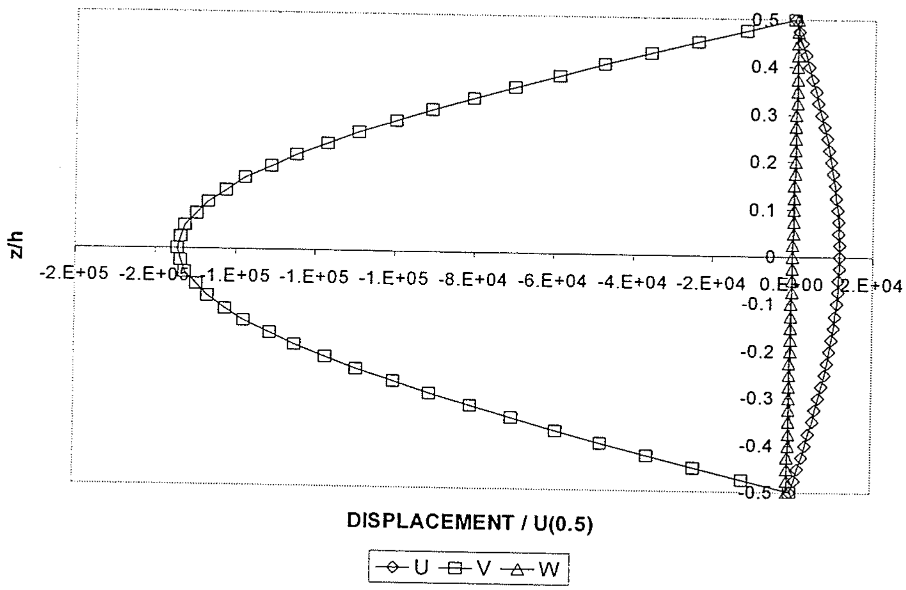

Figure 4.7a: Variations of displacements from first extensional, first non-zero frequency mode of vibration of orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

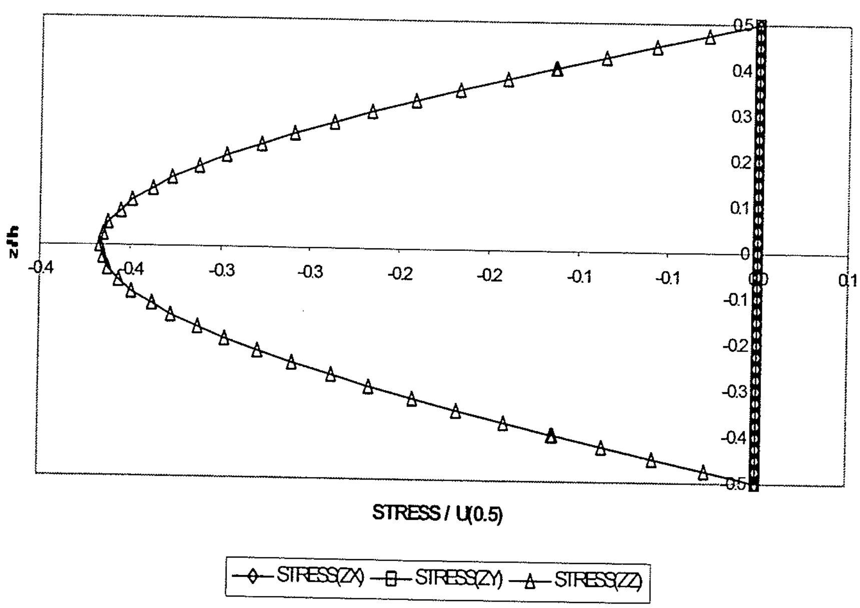

Figure 4.7b: Variations of stresses from first extensional, first non-zero frequency mode of vibration of orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

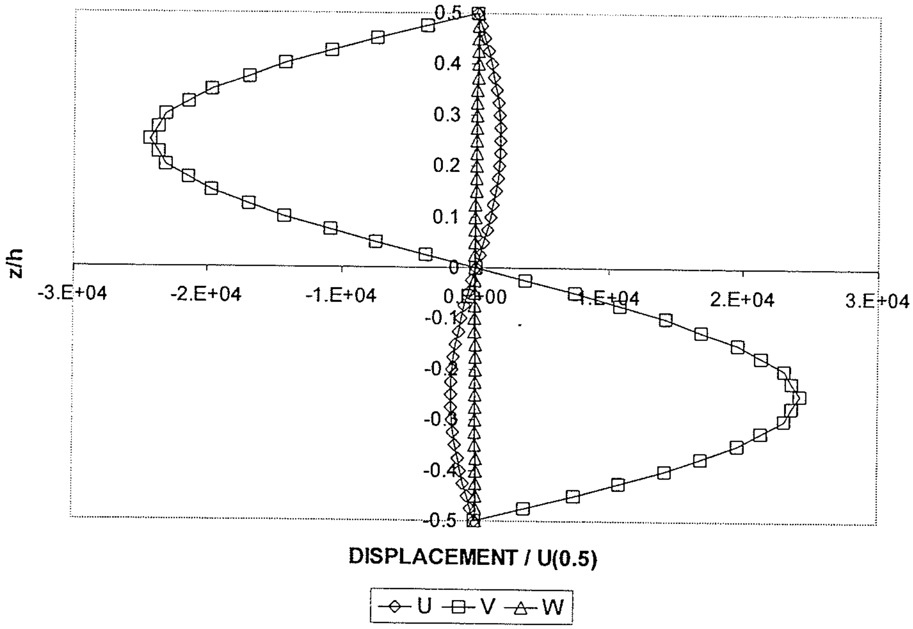

Figure 4.8a: Variations of displacements from first bending, second non-zero frequency mode of vibration of orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

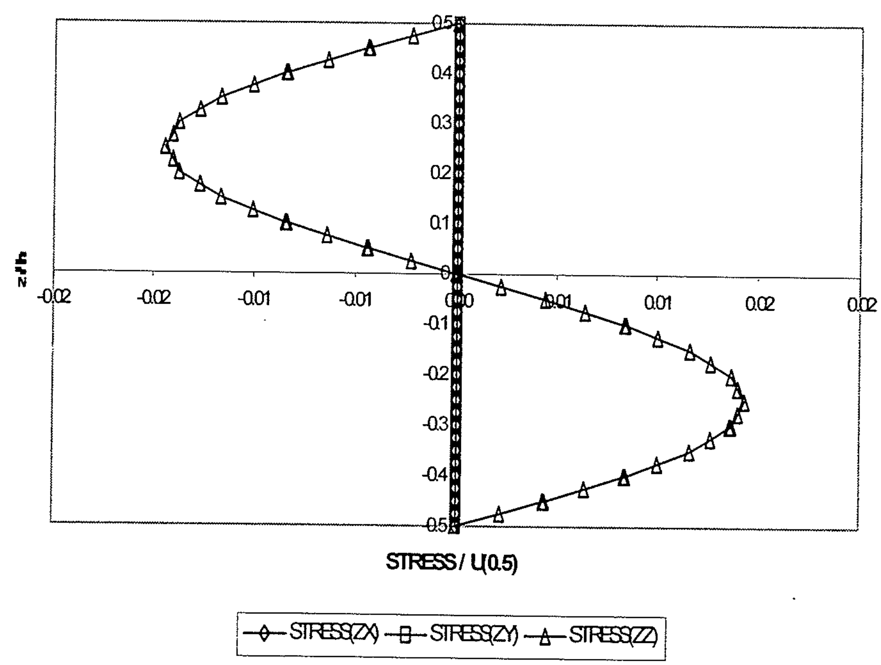

Figure 4.8b: Variations of stresses from first bending, second non-zero frequency mode of vibration of orthotropic homogeneous plate neglecting D_{44} and D_{55} in Example 6

Chapter 5

Conclusions and Recommendations

Four different models of composite, laminated plate vibration were investigated. The novel solution approach, an exact method using the propagator matrix method, was developed and numerical results agree with other published exact solutions. This method is also computationally efficient because only a linear increase in computational effort occurs with an increase in the number of layers. This is in contrast to the Raleigh-Ritz procedure, as well the traditional formulation of solving the eigenvalue problem, both of which yield an increase in computational effort that is proportional to the square of the number of layers or elements analyzed. Classical laminate plate theory, although computationally the most efficient, yields poor estimates of the lowest natural frequency which is a bending mode, but it is as accurate as the first order shear deformation theory for the natural frequencies corresponding to the extensional modes. Solutions from the first order shear deformation theory yield good estimates of the lowest 3 frequencies for a homogeneous plate. In general, equivalent single layer theories are more accurate when either a homogeneous plate or a plate with many laminates is studied. Their accuracy is higher for thin plates rather than thick plates. The same is true for the results from the Raleigh-Ritz type approximation. Accuracy for the approximate numerical methods decreases when the material stiffness in one direction is very large in proportion to material stiffness in another direction, or when the relative stiffnesses of different material layers vary. For a sandwich plate, the accuracy of the equivalent single layer theories increases as the plate becomes more uniform when the ratio t_c / t_f increases. Varying the aspect ratio a / b of a plate does not seem to affect the accuracy of the solutions for the approximate theories. A zero fundamental frequency, corresponding to a rigid body displacement eigenvector, was discovered for an orthotropic material with negligible shear material constants, D_{44} and D_{55}.

The results from the exact theory, using the propagator matrix method, should be compared to experimental results for validation. A problem proposed by Rao [26] i.e. - a

($0^{\circ}/90^{\circ}$/core/$0^{\circ}/90^{\circ}$) sandwich plate where the 0° and 90° orthotropic layers have negligible shear material constants, should be solved.

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Appendix A

Transformation of material coefficients

A.1 Three dimensional strain-stress to stress-strain material coefficients relations

A.1.1 Orthotropic material

The strain-stress elastic constitutive relationship for an orthotropic material is given by Jones, [3] as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & & & \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & & & \\ -\frac{\nu_{21}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{13}} & \\ & & & & & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \tag{A.1}$$

where subscripts 1, 2 and 3 denote the x, y and z directions, respectively. Constants E_1, E_2 and E_3 are Young's moduli in the x, y and z directions, respectively.. Constants ν_{ij} are Poisson's ratios, which are defined as the ratio of the transverse strain in the j th direction to the axial strain in the i th direction, when stressed in the i th direction, or $-\frac{\varepsilon_{jj}}{\varepsilon_{ii}}$ for an applied stress σ_{ii}. Constants G_{23}, G_{13} and G_{12} are shear moduli in the yz, xz and xy planes, respectively. Additionally, the following reciprocal relationship holds among the Young's moduli E_i and Poisson's ratios ν_{ij}:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \ . \tag{A.2}$$

The stress-strain elastic constants, D_{ij}, for an orthotropic material are given by Reddy, [37] as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ D_{12} & D_{22} & D_{23} & & & \\ D_{13} & D_{23} & D_{33} & & & \\ & & & D_{44} & & \\ & & & & D_{55} & \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}. \tag{A.3}$$

The strain-stress material constants given in the square matrix in equation (A.1) can be transformed to the stress-strain material constants, D_{ij}, given in equation (A.3) by using the following relationships given by Reddy, [37]

$$D_{11} = E_1 \frac{1 - \nu_{23}\nu_{32}}{\Delta} \tag{A.4a}$$

$$D_{12} = E_1 \frac{\nu_{21} + \nu_{31}\nu_{23}}{\Delta} = E_2 \frac{\nu_{12} + \nu_{32}\nu_{13}}{\Delta} \tag{A.4b}$$

$$D_{13} = E_1 \frac{\nu_{31} + \nu_{21}\nu_{32}}{\Delta} = E_3 \frac{\nu_{13} + \nu_{12}\nu_{23}}{\Delta} \tag{A.4c}$$

$$D_{22} = E_2 \frac{1 - \nu_{13}\nu_{31}}{\Delta} \tag{A.4d}$$

$$D_{23} = E_2 \frac{\nu_{32} + \nu_{12}\nu_{31}}{\Delta} = E_3 \frac{\nu_{23} + \nu_{21}\nu_{13}}{\Delta} \tag{A.4e}$$

$$D_{33} = E_3 \frac{1 - \nu_{12}\nu_{21}}{\Delta} \tag{A.4f}$$

$$D_{44} = G_{23} \tag{A.4g}$$

$$D_{55} = G_{13} \tag{A.4i}$$

$$D_{66} = G_{12} \tag{A.4j}$$

where

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13} \,. \tag{A.5}$$

A.1.2 Isotropic material

The strain-stress elastic constitutive relationship for an isotropic material is given as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{\mu} & & \\ & & & & \frac{1}{\mu} & \\ & & & & & \frac{1}{\mu} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \tag{A.6}$$

where E is Young's modulus. It is the same in the x, y and z directions. Constant ν is Poisson's ratio, which is defined as the ratio of transverse strain to the axial strain when stressed in the axial direction. Constant μ is the shear modulus and it is the same in the yz, xz and xy planes. For an isotropic material, there are only two independent constants. The above three constants described can be related to one another by the following equation:

$$\mu = \frac{E}{2(1+\nu)} \ . \tag{A.7}$$

The coefficients, D_{ij}, used in equation (A.3) can be stated in terms of the constants, ν and μ, as

$$D_{11} = D_{22} = D_{33} = \frac{2\mu(1-\nu)}{(1-2\nu)} \tag{A.8a}$$

$$D_{12} = D_{13} = D_{23} = \frac{2\mu\nu}{(1-2\nu)} \tag{A.8b}$$

$$D_{44} = D_{55} = D_{66} = \mu \ . \tag{A.8c}$$

A.2 Two dimensional strain-stress to stress-strain material coefficients relations

A.2.1 Orthotropic material in plane stress

The strain-stress elastic constitutive relationship for an orthotropic material in plane stress is given by Reddy, [37] as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & \\ & & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{A.9}$$

where the stresses σ_{zz}, σ_{zx} and σ_{zy} are zero and the strains γ_{zx} and γ_{zy} are also zero. Strain ε_{zz} is not zero but it is given by

$$\varepsilon_{zz} = -\frac{\nu_{13}}{E_1}\sigma_{xx} - \frac{\nu_{23}}{E_2}\sigma_{yy} . \tag{A.10}$$

The strain-stress relationship in equation (A.9) can be inverted to the following stress-strain relationship for an orthotropic material in a state of plane stress

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & \\ Q_{12} & Q_{22} & \\ & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{A.11}$$

where the material constants Q_{ij}, called the plane-stress reduced stiffnesses, are given by Reddy, [37] as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = D_{11} - \frac{D_{13}^{\ 2}}{D_{33}} \tag{A.12a}$$

$$Q_{12} = \frac{\nu_{12}E_1}{1-\nu_{12}\nu_{21}} = D_{12} - \frac{D_{13}D_{23}}{D_{33}} \tag{A.12b}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = D_{22} - \frac{D_{23}^{\ 2}}{D_{33}} \tag{A.12c}$$

$$Q_{66} = G_{12} = D_{66} . \tag{A.12d}$$

When the transverse shear stresses σ_{zx} and σ_{zy} are included, equation (A.11) should be appended with, [37]

$$\begin{Bmatrix} \sigma_{zy} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{44} & \\ & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{zy} \\ \gamma_{zx} \end{Bmatrix} \tag{A.13}$$

where

$$Q_{44} = G_{23} = D_{44} \tag{A.14a}$$

$$Q_{55} = G_{13} = D_{55} \ . \tag{A.14b}$$

A.3 Transformation of material coefficients for transverse fiber orientation

A.3.1 Orthotropic material

Material coefficients, D_{ij}, for an orthotropic material in the x' axis can be transformed to the material coefficients, $\overline{D}_{ij}$, for the material referenced to the x axis. The positive rotation about the z axis, where the angle of the rotation is θ, is illustrated in Figure A.1.

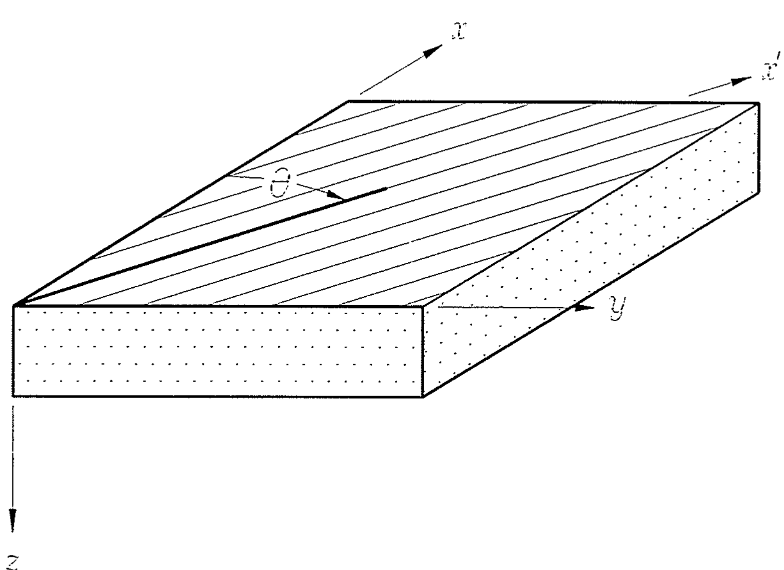

Figure A.1: Laminate with off-axis fiber orientation

The following formulas, which transform the material coefficients D_{ij} to $\overline{D}_{ij}$, are given by Reddy, [37] as

$$\overline{D}_{11} = D_{11}\cos^4\theta + 2(D_{12} + 2D_{66})\cos^2\theta\sin^2\theta + D_{22}\sin^4\theta \qquad \text{(A.15a)}$$

$$\overline{D}_{12} = D_{12}\cos^4\theta + 2(D_{11} + D_{22} - 4D_{66})\cos^2\theta\sin^2\theta + D_{12}\sin^4\theta \qquad \text{(A.15b)}$$

$$\overline{D}_{13} = D_{13}\cos^2\theta + D_{23}\sin^2\theta \qquad \text{(A.15c)}$$

$$\overline{D}_{16} = (D_{11} - D_{12} - 2D_{66})\cos^3\theta\sin\theta + (2D_{66} + D_{12} - D_{22})\cos\theta\sin^3\theta \qquad \text{(A.15d)}$$

$$\overline{D}_{22} = D_{22}\cos^4\theta + 2(D_{12} + 2D_{66})\cos^2\theta\sin^2\theta + D_{11}\sin^4\theta \qquad \text{(A.15e)}$$

$$\overline{D}_{23} = D_{23}\cos^2\theta + D_{13}\sin^2\theta \qquad \text{(A.15f)}$$

$$\overline{D}_{26} = (D_{12} - D_{22} + 2D_{66})\cos^3\theta\sin\theta + (D_{11} - D_{12} - 2D_{66})\cos\theta\sin^3\theta \qquad \text{(A.15g)}$$

$$\overline{D}_{33} = D_{33} \qquad \text{(A.15i)}$$

$$\overline{D}_{36} = (D_{13} - D_{23})\cos\theta\sin\theta \qquad \text{(A.15j)}$$

$$\overline{D}_{44} = D_{44}\cos^2\theta + D_{55}\sin^2\theta \qquad \text{(A.15k)}$$

$$\overline{D}_{45} = (D_{55} - D_{44})\cos\theta\sin\theta \qquad \text{(A.15l)}$$

$$\overline{D}_{55} = D_{55}\cos^2\theta + D_{44}\sin^2\theta \qquad \text{(A.15m)}$$

$$\overline{D}_{66} = (D_{11} + D_{22} - 2D_{12} - 2D_{66})\cos^2\theta\sin^2\theta + D_{66}(\cos^4\theta + \sin^4\theta)\ . \qquad \text{(A.15n)}$$

For a cross-ply plate, when the laminates are rotated at a right angle, $\theta = 90^o$ and the above material coefficients simplify to

$$\overline{D}_{11} = D_{22} \qquad \text{(A.16a)}$$

$$\overline{D}_{12} = D_{12} \qquad \text{(A.16b)}$$

$$\overline{D}_{13} = D_{23} \qquad \text{(A.16c)}$$

$$\overline{D}_{16} = 0 \qquad \text{(A.16d)}$$

$$\overline{D}_{22} = D_{11} \qquad \text{(A.16e)}$$

$$\overline{D}_{23} = D_{13} \qquad \text{(A.16f)}$$

$$\overline{D}_{26} = 0 \qquad \text{(A.16g)}$$

$$\overline{D}_{33} = D_{33} \qquad \text{(A.16h)}$$

$$\overline{D}_{36} = 0 \qquad \text{(A.16i)}$$

$$\overline{D}_{44} = D_{55} \tag{A.16j}$$

$$\overline{D}_{45} = 0 \tag{A.16k}$$

$$\overline{D}_{55} = D_{44} \tag{A.16l}$$

$$\overline{D}_{66} = D_{66} \ . \tag{A.16m}$$

A.3.2 Orthotropic material in plane stress

For an orthotropic material in a state of plane stress the following formulas, which transform the material coefficients Q_{ij} to $\overline{Q}_{ij}$, are given by Reddy, [37] as

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \tag{A.17a}$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right) \tag{A.17b}$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \tag{A.17c}$$

$$\overline{Q}_{22} = Q_{22}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{11}\sin^4\theta \tag{A.17d}$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \tag{A.17e}$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right) \tag{A.17f}$$

$$\overline{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \tag{A.17g}$$

$$\overline{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta \tag{A.17h}$$

$$\overline{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \ . \tag{A.17i}$$

For a cross-ply plate, when the laminates are rotated at a right angle, $\theta = 90^o$ and the above material coefficients reduce to

$$\overline{Q}_{11} = Q_{22} \tag{A.18a}$$

$$\overline{Q}_{12} = Q_{12} \tag{A.18b}$$

$$\overline{Q}_{16} = 0 \tag{A.18c}$$

$$\overline{Q}_{22} = Q_{11} \tag{A.18d}$$

$$\overline{Q}_{26} = 0 \tag{A.18e}$$

$$\overline{Q}_{66} = Q_{66} \tag{A.18f}$$

$$\overline{Q}_{44} = Q_{55} \tag{A.18g}$$

$$\overline{Q}_{45} = 0 \tag{A.18h}$$

$$\overline{Q}_{55} = Q_{44} \ . \tag{A.18i}$$

Appendix B

Supplement to Chapter 2 - Analytical method

Equations pertinent to Chapter 2 are presented here. This appendix contains intermediate equations and steps that are referenced in Chapter 2.

Substituting equations (2.2), (2.3), and (2.4) into the equilibrium equation (2.1) yields

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} & & & \\ D_{12} & D_{22} & D_{23} & & & \\ D_{13} & D_{23} & D_{33} & & & \\ & & & D_{44} & & \\ & & & & D_{55} & \\ & & & & & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} - \rho \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{B.1}$$

The equation above reduces to the system of differential equations given by equation (2.5).

From equation (2.14), setting χ_n to

$$\chi_n = -\begin{vmatrix} d_1 + d_2 c_n^2 & d_3 \\ d_3 & d_5 + d_6 c_n^2 \end{vmatrix} = -\left(d_1 + d_2 c_n^2\right)\left(d_5 + d_6 c_n^2\right) + d_3^2 \tag{B.2a}$$

and solving for ϕ_n and ψ_n yields

$$\phi_n = -\begin{vmatrix} -d_4 c_n & d_3 \\ -d_7 c_n & d_5 + d_6 c_n \end{vmatrix} = d_4 c_n \left(d_5 + d_6 c_n^2\right) - c_n d_3 d_7 \tag{B.2b}$$

$$\psi_n = -\begin{vmatrix} d_1 + d_2 c_n^2 & -d_4 c_n \\ d_3 & -d_7 c_n \end{vmatrix} = d_7 c_n \left(d_1 + d_2 c_n^2\right) - c_n d_3 d_4 \ . \tag{B.2c}$$

Assembling equations (B.2a), (B.2b), and (B.2c) into vector form yields equation (2.15).

To construct vector $\{B_i\}$, the displacement components of equation (2.7) can be expressed in matrix form as

$$\begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} = \begin{bmatrix} \cos(\hat{M}x)\sin(\hat{N}y) & & \\ & \sin(\hat{M}x)\cos(\hat{N}y) & \\ & & \sin(\hat{M}x)\sin(\hat{N}y) \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix}. \quad \text{(B.3)}$$

The three components of stress contained in $\{B_i\}$ can be expressed, according to equation (2.3), as

$$\begin{Bmatrix} \sigma_{zzi} \\ \sigma_{zxi} \\ \sigma_{zyi} \end{Bmatrix} = \begin{bmatrix} D_{13} & D_{23} & D_{33} & & \\ & & & & D_{55} \\ & & & D_{44} & \end{bmatrix} \begin{Bmatrix} \varepsilon_{xxi} \\ \varepsilon_{yyi} \\ \varepsilon_{zzi} \\ \gamma_{zyi} \\ \gamma_{zxi} \end{Bmatrix} \quad \text{(B.4)}$$

with the strain vector defined according to equation (2.4) as

$$\begin{Bmatrix} \varepsilon_{xxi} \\ \varepsilon_{yyi} \\ \varepsilon_{zzi} \\ \gamma_{zyi} \\ \gamma_{zxi} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & & \\ & \frac{\partial}{\partial y} & \\ & & \frac{\partial}{\partial z} \\ & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}. \quad \text{(B.5)}$$

Substituting equation (B.3) into (B.5) and the resulting equation (B.5) into (B.4) yields

$$\begin{Bmatrix} \sigma_{zzi} \\ \sigma_{zxi} \\ \sigma_{zyi} \end{Bmatrix} = \begin{bmatrix} -D_{13}\hat{M}\sin(\hat{M}x)\sin(\hat{N}y) & -D_{23}\hat{N}\sin(\hat{M}x)\sin(\hat{N}y) & D_{33}\sin(\hat{M}x)\sin(\hat{N}y)\frac{\partial}{\partial z} \\ D_{55}\cos(\hat{M}x)\sin(\hat{N}y)\frac{\partial}{\partial z} & & D_{55}\hat{M}\cos(\hat{M}x)\sin(\hat{N}y) \\ & D_{44}\sin(\hat{M}x)\cos(\hat{N}y)\frac{\partial}{\partial z} & D_{44}\hat{N}\sin(\hat{M}x)\cos(\hat{N}y) \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix}.$$

(B.6)

Constructing vector $\{B_i\}$ defined in equation (2.17) by substituting equation (B.3) and (B.6) into equation (2.17) yields

$$\begin{Bmatrix} u_i \\ v_i \\ \sigma_{zzi} \\ \sigma_{zxi} \\ \sigma_{zyi} \\ w_i \end{Bmatrix} = \begin{bmatrix} \cos(\hat{M}x)\sin(\hat{N}y) & 0 & 0 \\ 0 & \sin(\hat{M}x)\cos(\hat{N}y) & 0 \\ -D_{13}\hat{M}\sin(\hat{M}x)\sin(\hat{N}y) & -D_{23}\hat{N}\sin(\hat{M}x)\sin(\hat{N}y) & D_{33}\sin(\hat{M}x)\sin(\hat{N}y)\dfrac{\partial}{\partial z} \\ D_{55}\cos(\hat{M}x)\sin(\hat{N}y)\dfrac{\partial}{\partial z} & 0 & D_{55}\hat{M}\cos(\hat{M}x)\sin(\hat{N}y) \\ 0 & D_{44}\sin(\hat{M}x)\cos(\hat{N}y)\dfrac{\partial}{\partial z} & D_{44}\hat{N}\sin(\hat{M}x)\cos(\hat{N}y) \\ 0 & 0 & \sin(\hat{M}x)\sin(\hat{N}y) \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix}. \tag{B.7}$$

Transferring the $\dfrac{\partial}{\partial z}$ operator from the matrix to the right hand side vector in equation (B.7) and canceling the transcendental functions yields equation (2.18).

From equation (2.56), the corresponding stress and displacement eigenvectors can be obtained by letting u_1 equal unity. Consequently

$$\begin{bmatrix} P_{42}^G & P_{46}^G \\ P_{52}^G & P_{56}^G \end{bmatrix} \begin{Bmatrix} v_1 \\ w_1 \end{Bmatrix} = \begin{Bmatrix} -P_{41}^G \\ -P_{51}^G \end{Bmatrix} \tag{B.8}$$

so that, solving for v_1 and w_1, yields

$$v_1 = \frac{\begin{vmatrix} -P_{41}^G & P_{46}^G \\ -P_{51}^G & P_{56}^G \end{vmatrix}}{\begin{vmatrix} P_{42}^G & P_{46}^G \\ P_{52}^G & P_{56}^G \end{vmatrix}} = \frac{P_{46}^G P_{51}^G - P_{41}^G P_{56}^G}{P_{42}^G P_{56}^G - P_{46}^G P_{52}^G} \tag{B.9a}$$

and

$$w_1 = \frac{\begin{vmatrix} P_{42}^G & -P_{41}^G \\ P_{52}^G & -P_{51}^G \end{vmatrix}}{\begin{vmatrix} P_{42}^G & P_{46}^G \\ P_{52}^G & P_{56}^G \end{vmatrix}} = \frac{P_{41}^G P_{52}^G - P_{42}^G P_{51}^G}{P_{42}^G P_{56}^G - P_{46}^G P_{52}^G} . \tag{B.9b}$$

Substituting equations (B.9a) and (B.9b) and $(u_1 = 1)$ into the vector of stresses and displacements in equation (2.17) at the first interface $(i = 1)$ yields equation (2.58).

Appendix C

Derivation of propagator matrix for isotropic material

Elastic material constants in the stress-strain relationship given in equation (2.3) for an isotropic material can be reduced to

$$D_{11} = D_{22} = D_{33} = \frac{2\mu(1-\nu)}{(1-2\nu)} \tag{C.1a}$$

$$D_{12} = D_{13} = D_{23} = \frac{2\mu\nu}{(1-2\nu)} \tag{C.1b}$$

$$D_{44} = D_{55} = D_{66} = \mu \ . \tag{C.1c}$$

Substituting these relations into the governing system of differential equations (2.9) yields, after simplification [44],

$$\begin{bmatrix} L^2+\lambda^2-g^2-\dfrac{\hat{M}^2}{1-2\nu} & \dfrac{-\hat{M}\hat{N}}{1-2\nu} & \dfrac{\hat{M}L}{1-2\nu} \\ \dfrac{-\hat{M}\hat{N}}{1-2\nu} & L^2+\lambda^2-g^2-\dfrac{\hat{N}^2}{1-2\nu} & \dfrac{\hat{N}L}{1-2\nu} \\ \dfrac{-\hat{M}L}{1-2\nu} & \dfrac{-\hat{N}L}{1-2\nu} & L^2+\lambda^2-g^2+\dfrac{L^2}{1-2\nu} \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{C.2}$$

where ν is Poisson's ratio, μ is the shear modulus, L is $\dfrac{d}{dz}$, and

$$g = \sqrt{\left(\hat{M}^2 + \hat{N}^2\right)} \tag{C.3a}$$

$$\lambda = \omega\sqrt{\frac{\rho}{\mu}} . \tag{C.3b}$$

Taking the determinant of the left hand side matrix in equation (C.2) and setting it to zero yields two single $\left(\pm s = \sqrt{g^2 - \lambda^2(1-2\nu)/(2-2\nu)}\right)$ and two repeated roots $\left(\pm r = \sqrt{g^2-\lambda^2}\right)$ of L for the solution to the characteristic equation. To obtain the eigenvector corresponding to the repeated root r, assume a solution in the form

$$\begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} \phi_r \\ \psi_r \\ \chi_r \end{Bmatrix} e^{rz}. \tag{C.4}$$

Substitution into equation (C.2) in order to obtain the eigenvector yields, after simplification,

$$\begin{bmatrix} \hat{M} & \hat{N} & -r \\ \hat{M} & \hat{N} & -r \\ \hat{M} & \hat{N} & -r \end{bmatrix} \begin{Bmatrix} \phi_r \\ \psi_r \\ \chi_r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{C.5}$$

The rank of the left hand side matrix is one and, therefore, two solutions for the eigenvectors are possible.

$$\begin{Bmatrix} \phi_r \\ \psi_r \\ \chi_r \end{Bmatrix} = \begin{Bmatrix} r\hat{M} \\ r\hat{N} \\ g^2 \end{Bmatrix} \text{ and } \begin{Bmatrix} \hat{N} \\ \hat{M} \\ 0 \end{Bmatrix} \tag{C.6}$$

By changing $+r$ to $-r$ in equation (C.4), the eigenvectors are

$$\begin{Bmatrix} \phi_{-r} \\ \psi_{-r} \\ \chi_{-r} \end{Bmatrix} = \begin{Bmatrix} -r\hat{M} \\ -r\hat{N} \\ g^2 \end{Bmatrix} \text{ and } \begin{Bmatrix} \hat{N} \\ \hat{M} \\ 0 \end{Bmatrix}. \tag{C.7}$$

A similar routine is followed for obtaining the eigenvectors corresponding to the single double root $\pm s$ after adopting the following solution

$$\begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} \phi_s \\ \psi_s \\ \chi_s \end{Bmatrix} e^{sz}. \tag{C.8}$$

Substituting the above equation into equation (C.2) yields

$$\begin{bmatrix} \left(g^2 - s^2 - \hat{M}^2\right) & -\hat{M}\hat{N} & \hat{M}s \\ -\hat{M}\hat{N} & \left(g^2 - s^2 - \hat{N}^2\right) & \hat{N}s \\ \hat{M}s & \hat{N}s & -g^2 \end{bmatrix} \begin{Bmatrix} \phi_s \\ \psi_s \\ \chi_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{C.9}$$

The rank of the above left hand side matrix is two so that only one eigenvector is possible. That is,

$$\begin{Bmatrix} \phi_s \\ \psi_s \\ \chi_s \end{Bmatrix} = \begin{Bmatrix} \hat{M} \\ \hat{N} \\ s \end{Bmatrix}. \tag{C.10}$$

By changing $+s$ to $-s$ in equation (C.8), the eigenvector corresponding to $-s$ is

$$\begin{Bmatrix} \phi_{-s} \\ \psi_{-s} \\ \chi_{-s} \end{Bmatrix} = \begin{Bmatrix} \hat{M} \\ \hat{N} \\ -s \end{Bmatrix}. \tag{C.11}$$

Therefore, the vector of functions of z contained in equation (2.21) can be rewritten as

$$\{f(z)\} = \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \\ \dfrac{\partial\phi(z)}{\partial z} \\ \dfrac{\partial\psi(z)}{\partial z} \\ \dfrac{\partial\chi(z)}{\partial z} \end{Bmatrix} = [H][E(z)]\{A\} \tag{C.12}$$

where matrices $[H]$ and $[E(z)]$ are redefined as

$$[H] = \begin{bmatrix} r\hat{M} & -r\hat{M} & \hat{N} & \hat{N} & \hat{M} & \hat{M} \\ r\hat{N} & -r\hat{N} & -\hat{M} & -\hat{M} & \hat{N} & \hat{N} \\ g^2 & g^2 & 0 & 0 & s & -s \\ r^2\hat{M} & r^2\hat{M} & r\hat{N} & -r\hat{N} & s\hat{M} & -s\hat{M} \\ r^2\hat{N} & r^2\hat{N} & -r\hat{M} & r\hat{M} & s\hat{N} & -s\hat{N} \\ rg^2 & -rg^2 & 0 & 0 & s^2 & s^2 \end{bmatrix} \tag{C.13}$$

$$[E(z)] = \begin{bmatrix} e^{rz} & & & & & \\ & e^{-rz} & & & & \\ & & e^{rz} & & & \\ & & & e^{-rz} & & \\ & & & & e^{sz} & \\ & & & & & e^{-sz} \end{bmatrix}. \tag{C.14}$$

Vector $\{A\}$ is the same as that used in equation (2.24) of Chapter 2. It is defined as

$$\{A\}^T = \langle A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5 \quad A_6 \rangle. \tag{C.15}$$

Matrix $[G]$ is the same as that defined in equation (2.20) of Chapter 2. Using the elastic constants, normalized to the shear modulus μ contained in equations (C.1a) through (C.1c), matrix $[G]$ can be redefined as

$$[G]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{2\nu}{(1-2\nu)}\hat{M} & -\dfrac{2\nu}{(1-2\nu)}\hat{N} & 0 & 0 & 0 & \dfrac{2(1-\nu)}{(1-2\nu)} \\ 0 & 0 & \hat{M} & 1 & 0 & 0 \\ 0 & 0 & \hat{N} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{C.16}$$

The definition of the propagator matrix for the $i\ th$ sub-layer for an isotropic material is the same as that defined in equations (2.28) and (2.29). Hence

$$\{\mathrm{B}_{i+1}\}=[P_i]\{B_i\} \tag{C.17}$$

where

$$[P_i]=[G][H][E(z)]\big|_{z=z_i+h_i}[E(z)]^{-1}\big|_{z=z_i}[[G][H]]^{-1}=[G][H][E(z)]\big|_{z=h_i}[[G][H]]^{-1} \tag{C.18}$$

and

$$\{B_i\}^T=\langle u_i \quad v_i \quad \sigma_{zzi} \quad \sigma_{zxi} \quad \sigma_{zyi} \quad w_i\rangle . \tag{C.19}$$

The matrices $[H]$, $[E(z)]$, and $[G]$ are defined in equations (C.13), (C.14), and (C.16), respectively. Multiplying matrices $[G]$ and $[H]$ yields

$$[G][H]=\begin{bmatrix} r\hat{M} & -r\hat{M} & \hat{N} & \hat{N} & \hat{M} & \hat{M} \\ r\hat{N} & -r\hat{N} & -\hat{M} & -\hat{M} & \hat{N} & \hat{N} \\ 2g^2r & -2g^2r & 0 & 0 & r^2+g^2 & r^2+g^2 \\ (r^2+g^2)\hat{M} & (r^2+g^2)\hat{M} & r\hat{N} & -r\hat{N} & 2s\hat{M} & -2s\hat{M} \\ (r^2+g^2)\hat{N} & (r^2+g^2)\hat{N} & -r\hat{M} & r\hat{M} & 2s\hat{N} & -2s\hat{N} \\ g^2 & g^2 & 0 & 0 & s & -s \end{bmatrix}. \tag{C.20}$$

The six by six propagator matrix in equation (C.18) for an isotropic material can be multiplied with the propagator matrix in equation (2.29) for an orthotropic material to get the frequency equation and eigenvectors for a plate with both isotropic and orthotropic sub-layers. The frequency equation is the same as equation (2.33) and the initial eigenvector is the same as equation (2.34).

Appendix D

Derivation of frequency equation and propagator matrix for orthotropic material having negligable D_{44} and D_{55}

D.1 Governing equations

The governing set of differential equations is given in equation (2.9). Taking D_{44} and D_{55} as zero, the following constants in equations (2.10) are redefined as

$$d_1 = \rho\omega^2 - D_{11}\hat{M}^2 - D_{66}\hat{N}^2 \tag{D.1a}$$

$$d_3 = -\hat{M}\hat{N}\left(D_{12} + D_{66}\right) \tag{D.1b}$$

$$d_4 = \hat{M}D_{13} \tag{D.1c}$$

$$d_5 = \rho\omega^2 - D_{66}\hat{M}^2 - D_{22}\hat{N}^2 \tag{D.1d}$$

$$d_7 = \hat{N}D_{23} \tag{D.1e}$$

$$d_8 = \rho\omega^2 \tag{D.1f}$$

$$d_9 = D_{33}\,. \tag{D.1g}$$

Substituting of the above constants into the governing differential equation (2.9) yields the following

$$\begin{bmatrix} d_1 & d_3 & d_4 L \\ d_3 & d_5 & d_7 L \\ -d_4 L & -d_7 L & d_8 + d_9 L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{D.2}$$

Multiplying the first row by d_7 and the second row by d_4 and subtracting the second row from the first row yields

$$\psi(z) = R_1\phi(z) \tag{D.3}$$

where

$$R_1 = \frac{\left(d_1 d_7 - d_3 d_4\right)}{\left(d_4 d_5 - d_3 d_7\right)}. \tag{D.4}$$

Substitution of equation (D.3) into (D.2) yields

$$\begin{bmatrix} R_1 & -1 & 0 \\ d_3 & d_5 & d_7 L \\ -d_4 L & -d_7 L & d_8 + d_9 L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \psi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{D.5}$$

The displacement solutions (in terms of u, v and w) are those prescribed in equation (2.7). Substitution of row 1 into rows 2 and 3 reduces this system to

$$\begin{bmatrix} d_3 + R_1 d_5 & d_7 L \\ -(d_4 + R_1 d_7) L & d_8 + d_9 L^2 \end{bmatrix} \begin{Bmatrix} \phi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{D.6}$$

Taking the determinant of the matrix in the above equation yields the following characteristic equation

$$\left(\mathrm{d}_3 d_9 + R_1 d_5 d_9 + d_7 d_4 + R_1 d_7^{\,2}\right) L^2 + \left(d_3 d_8 + R_1 d_5 d_8\right) = 0 \tag{D.7}$$

which has two roots $\pm s$ where s is given as

$$s = \frac{\sqrt{d_8}\sqrt{-(d_3 + R_1 d_5)}}{\sqrt{\left(\mathrm{d}_3 d_9 + R_1 d_5 d_9 + d_7 d_4 + R_1 d_7^{\,2}\right)}}. \tag{D.8}$$

The following displacement solutions are proposed through the thickness

$$\begin{Bmatrix} \phi(z) \\ \chi(z) \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \chi_1 \end{Bmatrix} C_1 e^{sz} + \begin{Bmatrix} \phi_2 \\ \chi_2 \end{Bmatrix} C_2 e^{-sz} \tag{D.9}$$

where the expression for the displacement function, $\psi(z)$, is given in equation (D.3). Substitution of roots s, $-s$ into equation (D.6) leads to the following solutions for the eigenvectors

$$\begin{Bmatrix} \phi_1 \\ \chi_1 \end{Bmatrix} = \begin{Bmatrix} s d_7 \\ -(d_3 + R_1 d_5) \end{Bmatrix} \tag{D.10a}$$

and

$$\begin{Bmatrix} \phi_1 \\ \chi_1 \end{Bmatrix} = \begin{Bmatrix} -s d_7 \\ -(d_3 + R_1 d_5) \end{Bmatrix}. \tag{D.10b}$$

Expressions for $\phi(z)$ and $\chi(z)$ can be written as

$$\phi(z) = s d_7 \left(C_1 e^{sz} - C_2 e^{-sz}\right) \tag{D.11a}$$

and

$$\chi(z) = -(d_3 + R_1 d_5)\left(C_1 e^{sz} + C_2 e^{-sz}\right). \tag{D.11b}$$

The expression for the stress, $\sigma_{zz}(z)$, is given as

$$\sigma_{zz} = D_{13}\frac{\partial u}{\partial x} + D_{23}\frac{\partial v}{\partial y} + D_{33}\frac{\partial w}{\partial z}. \tag{D.12}$$

Substituting equation (D.3) into the above equation results in

$$\sigma_{zz}(z) = -\left(D_{13}\hat{M} + D_{23}\hat{N}R_1\right)\phi(z) + D_{33}\frac{d\chi(z)}{dz} \tag{D.13}$$

where $\frac{d\chi(z)}{dz}$ can be written with constant d_7 factored in as

$$\frac{d\chi(z)}{dz} = -\frac{(d_3 + R_1 d_5)}{d_7}\left\{sd_7\left(C_1 e^{sz} - C_2 e^{-sz}\right)\right\} = -\frac{(d_3 + R_1 d_5)}{d_7}\{\phi(z)\}. \tag{D.14}$$

Using the above relation, the stress $\sigma_{zz}(z)$ can be rewritten in terms of the displacement, $\phi(z)$, as

$$\sigma_{zz}(z) = \left[-\left(D_{13}\hat{M} + D_{23}\hat{N}R_1\right) - D_{33}\frac{(d_3 + R_1 d_5)}{d_7}\right]\{\phi(z)\} = R_2\phi(z) \tag{D.15}$$

where

$$R_2 = -\left(D_{13}\hat{M} + D_{23}\hat{N}R_1\right) - D_{33}\frac{(d_3 + R_1 d_5)}{d_7}. \tag{D.16}$$

Therefore, the stress, $\sigma_{zz}(z)$, and the displacements, $\phi(z)$ and $\chi(z)$, depend on one another and any one of these functions can be expressed in terms of another function multiplied by a constant. Since stress free boundary conditions are required, the stress, $\sigma_{zz}(z)$, is used as the main function and the displacements, $\phi(z)$ and $\chi(z)$, can be expressed as

$$\phi(z) = \frac{\sigma_{zz}(z)}{R_2} \tag{D.17a}$$

$$\psi(z) = \frac{R_1}{R_2}\sigma_{zz}(z). \tag{D.17b}$$

Continuity in the stress, $\sigma_{zz}(z)$, automatically ensures continuity in the displacements, $\phi(z)$ and $\chi(z)$.

D.2 Propagator matrix formulation

By evaluating the stress, $\sigma_{zz}(z)$, and displacement, $\chi(z)$, at $z = z_i$ and $z = z_i + h_i$ for a particular sub-layer and by eliminating the vector of the two unknown constants, C_1 and C_2, the following relation can be obtained

$$\{\mathrm{b}_{i+1}\} = [p_i]\{b_i\} \tag{D.18}$$

where

$$\{b_i\}^T = \langle w_i \quad \sigma_{zzi} \rangle. \tag{D.19}$$

Vector $\{b_i\}$ can be written in terms of the undetermined constants C_1 and C_2 in the following way

$$\{b_i\} = [J][E(z)]\{C\} \tag{D.20}$$

where

$$[E(z)] = \begin{bmatrix} e^{sz_i} & \\ & e^{-sz_i} \end{bmatrix} \tag{D.21}$$

$$[J] = \begin{bmatrix} -(d_3 + R_1 d_5) & -(d_3 + R_1 d_5) \\ R_2 s d_7 & -R_2 s d_7 \end{bmatrix} \tag{D.22}$$

and

$$\{C\}^T = \langle C_1 \quad C_2 \rangle. \tag{D.23}$$

Thus, equation (D.20) is an expression for the stresses and displacements for the i *th* interface in terms of the undetermined coefficients C_1 and C_2. Since it is required to relate the stresses and displacements at the i *th* interface to the stresses and displacements at the $(i+1)$ *th* interface with a propagator matrix for the i *th* sub-layer, assume the local coordinate $z = z_i$ for $\{b_i\}$ and $z = z_i + h_i$ for $\{\mathrm{b}_{i+1}\}$. Solving matrix equation (D.19) for $\{C\}$ gives

$$\{C\} = [E(z)]^{-1}\Big|_{z=z_i} [J]^{-1}\{b_i\} \tag{D.24}$$

where

$$[J]^{-1} = \begin{bmatrix} \dfrac{-1}{2(d_3 + R_1 d_5)} & \dfrac{1}{2sd_7 R_1} \\ \dfrac{-1}{2(d_3 + R_1 d_5)} & \dfrac{-1}{2sd_7 R_1} \end{bmatrix}. \tag{D.25}$$

Substitution of $z = z_i + h_i$ into matrix $[E(z)]$ and equation (D.23) into the stress and displacement vector at the $(i+1)\ th$ interface, $\{\mathrm{b}_{i+1}\}$, yields

$$\{\mathrm{b}_{i+1}\} = [P_i]\{b_i\} \tag{D.26}$$

where the $i\ th$ propagator matrix is given by

$$[p_i] = [J][E(z)]\big|_{z=z_i+h_i}[E(z)]^{-1}\big|_{z=z_i}[J]^{-1} = [J][E(z)]\big|_{z=h_i}[J]^{-1} \tag{D.27}$$

and

$$[p_i] = \begin{bmatrix} \dfrac{1}{2}\left(e^{sh_i} + e^{-sh_i}\right) & -\dfrac{(d_3 + R_1 d_5)}{2R_2 sd_7}\left(e^{sh_i} - e^{-sh_i}\right) \\ -\dfrac{R_2 sd_7}{2(d_3 + R_1 d_5)}\left(e^{sh_i} - e^{-sh_i}\right) & \dfrac{1}{2}\left(e^{sh_i} + e^{-sh_i}\right) \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \tag{D.28}$$

Matrix $[p_i]$ and equations (D.16a) and (D.16b) can be used to construct components for a 4 by 4 propagator matrix in which the displacements, u_i and v_i, are included in order that continuity of all the displacements is maintained. A vector involving the stresses and displacements is now proposed as

$$\{b^*{}_i\}^T = \langle u_i \quad v_i \quad w_i \quad \sigma_{zzi} \rangle \tag{D.29}$$

where the 4 by 4 propagator matrix becomes

$$[p^*{}_i] = \begin{bmatrix} p_{22} & 0 & \dfrac{p_{21}}{R_2} & 0 \\ 0 & p_{22} & \dfrac{R_1}{R_2} p_{21} & 0 \\ 0 & 0 & p_{11} & p_{12} \\ 0 & 0 & p_{21} & p_{22} \end{bmatrix}. \tag{D.30}$$

The global propagator matrix $\left[p^{*G}\right]$ is obtained by repeated application of equation (D.25). This results in

$$\{b^*{}_{N+1}\} = \left[p^{*G}\right]\{b^*{}_1\} \tag{D.31}$$

where

$$\left[p^{*G}\right]=\left[p^{*}{}_{N}\right]\left[p^{*}{}_{N-1}\right]\ldots\left[p^{*}{}_{i}\right]\ldots\left[p^{*}{}_{2}\right]\left[p^{*}{}_{1}\right]. \tag{D.32}$$

The repeated application of the above equation ensures that continuity of displacements and stresses are maintained at the interfaces between sub-layers.

D.3 Frequency equation

Denoting the elements of the 4 by 4 global propagator matrix, $\left[p^{*G}\right]$, by p^{*G}_{ij} and invoking zero traction conditions in terms of σ_{zz} at interfaces 1 and $(N+1)$, the following equation can be obtained from equation (D.30)

$$p^{*G}{}_{43}\{w_1\}=0. \tag{D.33}$$

The exact frequency equation for the plate is obtained by setting the global propagator matrix component, $p^{*G}{}_{43}$, to zero, i.e.

$$f(\omega,\hat{M},\hat{N})=p^{*G}{}_{43}=0. \tag{D.34}$$

The above equation can be solved for ω given $\hat{M}$ and $\hat{N}$.

Once the roots of the dispersion equation are determined from equation (D.33), the corresponding stress and displacement eigenvectors can be obtained by letting w_1 equal unity in equation (D.32) and letting stress the σ_{zz1} equal zero. Since the displacements, u and v, depend on the stress σ_{zz}, the displacements u and v must also be zero. The eigenvector of the stresses and displacements at the first interface becomes

$$\{b^{*}{}_{1}\}^{T}=\langle 0 \quad 0 \quad 1 \quad 0\rangle. \tag{D.35}$$

The eigenvector of the stresses and displacements at the $i\,th$ interface can then be calculated by

$$\{b^{*}{}_{i}\}=\left[\left[p^{*}{}_{i-1}\right]\left[p^{*}{}_{i-2}\right]\ldots\left[p^{*}{}_{2}\right]\left[p^{*}{}_{1}\right]\right]\{b^{*}{}_{1}\}. \tag{D.36}$$

D.4 Frequency equation and displacement solutions for homogeneous plate

If the roots of the characteristic equation are pure imaginary, then root s can be written as

$$\pm s = \pm i\bar{s} \tag{D.37}$$

$$\bar{s} = \frac{\sqrt{d_8}\sqrt{(d_3 + R_1 d_5)}}{\sqrt{\left(\mathrm{d}_3 d_9 + R_1 d_5 d_9 + d_7 d_4 + R_1 d_7{}^2\right)}}. \tag{D.38}$$

From equations (D.27), (D.29) and (D.33) the frequency equation for a homogeneous plate can be written as

$$f(\omega, \hat{M}, \hat{N}) = p^{*G}{}_{43} = p_{21} = -\frac{R_2 s d_7}{2\left(d_3 + R_1 d_5\right)}\left(e^{sH} - e^{-sH}\right) = 0 \tag{D.39}$$

where H is the total thickness of the plate. Cancelling out constant terms, the above equation can be reduced to

$$\frac{s}{2}\left(e^{sH} - e^{-sH}\right) = 0. \tag{D.40}$$

Using hyperbolic and trigonometric identities, this last equation can be reduced to

$$s\left(\sinh(sH)\right) = s\left(-i\sin(isH)\right) = 0. \tag{D.41}$$

If $s = i\bar{s}$, then the above frequency equation can be reduced further to

$$i\bar{s}\left(-i\sin(i^2\bar{s}H)\right) = \bar{s}\left(\sin(-\bar{s}H)\right) = -\bar{s}\sin(\bar{s}H) = 0. \tag{D.42}$$

Canceling out the term $-\bar{s}$ results in

$$\sin(\bar{s}H) = 0\ . \tag{D.43}$$

Thus, the frequency equation for a homogeneous plate reduces to the simple form

$$\bar{s}H = n\pi\ ; \qquad \text{where } n = 0, 1, 2, \ldots \tag{D.44}$$

and $\bar{s}$ is a function of frequency, ω. The first root of this equation for $n = 0$ is $\bar{s} = 0$. For this first root to be zero, equation (D.37) shows that the frequency, ω, contained in d_8 must be zero.

Solutions for the displacement, w, and the stress, σ_{zz}, from equations (D.11b) and (D.14) if $s = i\bar{s}$, reduce to

$$w(z) = \chi(z) = -(d_3 + R_1 d_5)\{(C_1 + C_2)\cos(\bar{s}z) + (C_1 - C_2)i\sin(\bar{s}z)\} \tag{D.45}$$

$$\sigma_{zz}(z) = R_2 \bar{s} d_7 \{i(C_1 + C_2)\cos(\bar{s}z) - (C_1 - C_2)\sin(\bar{s}z)\}. \tag{D.46}$$

If $s = i\bar{s}$, solutions for the displacements, u and v, can be written in terms of the stress, σ_{zz}, as

$$u(z) = \phi(z) = \frac{\sigma_{zz}(z)}{R_2} = \bar{s} d_7 \{i(C_1 + C_2)\cos(\bar{s}z) - (C_1 - C_2)\sin(\bar{s}z)\} \tag{D.47}$$

$$\psi(z) = \frac{R_1}{R_2}\sigma_{zz}(z) = R_1 \bar{s} d_7 \{i(C_1 + C_2)\cos(\bar{s}z) - (C_1 - C_2)\sin(\bar{s}z)\}. \tag{D.48}$$

Looking at the real parts of the displacements and stresses, and substituting in the frequency equation (D.43) in place of $\bar{s}$ yields the following real functions

$$w(z) = \chi(z) = -(d_3 + R_1 d_5)\left\{(C_1 + C_2)\cos\left(\frac{n\pi z}{H}\right)\right\} \tag{D.49}$$

$$\sigma_{zz}(z) = -R_2\left(\frac{n\pi}{H}\right)d_7\left\{(C_1 - C_2)\sin\left(\frac{n\pi z}{H}\right)\right\}. \tag{D.50}$$

Solutions to the displacements, u and v, can be written in terms of the stress, σ_{zz}, as

$$u(z) = \phi(z) = \frac{\sigma_{zz}(z)}{R_2} = -\left(\frac{n\pi}{H}\right)d_7\left\{(C_1 - C_2)\sin\left(\frac{n\pi z}{H}\right)\right\} \tag{D.51}$$

$$v(z) = \psi(z) = \frac{R_1}{R_2}\sigma_{zz}(z) = -R_1\left(\frac{n\pi}{H}\right)d_7\left\{(C_1 - C_2)\sin\left(\frac{n\pi z}{H}\right)\right\}. \tag{D.52}$$

The above displacement solutions show that the first frequency is zero when $n = 0$ and the displacements, u and v, as well as the stress, σ_{zz}, are zero throughout the thickness. Displacement w in this case is not zero but it is constant through the thickness. For the first non-zero frequency (i.e. when $n = 1$), displacements u and v and stress σ_{zz} are zero at the free surface and they all have a sine-like variation through the thickness. Displacement w is not zero at the free surface but varies cosine-like through the thickness. Subsequent higher frequencies contain equally spaced nodes through the thickness where the displacements u and v and the stress σ_{zz} are zero but the displacement w is not zero at these nodes.

Appendix E

Supplement to § 3.5 Raleigh-Ritz approximation

Equations pertinent to § 3.5 are presented here in order to detail the development of the constants, the intermediate equations and the steps given in § 3.5.

The three components in the strain-displacement relations are given from equation (3.84) as

$$\frac{\partial}{\partial x}\begin{Bmatrix} u \\ 0 \\ 0 \\ 0 \\ w \\ v \end{Bmatrix} = [b]\{q'\}; \qquad \frac{\partial}{\partial y}\begin{Bmatrix} 0 \\ v \\ 0 \\ w \\ 0 \\ u \end{Bmatrix} = [c]\{\dot{q}\}; \qquad \frac{\partial}{\partial z}\begin{Bmatrix} 0 \\ 0 \\ w \\ v \\ u \\ 0 \end{Bmatrix} = [a]\{q\} \tag{E.1}$$

where q' denotes $\frac{\partial q}{\partial x}$, $\dot{q}$ denotes $\frac{\partial q}{\partial y}$, and $\hat{N}_i$ denotes $\frac{\partial N_i}{\partial z}$. The matrices $[b], [c]$, and $[a]$ are given as

$$[b] = \begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i & N_j & N_k \\ 0 & 0 & 0 & N_i & N_j & N_k & 0 & 0 & 0 \end{bmatrix} \tag{E.2}$$

$$[c] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i & N_j & N_k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ N_i & N_j & N_k & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E.3}$$

$$[a]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{N}_i & \hat{N}_j & \hat{N}_k \\ 0 & 0 & 0 & \hat{N}_i & \hat{N}_j & \hat{N}_k & 0 & 0 & 0 \\ \hat{N}_i & \hat{N}_j & \hat{N}_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E.4}$$

where $\hat{N}_i$, $\hat{N}_j$, and $\hat{N}_k$ denote $\dfrac{\partial N_i}{\partial z}$, $\dfrac{\partial N_j}{\partial z}$, and $\dfrac{\partial N_k}{\partial z}$, respectively.

Matrix $[n]$ is given as

$$[n]=\begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i & N_j & N_k \end{bmatrix}. \tag{E.5}$$

From equation (3.91), the two terms expanded in terms of the matrices $[a]$, $[b]$, $[c]$, and $[n]$ are

$$\rho\left\{\overset{o}{U}\right\}^T\left\{\overset{o}{U}\right\}=\rho\left\{\overset{o}{q}\right\}^T[n]^T[n]\left\{\overset{o}{q}\right\} \tag{E.6}$$

and

$$\begin{aligned}\{\varepsilon\}^T[D]\{\varepsilon\}=&\{q'\}^T[b]^T[D][b]\{q'\}+\{q'\}^T[b]^T[D][c]\{\dot{q}\}+\{q'\}^T[b]^T[D][a]\{q\}\\&+\{\dot{q}\}^T[c]^T[D][b]\{q'\}+\{\dot{q}\}^T[c]^T[D][c]\{\dot{q}\}+\{\dot{q}\}^T[c]^T[D][a]\{q\}\\&+\{q\}^T[a]^T[D][b]\{q'\}+\{q\}^T[a]^T[D][c]\{\dot{q}\}+\{q\}^T[a]^T[D][a]\{q\}.\end{aligned} \tag{E.7}$$

Substituting equations (E.6) and (E.7) into equation (3.91) yields

$$L=\frac{1}{2}\int_t\int_y\int_x\left[\int_{z_i}^{z_i+h_i}\begin{pmatrix}\rho\left\{\overset{o}{q}\right\}^T[n]^T[n]\left\{\overset{o}{q}\right\}-\{q'\}^T[b]^T[D][b]\{q'\}-\{q'\}^T[b]^T[D][c]\{\dot{q}\}\\-\{q'\}^T[b]^T[D][a]\{q\}-\{\dot{q}\}^T[c]^T[D][b]\{q'\}-\{\dot{q}\}^T[c]^T[D][c]\{\dot{q}\}\\-\{\dot{q}\}^T[c]^T[D][a]\{q\}-\{q\}^T[a]^T[D][b]\{q'\}-\{q\}^T[a]^T[D][c]\{\dot{q}\}\\-\{q\}^T[a]^T[D][a]\{q\}\end{pmatrix}dz\right]dxdydt. \tag{E.8}$$

Define the following constants

$$[m] = \int_{z_i}^{z_i+h_i} \rho[n]^T[n]dz \tag{E.9a}$$

$$[k_1] = \int_{z_i}^{z_i+h_i} [b]^T[D][b]dz \tag{E.9b}$$

$$[k_2] = \int_{z_i}^{z_i+h_i} [b]^T[D][a]dz \tag{E.9c}$$

$$[k_2]^T = \int_{z_i}^{z_i+h_i} [a]^T[D][b]dz \tag{E.9d}$$

$$[k_3] = \int_{z_i}^{z_i+h_i} [a]^T[D][a]dz \tag{E.9e}$$

$$[k_4] = \int_{z_i}^{z_i+h_i} [c]^T[D][b]dz \tag{E.9f}$$

$$[k_4]^T = \int_{z_i}^{z_i+h_i} [b]^T[D][c]dz \tag{E.9g}$$

$$[k_5] = \int_{z_i}^{z_i+h_i} [c]^T[D][c]dz \tag{E.9h}$$

$$[k_6] = \int_{z_i}^{z_i+h_i} [c]^T[D][a]dz \tag{E.9i}$$

and

$$[k_6]^T = \int_{z_i}^{z_i+h_i} [a]^T[D][c]dz\,. \tag{E.9j}$$

Substituting the above constants into the total energy equation (E.8) yields equation (3.93).

Apply integration by parts to factor out the term , $\{\dot{q}\}^T$, in equation (3.93). The following generic formula is also useful

$$a\frac{\partial b}{\partial x}=\frac{\partial(ab)}{\partial x}-b\frac{\partial a}{\partial x}. \tag{E.10}$$

Applying equation (E.10) to each term in equation (3.93) yields

$$\int_t \left\{\overset{o}{q}\right\}^T [m]\left\{\overset{o}{q}\right\}=\{q\}^T[m]\left\{\overset{o}{q}\right\}\Big|_t-\int_t \{q\}^T[m]\left\{\overset{oo}{q}\right\} \tag{E.11a}$$

$$\int_x -\{q'\}^T[k_1]\{q'\}=-\{q\}^T[k_1]\{q'\}\big|_x+\int_x \{q\}^T[k_1]\{q''\} \tag{E.11b}$$

$$\int_x -\{q'\}^T[k_2]\{q\}=-\{q\}^T[k_2]\{q\}\big|_x+\int_x \{q\}^T[k_2]\{q'\} \tag{E.11c}$$

$$\int_y -\{\dot{q}\}^T[k_4]\{q'\}=-\{q\}^T[k_4]\{q'\}\big|_y+\int_y \{q\}^T[k_4]\{\dot{q}'\} \tag{E.11d}$$

$$\int_x -\{q'\}^T[k_4]^T\{\dot{q}\}=-\{q\}^T[k_4]^T\{\dot{q}\}\big|_x+\int_x \{q\}^T[k_4]^T\{\dot{q}'\} \tag{E.11e}$$

$$\int_y -\{\dot{q}\}^T[k_5]\{\dot{q}\}=-\{q\}^T[k_5]\{\dot{q}\}\big|_y+\int_y \{q\}^T[k_5]\{\ddot{q}\} \tag{E.11f}$$

and

$$\int_y -\{\dot{q}\}^T[k_6]\{q\}=-\{q\}^T[k_6]\{q\}\big|_y+\int_y \{q\}^T[k_6]\{\dot{q}\}. \tag{E.11g}$$

Substituting equations (E.11a) through (E.11g) into equation (3.93) yields equation (3.94).

The matrices $[m]$ through $\left[[k_6]-[k_6]^T\right]$ contained in equation (3.95), can be simplified by defining the following vectors:

$$\{\vec{n}\}^T=\langle N_i \quad N_j \quad N_k\rangle \tag{E.12a}$$

$$\left\{\hat{\vec{n}}\right\}^T=\langle \hat{N}_i \quad \hat{N}_j \quad \hat{N}_k\rangle=\left\{\frac{d\vec{n}}{dz}\right\}^T=\left\{\frac{d\vec{n}}{d\xi}\right\}^T\frac{d\xi}{dz}=\frac{2}{h_i}\left\{\frac{d\vec{n}}{d\xi}\right\}^T. \tag{E.12b}$$

Defining

$$[A]=\int_{z_i}^{z_i+h_i}\{\vec{n}\}\{\vec{n}\}^T dz=\frac{h_i}{2}\int_{-1}^{1}\{\vec{n}\}\{\vec{n}\}^T d\xi=\frac{h_i}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix} \tag{E.13a}$$

$$[B] = \int_{z_i}^{z_i+h_i} \{\hat{\bar{n}}\}\{\hat{\bar{n}}\}^T dz = \frac{h_i}{2}\left(\frac{2}{h_i}\right)\left(\frac{2}{h_i}\right)\int_{-1}^{1}\left\{\frac{d\bar{n}}{d\xi}\right\}\left\{\frac{d\bar{n}}{d\xi}\right\}^T d\xi = \frac{1}{3h_i}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \quad \text{(E.13b)}$$

and

$$[E] = \int_{z_i}^{z_i+h_i} \{\bar{n}\}\{\hat{\bar{n}}\}^T dz = \frac{h_i}{2}\left(\frac{2}{h_i}\right)\int_{-1}^{1}\{\bar{n}\}\left\{\frac{d\bar{n}}{d\xi}\right\}^T d\xi = \frac{1}{6}\begin{bmatrix} -3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} \quad \text{(E.13c)}$$

The nine by nine matrices $[m]$ through $\left[[k_6]-[k_6]^T\right]$ used in equation (3.95) can be defined now in terms of the three by three matrices $[A]$, $[B]$, and $[E]$ in the following fashion

$$[m] = \rho\begin{bmatrix} [A] & & \\ & [A] & \\ & & [A] \end{bmatrix} \quad \text{(E.14a)}$$

$$[k_1] = \begin{bmatrix} D_{11}[A] & & \\ & D_{66}[A] & \\ & & D_{55}[A] \end{bmatrix} \quad \text{(E.14b)}$$

$$\left[[k_2]-[k_2]^T\right] = \begin{bmatrix} & & D_{13}[E]-D_{55}[E]^T \\ & & \\ D_{55}[E]-D_{13}[E]^T & & \end{bmatrix} \quad \text{(E.14c)}$$

$$[k_3] = \begin{bmatrix} D_{55}[B] & & \\ & D_{44}[B] & \\ & & D_{33}[B] \end{bmatrix} \quad \text{(E.14d)}$$

$$\left[[k_4]+[k_4]^T\right] = \begin{bmatrix} & (D_{21}+D_{66})[A] & \\ (D_{21}+D_{66})[A] & & \\ & & \end{bmatrix} \quad \text{(E.14e)}$$

$$[k_5] = \begin{bmatrix} D_{66}[A] & & \\ & D_{22}[A] & \\ & & D_{44}[A] \end{bmatrix} \quad \text{(E.14f)}$$

and

$$\left[[k_6]-[k_6]^T\right]=\begin{bmatrix} & & \\ & & D_{23}[E]-D_{44}[E]^T \\ & D_{44}[E]-D_{23}[E]^T & \end{bmatrix}. \tag{E.14g}$$

Substitute the above equations (E.14a) through (E.14g) into equation (3.95) and perform the following multiplications for the matrix terms within the eigenvalue equation (3.95) yields

$$-[m]\{\ddot{q}\}=\begin{Bmatrix}\rho[A]\{\bar{u}\}\\ \\ \end{Bmatrix}\omega^2\cos(\hat{M}x)\sin(\hat{N}y)+\begin{Bmatrix} \\ \rho[A]\{\bar{v}\}\\ \end{Bmatrix}\omega^2\sin(\hat{M}x)\cos(\hat{N}y)+\begin{Bmatrix} \\ \\ \rho[A]\{\bar{w}\}\end{Bmatrix}\omega^2\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15a}$$

$$[k_1]\{q''\}=-\begin{Bmatrix}D_{11}[A]\{\bar{u}\}\\ \\ \end{Bmatrix}\hat{M}^2\cos(\hat{M}x)\sin(\hat{N}y)-\begin{Bmatrix} \\ D_{66}[A]\{\bar{v}\}\\ \end{Bmatrix}\hat{M}^2\sin(\hat{M}x)\cos(\hat{N}y)-\begin{Bmatrix} \\ \\ D_{55}[A]\{\bar{w}\}\end{Bmatrix}\hat{M}^2\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15b}$$

$$\left[[k_2]-[k_2]^T\right]\{q'\}=\begin{Bmatrix}\left[D_{13}[E]-D_{55}[E]^T\right]\{\bar{w}\}\\ \\ \end{Bmatrix}\hat{M}\cos(\hat{M}x)\sin(\hat{N}y)+\begin{Bmatrix}0\\0\\0\end{Bmatrix}\sin(\hat{M}x)\cos(\hat{N}y)+\begin{Bmatrix} \\ \\ \left[D_{55}[E]-D_{13}[E]^T\right]\{\bar{u}\}\end{Bmatrix}(-\hat{M})\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15c}$$

$$[k_3]\{q\}=-\begin{Bmatrix}D_{55}[B]\{\bar{u}\}\\ \\ \end{Bmatrix}\cos(\hat{M}x)\sin(\hat{N}y)-\begin{Bmatrix} \\ D_{44}[B]\{\bar{v}\}\\ \end{Bmatrix}\sin(\hat{M}x)\cos(\hat{N}y)-\begin{Bmatrix} \\ \\ D_{33}[B]\{\bar{w}\}\end{Bmatrix}\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15d}$$

$$\left[[k_4]+[k_4]^T\right]\{\dot{q}'\}=-\begin{Bmatrix}(D_{21}+D_{66})[A]\{\bar{v}\}\\ \\ \end{Bmatrix}\hat{M}\hat{N}\cos(\hat{M}x)\sin(\hat{N}y)-\begin{Bmatrix} \\ (D_{21}+D_{66})[A]\{\bar{u}\}\\ \end{Bmatrix}\hat{M}\hat{N}\sin(\hat{M}x)\cos(\hat{N}y)+\begin{Bmatrix}0\\0\\0\end{Bmatrix}\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15e}$$

$$[k_5]\{\ddot{q}\}=-\begin{Bmatrix}D_{66}[A]\{\bar{u}\}\\ \\ \end{Bmatrix}\hat{N}^2\cos(\hat{M}x)\sin(\hat{N}y)-\begin{Bmatrix} \\ D_{22}[A]\{\bar{v}\}\\ \end{Bmatrix}\hat{N}^2\sin(\hat{M}x)\cos(\hat{N}y)-\begin{Bmatrix} \\ \\ D_{44}[A]\{\bar{w}\}\end{Bmatrix}\hat{N}^2\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15f}$$

$$\left[[k_6]-[k_6]^T\right]\{\dot{q}\}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}\cos(\hat{M}x)\sin(\hat{N}y)+\begin{Bmatrix} \\ \left[D_{23}[E]-D_{44}[E]^T\right]\{\bar{w}\}\\ \end{Bmatrix}\hat{N}\sin(\hat{M}x)\cos(\hat{N}y)-\begin{Bmatrix} \\ \\ \left[D_{44}[E]-D_{23}[E]^T\right]\{\bar{v}\}\end{Bmatrix}\hat{N}\sin(\hat{M}x)\sin(\hat{N}y) \tag{E.15g}$$

Summing terms (E.15a) through (E.15g) yields equations (3.101a), (3.101b) and (3.101c).

The element stiffness matrix, $[K]$, and element mass matrix, $[M]$, from equation (3.102) can be rearranged to give $\left[K^*\right]$ and $\left[M^*\right]$ by using the following operations in equation (E.16)

- impose column 1 of $[K]$ and $[M]$ into column 1 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 2 of $[K]$ and $[M]$ into column 4 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 3 of $[K]$ and $[M]$ into column 7 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 4 of $[K]$ and $[M]$ into column 2 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 5 of $[K]$ and $[M]$ into column 5 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 6 of $[K]$ and $[M]$ into column 8 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 7 of $[K]$ and $[M]$ into column 3 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 8 of $[K]$ and $[M]$ into column 6 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively
- impose column 9 of $[K]$ and $[M]$ into column 9 of $\left[K^*\right]$ and $\left[M^*\right]$, respectively.

(E.16)

Applying of these operations to the rows and columns of $[K]$ and $[M]$ will yield the element stiffness matrix, $\left[K^*\right]$, and element mass matrix, $\left[M^*\right]$, in equation (3.109). These matrices are given as

$$
\left[K^{*}\right]=\begin{bmatrix}
K11_{11} & K12_{11} & K13_{11} & K11_{12} & K12_{12} & K13_{12} & K11_{13} & K12_{13} & K13_{13} \\
 & K22_{11} & K23_{11} & K21_{12} & K22_{12} & K23_{12} & K21_{13} & K22_{13} & K23_{13} \\
 & & K33_{11} & K31_{12} & K32_{12} & K33_{12} & K31_{13} & K32_{13} & K33_{13} \\
 & & & K11_{22} & K12_{22} & K13_{22} & K11_{23} & K12_{23} & K13_{23} \\
 & & & & K22_{22} & K23_{22} & K21_{23} & K22_{23} & K23_{23} \\
 & & & & & K33_{22} & K31_{23} & K32_{23} & K33_{23} \\
 & & \text{(Symmetric)} & & & & K11_{33} & K12_{33} & K13_{33} \\
 & & & & & & & K22_{33} & K23_{33} \\
 & & & & & & & & K33_{33}
\end{bmatrix}
\tag{E.17}
$$

and

$$
\left[M^{*}\right]=\rho\begin{bmatrix}
A_{11} & & & A_{12} & & & A_{13} & & \\
 & A_{11} & & & A_{12} & & & A_{13} & \\
 & & A_{11} & & & A_{12} & & & A_{13} \\
 & & & A_{22} & & & A_{23} & & \\
 & & & & A_{22} & & & A_{23} & \\
 & & & & & A_{22} & & & A_{23} \\
 & & \text{(Symmetric)} & & & & A_{33} & & \\
 & & & & & & & A_{33} & \\
 & & & & & & & & A_{33}
\end{bmatrix}.
\tag{E.18}
$$