OPERATIONS RESEARCH APPROACH FOR CERTAIN FINANCIAL DECISION MAKING PROBLEMS UNDER FUZZY ENVIRONMENT

BY

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A Thesis/Practicum submitted to the Faculty of Graduate Studies of The University of Manitoba in partial fulfillment of the requirements of the degree

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MASTER OF SCIENCE

TO MY LATE MUM AND DAD

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I

ABSTRACT

Fuzzy set theory has proved to be very efficient to handle certain type of uncertainties encountered in a variety of areas. It provides an excellent mathematical framework for explicitly incorporating imprecision and vagueness into the decision making models, especially when the system involves human subjectivity. Uncertainty is present in virtually all walks of life.

In this thesis we make use of fuzzy sets theory to model imprecision in a net present value problem, capital asset pricing model and the scoring model decision making problem under fuzzy environment. Since, triangular fuzzy numbers provides a good compromise between computational efficiency and realistic modeling of imprecision, therefore, use of such numbers is made throughout the thesis. We model a number of investment decision making problems in the field of finance and actuarial science under fuzzy environment. We also discuss the methods to obtain their solutions and interpretation to those solutions.

Chapter 1 provides an introduction to the concepts of decision making and presents introduction, prerequisites, the motivations and the needs for a comprehensive methodology for analyzing certain problems in the areas of finance and actuarial science under fuzzy environment. Chapter 2 deals with the literature review of the related work done by other researchers relevant to this research. Net present value problem under fuzzy environment, is considered in Chapter 3. Chapter 4 deals with the capital asset pricing model under fuzzy environment and makes its use in the net present value problem when the returns on investments is governed by the capital asset pricing model under fuzzy environment. Chapter 5 deals with the scoring models under fuzzy environment. Finally, the conclusion and the discussion on the contributions made by the thesis, along with some recommendations for further research, are given in Chapter 6.

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Chapter 1

INTRODUCTION

In today's world, the pace of change is so rapid and its nature is often unanticipated and unpredictable. Uncertainty is present in most decision making problems due to unknown future events, such as revenue streams, costs, future interest rates and inflation. One's ability to respond to rapid and unanticipated change by classical methods has not been effective so far [6, 17, 33]. Often the business world is looking for methods and tools to help configure their processes in order to respond effectively to unanticipated and unpredictable changes. It has been noticed [7] that most investment projects are typically chosen on the basis of some kind of restricted information. Furthermore, the volatility literature claims that prices are too volatile to accord with simple net present value models [33, 58, 59]. The actuarial and financial community do recognize the need for formal models to represent imprecise, vague, and uncertain information in a positive manner [47]. At the same time the incapability of the classical mathematical models to render a satisfactory solution to these problems is generally accepted. Moreover the monopoly position of probability theory to treat every kind of uncertainty has become highly questionable. As a consequence, the scientific literature of the past three decades has offered some new models and techniques to describe and to transfer vagueness, imprecision and uncertainty in a useful manner. Undoubtedly, one of those is the theory of fuzzy sets that was introduced in 1965 by the papers of Lofti A. Zadeh [62, 63].

Fuzzy sets theory is a relatively new mathematical tool to deal with vagueness, imprecision uncertainty [34, 35, 65] and is highly suitable for application areas where an expert's subjective judgment is involved. Used to represent uncertain reasoning, it mimics the human ability to take rational decisions in an ambiguous and inexact environment.

1.1 Fuzzy Sets Theory

We now introduce certain terminology, notation, definitions and prerequisites that will be used in the sequel.

Fuzzy Set

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x. The membership in a crisp subset of X is viewed as characteristic function $\mu_A(x)$ from X to {0, 1} such that:

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \notin A \\ 1 & \text{for } x \in A \end{cases}$$

where {0, 1} is called a valuation set [7, 34, 63].

If the valuation set is allowed to be the closed real interval [0, 1], A is called a fuzzy set as proposed by Zadeh [61], with $\mu_A(x)$ as the degree of membership (degree of belonging) of x in A. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Therefore, A is completely characterized by the set of ordered pairs:

$A = \{(x, \mu_A(x)/ x \in X\}$

where $\mu_A(x)$ maps X to the membership space [0,1]. Elements with zero degree of membership are usually not listed. If Sup $\mu_A(x) = 1$, $\forall\, x \in R$, then the fuzzy set A is called a normal fuzzy set in R. A fuzzy set that is not normal is called *subnormal* fuzzy set.

α – Cut or α – level Cut

An α – cut denoted by A_α is the crisp set of element x in R whose degree of belonging to the fuzzy set A is at least $\alpha \in [0,1]$. This means $A_\alpha = \{x \in X \mid \mu(x) \geq \alpha, \alpha \in [0,1]\}$.

The α – cut or α – level set is the crisp set A_α that contains all elements of the universal set X $\in$ R whose membership grades in A are greater than or equal to the specified value of α, $\alpha \in [0,1]$.

Support of a Fuzzy Set

The support of a fuzzy set A is a set S(A) such that $x \in S(A) \Leftrightarrow \mu_A(x) > 0$. If $\mu_A(x))$ is constant over S(A), then A is non–fuzzy.

Intersection of Fuzzy Sets

Intersection of two fuzzy sets A and B is a fuzzy set C denoted by $C = A \cap B$, whose membership function is related to those of A and B by

$\mu_C(x) = \text{Min}\,[\,\mu_A(x),\ \mu_B(x)\,]\quad \forall\ x \in X$

1.2 Algebraic Operations on Fuzzy Sets

In addition to the set theoretic operations, we can also define a number of combinations fuzzy sets and relate them to one another. Here we present some more important operations among them. Algebraic product of two fuzzy sets A and B, is A (.) B whose membership function is

$$\mu_{A(.)B}(x) = [\,\mu_A(x)\,.\,\mu_B(x)\,] \quad \forall\ x \in X$$

Algebraic Sum of two fuzzy sets A and B, is A (+) B whose membership function is

$$\mu_{A+B}(x) = [\,\mu_A(x)\ (+)\ \mu_B(x)\,] \quad \forall\ x \in X$$

provided $0 \le \mu_A(x)\ (+)\ \mu_B(x) \le 1$

1.3 Convexity of Fuzzy Sets

The notion of convexity can be extended to fuzzy sets in such a way as to preserve many of the properties that it has in case of crisp sets. In what follows, we assume that X is the n-dimensional space R^n. We have the following two definitions of convexity of a fuzzy sets.

Convex Fuzzy Set

A fuzzy set A is convex if and only if the sets $A_\alpha = \{\, x \in X \mid \mu_A(x) \ge \alpha \,\}$ for all $\alpha \in [0,1]$ is a convex set.

The second definition of convexity of a fuzzy set is as follows:

A fuzzy set A is said to be a convex set if

$$\mu(\,\lambda x_1 + (1-\lambda)\,x_2\,) \ \ge\ \mathrm{Min}\,(\,\mu(x_1),\ \mu(x_2)\,),\ x_1, x_2 \in X,\ \ \lambda \in [0,1]\,.$$

The definition of a fuzzy set leads us to the following definition of a fuzzy number.

1.4 Fuzzy Arithmetic

The first definition of a fuzzy set allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

Fuzzy Number

A Fuzzy Number A is a fuzzy set on the real line R, that possesses the following properties:

(1). A is a normal, convex fuzzy set on R,

(2). the α –level set A_α is a closed interval for every $\alpha \in [0, 1]$, and

(3). the support of A, $S(A) = \{x \mid \mu_A(x) > 0\}$, is bounded.

Fuzzy arithmetic is based on two properties of fuzzy numbers:

- Each fuzzy set and thus, each fuzzy number can be fully and uniquely represented by its α –level sets.
- α –level sets of each fuzzy number are closed intervals of real numbers for all $\alpha \in [0,1]$.

These properties enable us to define an arithmetic operation on fuzzy numbers in terms of arithmetic operations on their α –level sets. (i.e. arithmetic operations on closed intervals).

1.5 Fuzzy Arithmetic Based on Operations on Closed Intervals.

A fuzzy number can be characterized by an interval of confidence at level α, [34 ,35], as follows.

$A_\alpha = [a_1^{(\alpha)},\ a_2^{(\alpha)}]$

which has the property

$\alpha \leq \alpha' \quad \Rightarrow A_{\alpha'} \subset A_{\alpha}$

Let A = [a, b] ∈ R and B = [c, d] ∈ R be two fuzzy numbers then we define the arithmetic operations on them as follows.

Addition $A + B = [\, a + c, b + d \,]$

Subtraction $A - B = [\, a - d, b - c \,]$

Multiplication $A\,B = [\, \text{Min}\,(ac, ad, bc, bd),\ \text{Max}\,(ac, ad, bc, bd)\,]$

Inverse of A $A^{-1} = [\, \text{Min}\,(\frac{1}{a}, \frac{1}{b}), \text{Max}\,(\frac{1}{a}, \frac{1}{b})\,]$

Division $\frac{A}{B}$ $\frac{A}{B} = [\text{Min}\,(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}), \text{Max}\,(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d})]$

Minimum ($\wedge$) $A \wedge B = [a \wedge c, b \wedge d]$

Maximum ($\vee$) $A \vee B = [a \vee c, b \vee d]$

Let A and B be two fuzzy numbers such that $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ be the α – level set of A and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the α – level set of B.

Let * denote any of the arithmetic operations +, –, ., /, $\wedge$ and $\vee$ on fuzzy numbers.

Then, we define a fuzzy set A * B in R, by defining its α – level sets $(A * B)_\alpha$ as

$(A * B)_\alpha = A_\alpha * B_\alpha$ for any $\alpha \in [0,1]$

Since $(A * B)_\alpha$ is a closed interval for each $\alpha \in [0,1]$ and A and B are fuzzy numbers, A * B is also a fuzzy number.

The multiplication of fuzzy number $A \subset R$ by an ordinary number $k \in R^+$ is defined as

$(k^* A_\alpha) = k\,(\circ)\,A_\alpha = [\, ka_1^{(\alpha)},\ ka_2^{(\alpha)}\,]$

or equivalently, $\mu_{k.A}(x) = \mu_A(\frac{x}{k}) \quad \forall\, x \in R$

Triangular Fuzzy Number

A triplet (a, b, c) is defined as a triangular fuzzy number (T.F.N.) if its membership function is defined as

$$\mu_{A(x)} = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ \dfrac{x-c}{b-c} & b \leq x \leq c \\ 0 & x > c \end{cases}$$

Alternatively [35 , pp. 26, 27], in terms of confidence at level α we characterize the T.F.N. (a_1, a_2, a_3) as,

$$A(\alpha) = \left[(a_2 - a_1)\alpha + a_1, \; -(a_2 - a_3)\alpha + a_3 \right] \qquad \forall\ \alpha \in [0,1]$$

Associated Ordinary Number

Let $A_i = (a_{i1}, a_{i2}, a_{i3})$ be a triangular fuzzy number. Then, an associated ordinary number (AON) corresponding to A is a crisp number and is defined as follows.

$$(AON)_i = \frac{a_{i1} + 2a_{i2} + a_{i3}}{4}$$

Trapezoidal Fuzzy Number

A triplet (a, b, c, d) is defined as a trapezoidal fuzzy number (Tr.F.N.) if its membership function is defined as

$$\mu_{A(x)} = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b \le x \le c \\ \dfrac{x-d}{c-d} & c \le x \le d \\ 0 & x > d \end{cases}$$

Alternatively, in terms of confidence at level α we characterize the Tr.F.N. (a_1, a_2, a_3, a_4) as,

$$A(\alpha) = \left[(a_2 - a_1)\alpha + a_1,\ (a_3 - a_4)\alpha + a_4\right] \qquad \forall\ \alpha \in [0,1]$$

1.6 Kaufmann and Gupta Error Analysis with T.F.N. Approximation.

Using Kaufmann and Gupta [32] notation, suppose we have a fuzzy number A whose α – cut is given explicitly by

$$A(\alpha) = \left[A(\alpha_L), A(\alpha_R)\right]$$

where $A(\alpha_L) =$ represents the α – cut of left segment of the exact fuzzy number A.

$A(\alpha_R) =$ represents the α – cut of right segment of the exact fuzzy number A.

Suppose P, a T.F.N., is obtained as an approximation of the fuzzy number A, such that the α – cut of P is given by

$$P(\alpha) = \left[P(\alpha_L),\ P(\alpha_R)\right]$$

where $P(\alpha_L)$ = represents the α – cut of left segment of the approximate fuzzy number P.

$P(\alpha_R)$ = represents the α – cut of right segment of the approximate fuzzy number P.

The left divergence $\epsilon_{l\alpha}$ is given by [35],

$$\epsilon_{l\alpha} = A(\alpha_L) - P(\alpha_L). \tag{1.6.1}$$

The right divergence $\epsilon_{r\alpha}$ is given by [35],

$$\epsilon_{r\alpha} = A(\alpha_R) - P(\alpha_R). \tag{1.6.2}$$

According to Kaufmann and Gupta [35],

If each of the right and left divergence is small, then P can be considered as a good approximation of the fuzzy number A, and one can use P in place of A to obtain certain interesting results.

Kaufmann and Gupta [35] show that if the fuzzy number A is obtained as a result of multiplication of two fuzzy numbers, then the maximum value of both $\epsilon_{l\alpha}$ and $\epsilon_{r\alpha}$ occurs at $\alpha = \frac{1}{2}$.

1.7 Ranking of Fuzzy Numbers.

Fuzzy numbers are convenient for representing imprecise numerical quantities in a vague environment, and for purpose of application their comparison or ranking is very important. Also, ranking fuzzy numbers is a fundamental problem of fuzzy decision making. Since fuzzy numbers do not form a natural linear order (for example, real numbers form a natural linear order), different comparison methods have been developed by various researchers [7, 17, 35, 40]. Each of these methods has its own advantages and disadvantages and it would be a hard task to decide which one of them is the best. Despite many methods suggested in the literature [7, 17, 35, 40] there is no single measure which is, in case of fuzzy numbers, universally applicable to a wide variety of situations. Below we give, briefly few methods of ranking fuzzy numbers.

Brief Description of the Ranking and Ordering Indices.

Let $A_i = (a_{1i}, a_{2i}, a_{3i})$ for $i = 1, 2, 3, \ldots, n$, represent the triangular fuzzy numbers to be ranked. Researchers in [7, 17, 35, 40] develop a real number as an $(\text{Index})_i$ for each A_i as the ordering value or ranking value of A_i. The $(\text{Index})_i$ is treated as a fuzzy measure of A_i. Then, the fuzzy numbers are ranked according to the descending (or ascending) order of their corresponding indices, and the fuzzy number with the largest index is considered as the most preferred fuzzy number. Below we give some of the indices developed by various researchers.

1. **Chang's Ranking Index (Komolananij [40])**

$$(\text{Index})_i = \frac{(a_{3i} - a_{1i})(a_{1i} + a_{2i} + a_{3i})}{6} \qquad \text{for } i = 1, 2, \ldots, n.$$

2. **Chiu and Park Ranking Index** (Chiu and Park [17, 40])

Let w_{i1} and w_{i2} be the weights associated with the fuzzy number A_i whose ranking index has to be computed. Then,

$$(\text{Index})_i = \frac{(a_{1i} + a_{2i} + a_{3i})w_{i1}}{3} + w_{i2}a_{2i} \qquad \text{for } i = 1, 2, \ldots, n.$$

Chiu and Park [17] suggest that we let $w_{i1} = 1$ and w_{i2} be between [0.1 and 0.3].

3. **Kaufmann and Gupta Ranking Index** (Kaufmann and Gupta [34])

The Kaufmann and Gupta Ranking Method is a hierarchical test for which

$$(\text{Index})_i = \frac{(a_{1i} + 2a_{2i} + a_{3i})}{4} \qquad \text{for } i = 1, 2, \ldots, n.$$

Kaufmann and Gupta [35] suggest the if we have certain fuzzy numbers having the same indices, then the fuzzy number having the largest $[a_{2i}]$ is the best alternative. If an index and $[a_{2i}]$ are the same for a set of fuzzy numbers, then examine the range of $[a_{1i} - a_{3i}]$. The fuzzy number with the largest $[a_{1i} - a_{3i}]$ is chosen as the best alternative.

4. **Fuzzy Weighted Methods** (Bortolan [7])

Let w_{i1} and w_{i2} represent the weights associated with the fuzzy number whose ranking index has to be computed. Then,

$$(\text{Index})_i = \frac{(a_{1i} + a_{3i})\, w_{i1}}{2} + w_{i2} a_{3i} \qquad \text{for } i = 1, 2, \ldots, n.$$

According to Botolan [6] w_{i1} and w_{i2} can be chosen as $w_{i2} = 1$, $w_{i1} = 0.5$.

1.8 Net Present Value

In this section, we introduce and define a few terms which will be used in developing various ideas in the thesis.

Discount Rate

Discount rate represents the opportunity cost of money and is often selected as the after–tax rate of return on an alternative investment or the cost of borrowing money. In the sequel we shall denote the discount rate by the symbol r.

Some other common terms for Discount Rate are [11, 32, 47].

(a) Rate of Return,
(b) Marginal Efficiency of Capital,
(c) Actuarial Return,
(d) Interest Rate of Return,
(e) Investor's Method,
(f) Yield per Unit Time,
(g) Money –Weighted Rate of Return,
(h) Yield to Redemption.

Since the discount rate reflects the future value of money, it typically has two components: an adjustment for inflation, and a risk-adjusted return on the use of the money. Since market forces typically incorporate inflation adjustments into investment returns and borrowing costs, often the discount rate is keyed to a standard reference rate.

1.9 Net Present Value

Net Present Value (NPV) Formula

Suppose

(i) A_j is the amount of money invested at the end of period j, $j = 0, 1, 2, \ldots, n$.

(ii) each A_j, $j = 0, 1, 2, \ldots, n$, is known in advance with a crisp value,

(iii) r is the rate of discount, which is same for every period j, $j = 1, 2, \ldots, n$, and is known in advance with a crisp value.

The NPV in finance [11, 32, 49] is given by the following formula:

$$NPV = -A_0 + \frac{A_1}{(1+r)} + \frac{A_2}{(1+r)^2} + \ldots + \frac{A_n}{(1+r)^n} \tag{1.9.1}$$

NPV compares the value of a dollar today versus the value of that same dollar in the future after taking inflation and return into account Kellison [38]. It is used to compare the value of different investment alternatives, and is a valuable tool for decision-making.

Taking the rate of discount as different for different periods, Kellison [38, pp. 28] presents a generalized form of (1.9.1) as follows.

$$NPV = -A_0 + \frac{A_1}{(1+r_1)} + \frac{A_2}{(1+r_1)(1+r_2)} + \ldots + \frac{A_n}{(1+r_1)(1+r_2)\ldots(1+r_n)} \tag{1.9.2}$$

where

(iv) $A_j, j = 0, 1, 2, \ldots, n$ is as in (i) and (ii) above,

(v) $r_j, j = 1, 2, \ldots, n$ is the rate of discount for period j, $j = 1, 2, \ldots, n$, and is known in advance with a crisp value.

A method called 'Investment Appraisal' is used to evaluate the viability of a proposed project by assessing the value of net cash flows that result from its implementation [12]. The future growth and direction of any company is largely determined by how and where capital is invested. As a result, it is generally recommended that financial appraisals be undertaken as part of an overall management practice [13].

Benefits Associated with NPV. The NPV has following benefits associated with its use.

(a) The NPV uses cash flows, and uses all the cash flows of the project.

(Other approaches [11, 13] ignore the cash flows beyond a certain date.)

(c) The NPV discounts the cash flows properly – by the rate of return on projects with comparable risk. (Other approaches [11, 13] ignore the time value of money.)

Assumptions on which NPV is based. NPV is based on the following assumptions [10, 12].

(a) Cash flows are known (no risks taken) and are in constant value dollars (no inflation).

(c) Interest rates are known.

(d) Taxes are not included in calculations.

(e) Intangibles are not considered.

(f) Funds are considered to exist at all times.

NPV Decision Making Rules Under Crisp Environment

Rule 1. An investment is worth making if it has a positive NPV, because

if NPV > 0, the firm overall value will increase.

In this case accept all independent investments having NPVs greater than 0.

Rule 2. Investments with negative NPVs should be rejected because

if NPV < 0, the firm overall value will decrease.

In this case reject all independent investments having NPVs less than 0.

Rule 3. If NPV = 0, the firm overall value will not change if the new project is adopted.

In this case the investors are more or less indifferent about the project.

According to Brealey [10], NPV is the best way to evaluate capital budgeting projects.

1.10 Various Types of Risk and Various Types of Investors.

Risk is defined as:

(a) The possibility that some invested funds will be lost through a decline in the value of an investment.

(b) Degree of uncertainty of return of asset.

Risk involves positive and negative consequences. Positive risks are called opportunities and negative risks are called risks. Investment activities always gives rise to risk due to uncertainty. Investors cannot predict exactly what will happen or guarantee outcomes with reasonable certainty [11].

Some synonymous terms used for risk in the corporate world are:

(a) Uncertainty, (b) Exposure, (c) Variance, (d) Volatility

In investment the term 'risk' is generally used to mean the probability that the actual return on an investment will be somewhat different from the anticipated return. The most common method of measuring risk is to calculate the volatility of returns, and the common measure of volatility is the standard deviation of returns.

Risk Free Rate

Risk free rate is mathematically represented by the notation R_f. It is a theoretical interest rate at which an investment capital may earn interest without taking any risk whatsoever. This notion is extensively used in capital asset pricing model and option pricing theory. Risk free rate depends on two important factors:

(a) Premuim for delayed gratification, and (b) Inflation factor.

Market Risk

Market risk, mathematically represented by the notation R_m, can be defined as the exposure to adverse movements in market prices or rates. It can be split into risk arising from two different and distinct entities namely:

(a) Interest rates, and (b) Equity movements.

Market risk functions are usually split along product and geographic lines, and the market risk can be monitored on

a number of levels, including individual transaction, portfolio and the firm as a whole.

Risk Premium

Risk Premium, mathematically expressed as the difference between the return on a risky asset and a risk free asset with identical life spans, is defined as the additional expected return required to entice investors into providing funds for risky investments rather than a safe one. Mathematically, Risk Premium $= [R_m - R_f]$.

Systematic Risk

Systematic risk is the risk that is due to general market–wide factors, such as a severe recession, or a natural disaster like an earthquake. This type of risk is also known as unavoidable risk or non–diversified risk, since no amount of diversification can reduce it. The systematic risk principle states that the reward for bearing risk depends on the systematic risk of an investment, and the expected returns on investments can reward investors for enduring systematic risks.

β, a Measure of Systematic Risk

β is a measure of systematic risk and is the overall risk in investing in a large market and company also has its own β. A company's β is that company's risk compared to the risk of the overall market. If a company, for example, has a β of 2.0, then this company is said to be 2 times riskier than the overall market. β times the market risk premium determines the additional risk premium over risk free rate that is demanded by investors. β is assumed to be a predictor of future market behavior. If for an investment β is negative, the tendency of the stock is to move in the opposite direction to that of the market. β is calculated using historical information of market movements in relationship to the movements in the security or portfolio. An investment with a β of 1 means it follows the market perfectly. β is generally estimated from historical data and applied to a future period. If we assume that

n is the number of periods under consideration,

R_i represents the actual return through different periods i of time, $i = 1, 2, \ldots, n$,

$\text{cov}(R_i, R_m)$ is the covariance between the actual return R_i for period i and the market return R_m, and

$\text{var}(R_m)$ is the variance of the market return R_m,

then, β the measure of systematic risk, is given by

$$\beta = \frac{\text{cov}\,(R_i, R_m)}{\text{var}\,(R_m)} \tag{1.10.1}$$

Unsystematic Risk

Unsystematic risk is a function of the characteristics of the industry, the individual company, and the type of investment interest. Also called the diversified risk, residual risk, or company specific risk, this type of risk is unique to a company and usually is associated with such events as a strike, the outcome of unfavorable litigation, or a natural catastrophe. Unsystematic risk can easily be eliminated by constructing suitably large well-diversified portfolios.

Diversification and its Merits.

Diversification is the process of dividing an investment among a variety of assets. It has been proved and shown by Ross [48, 49] that diversification is fundamental in reducing risk. Risk is difficult to avoid no matter how an investor chooses to invest and investors are made aware that they must take greater risks to achieve higher returns. Unsystematic risks can be avoided by diversifying among different companies rather that just investing in the same one. Diversification across asset classes provides a cushion against market tremors because each asset class has different risks, rewards and tolerance to economic events. By selecting investments from different asset classes, one can minimize risk. Diversification among companies, industries and asset classes affords the investor the greatest protection against volatility.

Types of Investors

Investors can be classify as having three types of risk preferences namely risk loving, risk neutral and risk averse.

(i) A **risk loving investor** strictly prefers a risky investment opportunity to a riskless opportunity.

(ii) A **risk neutral investor** is indifferent to the risk of an investment opportunity. (i.e. a risk neutral investor is indifferent between a risky investment and a riskless investment opportunity.)

(iii) A **risk averse investor** strictly prefers a riskless investment opportunity to a risky opportunity.

The Capital Asset Pricing Model.

The Capital Asset Pricing Model (CAPM) was developed in the mid –1960s by William Sharp, John Lintner and Jack Treynor [43, 54]. Applications of this model have found acceptance in various of economic forecasting, applied finance as well as in actuarial sciences. The CAPM describes the unique relationship that exists between risk and reward of an investment, and asserts that rational investors can expect a return for taking on systematic risk, or the set of risks to which the entire identified market is subjected because that risk cannot be completely eliminated through a diversification of holdings. Fama and French [23, 27] conclude that the CAPM does not hold empirically and should be disregarded entirely. They claim that the CAPM is just a model that provides a convenient way of thinking about risks and returns. It is not an economic law and, therefore, is subject to criticism. Despite years of attempts to verify or refute the Capital Asset Pricing Model, there has been no consensus on its legitimacy. According to CAPM, the marketplace compensates rational investors for taking systematic risk, but not for taking unsystematic risk. Even after careful diversification, some risk associated with the market as a whole will still remain, and cannot be neutralized through diversification alone. CAPM is based on certain assumptions as stated below.

Assumptions underlying CAPM.

The CAPM is a *ceteris paribus* model. It is valid within the following set of assumptions [11, 32, 44].

(a) Investors are risk averse individuals.

(b) Investors have homogenous expectations (beliefs) about asset returns.

(c) Asset returns are distributed by the normal distribution.

(d) There exists a risk free asset and investors may borrow or lend unlimited amounts of this asset at a constant rate.

(e) There are a definite number of assets and their quantities are fixed within the one period world.

(f) All assets are perfectly divisible and priced in a perfectly competitive market.

(g) Asset markets are frictionless and information is costless and simultaneously available to all investors.

(h) There are no market imperfections such as taxes, regulations, or restrictions on short selling.

(i) Everybody will hold a portfolio of assets which is as diversified as possible.

(k) All investor are mean variance optimizers.

(l) All investors have the same expectations.

CAPM under Crisp Environment

Below, we now provide the CAPM under crisp environment. In Chapter 4, we extend it under fuzzy environment.

Let

n = the number of periods under consideration,

R = return on an investment under the CAPM,

R_f = risk-free rate of return,

β = the amount of systematic risk,

R_m = the market return on the investment, and

$(R_m - R_f)$ = market risk premium.

Then,

$$\text{Total Risk} = [\text{SystematicRisk} + \text{Unsystematic Risk}]$$

According to Brealey [11], the return on investment in the CAPM is given by,

$$R = (\text{Risk free interest rate}) + [\text{Amount of Systematic Risk}]\begin{bmatrix}\text{Risk Premium needed to} \\ \text{compensate} \\ \text{unit of Systematic risk}\end{bmatrix}$$

$$= R_f + (\beta \text{ of the asset})[\text{Expected return on the market portfolio} - R_f]$$

Therefore,

$$R = R_f + \beta (R_m - R_f) \tag{1.10.2}$$

where β represents the amount of systematic risk, and is given by

$$\beta = \frac{\sum_{i=1}^{n} (R_i - \bar{R}_i)(R_{mi} - \bar{R}_{mi})}{\sum_{i=1}^{n} (R_{mi} - \bar{R}_{mi})^2}, \qquad i = 1, 2, \ldots, n \tag{1.10.3}$$

The CAPM Return as Discount Rate

Generally, the CAPM return is often used in the appraisal of capital investment by estimating a β for a project and using the estimate in the CAPM to give a discount rate for use in the calculation of net present value (NPV). The CAPM also offers a useful, simple, and applicable approach to investment appraisal in risky environments.

1.11 Scoring Models

Project selection models first introduced by Keeney and Raiffa [36, 37] are based on scoring or ranking attempt to broaden the base on which the selection of various decision alternatives takes place. This is not to say that they ignore questions of cost and profit, but rather they seek to include other factors so as to get a wider perspective on the merits of a decision alternative. These other factors are often chosen so as to reflect company strategies or goals. Scoring models, first introduced by Keeney and Raiffa [36, 37], are based on an analytical tool and often help to highlight the level of compatibility that a project might have with the firms overall strategic aims. The analytical tool used enables one to explicitly rank tangible and intangible factors, having certain degree of importance, against each other for the purpose of resolving conflict or setting priorities. It is a multi-attribute modelling methodology, that has helped various researchers to investigate successfully problems of multi-decision modelling [3, 31, 36, 37, 46, 60].

Weighted Scoring Model under Crisp Environment

The weighted scoring model under crisp environment, as stated by Keeney and Raiffa [36, 37], is stated as follows.

Suppose we have n decision alternatives and m type of different criteria. Let

w_i = the weight associated with the i-th criterion, $i = 1, 2, \ldots, m$

s_{ij} = the score associated with i-th criterion and j-th decision alternative,
$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$

S_j = the total score associated with the j-th decision alternative, $j = 1, 2, \ldots, n$

then according to [52] the total score S_j, in the scoring model is given by

$$S_j = \sum_{i=1}^{n} w_i\, s_{ij} \qquad \text{for } j = 1, 2, \ldots, n. \tag{1.11.1}$$

This decision alternative selection model uses a scoring system [34, 59] in which the designers of the model are required to give i-th criterion a weight w_i. The weights, w_i may be generated by various techniques. One effective and widely used procedure for this purpose is the Delphi technique [34, 59], however, any weighting which is considered to reflect accurately the organization's priorities can be used. This weighting is chosen to reflect the importance of that particular criterion from the point of view of management, or the importance of the criterion with respect to achieving the company's strategic goals, and is usually allocated a value belonging to the interval [1, 100] depending upon the importance of the criterion. Often, for convenience, the interval [1, 100] is also replaced by [1, 10].

Net Present Value versus Scoring Models

Net Present Value is not a multi–criteria decision making tool and is not meant for multi criteria decision making problems [1, 18]. It is not suitable for projects that compete for the same resources. If we rank multi criteria decision according to their Net Present Values we do not yield the right selection, because either the method ignores resource constraints or other exogenous factors. Weighted Scoring models are valuable decision making tools for evaluating multi criteria decision projects.

1.12 Organization of the Thesis

In the present thesis, we use operations research approach to model a number of investment decision making problems in the field of finance and actuarial science under fuzzy environment. We also discuss the methods to obtain their solutions and interpretation to the solutions.

Chapter 1 provides an introduction to the concepts of decision making and presents the motivations and the needs for a comprehensive methodology for analyzing investment decision making. Chapter 2 deals with the literature review of the related work done by other researchers relevant to this research. NPV problem under fuzzy environment, is considered in Chapter 3 and various special cases are discussed. Chapter 4 deals with the fuzzy NPV problem when the return on investments is governed by the capital asset pricing model under fuzzy environment. Chapter 5 deals with the weighted scoring model under fuzzy environment with an application to a bond selection problem. Finally, the conclusion and the discussion on the contributions made by the thesis, along with some recommendations for further research, are given in Chapter 6.

Chapter 2

LITERATURE SURVEY

The main objective of this chapter is to provide a literature survey of the literature dealing with Net Present Value, Capital Asset Pricing Model and the Weighted Scoring Model.

2.1 Fuzzy Net Present Value

A number of capital budgeting problems in finance and actuarial science deal with future and uncertain data that does not properly follow any probability distribution. To cope with this uncertainty, fuzzy sets theory has been applied to the Net Present Value problem by treating uncertain data as fuzzy data.

Applying fuzzy techniques to cash flow analysis has been a subject of extensive study by a number of researchers [6, 14, 17, 33, 58, 59]. In 1985, Ward [58] published a paper entitled, "Discounted fuzzy cash flow analysis", using the notion of fuzzy cash flow. This paper gave rise to an enormous interest among researchers in Finance, Management Science and Actuarial Science ([for example, see [6]). There has also been some interest in applying fuzzy logic and artificial neural networks, to Net Present Value. Ward [58, 59] discussed using fuzzy composition to estimate NPV after specifying the membership functions for future cash flows. In a subsequent article, Chui and Park [17] work in a broader framework and consider general accumulation and discount models characterized by their properties rather than specific functions. They derive the necessary and sufficient conditions for the future and present values on fuzzy intervals. Further, Buckley [14] presents the natural extension of the accumulation and discount models to the cash flow case, with a different discount rate. Buckley [14] uses fuzzy algebra in cash flow analysis.

Bojadziev and Bojadziev [6] demonstrate a number of potential applications of fuzzy techniques in finance. Fuzzy zero-based budgeting is implemented to construct a conservative or an optimistic budget. The authors use a fuzzy interest rate and fuzzy cash amounts and apply the standard arithmetic of fuzzy numbers. Further, they derive fuzzy future and present values and fuzzy annuities under a fuzzy number of interest periods. The effort of Karsak [33] is an illustration of how successfully fuzzy set theory can be applied to decision making.

2.2 Fuzzy Capital Asset Pricing Model.

In 1990, Sharpe's role in developing the Capital Asset Pricing Model was recognized by the Nobel Prize committee [54]. Ross [49, 50] presents the various elements of the Capital Asset Pricing Model within a unified framework and summarizes a vast body of literature that has grown steadily over the years. Paper by Chui and Park [17] on cash flow analysis includes some level of fuzziness in the discount rates in calculating the present worth of an investment. In spite of the considerable effort made by Chui and Park [17] to fuzzify the discount rates, no dominating theory or conclusive empirical evidence emerged to give a satisfactory result. Quite often the models suggested contain contradictory elements and suggestions, and the method appear to be weak, ambiguous and confusing. This may be partly due to the fact that the determinants of the discount rates are not obvious. The variables that determine the discount rates may vary significantly depending on the type of model used. In Chapter 4 we present a fuzzy version of the capital asset pricing model to obtain the fuzzy return, and then use the

.fuzzy return as fuzzy discount rate to evaluate fuzzy net present value in a net present value problem. Also, we provide a numerical example to demonstrate the use of fuzzy return, obtained from fuzzy capital asset pricing model, as the discount rate in obtaining the fuzzy net present value.

2.3 Fuzzy Weighted Scoring Models.

Various methods are cited in the literature to solve multicriteria selection problems under crisp environment (for example see, [4, 20]). Most of the those methods make no provision for the inclusion of uncertain, vague and imprecise data in the model. The weighted scoring model is a multi-objective decision making tool which was introduced by Keeney and Raiffa [36, 37] as a selection technique process to overcome the common inconsistency in human judgment. It is considered as a powerful decision making tool in many disciplines, and is a promising and important field of study in the early 1970's. A classical crisp weighted scoring model, as introduced by Keeney and Raiffa [36, 37] is stated as follows.

Suppose we have n decision alternatives and m criteria such that the m criteria are used on each of the n projects.

We take

s_{ij} = the score associated with the i-th criterion and j-th decision alternative, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, and

w_i = the weight of the i-th criterion, $i = 1, 2, \ldots, m$,

such that S_j, the total weighted score of the j-th decision alternative, $j = 1, 2, \ldots, n$, is given by

$$S_j = \sum_{i=1}^{m} w_i s_{ij} \qquad j = 1, 2, \ldots, n.$$

The process of deriving weights is fundamental to the effectiveness of the model's application. As already stated in Section 1.11, the weights w_i may be generated by various techniques. One effective and widely used procedure for this purpose is the Delphi technique [34, 59], however, any weighting which is considered to reflect accurately the organization's priorities can be used. A weighted scoring model allows the relative importance of the criteria to be modelled. The degree to which a criterion satisfies the relevant goal is modelled and the scores are then summed. More preferred decision alternatives score higher on the scale, and less preferred decision alternatives score lower. In practice, scales extending from 1 to 100 are often used, where 1 represents a real or hypothetical least preferred decision alternatives, and 100 is associated with a real or hypothetical most preferred decision alternatives. Often, for convenience, scales belonging to the interval [1,10] are also used. The most common way to combine scores on criteria, and relevant weights between criteria, is to calculate a simple weighted average of scores. Use of such weighted averages depends on the assumption of mutual independence of preferences. This means that the judged strength of preference for a decision alternative on one criterion will be independent of its judged strength of preference on another. Constraints can also be modelled into this procedure to give the constrained weighted scoring model. Using a variety of techniques, certain type of fuzzy decision making problems have been studied by Kaufmann and Gupta [34, 35], Zadeh [63], and Zimmermann [64, 65].

2.4 Summary of the Thesis

The research and the results conducted in this thesis are contained in Chapter 3 – 5. summarize them as follows:

Chapter 3 Fuzzy Net Present Value.

In this chapter, we compute the net present value under a variety of scenarios incorporating uncertainty in almost all of the parameters involved. We fuzzify the selection criteria for net present value under independent and mutually exclusive projects. This extends the results proved by Kaufmann and Gupta [34, 35]. The advantage of using fuzzy mathematics in the model is that it gives decision-maker flexibility and quantifies the uncertainty involved in the problem.

Chapter 4 Fuzzy Net Present Value with Risk Discount Rate.

In the present chapter under a variety of assumptions, we compute both the fuzzy systematic risk and fuzzy return using the CAPM. Furthermore, we suggest steps to use the fuzzy return, obtained using the CAPM, as the fuzzy discount rate in the NPV problem under a variety of assumptions. Also, we consider a numerical example and compare FNPV so obtained with the associated ordinary net present value (AONPV) and observe that the results obtained using fuzzy assumptions on various parameters involved in the CAPM give highly flexible results.

Chapter 5 Fuzzy Weighted Scoring Models

In this chapter, we consider the weighted scoring model of Keeney and Raiffa [36, 37] under fuzzy environment and apply it to a bonds selection problem.

Chapter 6 Conclusion, Contribution and Recommendations.

In this chapter, we present the contributions made in the thesis, conclusion along with some recommendations for further research on the problems considered in the dissertation.

Chapter 3

NET PRESENT VALUE UNDER FUZZY ENVIRONMENT

In the present chapter, we consider the Net Present Value problem under fuzzy environment. We reformulate the Net Present Value problem, as discussed by Kaufmann and Gupta [34], under fuzzy environment by implementing fuzziness in the parameters. Triangular fuzzy numbers are assumed for the parameters though other forms of fuzzy numbers could also be considered. We develop formulas to compute the Net Present Value of an investment under fuzzy environment. Further, we provide numerical examples in order to exhibit the salient points in the model.

3.1 Introduction.

Fuzzy Discounted Cash flow analysis has been the subject of study of Kaufmann and Gupta [34]. In this chapter we extend Fuzzy Net Present Value (FNPV) problem using some of the concepts laid down by Kaufmann and Gupta [34].

3.2 Net Present Value Under Fuzzy Data

We assume that each A_j, the fuzzy net investment made at the end of period j, is represented by a triangular fuzzy number.

$A_j = (a_{j1}, a_{j2}, a_{j3}) \quad j = 0, 1, 2, \ldots, n$

whose α – cut is given by

$$A_j(\alpha) = [a_{j1} + (a_{j2} - a_{j1})\,\alpha\,,\ a_{j3} - (a_{j3} - a_{j2})\,\alpha]$$

$$= (\,P_j(\alpha), Q_j(\alpha)\,) \quad j = 0, 1, 2, \dots, n$$

where, $P_j(\alpha) = a_{j1} + (a_{j2} - a_{j1})\,\alpha\,,\ Q_j(\alpha) = a_{j3} - (a_{j3} - a_{j2})\,\alpha,\ \ 0 \le \alpha \le 1,\ j = 0, 1, 2, \dots, n$

Let r_j represent the fuzzy discount rates for $j = 0, 1, 2, \dots, n$

such that each r_j is represented by a triangular fuzzy number

$$r_j = (r_{j1}, r_{j2}, r_{j3}) \quad j = 0, 1, 2, \dots, n$$

whose α –cut is given by

$$r_j(\alpha) = [r_{j1} + (r_{j2} - r_{j1})\,\alpha\,,\ r_{j3} - (r_{j3} - r_{j2})\,\alpha] \quad 0 \le \alpha \le 1$$

Let us use the notation

$$p_j(\alpha) = r_{j1} + (r_{j2} - r_{j1})\,\alpha \qquad 0 \le \alpha \le 1$$

and

$$q_j(\alpha) = r_{j3} - (r_{j3} - r_{j2})\,\alpha \qquad 0 \le \alpha \le 1$$

then we have

$$r_j(\alpha) = (p_j(\alpha), q_j(\alpha)) \qquad 0 \le \alpha \le 1$$

and $\dfrac{1}{1 \oplus r_j(\alpha)} = \left[\dfrac{1}{1 \oplus q_j(\alpha))} . \dfrac{1}{1 \oplus p_j(\alpha)}\right]$

3.3 Theorem 3.1

Let

1. $A_j = [P_j(\alpha), Q_j(\alpha)]$ $0 \leq \alpha \leq 1$ $j = 0,1,2,\ldots,n$ be the fuzzy net investment made at the end of period j, and

2. $R(\alpha) = [p_j(\alpha), q_j(\alpha)]$ $0 \leq \alpha \leq 1$ $j = 0,1,2,\ldots,n$ be the fuzzy returns over the period j then, the fuzzy net present value (FNPV) is given by.

$$FNPV(\alpha) = [-Q_0(\alpha), -P_0(\alpha)] \, (+) \left[\frac{P_1(\alpha)}{1+q_1(\alpha)}, \frac{Q_1(\alpha)}{1+p_1(\alpha)} \right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+q_1(\alpha)][1+q_2(\alpha)]}, \frac{Q_2(\alpha)}{[1+p_1(\alpha)][1+p_2(\alpha)]} \right] (+) \ldots (+)$$

$$\left[\frac{P_n(\alpha)}{[1+q_1(\alpha)][1+q_2(\alpha)]\ldots[1+q_n(\alpha)].}, \frac{Q_n(\alpha)}{[1+p_1(\alpha)][1+p_2(\alpha)]\ldots[1+p_n(\alpha)]} \right] \quad (3.3.1)$$

PROOF

In fuzzy notation , the net present value formula (1.9.2) can be rewritten as

$$\mathbf{FNPV}(\alpha) = (-Q_0(\alpha), -P_0(\alpha)) \, (+) \left[\frac{(P_1(\alpha), Q_1(\alpha))}{(1(+)([p_1(\alpha), q_1(\alpha)]))} \right]$$

$$(+) \left[\frac{(P_2(\alpha), Q_2(\alpha))}{(1(+)([p_1(\alpha), q_1(\alpha)]))(\bullet)(1(+)([p_2(\alpha), q_2(\alpha)]))} \right]$$

$$(+) \ldots (+) \left[\frac{(P_n(\alpha), Q_n(\alpha))}{(1(+)([p_1(\alpha), q_1(\alpha)]))\ldots(1(+)([p_n(\alpha), q_n(\alpha)]))} \right]$$

Using this relationship for fuzzy numbers

$$\frac{1}{[1 + R(\alpha)]} = \left[\frac{1}{1+q_j(\alpha)}, \frac{1}{1+p_j(\alpha)} \right] \qquad 0 \leq \alpha \leq 1 \qquad (3.3.2)$$

$$\frac{(P_n(\alpha), Q_n(\alpha))}{[1 + R(\alpha)]} = \left[\frac{P_n(\alpha)}{1 + q_j(\alpha)}, \frac{Q_n(\alpha)}{[1 + p_j(\alpha)} \right] \qquad 0 \le \alpha \le 1 \qquad (3.3.3)$$

and $\quad -(P_n(\alpha), Q_n(\alpha)) = (-Q_n(\alpha), \; -P_n(\alpha)) \qquad 0 \le \alpha \le 1 \qquad (3.3.4)$

Using (3. 3.2), (3.3.3) and (3.3.4) we obtain the result.

3.4 Special Cases

In Table 3.1 below we provide a summary of the different combinations that are possible in Theorem 3.1

Table 3.1:Summary of the different combinations that are possible in Theorem 3.1

Cases	Initial Investment	Future Earnings	Discount Rate
1	Fuzzy	Fuzzy and Unequal	Fuzzy and Unequal
2	Fuzzy	Fuzzy and Unequal	Fuzzy and Equal
3	Fuzzy	Fuzzy and Equal	Fuzzy and Unequal
4	Fuzzy	Fuzzy and Equal	Fuzzy and Equal
5	Crisp	Fuzzy and Equal	Fuzzy and Equal
6	Crisp	Fuzzy and Equal	Fuzzy and Unequal
7	Crisp	Fuzzy and Unequal	Fuzzy and Unequal
8	Crisp	Fuzzy and Unequal	Fuzzy and Equal
9	Crisp	Crisp	Crisp

3.5 Derivation of Some of the Special Cases

CASE 1. In (3.3.1) when

$A_j(\alpha) = (P_j(\alpha), Q_j(\alpha)) \quad 0 \le \alpha \le 1 \; \forall \; j = 0, 1, 2, \ldots, n$ then it reduces to ,

$$\text{FNPV}(\alpha) = [-Q_0(\alpha), -P_0(\alpha)]\ (+) \left[\frac{P_1(\alpha)}{1+q_1(\alpha)}, \frac{Q_1(\alpha)}{1+p_1(\alpha)}\right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+q_1(\alpha)][1+q_2(\alpha)]}, \frac{Q_2(\alpha)}{[1+p_1(\alpha)][1+p_2(\alpha)]}\right] (+)\ldots(+)$$

$$\left[\frac{P_n(\alpha)}{[1+q_1(\alpha)][1+q_2(\alpha)]\ldots[1+q_n(\alpha)]}, \frac{Q_n(\alpha)}{[1+p_1(\alpha)][1+p_2(\alpha)]\ldots[1+p_n(\alpha)]}\right] \quad (3.5.1)$$

This is the case when the capital investment, expected earnings and discount rate are all fuzzy and different.

CASE 2. In (3.3.1) when

$A_j(\alpha) = (P_j(\alpha), Q_j(\alpha))\ 0 \le \alpha \le 1 \quad \forall\ j = 0, 1, 2, \ldots, n$

and $r_j(\alpha) = (p_j(\alpha), q_j(\alpha))$

$= (p(\alpha), q(\alpha))$

$= r(\alpha) \qquad 0 \le \alpha \le 1 \quad \forall\ j = 0, 1, 2, \ldots, n$

then it reduces to ,

$$\text{FNPV}(\alpha) = [-Q_0(\alpha), -P_0(\alpha)]\ (+) \left[\frac{P_1(\alpha)}{1+q(\alpha)}, \frac{Q_1(\alpha)}{1+p(\alpha)}\right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+q(\alpha)]^2}, \frac{Q_2(\alpha)}{[1+p(\alpha)]^2}\right] (+)\ldots(+)$$

$$\left[\frac{P_n(\alpha)}{[1+q(\alpha)]^n}, \frac{Q_n(\alpha)}{[1+p(\alpha)]^n}\right] \quad (3.5.2)$$

This is the case when the capital investment and expected earnings are all fuzzy and different. The discount rate being fuzzy and identical throughout the life of the investment.

CASE 3. In (3.3.1) when

$A_j(\alpha) = (P_j(\alpha), Q_j(\alpha)) \quad 0 \le \alpha \le 1 \qquad \forall\ j = 0, 1, 2, \ldots, n$

and " r " the return being non–fuzzy and equal throughout the life of the investment

then it reduces to,

$$\text{FNPV}(\alpha) = [-Q_0(\alpha), -P_0(\alpha)]\ (+) \left[\frac{P_1(\alpha)}{1+r}, \frac{Q_1(\alpha)}{1+r}\right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+r]^2}, \frac{Q_2(\alpha)}{[1+r]^2}\right](+)\ldots(+)\left[\frac{P_n(\alpha)}{[1+r]^n}, \frac{Q_n(\alpha)}{[1+r]^n}\right] \tag{3.5.3}$$

Case 4. In (3.3.1) when we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be different but non-fuzzy, and obtain

$$\text{FNPV}(\alpha) = [-Q_0(\alpha),\ -P_0(\alpha)]\ (+) \left[\frac{P_1(\alpha)}{1+r_1}, \frac{Q_1(\alpha)}{1+r_1}\right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+r_1][1+r_2]}, \frac{Q_2(\alpha)}{[1+r_1][1+r_2]}\right](+)\ldots(+)$$

$$\left[\frac{P_n(\alpha)}{[1+r_1]\ldots[1+r_n]}, \frac{Q_n(\alpha)}{[1+r_1]\ldots[1+r_n]}\right] \tag{3.5.4}$$

CASE 5. In (3.3.1) when $A_j(\alpha) = A_j \quad \forall\ j = 0, 1, 2, \ldots, n$ and non fuzzy but the discount rate is fuzzy and equal throughout the life of the investment

then it reduces to,

$$\mathrm{FNPV}(\alpha) = [-A_0, -A_0]\ (+) \left[\frac{A_1}{1+q_1(\alpha)}, \frac{A_1}{1+p_1(\alpha)}\right] (+)$$

$$\left[\frac{A_2}{[1+q_1(\alpha)][1+q_2(\alpha)]}, \frac{A_2}{[1+p_1(\alpha)][1+p_2(\alpha)]}\right] (+) \ldots (+)$$

$$\left[\frac{A_n}{[1+q_1(\alpha)][1+q_2(\alpha)]\ldots[1+q_n(\alpha)].}, \frac{A_n}{[1+p_1(\alpha)][1+p_2(\alpha)]\ldots[1+p_n(\alpha)]}\right] \qquad (3.5.5)$$

CASE 6. In (3.3.1) we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be equal and fuzzy, and A_j , $j = 0, 1, 2, \ldots, n$ different and non fuzzy, we obtain

$A_j(\alpha) = A_j \quad \forall\ j = 0, 1, 2, \ldots, n$

$$\mathrm{FNPV}(\alpha) = [-A_0, -A_0]\ (+) \left[\frac{A_1}{1+q_1(\alpha)}, \frac{A_1}{1+p_1(\alpha)}\right] (+)$$

$$\left[\frac{A_2}{[1+q_1(\alpha)]^2}, \frac{A_2}{[1+p_1(\alpha)]^2}\right] (+) \ldots (+)$$

$$\left[\frac{A_n}{[1+q_1(\alpha)]^n.}, \frac{A_n}{[1+p_1(\alpha)]^n}\right] \qquad (3.5.6)$$

Case 7. When in (3.3.1) we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be different but non-fuzzy, and A_j, $j = 0, 1, 2, \ldots, n$ different and non fuzzy, we obtain the result.

Case 8. When in (3.3.1) we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be different and fuzzy, and A_j, $j = 0, 1, 2, \ldots, n$ different but non–fuzzy, we obtain the result given in Kaufmann and Gupta [35].

Case 9. When in (3.3.1) we take the discount rate r_j , $j = 1, 2, . . . , n$ to be crisp values and all A_j, $j = 0, 1, 2, . . ., n$ to be crisp values, we obtain the result given in Kellison [36].

3.6 Numerical Example

We now consider numerical examples to illustrate some of the various cases considered above Numerical computation along with their corresponding graph are provided for each case discussed.

Case 1. In this case all of the investment capital, future earnings and the discount rates are fuzzy and unequal.

Let

$r_1 = (8 , 10 , 13)$ in percent

$r_2 = (9 , 12 , 15)$ in percent

$r_3 = (7 , 10 , 12)$ in percent

For $0 \le \alpha \le 1$, we obtain

$r_1(\alpha) = (0.08 + 0.02\,\alpha,\ 0.13 - 0.03\,\alpha)$

$r_2(\alpha) = (0.09 + 0.03\alpha,\ 0.15 - 0.03\alpha)$

$r_3(\alpha) = (0.07 + 0.03\alpha, 0.12 - 0.02\alpha)$

Let

$A_0 = (1000, 2000, 3000)$

$A_1 = (2000, 3000, 4000)$

$A_2 = (1000, 2000, 2500)$

$A_3 = (1000, 1500, 2000)$

$A_0\ (\alpha) = (1000 + 1000\alpha\ , 3000 - 1000\alpha)$

$A_1\ (\alpha) = (2000 + 1000\alpha, 4000 - 1000\alpha)$

$A_2\ (\alpha) = (1000 + 1000\alpha, 2500 - 500\alpha)$

$A_3\ (\alpha) = (1000 + 500\alpha, 2000 - 500\alpha)$

Plugging these values in (3.5.1), for $0 \leq \alpha \leq 1$ we obtain,

$$\mathrm{FNPV}(\alpha) = [-3000 + 1000\alpha, -1000 - 1000\alpha]\ (+)\ \left[\frac{2000 + 1000\alpha}{1.13 - 0.03\alpha}, \frac{4000 - 1000\alpha}{1.08 + 0.02\alpha}\right] (+)$$

$$\left[\frac{1000 + 1000\alpha}{[1.13 - 0.03\alpha]\bullet[1.15 - 0.03\alpha]}, \frac{2500 - 500\alpha}{[1.08 + 0.02\alpha]\bullet[1.09 + 0.03\alpha]}\right] (+)$$

$$\left[\begin{array}{c} \dfrac{1000 + 500\alpha}{[1.13 - 0.03\alpha][1.15 - 0.03\alpha][1.12 - 0.02\alpha]}, \\ \dfrac{2000 - 500\alpha}{[1.08 + 0.02\alpha][1.09 + 0.03\alpha][1.07 + 0.03\alpha]} \end{array}\right]$$

Below we now give detailed numerical computations for values of $0 \leq \alpha \leq 1$.

Table 3.2. Computation of FNPV Case1

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	226.5	6415.2
0.1	540.8	6112.8
0.2	857.1	5811.9
0.3	1175.2	5512.5
0.4	1495.3	5214.6
0.5	1817.3	4918.2
0.6	2141.3	4623.2
0.7	2467.3	4329.7
0.8	2795.3	4037.6
0.9	3125.4	3746.8
1.0	3457.5	3457.5

The graph corresponding to Table 3.2 is as follows. It may be observed that the graph may not necessarily be a

triangular.

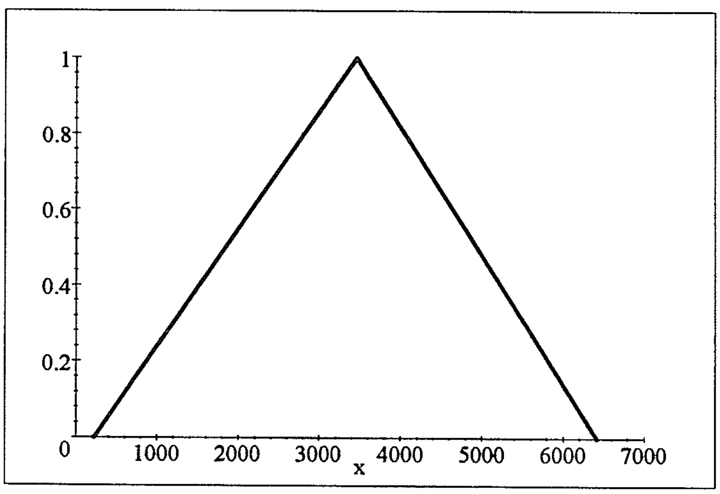

Fig 3.1: Fuzzy Net Present Value Case1

Case 2. In this case all the investment capital, future earnings and the discount rate are fuzzy. However, the discount rates are equal throughout the term of the investments.

$r_1 = (8\ ,\ 10\ ,\ 13\ )$ in per cent

$r_1(\alpha) = (0.08 + 0.02\ \alpha,\ 0.13 - 0.03\ \alpha)$

$A_0 = (1000,\ 2000,\ 3000)$

$A_1 = (2000,\ 3000,\ 4000)$

$A_2 = (1000,\ 2000,\ 2500)$

$A_3 = (1000,\ 1500,\ 2000)$

$A_0\ (\alpha) = (1000 + 1000\alpha\ ,\ 3000 - 1000\alpha)$

$A_1\ (\alpha) = (2000 + 1000\alpha,\ 4000 - 1000\alpha)$

$A_2\ (\alpha) = (1000 + 1000\alpha,\ 2500 - 500\alpha)$

$A_3\ (\alpha) = (1000 + 500\alpha,\ 2000 - 500\alpha)$

Plugging these values in (3.5.2), for $0 \le \alpha \le 1$ we obtain,

$$\text{FNPV}(\alpha) = [-3000 + 1000\alpha, -1000 - 1000\alpha]\ (+)\ \left[\frac{2000 + 1000\alpha}{1.13 - 0.03\alpha}, \frac{4000 - 1000\alpha}{1.08 + 0.02\alpha}\right]\ (+)$$

$$\left[\frac{1000 + 1000\alpha}{[1.13 - 0.03\alpha]^2}, \frac{2500 - 500\alpha}{[1.08 + 0.02\alpha]^2}\right](+)$$

$$\left[\frac{1000 + 500\alpha}{[1.13 - 0.03\alpha]^3}, \frac{2000 - 500\alpha}{[1.08 + 0.02\alpha]^3}\right]$$

Below we now give detailed numerical computations for values of $0 \le \alpha \le 1$.

Table 3.3. Computation of FNPV Case 2

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	241.1	6434.7
0.1	562.9	6136.6
0.2	881.8	5839.6
0.3	1202.6	5543.9
0.4	1525.5	5249.4
0.5	1850.4	4956.1
0.6	2177.5	4664.0
0.7	2506.7	4373.0
0.8	2838.0	4083.2
0.9	3171.5	3794.6
1.0	3507.1	3507.1

The graph corresponding to Table 3.3 is as follows. It may be observed that the graph may not necessarily be a triangular.

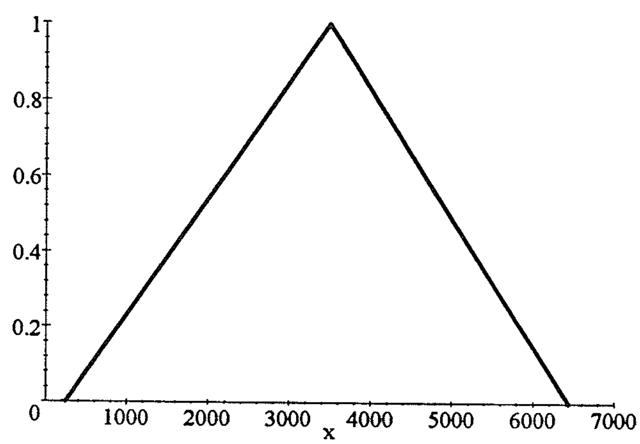

Fig 3.2: Fuzzy Net Present Value Case 2

Case 3. In this case we take all of the investment capital and future earnings fuzzy and unequal. However, the discount rates are crisp and equal through the life of the investment.

$r = 10$ in percent

$A_0 = (1000, 2000, 3000)$

$A_1 = (2000, 3000, 4000)$

$A_2 = (1000, 2000, 2500)$

$A_3 = (1000, 1500, 2000)$

$A_0\ (\alpha) = (1000 + 1000\alpha\ , 3000 - 1000\alpha)$

$A_1\ (\alpha) = (2000 + 1000\alpha, 4000 - 1000\alpha)$

$A_2\ (\alpha) = (1000 + 1000\alpha, 2500 - 500\alpha)$

$A_3\ (\alpha) = (1000 + 500\alpha, 2000 - 500\alpha)$

Plugging these values in (3.5.3), for $0 \leq \alpha \leq 1$ we obtain,

$$\mathrm{FNPV}(\alpha) = [-3000 + 1000\alpha, -1000 - 1000\alpha]\ (+) \left[\frac{2000 + 1000\alpha}{(1.1)}, \frac{4000 - 1000\alpha}{(1.1)}\right] (+)$$

$$\left[\frac{1000 + 1000\alpha}{(1.1)^2}, \frac{2500 - 500\alpha}{(1.1)^2}\right] (+) \left[\frac{1000 + 500\alpha}{(1.1)^3}, \frac{2000 - 500\alpha}{(1.1)^3}\right]$$

Below we now give detailed numerical computations for values of $0 \leq \alpha \leq 1$.

Table 3.4: Computation of FNPV Case 3

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	395.9	6205.1
0.1	707.1	5935.3
0.2	1018.2	5665.5
0.3	1329.3	5395.7
0.4	1640.4	5125.9
0.5	1951.5	4856.1
0.6	2262.7	4586.3
0.7	2573.8	4316.5
0.8	2884.9	4046.7
0.9	3196.0	3776.9
1.0	3507.1	3507.1

The graph corresponding to Table 3.4 is as follows. It may be observed that the graph may not necessarily be a triangular.

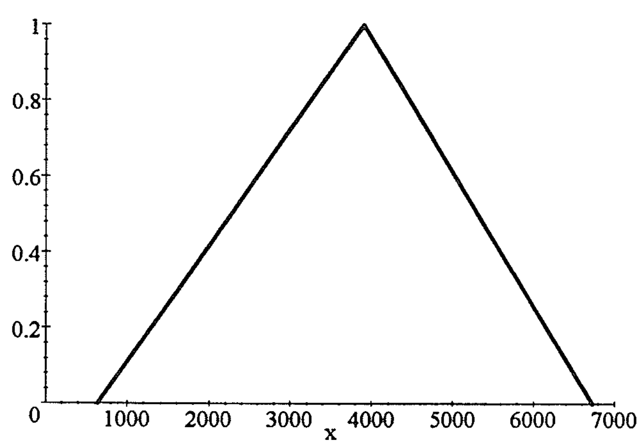

Fig 3.3: Fuzzy Net Present Value Case 3

Case 4. In this case we have all of the investment capital, future earnings and the discount rates are fuzzy and unequal throughout the life of the investment. However, each discount rate, though fuzzy, is transformed to a crisp associated ordinary numbers (AON). An AON for $A_i = (a_i, b_i, c_i)$ is defined as $(AON)_i = \frac{a_i + 2b_i + c_i}{4}$. From Case1, where we have discounted rates as $r_1 = (8, 10, 13)$ in percent,

$r_2 = (9, 12, 15)$ in percent, and $r_3 = (7, 10, 12)$ in percent, we obtain the discount rates in terms of AON's as

$r_1 = 10.25$ in per cent, $r_2 = 12$ in per cent, $r_3 = 9.75$ in per cent.

$A_0 = (1000, 2000, 3000)$

$A_1 = (2000, 3000, 4000)$

$A_2 = (1000, 2000, 2500)$

$A_3 = (1000, 1500, 2000)$

$A_0(\alpha) = (1000 + 1000\alpha, 3000 - 1000\alpha)$

$A_1(\alpha) = (2000 + 1000\alpha, 4000 - 1000\alpha)$

$A_2(\alpha) = (1000 + 1000\alpha, 2500 - 500\alpha)$

$A_3(\alpha) = (1000 + 500\alpha, 2000 - 500\alpha)$

Plugging these values in (3.5.4), for $0 \le \alpha \le 1$ we obtain,

$$FNPV(\alpha) = [-3000 + 1000\alpha, -1000 - 1000\alpha] (+) \left[\frac{2000 + 1000\alpha}{(1.1025)}, \frac{4000 - 1000\alpha}{(1.1025)}\right] (+)$$

$$\left[\frac{1000 + 1000\alpha}{(1.1025)(1.12)}, \frac{2500 - 500\alpha}{(1.1025)(1.12)}\right] (+)$$

$$\left[\frac{1000 + 500\alpha}{(1.1025)(1.12)(1.0975)}, \frac{2000 - 500\alpha}{(1.1025)(1.12)(1.0975)}\right]$$

Below we now give detailed numerical computations for values of $0 \le \alpha \le 1$.

Table 3.5. Computation of FNPV Case 4

α	$\mathbf{FNPV_L}(\alpha)$	$\mathbf{FNPV_R}(\alpha)$
0	361.8	6128.5
0.1	670.4	5860.5
0.2	979.0	5592.4
0.3	1287.6	5324.3
0.4	1596.1	5056.2
0.5	1904.7	4788.1
0.6	2213.3	4520.0
0.7	2521.9	4251.9
0.8	2830.5	3983.8
0.9	3139.1	3715.7
1.0	3447.6	3447.6

The graph corresponding to Table 3.5 is as follows. It may be observed that the graph may not necessarily be a triangular.

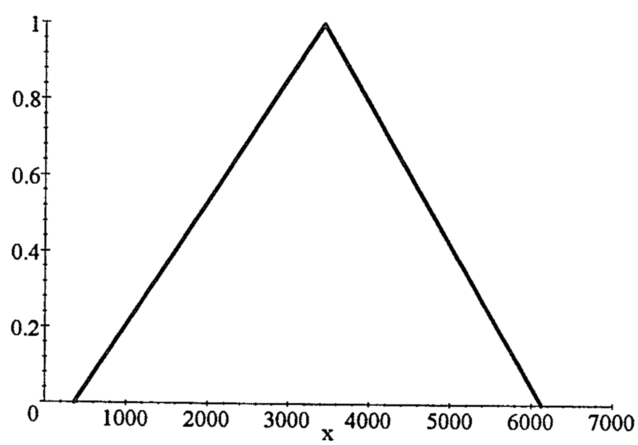

Fig 3.4: Fuzzy Net Present Value Case 4

3.7 Conclusion

In this we derive a formula to compute fuzzy net present value making various assumptions on the investment capital, future earnings and the interest rates at the end of period j, $j = 0, 1, 2, \ldots, n$. A number of special cases are deduced from the main formula and few numerical examples are considered to demonstrate the use of the main formula. Fuzzy net present value evaluations are only as good as the estimates of the parameters used to calculate them. Clearly, inaccurate estimates could show unrealistic results. We believe that fuzzy sets theory is a positive addition to NPV computation. Numerical results for rest of the cases can be discussed on similar lines.

Chapter 4

FUZZY NET PRESENT VALUE WITH FUZZY RISK DISCOUNT RATE

In the present chapter we consider the Capital Asset Pricing Model (CAPM), in which all parameters involved are assumed to be fuzzy, and compute the fuzzy return. We then use the fuzzy return as a discount rate in the net present value (NPV) problem and compute the fuzzy net present value (FNPV). Numerical example is provided to exhibit the use of the results developed in the chapter.

4.1 Introduction

CAPM is often used by investors for making important investment decisions and is probably one of the most well known of all financial models available. Based on the accepted crisp theory that the higher the risk associated with an investment the higher is the required return, the model shows the relationship that exists between expected risk and expected return on an investment. In a world of uncertainty, financial investments should on average, offer a return that is high enough to compensate investors for the level of risk that they bear. One of the main insights of modern capital market theory is that investors need only be compensated for the portion of risk that cannot be diversified by the market as a whole [11, 23, 27]. The main objective of this chapter is to show that by introducing fuzziness in the parameters of the CAPM we arrive at the return in the form of a fuzzy number which is relatively more flexible than the one obtained under the crisp environment. We further exhibit the use of the fuzzy return as a fuzzy discount rate to compute the FNPV of an investment under fuzzy environment.

4.2 Computation of β, the Systematic Risk Under Crisp and Fuzzy Environment.

Crisp Environment

In Chapter 1, we stated that the CAPM return under crisp environment for the period i is given by

$$R = R_f + \beta\,(R_m - R_f). \tag{4.2.1}$$

where β represents the amount of systematic risk, and is computed by using

$$\beta = \frac{\sum_{i=1}^{n} (R_i - \bar{R}_i)\,(R_{mi} - \bar{R}_{mi})}{\sum_{i=1}^{n} (R_{mi} - \bar{R}_{mi})^2} \tag{4.2.2}$$

where,

n = the number of periods under consideration,

R = return on investment under CAPM.

R_i = represents the actual return on investment under CAPM during period i, $i = 1, 2, \ldots, n$,

R_f = risk-free rate of return

R_{fi} = risk-free rate of return for the period i, $i = 1, 2, \ldots, n$,

β = the amount of systematic risk,

R_m = represent the market return on the investment, and

R_{mi} = represent the market return on the investment during period i, $i = 1, 2, \ldots, n$,

and $(R_{mi} - R_{fi})$. = Market risk premium, $i = 1, 2, \ldots, n$.

CAPM under Fuzzy Environment

We now write the fuzzy analogue of (4.2.1) as follows.

$$R(\mathit{fuzzy}) = R_f(\mathit{fuzzy}) + \beta(\mathit{fuzzy})\,[R_m(\mathit{fuzzy}) - R_f(\mathit{fuzzy})] \tag{4.2.3}$$

whose α –cut is

$$R(\alpha) = R_f(\alpha) + \beta(\alpha)\,(R_m(\alpha) - R_f(\alpha)), \tag{4.2.4}$$

where, $\beta(\alpha)$ is computed by using the following fuzzy analogue

$$\beta(\alpha) = \frac{\sum_{i=1}^{n} [R_i(\alpha) - \bar{R}_i(\alpha)]\,[R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]}{\sum_{i=1}^{n} [R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]^2} \tag{4.2.5}$$

of β in (4.2.2).

In the present thesis, we work on the fuzzy CAPM (4.2.3) under the assumption that R_f, R_i and R_{mi} are T.F.N.'s as follows.

$$R_f = (r_{f1}, r_{f2}, r_{f3}) \tag{4.2.6}$$

so that

$$R_f(\alpha) = [r_{f1} + (r_{f2} - r_{f1})\alpha,\ r_{f3} + (r_{f2} - r_{f3})\alpha] \qquad 0 \le \alpha \le 1. \tag{4.2.7}$$

and

$$R_i = (r_{i1}, r_{i2}, r_{i3}) \qquad i = 1, 2, \ldots, n \tag{4.2.8}$$

so that

$$R_i(\alpha) = [r_{i1} + (r_{i2} - r_{i1})\alpha,\ r_{i3} + (r_{i2} - r_{i3})\alpha] \tag{4.2.9}$$

$$\bar{R}_i = \left[\sum_{i=1}^{n} \frac{r_{i1}}{n}, \sum_{i=1}^{n} \frac{r_{i2}}{n}, \sum_{i=1}^{n} \frac{r_{i3}}{n}\right]$$

$$\bar{R}_i = [\bar{r}_{i1}, \bar{r}_{i2}, \bar{r}_{i3}]$$

where, $\bar{r}_{i1} = \sum_{i=1}^{n} \frac{r_{i1}}{n}, \quad \bar{r}_{i2} = \sum_{i=1}^{n} \frac{r_{i2}}{n}, \quad \bar{r}_{i3} = \sum_{i=1}^{n} \frac{r_{i3}}{n}$ (4.2.10)

$$\bar{R}_i(\alpha) = \left[\bar{r}_{i1} + (\bar{r}_{i2} - \bar{r}_{i1})\alpha,\ \bar{r}_{i3} + (\bar{r}_{i2} - \bar{r}_{i3})\alpha\right]. \quad (4.2.11)$$

Also,

$$R_{mi} = (r_{mi1}, r_{mi2}, r_{mi3}) \qquad i = 1, 2, \ldots, n \quad (4.2.12)$$

so that

$$R_{mi}(\alpha) = \left[r_{mi1} + (r_{mi2} - r_{mi1})\alpha,\ r_{mi3} + (r_{mi2} - r_{mi3})\alpha\right] \quad \text{for } i = 1, 2, \ldots, n \quad (4.2.13)$$

Now, $\bar{R}_{mi} = \left[\sum_{i=1}^{n} \frac{r_{mi1}}{n}, \sum_{i=1}^{n} \frac{r_{mi2}}{n}, \sum_{i=1}^{n} \frac{r_{mi3}}{n}\right]$

$$= \left[\bar{r}_{m1}, \bar{r}_{m2}, \bar{r}_{m3}\right]$$

so that

$$\bar{R}_{mi}(\alpha) = \left[\bar{r}_{m1} + (\bar{r}_{m2} - \bar{r}_{m1})\alpha,\ \bar{r}_{m3} + (\bar{r}_{m2} - \bar{r}_{m3})\alpha\right] \quad (4.2.14)$$

From (4.2.5) we have

$$\beta(\alpha) = \frac{\sum_{i=1}^{n} [R_i(\alpha) - \bar{R}_i(\alpha)]\,[R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]}{\sum_{i=1}^{n} [R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]^2}$$

$$= \frac{N(\alpha)}{D(\alpha)} \quad (4.2.15)$$

where $N(\alpha) = \sum_{i=1}^{n} [R_i(\alpha) - \bar{R}_i(\alpha)]\,[R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)] = \sum_{i=1}^{n} A_i(\alpha)\, B_{mi}(\alpha)$

$$D(\alpha) = \sum_{i=1}^{n} [B_{mi}(\alpha)]^2$$

and $A_i(\alpha) = [R_i(\alpha) - \bar{R}_i(\alpha)]$, $B_{mi}(\alpha) = [R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]$,

Now

$$A_i(\alpha) = [R_i(\alpha) - \bar{R}_i(\alpha)]$$

$$= \left[r_{i1} + (r_{i2} - r_{i1})\alpha,\ r_{i3} + (r_{i2} - r_{i3})\alpha\right] - \left[\bar{r}_{i1} + (\bar{r}_{i2} - \bar{r}_{i1})\alpha,\ \bar{r}_{i3} + (\bar{r}_{i2} - \bar{r}_{i3})\alpha\right].$$

$$= \left[p_{i1} + (p_{i2} - p_{i1})\alpha,\ p_{i3} + (p_{i2} - p_{i3})\alpha \right] \tag{4.2.16}$$

where, $\quad p_{i1} = r_{i1} - \bar{r}_{i3}, \quad p_{i2} = r_{i2} - \bar{r}_{i2}, \quad$ and $\quad p_{i3} = r_{i3} - \bar{r}_{i1} \qquad (4.2.17)$

Similarly, we have

$$B_{mi}(\alpha) = [R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]$$

$$= \left[q_{mi1} + (q_{mi2} - q_{mi1})\alpha,\ q_{mi3} + (q_{mi2} - q_{mi3})\alpha \right] \tag{4.2.18}$$

where, $\quad q_{i1} = r_{mi1} - \bar{r}_{mi3}, \quad q_{mi2} = r_{mi2} - \bar{r}_{mi2}, \quad$ and $\quad q_{mi3} = r_{mi3} - \bar{r}_{mi3} \qquad (4.2.19)$

Therefore, using (4.2.16) and (4.2.18), we have

$$N(\alpha) = \sum_{i=1}^{n} A_i(\alpha)\, B_{mi}(\alpha)$$

$$= \sum_{i=1}^{n} \left[p_{i1} + (p_{i2} - p_{i1})\alpha,\ p_{i3} + (p_{i2} - p_{i3})\alpha \right]$$

$$\otimes \left[q_{mi1} + (q_{mi2} - q_{mi1})\alpha,\ q_{mi3} + (q_{mi2} - q_{mi3})\alpha \right]$$

$$= \begin{bmatrix} \sum_{i=1}^{n} (p_{i1} q_{mi1} + \{p_{i1}(q_{mi2} - q_{mi1}) + q_{mi1}(p_{i2} - p_{i1})\}\alpha \\ +(p_{i2} - p_{i1})(q_{mi2} - q_{mi1})\alpha^2), \\ \sum_{i=1}^{n} (p_{i3} q_{mi3} + \{p_{i3}(q_{mi2} - q_{mi3}) + q_{mi3}(p_{i2} - p_{i3})\}\alpha \\ +(p_{i2} - p_{i3})(q_{mi2} - q_{mi3})\alpha^2) \end{bmatrix} \tag{4.2.20}$$

Using (4.2.18), we have

$$D(\alpha) = \sum_{i=1}^{n} [B_{mi}(\alpha)]^2 = \sum_{i=1}^{n} \left[q_{mi1} + (q_{mi2} - q_{mi1})\alpha,\ q_{mi3} + (q_{mi2} - q_{mi3})\alpha \right]^2$$

$$= \sum_{i=1}^{n} \left[q_{mi1} + (q_{mi2} - q_{mi1})\alpha,\ q_{mi3} + (q_{mi2} - q_{mi3})\alpha \right]$$

$$\otimes \left[q_{mi1} + (q_{mi2} - q_{mi1})\alpha,\ q_{mi3} + (q_{mi2} - q_{mi3})\alpha \right]$$

$$= \begin{bmatrix} \sum_{i=1}^{n}((q_{mi1})^2 + 2\, q_{mi1}(q_{mi2} - q_{mi1})\alpha + (q_{mi2} - q_{mi1})^2\alpha^2), \\ \sum_{i=1}^{n}((q_{mi3})^2 + 2\, q_{mi3}(q_{mi2} - q_{mi3})\alpha + (q_{mi2} - q_{mi3})^2\alpha^2) \end{bmatrix} \qquad (4.2.21)$$

Using (4.2.20) and (4.2.21) in (4.2.5), we obtain $\beta(\alpha)$.

In (4.2.3), if we take R_f (*fuzzy*) and R_m(*fuzzy*) as triangular fuzzy numbers (T.F.N.'s), we get

$$R(fuzzy) = R_f(TFN) + \beta\,(fuzzy)\,(R_m(TFN) - R_f(TFN)) \qquad (4.2.22)$$

Furthermore, in (4.2.22), if we approximate β (*fuzzy*) also by another T.F.N., β_{TFN} say, then (4.2.22) becomes

$$R(fuzzy) = R_f(TFN) + \beta_{TFN}\,(R_m(TFN) - R_f(TFN)) \qquad (4.2.23)$$

Note that in β(*fuzzy*), the fuzzy systematic risk in (4.2.3), associated with $\beta(\alpha)$ in equation (4.2.4) and (4.2.5), is represented by a fuzzy number (obtained by using (4.2.20) and (4.2.21) in (4.2.15)) having a non-zero curvature and a curvilinear membership function. It may be observed that T.F.N. in (4.2.23) can only approximate β(*fuzzy*). However, this approximation of β(*fuzzy*) by a T.F.N. can sometimes be quite poor and may lead to erroneous results when applied to certain financial or engineering problems. Therefore, as suggested by Kaufmann and Gupta [35], it is important to perform error analysis on a fuzzy number approximation. This means that in (4.2.22), β(*fuzzy*) can be approximated by a triangular fuzzy number provided the left and right divergences of the triangular fuzzy number relative to the curvilinear fuzzy number are small. Kaufmann and Gupta [34, 35] demonstrate that if the above criteria are not satisfied, the approximated fuzzy number cannot be considered as a good and legitimate approximation. Therefore, the equation (4.2.23) is acceptable only if the error analysis reveals that the 'error' and the 'percentage error' are small.

4.3 Use of R(*fuzzy*) as Discount Rate in Net Present Value Problem.

Step1. Consider $R(fuzzy) = R_f(TFN) + \beta\,(fuzzy)\,[R_m(TFN) - R_f(TFN)]$

and obtain $R(\alpha)$, the α – cut for R(*fuzzy*).

Step 2. Use $R(\alpha)$ as a discount rate in the net present value problem.

Step 3. Compute $1 + R(\alpha)$ and set $1 + R(\alpha) = [1 + p(\alpha)\ ,\ 1 + q(\alpha)]$.

Step 4. Compute $\frac{1}{1+R(\alpha)} = \left[\frac{1}{1+q_1(\alpha)}, \frac{1}{1+p_1(\alpha)}\right]$, and use it to

compute FNPV on the lines similar to Case 2 in Section 3.5 of Chapter 3.

4.4 Numerical Example.

Below, we illustrate the computation of β, $\beta(\alpha)$, and $R(\alpha)$ with the help of a numerical example based on the data provided in Appendix 2 (pp135-137), and Appendix 3 (pp138-151)

4.4.1. Computation of β.

Under crisp environment, using (4.2.2) and the data from Appendix 2, β the systematic risk is given by

$$\beta = \frac{\sum_{i=1}^{n} (R_i - \bar{R}_i))\,(R_{mi} - \bar{R}_{mi})}{\sum_{i=1}^{n} (R_{mi} - \bar{R}_{mi})^2} \qquad i = 1, 2, \ldots, n,$$

$$= 1.66 \tag{4.4.1}$$

4.4.2. Computation of $\beta(\alpha)$ Under Fuzzy Environment.

From (4.2.5), α-cut of the systematic risk β(*fuzzy*) is given by

$$\beta(\alpha) = \frac{\sum_{i=1}^{n} (R_i(\alpha) - \bar{R}_i(\alpha))\,(R_{mi}(\alpha) - \bar{R}_{mi}(\alpha))}{\sum_{i=1}^{n} (R_{mi}(\alpha) - \bar{R}_{mi}(\alpha))^2} = \frac{N(\alpha)}{D(\alpha)}$$

$$N(\alpha) = \sum_{i=1}^{n} (R_i(\alpha) - \bar{R}_i(\alpha))\,(R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)) = \begin{bmatrix} 2572.181 - 564.688\alpha + 276.471\alpha^2 \\ , \\ 2541.305 - 533.472\alpha + 276.131\alpha^2 \end{bmatrix}$$

and $$D(\alpha) = \sum_{i=1}^{n} (R_{mi}(\alpha) - \bar{R}_{mi}(\alpha))^2 = \begin{bmatrix} 1517.138 - 329.679\alpha + 185.955\alpha^2 \\ , \\ 1569.852 - 379.418\alpha + 182.981\alpha^2 \end{bmatrix}$$

such that the α –cut of the nonlinear fuzzy number that represents the systematic risk $\beta(fuzzy)$ is given by

$$\beta(\alpha) = \begin{bmatrix} \dfrac{(2572.181 - 564.688\alpha + 276.471\alpha^2)}{(1517.138 - 329.679\alpha + 185.955\alpha^2)}, \\ \dfrac{(2541.305 - 533.472\alpha + 276.131\alpha^2)}{(1569.852 - 379.418\alpha + 182.981\alpha^2)} \end{bmatrix} \tag{4.4.2}$$

Thus, from (4.4.2),

$\beta(fuzzy)$ = (1.6385, 1.663, 1.6751), which is not necessarily a triangular fuzzy number (T.F.N.).

The membership function $\mu(\beta(fuzzy))$ for $\beta(fuzzy)$ is obtained by setting

$$\frac{(2572.181 - 564.688\alpha + 276.471\alpha^2)}{(1517.138 - 329.679\alpha + 185.955\alpha^2)} = \beta$$

and

$$\frac{(2541.305 - 533.472\alpha + 276.131\alpha^2)}{(1569.852 - 379.418\alpha + 182.981\alpha^2)} = \beta$$

and solving those two equations for α in terms of β.

Thus, the membership function $\mu(\beta_{fuzzy})$ is as given on next page.

$$
\mu(\beta(\mathit{fuzzy})) = \begin{cases}
0 & \text{if} \quad \beta \le 1.638 \\[2ex]
\dfrac{(-564.688 + 3.29679\beta) + \sqrt{\begin{array}{c}(-564.688 + 329.679\beta)^2 - \\ 4(2.76.471 - 185.955\beta) \\ *(2572.181 - 1517.138\beta)\end{array}}}{2(2.76.471 - 185.955\beta)} & \text{if} \quad 1.638 \le \beta \le 1.663 \\[2ex]
\dfrac{(-5.33.472 + 379.418\beta) - \sqrt{\begin{array}{c}(-533.472 + 379.418\beta)^2 - \\ 4(2.76.131 - 182.981\beta) \\ *(2541.305 - 1569.852\beta)\end{array}}}{2(2.76.131 - 182.981\beta)} & \text{if} \quad 1.663 \le \beta \le 1.675 \\[2ex]
0 & \text{if} \quad \beta \ge 1.675
\end{cases}
$$

The graph of the membership function $\mu(\beta(fuzzy))$ is obtained below using the Table 4.1.

Table 4.1: left and Right Segment of $\beta(fuzzy)$

α	Left $\beta(\alpha)$	Right $\beta(\alpha)$
0	1.638	1.675
0.1	1.642	1.676
0.2	1.646	1.677
0.3	1.649	1.677
0.4	1.652	1.676
0.5	1.654	1.676
0.6	1.657	1.674
0.7	1.659	1.672
0.8	1.661	1.670
0.9	1.662	1.667
1	1.663	1.663

and is shown in Figure 4.1.

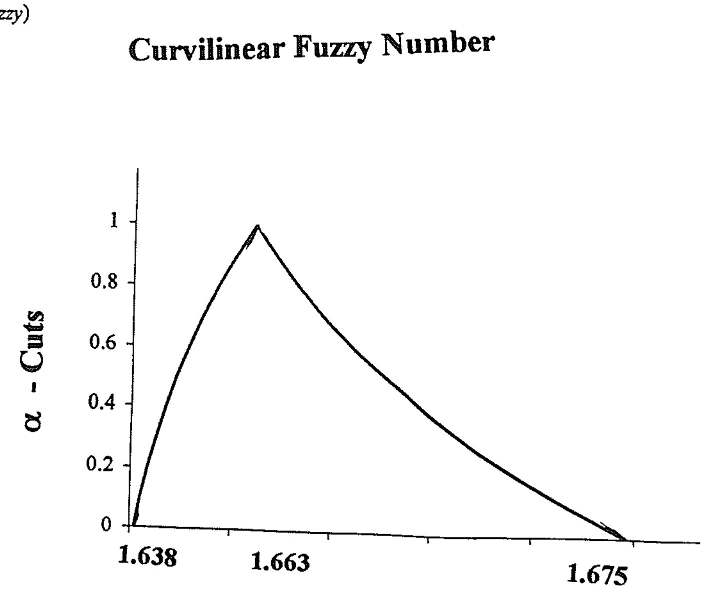

Figure 4.1

We now approximate the $\beta(fuzzy)$ by β_{TFN}. Thus,

$$\beta_{TFN} = (1.6385, 1.6630,\ 1.6751) \tag{4.4.3}$$

for which the α-cut is given by

$$\beta_{TFN}(\alpha) = (\ 1.6385 + 0.0215\alpha,\ 1.6751 - 0.0121\alpha\ ) \quad \forall\ \alpha \in [0,\ 1] \tag{4.4.4}$$

Using (4.4.4), we have the membership function for β_{TFN} as

$$\mu(\beta_{TFN}) = \begin{cases} 0 & \text{if } \beta \le 1.6385 \\ \dfrac{\beta - 1.6385}{0.0245} & \text{if } 1.6385 \le \beta \le 1.6630 \\ \dfrac{1.6751 - \beta}{0.0121} & \text{if } 1.6630 \le \beta \le 1.6751 \\ 0 & \text{if } \beta \ge 1.6751 \end{cases}$$

The graph, shown in Figure 4.2, of the membership function $\mu(\beta_{TFN})$ is obtained using the Table 4.2.

Table 4.2 :Left and Right Sement of $\beta_{T.F.N}$

α	Left β	Right β
-	TFN	TFN
0	1.638	1.675
0.1	1.641	1.674
0.2	1.643	1.673
0.3	1.646	1.671
0.4	1.648	1.670
0.5	1.651	1.669
0.6	1.653	1.668
0.7	1.656	1.667
0.8	1.658	1.665
0.9	1.661	1.664
1.0	1.663	1.663

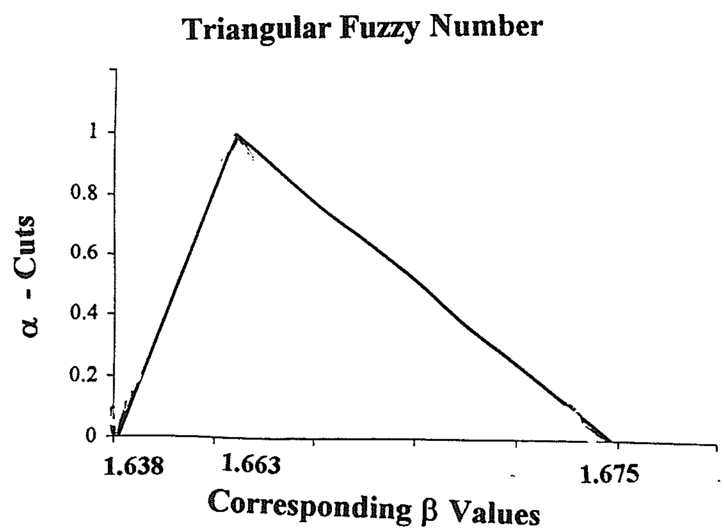

Figure 4.2

To see if the triangular approximation is a suitable approximation to the curvilinear fuzzy number in the problem discussed above, we conduct the errors analysis (Kauffmann and Gupta [35]) as in 4.3.

Table 4.3: Error Analysis for Fuzzy Systematic Risk β

-	1	2	3	4	5	6	7	8
α	Left β	Right β	Left β	Right β	Error	Error	%Error	%Error
-	Curvature	Curvature	TFN	TFN	(3 – 1)	(4 – 2)	-	-
0	1.638	1.675	1.638	1.675	0.000	0.000	0.000	0.000
0.1	1.642	1.676	1.641	1.674	0.001	0.002	0.068	0.133
0.2	1.645	1.677	1.643	1.673	0.002	0.004	0.126	0.240
0.3	1.649	1.677	1.646	1.671	0.003	0.005	0.172	0.318
0.4	1.652	1.676	1.648	1.670	0.003	0.006	0.204	0.366
0.5	1.654	1.675	1.651	1.669	0.004	0.006	0.220	0.384
0.6	1.657	1.674	1.653	1.668	0.004	0.006	0.218	0.370
0.7	1.659	1.672	1.656	1.667	0.003	0.005	0.197	0.324
0.8	1.661	1.670	1.658	1.665	0.003	0.004	0.154	0.246
0.9	1.662	1.666	1.661	1.664	0.001	0.002	0.089	0.318
1.0	1.663	1.663	1.663	1.663	0.000	0.000	0.000	0.000

The following graph, shown in figure 4.3, illustrate the 'error analysis' performed in Table 4.3.

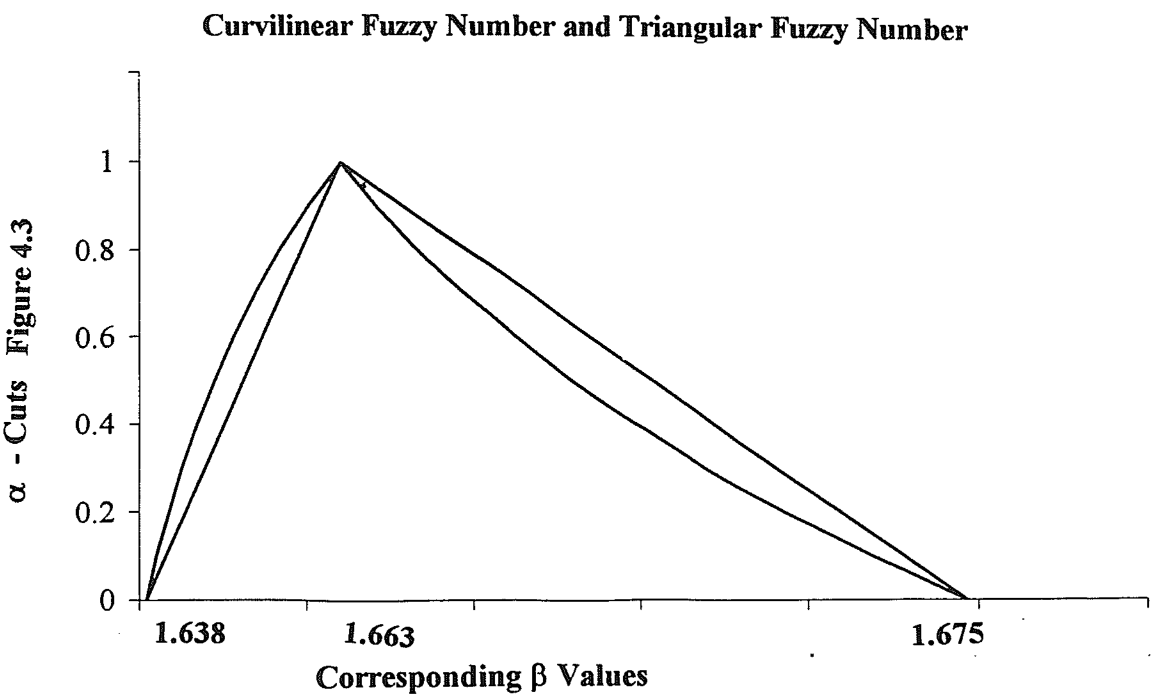

Figure 4.3

From the Table 4.3, we observe that the errors and the percentage errors are small, therefore, we conclude that in the numerical problem discussed above it is suitable to approximate the curvilinear fuzzy number by a T.F.N. (Kaufmann and Gupta [34, 35]). Had the errors and the percentage errors been large, the T.F.N. β_{TFN} would not have been an acceptable alternative to the curvilinear fuzzy number $\beta(fuzzy)$. Comparing (4.3.1) with (4.3.3) we see that the value of the interior point of β_{TFN}, at which the membership function $\mu(\beta_{TFN}) = 1$, is exactly the same as the crisp value of β in (4.3.1), however, we have more flexibility for dealing with the systematic risk in (4.3.3) as is seen in the Table 4.3 and the graph (Fig 4.1) corresponding to the Table 4.3.

4.4.3. Computation of Return $R(\alpha)$ Under Fuzzy Environment

We now use the $\beta(\alpha)$ computed in (4.3.2) to compute the CAPM return $R(\alpha)$.

From (4.2.3) we have

$$R(fuzzy) = R_f(fuzzy) + \beta(fuzzy)\,[R_m(fuzzy) - R_f(fuzzy)]$$

whose α-cut is

$$R(\alpha) = R_f(\alpha) + \beta(\alpha)\,[R_m(\alpha) - R_f(\alpha)], \tag{4.4.5}$$

We now assume that the risk free rate R_f is represented by the T.F.N.

$$R_f = (r_{f1}, r_{f2}, r_{f3}) = (2.68, 3.73, 4.73) \tag{4.4.6}$$

and the market risk R_m is represented by

$$R_m = (r_{m1}, r_{m2}, r_{m3}) = (9.93, 13, 16.35). \tag{4.4.7}$$

This yields,

$$R_f(\alpha) = (0.0268 + 0.0105\alpha,\ 0.0473 - 0.01\alpha), \tag{4.4.8}$$

$$R_m(\alpha) = (0.0993 + 0.0307\alpha,\ 0.1635 - 0.0335\alpha), \tag{4.4.9}$$

and

$$(R_m(\alpha) - R_f(\alpha)) = (0.052 + 0.0407\alpha,\ 0.1367 - 0.044\alpha) \tag{4.4.10}$$

Using (4.4.8), (4.4.10) and (4.4.2) in (4.4.5), we get

$$R(\alpha) = (0.0268 + 0.0105\alpha,\ 0.0473 - 0.01\alpha)$$

$$+\left[\begin{array}{c} \dfrac{(2572.181 - 564.688\alpha + 276.471\alpha^2)}{(1569.852 - 379.418\alpha + 182.981\alpha^2)}, \\ \dfrac{(2541.305 - 533.472\alpha + 276.131\alpha^2)}{(1517.1376 - 329.6785\alpha + 185.9555\alpha^2)} \end{array}\right] *(0.052 + 0.0407\alpha,\ 0.1367 - 0.044\alpha)$$

That is

$$R(\alpha) = \begin{bmatrix} \dfrac{175.8255 + 81.6390\alpha - 7.6863\alpha^2 + 13.1737\alpha^3}{1569.8517 - 379.4181\alpha + 182.9810\alpha^2}, \\ \dfrac{419.1570 - 215.5082\alpha + 73.3124\alpha^2 - 14.0093\alpha^3}{1517.1376 - 329.6785\alpha + 185.9555\alpha^2} \end{bmatrix} \quad \text{where } 0 \le \alpha \le 1.$$

This gives

R(*fuzzy*) = (0.1120, 0.1915, 0.2763)

the graph (fig 4.2) for whose membership function provided below using the Table 4.4

for $0 \le \alpha \le 1$.

Table 4.4: Left and Right Segment of R(*fuzzy*)

α	Left R(α)	Right R(α)
0	0.1120	0.2763
0.1	0.1199	0.2680
0.2	0.1279	0.2597
0.3	0.1358	0.2514
0.4	0.1438	0.2430
0.5	0.1517	0.2345
0.6	0.1597	0.2259
0.7	0.1677	0.2174
0.8	0.1756	0.2088
0.9	0.1836	0.2001
1.0	0.1915	0.1915

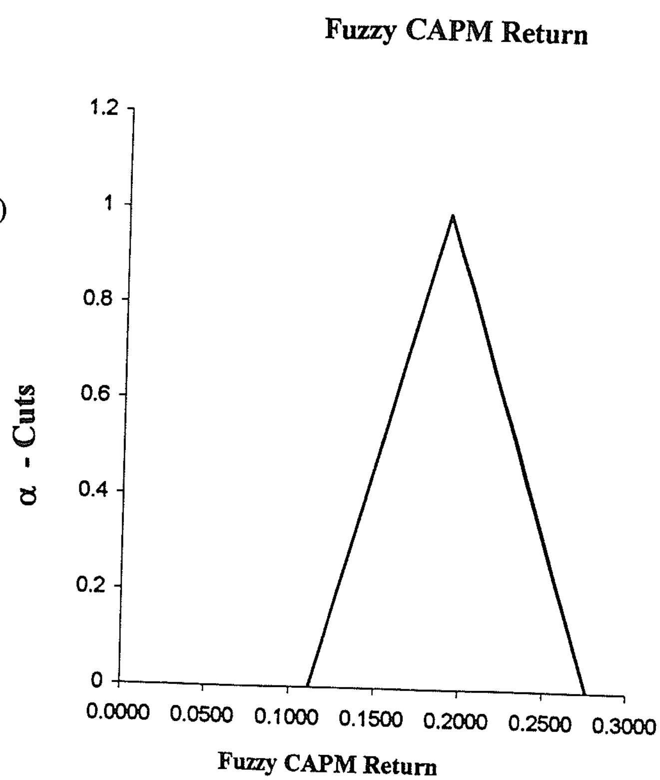

Figure 4.4

We observe that R(*fuzzy*) = (0,1120, 0.1915, 0.2763) is not necessarily a T.F.N. However, to see if it can be approximated as a T.F.N., we can conduct 'error analysis' similar to the one conducted for β(*fuzzy*) in Table 4.3.

4.4.4. Computation of FNPV(α).

We now use R(α), as explained in Steps 1-3 in Section 4.3, to compute the FNPV(α).

$$1+\mathrm{R}(\alpha) = \left[\begin{array}{c} \dfrac{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3}{1569.851663-379.4181059\alpha+182.980\alpha^2}, \\ \dfrac{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}{1517.137-329.678\alpha+185.955\alpha^2} \end{array}\right]$$

$$= [1+\mathrm{p}_1(\alpha), 1+\mathrm{q}_1(\alpha)]$$

where,

$$1+\mathrm{p}_1(\alpha) = \frac{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3}{1569.851663-379.4181059\alpha+182.980\alpha^2},$$

$$1+\mathrm{q}_1(\alpha) = \frac{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}{1517.137-329.678\alpha+185.955\alpha^2}$$

Therefore,

$$\frac{1}{1+\mathrm{R}(\alpha)} = \left[\frac{1}{1+\mathrm{q}_1(\alpha)}, \frac{1}{1+\mathrm{p}_1(\alpha)}\right]$$

$$= \left[\begin{array}{c} \dfrac{1517.137-329.678\alpha+185.955\alpha^2}{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}, \\ \dfrac{1569.851663-379.4181059\alpha+182.980\alpha^2}{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3} \end{array}\right]$$

where,

$$\frac{1}{1+\mathrm{q}_1(\alpha)} = \frac{1517.137-329.678\alpha+185.955\alpha^2}{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}$$

$$\frac{1}{1+\mathrm{p}_1(\alpha)} = \frac{1569.851663-379.4181059\alpha+182.980\alpha^2}{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3}$$

Using the same notation as in Case 2 in Section 3.5 of Chapter 3, the FNPV(α) is given as

$$\mathrm{FNPV}(\alpha) = [-Q_0(\alpha),\ -P_0(\alpha)]\ (+) \left[\frac{P_1(\alpha)}{1+q_1(\alpha)}, \frac{Q_1(\alpha)}{1+p_1(\alpha)}\right] (+)$$

$$\left[\frac{P_2(\alpha)}{[1+q_1(\alpha)]^2}, \frac{Q_2(\alpha)}{[1+p_1(\alpha)]^2}\right] (+) \left[\frac{P_3(\alpha)}{[1+q_1(\alpha)]^3.}, \frac{Q_3(\alpha)}{[1+p_1(\alpha)]^3}\right]$$

(4.4.11)

We now assume

$A_0 = (2000,\ 2500,\ 3000)$

$A_1 = (1600,\ 1700,\ 2000)$

$A_2 = (1200,\ 1400,\ 1800)$

$A_3 = (1300,\ 1700,\ 1900)$

where A_0, A_1, A_2 and A_3 are as defined in Section 3.2 in Chapter 3. Thus, for $0 \le \alpha \le 1$ we have

$A_0(\alpha) = (2000 + 500\alpha,\ 3000 - 500\alpha) = [P_0(\alpha), Q_0(\alpha)]$

$-A_0(\alpha) = (-3000 + 500\alpha,\ -2000 - 500\alpha)$

$A_1(\alpha) = (1600 + 200\alpha,\ 2000 - 300\alpha) = [P_1(\alpha), Q_1(\alpha)]$

$A_2(\alpha) = (1200 + 200\alpha,\ 1800 - 400\alpha) = [P_2(\alpha), Q_2(\alpha)]$

$A_3(\alpha) = (1300 + 400\alpha,\ 1900 - 200\alpha) = [P_3(\alpha), Q_3(\alpha)]$

Plugging these in equation (4.4.11), we obtained the fuzzy net present value (FNPV) as

$\mathrm{FNPV}(\alpha) = (-3000 + 500\alpha,\ -2000 - 500\alpha) +$

$$\begin{bmatrix} 1600+200\alpha, \\ 2000-300\alpha \end{bmatrix} \otimes \begin{bmatrix} \dfrac{1517.137-329.678\alpha+185.955\alpha^2}{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}, \\ \dfrac{1569.851663-379.4181059\alpha+182.980\alpha^2}{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3} \end{bmatrix} +$$

$$\begin{bmatrix} 1200+200\alpha, \\ 1800-400\alpha \end{bmatrix} \otimes \begin{bmatrix} \dfrac{1517.137-329.678\alpha+185.955\alpha^2}{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}, \\ \dfrac{1569.851663-379.4181059\alpha+182.980\alpha^2}{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3} \end{bmatrix}^2 +$$

$$\begin{bmatrix} 1300+400\alpha, \\ 1900-200\alpha \end{bmatrix} \otimes \begin{bmatrix} \dfrac{1517.137-329.678\alpha+185.955\alpha^2}{1936.29-545.186\alpha+259.26\alpha^2-14.00\alpha^3}, \\ \dfrac{1569.851663-379.4181059\alpha+182.980\alpha^2}{1745.67-297.77\alpha+175.29\alpha^2+13.17\alpha^3} \end{bmatrix}^3$$

By varying α between 0 and 1, we obtain Table 4.5 and fig 4.3 representing the FNPV.

Table 4.5: Left and Right Segment of FNPV(α)

α	Left FNPV(α)	Right FNPV(α)
0	-384.340	2636.002
0.1	-264.395	2450.752
0.2	-142.370	2268.642
0.3	-18.183	2089.615
0.4	108.249	1913.615
0.5	237.003	1740.589
0.6	368.159	1570.481
0.7	501.781	1403.239
0.8	637.946	1238.807
0.9	776.713	1077.127
1	918.141	918.141

4.4.5. Computation of Return R(α) Under Fuzzy Environment When the Systematic Risk β(*fuzzy*) is Approximated by a T.F.N..

Since, from the 'error analysis' and 'percent error analysis' in Table 4.1 we concluded that the approximation of β(*fuzzy*) by a T.F.N. is justified, therefore, in this case using (4.4.4), (4.4.6) and (4.4.7) in (4.4.5) we obtain

$$R(\alpha) = [0.1120 + 0.0785\alpha + 0.0010\alpha^2, 0.2763 - 0.0854\alpha + 0.0005\alpha^2].$$

This yields

$$1 + R(\alpha) = \begin{bmatrix} 1.1120 + 0.0785\alpha + 0.0010\alpha^2, \\ 1.2763 - 0.0854\alpha + 0.0005\alpha^2 \end{bmatrix}$$

$$= \left[1 + p_1(\alpha),\ 1 + q_1(\alpha)\right]$$

where,

$$1 + p_1(\alpha) = 1.1120 + 0.0785\alpha + 0.0010\alpha^2$$

$$1 + q_1(\alpha) = 1.2763 - 0.0854\alpha + 0.0005\alpha^2$$

$$\frac{1}{1 + R(\alpha)} = \begin{bmatrix} \dfrac{1}{1.2763 - 0.0854\alpha + 0.0005\alpha^2}, \\ \dfrac{1}{1.1120 + 0.0785\alpha + 0.0010\alpha^2} \end{bmatrix}$$

We assume that

$A_0 = (2000,\ 2500,\ 3000)$, $\quad A_1 = (1600,\ 1700,\ 2000)$

$A_2 = (1200,\ 1400,\ 1800)$, $\quad A_3 = (1300,\ 1700,\ 1900)$

As in Section 4.4.4., for $0 \leq \alpha \leq 1$ we obtain the fuzzy net present value and denote it by ($FNPV_{TFN}$).

$$FNPV_{TFN}(\alpha) = (-3000 + 500\alpha,\ -2000 - 500\alpha) +$$

$$\begin{bmatrix} 1600 + 200\alpha, \\ 2000 - 300\alpha \end{bmatrix} \begin{bmatrix} \dfrac{1}{1.2763 - 0.0854\alpha + 0.0005\alpha^2}, \\ \dfrac{1}{1.1120 + 0.0785\alpha + 0.0010\alpha^2} \end{bmatrix} +$$

$$\begin{bmatrix} 1200+200\alpha, \\ 1800-400\alpha \end{bmatrix} \begin{bmatrix} \dfrac{1}{1.2763-0.0854\alpha+0.0005\alpha^2}, \\ \dfrac{1}{1.1120+0.0785\alpha+0.0010\alpha^2} \end{bmatrix}^2 +$$

$$\begin{bmatrix} 1300+400\alpha, \\ 1900-200\alpha \end{bmatrix} \begin{bmatrix} \dfrac{1}{1.2763-0.0854\alpha+0.0005\alpha^2}, \\ \dfrac{1}{1.1120+0.0785\alpha+0.0010\alpha^2} \end{bmatrix}^3$$

As in Section 4.4.4, we obtain the Table 4.6 and the graph for $\text{FNPV}_{\text{TFN}}(\alpha)$

Table 4.6: Left and Right segment of FNPV(α)

α	Left $\text{FNPV}_{\text{TFN}}(\alpha)$	Right $\text{FNPV}_{\text{TFN}}(\alpha)$
0	-384.340	2636.002
0.1	-263.286	2451.230
0.2	-140.361	2269.650
0.3	-15.506	2090.906
0.4	111.344	1915.185
0.5	240.252	1742.317
0.6	371.287	1572.225
0.7	504.519	1404.832
0.8	640.021	1240.068
0.9	777.869	1077.860
1	918.141	918.141

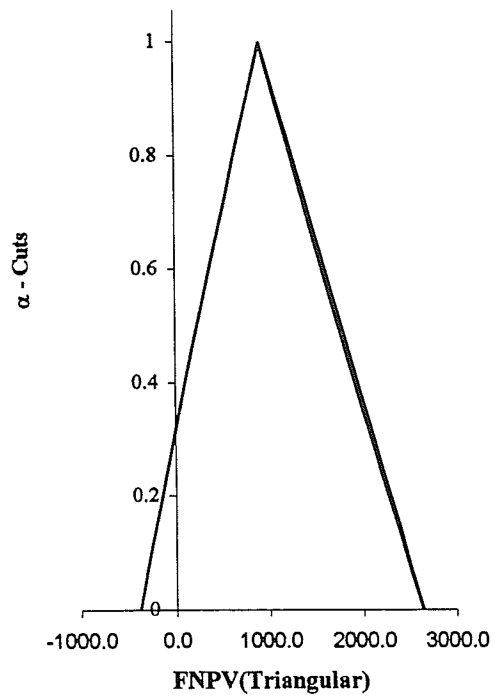

Figure 4.5

On page 65, Table 4.7 we perform the 'error analysis' to examine as to how close the FNPV_{TFN} and the FNPV are.

Table 4.7: Error Analysis for FNPV

-	1	2	3	4	5	6	7	8
α	Left	Right)	Left	Right	Left r	Right	Left	Right
-	FNPV(α)	FNPV(α	FNPV	FNPV	Error	Error	%Error	%Error
-	Curvature	Curvature	TFN	TFN	$(3-1)$	$(4-2)$	-	-
0	-384.340	2336.002	-384.340	2636.002	0.0000	0.0000	0.000	0.000
).1	-264.395	2450.752	-263.286	2451.230	1.1090	0.4785	0.419	0.020
).2	-142.370	2268.642	-140.361	2269.650	2.0089	0.9179	1.411	0.040
).3	-18.183	2089.615	-15.506	2090.906	2.6773	1.2907	1.724	0.062
).4	108.249	1913.615	111.344	1915.185	3.0954	1.5696	2.860	0.082
).5	237.003	1740.589	240.252	1742.317	3.2493	1.7285	1.371	0.099
).6	368.157	1570.481	371.287	1572.225	3.1308	1.7434	0.850	0.111
).7	501.781	1403.239	504.519	1404.832	2.7378	1.5934	0.546	0.114
).8	637.946	1238.807	640.021	1240.068	2.0755	1.2608	0.325	0.102
).9	776.713	1077.127	777.869	1077.860	1.1562	0.7325	0.149	0.068
l.0	918.141	918.141	918.141	918.141	0.0000	0.0000	0.000	0.000

As before from Table 4.7, we observe that both the 'error' and the 'percent error' are small. Therefore, in this case.FNPV_{TFN} can be considered as a good approximation for FNPV.

4.4.6 **Computing R using Associated Ordinary Numbers.**

From (4.2.1), we have that under crisp environment the CAPM return is given by

$$R = R_f + \beta\,(R_m - R_f)$$

We rewrite it as

$$R = R_{fA} + \beta_{\text{TFNA}}\,(R_{mA} - R_{fA}) \qquad (4.4.12)$$

where we assume that R_{fA} and R_{mA} are the associated ordinary numbers (AON's) of R_f and R_m. Since we have already justified in Section 4.4.2 that β_{TFN} is a good approximation for β(*fuzzy*), therefore, we take β_{TFNA} as the associated ordinary number corresponding to β_{TFN}.

From (4.4.3), (4.4.6) and (4.4.7), we have

$\beta_{TFNA} = 1.66$, $R_{fA} = 3.72$, $R_{mA} = 13.07$ respectively.

Therefore, from (4.4.12), we now have

$$R = 3.72 + 1.66(13.07 - 3.72)$$

$$= 19.24$$

Taking $A_0 = 2500$ $A_1 = 1750$ $A_2 = 1450$ and $A_3 = 1650$, and using the above $R = 19.24$ as the discount rate in the NPV problem, we have now the Associated Ordinary Net Present Value (AONPV) as

$$\text{AONPV} = -2500 + \frac{1750}{(1.1924)} + \frac{1450}{(1.1924)^2} + \frac{1650}{(1.1924)^3}$$

That is $\quad \text{AONPV} = 960.63$

Comparing AONPV = 960.63, FNPV and FNPV_{TFN} each given by (-384.340, 918.141, 2636.002), we observe that FNPV and FNPV_{TFN} are more flexible in the sense that each of them gives us a range of values as compared with a single value given by AONPV. Another advantage of the approach developed in this chapter is that the approach developed in Chapter 3 does not capture the fuzzy systematic risk level involved in an investment problem. One can further extend the above approach for the problem in which the fuzzy data is contained in sheaves (see page 92 of [33]) in the form of T.F.N.'s. or in the form of trapezoidal fuzzy numbers. From these sheaves one can find the fuzzy systematic risk level as well as the FNPV.

4.7 Conclusion

In the present chapter under a variety of assumptions, we compute both the fuzzy systematic risk and fuzzy return using the CAPM. Furthermore, we suggest steps to use the fuzzy return, obtained using the CAPM, as the fuzzy discount rate in the NPV problem under a variety of assumptions. Also, we consider a numerical example and compare FNPV so obtained with the AONPV and observe that the results obtained using fuzzy assumptions on various parameters involved in the CAPM give more flexible results.

Chapter 5

FUZZY SCORING MODELS

The purpose of this chapter is to develop the scoring models under fuzzy environment and apply them to assess the best decision alternative from a multicriteria decision problem. We call such models as fuzzy scoring models (FSM).

5.1 Introduction

In modelling real world problems we often encounter multicriteria decision making problems with interdependent objectives. For example, project selection problem, which is a multicriteria decision making problem, is the process of choosing a project or set of projects to be implemented by an organization. Since projects in general require a substantial investment in terms of money and resources, both of which are limited, it is of vital importance that the projects that an organization selects provide good returns on the resources and capital invested. A scoring model introduced by Keeney and Raiffa [36, 37] is a relatively quick and easy way to identify the best decision alternatives in such problems [1]. It is an analytical tool that enables one to explicitly rank tangible and intangible factors against each other for the purpose of resolving conflict or setting priorities. In this chapter we consider a scoring model under fuzzy environment and show how such a model can assist in analyzing a multicriteria decision making problem when the information available is vague, imprecise and subjective. Numerical examples, with both crisp and fuzzy cases, are provided at the end of the chapter to demonstrate the application of the model in analyzing a Bond Selection problem. It is believed that FSM developed here will find its applications in a wide variety of areas like fuzzy facility location problem, fuzzy part family formation in a production system, fuzzy vendor selection problem, and a fuzzy job selection problem etc.

5.2 Fuzzy Scoring Model.

Assume that

s_{ij} = the rating associated with the criterion i and decision alternative j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$

is a T.F.N. such that,

$s_{ij} = (s_{ij2}, s_{ij2}, s_{ij3})$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$

whose α –cut is given by

$$s_{ij}(\alpha) = \left[s_{ij1} + (s_{ij2} - s_{ij1})\alpha,\ s_{ij3} - (s_{ij3} - s_{ij2})\alpha\right] \qquad (5.2.1)$$

for $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$ and $0 \le \alpha \le 1$.

Let

w_i = the weight of the criterion i for $i = 1, 2, 3, \ldots, m$

be a T.F.N. such that

$w_i = (w_{i1}, w_{i2}, w_{i3})$ for $i = 1, 2, 3, \ldots, m$

whose α –cut is given by

$$w_i(\alpha) = (w_{i1} + (w_{i2} - w_{i1})\alpha,\ w_{i3} - (w_{i3} - w_{i2})\alpha) \qquad (5.2.2)$$

for $i = 1, 2, 3, \ldots, m$ and $0 \le \alpha \le 1$.

If a research organization or program pursues several objectives simultaneously, it needs to decide on the relative importance of the different criteria under consideration and the weights have to be defined properly in relation with the variability in the different scores. If one criterion is measured in very small units, and another criterion in very large units, equal weights for the two criteria would be unfavorable to the second criterion. Weighting is further complicated if the dimensions of the criteria are different, for example when qualitative and quantitative criteria are being compared. As already stated in Chapter 2, one approach is to generate the weights w_i by the Delphi technique [34, 59], along with the use of sensitivity analysis. The significance of a different weighting system is often not obvious to people. Most people cannot relate well to the abstract weight. Thus, under such circumstances, it is more appropriate if the weighting is done using fuzzy numbers which allow more subjectivity and add a touch of flexibility. In practice, scales extending from 1 to 100 are often used for each component of the T.F.N.

$w_i = (w_{i1}, w_{i2}, w_{i3})$, $i = 1, 2, \ldots, m$, where 1 represents a real or hypothetical least preferred decision alternatives, and 100 is associated with a real or hypothetical most preferred decision alternatives. Often, for convenience, scales belonging to the interval [1,10] are also used.

We now let

$S_j =$ fuzzy score for decision alternative j, $j = 1, 2, \ldots, n$.

In view of (5.2.1) and (5.2.2), we have

$$S_j(\alpha) = \sum_{i=1}^{m} w_i(\alpha) \otimes s_{ij}(\alpha) \tag{5.2.3}$$

$$= \sum_{i=1}^{m} [(w_{i1} + (w_{i2} - w_{i1})\alpha, w_{i3} - (w_{i3} - w_{i2})\alpha)] \otimes$$

$$[s_{ij1} + (s_{ij2} - s_{ij1})\alpha,\ s_{ij3} - (s_{ij3} - s_{ij2})\alpha] \tag{5.2.4}$$

where $\otimes$ denotes the fuzzy multiplication.

$$S_j(\alpha) = (\sum_{i=1}^{m} w_{i1} s_{ij1} + [\sum_{i=1}^{m} w_{i1} s_{ij2} - 2\sum_{i=1}^{m} w_{j1} s_{ij1} + \sum_{i=1}^{m} w_{i2} s_{ij1}]\alpha$$

$$+\left[\sum_{i=1}^{m} w_{i2}s_{ij2} - \sum_{i=1}^{m} w_{i2}s_{ij1} - \sum_{i=1}^{m} w_{i1}s_{ij2} + \sum_{i=1}^{m} w_{i1}s_{ij1}\right]\alpha^2,$$

$$\sum_{i=1}^{m} w_{i3}s_{ij3} + \left[\sum_{i=1}^{m} w_{i3}s_{ij2} - 2\sum_{i=1}^{m} w_{i3}s_{ij3} + \sum_{i=1}^{m} w_{i2}s_{ij3}\right]\alpha$$

$$+\left[\sum_{i=1}^{m} w_{i3}s_{ij3} - \sum_{i=1}^{m} w_{i3}s_{ij2} - \sum_{i=1}^{m} w_{i2}s_{ij3} + \sum_{i=1}^{m} w_{i2}s_{ij2}\right]\alpha^2 \qquad (5.2.5)$$

The fuzzy number S_j corresponding to $S_j(\alpha)$ is obtained by setting $\alpha = 0$ for obtaining end values, and $\alpha = 1$,

for obtaining the interior value. Therefore,

$$S_j = \left(\sum_{i=1}^{m} w_{i1}s_{ij1},\ \sum_{i=1}^{m} w_{i2}s_{ij2},\ \sum_{i=1}^{m} w_{i3}s_{ij3}\right), \quad j = 1, 2, \ldots, n.$$

for which the membership function $\mu_{S_j}(x)$ for $j = 1, 2, \ldots, n$, is obtained by setting

$$\sum_{i=1}^{m} w_{i1}s_{ij1} + \left[\sum_{i=1}^{m} w_{i1}s_{ij2} - 2\sum_{i=1}^{m} w_{j1}s_{ij1} + \sum_{i=1}^{m} w_{i2}s_{ij1}\right]\alpha$$

$$+\left[\sum_{i=1}^{m} w_{i2}s_{ij2} - \sum_{i=1}^{m} w_{i2}s_{ij1} - \sum_{i=1}^{m} w_{i1}s_{ij2} + \sum_{i=1}^{m} w_{i1}s_{ij1}\right]\alpha^2 = x,$$

and

$$\sum_{i=1}^{m} w_{i3}s_{ij3} + \left[\sum_{i=1}^{m} w_{i3}s_{ij2} - 2\sum_{i=1}^{m} w_{i3}s_{ij3} + \sum_{i=1}^{m} w_{i2}s_{ij3}\right]\alpha$$

$$+\left[\sum_{i=1}^{m} w_{i3}s_{ij3} - \sum_{i=1}^{m} w_{i3}s_{ij2} - \sum_{i=1}^{m} w_{i2}s_{ij3} + \sum_{i=1}^{m} w_{i2}s_{ij2}\right]\alpha^2 = x$$

respectively, and solving each of them for α in terms of x. Thus, we have the membership function as given on the next page.

$$
\mu_{s(x)} = \begin{cases}
0 & \text{if } x \le \sum_{i=1}^{m} w_{i1} s_{ij1} \\[2ex]
\dfrac{-\left[\sum_{i=1}^{m} w_{i1} s_{ij2} - 2\sum_{i=1}^{m} w_{i1} s_{ij1} + \sum_{i=1}^{m} w_{i2} s_{ij1}\right] + \sqrt{\begin{array}{c} 4\left(\left[\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij1} - \sum_{i=1}^{m} w_{i1} s_{ij2} + \sum_{i=1}^{m} w_{i1} s_{ij1}\right]^2\right)x \\ + \left[\sum_{i=1}^{m} w_{i1} s_{ij2} - 2\sum_{i=1}^{m} w_{i1} s_{ij1} + \sum_{i=1}^{m} w_{i2} s_{ij1}\right]^2 \\ -4\left[\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij1} - \sum_{i=1}^{m} w_{i1} s_{ij2} + \sum_{i=1}^{m} \pi w_{i1} s_{ij1}\right]^2 \left[\sum_{i=1}^{m} w_{i1} s_{ij1}\right] \end{array}}}{2\left(\left[\sum_{i=1}^{m} w_{i1} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij1} - \sum_{i=1}^{m} w_{i1} s_{ij2} + \sum_{i=1}^{m} w_{i1} s_{ij1}\right]\right)} & \text{if } \sum_{i=1}^{m} w_{i1} s_{ij1} \le x \le \sum_{i=1}^{m} w_{i2} s_{ij2} \\[2ex]
\dfrac{-\left[\sum_{j=1}^{n} w_{i3} s_{ij2} - 2\sum_{j=1}^{n} w_{i3} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij3}\right] \sqrt{\begin{array}{c} 4\left(\left[\sum_{j=1}^{n} w_{i3} s_{ij3} - \sum_{j=1}^{n} w_{i3} s_{ij2} - \sum_{j=1}^{n} w_{i2} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij2}\right]^2\right)x \\ + \left[\sum_{j=1}^{n} w_{i3} s_{ij2} - 2\sum_{j=1}^{n} w_{i3} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij3}\right]^2 \\ -4\left[\sum_{j=1}^{n} w_{i3} s_{ij3} - \sum_{j=1}^{n} w_{i3} s_{ij2} - \sum_{j=1}^{n} w_{i2} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij2}\right]^2 \left[\sum_{j=1}^{n} w_{i3} s_{ij3}\right] \end{array}}}{2\left(\left[\sum_{j=1}^{n} w_{i3} s_{ij3} - \sum_{j=1}^{n} w_{i3} s_{ij2} - \sum_{j=1}^{n} w_{i2} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij2}\right]\right)} & \text{if } \sum_{i=1}^{m} w_{i2} s_{1i2} \le x \le \sum_{i=1}^{m} w_{i3} s_{1i3} \\[2ex]
0 & \text{if } x \ge \sum_{i=1}^{m} w_{i3} s_{1i3}
\end{cases}
$$

5.3 The Approximate Fuzzy Scoring Model.

If we approximate the above fuzzy number by a T.F.N., and denote it by AS_j, then AS_j is given by

$$AS_j = \left[\sum_{i=1}^{m} w_{i1}\, s_{ij1},\ \sum_{i=1}^{m} w_{i2}\, s_{ij2}\, ,\ \sum_{i=1}^{m} w_{i3} s_{ij3}\right], \qquad j = 1, 2, \ldots, n \qquad (5.3.1)$$

and we call it the approximate fuzzy score (AFS). The corresponding confidence interval at α –level of the

approximated T.F.N. $AS_j(\alpha)$ is given by

$$AS_j(\alpha) = \Big(\sum_{i=1}^{m} w_{i1} s_{ij1} + \Big[\sum_{i=1}^{m} w_{i1} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij1}\Big]\alpha,$$

$$\sum_{i=1}^{m} w_{i3} s_{ij3} + \Big[\sum_{i=1}^{m} w_{i3} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij3}\Big]\Big). \quad (5.3.2)$$

The approximate membership function of the AFS model is as given below.

$$\mu_{AS_j}(x) = \begin{cases} 0 & \text{if } x \le \sum_{i=1}^{m} w_{i1} s_{ij1} \\ \dfrac{x - \sum_{i=1}^{m} w_{i1} s_{ij1}}{\left(\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i1} s_{ij1}\right)} & \text{if } \sum_{i=1}^{m} w_{i1} s_{ij1} \le x \le \sum_{i=1}^{m} w_{i2} s_{ij2} \\ \dfrac{x - \sum_{i=1}^{m} w_{i3} s_{ij3}}{\left(\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i3} s_{ij3}\right)} & \text{if } \sum_{i=1}^{m} w_{i2} s_{ij2} \le x \le \sum_{i=1}^{m} w_{i3} s_{ij3} \\ 0 & \text{if } x \ge \sum_{i=1}^{m} w_{i3} s_{ij3} \end{cases}$$

where "x" represent the scoring value.

5.4 Error Analysis of TFN

Unfortunately the triangular fuzzy numbers are not closed under multiplication and division and it has been pointed out by Kaufmann and Gupta [34, 35] that the result of these operators is a polynomial membership function, and the triangular shape only approximates the actual result. It is a common mistake to blindly assume that the error introduced by the approximations is small and acceptable. The linear approximation can be quite poor and can lead to incorrect results when used in a variety of applications. Thus, in this case, it is important to perform error analysis to check the accuracy of the estimate. Error of approximation between the approximated membership function and the actual membership function can be sufficiently large to produce erroneous results. Each triangular or polynomial fuzzy number can be separated into a left and right segment.

Using the notation of Kaufmann and Gupta [34, 35], the actual fuzzy number for the value x, at a given α is defined as T_L for the left segment and T_R for the right segment. The standard approximation for value x at a given α is defined as P_L and P_R for the left and right segments respectively. This allows us to separately analyze the left and right portions of the membership curve [34, 35].

The left segment error, ε_L, is

$$\varepsilon_L = P_L - T_L \tag{5.4.1}$$

and the right segment error, ε_R, is

$$\varepsilon_R = P_R - T_R \tag{5.4.2}$$

A more meaningful measure of the error can be obtained by taking the absolute percent error. The absolute percent error with respect to the actual value is defined for the left segment as

$$\%\,\varepsilon_L = \left|\frac{P_L - T_L}{T_L}\right| .100 \tag{5.4.3}$$

and the right segment error is defined as

$$\%\,\varepsilon_R = \left|\frac{P_R - T_R}{T_R}\right| .100. \tag{5.4.4}$$

These expressions require knowledge of both the approximation and the actual value at every α – cut to have any utility.

If the above errors are small, we can take the approximate triangular fuzzy number as a legitimate approximation (Kaufmann and Gupta [35]).

5.5. Kaufmann and Gupta Error Analysis with T.F.N. Approximation.

Using Kaufmann and Gupta [35] notation, let the fuzzy number A be obtained as the multiplication

of the two T.F.N.'s such that the α –cut of A is given explicitly by

$A(\alpha) = \left[A(\alpha_L), A(\alpha_R)\right]$

where $A(\alpha_L) =$ represents the α – cut of left segment of the exact fuzzy number A.

$A(\alpha_R) =$ represents the α – cut of right segment of the exact fuzzy number A.

Suppose P, a T.F.N., is obtained as an approximation of the fuzzy number A, such that the α – cut

of P is given by

$P(\alpha) = \left[P(\alpha_L), P(\alpha_R)\right]$

where $P(\alpha_L) =$ represents the α – cut of left segment of the approximate fuzzy number P.

$P(\alpha_R) =$ represents the α –cut of right segment of the approximate fuzzy number P.

The left divergence $\epsilon_{l\alpha}$ is given by [35],

$$\epsilon_{l\alpha} = A(\alpha_L) - P(\alpha_L). \qquad (5.5.1)$$

The right divergence $\epsilon_{R\alpha}$ is given by [35],

$$\epsilon_{r\alpha} = A(\alpha_R) - P(\alpha_R). \tag{5.5.2}$$

According to Kaufmann and Gupta [35],

the maximum left and right divergences occur at $\alpha = \frac{1}{2}$

If each of the right and left divergence is small, then P can be considered as a good approximation of the product of the two fuzzy numbers.

5.6 Left and Right Divergence of the Parabolic Fuzzy Scoring Number.

From (5.2.5), we have the parabolic fuzzy number of the scoring number as

$$\begin{aligned} S_j(\alpha) = \Big(&\textstyle\sum_{i=1}^m w_{i1}s_{ij1} + \big[\sum_{i=1}^m w_{i1}s_{ij2} - 2\sum_{i=1}^m w_{j1}s_{ij1} + \sum_{i=1}^m w_{i2}s_{ij1}\big]\alpha \\ &+ \textstyle\big[\sum_{i=1}^m w_{i2}s_{ij2} - \sum_{i=1}^m w_{i2}s_{ij1} - \sum_{i=1}^m w_{i1}s_{ij2} + \sum_{i=1}^m w_{i1}s_{ij1}\big]\alpha^2, \\ &\textstyle\sum_{i=1}^m w_{i3}s_{ij3} + \big[\sum_{i=1}^m w_{i3}s_{ij2} - 2\sum_{i=1}^m w_{i3}s_{ij3} + \sum_{i=1}^m w_{i2}s_{ij3}\big]\alpha \\ &+ \textstyle\big[\sum_{i=1}^m w_{i3}s_{ij3} - \sum_{i=1}^m w_{i3}s_{ij2} - \sum_{i=1}^m w_{i2}s_{ij3} + \sum_{i=1}^m w_{i2}s_{ij2}\big]\alpha^2\Big) \end{aligned}$$

and from (5.3.2) we have

$$\begin{aligned} AS_j(\alpha) = \Big(&\textstyle\sum_{i=1}^m w_{i1}s_{ij1} + \big[\sum_{i=1}^m w_{i1}s_{ij2} + \sum_{i=1}^m w_{i2}s_{ij1}\big]\alpha, \\ &\textstyle\sum_{i=1}^m w_{i3}s_{ij3} + \big[\sum_{i=1}^m w_{i3}s_{ij2} + \sum_{i=1}^m w_{i2}s_{ij3}\big]\Big). \end{aligned}$$

According to Kaufmann and Gupta [32] the left Divergence for $j = 1, 2, \ldots, n$ is given by using equation (5.6.1)

$$\varepsilon_{Lj} = \left[\frac{\begin{array}{c}\sum_{i=1}^{m} w_{i1} s_{ij1} + [\sum_{i=1}^{m} w_{i1} s_{ij2} - 2\sum_{i=1}^{m} w_{i1} s_{ij1} + \sum_{i=1}^{m} w_{i2} s_{ij1}]\alpha + \\ [\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij1} - \sum_{i=1}^{m} w_{i1} s_{ij2} + \sum_{i=1}^{m} w_{i1} s_{ij1}]\alpha^2\end{array}}{\sum_{i=1}^{m} w_{i1} s_{ij1} + (\sum_{i=1}^{m} w_{i2} s_{ij2} - \sum_{i=1}^{m} w_{i1} s_{ij1})\alpha} \right] \quad (5.6.1)$$

The maximum left divergences occurs at $\alpha = \frac{1}{2}$, therefore, substitute $\alpha = \frac{1}{2}$ in (5.6.1).

The maximum left divergence for j = 1, 2, . . . , n is given by

$$\text{Max\_}\varepsilon_{Lj} = \frac{\sum_{i=1}^{m} w_{i1} s_{ij2} + \varepsilon_{Lj} w_{i2} s_{ij1} + \sum_{i=1}^{m} w_{i2} s_{ij2} + \sum_{i=1}^{m} w_{i1} s_{ij1}}{4} \quad (5.6.2)$$

Similarly, the right divergence is given by (5.6.2).

$$\varepsilon_{Rj} = \left[\frac{\begin{array}{c}\sum_{i=1}^{m} w_{i3} s_{ij3} \\ + \\ [\sum_{i=1}^{m} w_{i3} s_{ij2} - 2\sum_{i=1}^{m} w_{i3} s_{ij3} + \sum_{i=1}^{m} w_{i2} s_{ij3}]\alpha \\ + \\ [\sum_{i=1}^{m} w_{i3} s_{ij3} - \sum_{i=1}^{m} w_{i3} s_{ij2} - \sum_{i=1}^{m} w_{i2} s_{ij3} + \sum_{i=1}^{m} w_{i2} s_{ij2}]\alpha^2)\end{array}}{[\sum_{i=1}^{m} w_{i3} s_{ij3} - (\sum_{i=1}^{m} w_{i3} s_{ij3} - \sum_{i=1}^{m} w_{i2} s_{ij2})\alpha]} \right] \quad (5.6.3)$$

The maximum right divergences occurs at $\alpha = \frac{1}{2}$. Therefore, substitute $\alpha = \frac{1}{2}$ in (5.7.3)

The maximum right divergence is given by

$$\text{Max\_}\varepsilon_{R} = \frac{\sum_{j=1}^{n} w_{i3} s_{ij1} + \sum_{j=1}^{n} w_{i2} s_{ij3} + \sum_{j=1}^{n} w_{i2} s_{ij2} + \sum_{j=1}^{n} w_{i3} s_{ij3}}{4} \quad (5.7.4)$$

We will be using Max_ε_{Lj} and Max_ε_{Rj} for j = 1, 2, . . . , n, in the example below.

5.7 Bond Investment Problem: A Crisp Scoring Model Application

For the purpose of comparing the crisp approach with the fuzzy approach we first consider a numerical example under crisp environment.

Assuming that an investor with a reasonable investment capital decided to invest in the following long terms instruments.

- Discount Saving Bonds (DSB)
- Federal Bonds (FB)
- Corporate Bonds (CB)

The scoring method developed earlier is use to select the best investment opportunity.

A summary of the ratings assigned to the different criteria and decision alternatives is provided in Table 5.1,

and a summary of the weights assigned to different criteria are provided in the Table 5.2.

A BOND is a certificate representing an agreement by a company, federal government or municipality to pay a certain amount for the "loan" you have given the entity. Further, at a certain period in time, the company, federal government or municipality agrees to return your principal- the maturity date. Bonds are far from being the safe and non-volatile investment that many people thought of. Their volatility has increased tremendously and the correlation between stocks and bonds is a highly positive one.

The terms and Availability of a bond is a common term referring to the amount of time left until maturity. The price of a bond swings more violently with interest rates the longer the maturity of the bond is.

Quality: Refers to the quality of a bond. Some bonds are superior that others. Quality are not absolute measures it takes into consideration such factors as the issuer's past earnings record and future earnings expectations, the financial condition of the issuer, the nature of the issuer's business, the backing for a particular issue, and the rating agency's appraisals of the issuer's management.

Backed by: We say that a bond is 'backed by ' if some municipalities, provinces and federal government support the issuance of those bonds for economic and safety reasons.

Liquidity: It is the ease with which an investment can be converted to cash. An asset is liquid if it can be bought or sold quickly with relatively small priced changes.

Income Frequency: Refers to the number of dividend payment yielded by a bond during a specific period of time.

Trade Denominations: This term refers to the bond trading scheme on the market. For example trade denomination can be $100, $500, $1000 or $5000.

Taxation: Refers to some tax advantage or break that an investor is entitled for his or her investment capital.

Other Characteristics: There are some bonds that are pigmented and flavored with some stock characteristics. Such characteristics are termed as 'other characteristics'.

Recommended for: Some bonds are recommended while others are not recommended for investment to potential investors.

Table 5.1. Crisp Ratings for Each Bond Criterion

Criterion	Discount Saving Bonds	Federal Bonds	Corporate Bonds
Terms & Availability	5	8	8
Quality	8	10	6
Backed By	9	8	6
Liquidity	6	8	6
Income Frequency	5	7	7
Trade Denominations	7	7	7
Taxation	5	5	5
Other Characteristics	5	5	5
Recomended For	4	8	5

Table 5.2. Crisp Weight for Each Bond Criterion

Criterion	Level of Importance	Weight
Terms & Availability	Somewhat Important	3
Quality	Important	6
Backed By	Very Important	7
Liquidity	Important	6
Income Frequency	Very important	7
Trade Denominations	Somewhat Unimportant	2
Taxation	Average Importance	4
Other Characteristics	Somewhat Important	3
Recomended For	Average Importance	4

Table 5.3 provides a summary of the score value for each of the sub-criterion for the discount saving bond.

Table 5.3. Computation of Crisp Scores for Discount Saving Bonds (DSB)

Criterion	Weight	Rating	Score Value
Terms & Availability	3	5	15
Quality	6	8	48
Backed By	7	9	63
Liquidity	6	6	36
Income Frequency	7	5	35
Trade Denominations	2	7	14
Taxation	4	5	20
Other Characteristics	3	5	15
Recomended For	4	4	16
Total Score	-	-	262

Table 5.4 provides a summary of the crisp scores for each of the sub-criterion for the federal bond.

Table 5.4. Computation of Crisp Scores for Federal Bond(FB)

Criterion	Weight	Rating	Score
Terms & Availability	3	8	24
Quality	6	10	60
Backed By	7	8	56
Liquidity	6	8	48
Income Frequency	7	7	49
Trade Denominations	2	7	14
Taxation	4	5	20
Other Characteristics	3	5	15
Recomended For	4	8	32
Total Score	-	-	318

Table 5.5 provides a summary of the score value for each of the sub-criterion for the corporate bond is provided.

Table 5.5. Computation of Crisp Scores for Corporate Bonds (CB)

Criterion	Weight	Rating	Score
Terms & Availability	3	8	24
Quality	6	6	36
Backed By	7	6	42
Liquidity	6	6	36
Income Frequency	7	7	49
Trade Denominations	2	7	14
Taxation	4	5	20
Other Characteristics	3	5	15
Recomended For	4	5	20
Total Score	-	-	256

Comparing the results from Table 5.3 – 5.5, we see that the highest crisp score of 318 corresponds to the Federal Bonds. Thus, under crisp environment, the Federal Bonds is the recommended decision alternative.

5.8 Bond Investment Problem: A Fuzzy Scoring Model Application

We now consider the model in which each of the discount saving bonds (DSB), federal bonds (FB), and corporate bonds (CB) have their weights and ratings given in terms of T.F.N.'s in place of crisp value. We consider each of them one by one. Various ranking procedures [7, 17, 35, 41] are used to rank the different investment alternatives in fuzzy environment.

Case 1. Discount Saving Bonds (DSB). We assume that the fuzzy ratings and the fuzzy weights for the DSB are given in the form of T.F.N.'s in Tables 5.6 and 5.7 respectively, along with their α –cuts.

Table 5.6. Fuzzy Rating for Discount Saving Bonds(DSB)

Criterion	Rating	Rating(α)
Terms & Availability(α)	(4 ,5, 6)	$(4+\alpha,\ 6-\alpha)$
Quality(α)	(7 ,8, 9)	$(7+\alpha,\ 9-\alpha)$
Backed By(α)	(8 ,9, 10)	$(8+\alpha,\ 10-\alpha)$
Liquidity(α)	(5 ,6, 7)	$(5+\alpha,\ 7-\alpha)$
Income Frequency(α)	(4 ,5, 6)	$(4+\alpha,\ 6-\alpha)$
Trade Denominations(α)	(6 ,7, 8)	$(6+\alpha,\ 8-\alpha)$
Taxation(α)	(4 ,5, 6)	$(4+\alpha,\ 6-\alpha)$
Other Characteristics(α)	(4 ,5, 6)	$(4+\alpha,\ 6-\alpha)$
Recomended For(α)	(3 ,4, 5)	$(3+\alpha,\ 5-\alpha)$

Table 5.7. Fuzzy Weights for Each Bond Criterion

Criterion	Level of Importance	Weight	Weight(α)
Terms & Availability	Somewhat Important	(2, 3, 4)	$(2+\alpha,\ 4-\alpha)$
Quality	Important	(5, 6, 7)	$(5+\alpha,\ 7-\alpha)$
Backed By	Very Important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Liquidity	Important	(5, 6, 7)	$(5+\alpha,\ 7-\alpha)$
Income Frequency	Very important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Trade Denominations	Somewhat Unimportant	(1, 2, 3)	$(1+\alpha,\ 3-\alpha)$
Taxation	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$
Other Characteristics	Somewhat Important	(2, 3, 4)	$(2+\alpha,\ 4-\alpha)$
Recomended For	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$

If we plug in values of the rating and weight in the formula developed in (5.2.7), we end up with values in Table 5.8

Table 5.8. Computation of Fuzzy Scores for Discount Saving Bonds (DSB)

Criterion	Weight (α)	Rating (α)	Score (α) = Weight (α)⊗Rating (α)
Terms & Availability	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Quality	$(5+\alpha,\ 7-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(35+12\alpha+\alpha^2,\ 63-16\alpha+\alpha^2)$
Backed By	$(6+\alpha,\ 8-\alpha)$	$(8+\alpha,\ 10-\alpha)$	$(48+14\alpha+\alpha^2,\ 80-18\alpha+\alpha^2)$
Liquidity	$(5+\alpha,\ 7-\alpha)$	$(5+\alpha,\ 7-\alpha)$	$(25+10\alpha+\alpha^2,\ 49-14\alpha+\alpha^2)$
Income Frequency	$(6+\alpha,\ 8-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(24+10\alpha+\alpha^2,\ 48-14\alpha+\alpha^2)$
Trade Denominations	$(1+\alpha,\ 3-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(6+7\alpha+\alpha^2,\ 24-11\alpha+\alpha^2)$
Taxation	$(3+\alpha,\ 5-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(12+7\alpha+\alpha^2,\ 30-11\alpha+\alpha^2)$
Other Characteristics	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Recomended For	$(3+\alpha,\ 5-\alpha)$	$(3+\alpha,\ 5-\alpha)$	$(9+6\alpha+\alpha^2,\ 25-10\alpha+\alpha^2)$
Total Fuzzy Score (α) =	Weight(α)	Rating(α)	$(175+78\alpha+9\alpha^2,\ 367-114\alpha+9\alpha^2)$

Defining the interval of confidence at level α, the total fuzzy score is a parabolic fuzzy number as follows.

Total Fuzzy Score$_{\mathrm{DSB}}(\alpha) = (175+78\alpha+9\alpha^2,\ 367-114\alpha+9\alpha^2) \quad \forall\ \alpha \in [0,\ 1]$ (5.10.1)

The membership function for the Discount Saving Bond with parabolic curvature is given as shown below.

$$\mu_{DSB}(\text{Score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \leq 175 \\ \dfrac{-39+\sqrt{9\ \text{Score}-54}}{9} & \text{if} \quad 175 \leq \text{Score} \leq 262 \\ \dfrac{57-\sqrt{9\ \text{Score}-54}}{9} & \text{if} \quad 262 \leq \text{Score} \leq 367 \\ 0 & \text{if} \quad \text{Score} \geq 367 \end{cases}$$

Now if we let $\alpha = 0$ and $\alpha = 1$, in the equation (5.10.1), we get in terms of a fuzzy number the total score for the DSB.

Total Fuzzy Score$_{\mathrm{DSB}} = (175,\ 262,\ 367)$

Approximating the **Total Fuzzy Score**$_{DSB}$ by a T.F.N., we obtain the following α-cut.

$$\textbf{Total Fuzzy Score}_{DSB}\ (\alpha) = (175 + 87\alpha,\ 367 - 105\alpha) \qquad \forall\ \alpha \in [0,\ 1] \quad (5.10.2)$$

The membership function for the approximated **Total Fuzzy Score**$_{DSB}$ is given as below.

$$\mu_{DSB}(\text{Score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \le 175 \\ \dfrac{(\text{Score} - 175)}{87} & \text{if} \quad 175 \le \text{Score} \le 262 \\ \dfrac{(367 - (\text{Score})}{105} & \text{if} \quad 262 \le \text{Score} \le 36 \\ 0 & \text{if} \quad \text{Score} \ge 367. \end{cases}$$

In Table 5.9, columns 1-8 are interpreted as follows.

Column 1 = the left part of equation (5.10.1) for $\forall\ \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 2 = values of the right part of equation 5.10.1 for $\forall\ \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 3 = values of the left part of equation 5.10.2 for $\forall\ \alpha \in [0,\ 1]$ and gives zero curvature.

Column 4 = values of the right part of equation 5.10.2 for $\forall\ \alpha \in [0,\ 1]$ and gives zero curvature.

Column 5 = the difference between column 3 and column 1.

Column 6 = the difference between column 4 and column 2..

Column 7 = computation of left percentage error $\%\ \varepsilon_L = \left| \dfrac{\text{Column 3 - Columnn 1}}{\text{columnn 1}} \right| .100$

Column 8 = computation of right percentage error $\% \varepsilon_R = \left| \frac{\text{Column 4 - Columnn 2}}{\text{Columnn 2}} \right| .100.$

Table 5.9: Error Analysis for Discount Saving Bonds(DSB)

.	1	2	3	4	5	6	7	8
α	Left Scoring	Right Scoring	Left Scoring	Right Scoring	Error	Error	% Error	% Error
-	Curvature	Curvature	TFN	TFN	(3 – 1)	(4 – 2)	-	-
0	175	367	175	367	0	0	0	0
0.1	182.89	355.69	183.7	356.5	0.81	0.81	0.4	0.2
0.2	190.96	344.56	192.4	346	1.44	1.44	0.8	0.4
0.3	199.21	333.61	201.1	335.5	1.89	1.89	0.9	0.6
0.4	207.64	322.84	209.8	325	2.16	2.16	1.0	0.7
0.5	216.25	312.25	218.5	314.5	2.25	2.25	1.0	0.7
0.6	225.04	301.84	227.2	304	2.16	2.16	1.0	0.7
0.7	234.01	291.61	235.9	293.5	1.89	1.89	0.8	0.6
0.8	243.16	281.56	244.6	283	1.44	1.44	0.6	0.5
0.9	253.49	271.69	253.3	272.5	0.81	0.81	0.3	0.3
1.0	262	262	262	262	0	0	0	0

In this case we see that the left and right maximum percent error is small. Therefore, using the T.F.N. approximation in this case is useful.

Case 2. Federal Bonds (FB). We assume that the fuzzy ratings and the fuzzy weights for the FB are given in the form of T.F.N.'s in Tables 5.10 and 5.11 respectively, along with their α –cuts.

Table 5.10. Fuzzy Rating for Federal Bonds (FB)

Criterion	**Rating**	**Rating(α)**
Terms & Availability(α)	(7, 8, 9)	$(7+\alpha,\ 9-\alpha)$
Quality(α)	(9, 10, 1 1)	$(9+\alpha,\ 11-\alpha)$
Backed By(α)	(7, 8, 9)	$(7+\alpha,\ 9-\alpha)$
Liquidity(α)	(7, 8, 9)	$(7+\alpha,\ 9-\alpha)$
Income Frequency(α)	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Trade Denominations(α)	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Taxation(α)	(4, 5, 6)	$(4+\alpha,\ 6-\alpha)$
Other Characteristics(α)	(4, 5, 6)	$(4+\alpha,\ 6-\alpha)$
Recomended For(α)	(7, 8, 9)	$(7+\alpha,\ 9-\alpha)$

Table 5.11. Fuzzy Weight for Federal Bonds (FB)

Criterion	**Level of Importance**	**Weight**	**Weight(α)**
Terms & Availability	Somewhat Important	(2, 3, 4)	$(2+\alpha,\ 4-\alpha)$
Quality	Important	(5, 6 ,7)	$(5+\alpha,\ 7-\alpha)$
Backed By	Very Important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Liquidity	Important	(5, 6, 7)	$(5+\alpha,\ 7-\alpha)$
Income Frequency	Very important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Trade Denominations	Somewhat Unimportant	(1, 2, 3)	$(1+\alpha,\ 3-\alpha)$
Taxation	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$
Other Characteristics	Somewhat Important	(2, 3 ,4)	$(2+\alpha,\ 4-\alpha)$
Recomended For	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$

Table 5.12: Computation of total fuzzy scores for Federal Bonds (FB)

Criterion	Weight(α)	Rating(α)	Score (α) = Weight (α)$\otimes$Rating (α)
Terms & Availability	$(2+\alpha,\ 4-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(14+9\alpha+\alpha^2,\ 36-13\alpha+\alpha^2)$
Quality	$(5+\alpha,\ 7-\alpha)$	$(9+\alpha,\ 11-\alpha)$	$(45+14\alpha+\alpha^2,\ 77-18\alpha+\alpha^2)$
Backed By	$(6+\alpha,\ 8-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(42+13\alpha+\alpha^2,\ 72-17\alpha+\alpha^2)$
Liquidity	$(5+\alpha,\ 7-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(35+12\alpha+\alpha^2,\ 63-16\alpha+\alpha^2)$
Income Frequency	$(6+\alpha,\ 8-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(36+12\alpha+\alpha^2,\ 64-16\alpha+\alpha^2)$
Trade Denominations	$(1+\alpha,\ 3-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(6+7\alpha+\alpha^2,\ 24-11\alpha+\alpha^2)$
Taxation	$(3+\alpha,\ 5-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(12+7\alpha+\alpha^2,\ 30-11\alpha+\alpha^2)$
Other Characteristics	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Recomended For	$(3+\alpha,\ 5-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(21+10\alpha+\alpha^2,\ 45-14\alpha+\alpha^2)$
Total Fuzzy Score	-	-	$(219+90\alpha+9\alpha^2,\ 435-126\alpha+9\alpha^2)$

Defining the interval of confidence at level α, the total fuzzy score is a parabolic fuzzy number and is given as follows.

Total Fuzzy Score$_{FB}$ $(\alpha) = (219+90\alpha+9\alpha^2,\ 435-126\alpha+9\alpha^2)$ $\forall\ \alpha \in [0,1]$ (5.10.3)

The membership function for the Federal Bonds with parabolic curvature is given as shown below.

$$\mu_{FB}(\text{score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \le 219 \\ \dfrac{-15+\left[\sqrt{\text{Score}+6}\right]}{3} & \text{if} \quad 219 \le \text{Score} \le 318 \\ \dfrac{126-\left[\sqrt{36\ \text{Score}+215}\right]}{117} & \text{if} \quad 318 \le \text{Score} \le 435 \\ 0 & \text{if} \quad \text{Score} \ge 435 \end{cases}$$

Now if we let $\alpha = 0$ and $\alpha = 1$, in the equation (5.10.3), we get in terms of a fuzzy number the total fuzzy score for the DSB.

Total Fuzzy Score$_{FB}$ = (219, 318, 435)

Approximating the **Total Fuzzy Score**$_{FB}$ by a T.F.N., we obtain the following α –cut.

$$P_{FB}(\alpha) = (219 + 99\alpha, 435 - 117\alpha) \qquad (5.10.4)$$

The membership function for the approximated **Total Fuzzy Score**$_{FB}$ is given as

$$\mu_{FB}(\text{score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \le 219 \\ \dfrac{\text{Score} - 219}{99} & \text{if} \quad 219 \le \text{Score} \le 318 \\ \dfrac{435 - \text{Score}}{117} & \text{if} \quad 318 \le \text{Score} \le 435 \\ 0 & \text{if} \quad \text{Score} \ge 435. \end{cases}$$

In Table 5.12 below, Columns 1-8 are interpreted as follows.

Column 1 = values of the left part of equation 5.11.1 for $\forall\, \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 2 = values of the right part of equation 5.11.1 for $\forall\, \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 3 = values of the left part of equation 5.11.2 for $\forall\, \alpha \in [0,\ 1]$ and gives zero curvature.

Column 4 = values of the right part of equation 5.11.2 for $\forall\, \alpha \in [0,\ 1]$ and gives zero curvature.

Column 5 = the difference between column 3 and column 1.

Column 6 = the difference between column 4 and column 2..

Column 7 = computation of left percentage error $\%\varepsilon_L = \left| \dfrac{\text{Column 3 - Columnn 1}}{\text{Columnn 1}} \right| .100$

Column 8 = computation of right percentage error $\%\varepsilon_R = \left| \dfrac{\text{Column 4 - Columnn 2}}{\text{Columnn 2}} \right| .100$

Table 5.13. Error Analysis for Fuzzy Scores Federal Bonds(FB)

-	1	2	3	4	5	6	7	8
α	Left Scoring	Right Scoring	Left Scoring	Right Scoring	Error	Error	%Error	%Error
-	Curvature	Curvature	TFN	TFN	$(3-1)$	$(4-2)$	-	-
0	219	435	219	435	0	0	0	0
0.1	228.09	422.49	228.9	423.3	0.81	0.81	0.35	0.19
0.2	237.36	410.16	238.8	411.6	1.44	1.44	0.60	0.35
0.3	246.81	398.01	248.7	399.9	1.89	1.89	0.76	0.47
0.4	256.44	366.04	258.6	388.2	2.16	2.16	0.84	0.55
0.5	266.25	374.25	268.5	376.5	2.25	2.25	0.84	0.60
0.6	276.24	362.64	278.4	364.8	2.16	2.16	0.78	0.59
0.7	286.41	351.21	288.3	353.1	1.89	1.89	0.65	0.53
0.8	296.76	339.96	298.2	341.4	1.44	1.44	0.48	0.42
0.9	307.29	328.89	308.1	329.7	0.81	0.81	0.26	0.24
1.0	318	318	318	318	0	0	0	0

In this case we see that the left and right maximum percent error is small. Therefore, using the T.F.N. approximation in this case is useful.

Case 3. Corporate Bonds (CB). We assume that the fuzzy ratings and the fuzzy weights for the CB are given in the form of T.F.N.'s in Tables 5.14 and Tables 5.15 respectively, along with their α –cuts.

Table 5.14. Fuzzy Rating For Corporate Bonds

Criterion	Rating	Rating(α)
Terms & Availability(α)	(7 ,8, 9)	$(7+\alpha, 9-\alpha)$
Quality(α)	(5 ,6, 7)	$(5+\alpha, 7-\alpha)$
Backed By(α)	(5 ,6, 7)	$(5+\alpha, 7-\alpha)$
Liquidity(α)	(5 ,6, 7)	$(5+\alpha, 7-\alpha)$
Income Frequency(α)	(6 ,7, 8)	$(6+\alpha, 8-\alpha)$
Trade Denominations(α)	(6 ,7, 8)	$(6+\alpha, 8-\alpha)$
Taxation(α)	(4 ,5, 6)	$(4+\alpha, 6-\alpha)$
Other Characteristics(α)	(4 ,5, 6)	$(4+\alpha, 6-\alpha)$
Recomended For(α)	(4 ,5, 6)	$(4+\alpha, 6-\alpha)$

Table 5.15. Fuzzy Weight For Corporate Bonds

Criterion	Level of Importance	Weight	Weight(α)
Terms & Availability	Somewhat Important	(2, 3, 4)	$(2+\alpha,\ 4-\alpha)$
Quality	Important	(5, 6 ,7)	$(5+\alpha,\ 7-\alpha)$
Backed By	Very Important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Liquidity	Important	(5, 6, 7)	$(5+\alpha,\ 7-\alpha)$
Income Frequency	Very important	(6, 7, 8)	$(6+\alpha,\ 8-\alpha)$
Trade Denominations	Somewhat Unimportant	(1, 2, 3)	$(1+\alpha,\ 3-\alpha)$
Taxation	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$
Other Characteristics	Somewhat Important	(2, 3 ,4)	$(2+\alpha,\ 4-\alpha)$
Recomended For	Average Importance	(3, 4, 5)	$(3+\alpha,\ 5-\alpha)$

Table 5.16: Computation of Total Fuzzy Score for Corporate Bonds(CB)

Criterion	Weight (α)	Rating (α)	Score (α) = Weight (α)$\otimes$Rating (α)
Terms & Availability	$(2+\alpha,\ 4-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(14+9\alpha+\alpha^2,\ 36-13\alpha+\alpha^2)$
Quality	$(5+\alpha,\ 7-\alpha)$	$(5+\alpha,\ 7-\alpha)$	$(25+10\alpha+\alpha^2,\ 49-14\alpha+\alpha^2)$
Backed By	$(6+\alpha,\ 8-\alpha)$	$(5+\alpha,\ 7-\alpha)$	$(30+11\alpha+\alpha^2,\ 56-15\alpha+\alpha^2)$
Liquidity	$(5+\alpha,\ 7-\alpha)$	$(5+\alpha,\ 7-\alpha)$	$(25+10\alpha+\alpha^2,\ 49-14\alpha+\alpha^2)$
Income Frequency	$(6+\alpha,\ 8-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(36+12\alpha+\alpha^2,\ 64-16\alpha+\alpha^2)$
Trade Denominations	$(1+\alpha,\ 3-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(6+7\alpha+\alpha^2,\ 24-11\alpha+\alpha^2)$
Taxation	$(3+\alpha,\ 5-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(12+7\alpha+\alpha^2,\ 30-11\alpha+\alpha^2)$
Other Characteristics	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Recomended For	$(3+\alpha,\ 5-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(12+7\alpha+\alpha^2,\ 30-11\alpha+\alpha^2)$
Total Fuzzy Score	-	-	$(168+79\alpha+9\alpha^2,\ 362-115\alpha+9\alpha^2)$

Defining the interval of confidence at level α, the total fuzzy score is a parabolic fuzzy number and is given as follows.

Total Fuzzy Score$_{\text{CB}}$ $(\alpha) = (168+79\alpha+9\alpha^2,\ 362-115\alpha+9\alpha^2)\ \forall\ \alpha \in [0,\ 1]$ (5.10.5)

The membership function for the Corporate Bonds with parabolic curvature is given as shown below.

$$\mu_{\text{CB}}(\text{score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \le 168 \\ \dfrac{-79+\left[\sqrt{36\,(\text{Score})+193}\right]}{18} & \text{if} \quad 168 \le \text{Score} \le 256 \\ \dfrac{115-\left[\sqrt{36\,(\text{Score})+193}\right]}{18} & \text{if} \quad 256 \le \text{Score} \le 362 \\ 0 & \text{if} \quad \text{Score} \ge 362 \end{cases}$$

Now if we let $\alpha = 0$ and $\alpha = 1$, in the equation (5.10.3), we get in terms of a fuzzy number the total fuzzy score for the CB.

$P_{\text{CB}} = (168,\ 256,\ 362)$

Approximating the **Total Fuzzy Score**$_{\text{CB}}$ by a T.F.N., we obtain the following α-cut.

$$P_{CB}(\alpha) = (168 + 88\alpha, 362 - 106\alpha) \tag{5.10.6}$$

The membership function for the approximated **Total Fuzzy Score**$_{CB}$ is given by:

$$\mu_{CB}(\text{score}) = \begin{cases} 0 & \text{if} \quad \text{Score} \le 168 \\ \dfrac{\text{Score}-168}{88} & \text{if} \quad 168 \le \text{Score} \le 256 \\ \dfrac{362 - \text{Score}}{106} & \text{if} \quad 256 \le \text{Score} \le 362 \\ 0 & \text{if} \quad \text{Score} \ge 362 \end{cases}$$

In Table 5.17 below, columns 1-8 are interpreted as follows.

Column 1 = values of the left part of equation 5.12.1 for $\forall\ \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 2 = values of the right part of equation 5.12.1 for $\forall\ \alpha \in [0,\ 1]$ and gives parabolic curvature.

Column 3 = values of the left part of equation 5.12.2 for $\forall\ \alpha \in [0,\ 1]$ and gives zero curvature.

Column 4 = values of the right part of equation 5.12.2 for $\forall\ \alpha \in [0,\ 1]$ and gives zero curvature.

Column 5 = the difference between column 3 and column 1.

Column 6 = the difference between column 4 and column 2..

Column 7 = computation of left percentage error $\%\varepsilon_L = \left|\dfrac{\text{Column 3} - \text{Columnn 1}}{\text{Columnn 1}}\right|.100$

Column 8 = computation of right percentage error $\%\varepsilon_R = \left|\dfrac{\text{Column 4} - \text{Columnn 2}}{\text{Columnn 2}}\right|.100$

Table 5.17: Error Analysis for Total Fuzzy Scores Corporate Bonds(CB)

-	1	2	3	4	5	6	7	8
α	Left Scoring	Right Scoring	Left Scoring	Right Scoring	Error	Error	%Error	%Error
-	Curvature	Curvature	TFN	TFN	(3 – 1)	(4 – 2)	-	-
0	168	362	168	362	0	0	0	0.0
0.1	175.99	350.39	176.8	351.4	0.81	0.81	0.5	0.2
0.2	184.16	339.36	185.6	340.8	1.44	1.44	0.8	0.4
0.3	192.51	328.31	194.4	330.2	1.89	1.89	1.0	0.6
0.4	201.75	317.44	203.2	319.6	2.16	2.16	1.1	0.7
0.5	218.64	306.75	212	309	2.25	2.25	1.1	0.7
0.6	227.71	296.24	220.8	298.4	2.16	2.16	1.0	0.7
0.7	236.96	285.91	229.6	287.8	1.89	1.89	0.8	0.7
0.8	246.96	275.76	238.4	277.2	1.44	1.44	0.6	0.5
0.9	246.39	265.79	247.2	266.6	0.81	0.81	0.3	0.3
1.0	256	256	256	256	0	0	0.0	0.0

In this case we see that both the left and right maximum percent errors are small. Therefore, using the T.F.N. approximation in this case is useful.

5.9 Bond Selection for Investment

Since the error analysis clearly shows that the maximum values of errors and the percentage errors between the parabolic fuzzy numbers and the approximated fuzzy numbers are small, therefore, for selecting the bond in which investment could be made, we rank the approximated T.F.N.'s using various ranking techniques given in Chapter 1.

Ranking Method 1. Kaufmann & Gupta [35]

$$(AON)_i = \frac{a_{i1} + 2a_{i2} + a_{i3}}{4} \qquad \text{for } i = DSB, FB, CB.$$

Table 5.18. Rating for Each Decision Criteria and Each Decision Alternative Combination

Criterion	Fuzzy Number	AON
Discount Saving Bonds (DSB).	(175, 262, 367)	266.5
Federal Bonds (FB).	(219, 318, 435)	322.5
Corporate Bonds (CB).	(168, 256, 362)	273.25

We observe that in this case the FB are the bonds with the highest index 322.5.

Ranking Method 2. Chang's Ranking Index (Komolananij [40])

$$\text{Index}_i = \frac{(a_{3i} - a_{1i})(a_{1i} + a_{2i} + a_{3i})}{6} \qquad \text{for } i = \text{DSB, FB, CB}$$

Table 5.19. Rating for Each Decision Criteria and Each Decision Alternative Combination

Criterion	Fuzzy Number	Index_i
Discount Saving Bonds (DSB)	(175, 262, 367)	28928
Federal Bonds (FB)	(219, 318, 435)	34992
Corporate Bonds (CB)	(168, 256, 362)	25414

We observe that in this case the FB are the bonds with the highest index 34992.

Ranking Method 3. Chiu and Park Ranking Index (Chiu and Park [17])

Let w_1 and w_2 represents the weights associated with the fuzzy number whose ranking index has to be computed. Then,

$$\text{Index}_i = \frac{(a_{1i} + a_{2i} + a_{3i})w_{i1}}{3} + w_{i2}\, a_{2i} \qquad \text{for } i = \text{DSB, FB. CB.}$$

w_1 and w_2 are the weights.

Chui and Park believe that the weight w_{i2} should be taken as 1 and the weight w_{i1} should range between 0 and 1 and the most likely possible value is 0.5. Index_i can be rewritten as

$$\text{Index}_i = 0.5\left[\frac{(a_{1i} + a_{2i} + a_{3i})}{3}\right] + a_{3i}$$

Table 5.20. Rating for Each Decision Criteria and Each Decision Alternative Combination

Criterion	Fuzzy Number	Index_i
Discount Saving Bonds (DSB)	(175, 262, 367)	396
Federal Bonds (FB)	(219, 318, 435)	480
Corporate Bonds (CB)	(168, 256, 362)	387

In this case we observe that the FB are the bonds with the highest index of 480.

Ranking Method 4. Chui and Park weighted Ranking Index 2 [17]

$$\text{Index}_{i1} = w_{i1}\left[\frac{a_{1i} + a_{3i}}{2}\right] + w_{i2}\ a_{2i}$$

where w_{i1} represents the weight associated with the values of a_{1i}, a_{3i} and w_{i2} represents the weight associated with a_{3i}.

Chui and Park believe that the weight w_{i2} should be taken as 1 and the weight w_{i1} should range between 0 and 1 and the most likely possible value is 0.5. In this case, Index_{i1} can be rewritten as

$$\text{Index}_{i2} = .5\left[\frac{a_{1i} + a_{3i}}{2}\right] + a_{2i}$$

Table 5.21. Rating for Each Decision Criteria and Each Decision Alternative Combination

Criterion	Fuzzy Number	Index_{i2}
Discount Saving Bonds (DSB).	(175, 262, 367)	397.5
Federal Bonds (FB).	(219, 318, 435)	481.5
Corporate Bonds (CB).	(168, 256, 362)	388.5

In this case we observe that the FB are the bonds with the highest index of 481.5.

Thus, keeping in view the fact that according to the ranking criteria, the T.F.N. with the highest rank is the best alternative, we observe that all the four methods used above yield that the Federal Bonds are the most attractive bonds.

5.10 Conclusion

Most scoring problems deal with future and often uncertain and imprecise data. To cope this uncertainty, fuzzy sets theory has been applied to the traditional deterministic approach for scoring models by dealing with uncertain data using triangular fuzzy numbers. The main contribution of this chapter is the application of fuzzy sets theory to the scoring problem and their application in Bond Selection Problem. An numerical example of a bond selection problem with fuzzy data is considered using the fuzzy scoring model.

Chapter 6

CONCLUSION, CONTRIBUTION AND RECOMMENDATIONS

In the present chapter, we summarize the contributions and conclusions of this dissertation. Finally, we give some recommendations for further research on the problems considered in this dissertation.

6.1 Conclusion and Contribution

Probabilistic models and the models using fuzzy sets and fuzzy logic describe different aspects of uncertainty. Probabilistic models primarily describe random variability in parameters. In contrast, models using fuzzy sets and fuzzy logic incorporate vagueness, imprecision, and subjective judgement. The data set of forecasted values rarely if ever turn out to be crisply correct. Therefore, the models based on precise knowledge of forecasted values have restricted practical applications. This thesis deals with such a problem through fuzzy set theory approach. Based on the results presented in this thesis, we conclude that fuzzy sets theory approach offers added advantage of flexibility of dealing with uncertainty involved in the net present value problem, capital asset pricing models, and the weighted scoring models. We propose various fuzzy techniques to certain existing deterministic models and arrive at more flexible solutions than normally obtained in their counterparts under crisp environment. It appears that the fuzzy ranking method is significantly more flexible than the non-fuzzy based prediction methods. Thus, the major contribution of the present thesis is that the results obtained here, using the theory of fuzzy sets and fuzzy logic, provide more flexible and satisfactory solutions versus the results obtaibed using their risp counterparts.

Chapter 3. Kellison [38] obtained the net present value formula under crisp environment Iowever, one underlying assumption in the model by Kellison [38] and other researchers, is that he underlying data is deterministically known. Under fuzzy environments, the net present value ormula is formulated on the line of Kaufmann and Gupta [34, 35]. We extend the existing ormulation of net present value by Kaufmann and Gupta [31] to incorporate fuzziness in all the parameters.

Chapter 4. If there is limited information about variability, it is possible to use probabilistic nodels by making suitable assumptions on the statistics of the variability. However, it has been repeatedly shown that using these assumptions often entail serious errors in the final results. Fuzzy set models, appear to be well suited for problems where little is known about the uncertainty involving vagueness, imprecision, and subjectivity. To capture uncertainty and risk in the net present value model considered in this thesis, we use the fuzzy CAPM to obtain the fuzzy return and subsequently use it as the fuzzy discount rate in the fuzzy net present value problem. Several studies have compared fuzzy cash flows and probabilistic cash flow under uncertainty (for example, see Brealey [11], Chiu and Park [17], Ross et al. [49] etc.) . However, no study has used the fuzzy return from the fuzzy CAPM as the fuzzy discount rate in the fuzzy NPV problem. This is the main achievement of this chapter.

Chapter 5. We develop a fuzzified version of the weighted scoring model formula. Based on the observations in the chapter we conclude that the fuzzy approach presented in this chapter offers more flexibility and usefulness for the decision making process than the crisp approach formulated by Keeney and Raiffa [36, 37]. We observe that when there is imprecise and vague information available, a fuzzy weighted scoring model provides a better and more flexible decision in certain multicriteria decision making problems like Bond Selection Problem..

6.2 Recommendations for Future Research

A number of extensions are possible to the net present value problems addressed in Chapters 3 and 4. The problem can be further extended to the case where the parameters involved are taken as trapezoidal fuzzy numbers (Tr.F.N.'s) instead of triangular fuzzy number (T.F.N.'s), and also if the data available is in terms of sheaves. Fuzzy selection processes are, at present, largely confined to academic literature or to experimental applications, although ideas about certain type of multi-criteria analysis based on fuzzy sets have been discussed by many academics [7, 16, 17, 29, 33, 34, 35, 41, 42, 58, 59, 64, 65]. for more than a decade. Since it has been shown that fuzzy sets provide an explicit way of representing vagueness and subjectivity in the decision makers mind , fuzzy systems can also be easily implemented to decision support systems in the areas of bid preparation, investment portfolio, stock market analysis, database management, medical diagnostic systems, pattern recognition, production scheduling, cost-volume-profit analysis and counterfeit bank note detectors. Various other results that exist in the areas of investments, portfolio selection, purchases with credit and many other financial operations can be generalized using fuzzy sets and fuzzy logic.

Chapter 3 and 4 could also be considered by incorporating the prior information using a distribution with parameter α and then updating the current information in such a way that the resulting value of the membership function is derived from a posterior Bayesian setup in statistics (Thavaneswaran and Heyde [56]), because a posterior Bayes' estimate is superior to the Minimum Mean Square Estimate (MMSE) estimate as established and proved by Thavaneswaran and Heyde [56].

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APPENDIX 1

Fuzzy Net Present Value

Output For Fuzzy Net Present Value

CASE 1.

This is the case when the capital investment, expected earnings and discount rate are all fuzzy and different.

α	$\frac{1}{(1.13 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)}$
0	0.8850	0.9259
0.1	0.8873	0.9242
0.2	0.8897	0.9225
0.3	0.8921	0.9208
0.4	0.8945	0.9191
0.5	0.8969	0.9174
0.6	0.8993	0.9158
0.7	0.9017	0.9141
0.8	0.9042	0.9124
0.9	0.9066	0.9107
1	0.9091	0.9091

α	$\frac{1}{(1.13 - 0.03\alpha)(1.15 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)(1.09 + 0.02\alpha)}$
0	0.769527	0.8495
0.1	0.773593	0.8456
0.2	0.777692	0.8417
0.3	0.781824	0.8379
0.4	0.785988	0.8340
0.5	0.790186	0.8303
0.6	0.794417	0.8265
0.7	0.798683	0.8228
0.8	0.802983	0.8190
0.9	0.807318	0.8154
1	0.811688	0.8117

α	$\dfrac{1}{(1.13 - 0.03\alpha)(1.15 - 0.03\alpha)(1.12 - 0.02\alpha)}$
0	0.68708
0.1	0.69194
0.2	0.69686
0.3	0.70182
0.4	0.70682
0.5	0.71188
0.6	0.71698
0.7	0.72214
0.8	0.72734
0.9	0.73259
1	0.73790

α	$\dfrac{1}{(1.08 + 0.02\alpha)(1.09 + 0.02\alpha)(1.07 + 0.03\alpha)}$
0	0.79390
0.1	0.78805
0.2	0.78225
0.3	0.77652
0.4	0.77084
0.5	0.76521
0.6	0.75964
0.7	0.75413
0.8	0.74866
0.9	0.74326
1	0.73790

α	$-3000 + 1000\alpha$	$2000 + 1000\alpha$
0	-3000.0	2000.0
0.1	-2900.0	2100.0
0.2	-2800.0	2200.0
0.3	-2700.0	2300.0
0.4	-2600.0	2400.0
0.5	-2500.0	2500.0
0.6	-2400.0	2600.0
0.7	-2300.0	2700.0
0.8	-2200.0	2800.0
0.9	-2100.0	2900.0
1	-2000.0	3000.0

α	$1000 + 1000\alpha$	$1000 + 500\alpha$
0	1000.0	1000.0
0.1	1100.0	1050.0
0.2	1200.0	1100.0
0.3	1300.0	1150.0
0.4	1400.0	1200.0
0.5	1500.0	1250.0
0.6	1600.0	1300.0
0.7	1700.0	1350.0
0.8	1800.0	1400.0
0.9	1900.0	1450.0
1	2000.0	1500.0

α	$1000 + 1000\alpha$	$1000 + 500\alpha$
0	1000.0	1000.0
0.1	1100.0	1050.0
0.2	1200.0	1100.0
0.3	1300.0	1150.0
0.4	1400.0	1200.0
0.5	1500.0	1250.0
0.6	1600.0	1300.0
0.7	1700.0	1350.0
0.8	1800.0	1400.0
0.9	1900.0	1450.0
1	2000.0	1500.0

α	$-1000 - 1000\alpha$	$4000 - 1000\alpha$
0	-1000.0	4000.0
0.1	-1100.0	3900.0
0.2	-1200.0	3800.0
0.3	-1300.0	3700.0
0.4	-1400.0	3600.0
0.5	-1500.0	3500.0
0.6	-1600.0	3400.0
0.7	-1700.0	3300.0
0.8	-1800.0	3200.0
0.9	-1900.0	3100.0
1	-2000.0	3000.0

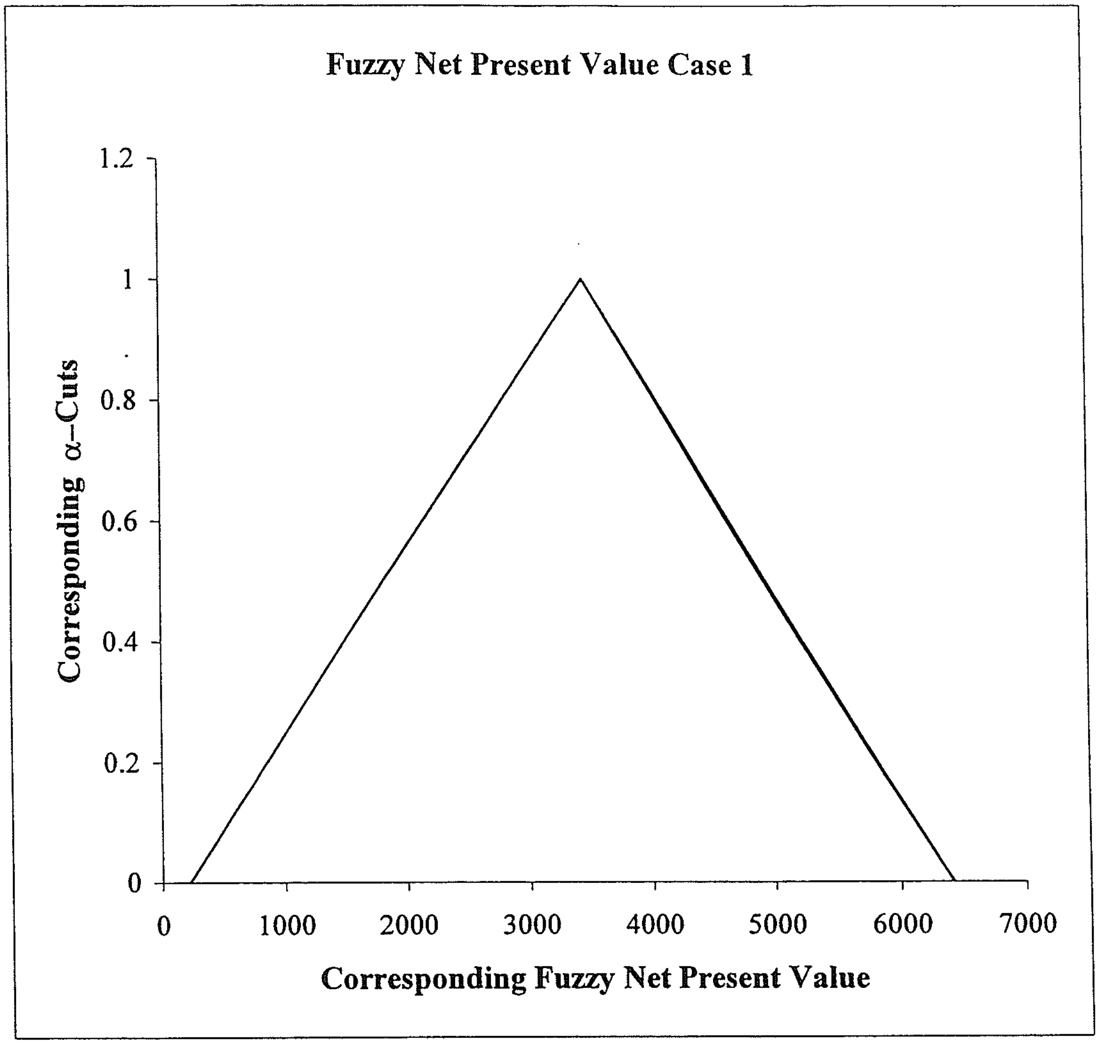
α	$2500 - 500\alpha$	$2000 - 500\alpha$
0	2500.0	2000.0
0.1	2450.0	1950.0
0.2	2400.0	1900.0
0.3	2350.0	1850.0
0.4	2300.0	1800.0
0.5	2250.0	1750.0
0.6	2200.0	1700.0
0.7	2150.0	1650.0
0.8	2100.0	1600.0
0.9	2050.0	1550.0
1	2000.0	1500.0

α	$2500-500\alpha$	$2000-500\alpha$
0	2500.0	2000.0
0.1	2450.0	1950.0
0.2	2400.0	1900.0
0.3	2350.0	1850.0
0.4	2300.0	1800.0
0.5	2250.0	1750.0
0.6	2200.0	1700.0
0.7	2150.0	1650.0
0.8	2100.0	1600.0
0.9	2050.0	1550.0
1	2000.0	1500.0

α	$1000+500\alpha$	$2000-500\alpha$
0	1000.0	2000.0
0.1	1050.0	1950.0
0.2	1100.0	1900.0
0.3	1150.0	1850.0
0.4	1200.0	1800.0
0.5	1250.0	1750.0
0.6	1300.0	1700.0
0.7	1350.0	1650.0
0.8	1400.0	1600.0
0.9	1450.0	1550.0
1	1500.0	1500.0

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	226.5	6415.2
0.1	540.8	6112.8
0.2	857.1	5811.9
0.3	1175.2	5512.5
0.4	1495.3	5214.6
0.5	1817.3	4918.2
0.6	2141.3	4623.2
0.7	2467.3	4329.7
0.8	2795.3	4037.6
0.9	3125.4	3746.8
1	3457.5	3457.5

Fig A1.1: FNPV (α), Case 1



CASE 2.

This is the case when the capital investment and expected earnings are all fuzzy and different. The discount rate being fuzzy and identical throughout the life of the investment.

α	$\frac{1}{(1.13 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)}$
0	0.8850	0.9259
0.1	0.8873	0.9242
0.2	0.8897	0.9225
0.3	0.8921	0.9208
0.4	0.8945	0.9191
0.5	0.8969	0.9174
0.6	0.8993	0.9158
0.7	0.9017	0.9141
0.8	0.9042	0.9124
0.9	0.9066	0.9107
1	0.9091	0.9091

α	$\frac{1}{(1.13 - 0.03\alpha)^2}$	$\frac{1}{(1.08 + 0.02\alpha)^2}$
0	0.783147	0.8573
0.1	0.787322	0.8542
0.2	0.791530	0.8510
0.3	0.795772	0.8479
0.4	0.800049	0.8448
0.5	0.804360	0.8417
0.6	0.808706	0.8386
0.7	0.813087	0.8355
0.8	0.817504	0.8325
0.9	0.821957	0.8295
1	0.826446	0.8264

α	$\frac{1}{(1.13-0.03\alpha)^3}$	$\frac{1}{(1.08+0.02\alpha)^3}$
0	0.69305	0.7938
0.1	0.69860	0.7894
0.2	0.70421	0.7851
0.3	0.70988	0.7807
0.4	0.71561	0.7764
0.5	0.72140	0.7722
0.6	0.72725	0.7679
0.7	0.73317	0.7637
0.8	0.73915	0.7596
0.9	0.74520	0.7554
1	0.75131	0.7513

α	$-3000+1000\alpha$	$2000+1000\alpha$
0	-3000.0	2000.0
0.1	-2900.0	2100.0
0.2	-2800.0	2200.0
0.3	-2700.0	2300.0
0.4	-2600.0	2400.0
0.5	-2500.0	2500.0
0.6	-2400.0	2600.0
0.7	-2300.0	2700.0
0.8	-2200.0	2800.0
0.9	-2100.0	2900.0
1	-2000.0	3000.0

α	$1000+1000\alpha$	$1000+500\alpha$
0	1000.0	1000.0
0.1	1100.0	1050.0
0.2	1200.0	1100.0
0.3	1300.0	1150.0
0.4	1400.0	1200.0
0.5	1500.0	1250.0
0.6	1600.0	1300.0
0.7	1700.0	1350.0
0.8	1800.0	1400.0
0.9	1900.0	1450.0
1	2000.0	1500.0

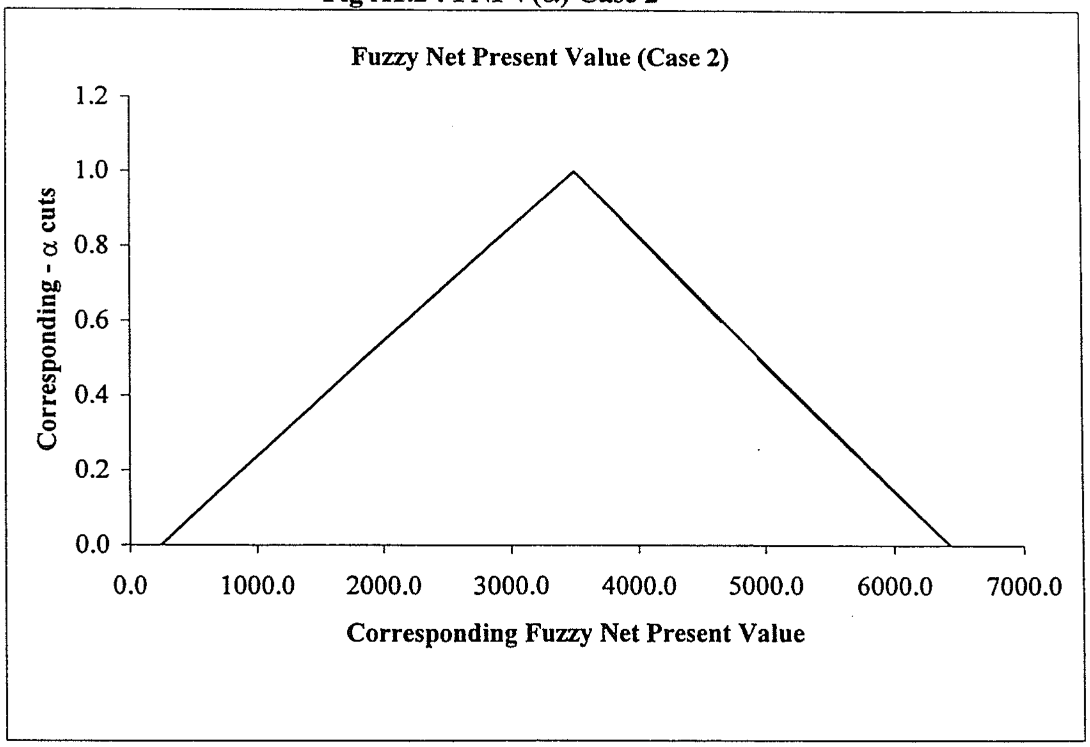
α	$-1000 - 1000\alpha$	$4000 - 1000\alpha$
0	-1000.0	4000.0
0.1	-1100.0	3900.0
0.2	-1200.0	3800.0
0.3	-1300.0	3700.0
0.4	-1400.0	3600.0
0.5	-1500.0	3500.0
0.6	-1600.0	3400.0
0.7	-1700.0	3300.0
0.8	-1800.0	3200.0
0.9	-1900.0	3100.0
1	-2000.0	3000.0

α	$2500 - 500\alpha$	$2000 - 500\alpha$
0	2500.0	2000.0
0.1	2450.0	1950.0
0.2	2400.0	1900.0
0.3	2350.0	1850.0
0.4	2300.0	1800.0
0.5	2250.0	1750.0
0.6	2200.0	1700.0
0.7	2150.0	1650.0
0.8	2100.0	1600.0
0.9	2050.0	1550.0

Fig A1.2: FNPV (α), Case 2

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	246.1	6434.7
0.1	562.9	6136.6
0.2	881.8	5839.6
0.3	1202.6	5543.9
0.4	1525.5	5249.4
0.5	1850.4	4956.1
0.6	2177.5	4664.0
0.7	2506.7	4373.0
0.8	2838.0	4083.2
0.9	3171.5	3794.6
1	3507.1	3507.1

Fig A1.2 : FNPV(α) Case 2



CASE 3.

In (3.3.1) when $A_j(\alpha) = (P_j(\alpha), Q_j(\alpha))$ $0 \le \alpha \le 1$ $\forall$ $j = 0, 1, 2, \ldots, n$

and " r " the return being non–fuzzy and equal throughout the life of the investment.

α	$-3000 + 1000\alpha$	$2000 + 1000\alpha$
0	-3000.0	2000.0
0.1	-2900.0	2100.0
0.2	-2800.0	2200.0
0.3	-2700.0	2300.0
0.4	-2600.0	2400.0
0.5	-2500.0	2500.0
0.6	-2400.0	2600.0
0.7	-2300.0	2700.0
0.8	-2200.0	2800.0
0.9	-2100.0	2900.0
1	-2000.0	3000.0

α	$1000 + 1000\alpha$	$1000 + 500\alpha$
0	1000.0	1000.0
0.1	1100.0	1050.0
0.2	1200.0	1100.0
0.3	1300.0	1150.0
0.4	1400.0	1200.0
0.5	1500.0	1250.0
0.6	1600.0	1300.0
0.7	1700.0	1350.0
0.8	1800.0	1400.0
0.9	1900.0	1450.0
1	2000.0	1500.0

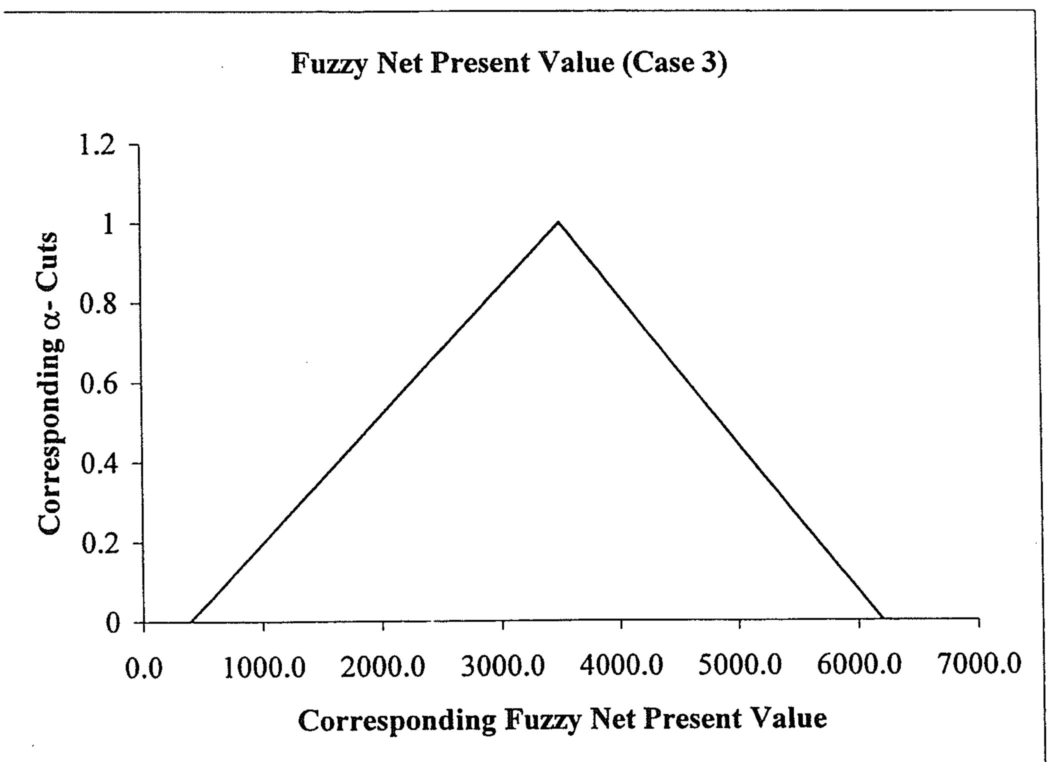
α	$-1000 - 1000\alpha$	$4000 - 1000\alpha$
0	-1000.0	4000.0
0.1	-1100.0	3900.0
0.2	-1200.0	3800.0
0.3	-1300.0	3700.0
0.4	-1400.0	3600.0
0.5	-1500.0	3500.0
0.6	-1600.0	3400.0
0.7	-1700.0	3300.0
0.8	-1800.0	3200.0
0.9	-1900.0	3100.0
1	-2000.0	3000.0

α	$2500 - 500\alpha$	$2000 - 500\alpha$
0	2500.0	2000.0
0.1	2450.0	1950.0
0.2	2400.0	1900.0
0.3	2350.0	1850.0
0.4	2300.0	1800.0
0.5	2250.0	1750.0
0.6	2200.0	1700.0
0.7	2150.0	1650.0
0.8	2100.0	1600.0
0.9	2050.0	1550.0
1	2000.0	1500.0

$r_1 = 1.1025$ $\quad r_2 = 1.1200$ $\quad r_3 = 1.0975$

Fig A1.3: FNPV (α), Case 3

α		
0	395.9	6205.1
0.1	707.1	5935.3
0.2	1018.2	5665.5
0.3	1329.3	5395.7
0.4	1640.4	5125.9
0.5	1951.5	4856.1
0.6	2262.7	4586.3
0.7	2573.8	4316.5
0.8	2884.9	4046.7
0.9	3196.0	3776.9
1	3507.1	3507.1



CASE 4.

This is the case when we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be different but non-fuzzy,.

α	$-3000 + 1000\alpha$	$2000 + 1000\alpha$
0	-3000.0	2000.0
0.1	-2900.0	2100.0
0.2	-2800.0	2200.0
0.3	-2700.0	2300.0
0.4	-2600.0	2400.0
0.5	-2500.0	2500.0
0.6	-2400.0	2600.0
0.7	-2300.0	2700.0
0.8	-2200.0	2800.0
0.9	-2100.0	2900.0
1	-2000.0	3000.0

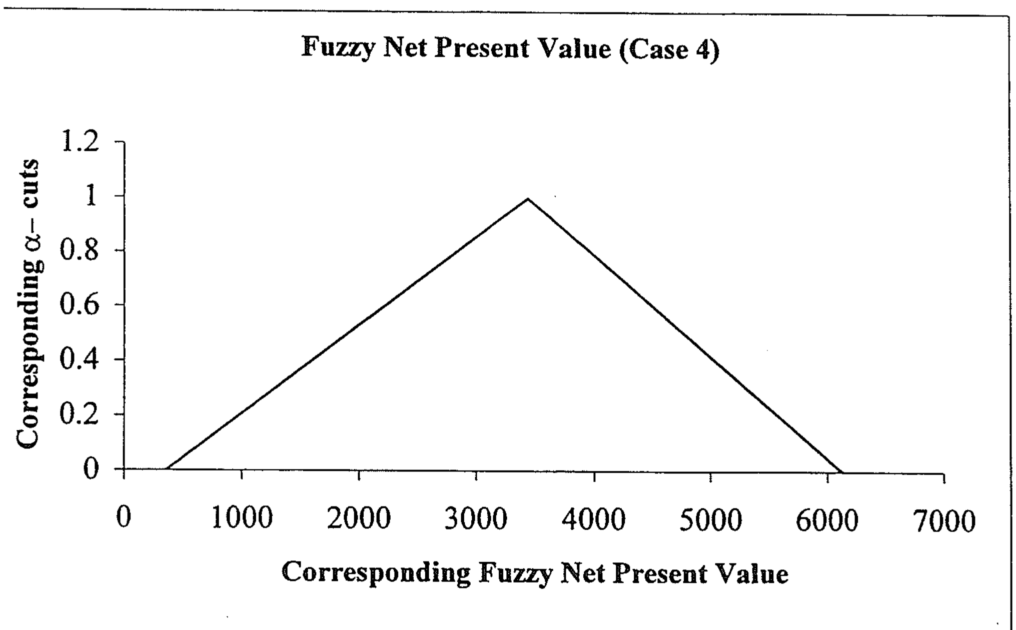
α	$1000 + 1000\alpha$	$1000 + 500\alpha$
0	1000.0	1000.0
0.1	1100.0	1050.0
0.2	1200.0	1100.0
0.3	1300.0	1150.0
0.4	1400.0	1200.0
0.5	1500.0	1250.0
0.6	1600.0	1300.0
0.7	1700.0	1350.0
0.8	1800.0	1400.0
0.9	1900.0	1450.0
1	2000.0	1500.0

α	$-1000 - 1000\alpha$	$4000 - 1000\alpha$
0	-1000.0	4000.0
0.1	-1100.0	3900.0
0.2	-1200.0	3800.0
0.3	-1300.0	3700.0
0.4	-1400.0	3600.0
0.5	-1500.0	3500.0
0.6	-1600.0	3400.0
0.7	-1700.0	3300.0
0.8	-1800.0	3200.0
0.9	-1900.0	3100.0
1	-2000.0	3000.0

α	$2500 - 500\alpha$	$2000 - 500\alpha$
0	2500.0	2000.0
0.1	2450.0	1950.0
0.2	2400.0	1900.0
0.3	2350.0	1850.0
0.4	2300.0	1800.0
0.5	2250.0	1750.0
0.6	2200.0	1700.0
0.7	2150.0	1650.0
0.8	2100.0	1600.0
0.9	2050.0	1550.0
1	2000.0	1500.0

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	361.8	6128.5
0.1	670.4	5860.5
0.2	979.0	5592.4
0.3	1287.6	5324.3
0.4	1596.1	5056.2
0.5	1904.7	4788.1
0.6	2213.3	4520.0
0.7	2521.9	4251.9
0.8	2830.5	3983.8
0.9	3139.1	3715.7
1	3447.6	3447.6

Fig A1.4: FNPV (α), Case 4



CASE 5.

This is the case when $A_j(\alpha) = A_j \quad \forall\ j = 0, 1, 2, \ldots, n$ and non fuzzy but the discount rate is fuzzy and equal throughout the life of the investment.

α	$\frac{1}{(1.13 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)}$
0	0.8850	0.9259
0.1	0.8873	0.9242
0.2	0.8897	0.9225
0.3	0.8921	0.9208
0.4	0.8945	0.9191
0.5	0.8969	0.9174
0.6	0.8993	0.9158
0.7	0.9017	0.9141
0.8	0.9042	0.9124
0.9	0.9066	0.9107
1	0.9091	0.9091

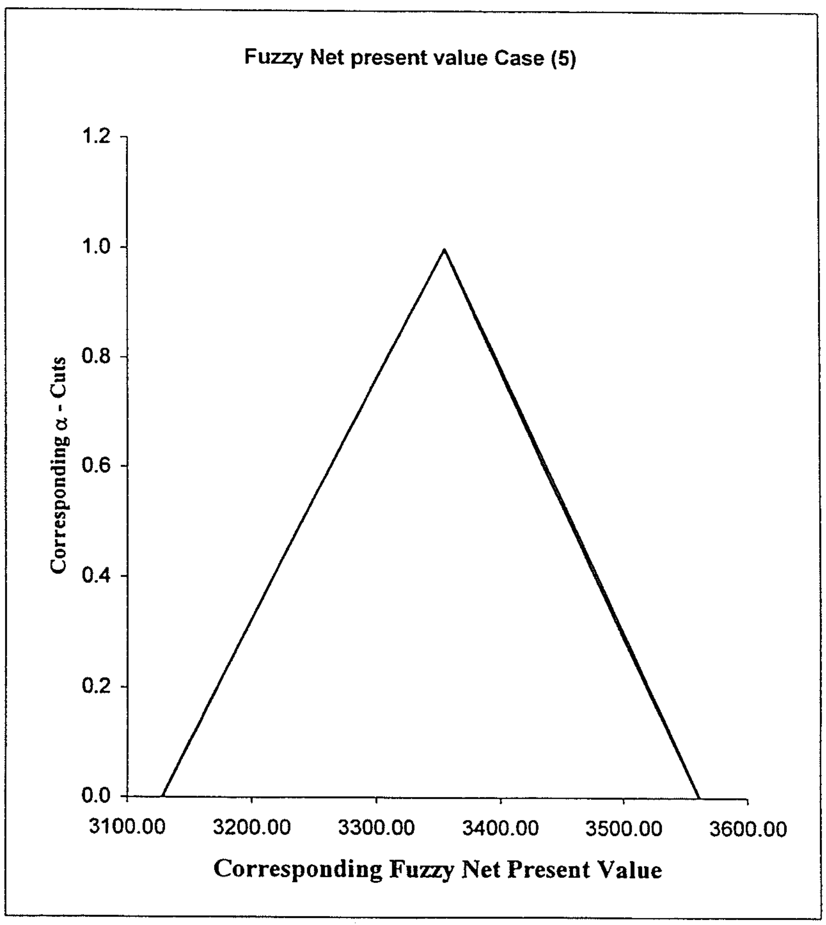
α	$\frac{1}{(1.13 - 0.03\alpha)(1.15 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)(1.09 + 0.02\alpha)}$
0	0.769527	0.8495
0.1	0.773593	0.8456
0.2	0.777692	0.8417
0.3	0.781824	0.8379
0.4	0.785988	0.8340
0.5	0.790186	0.8303
0.6	0.794417	0.8265
0.7	0.798683	0.8228
0.8	0.802983	0.8190
0.9	0.807318	0.8154
1	0.811688	0.8117

α	$\frac{1}{(1.13 - 0.03\alpha)(1.15 - 0.03\alpha)(1.12 - 0.02\alpha)}$
0	0.68708
0.1	0.69194
0.2	0.69686
0.3	0.70182
0.4	0.70682
0.5	0.71188
0.6	0.71698
0.7	0.72214
0.8	0.72734
0.9	0.73259
1	0.73790

α	$\frac{1}{(1.08 + 0.02\alpha)(1.09 + 0.02\alpha)(1.07 + 0.03\alpha)}$
0	0.79390
0.1	0.78805
0.2	0.78225
0.3	0.77652
0.4	0.77084
0.5	0.76521
0.6	0.75964
0.7	0.75413
0.8	0.74866
0.9	0.74326
1	0.73790

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	3128.3	3561.4
0.1	3150.3	3540.2
0.2	3172.5	3519.1
0.3	3194.8	3498.2
0.4	3217.3	3477.4
0.5	3240.0	3456.8
0.6	3262.8	3436.4
0.7	3285.9	3416.1
0.8	3309.1	3395.9
0.9	3332.5	3375.9
1	3356.0	3356.0

Fig A1.5: FNPV (α), Case 5



CASE 6.

This is the case when we take the discount rate r_j , $j = 1, 2, \ldots, n$ to be equal and fuzzy, and A_j , $j = 0, 1, 2, \ldots, n$ different and non fuzzy.

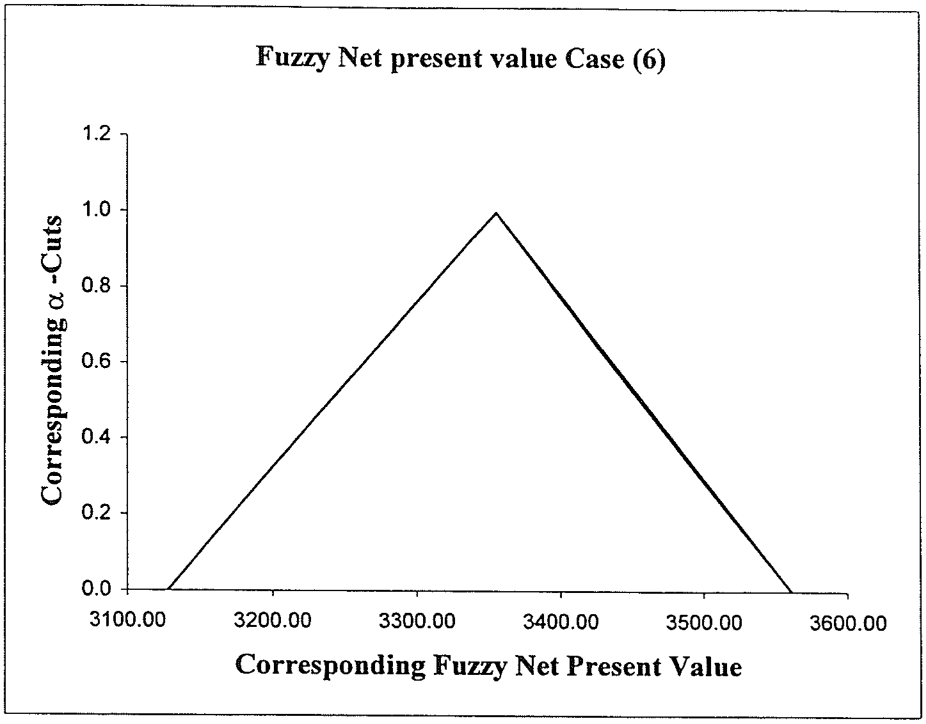
α	$\frac{1}{(1.13 - 0.03\alpha)}$	$\frac{1}{(1.08 + 0.02\alpha)}$
0	0.8850	0.9259
0.1	0.8873	0.9242
0.2	0.8897	0.9225
0.3	0.8921	0.9208
0.4	0.8945	0.9191
0.5	0.8969	0.9174
0.6	0.8993	0.9158
0.7	0.9017	0.9141
0.8	0.9042	0.9124
0.9	0.9066	0.9107
1	0.9091	0.9091

α	$\frac{1}{(1.13 - 0.03\alpha)^2}$	$\frac{1}{(1.08 + 0.02\alpha)^2}$
0	0.7831	0.8573
0.1	0.7873	0.8542
0.2	0.7915	0.8510
0.3	0.7958	0.8479
0.4	0.8000	0.8448
0.5	0.8044	0.8417
0.6	0.8087	0.8386
0.7	0.8131	0.8355
0.8	0.8175	0.8325
0.9	0.8220	0.8295
1	0.8264	0.8264

α	$\frac{1}{(1.13 - 0.03\alpha)^3}$	$\frac{1}{(1.08 + 0.02\alpha)^3}$
0	0.6931	0.7938
0.1	0.6986	0.7894
0.2	0.7042	0.7851
0.3	0.7099	0.7807
0.4	0.7156	0.7764
0.5	0.7214	0.7722
0.6	0.7273	0.7679
0.7	0.7332	0.7637
0.8	0.7392	0.7596
0.9	0.7452	0.7554
1	0.7513	0.7513

α	$FNPV_L(\alpha)$	$FNPV_R(\alpha)$
0	3162.8	3576.0
0.1	3186.1	3558.4
0.2	3209.5	3540.8
0.3	3233.1	3523.3
0.4	3256.9	3506.0
0.5	3280.9	3488.7
0.6	3305.0	3471.5
0.7	3329.4	3454.5
0.8	3354.0	3437.5
0.9	3378.8	3420.6
1	3403.8	3403.8

Fig A1.6: FNPV (α), Case 6



APPENDIX 2

Case 2 : Crisp

(b) Output For β the Crisp Systematic Risk Parameter

Computation of β, the Systematic Risk Parameter.

$$\beta = \frac{\sum_{i=1}^{n} (R_i - \bar{R}_i)) (R_{mi} - \bar{R}_{mi})}{\sum_{i=1}^{n} (R_{mi} - \bar{R}_{mi})^2}, \qquad i = 1, 2, \ldots, n$$

where,

n = the number of periods under consideration,

R_i = Represent the actual return through different periods i of time, $i = 1, 2, \ldots, n$,

β = Beta represent the amount of systematic risk,

R_{mi} = Represent the market return on the investment during period i, $i = 1, 2, \ldots, n$,

$\bar{R}_i$ = Represent the mean value of the actual return through different periods i of time, $i = 1, 2, \ldots, n$,

$\bar{R}_{mi}$ = Represent the mean value of the market return on the investment during period i, $i = 1, 2, \ldots, n$,

Observed stock returns and market returns for period i =1 to i = 60

Period	Rm	R_i
1	-4.6547	-3.4803
2	2.6716	-9.1346
3	4.5772	-2.6455
4	-0.7914	-8.1522
5	3.9531	14.497
6	1.7648	-0.2584
7	1.9416	-8.8083
8	-2.1311	-11.3636
9	0.2889	0.3205
10	-1.5584	5.1118
11	4.5304	41.6413
12	1.1233	-3.0043
13	5.4076	21.9027
14	-0.6986	-2.1779

15	0.7522	13.5436
16	3.5334	14.7712
17	1.9417	11.6173
18	-3.8565	-8.9286
19	-2.2783	-33.1092
20	4.3472	31.742
21	2.8705	9.4723
22	5.8164	-1.2776
23	7.4632	-5.1765
24	-1.4899	-3.598
25	3.0800	20.3346
26	0.7899	-7.0588
27	-4.9956	-9.0909
28	2.1608	12.9114
29	6.7846	23.3184
30	0.8715	7.6364
31	6.8338	21.6216
32	-3.8660	-12.1528
33	6.4799	30.751
34	-2.8106	-9.1294
35	-4.8168	-19.2282
36	2.8661	-17.5453
37	0.0113	-24.0759
38	5.8549	-12.2368
39	6.5705	13.1934
40	1.4089	10.9934
41	-0.9812	-1.6706
42	-2.9367	-14.6845
43	-5.9110	-10.0996
44	-20.2082	-6.8038
45	1.5081	-6.7912
46	10.5833	15.4827
47	2.1840	41.9558
48	2.2395	3.5556
49	3.7561	14.3777
50	-6.1946	-30.863
51	4.5163	27.5441
52	6.3189	14.8936
53	-2.4648	-23.1481
54	2.4594	1.2048
55	1.0123	-4.7619
56	-1.5566	2.5
57	-0.1878	-7.439
58	4.2902	-24.5059
59	3.6797	16.2304
60	11.7349	-0.7508

Mean	1.276	1.666
St Dev	4.825	16.589

β	1	1.663

APPENDIX 3

Case 2 : Fuzzy

(b) Output For β the Fuzzy Systematic Risk Parameter

Computation of $[R_{mi}(\alpha) - R_{mi}(\alpha)]$, the difference between the α - cuts for market returns and the α-cuts for the average market returns.

		Column A	Column B	$[R_{mi}(\alpha) - \overline{R_{mi}}(\alpha)]$ Column C
Period	CrispMonthly market returns	Fuzzy triangular representation of the market returns	α - Cut for market returns	α - Cut for the difference between the market returns and the α - cuts for the average market returns.
1	–4.655	$(-4.855,\ -4.455,\ -4.847)$	$(-4.855 + 0.2\alpha,\ -4.4847 - 0.2\alpha)$	$(-7.020 + 1.089\alpha,\ -4.4847 - 1.038\alpha)$
2	2.672	$(1.672, 2.672, 3.672)$	$(1.672 + \alpha,\ 3.62 - \alpha)$	$(-0.494\ 1.889\alpha,\ 3.233 - 1.838\alpha)$
3	4.577	$(4.077,\ -4.577,\ -5.577)$	$(4.077 + 0.5\alpha,\ -5.577 - \alpha)$	$(1.912 + 1.389\alpha,\ -5.139 - 1.838\alpha)$
4	–0.791	$(-0.991,\ -0.791,\ 0.209)$	$(-0.991 + 1\alpha,\ -0.209 - \alpha)$	$(-3.157 + 1.089\alpha,\ -0.230 - 1.838\alpha)$
5	3.953	$(3.753,\ 3.953,\ 4.953)$	$(3.753 + 0.2\alpha,\ 4.593 - \alpha)$	$(1.588 + 1.089\alpha,\ -4.515 - 1.838\alpha)$
6	1.765	$(1.515, 1.765, 2.125)$	$(1.515 + 0.25\alpha,\ 2.125 - 0.36\alpha)$	$(-0.650 + 1.139\alpha,\ 1.687 - 1.198\alpha)$
7	1.942	$(0.942,\ 1.942,\ 2.942)$	$(0.942 + \alpha,\ 2.942 - \alpha)$	$(-1.224 + 1.889\alpha,\ 2.503 - 1.838\alpha)$
8	–2.131	$(-3.131,\ -2.131,\ -1.131)$	$(-3.131 + \alpha,\ -1.231 - \alpha)$	$(-5.296 + 1.889\alpha,\ -1.569 - 1.838\alpha)$
9	0.289	$(-0.711,\ 0.289,\ 1.289)$	$(-0.711 + \alpha,\ 1.289 - \alpha)$	$(-2.876 + 1.889\alpha,\ 0.851 - 1.838\alpha)$
10	–1.558	$(-0.275,\ -1.558,\ -0.558)$	$(-0.275 + 1.2\alpha,\ -0.558 - 1\alpha)$	$(-4.924 + 2.089\alpha,\ -0.997 - 1.838\alpha)$
11	4.530	$(-3.530,\ 4.530,\ 5.530)$	$(-3.530 + \alpha,\ 5.530 - \alpha)$	$(1.365 + 1.889\alpha,\ 5.092 - 1.838\alpha)$
12	1.123	$(0.123,\ 1.123,\ 2.123)$	$(0.123 + \alpha,\ 2.123 - \alpha$	$(-2.042 + 1.889\alpha,\ 1.685 - 1.838\alpha$
13	5.408	$(4.408,\ 5.408, 6.408)$	$(4.408 + \alpha,\ 6.408 - \alpha)$	$(2.242 + 1.889\alpha,\ 5.969 - 1.838\alpha)$
14	–0.699	$(-1.199,\ -0.699,\ 0.301)$	$(-1.199 + 0.5\alpha,\ 0.301 - \alpha)$	$(-3.364 + 1.889\alpha,\ -0.137 - 1.838\alpha)$
15	0.752	$(-0.248,\ 0.752, 1.752)$	$(-0.248 + \alpha,\ 1.752 - \alpha)$	$(-2.413 + 1.889\alpha,\ 1.314 - 1.838\alpha)$
16	3.533	$(3.508,\ 3.533,\ 3.558)$	$(3.508 + 0.25\alpha,\ 3.558 - 0.25\alpha)$	$(1.343 + 0.914\alpha,\ 3.120 - 0.863\alpha)$
17	1.942	$(0.942,\ 1.942,\ 2.942)$	$(0.942 + \alpha,\ 2.942 - \alpha)$	$(-1.224 + 1.889\alpha,\ 2.503 - 1.838\alpha)$
18	–3.857	$(-4.457,\ -3.857,\ -2.857)$	$(-4.457 + 0.6\alpha,\ -2.857 - \alpha)$	$(-6.622 + 1.889\alpha,\ -3.295 - 1.838\alpha)$
19	–2.278	$(-3.278,\ -2.278,\ -1.278)$	$(-3.278 + \alpha,\ -1.278 - \alpha)$	$(-5.444 + 1.889\alpha,\ -1.717 - 1.838\alpha)$
20	4.347	$(3.347,\ 4.347,\ 5.347)$	$(3.347 + \alpha,\ 5.347 - \alpha)$	$(1.182 + 1.889\alpha,\ 4.909 - 1.838\alpha)$
21	2.870	$(2.370,\ 2.870, 3.870)$	$(2.370 + 0.5\alpha,\ 3.870 - \alpha)$	$(0.205 + 1.389\alpha,\ 3.432 - 1.838\alpha)$
22	5.816	$(3.816,\ 5.816,\ 6.816)$	$(3.816 + 2\alpha,\ 6.816 - \alpha)$	$(1.651 + 2.889\alpha,\ 6.378 - 1.838\alpha)$
23	7.463	$(6.963,\ 7.463, 7.963)$	$(6.963 + 0.5\alpha,\ 7.963 - 0.5\alpha)$	$(4.798 + 1.389\alpha,\ 7.525 - 1.338\alpha)$
24	–1.490	$(-2.490,\ -1.490,\ -0.490)$	$(-2.490 + \alpha,\ -0.490 - \alpha)$	$(-4.655 + 1.889\alpha,\ -0.928 - 1.838\alpha)$

27	−4.996	(−5.996, − 4.996, − 3.996)	(−5.996+ α, − 3.996−α)	(−8.161+ 1.889α, − 4.434−1.838α)
28	2.161	(1.661, 2.161, 2.661)	(1.661+0.5 α, 2.661−0.5α)	(0.504+1.389α, 2.223−1.338α)
29	6.785	(5.785, 6.785, 7.785)	(5.785+ α, 7.785−α)	(3.619+1.889α, 7.346−1.838α)
30	0.871	(−0.129, 0.871, 1.871)	(−0.129+ α, 1.871−α)	(−2.294+ 1.889α, 1.433−1.838α)
31	6.834	(5.834, 6.834, 7.834)	(5.834+ α, 7.834−α)	(3.669+ 1.889α, 7.396−1.838α)
32	−3.866	(− 4.366, − 3.866, − 3.366)	(− 4.366+ 0.5α, − 3.366−0.5α)	(− 6.531+ 1.389α, − 3.804−1.838α)
33	6.480	(5.840, 6.480, 7.480)	(5.840+ α, 7.480−α)	(3.315+ 1.889α, 7.042−1.838α)
34	−2.811	(− 3.811, − 2.811, − 1.811)	(− 3.811+ α, − 1.811−α)	(− 5.976+ 1.889α, − 2.249−1.838α)
35	−4.817	(−5.817, − 4.817, − 3.817)	(−5.817+ α, − 3.817−α)	(−7.982+1.889α, − 4.255−1.838α)
36	2.866	(1.866, 2.866, 3.866)	(1.866+ α, 3.866−α)	(−0.299+ 1.889α, 3.428−1.838α)
37	0.011	(−0.989, 0.011, 1.011)	(−0.989+ α, 1.011−α)	(−3.154+ 1.889α, 0.573−1.838α)
38	5.855	(4.855, 5.855, 6.855)	(4.855+ α, 6.855−α)	(2.690+ 1.889α, 6.417−1.838α)
39	6.570	(6.270, 6.570, 6.970)	(6.270+ 0.3α, 6.970−0.4α)	(4.105+ 1.189α, 6.532−1.238α)
40	1.409	(0.4089, 1.409, 2.409)	(0.4089+ α, 2.409−α)	(−1.756+ 1.889α, 1.971−1.838α)
41	−0.981	(− 1.281, − 0.981, − 0.681)	(− 1.281+ 0.3α, − 0.681− 0.3α)	(− 3.446+ 1.189α, − 1.119− 1.138α)
42	−2.937	(− 3.9367, − 2.937, − 1.937)	(− 3.9367+ α, − 1.937− α)	(− 6.102+ 1.889α, − 2.375− 1.838α)
43	−5.911	(− 6.9110, − 5.911, − 4.911)	(− 6.9110+ α, − 4.911− α)	(− 9.076+ 1.889α, − 5.349− 1.838α)
44	−20.208	(− 20.708, − 20.208, − 19.208)	(− 20.708+ 0.5α, − 19.208− α)	(− 22.873+ 1.389α, − 19.646− 1.838α)
45	1.508	(1.008, 1.5081, − 4.455)	(1.008+0.5 α, − 4.455− 0.5α)	(−1.157+1.389α, − 1.570− 1.138α)
46	10.583	(9.583, 10.5833, − 4.455)	(9.583+ α, − 4.455− α)	(7.418+ 1.889α, − 11.145− 1.838α)
47	2.184	(1.184, 2.1840, − 4.455)	(1.184+ α, − 4.455− α)	(−0.981+ 1.889α, 2.746− 1.838α)
48	2.239	(1.2395, 2.239, 3.239)	(1.2395+ α, 3.239− α)	(−0.926+ 1.889α, 2.801−1.838 α)
49	3.756	(2.756, 3.7561, 4.756)	(2.756+ α, 4.756− α)	(0.591+ 1.889α, 4.318−1.838 α)
50	−6.195	(−6.215, − 6.1946, − 6.115)	(−6.215+ 0.02α, − 6.115−0.04 α)	(−8.380+ 0.909α, − 6.596−0.878 α)
51	4.516	(3.561, 4.5163, 5.516)	(3.561+ α, 5.516− α)	(1.351+ α, 5.078− 1.838α)
52	6.319	(5.319, 6.3189, 7.319)	(5.319+ α, 7.319− α)	(3.154+ 1.889α, 6.881−1.838α)
53	−2.465	(− 3.465, − 2.465, − 1.456)	(− 3.465+ α, − 1.456− α)	(5.630+ 1.889α, − 1.903−1.838α)
54	2.459	(1.459, 2.459, − 3.459)	(1.459+ α, − 3.459− α)	(−0.706+1.889α, − 3.021− 1.838α)
55	1.012	(0.012, 1.0123, 2.012)	(0.012+ α, 2.012− α)	(−2.153+ 1.889α, 1.574− 1.838α)
56	−1.557	(− 2.5566, − 1.557, − 0.557)	(− 2.5566+ α, − 0.557− α)	(− 4.722+ 1.889α, − 0.995− 1.838α)
57	−0.188	(−1.188, − 0.188, 0.812)	(−1.188+ α, 0.812− α)	(−3.353+ 1.889α, 0.374− 1.838α)
58	4.290	(3.290, 4.2902, 5.290)	(3.290+ α, 5.290− α)	(1.125+ 1.889α, 4.852− 1.838α)
59	3.680	(2.680, 3.680, 4.680)	(2.680+ α, 4.680− α)	(0.515+ 1.889α, 4.242− 1.838α)
60	11.735	(10.735, 11.7349, 12.735)	(10.735+ α, 12.735− α)	(8.570+ 1.889α, 12.297−1.838 α)

Fuzzy Number Analysis

Computation of $[R_{mi}(\alpha)-R_{mi}(\alpha)]^2$, the square of the difference between the α-cuts for market returns and the α-cuts for the average market returns.

$[\ R_{mi}(\alpha) - R_{mi}(\alpha)\ ]^2$

Months	α - Cut for the square of the difference between the market returns and the average market returns.
1	$(49.279 - 15.286\alpha + 1.185\alpha^2 , 23.941 + 10.160\alpha + 1.077\alpha^2)$
2	$(0.244 + 1.865\alpha + 3.567\alpha^2 , 10.454 - 11.887\alpha + 3.379\alpha^2)$
3	$(3.656 + 5.310\alpha + 1.929\alpha^2 , 26.409 - 18.893\alpha + 3.379\alpha^2)$
4	$(9.965 + 6.874\alpha + 1.185\alpha^2 , 0.053 + 0.844\alpha + 3.379\alpha^2)$
5	$(2.521 + 3.457\alpha + 1.185\alpha^2 , 20.384 - 16.599\alpha + 3.379\alpha^2)$
6	$(0.423 + 1.481\alpha + 1.297\alpha^2 , 2.845 - 4.042\alpha + 1.435\alpha^2)$
7	$(1.497 + 4.622\alpha + 3.567\alpha^2 , 6.267 - 9.204\alpha + 3.379\alpha^2)$
8	$(28.051 + 20.007\alpha + 3.567\alpha^2 , 2.463 + 5.770\alpha + 3.379\alpha^2)$
9	$(8.273 + 10.865\alpha + 3.567\alpha^2 , 0.724 - 3.128\alpha + 3.379\alpha^2)$
10	$(24.242 + 20.568\alpha + 4.363\alpha^2 , 0.993 + 3.664\alpha + 3.379\alpha^2)$
11	$(1.864 + 5.157\alpha + 3.567\alpha^2 , 25.931 - 18.722\alpha + 3.379\alpha^2)$
12	$(4.169 + 7.713\alpha + 3.567\alpha^2 , 2.840 - 6.195\alpha + 3.379\alpha^2)$
13	$(5.028 + 8.471\alpha + 3.567\alpha^2 , 35.633 - 21.946\alpha + 3.379\alpha^2)$
14	$(11.315 + 9.343\alpha + 1.929\alpha^2 , 0.019 + 0.503\alpha + 3.379\alpha^2)$
15	$(5.823 + 9.115\alpha + 3.567\alpha^2 , 1.726 - 4.831\alpha + 3.379\alpha^2)$
16	$(1.804 + 2.455\alpha + 0.835\alpha^2 , 9.736 - 5.387\alpha + 0.745\alpha^2)$
17	$(1.497 + 4.622\alpha + 3.567\alpha^2 , 6.267 - 9.204\alpha + 3.379\alpha^2)$

18	$(45.648 + 19.710\alpha + 2.210\alpha^2, 10.650 + 12.115\alpha + 3.379\alpha^2)$
19	$(29.632 + 20.563\alpha + 3.567\alpha^2, 2.947 + 6.311\alpha + 3.379\alpha^2)$
20	$(1.397 + 4.465\alpha + 3.567\alpha^2, 24.098 - 18.048\alpha + 3.379\alpha^2)$
21	$(0.042 + 0.570\alpha + 1.929\alpha^2, 11.780 - 12.619\alpha + 3.379\alpha^2)$
22	$(2.726 + 9.540\alpha + 8.345\alpha^2, 40.681 - 23.449\alpha + 3.379\alpha^2)$
23	$(23.021 + 13.326\alpha + 1.929\alpha^2, 56.625 - 20.141\alpha + 1.790\alpha^2)$
24	$(21.670 + 17.585\alpha + 3.567\alpha^2, 0.861 + 3.412\alpha + 3.379\alpha^2)$
25	$(0.007 + 0.322\alpha + 3.567\alpha^2, 13.262 - 13.389\alpha + 3.379\alpha^2)$
26	$(5.642 + 8.973\alpha + 3.567\alpha^2, 1.827 - 4.969\alpha + 3.379\alpha^2)$
27	$(66.599 + 30.827\alpha + 3.567\alpha^2, 19.659 + 16.301\alpha + 3.379\alpha^2)$
28	$(0.254 + 1.401\alpha + 1.929\alpha^2, 4.940 - 5.949\alpha + 1.790\alpha^2)$
29	$(13.100 + 13.672\alpha + 3.567\alpha^2, 53.969 - 27.009\alpha + 3.379\alpha^2)$
30	$(5.261 + 8.665\alpha + 3.567\alpha^2, 2.054 - 5.269\alpha + 3.379\alpha^2)$
31	$(13.458 + 13.858\alpha + 3.567\alpha^2, 54.694 - 27.190\alpha + 3.379\alpha^2)$
32	$(42.657 + 18.140\alpha + 1.929\alpha^2, 14.472 + 10.182\alpha + 1.709\alpha^2)$
33	$(10.987 + 12.521\alpha + 3.567\alpha^2, 49.586 - 25.889\alpha + 3.379\alpha^2)$
34	$(35.710 + 22.574\alpha + 3.567\alpha^2, 5.057 + 8.268\alpha + 3.379\alpha^2)$
35	$(63.712 + 30.152\alpha + 3.567\alpha^2, 18.105 + 15.643\alpha + 3.379\alpha^2)$
36	$(0.090 + 1.130\alpha + 3.567\alpha^2, 11.750 - 12.602\alpha + 3.379\alpha^2)$
37	$(9.947 + 11.914\alpha + 3.567\alpha^2, 0.328 - 2.107\alpha + 3.379\alpha^2)$
38	$(7.234 + 10.160\alpha + 3.567\alpha^2, 41.174 - 23.591\alpha + 3.379\alpha^2)$
39	$(16.853 + 9.760\alpha + 1.413\alpha^2, 42.670 - 16.177\alpha + 1.533\alpha^2)$
40	$(3.085 + 6.635\alpha + 3.567\alpha^2, 3.883 - 7.245\alpha + 3.379\alpha^2)$
41	$(11.878 + 8.194\alpha + 1.413\alpha^2, 1.253 + 2.548\alpha + 1.295\alpha^2)$
42	$(37.234 + 23.050\alpha + 3.567\alpha^2, 5.640 + 8.731\alpha + 3.379\alpha^2)$
43	$(82.379 + 34.286\alpha + 3.567\alpha^2, 28.615 + 19.667\alpha + 3.379\alpha^2)$
44	$(523.196 + 63.531\alpha + 1.929\alpha^2, 385.984 + 72.230\alpha + 3.379\alpha^2)$
45	$(1.339 + 3.214\alpha + 1.929\alpha^2, 2.465 - 4.202\alpha + 1.790\alpha^2)$
46	$(55.028 + 28.022\alpha + 3.567\alpha^2, 124.213 - 40.975\alpha + 3.379\alpha^2)$

[illegible]	[illegible]
48	$(0.857+ 3.497\alpha+ 3.567\alpha^2\ ,\ 7.847 - 10.299\alpha+3.379\alpha^2)$
49	$(0.349 + 2.232\alpha+ 3.567\alpha^2\ ,\ 18.644 - 15.875\alpha+ 3.379\alpha^2)$
50	$(70.222 + 15.230\alpha+ 0.826\alpha^2\ ,\ 43.466 + 11.580\alpha+ 0.771\alpha^2)$
51	$(1.825 + 5.104\alpha+ 3.567\alpha^2\ ,\ 25.787 - 18.669\alpha+3.379\alpha^2)$
52	$(9.946 + 11.913\alpha+ 3.567\alpha^2\ ,\ 47.344 - 25.297\alpha+ 3.379\alpha^2)$
53	$(31.698 + 21.268\alpha+ 3.567\alpha^2\ ,\ 3.622 + 6.997\alpha+3.379\alpha^2)$
54	$(0.498 + 2.666\alpha+3.567\alpha^2\ ,\ 9.128 - 11.107\alpha+ 3.379\alpha^2)$
55	$(4.635 + 8.133\alpha+ 3.567\alpha^2\ ,\ 2.478 - 5.787\alpha+ 3.379\alpha^2)$
56	$(22.295+ 17.837\alpha+ 3.567\alpha^2\ ,\ 0.990 + 3.657\alpha+ 3.379\alpha^2)$
57	$(11.243 + 12.666\alpha+ 3.567\alpha^2\ ,\ 0.140 - 1.375\alpha+ 3.379\alpha^2)$
58	$(1.266+ 4.250\alpha+ 3.567\alpha^2\ ,\ 23.542 - 17.838\alpha+ 3.379\alpha^2)$
59	$(0.265 + 1.944\alpha+ 3.567\alpha^2\ ,\ 17.990 - 15.594\alpha+ 3.379\alpha^2)$
60	$(- 7.020 + 32.372\alpha+ 3.567\alpha^2\ ,\ 151.207 - 45.209\alpha+ 3.379\alpha^2)$

Computation of $[R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]^2$, the square of the difference between the α-cuts for market returns and the α-cuts for the average market returns.

$$[R_{mi}(\alpha) - \bar{R}_{mi}(\alpha)]^2 = (1517.138 - 329.679\alpha + 185.955\alpha^2,\ 1569.852 - 379.418\alpha + 182.981\alpha^2)$$

Computation of $[R_i(\alpha) - R_i(\alpha)]$, the square of the difference between the α-cuts for stocks returns and the α-cuts for the average stocks returns.

				$[R_i(\alpha) - \bar{R}_i(\alpha)]$
		Column A		Column B
Months	CrispMonthly stock returns	Fuzzy triangular representation of the stock returns	α - Cut for stock returns	α - Cut for the difference between the stock returns and the α - cuts for the average stock returns.
1	−3.4803	(−8.4803, −3.4803, 2.0197)	(−8.4803+ 5α, 2.0197 −5.5α)	(− 11.723 + 6.577α, −1.462 −6.608α)
2	−9.1346	(−10.135, −9.1346, −7.6346)	(−10.135+ α, −7.6346 −α)	(−13.377+ 2.577α, −8.192 −2.608α)
3	−2.6455	(−3.646,−2.6455, −1.1455)	(−3.646+ α,−1.1455 −α)	(−6.888 + 2.577α, − 1.703 −2.608α)
4	−8.1522	(−9.152,−8.1522, −6.6522)	(−9.152+ α, −6.6522 −α)	(−12.395 +2.577 α, − 7.210 −2.608α)
5	14.497	(13.497,14.497, 15.997)	(13.497+ α,15.997 −α)	(10.254 + 2.577α, 15.439 −2.608α)
6	−0.2584	(−1.258,−0.2584, 1.2416)	(−1.258+ α, 1.2416 −α)	(−4.501 +2.577 α, 0.684 −2.608α)
7	−8.8083	(−9.808,−8.8083, −7.3083)	(−9.808+ α, −7.3083 −α)	(−13.051 +2.577 α, −7.866 −2.608α)
8	−11.3636	(−12.364,−11.3636,−9.8636)	(−12.364+ α,−9.8636 −α)	(− 15.606+ 2.577α, −10.421 −2.608α)
9	0.3205	(−0.680,0.3205,1.3205)	(−0.680+ α, 1.3205 −α)	(− 3.922 + 2.577α, 0.763 −2.608α)
10	5.1118	(4.1118,5.1118,6.6118)	(4.1118+ α, 6.6118 −α)	(0.869 +2.577 α, 6.054−2.608α)
11	41.6413	(40.6413,41.6413,43.1413)	(40.6413+ α,43.1413 −α)	(37.398+ 2.577α, 42.583−2.608α)
12	−3.0043	(−4.0043,−3.0043,−1.5043)	(−4.0043+ α,−1.5043 −α)	(−7.247+ 2.577α, −2.062−2.608α
13	21.9027	(20.9027,21.9027,23.4027)	(20.9027+ α, 23.4027 −α)	(17.660+ 2.577α, 22.845− 2.608α)
14	−2.1779	(−3.1779,−2.1779, −0.6779)	(−3.1779+ α, −0.6779 −α)	(−6.421+ 2.577α, −1.236−2.608α)
15	13.5436	(12.5436,13.5436, 15.0436)	(12.5436+ α, 15.0436 −α)	(9.301+ 2.577α, 14.486−2.608α)
16	14.7712	(13.7712,14.7712,16.2712)	(13.7712+ α,16.2712 −α)	(10.528+ 2.577α, 15.713−2.608 α)
17	11.6173	(10.6173,11.6173, 13.1173)	(10.6173+ α, 13.1173 −α)	(7.374+ 2.577α, 12.559−2.608α)
18	−8.9286	(−9.9286,−8.9286, −7.4286)	(−9.9286+ α,, −7.4286 −α)	(− 13.171+ 2.577α, − 7.986−2.608α)
19	−33.1092	(−34.1092,−33.1092, −31.6092)	(−34.1092+ α, −31.6092 −α)	(− 37.352+ 2.577α, − 32.167−2.608α)
20	31.742	(30.742,31.742, 33.242)	(30.742+ α 33.242 −α)	(27.499+ 2.577α, 32.684−2.608α)
21	9.4723	(8.4723,9.4723, 10.9723)	(8.4723+ α,10.9723 −α)	(5.229+ 2.577α, 10.414−2.608α)
22	−1.2776	(−2.2776,−1.2776, 0.2224)	(−2.2776+ α, 0.2224 −α)	(−5.520+ 2.577α, −0.335−2.608α)
23	−5.1765	(−5.3765,−5.1765, −4.8765)	(−5.3765+ α, −4.8765 −α)	(−8.619+ 2.577α, −5.434 −2.608α)
24	−3.598	(−8.598,−3.598,1.402)	(−8.598+ α,−1.402 −α)	(− 11.841+2.577α, 0.844−2.608α)
25	20.3346	(19.3346,20.3346, 21.834)	(19.3346+ α, 21.834 −α)	(16.092+ 2.577α, 21.277−2.608α)

26	−7.0588	(−8.0588,−7.0588, −5.5588)	(−8.0588+ α, −5.5588 −α)	(− 11.302+ 2.577α, −6.117−2.608α)
27	−9.0909	(−10.0909,−9.0909,−7.5909)	(−10.0909+ α,−7.5909 −α)	(−13.334+ 2.577α, − 8.149−2.608α)
28	12.9114	(11.9114,12.9114,14.4114)	(11.9114+ α,14.4114 −α)	(8.669+2.577 α, 12.854−2.608α)
29	23.3184	(23.0184,23.3184,23.6184)	(23.0184+ α,23.6184 −α)	(19.776+ 1.877α, 23.0612.608−α)
30	7.6364	(6.6364,7.6364, 9.1364)	(6.6364+ α,9.1364 −α)	(3.394+ 2.577α, 8.579−2.608α)
31	21.6216	(20.6216,21.6216,23.1216)	(20.6216+ α,23.1216 −α)	(17.379+ 2.577α, 22.564−2.608α)
32	−12.1528	(−13.1528,−12.1528,−10.6528)	(−13.1528+ α,−10.6528 −α)	(− 16.396+ 2.577α, − 11.211−2.608α)
33	30.751	(29.751,30.751,32.251)	(29.751+ α,32.251 −α)	(26.508+ 2.577α, 31.693−2.608α)
34	−9.1294	(−10.1294,−9.1294,−7.6294)	(−10.1294+ α,−7.6294 −α)	(− 13.372+2.577 α, − 8.187−2.608α)
35	−19.2282	(−20.2282,−19.2282,−17.7282)	(−20.2282+ α,−17.7282 −α)	(−23.471+ 2.577α, − 18.2862.608α)
36	−17.5453	(−18.5453,−17.5453,−16.0453)	(−18.5453+ α,−16.0453 −α)	(−21.788+ 2.577α, −16.603−2.608α)
37	−24.0759	(−25.0759,−24.0759, −22.5759)	(−25.0759+ α, −22.5759 −α)	(−28.319+2.577 α, −23.134−2.608α)
38	−12.2368	(−13.2368, −12.2368, −10.7368)	(−13.2368+ α, −10.7368 −α)	(−16.480+ 2.577α, −11.295−2.608α)
39	13.1934	(12.1934,13.1934, 14.6934)	(12.1934+ α, 14.6934 −α)	(8.951+ 2.577α, 14.136−2.608α)
40	10.9934	(9.9934,10.9934,12.4934)	(9.9934+ α,,12.4934 −α)	(6.751+ 2.577α, 11.936−2.608α)
41	−1.6706	(−2.6706,−1.6706, −0.1706)	(−2.6706+ α, −0.1706 −α)	(− 5.913+ 2.577α, − 0.728− 2.608α)
42	−14.6845	(−15.6845,−14.6845, −13.1845)	(−15.6845+ α, −13.1845 −α)	(− 18.927+2.577α, − 1.3.742−2.608 α)
43	−10.0996	(−11.0996,−10.0996, −8.5996)	(−11.0996+ α, −8.5996 −α)	(− 14.342+ 2.577α, − 9.157−2.608 α)
44	−6.8038	(−7.8038,−6.8038,−5.3038)	(−7.8038+ α, −5.3038 −α)	(− 11.047+ 2.577α, − 5.862− 2.608α)
45	−6.7912	(−7.7912,−6.7912, −5.2912)	(−7.7912+ α, −5.2912 −α)	(−11.034+2.577 α, − 5.849− 2.608α)
46	15.4827	(14.4827,15.4827,16.9827)	(14.4827+ α,16.9827 −α)	(11.240+ 2.577α, − 16.425− 2.608α)
47	41.9558	(40.9558,41.9558, 43.4558)	(40.9558+ α, 43.4558 −α)	(37.713+2.577 α, − 42.898− 2.608α)
48	3.5556	(2.5556,3.5556, 5.0556)	(2.5556+ α, 5.0556 −α)	(−0.687+ 2.577α, 4.498−2.608 α)
49	14.3777	(13.3777,14.3777,15.8777)	(13.3777+ α,15.8777 −α)	(10.135+ 2.577α, 15.320− 2.608α)
50	−30.863	(−31.863,−30.863,−29.363)	(−31.863+ α,−29.363 −α)	(−35.106+ 2.577α, − 29.921−2.608 α)
51	27.5441	(26.5441,27.5441,29.0441)	(26.5441+ α,29.0441 −α)	(23.301+ 2.577α, 28.486− 2.608α)
52	14.8936	(13.8936,14.8936, 16.3936)	(13.8936+ α,16.3936 −α)	(10.651+ 2.577α, 15.836− 2.608α)
53	−23.1481	(−24.1481,−23.1481,−21.6481)	(−24.1481+ α,−21.6481 −α)	(− 27.391+2.577 α, − 22.206−2.608 α)
54	1.2048	(0.2048,1.2048, 2.7048)	(0.2048+ α,2.7048 −α)	(−3.038+ 2.577α, 2.147− 2.608α)
55	−4.7619	(−5.7619,−4.7619, −3.2619)	(−5.7619+ α, −3.2619 −α)	(−9.005+ 2.577α, −3.820− 2.608α)
56	2.5	(1.5,2.500, 4.000)	(1.5+ α, 4.000 −α)	(− 1.743+ 2.577α, 3.442−2.608 α)
57	−7.439	(−8.439,−7.439, −5.939)	(−8.439+ α, −5.939 −α)	(−11.682+ 2.577α, −6.497−2.608 α)
58	−24.5059	(−25.5059,−24.5059, −23.0059)	(−25.5059+ α,−23.0059 −α)	(−28.749+ 2.577α, −23.564−2.608 α)
59	16.2304	(15.2304,16.2304, 17.7304)	(15.2304+ α, 17.7304 −α)	(11.988+ 2.577α, 17.173− 2.608α)
60	−0.7508	(−1.7508,−0.7508,0.7492)	(−1.7508+ α,0.7492 −α)	(4.994+ 2.577α, 0.191−2.608 α)

he fuzzy number to represent the average stock returns is explicitly given by

$$_i = \left[\sum_{i=1}^{n} \frac{r_{i1}}{n}, \sum_{i=1}^{n} \frac{r_{i2}}{n}, \sum_{i=1}^{n} \frac{r_{i3}}{n}\right]$$

$$_i = (\,0.558,\ 1.666,\ 3.243\,)$$

hose α - Cut to represent the average of all the stocks returns under considerations is given by

$$_i(\alpha) = \left[\sum_{i=1}^{n} \frac{r_{i1}}{n} + \left(\sum_{i=1}^{n} \frac{r_{i2}}{n} - \sum_{i=1}^{n} \frac{r_{i1}}{n}\right)\alpha,\ \sum_{i=1}^{n} \frac{r_{i3}}{n} - \left(\sum_{i=1}^{n} \frac{r_{i3}}{n} - \sum_{i=1}^{n} \frac{r_{i2}}{n}\right)\alpha\right]$$

$$_i(\alpha) = (\,0.438 + 0.838\alpha,\ 1.276 - 0.889\alpha\,)$$

$[R_i(\alpha)-\bar{R}_i(\alpha)][R_{mi}(\alpha)-\bar{R}_{mi}(\alpha)]$, the square of the difference between the α - cuts for market returns and the α-cuts for the average market returns.

	$[R_i(\alpha)-\bar{R}_i(\alpha)]$	$[R_{mi}(\alpha)-\bar{R}_{mi}(\alpha)]$	$[R_i(\alpha)-\bar{R}_i(\alpha)][R_{mi}(\alpha)-\bar{R}_{mi}(\alpha)]$
	Column B	**Column C**	**Column C**
Months	α - Cut for the difference between the stock returns and the α - cuts for the average stock returns.	α - Cut for the difference between the market returns and the α - cuts for the average market returns.	α - Cut for the difference between the market returns and the α - cuts for the average market returns.
1	$(-11.723 + 6.577\alpha, -1.462 -6.608\alpha)$	$(-7.020 + 1.089\alpha, -4.4847 -1.038\alpha)$	$(82.295 -58.931\alpha+7.160\alpha^2, -7.152+30.816\alpha+6.861\alpha^2)$
2	$(-13.377+ 2.577\alpha, -8.192 -2.608\alpha)$	$(-0.494\ 1.889a, 3.233 -1.838\alpha)$	$(6.603 -26.538\alpha+4.867\alpha^2, -26.489 +6.626\alpha+4.794\alpha^2)$
3	$(-6.888 + 2.577\alpha, -1.703 -2.608\alpha)$	$(1.912 + 1.389a, -5.139 -1.838\alpha)$	$(-13.17 -4.639\alpha +3.578\alpha2, -8.753 -10.272\alpha+4.794\alpha^2)$
4	$(-12.395 +2.577\alpha, -7.210 -2.608\alpha)$	$(-3.157 + 1.089\alpha, -0.230 -1.838\alpha)$	$(39.127 -21.628\alpha+2.805\alpha^2, 1.656 +13.853\alpha+4.794\alpha^2)$
5	$(10.254 + 2.577\alpha, 15.439 -2.608\alpha)$	$(1.588+ 1.089\alpha, -4.515 -1.838\alpha)$	$(16.281+15.255\alpha+2.805\alpha^2, 69.704 -40.157\alpha+4.794\alpha^2)$
6	$(-4.501 +2.577\alpha, 0.684 -2.608\alpha)$	$(-0.650 + 1.139a, 1.687 -1.198\alpha)$	$(2.927 -6.801\alpha+2.934\alpha^2, 1.153 -5.128\alpha+3.125\alpha^2)$
7	$(-13.051 +2.577\alpha, -7.866 -2.608\alpha)$	$(-1.224 +1.889a, 2.503 -1.838\alpha)$	$(15.969 -27.803\alpha+4.867\alpha^2, -19.692 +7.930\alpha+4.794\alpha^2)$
8	$(-15.606+ 2.577\alpha, -10.421 -2.608\alpha)$	$(-5.296 + 1.889\alpha, -1.569 -1.838\alpha)$	$(82.656 -43.123\alpha+4.867\alpha^2, 16.354 +23.250\alpha+4.794\alpha^2)$
9	$(-3.922 + 2.577\alpha, 0.763 -2.608\alpha)$	$(-2.876+ 1.889\alpha, 0.851 -1.838\alpha)$	$(11.281 -14.819\alpha+4.867\alpha^2, 0.48 -3.195\alpha+3.875\alpha^2)$
10	$(0.869 +2.577\alpha, 6.054-2.608\alpha)$	$(-4.924 + 2.089\alpha, -0.997 -1.838\alpha)$	$(-4.278 -10.871\alpha +5.382\alpha^2 - 6.033 -8.529\alpha+4.794\alpha^2)$
11	$(37.398+ 2.577\alpha, 42.583-2.608\alpha)$	$(1.365+1.889\alpha, 5.092-1.838\alpha)$	$(51.056 + 74.153\alpha+4.867\alpha^2, 216.843 -91.561\alpha+4.794\alpha^2)$
12	$(-7.247+ 2.577\alpha, -2.062-2.608\alpha$	$(-2.042+ 1.889\alpha, 1.685-1.838\alpha$	$(14.798 -18.949\alpha+4.867\alpha^2, -3.474 -0.604\alpha+4.794\alpha^2)$
13	$(17.660+ 2.577\alpha, 22.845- 2.608\alpha)$	$(2.242+ 1.889\alpha, 5.969- 1.838\alpha)$	$(39.599 + 39.132\alpha+4.867\alpha^2, 136.369 -57.564\alpha+4.794\alpha^2)$
14	$(-6.421+ 2.577\alpha, -1.236-2.608\alpha)$	$(-3.364+ 1.889\alpha, -0.137-1.838\alpha)$	$(21.598 -17.584\alpha +3.578\alpha^2, 0.169 +2.629\alpha+4.794\alpha^2)$
15	$(9.301+ 2.577\alpha, 14.486-2.608\alpha)$	$(-2.413+ 1.889\alpha, 1.314-1.838\alpha)$	$(-22.443 + 11.349\alpha+4.867\alpha^2, 19.033 -30.055\alpha+4.794\alpha^2)$
16	$(10.528+ 2.577\alpha, 15.713- 2.608\alpha)$	$(1.343+ 0.914\alpha, 3.120-0.863\alpha)$	$(14.141 + 13.081\alpha+2.354\alpha^2 49.028 -21.702\alpha+2.251\alpha^2)$
17	$(7.374+ 2.577\alpha, 12.559- 2.608\alpha)$	$(-1.224+1.889\alpha, 2.503-1.838\alpha)$	$(-9.022 + 10.775\alpha+4.867\alpha^2, 31.442 -29.617\alpha+4.794\alpha^2)$
18	$(-13.171+ 2.577\alpha, -7.986-2.608\alpha)$	$(-6.622+1.889\alpha, -3.295-1.838\alpha)$	$(87.218 -36.671\alpha +3.836\alpha^2, 26.313 +23.275\alpha+4.794\alpha^2)$
19	$(-37.352+ 2.577\alpha, -32.167-2.608\alpha)$	$(-5.444+ 1.889\alpha, -1.717-1.838\alpha)$	$(203.328 -84.574\alpha +4.867\alpha^2, 55.216 +63.608\alpha+4.794\alpha^2)$
20	$(27.499+ 2.577\alpha, 32.684-2.608\alpha)$	$(1.182+ 1.889\alpha, 4.909-1.838\alpha)$	$(32.502 + 54.984\alpha+4.867\alpha^2, 160.445 -72.885\alpha+4.794\alpha^2)$
21	$(5.229+ 2.577\alpha, 10.414-2.608\alpha)$	$(0.205+ 1.389\alpha, 3.432-1.838\alpha)$	$(1.0731 + 7.791\alpha+3.578\alpha^2, 35.745 -28.096\alpha+4.794\alpha^2)$
22	$(-5.520+ 2.577\alpha, -0.335-2.608\alpha)$	$(1.651+ 2.889\alpha, 6.378-1.838\alpha)$	$(-9.115 -11.692\alpha+4.867\alpha^2, -2.139 -16.019\alpha+4.794\alpha^2)$
23	$(-8.619+ 2.577\alpha, -5.434 -2.608\alpha)$	$(4.798+ 1.389\alpha, 7.525 -1.338\alpha)$	$(-41.335 -3.445\alpha+2.467\alpha^2, -40.893 -3.325\alpha+1.884\alpha^2)$
24	$(-11.841+2.577\alpha, 0.844-2.608\alpha)$	$(-4.655+ 1.889\alpha, -0.928-1.838\alpha)$	$(55.120 -52.979\alpha+12.422\alpha^2, -0.783 +4.117\alpha+11.228\alpha^2)$

25	$(16.092+2.577\alpha,\ 21.277-2.608\alpha)$	$(0.085+1.889\alpha,\ 3.642-1.838\alpha)$	$(-1.372+30.173\alpha+4.867\alpha^2,\ 77.483-48.10\alpha+4.794\alpha^2)$
26	$(-11.302+2.577\alpha,\ -6.117-2.608\alpha)$	$(-2.375+1.889\alpha,\ 1.352-1.838\alpha)$	$(26.845-27.466\alpha+4.867\alpha^2,\ -8.267+7.718\alpha+4.794\alpha^2)$
27	$(-13.334+2.577\alpha,\ -8.149-2.608\alpha)$	$(-8.161+1.889\alpha,\ -4.434-1.838\alpha)$	$(108.814-46.211\alpha+4.867\alpha^2,\ 36.130+26.544\alpha+4.794\alpha^2)$
28	$(8.669+2.577\alpha,\ 12.854-2.608\alpha)$	$(0.504+1.389\alpha,\ 2.223-1.338\alpha)$	$(-4.372+10.738\alpha+3.578\alpha^2,\ 30.789-24.337\alpha+3.490\alpha^2)$
29	$(19.776+1.877\alpha,\ 23.0612.608--\alpha)$	$(3.619+1.889\alpha,\ 7.346-1.838\alpha)$	$(71.574+44.143\alpha+3.545\alpha^2,\ 169.410-52.737\alpha+2.588\alpha^2)$
30	$(3.394+2.577\alpha,\ 8.579-2.608\alpha)$	$(-2.294+1.889\alpha,\ 1.433-1.838\alpha)$	$(-7.783+0.499\alpha+4.867\alpha^2,\ 12.295-19.507\alpha+4.794\alpha^2)$
31	$(17.379+2.577\alpha,\ 22.564-2.608\alpha)$	$(3.669+1.889\alpha,\ 7.396-1.838\alpha)$	$(63.754+42.276\alpha+4.867\alpha^2,\ 166.870-60.768\alpha+4.794\alpha^2)$
32	$(-16.396+2.577\alpha,\ -11.211-2.608\alpha)$	$(-6.531+1.389\alpha,\ -3.804-1.838\alpha)$	$(107.083-39.598\alpha+3.578\alpha^2,\ 42.647+24.925\alpha+3.490\alpha^2)$
33	$(26.508+2.577\alpha,\ 31.693-2.608\alpha)$	$(3.315+1.889\alpha,\ 7.042-1.838\alpha)$	$(87.866+58.608\alpha+4.867\alpha^2,\ 223.173-76.626\alpha+4.794\alpha^2)$
34	$(-13.372+2.577\alpha,\ -8.187-2.608\alpha)$	$(-5.976+1.889\alpha,\ -2.249-1.838\alpha)$	$(79.910-40.654\alpha+4.867\alpha^2,\ 18.411+20.915\alpha+4.794\alpha^2)$
35	$(-23.471+2.577\alpha,\ -18.2862.608\alpha)$	$(-7.982+1.889\alpha,\ -4.255-1.838\alpha)$	$(187.346-64.897\alpha+4.867\alpha^2,\ 77.807+44.712\alpha+4.794\alpha^2)$
36	$(-21.788+2.577\alpha,\ -16.603-2.608\alpha)$	$(-0.299+1.889\alpha,\ 3.428-1.838\alpha)$	$(6.518-41.923\alpha+4.867\alpha^2,\ -56.912+21.579\alpha+4.794\alpha^2)$
37	$(-28.319+2.577\alpha,\ -23.134-2.608\alpha)$	$(-3.154+1.889\alpha,\ 0.573-1.838\alpha)$	$(89.314-61.614\alpha+4.867\alpha^2,\ -13.258+41.030\alpha+4.794\alpha^2)$
38	$(-16.480+2.577\alpha,\ -11.295-2.608\alpha)$	$(2.690+1.889\alpha,\ 6.417-1.838\alpha)$	$(-44.324-24.195\alpha+4.867\alpha^2,\ -72.474+4.025\alpha+4.794\alpha^2)$
39	$(8.951+2.577\alpha,\ 14.136-2.608\alpha)$	$(4.105+1.189\alpha,\ 6.532-1.238\alpha)$	$(36.743+21.217\alpha+3.063\alpha^2,\ 92.336-34.541\alpha+3.229\alpha^2)$
40	$(6.751+2.577\alpha,\ 11.936-2.608\alpha)$	$(-1.756+1.889\alpha,\ 1.971-1.838\alpha)$	$(-11.856+8.224\alpha+4.867\alpha^2,\ 23.520-27.080\alpha+4.794\alpha^2)$
41	$(-5.913+2.577\alpha,\ -0.728-2.608\alpha)$	$(-3.446+1.189\alpha,\ -1.119-1.138\alpha)$	$(-20.380-15.910\alpha+3.063\alpha^2,\ 0.816+3.749\alpha+2.968\alpha^2)$
42	$(-18.927+2.577\alpha,\ -13.742-2.608\alpha)$	$(-6.102+1.889\alpha,\ -2.375-1.838\alpha)$	$(115.493-51.471\alpha+4.867\alpha^2,\ 32.637+31.456\alpha+4.794\alpha^2)$
43	$(-14.342+2.577\alpha,\ -9.157-2.608\alpha)$	$(-9.076+1.889\alpha,\ -5.349-1.838\alpha)$	$(130.176-50.475\alpha+4.867\alpha^2,\ 48.985+30.786\alpha+4.794\alpha^2)$
44	$(-11.047+2.577\alpha,\ -5.862-2.608\alpha)$	$(-22.873+1.389\alpha,\ -19.646-1.838\alpha)$	$(252.676-74.278\alpha+3.578\alpha^2,\ 115.161+62.019\alpha+4.794\alpha^2)$
45	$(-11.034+2.577\alpha,\ -5.849-2.608\alpha)$	$(-1.157+1.389\alpha,\ -1.570-1.138\alpha)$	$(12.767-18.305\alpha+3.578\alpha^2,\ -9.182+3.732\alpha+3.490\alpha^2)$
46	$(11.240+2.577\alpha,\ -16.425-2.608\alpha)$	$(7.418+1.889\alpha,\ -11.145-1.838\alpha)$	$(83.377+40.343\alpha+4.867\alpha^2,\ 183.055-59.262\alpha+4.794\alpha^2)$
47	$(37.713+2.577\alpha,\ -42.898-2.608\alpha)$	$(-0.981+1.889\alpha,\ 2.746-1.838\alpha)$	$(-37.004+68.701\alpha+4.867\alpha^2,\ 117.788-86.019\alpha+4.794\alpha^2)$
48	$(-0.687+2.577\alpha,\ 4.498-2.608\alpha)$	$(-0.926+1.889\alpha,\ 2.801-1.838\alpha)$	$(0.636-3.683\alpha+4.867\alpha^2,\ 12.599-15.574\alpha+4.794\alpha^2)$
49	$(10.135+2.577\alpha,\ 15.320-2.608\alpha)$	$(0.591+1.889\alpha,\ 4.318-1.838\alpha)$	$(5.988+20.664\alpha+4.867\alpha^2,\ 66.149-39.424\alpha+4.794\alpha^2)$
50	$(-35.106+2.577\alpha,\ -29.921-2.608\alpha)$	$(-8.380+0.909\alpha,\ -6.596-0.878\alpha)$	$(294.181-53.494\alpha+2.342\alpha^2,\ 197.263+43.474\alpha+2.290\alpha^2)$
51	$(23.301+2.577\alpha,\ 28.486-2.608\alpha)$	$(1.351+\alpha,\ 5.078-1.838\alpha)$	$(31.481+47.491\alpha+4.867\alpha^2,\ 144.654-65.610\alpha+4.794\alpha^2)$
52	$(10.651+2.577\alpha,\ 15.836-2.608\alpha)$	$(3.154+1.889\alpha,\ 6.881-1.838\alpha)$	$(33.589+28.242\alpha+4.867\alpha^2,\ 108.960-47.057\alpha+4.794\alpha^2)$
53	$(-27.391+2.577\alpha,\ -22.206-2.608\alpha)$	$(5.630+1.889\alpha,\ -1.903-1.838\alpha)$	$(154.212-66.241\alpha+4.867\alpha^2,\ 42.259+45.783\alpha+4.794\alpha^2)$
54	$(-3.038+2.577\alpha,\ 2.147-2.608\alpha)$	$(-0.706+1.889\alpha,\ -3.021-1.838\alpha)$	$(2.144-7.556\alpha+4.867\alpha^2,\ 6.486-11.826\alpha+4.794\alpha^2)$
55	$(-9.005+2.577\alpha,\ -3.820-2.608\alpha)$	$(-2.153+1.889\alpha,\ 1.574-1.838\alpha)$	$(19.387-22.555\alpha+4.867\alpha^2,\ -6.012+2.916\alpha+4.794\alpha^2)$
56	$(-1.743+2.577\alpha,\ 3.442-2.608\alpha)$	$(-4.722+1.889\alpha,\ -0.995-1.838\alpha)$	$(8.229-15.458\alpha+4.867\alpha^2,\ -3.424-3.732\alpha+4.794\alpha^2)$
57	$(-11.682+2.577\alpha,\ -6.497-2.608\alpha)$	$(-3.353+1.889\alpha,\ 0.374-1.838\alpha)$	$(39.169-30.703\alpha+4.867\alpha^2,\ -2.429+10.967\alpha+4.794\alpha^2)$

58	$(-28.749+2.577\alpha,\ -23.564-2.608\alpha)$	$(1.125+1.889\alpha,\ 4.852-1.838\alpha)$	$(-32.341-51.400\alpha+4.867\alpha^2,\ -114.330+30.660\alpha+4.794\alpha^2)$
59	$(11.988+2.577\alpha,\ 17.173-2.608\alpha)$	$(0.515+1.889\alpha,\ 4.242-1.838\alpha)$	$(6.167+23.967\alpha+4.867\alpha^2,\ 72.837-42.630\alpha+4.794\alpha^2)$
60	$(4.994+2.577\alpha,\ 0.191-2.608\alpha)$	$(8.570+1.889\alpha,\ 12.297-1.838\alpha)$	$(-42.793+12.649\alpha+4.867\alpha^2,\ 2.352-32.42\alpha+4.794\alpha^2)$

From the tables

$$\sum_{i=1}^{n}(R_i(\alpha)-\bar{R}_i(\alpha))\,(R_{mi}(\alpha)-\bar{R}_{mi}(\alpha)) = [2572.181 - 564.688\alpha + 276.471\alpha^2,\ 2572.181 - 564.688\alpha + 276.471\alpha^2\ ]$$

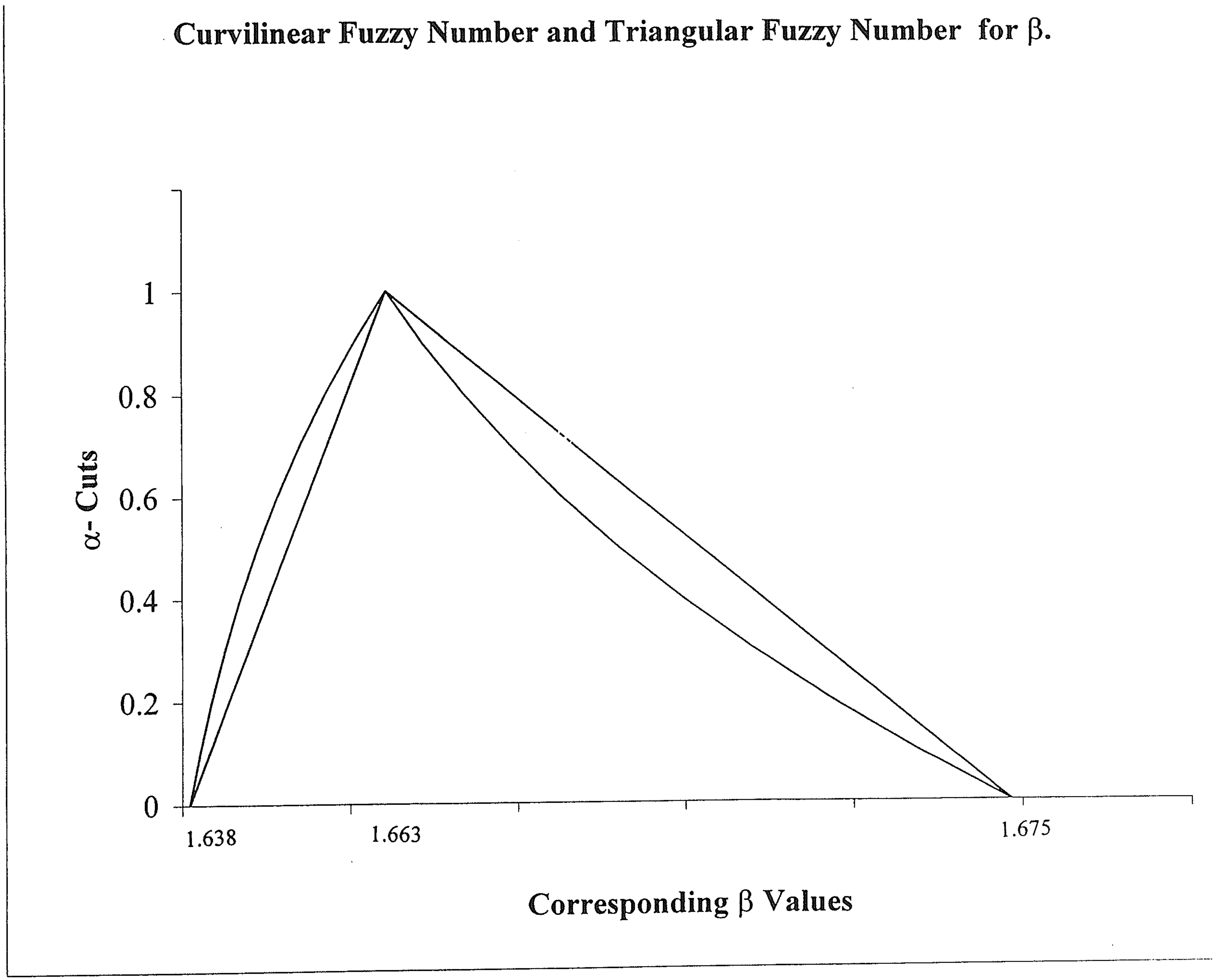
$$\sum_{i=1}^{n}(R_{mi}(\alpha)-\bar{R}_{mi}(\alpha))^2 = [1517.138 - 329.679\alpha + 185.955\alpha^2, 1569.852 - 379.418\alpha + 182.981\alpha^2]$$

$$\beta(\alpha) = \frac{\sum_{i=1}^{n}(R_i(\alpha) - \bar{R}_i(\alpha))\,(R_{mi}(\alpha) - \bar{R}_{mi}(\alpha))}{\sum_{i=1}^{n}(R_{mi}(\alpha) - \bar{R}_{mi}(\alpha))^2}$$

$$\beta(\alpha) = \left[\frac{2572.181 - 564.688\alpha + 276.471\alpha^2}{1569.852 - 379.418\alpha + 182.981\alpha^2}, \frac{2572.181 - 564.688\alpha + 276.471\alpha^2}{1517.138 - 329.679\alpha + 185.955\alpha^2}\right]$$

for $0 \le \alpha \le 1$

α	**Left $\beta(\alpha)$**	**Right $\beta(\alpha)$**
0.0	1.6385	1.6751
0.1	1.6421	1.6761
0.2	1.6455	1.6767
0.3	1.6487	1.6768
0.4	1.6517	1.6764
0.5	1.6544	1.6755
0.6	1.6568	1.6740
0.7	1.6589	1.6720
0.8	1.6606	1.6695
0.9	1.6620	1.6665
1.0	1.6630	1.6630



Curvilinear Fuzzy Number and Triangular Fuzzy Number for β.
1
0.8
0.6
0.4
0.2
0
α- Cuts
1.638
1.663
1.675
Corresponding β Values

APPENDIX 4

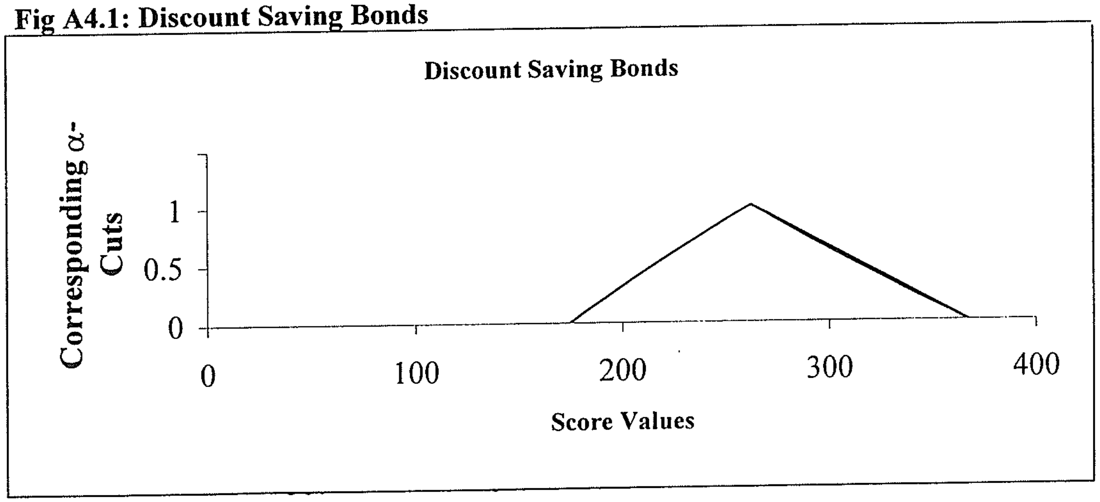
Scoring Model

Output For Fuzzy Weighted Scoing Model

Discount Saving Bonds

α	Left Scoring P- Curvature	Right Scoring P- Curvature	Left Scoring Zero- Curvature	Right Scoring Zero- Curvature	Left Error	Right Error	% Left Error	% Right Error
0.00	175.00	367.00	175.00	367.00	0.00	0.00	0.00	0.00
0.10	182.89	355.69	183.70	356.50	0.81	0.81	0.44	0.23
0.20	190.96	344.56	192.40	346.00	1.44	1.44	0.75	0.42
0.30	199.21	333.61	201.10	335.50	1.89	1.89	0.95	0.57
0.40	207.64	322.84	209.80	325.00	2.16	2.16	1.04	0.67
0.50	216.25	312.25	218.50	314.50	2.25	2.25	1.04	0.72
0.60	225.04	301.84	227.20	304.00	2.16	2.16	0.96	0.72
0.70	234.01	291.61	235.90	293.50	1.89	1.89	0.81	0.65
0.80	243.16	281.56	244.60	283.00	1.44	1.44	0.59	0.51
0.90	252.49	271.69	253.30	272.50	0.81	0.81	0.32	0.30
1.00	262.00	262.00	262.00	262.00	0.00	0.00	0.00	0.00

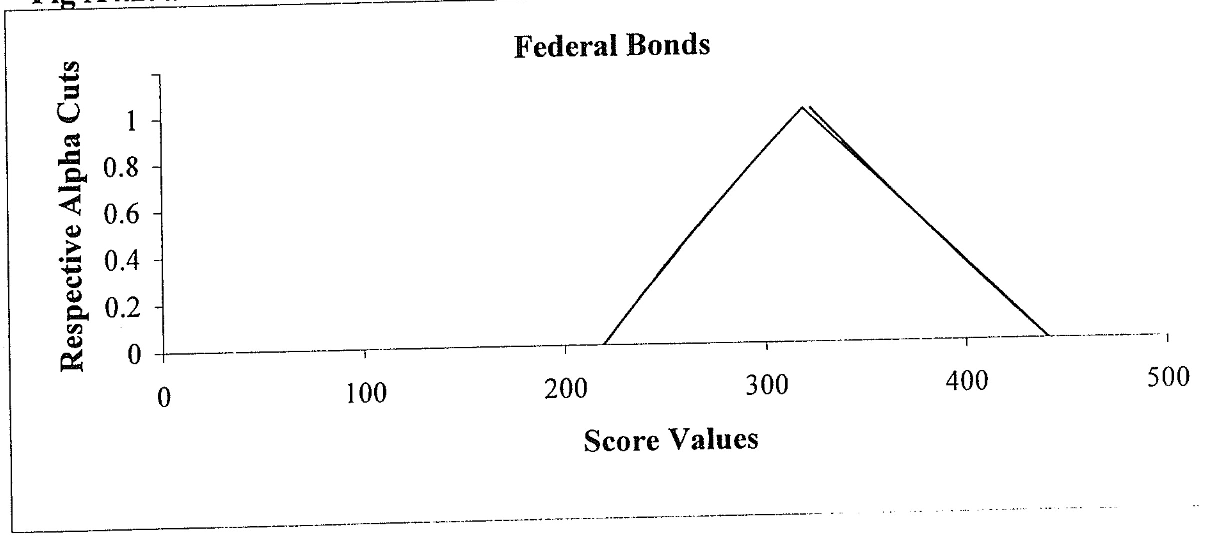
Fig A4.1: Discount Saving Bonds



Federal Bonds

α	Left Scoring P- Curvature	Right Scoring P- Curvature	Left Scoring Zero- Curvature	Right Scoring Zero- Curvature	Left Error	Right Error	% Left Error	% Right Error
0.0	219.00	435.00	219.00	435.00	0.00	0.00	0.00	0.00
0.1	228.09	422.49	228.90	423.30	0.81	0.81	0.36	0.19
0.2	237.36	410.16	238.80	411.60	1.44	1.44	0.61	0.35
0.3	246.81	398.01	248.70	399.90	1.89	1.89	0.77	0.47
0.4	256.44	386.04	258.60	388.20	2.16	2.16	0.84	0.56
0.5	266.25	374.25	268.50	376.50	2.25	2.25	0.85	0.60
0.6	276.24	362.64	278.40	364.80	2.16	2.16	0.78	0.60
0.7	286.41	351.21	288.30	353.10	1.89	1.89	0.66	0.54
0.8	296.76	339.96	298.20	341.40	1.44	1.44	0.49	0.42
0.9	307.29	328.89	308.10	329.70	0.81	0.81	0.26	0.25
1.0	318.00	318.00	318.00	318.00	0.00	0.00	0.00	0.00

Fig A4.2: Federal Bonds.



Corporate Bonds

α	Left Scoring P- Curvature	Right Scoring P- Curvature	Left Scoring Zero- Curvature	Right Scoring Zero- Curvature	Left Error	Right Error	% Left Error	% Right Error
0	168.00	362.00	168.00	362.00	0.00	0.00	0.00	0.00
0.1	175.99	350.59	176.80	351.40	0.81	0.81	0.46	0.23
0.2	184.16	339.36	185.60	340.80	1.44	1.44	0.78	0.42
0.3	192.51	328.31	194.40	330.20	1.89	1.89	0.98	0.58
0.4	201.04	317.44	203.20	319.60	2.16	2.16	1.07	0.68
0.5	209.75	306.75	212.00	309.00	2.25	2.25	1.07	0.73
0.6	218.64	296.24	220.80	298.40	2.16	2.16	0.99	0.73
0.7	227.71	285.91	229.60	287.80	1.89	1.89	0.83	0.66
0.8	236.96	275.76	238.40	277.20	1.44	1.44	0.61	0.52
0.9	246.39	265.79	247.20	266.60	0.81	0.81	0.33	0.30
1	256.00	256.00	256.00	256.00	0.00	0.00	0.00	0.00

Fig A4.3: Corporate Bonds.