Distribution of Increasing/Decreasing ℓ-sequences in Random Permutations

By

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A Thesis
Submitted to the Faculty of Graduate Studies
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Abstract

This thesis examines the distribution of increasing ℓ-sequences in a random permutation generated by the integers $1, \ldots, n$; an increasing ℓ-sequence being a sequence of ℓ consecutive integers. Known methods are reviewed and two new solutions to the problem are derived. An equation is obtained that is more efficient than existing methods. In addition, we derive the expectation and use this to show that, for $\ell > 2$, the distribution is degenerate about 0. The thesis concludes with a discussion of applications of these numbers and the description of an extension to the problem.

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A debt of gratitude also goes to my wife Catherine, my daughter Alberta and my son Cole. Their support has remained unwavering throughout my academic career. I would also like to thank Andrew Johnson for his assistance in automatically generating the LaTeX code for Appendix A. Friends, too numerous to mention, have also been a great source of encouragement and support. Last, but not least — Thanks Mom.

Summary of Notation

The following table summarizes the notation used throughout the thesis. The page references note where the notation is first used. I recommend photo-copying the next page and using it as a book-mark while reading this thesis.

Notation	Meaning	Page
$\lfloor x \rfloor$	largest integer less than or equal to x	2
$\lceil x \rceil$	least integer greater than or equal to x	2
$[\mathcal{S}]$	1 if statement $\mathcal{S}$ is true, 0 otherwise	2
$\#A$	cardinality of the set A	5
$C_{n,m}$	number of compositions of n into exactly m parts	10
$C_{n,m}(\ell)$	number of compositions of n into exactly m parts; none $> \ell$	10
D_n	derangement numbers	8
$D_{n,k}$	matching numbers	8
$\mathbb{N}$	the natural numbers $\{1, 2, \ldots\}$	1
$\mathbb{N}_n$	the natural numbers $\{1, 2, \ldots, n\}$	1
π	general element of $\mathbb{S}_n$	2
$\pi(j)$	j^{th} element of $\pi \in \mathbb{S}_n$	2
$p(n)$	partition numbers — number of partitions of n	9
$p(n, m)$	number of partitions of n, no summand larger than m	9
$\mathbb{R}$	the real numbers	1
$R_n^+(\pi, \ell)$	number of maximal increasing ℓ-sequences in π	5
$R_n^-(\pi, \ell)$	number of maximal decreasing ℓ-sequences in π	5
$R_n(\pi, \ell)$	number of maximal increasing/decreasing ℓ-sequences in π	5
$R_n^+(\pi)$	total number of maximal increasing sequences in π	5
$R_n^-(\pi)$	total number of maximal decreasing sequences in π	5
$R_n(\pi)$	total number of maximal increasing/decreasing sequences in π	5
$\mathbb{S}_n$	symmetric group on n elements	2
$X_n^+(\pi, \ell)$	number of increasing ℓ-sequences in π	5
$X_n^-(\pi, \ell)$	number of decreasing ℓ-sequences in π	5
$X_n(\pi, \ell)$	number of increasing/decreasing ℓ-sequences in π	5
$\left\langle {n \atop k} \uparrow \ell \right\rangle$	$\#\{\pi \in \mathbb{S}_n \mid X_n^+(\pi, \ell) = k\}$	6
$\left\langle {n \atop k} \downarrow \ell \right\rangle$	$\#\{\pi \in \mathbb{S}_n \mid X_n^-(\pi, \ell) = k\}$	6
$\left\langle {n \atop k} \updownarrow \ell \right\rangle$	$\#\{\pi \in \mathbb{S}_n \mid X_n(\pi, \ell) = k\}$	6
$\mathbb{Z}$	the integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$	1

Chapter 1

Introduction

The topic of this thesis is the distribution of the number of increasing ℓ-sequences (sequences of ℓ consecutive integers) in a random permutation of the elements $\{1, \ldots, n\}$. In this chapter, Section 1.1 introduces the terminology, definitions and notation to be used throughout the thesis; Section 1.2 reviews the existing literature on the subject; and Section 1.3 provides an overview of the remaining thesis.

1.1 Definitions and Notation

Throughout this thesis, we will make use of the following symbols for the common sets:

$\mathbb{N}$	the set of all natural numbers: $\{1, 2, 3, \cdots\}$;
$\mathbb{N}_n$	the set $\{1, 2, 3, \cdots, n\}$;
$\mathbb{Z}$	the set of all integers: $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$;
$\mathbb{R}$	the set of all real numbers;

For real x, we denote the largest integer less than or equal to x as $\lfloor x \rfloor$ and the least integer greater or equal to x as $\lceil x \rceil$. That is,

$$\lfloor x \rfloor = \max\{ n \in \mathbb{Z} \mid n \le x \} \quad \text{and} \quad \lceil x \rceil = \min\{ n \in \mathbb{Z} \mid n \ge x \}.$$

We also make use of the following indicator notation (Graham, Knuth, and Patashnik 1994): If $\mathcal{S}$ is any boolean statement then $[\mathcal{S}] = 1$ if $\mathcal{S}$ is true and $[\mathcal{S}] = 0$ if $\mathcal{S}$ is false. For example,

$$[x \le 1] = \begin{cases} 1, & \text{if } x \le 1; \\ 0, & \text{if } x > 1. \end{cases}$$

We assume that, if $[\mathcal{S}] = 0$, then $u[\mathcal{S}] = 0$ even when u is undefined.

We use the following formal definition for binomial coefficients:

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k!}, & k \ge 0; \\ 0, & k < 0. \end{cases} \qquad \text{for all } r \in \mathbb{R}, k \in \mathbb{Z}. \tag{1.1}$$

This definition allows the upper index to be any real number and, in particular, $r = -1$ occurs frequently throughout. When $r = n$ is a non-negative integer, the definition is consistent with the usual $\binom{n}{k} = n!/k!\,(n-k)!$. We also note that $\binom{r}{k} = \binom{r}{r-k}$ if and only if r is a non-negative integer. Restrictions may be required on other familiar binomial identities and care must be taken throughout.

We denote the symmetric group on n elements by $\mathbb{S}_n$, that is, $\mathbb{S}_n$ is the permutation group on $\mathbb{N}_n = \{1, \cdots, n\}$. As usual, the group operation is composition of functions. In general, we will denote the j^{th} element of $\pi \in \mathbb{S}_n$ as $\pi(j)$ for all $j \in \mathbb{N}_n$, hence $\pi \colon \mathbb{N}_n \to \mathbb{N}_n$ is a bijection. We denote a group of $\ell \in \mathbb{N}$ elements or sub-permutation in π by $\pi(j)\cdots\pi(j+\ell-1)$. It will be understood that, when arithmetic modulo n is called for on the indices or elements of $\pi \in \mathbb{S}_n$, we will identify the equivalence class $[0] = \{nz \mid z \in \mathbb{Z}\}$ with the label n instead of the usual 0. For convenience, if $\pi \in \mathbb{S}_n$, we assume $\pi(j)$ is undefined for all $j \leq 0$ and $j > n$.

We begin with the following two definitions:

Definition 1.1 (Increasing/Decreasing ℓ-Sequence) Let $n, j, \ell \in \mathbb{N}$ such that $1 \leq j \leq n - \ell + 1 \leq n$ and let $\pi \in \mathbb{S}_n$. Then $\pi(j) \cdots \pi(j + \ell - 1)$ is an *increasing ℓ-sequence* if and only if

$$\forall\, i\colon j < i < j + \ell, \qquad \pi(i) - \pi(i-1) = 1.$$

Similarly, we say $\pi(j) \cdots \pi(j + \ell - 1)$ is a *decreasing ℓ-sequence* if and only if

$$\forall\, i\colon j < i < j + \ell, \qquad \pi(i-1) - \pi(i) = 1.$$

Lastly, $\pi(j) \cdots \pi(j + \ell - 1)$ is an *ℓ-sequence* if and only if

$$\forall\, i\colon j < i < j + \ell, \qquad |\pi(i) - \pi(i-1)| = 1.$$

Definition 1.2 (Maximal Increasing/Decreasing ℓ-Sequence) Let $n, j, \ell \in \mathbb{N}$ as in the previous definition and let $\pi \in \mathbb{S}_n$. Then we say $\pi(j) \cdots \pi(j + \ell - 1)$ is a *maximal increasing ℓ-sequence* if and only if $\pi(j) \cdots \pi(j + \ell - 1)$ is an increasing ℓ-sequence and $\pi(j) \cdots \pi(j + \ell - 1)$ is maximal in the sense that it is not contained in a larger increasing ℓ-sequence, that is, $\pi(j-1) \neq \pi(j) - 1$ and $\pi(j + \ell - 1) \neq \pi(j + \ell) - 1$. *Maximal decreasing ℓ-sequences* are defined in an analogous manner. Similarly, by a *maximal ℓ-sequence*, we mean an ℓ-sequence which in not contained in a larger ℓ-sequence.

Table 1.1 illustrates the definitions for the example $\pi = (45621873) \in \mathbb{S}_8$. We also make the following remarks concerning these definitions:

Remark 1.3 The previous two definitions have the following consequences:

1. since $\pi(j)$ is undefined for $j \notin \mathbb{N}_n$, the restriction $1 \leq j \leq n - \ell + 1 \leq n$ is implicit in the definitions.
2. $\pi(j)$ is both an increasing and decreasing 1-sequence for all $j \in \mathbb{N}_n$;
3. increasing and/or decreasing ℓ-sequences may overlap while maximal increasing and/or decreasing ℓ-sequences are necessarily disjoint.

Table 1.1: Increasing/Decreasing ℓ-sequences in $\pi = (45621873)$.

Type of Sequence	$\ell = 1$	$\ell = 2$	$\ell = 3$
Increasing ℓ-Sequences	$4, 5, 6, 2, 1, 8, 7, 3$	$45, 56$	456
Decreasing ℓ-Sequences	$4, 5, 6, 2, 1, 8, 7, 3$	$21, 87$	
ℓ-Sequences	$4, 5, 6, 2, 1, 8, 7, 3$	$45, 56, 21, 87$	456
Maximal Increasing ℓ-Sequences	$2, 1, 8, 7, 3$		456
Maximal Decreasing ℓ-Sequences	$4, 5, 6, 3$	$21, 87$	
Maximal ℓ-Sequences	3	$21, 87$	456

For $n, \ell \in \mathbb{N}$ and $\pi \in \mathbb{S}_n$, define

$$X_n^+(\pi, \ell) = \text{number of increasing } \ell\text{-sequences in } \pi;$$

$$X_n^-(\pi, \ell) = \text{number of decreasing } \ell\text{-sequences in } \pi;$$

$$X_n(\pi, \ell) = X_n^+(\pi, \ell) + X_n^-(\pi, \ell).$$

More formally, we have

$$X_n^+(\pi, \ell) = \#\{\, j \in \mathbb{N}_n \mid \forall i\colon j < i < j + \ell \ (\pi(i) - \pi(i-1) = 1)\,\}; \tag{1.2}$$

$$X_n^-(\pi, \ell) = \#\{\, j \in \mathbb{N}_n \mid \forall i\colon j < i < j + \ell \ (\pi(i-1) - \pi(i) = 1)\,\}; \tag{1.3}$$

$$X_n(\pi, \ell) = \#\{\, j \in \mathbb{N}_n \mid \forall i\colon j < i < j + \ell \ (|\pi(i) - \pi(i-1)| = 1)\,\}. \tag{1.4}$$

For $n, \ell \in \mathbb{N}$ and $\pi \in \mathbb{S}_n$, define $R_n^+(\pi, \ell)$, $R_n^-(\pi, \ell)$ and $R_n(\pi, \ell)$ as the number of maximal increasing ℓ-sequences, maximal decreasing ℓ-sequences and maximal ℓ-sequences in π respectively. By definition, maximal (increasing/decreasing) ℓ-sequences are disjoint and we define

$$R_n^+(\pi) = \sum_{\ell=1}^{n} R_n^+(\pi, \ell), \qquad R_n^-(\pi) = \sum_{\ell=1}^{n} R_n^-(\pi, \ell), \qquad R_n(\pi) = \sum_{\ell=1}^{n} R_n(\pi, \ell). \tag{1.5}$$

Hence, $R_n^+(\pi)$ denotes the total number of maximal increasing sequences in π; $R_n^-(\pi)$ denotes the total number of maximal decreasing sequences in π; and $R_n(\pi)$ represents the total number of maximal sequences in π. Table 1.2 illustrates these definitions for our example permutation $\pi = (45621873)$.

Table 1.2: Examples of Defined Numbers for $\pi = (45621873)$.

	$\ell=1$	$\ell=2$	$\ell=3$	
$X_8^+(\pi,\ell)$	8	2	1	
$X_8^-(\pi,\ell)$	8	2	0	
$X_8(\pi,\ell)$	8	4	1	
$R_8^+(\pi,\ell)$	5	0	1	$R_8^+(\pi)=6$
$R_8^-(\pi,\ell)$	4	2	0	$R_8^-(\pi)=6$
$R_8(\pi,\ell)$	1	2	1	$R_8(\pi)=4$

It is easy to see the following relations:

$$\begin{aligned} n &= X_n^+(\pi,1) = X_n^-(\pi,1) \\ &= R_n^+(\pi) + X_n^+(\pi,2) \\ &= R_n^-(\pi) + X_n^-(\pi,2) \\ &= R_n(\pi) + X_n(\pi,2). \end{aligned} \tag{1.6}$$

Now, for $n,\ell \in \mathbb{N}$ and $0 \le k \in \mathbb{Z}$, define

$$\begin{aligned} \left\langle {n \atop k} \middle\uparrow \ell \right\rangle &= \sum_{\pi\in\mathbb{S}_n} [X_n^+(\pi,\ell)=k] = \#\{\,\pi\in\mathbb{S}_n \mid X_n^+(\pi,\ell)=k\,\}; \\ \left\langle {n \atop k} \middle\downarrow \ell \right\rangle &= \sum_{\pi\in\mathbb{S}_n} [X_n^-(\pi,\ell)=k] = \#\{\,\pi\in\mathbb{S}_n \mid X_n^-(\pi,\ell)=k\,\}; \\ \left\langle {n \atop k} \middle\updownarrow \ell \right\rangle &= \sum_{\pi\in\mathbb{S}_n} [X_n(\pi,\ell)=k] = \#\{\,\pi\in\mathbb{S}_n \mid X_n(\pi,\ell)=k\,\}. \end{aligned} \tag{1.7}$$

Hence, $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ is the number of permutations in $\mathbb{S}_n$ with exactly k increasing ℓ-sequences, $\left\langle {n \atop k} \middle\downarrow \ell \right\rangle$ is the number of permutations in $\mathbb{S}_n$ with exactly k decreasing ℓ-sequences, and $\left\langle {n \atop k} \middle\updownarrow \ell \right\rangle$ is the number of permutations in $\mathbb{S}_n$ with exactly k increasing and/or decreasing ℓ-sequences. By symmetry, we have

$$\left\langle {n \atop k} \middle\uparrow \ell \right\rangle = \left\langle {n \atop k} \middle\downarrow \ell \right\rangle.$$

We also note the special case

$$\left\langle {n \atop k} \middle\uparrow 1 \right\rangle = \left\langle {n \atop k} \middle\downarrow 1 \right\rangle = \left\langle {n \atop k} \middle\updownarrow 1 \right\rangle = n!\,[k = n], \qquad \text{for all } n \in \mathbb{N};$$

There are two related numbers that, while not the main subject of this thesis, appear frequently enough in the literature to warrant definition here.

Definition 1.4 (Increasing/Decreasing $\ell_\circ$-Sequence) Let $n, j, \ell \in \mathbb{N}$ such that $1 \leq \ell \leq n$ and $1 \leq j \leq n$ and let $\pi \in \mathbb{S}_n$. Then $\pi(j)\pi(j+1 \bmod n)\cdots\pi(j+\ell-1 \bmod n)$ is an *increasing $\ell_\circ$-sequence* if and only if

$$\forall\, i \colon j < i < j + \ell, \qquad \pi(i \bmod n) - \pi(i-1 \bmod n) = 1.$$

Recall that we use the label n for the equivalence class $[0]$. We similarly define *decreasing $\ell_\circ$-sequences* and *$\ell_\circ$-sequences.*

Definition 1.5 (Increasing/Decreasing $\ell_\star$-Sequence) Let $n, j, \ell \in \mathbb{N}$ such that $1 \leq j \leq n-\ell+1 \leq n$ and let $\pi \in \mathbb{S}_n$. Then $\pi(j)\cdots\pi(j+\ell-1)$ is an *increasing $\ell_\star$-sequence* if and only if

$$\forall\, i \colon j < i < j + \ell, \qquad \pi(i) - \pi(i-1) \equiv 1 \pmod{n}.$$

Again, we define *decreasing* $\ell_\star$*-sequences* and $\ell_\star$*-sequences* in an analogous manner and note that the equivalence class [0] is identified with n instead of 0.

In the $\ell_\circ$ (circular) case, we are imagining that $\pi(1)$ immediately follows $\pi(n)$. In the $\ell_\star$ (star) case, we imagine that 1 immediately follows n. When required, we will use the appropriate subscript on the parameter ℓ to denote these types of sequences. For example, $X_n^+(\pi, \ell_\star)$ will denote the number of increasing $\ell_\star$-sequences in π; $R_n^-(\pi, \ell_\circ)$ will denote the number of maximal decreasing $\ell_\circ$-sequences in π; and so on.

Derangement and Matching Numbers

Definition 1.6 (Fixed Point) Let $\pi \in \mathbb{S}_n$. We say $\pi(j)$ is a *fixed point* of π if and only if $\pi(j) = j$.

Definition 1.7 (Derangement) We say $\pi \in \mathbb{S}_n$ is a *derangement* if and only if π contains no fixed points.

The number of derangements of n elements is denoted by D_n or, in some literature, $n¡$ (read "n sub-factorial"). The general formula, given in (Whitworth 1901), is

$$D_n = n¡ = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor .$$

Where e is the base of the natural logarithm — hence $D_n = n¡$ is simply $n!/e$ rounded to the nearest integer.

In the more general case, we denote the number of permutations in $\mathbb{S}_n$ with exactly k fixed points as $D_{n,k}$. Thus, $D_{n,0} = D_n = n¡$. The numbers $D_{n,k}$ satisfy (Graham, Knuth, and Patashnik 1994)

$$D_{n,k} = \binom{n}{k} D_{n-k}.$$

We see that $D_{n,k}$ is the coefficient of z^k in the expansion of $(D + z)^n$ where we equate D^j with D_j.

The numbers $D_{n,k}$ are often referred to as *matching numbers* since they represent the number of ways of permuting n objects such that k of them remain in their original position (i.e. k matches).

Partitions and Compositions of Integers

Definition 1.8 Let n a positive integer and let $\xi = \{\xi_i\}_1^n$ be a sequence of n non-negative integers. Then, $\xi = \{\xi_i\}_1^n$ is a *partition of* n if and only if

$$\xi_1 + 2\xi_2 + \cdots + n\xi_n = n.$$

For a given n, we denote the set of all partitions of n by Ξ_n. The number of summands in the partition is $\xi_1 + \cdots + \xi_n$.

The cardinality of Ξ_n is denoted by $p(n)$ and is known as a *partition number*. The history of these numbers can be traced back to Leibniz, Bernoulli and Euler (cf. Dickson 1920). The generating function for $p(n)$ is well known to be

$$G_{p(n)}(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\cdots}.$$

While no known closed form for $p(n)$ exists, the values are easily computed using a recurrence. If we let $p(n, k)$ denote the number of partitions of n with no summand (part) larger that k, then

$$p(n, k) = p(n-k, k) + p(n, k-1),$$

with boundary conditions $p(n, 1) = 1$ for all $n \geq 0$ and $p(n, k) = 0$ for all $n < 0$. Thus, $p(n) = p(n, n)$.

By a composition of n into exactly m parts (n, m non-negative integers) we mean an ordered collection of positive integers $x_1, \ldots, x_m$ such that

$$x_1 + x_2 + \cdots + x_m = n.$$

For example, the compositions of 4 with exactly 2 parts are 13, 31 and 22. If the order is disregarded we obtain a partition of n.

We denote the number of possible compositions of n with exactly m parts as $C_{n,m}$. The generating function is

$$G_{C_m}(z) = (z + z^2 + z^3 + \cdots)^m = z^m(1-z)^{-m}, \qquad |z| < 1. \tag{1.8}$$

Writing $(1-z)^{-m}$ as $(-1)^m(z-1)^{-m}$ and making use of the Binomial Theorem, we obtain

$$G_{C_m}(z) = (-1)^m \sum_k \binom{-m}{k} z^{k+m}(-1)^{-k}. \tag{1.9}$$

Hence, the coefficient of z^n is $(-1)^{n-m}\binom{-m}{n-m} = \binom{n-1}{n-m}$ and we have

$$C_{n,m} = \binom{n-1}{n-m}, \qquad \text{integers } n, m \geq 0. \tag{1.10}$$

Note, when $n = m = 0$, we get $C_{0,0} = \binom{-1}{0} = (-1)^0 = 1$ which agrees with our generating function. In allowing $n = 0$, we must refrain from writing this as $\binom{n-1}{m-1}$ since the binomial reflection of $\binom{r}{k}$ is valid only for $0 \le r \in \mathbb{Z}$.

For non-negative integers n, m and ℓ, we denote the number of compositions of n with exactly m parts, none of which are greater than ℓ, by $C_{n,m}(\ell)$. The generating function is

$$G_{C_m}(z:\ell) = (z + z^2 + \cdots + z^\ell)^m = \left(\frac{z(1-z^\ell)}{1-z}\right)^m. \tag{1.11}$$

Hence

$$(1-z)G_{C_m}(z:\ell) = z(1-z^\ell)G_{C_{m-1}}(z:\ell). \tag{1.12}$$

Equating coefficients of z^n, we obtain the recursion

$$C_{n,m}(\ell) - C_{n-1,m}(\ell) = C_{n-1,m-1}(\ell) - C_{n-\ell-1,m-1}(\ell)$$

$$C_{n,m}(\ell) = C_{n-1,m-1}(\ell) - C_{n-\ell-1,m-1}(\ell) + C_{n-1,m}(\ell) \tag{1.13}$$

The boundary conditions are $C_{n,0}(\ell) = [m = 0]$ and $C_{0,m}(\ell) = [m = 0]$. In fact, $C_{n,m}(\ell) \neq 0$ implies $n \le \ell m \le \ell n$ and, if $\ell \ge n$, $C_{n,m}(\ell) = C_{n,m}$. If $\ell = 2$ then $C_{n,m}(2)$ is the coefficient of z^n in $(z + z^2)^m$ and we have the special case

$$C_{n,m}(2) = \binom{m}{n-m}. \tag{1.14}$$

From the generating function, $C_{n,m}(0) = [n = m = 0]$.

See Riordan (1958), Hall (1967) and Comtet (1974) for discussions of partitions, partition numbers and compositions as well as the related *denumerants*. The reader is also directed to the collection of papers by P. A. MacMahon in (Andrews 1978), in particular (MacMahon 1894, 1908).

1.2 Literature Review

The number of permutations in $\mathbb{S}_n$ without increasing 2-sequences first appears in (Whitworth 1901) where the solution D_{n+1}/n is given. Whitworth (1901) also considers the case of no increasing $2_\circ$-sequences.

Wolfowitz (1942) describes a non-parametric test of independence for paired samples using $R_n(\pi) = n - X_n(\pi, 2)$. He gives the numbers $\left\langle {n \atop k} \updownarrow 2 \right\rangle$ as the solution to a system of n equations in n unknowns — the equations easily solved by back-substitution. Wolfowitz (1944) derives the limiting distribution of $X_n(\pi, 2) = n - R_n(\pi)$ as Poisson with mean 2 while Kaplansky (1945) provides a more explicit result concerning the limiting distribution of $X_n(\pi, 2)$.

Kaplansky (1944) gives a symbolic solution to the n-kings problem: How many ways may n-kings be placed on an $n \times n$ chess board such that no two are in the same rank or file and no two attack each other. This is equivalent to determining the number of permutations in $\mathbb{S}_n$ with no increasing and/or decreasing 2-sequences since, if $\pi \in \mathbb{S}_n$ has no increasing and/or decreasing 2-sequences, we may use $\pi(1)$ for the rank of the king in the first file, $\pi(2)$ for the rank of the king in the second file, and so on up to $\pi(n)$. Riordan (1965) gives a recursive equation of order 5 for these numbers. Abramson and Moser (1966) provide an independent solution.

The number of permutations in $\mathbb{S}_n$ with k increasing 2, $2_\circ$ and $2_\star$-sequences are examined in (Roselle 1968; Roselle 1974; Tanny 1976; Reilly and Tanny 1979). The results are either explicitly expressed in terms of derangement numbers or can easily be shown to be equivalent. Roselle (1974) and Dymacek and Roselle (1978) also relate the numbers $\left\langle {n \atop k} \Big\uparrow 2_{\circ} \right\rangle$ and $\left\langle {n \atop k} \Big\uparrow 2_{\blacktriangledown} \right\rangle$ to the derangement numbers. Dwass (1973) follows a more probabilistic approach, making use of a Markov chain in determining the distribution of $X_n^+(\pi, 2)$ and showing its limiting distribution is Poisson with mean 1. More recently, Fu (1995) used a finite Markov chain embedding technique to give a simple matrix form for the distribution of $X_n^+(\pi, 2)$; to show that the asymptotic distribution is Poisson with mean 1; and, to show that the distribution of $X_n^+(\pi, \ell)$ is degenerate about 0 when $\ell > 2$.

Riordan (1945) uses combinatorics and the symbolic method to enumerate $\left\langle {n \atop k} \Big\uparrow 3 \right\rangle$ and derives the mean and variance of $X_n^+(\pi, 3)$.

The more general case of increasing ℓ-sequences ($\ell > 2$) is discussed in (Abramson and Moser 1967; Jackson and Reilly 1976; Jackson and Aleliunas 1977). Abramson and Moser (1967) make extensive use of combinatorics. Jackson and Reilly (1976) and Jackson and Aleliunas (1977) make use of generating functions to enumerate $\left\langle {n \atop k} \Big\uparrow \ell \right\rangle$. Jackson and Reilly (1976) further derive a recursion for the coefficients in the generating function and give a method to calculate $\left\langle {n \atop k} \Big\uparrow \ell \right\rangle$ in $O(n^3)$ time.

1.3 Overview

The remainder of this thesis is organized as follows: Chapter 2 reviews two known methods for the case $\ell = 2$. Chapter 3 contains the main results of the thesis — the general case $\left\langle {n \atop k} \Big\uparrow \ell \right\rangle$ is solved and the expectation of $X_n^+(\pi, \ell)$ is derived. From these two results, all known results for $\ell = 2$ and more efficient equations for the case $\ell = 3$

are derived. The chapter concludes with an alternate method for generating the numbers $\left\langle {n \atop k} \uparrow \ell \right\rangle$ and proving a closed form for the special case $\ell > \lfloor n/2 \rfloor$. Chapter 4 discusses some applications for the numbers $\left\langle {n \atop k} \uparrow \ell \right\rangle$ and the stochastic variable $X_n^+(\pi, \ell)$. Chapter 5 contains some concluding remarks and an extension of the problem. Appendix A enumerates the numbers $\left\langle {n \atop k} \uparrow \ell \right\rangle$ for $2 \leq \ell \leq n \leq 20$ and $0 \leq k \leq n - \ell + 1$.

For the impatient, dog-ear page x and turn to Table 3.1 on page 34.

Chapter 2

Increasing/Decreasing 2-sequences

This chapter reviews two methods for determining the numbers $\left\langle {n \atop k} \uparrow 2 \right\rangle$. We first apply the symbolic method to the determination of $\left\langle {n \atop k} \uparrow 2 \right\rangle$ and the related $\left\langle {n \atop k} \uparrow 2_{\circ} \right\rangle$. The second method makes use of the finite Markov chain embedding technique of Fu (1995).

2.1 Symbolic Approach

Riordan (1958) gives a formal description of the symbolic approach for solving a wide class of combinatorial problems. Kaplansky (1944) gives a description of the method and applies it to a number of problems involving permutations — in particular, the n-kings problem. Our approach follows closely that of Roselle (1974), who used this method to provide solutions for $\left\langle {n \atop 0} \uparrow 2 \right\rangle$, $\left\langle {n \atop 0} \uparrow 2_{*} \right\rangle$ and $\left\langle {n \atop 0} \updownarrow 2 \right\rangle$ as well as others.

Let $n \in \mathbb{N}$ given and $\mathbb{S}_n$ the symmetric group on n elements. Let $A_{i,i+1}$ denote the event that, in a random $\pi \in \mathbb{S}_n$, $i+1$ immediately follows i. Clearly, there are $n-1$ such events to consider: $A_{12}, A_{23}, \ldots, A_{(n-1),n}$. Let $A_{i_1}A_{i_2}\cdots A_{i_k}$ denote an arbitrary k element subset of these $n-1$ events and let $\phi_k = p(A_{i_1}A_{i_2}\cdots A_{i_k})$ denote the probability of the joint occurrence of $A_{i_1}\cdots A_{i_k}$; then, since k increasing 2-sequences leads to $n-k$ maximal increasing sequences which may be permuted in $(n-k)!$ ways, we have $\phi_k = p(A_{i_1}\cdots A_{i_k}) = (n-k)!/n!$. Thus, ϕ_k is symmetric in the sense that it is a function of k alone and never vanishes for $k \le n-1$. By Poincaré's formula, we have

$$\Pr[X_n^+(\pi,2)=0] = 1 - \sum_i p(A_i) + \sum_{i\neq j} p(A_iA_j) - \sum_{i\neq j\neq l} p(A_iA_jA_l) + \cdots .$$

Now, since a sum involving k events has $\binom{n-1}{k}$ terms, we have

$$\Pr[X_n^+(\pi,2)=0] = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{(n-k)!}{n!}, \tag{2.1}$$

which is the result in (Roselle 1974). If we let E such that $E^k\phi_0 = \phi_k$ then symbolically we have

$$\Pr[X_n^+(\pi,2)=0] = (1-E)^{n-1}\phi_0. \tag{2.2}$$

As an example, for $n=4$,

$$\begin{aligned}\Pr[X_n^+(\pi,2)=0] = (1-E)^3\phi_0 &= (1-3E+3E^2-E^3)\phi_0\\ &= \phi_0 - 3\phi_1 + 3\phi_2 - \phi_3\\ &= \frac{1}{4!}\left(1(4!) - 3(3!) + 3(2!) - 1\right)\\ &= \frac{11}{24};\end{aligned}$$

corresponding to the 11 permutations in $\mathbb{S}_4$ without increasing 2-sequences. We also have, by Poincaré's formula,

$$\begin{aligned}
\Pr[X_n^+(\pi,2)=r] &= \sum_{k=r}^{n-1}(-1)^{k-r}\binom{k}{k-r}\sum p(A_{i_1}\cdots A_{i_k}) \\
&= \sum_{k=r}^{n-1}(-1)^{k-r}\binom{k}{r}\binom{n-1}{k}\frac{(n-k)!}{n!} \\
&= \sum_{k=r}^{n-1}\frac{(-1)^{k-r}}{k!}\binom{k}{r}\left(\frac{n-k}{n}\right) \qquad (2.3)\\
&= \sum_{k=0}^{n-r-1}\frac{(-1)^k}{k!\,r!}\left(\frac{n-k-r}{n}\right) \\
&= \frac{1}{nr!}\left\{(n-r)\sum_{k=0}^{n-r}\frac{(-1)^k}{k!}+\sum_{k=0}^{n-r-1}\frac{(-1)^k}{k!}\right\} \\
&= \frac{1}{nr!}\left\{\frac{D_{n-r}}{(n-r-1)!}+\frac{D_{n-r-1}}{(n-r-1)!}\right\} \\
&= \frac{1}{n!}\binom{n-1}{r}\left(D_{n-r}+D_{n-r-1}\right).
\end{aligned}$$

Multiplication though by $n!$ and replacing r by k yields

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = \binom{n-1}{k}\left(D_{n-k}+D_{n-k-1}\right), \qquad (2.4)$$

which is the exact result appearing in (Tanny 1976) and (Reilly and Tanny 1979). Roselle (1968) gives the symbolic form

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = [x^k]\left\{(D+x)^n+(1-x)(D+x)^{n-1}\right\}, \qquad D^j \equiv D_j. \qquad (2.5)$$

The coefficient of x^k in this polynomial is

$$[x^k] = \binom{n}{k}D^{n-k}+\binom{n-1}{k}D^{n-k-1}-\binom{n-1}{k-1}D^{n-k}.$$

Equating D^j with D_j and noting that $\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}$ we see this is also equivalent to (2.4).

For a discrete random variable X, taking on values $0, 1, \ldots, n$ with probabilities $p_0, p_1, \ldots, p_n$, the ordinary generating function is given by

$$P(t) = \sum_{r=0}^{n} t^r p_r.$$

Evaluating at $t+1$ yields

$$P(t+1) = \sum_{r=0}^{n} (t+1)^r p_r = \sum_{r=0}^{n} p_r \sum_{k=0}^{r} \binom{r}{k} t^k = \sum_{k=0}^{n} t^k \sum_{r=k}^{n} \binom{r}{k} p_r = \sum_{k=0}^{n} \frac{t^k (m)_k}{k!};$$

where $(m)_k$ is the k^{th} factorial moment of X. Evaluating this last expression at $t-1$ gives

$$\begin{aligned} P(t) &= \sum_{k=0}^{n} \frac{(t-1)^k (m)_k}{k!} \\ &= \sum_{k=0}^{n} \frac{(m)_k}{k!} \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} t^r \\ &= \sum_{r=0}^{n} t^r \sum_{k=r}^{n} \binom{k}{r} \frac{(-1)^{k-r} (m)_k}{k!}. \end{aligned}$$

Hence,

$$p_r = \sum_{k=r}^{n} \binom{k}{r} \frac{(-1)^{k-r} (m)_k}{k!}.$$

Therefore, from (2.3), we see that the k^{th} factorial moment for $1 \leq k \leq n$ (higher factorial moments all vanish) is $(n-k)/n$ and we obtain

$$E[X_n^+(\pi, 2)] = \frac{n-1}{n} \quad \text{and} \quad Var[X_n^+(\pi, 2)] = 1 - \frac{1}{n} - \frac{1}{n^2}. \tag{2.6}$$

Since all factorial moments tend to 1 as $n \to \infty$ we have the limiting distribution of $X_n^+(\pi, \ell)$ as Poisson with mean 1.

For the case of increasing ℓ_*-sequences, we consider the addition event $A_{n,1}$, for a total of n possible events. As before, the probability of the joint occurrence of $k \leq n-1$ of these events is $(n-k)!/n!$ which is a function of k alone and never vanishes for $0 \leq k \leq n-1$ and we have the required symmetry. Since the probability of the joint occurrence of n of these events is 0, we may simply drop the last term in Poincaré's formula. The derivation is even simpler than for that of $\left\langle {n \atop k} \middle\uparrow 2 \right\rangle$. We have

$$\begin{aligned}
\left\langle {n \atop r} \middle\uparrow 2_* \right\rangle = n!\ \Pr[X_n^+(\pi, 2_*) = r] &= \sum_{k=r}^{n-1} (-1)^{k+r} \binom{k}{r} \binom{n}{k} (n-k)! \\
&= n! \sum_{k=r}^{n-1} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \qquad (2.7) \\
&= n! \sum_{k=0}^{n-r-1} \binom{k+r}{r} \frac{(-1)^k}{(k+r)!} \\
&= n! \sum_{k=0}^{n-r-1} \frac{(-1)^k}{k!r!} \\
&= n \binom{n-1}{r} D_{n-r-1}. \qquad (2.8)
\end{aligned}$$

We see immediately from (2.7) that the k^{th} factorial moment is 1 for $k = 1, \dots, n$ and 0 otherwise which is enough to verify that the limiting distribution is again Poisson with mean 1.

These results are equivalent to those of Roselle (1974); Tanny (1976); Dymacek and Roselle (1978); and, Reilly and Tanny (1979). Due to the following theorem, it suffices to determine $\left\langle {n \atop k} \uparrow 2_* \right\rangle$ only.

Theorem 2.1 *Let $\mathbb{S}_n$ the symmetric group on n elements, $\pi \in \mathbb{S}_n$ and π^{-1} its unique inverse. Then*

$$X_n^+(\pi, \ell_\circ) = X_n^+(\pi^{-1}, \ell_*) \tag{2.9}$$

Proof: Let $\pi \in \mathbb{S}_n$, then we clearly have, for all $1 \leq k < n$,

$$\pi(k) = m \text{ and } \pi(k+1) = m + 1 \bmod n$$

$$\Longleftrightarrow$$

$$\pi^{-1}(m) = k \text{ and } \pi^{-1}(m + 1 \bmod n) = k + 1.$$

Hence, every increasing ℓ_*-sequence maps onto a unique $\ell_\circ$-sequence which completes the proof. $\square$

In determining $\left\langle {n \atop 0} \updownarrow 2 \right\rangle$, Roselle (1974) uses the notion of *quasi-symmetry* — where $\phi_k = 0$ or is a function of k alone. The method is identical to that above except that all terms where ϕ_k vanishes must first be suppressed. Riordan (1945) uses the symbolic method to determine $\left\langle {n \atop k} \uparrow 3 \right\rangle$.

2.2 Finite Markov Chain Embedding Approach

Dwass (1973) made use of a non-homogeneous Markov chain to examine the distribution of $X_n^+(\pi, 2)$ and $X_n^+(\pi, 2_\circ)$. Fu (1994) formalized a finite Markov chain embedding technique for the study of the exact distribution of a specified pattern in a sequence of multi-state trials (cf. Fu and Koutras 1994; Fu 1996). Fu (1995) modifies this technique for use in determining the exact and limiting distributions of $X_n^+(\pi, 2)$.

The generation of a random permutation $\pi \in \mathbb{S}_n$ is equivalent to the stepwise insertion of n integers $\{1, \ldots, n\}$. For $\pi \in \mathbb{S}_n$, there is exactly $n+1$ positions to insert $n+1$ to obtain $\pi \in \mathbb{S}_{n+1}$. It is easy to see that this generation is unique in that, if at any stage we choose a different insertion position, the resulting permutation will be different. Let π_t denote the sub-permutation created after inserting the t^{th} integer. Consider the state space $\Omega = \{0, 1, \ldots, n-1\}$, the index set $\Gamma = \{0, 1, \ldots, n\}$ and a sequence of transformations Y_t: $\mathbb{S}_n \to \Omega$, $t = 1, \ldots, n$ where, for each $\pi \in \mathbb{S}_n$ and each $t = 1, 2, \ldots, n$,

$$Y_t(\pi) = X_t^+(\pi_t, 2). \tag{2.10}$$

For example, the permutation $\pi = (12376854)$ is created by the sequence of permutations

t	1	2	3	4	5	6	7	8
π_t	(1)	(12)	(123)	(1234)	(12354)	(123654)	(1237654)	(12376854)
$Y_t(\pi)$	0	1	2	3	2	2	2	2

In general, if $Y_{t-1}(\pi) = k$ for some $t = 2, \ldots, n$ and $k = 0, \ldots, t-2$ then $Y_t(\pi)$ can only be in states $k-1$, k or $k+1$. The integer t has equal probability of being inserted into any of the t available positions in $\pi \in \mathbb{S}_{t-1}$ and we have the following transition probabilities for $Y_t(\pi)$:

$$\Pr\left[\, Y_t = x \mid Y_{t-1} = k \,\right] = \begin{cases} k/t, & \text{if } x = k-1; \\ (t-k-1)/t, & \text{if } x = k; \\ 1/t, & \text{if } x = k+1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

Hence $\{\, Y_t : t \in \Gamma_n \}$ forms a non-homogeneous finite Markov chain on Ω with transition matrices $M_t(n) = p_{ij}(n : t)$, for $t = 1, \ldots, n$, where

$$p_{ij}(n : t) = \begin{cases} \Pr[\, Y_t = j \mid Y_{t-1} = i \,], & \text{for } i, j \in \{0, 1, \ldots, t-2\}; \\ 0, & \text{otherwise.} \end{cases}$$

Fu (1995) shows that

$$\Pr[X_n^+(\pi, 2) = i] = \Pr[\, Y_n = i \mid Y_0 = 0 \,] = a(0) \left(\prod_{t=1}^{n} M_t(n) \right) U'(i), \tag{2.12}$$

where $a(0) = (1, 0, 0, \ldots, 0)$ is an $1 \times n$ vector and $U'(i) = (0, \ldots, 0, 1, 0, \ldots, 0)'$ is an $n \times 1$ vector with a 1 at the i^{th} coordinate and 0's elsewhere.

Using these results, he further proves that

$$\lim_{n \to \infty} \Pr[X_n^+(\pi, 2) = x] = \frac{e^{-1}}{x!}, \qquad x = 0, 1, \ldots; \tag{2.13}$$

and, for $\ell \geq 3$,

$$\lim_{n \to \infty} \Pr[X_n^+(\pi, \ell) = x] = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \geq 1. \end{cases} \tag{2.14}$$

That is, for $\ell > 2$, the distribution of $X_n^+(\pi, \ell)$ has a degenerate limiting distribution at zero.

Chapter 3

Increasing/Decreasing ℓ-Sequences

In this chapter, we state and prove the main theorem, which gives a formula for $\left\langle {n \atop k} \uparrow \ell \right\rangle$ which may be evaluated in time of order $k(n-k)$ — a vast improvement over any known method. The corollaries yield a number of special cases and limiting distributions. We derive the expectation of $X_n^+(\pi, \ell)$ and show that the limiting distribution is degenerate for $\ell > 2$. In addition to the main results, we provide a second method for determining $\left\langle {n \atop k} \uparrow \ell \right\rangle$ based on partitions. The chapter concludes with the derivation a closed form for $\left\langle {n \atop k} \uparrow \ell \right\rangle$ when $\ell > \lfloor n/2 \rfloor$.

3.1 The Main Results

We prefer to save the discussion of the results until after all of the theorems and corollaries are stated and proved. In the following proofs, recall that $R_n^+(\pi, \ell)$ is the number of maximal ℓ-sequences in π and $R_n^+(\pi)$ is the total number of maximal increasing sequences in π. The proofs depend on the following identities:

$$C_{n,m}(2) = \binom{m}{n-m}. \qquad n, m \in \mathbb{Z},\ 0 \le m \le n; \tag{3.C}$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad n, k \in \mathbb{Z},\ n \ge 0; \tag{3.R}$$

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}, \qquad m, k \in \mathbb{Z}; \tag{3.T}$$

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \qquad n \in \mathbb{Z}; \tag{3.V}$$

and, for $n \in \mathbb{Z}$,

$$g(n) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k f(k) \iff f(n) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k g(k). \tag{3.I}$$

(3.C) is (1.14); (3.R) is the *binomial reflection;* (3.T) is known as the *trinomial revision*; (3.V) is *Vandermonde's convolution*; and, (3.I) is knows as the *binomial inversion.* The letters are intended as a mnemonic device.

Theorem 3.1 (Main Theorem) *Let* $n, \ell \in \mathbb{N}$ *and* $0 \le k \in \mathbb{Z}$. *Then the number of permutations in* $\mathbb{S}_n$ *with exactly* k *increasing* ℓ*-sequences is*

$$\left\langle {n \atop k} \middle\uparrow \ell \right\rangle = \sum_{p=1}^{n} \sum_{d=0}^{\min(k,p)} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \binom{p}{d}\binom{k-1}{k-d} C_{n-k-d(\ell-1),p-d}(\ell-1); \tag{3.1}$$

where $C_{n-k-d(\ell-1),p-d}(\ell-1)$ *is the number of compositions of* $n-k-d(\ell-1)$ *into exactly* $p-d$ *parts, none greater than* $\ell-1$, *and is calculated using (1.13). Furthermore, for* $\ell > 1$,

$$\left\langle {n \atop k} \middle\uparrow \ell \right\rangle = \sum_{p=p'}^{p^*} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \sum_{d=0}^{\min(k,p)} \binom{p}{d}\binom{k-1}{k-d} C_{n-k-d(\ell-1),p-d}(\ell-1), \tag{3.2}$$

where

$$p' = \left\lceil \frac{n-k}{\ell - 1} \right\rceil \qquad \text{and} \qquad p^* = n - (k + \ell - 2)[k \neq 0].$$

Proof: Let n, k and ℓ as given, let $p, d \in \mathbb{Z}$ such that $1 \leq p \leq n$ and $0 \leq d \leq n$ and let $S_n(k, \ell, p, d)$ denote the set of all $\pi \in \mathbb{S}_n$ such that

$$R_n^+(\pi) = p, \quad \sum_{\ell' \geq \ell} R_n^+(\pi, \ell') = d \quad \text{and} \quad X_n^+(\pi, \ell) = k. \tag{$*$}$$

That is, if $\pi \in S_n(k, \ell, p, d)$, then π has exactly p maximal increasing sequences (of lengths $\ell_1, \ldots, \ell_p$) and

i) $\exists\, I_d = \{i_1, \ldots, i_d\} \subseteq \{1, \ldots, p\}$ such that $\ell_j \geq \ell$ for all $j \in I_d$;

ii) $(\ell_{i_1} - \ell + 1) + (\ell_{i_2} - \ell + 1) + \cdots + (\ell_{i_d} - \ell + 1) = k$; and,

iii) $\ell_j < \ell$ for all $j \notin I_d$.

Now, there are $\binom{p}{d}$ ways to choose the $\{i_1, \ldots, i_d\}$ satisfying (i); for each of these, there are $\binom{k-1}{k-d}$ solutions satisfying (ii) [by (1.10)]; and, for each of these, there are $C_{n-k-d(\ell-1),p-d}(\ell - 1)$ solutions satisfying (iii) [since (ii) implies $\ell_{i_1} + \cdots + \ell_{i_d} = k + d(\ell - 1)$]. By definition, there are $\left\langle {p \atop 0} \middle\uparrow 2 \right\rangle$ ways to arrange these p maximal sequences such that $(*)$ is still satisfied and we have

$$\#S_n(k, \ell, p, d) = \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \binom{p}{d} \binom{k-1}{k-d} C_{n-k-d(\ell-1),p-d}(\ell - 1).$$

Given n, k and ℓ, the sets $S_n(k, \ell, p, d)$ are pairwise disjoint so that summing $\#S_n(k, \ell, p, d)$ over all possible p and d yields $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ and hence (3.1). For (3.2), we note that the upper bound on p is clearly $n-(k+\ell-2)[k\neq 0]$; for the lower bound, we have

$$C_{n-k-d(\ell-1),p-d}(\ell-1)\neq 0 \implies p-d\geq\left\lceil\frac{n-k-d(\ell-1)}{\ell-1}\right\rceil \implies p\geq\left\lceil\frac{n-k}{\ell-1}\right\rceil.$$

Also, $d\leq p$ and $d\leq k$ due to the binomial coefficients and (3.2) follows. $\square$

Remark 3.2 It is convenient to define, by convention, $\left\langle {0 \atop k} \middle\uparrow \ell \right\rangle = [k=0]$ for all $\ell\in\mathbb{N}$ and extend the double summation in (3.1) over all $p,d\geq 0$. The additional terms all equate to 0 and the resulting equations are easier to manipulate.

Corollary 3.3 *Let $n\in\mathbb{N}$ and $k\in\mathbb{Z}$ such that $0\leq k\leq n$. Then the number of permutations in $\mathbb{S}_n$ with exactly k increasing 2-sequences satisfies*

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = \binom{n-1}{k}\left\langle {n-k \atop 0} \middle\uparrow 2 \right\rangle, \tag{3.3}$$

and hence, for $k\geq 0$,

$$k\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = (n-1)\left\langle {n-1 \atop k-1} \middle\uparrow 2 \right\rangle. \tag{3.4}$$

Proof: Put $\ell=2$ into (3.1) and note that $C_{n-k-d,p-d}(1)=1$ when $p=n-k$ and 0 otherwise. We obtain

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = \left\langle {n-k \atop 0} \middle\uparrow 2 \right\rangle \sum_{d=0}^{d^*}\binom{n-k}{d}\binom{k-1}{k-d},$$

and (3.3) follows by (3.V). Equation (3.4) follows since

$$\begin{aligned}(n-1)\left\langle {n-1 \atop k-1} \middle\uparrow 2 \right\rangle &= (n-1)\binom{n-2}{k-1}\left\langle {n-k \atop 0} \middle\uparrow 2 \right\rangle \\ &= k\binom{n-1}{k}\left\langle {n-k \atop 0} \middle\uparrow 2 \right\rangle \\ &= k\left\langle {n \atop k} \middle\uparrow 2 \right\rangle. \quad \square\end{aligned}$$

Corollary 3.4 *The limiting distribution of* $X_n^+(\pi, 2)$ *is Poisson with mean 1. That is,*

$$\lim_{n\to\infty} \Pr[X_n^+(\pi,2) = k] = \frac{e^{-1}}{k!}. \tag{3.5}$$

Proof: It follows easily from recurrence (3.4) that the m^{th} factorial moment, for $1 \leq m \leq n$, of the distribution of $X_n^+(\pi, 2)$ is

$$\begin{aligned}E[X_n^+(\pi,2)\cdots(X_n^+(\pi,2)-m+1)] &= \frac{1}{n!}\sum_k k(k-1)\cdots(k-m+1)\left\langle {n \atop k} \middle\uparrow 2 \right\rangle \\ &= \frac{(n-m)(n-1)!}{n!} \\ &= \frac{n-m}{n}\end{aligned}$$

Since these tend to 1 as $n \to \infty$ the result follows immediately. $\square$

Corollary 3.5 *Let* $n \in \mathbb{N}$. *The number of permutations in* $\mathbb{S}_n$ *with no increasing 2-sequence is*

$$\left\langle {n \atop 0} \middle\uparrow 2 \right\rangle = \frac{D_{n+1}}{n} = \frac{1}{n}\left\lfloor \frac{(n+1)!}{e} + \frac{1}{2} \right\rfloor. \tag{3.6}$$

Proof: Let $n \in \mathbb{N}$. It is easy to see that summing $\left\langle {n \atop k} \middle\uparrow 2 \right\rangle$ over all k yields $n!$, hence

$$nn! = n\sum_{k=0}^{n}\left\langle {n \atop k}\middle\uparrow 2\right\rangle = n\sum_{k=0}^{n}\binom{n-1}{k}\left\langle {n-k \atop 0}\middle\uparrow 2\right\rangle$$
$$= \sum_{k=0}^{n} k\binom{n}{k}\left\langle {k \atop 0}\middle\uparrow 2\right\rangle.$$

Where we are replacing k by $n-k$ and $\binom{n-1}{n-k}$ by $\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$. We now make use of (3.I), with $g(n) = nn!$ and $f(k) = (-1)^k k\left\langle {k \atop 0}\middle\uparrow 2\right\rangle$, and obtain

$$(-1)^n n\left\langle {n \atop 0}\middle\uparrow 2\right\rangle = \sum_{k=0}^{n}\binom{n}{k}(-1)^k kk!.$$

Whence

$$n\left\langle {n \atop 0}\middle\uparrow 2\right\rangle = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}kk! = n!\sum_{k=0}^{n}\frac{k}{(n-k)!}(-1)^{n+k}$$
$$= n!\sum_{k=0}^{n}\frac{n-k}{k!}(-1)^k$$
$$= (n+1)!\sum_{k=0}^{n+1}\frac{(-1)^k}{k!}.$$

The last line follows from the fact that

$$-\sum_{k=0}^{n}\frac{k(-1)^k}{k!} = \sum_{k=0}^{n-1}\frac{(-1)^k}{k!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!} + (n+1)\frac{(-1)^{n+1}}{(n+1)!}.$$

We have shown that

$$\left\langle {n \atop 0}\middle\uparrow 2\right\rangle = \frac{(n+1)!}{n}\sum_{k=0}^{n+1}\frac{(-1)^k}{k!} = \frac{D_{n+1}}{n} = \frac{1}{n}\left\lfloor\frac{(n+1)!}{e} + \frac{1}{2}\right\rfloor,$$

which completes the proof. $\square$

Corollary 3.6 *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $0 \leq k \leq n$. Then the number of permutations in $\mathbb{S}_n$ with exactly k increasing 2-sequences is*

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = \binom{n}{k} \frac{D_{n-k+1}}{n} = \frac{1}{n}\binom{n}{k} \left\lfloor \frac{(n-k+1)!}{e} + \frac{1}{2} \right\rfloor. \tag{3.7}$$

Proof: Directly from Corollaries 3.3 and 3.5. □

Corollary 3.7 *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $0 \leq k \leq n$. Then the number of permutations in $\mathbb{S}_n$ with exactly k increasing 3-sequences is*

$$\left\langle {n \atop k} \middle\uparrow 3 \right\rangle = \sum_{p=k}^{\lfloor (n+k)/2 \rfloor} \left\langle {n-p \atop 0} \middle\uparrow 2 \right\rangle \binom{n-p}{p-k} \binom{p-1}{k}. \tag{3.8}$$

Proof: Put $\ell = 3$ into (3.1) and note that, by (3.C), we have $C_{n-k-2d,p-d}(2) = \binom{p-d}{n-k-d-p}$. Hence, by Remark 3.2, we have

$$\begin{aligned}
\left\langle {n \atop k} \middle\uparrow 3 \right\rangle &= \sum_{p \geq 0} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \sum_{d \geq 0} \binom{p}{d}\binom{k-1}{k-d}\binom{p-d}{n-k-d-p} && \text{(by 3.C)} \\
&= \sum_{p \geq 0} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \sum_{d \geq 0} \binom{k-1}{k-d}\binom{p}{p-d}\binom{p-d}{k+2p-n} && \text{(by 3.R)} \\
&= \sum_{p \geq 0} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \binom{p}{n-k-p} \sum_{d \geq 0} \binom{k-1}{k-d}\binom{n-k-p}{d} && \text{(by 3.T)} \\
&= \sum_{p=0}^{n} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \binom{p}{n-k-p}\binom{n-p-1}{k} && \text{(by 3.V)}
\end{aligned}$$

Replacing p with $n-p$ and noting that $\binom{n-p}{p-k} \neq 0$ implies $k \leq p \leq \lfloor (n+k)/2 \rfloor$ completes the proof. □

Corollary 3.8 *Let $n \in \mathbb{N}$. Then the number of permutations in $\mathbb{S}_n$ with no increasing 3-sequence is*

$$\left\langle {n \atop 0} \middle\uparrow 3 \right\rangle = \sum_{p=0}^{\lfloor n/2 \rfloor} \left\langle {n-p \atop 0} \middle\uparrow 2 \right\rangle \binom{n-p}{p}. \tag{3.9}$$

Proof: Put $k = 0$ into (3.8) and note that $\binom{n-p}{p} \neq 0$ implies $p \leq \lfloor n/2 \rfloor$. □

Corollary 3.9 *Let $n, \ell \in \mathbb{N}$ such that $\ell > 1$. Then the number of permutations in $\mathbb{S}_n$ with no increasing ℓ-sequence is*

$$\left\langle {n \atop 0} \middle\uparrow \ell \right\rangle = \sum_{p=p'}^{n} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle C_{n,p}(\ell - 1), \qquad p' = \left\lceil \frac{n}{\ell - 1} \right\rceil. \tag{3.10}$$

Proof: Put $k = 0$ in (3.2) and note that $d = 0$ only, hence

$$\binom{p}{d}\binom{k-1}{k-d} = \binom{p}{0}\binom{-1}{0} = 1$$

and the result follows. □

Corollary 3.10 *The number of summands required to calculate $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ is, at most,* $O(\,k(n-k)\,)$

Proof: Follows directly from the bounds on p and d in Theorem 3.1. □

Theorem 3.11 and it's corollaries concern the distribution of $X_n^+(\pi, \ell)$.

Theorem 3.11 *Let $n, \ell \in \mathbb{N}$ such that $\ell \leq n$. Then*

$$\sum_{k \geq 0} k \left\langle {n \atop k} \middle\uparrow \ell \right\rangle = (n - \ell + 1)(n - \ell + 1)!. \tag{3.11}$$

Proof: First note that, since $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle = \sum_{\pi\in\mathbb{S}_n}[X_n^+(\pi,\ell)=k]$, we may write

$$\sum_{k\geq 0} k \left\langle {n \atop k} \middle\uparrow \ell \right\rangle = \sum_{\pi\in\mathbb{S}_n}\sum_{k\geq 0} k[X_n^+(\pi,\ell)=k] = \sum_{\pi\in\mathbb{S}_n} X_n^+(\pi,\ell).$$

Now, given n and ℓ, let $\sigma_1, \sigma_2, \ldots, \sigma_{n!}$ a fixed enumeration of all permutations in $\mathbb{S}_n$ and define

$$m_{ij} = \begin{cases} \sigma_i(j), & \text{if } \sigma_i(j)\cdots\sigma_i(j+\ell-1) \text{ is an increasing } \ell\text{-sequence;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

For example, if $n = 8$, $\ell = 2$ and $\sigma_i = (12356478)$, then

$$(m_{i1}, \ldots, m_{in}) = (1, 2, 0, 5, 0, 0, 7, 0).$$

It is easy to see that $0 \leq m_{ij} \leq n - \ell + 1$ and, for all $j > n - \ell + 1$, that $m_{ij} = 0$. For each $i = 1, \ldots, n!$, we have

$$X_n^+(\sigma_i, \ell) = \sum_{j=1}^{n-\ell+1} [m_{ij} \neq 0].$$

Now, $[m_{ij} \neq 0] = [m_{ij} = 1] + [m_{ij} = 2] + \cdots + [m_{ij} = n - \ell + 1]$ and we have

$$\begin{aligned} \sum_{i=1}^{n!} X_n^+(\sigma_i, \ell) &= \sum_{i=1}^{n!}\sum_{j=1}^{n-\ell+1}\sum_{k=1}^{n-\ell+1} [m_{ij} = k] \\ &= \sum_{j=1}^{n-\ell+1}\sum_{k=1}^{n-\ell+1}\sum_{i=1}^{n!} [m_{ij} = k]. \end{aligned}$$

But $m_{ij} = k$ if and only if

$$\sigma_i(j) = k, \quad \sigma_i(j+1) = k+1, \quad \ldots, \quad \sigma_i(j+\ell-1) = k+\ell-1;$$

hence the inner summation is the number of permutations in $\mathbb{S}_n$ with ℓ elements fixed which is $(n-\ell)!$. We obtain

$$\begin{aligned}\sum_{i=1}^{n!} X_n^+(\sigma_i, \ell) &= \sum_{j=1}^{n-\ell+1}\sum_{k=1}^{n-\ell+1}\sum_{i=1}^{n!}[m_{ij}=k] \\ &= \sum_{j=1}^{n-\ell+1}\sum_{k=1}^{n-\ell+1}(n-\ell)! \\ &= (n-\ell+1)^2(n-\ell)!. \quad \square\end{aligned}$$

Remark 3.12 Intuitively, the left hand side of (3.11) is simply the totality of all possible increasing ℓ-sequences in $\mathbb{S}_n$. There is $(n-\ell+1)^2$ ways to fix the starting position and initial element of an increasing ℓ-sequence. Once fixed, this particular increasing ℓ-sequence will occur in $(n-\ell)!$ permutations, hence, there are $(n-\ell+1)^2(n-\ell)! = (n-\ell+1)(n-\ell+1)!$ increasing ℓ-sequences in $\mathbb{S}_n$.

Note that (3.11) directly leads to the following recursions

$$\sum_{k=0}^{n+m} k\left\langle {n+m \atop k}\middle\uparrow \ell+m\right\rangle = \sum_{k=0}^{n} k\left\langle {n \atop k}\middle\uparrow \ell\right\rangle, \qquad \forall m \in \mathbb{N}; \tag{3.13}$$

$$\sum_{k=0}^{n+m} k\left\langle {n+m \atop k}\middle\uparrow \ell\right\rangle = \sum_{k=0}^{n} k\left\langle {n \atop k}\middle\uparrow \ell-m\right\rangle, \qquad \forall m \in \mathbb{N}, m<\ell; \tag{3.14}$$

$$\sum_{k=0}^{n} k\left\langle {n \atop k}\middle\uparrow \ell+1\right\rangle = \sum_{k=0}^{n+1} k\left\langle {n+1 \atop k}\middle\uparrow \ell\right\rangle = \frac{(n-\ell+2)^2}{(n-\ell+1)}\sum_{k=0}^{n} k\left\langle {n \atop k}\middle\uparrow \ell\right\rangle. \tag{3.15}$$

Corollary 3.13 *Let $n, \ell \in \mathbb{N}$ such that $\ell \le n$. Then*

$$E[X_n^+(\pi, \ell)] = \frac{(n-\ell+1)(n-\ell+1)!}{n!}. \tag{3.16}$$

Proof: Directly from (3.11). Note that, for $\ell = 1$, we get $\mathsf{E}[X_n^+(\pi, 1)] = n$ as expected. $\square$

Note also that, from (3.13), we have

$$\mathsf{E}[X_n^+(\pi,\ell)] = \frac{\mathsf{E}[X_{n-1}^+(\pi,\ell-1)]}{n}, \qquad n, \ell > 1. \tag{3.17}$$

Corollary 3.14 *Let $\ell \in \mathbb{N}$ such that $\ell > 2$. Then*

$$\lim_{n\to\infty} \Pr[X_n^+(\pi,\ell) = 0] = 1.$$

That is, the limiting distribution of $X_n^+(\pi,\ell)$ is degenerate about 0 for $\ell > 2$.

Proof: By the Markov inequality, we have

$$\begin{aligned}\Pr[X_n^+(\pi,\ell) \geq 1] &\leq \mathsf{E}[X_n^+(\pi,\ell)] \\ &= \frac{(n-\ell+1)(n-\ell+1)!}{n!}\end{aligned}$$

Taking the limit of both sides as $n \to \infty$ completes the proof. $\square$

Discussion of the Main Results

Table 3.1 summarizes the main results. For the case $\ell = 2$, we note that $D_{n-k+1} = (n-k)(D_{n-k} + D_{n-k-1})$; hence, we may rewrite (3.7) as

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = \frac{n-k}{n}\binom{n}{k}(D_{n-k} + D_{n-k-1}) = \binom{n-1}{k}(D_{n-k} + D_{n-k-1}),$$

Table 3.1: Summary of Main Results for $n, \ell \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $0 \le k \le n$.

Boundary cases:

$$\left\langle {n \atop k} \middle| \ell \right\rangle = 0, \quad k+\ell-1>n; \qquad \left\langle {n \atop k} \middle| 1 \right\rangle = n!\,[k=n]; \qquad \left\langle {0 \atop k} \middle| \ell \right\rangle = [k=0].$$

General case:

$$\left\langle {n \atop k} \middle| \ell \right\rangle = \sum_{p=1}^{n} \sum_{d=0}^{\min(k,p)} \left\langle {p \atop 0} \middle| 2 \right\rangle \binom{p}{d} \binom{k-1}{k-d} C_{n-k-d(\ell-1),p-d}(\ell-1).$$

$$\sum_{k\ge 0} k \left\langle {n \atop k} \middle| \ell \right\rangle = (n-\ell+1)(n-\ell+1)!, \qquad \ell \le n.$$

Special cases:

$$\left\langle {n \atop 0} \middle| \ell \right\rangle = \sum_{p=p'}^{n} \left\langle {p \atop 0} \middle| 2 \right\rangle C_{n,p}(\ell-1), \qquad p' = \left\lceil \frac{n}{\ell-1} \right\rceil.$$

$$\left\langle {n \atop 0} \middle| 2 \right\rangle = \frac{D_{n+1}}{n} = \frac{1}{n} \left\lfloor \frac{(n+1)!}{e} + \frac{1}{2} \right\rfloor.$$

$$\left\langle {n \atop k} \middle| 2 \right\rangle = \binom{n-1}{k} \left\langle {n-k \atop 0} \middle| 2 \right\rangle = \binom{n}{k} \frac{D_{n-k+1}}{n} = \frac{1}{n} \binom{n}{k} \left\lfloor \frac{(n-k+1)!}{e} + \frac{1}{2} \right\rfloor.$$

$$\left\langle {n \atop k} \middle| 3 \right\rangle = \sum_{p=k}^{\lfloor (n+k)/2 \rfloor} \left\langle {n-p \atop 0} \middle| 2 \right\rangle \binom{n-p}{p-k} \binom{p-1}{k}.$$

$$\left\langle {n \atop 0} \middle| 3 \right\rangle = \sum_{p=0}^{\lfloor n/2 \rfloor} \left\langle {n-p \atop 0} \middle| 2 \right\rangle \binom{n-p}{p}.$$

Expectations and Limiting Distributions ($\ell \le n$):

$$E[X_n^+(\pi,\ell)] = \frac{(n-\ell+1)(n-\ell+1)!}{n!}; \qquad Var[X_n^+(\pi,2)] = 1 - \frac{1}{n} - \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \Pr[X_n^+(\pi,2)=k] = \frac{e^{-1}}{k!}; \qquad \lim_{n\to\infty} \Pr[X_n^+(\pi,\ell)=0] = 1, \quad \ell > 2.$$

which is consistent with the results in (Tanny 1976) and (Reilly and Tanny 1979). Roselle (1968) gives the symbolic form

$$\left\langle {n \atop k} \middle\uparrow 2 \right\rangle = [x^k]\left\{(D+x)^n + (1-x)(D+x)^{n-1}\right\}, \qquad D^j \equiv D_j. \tag{3.18}$$

The coefficient of x^k in this polynomial is

$$[x^k] = \binom{n}{k} D^{n-k} + \binom{n-1}{k} D^{n-k-1} - \binom{n-1}{k-1} D^{n-k}.$$

Equating D^j with D_j and noting that $\binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}$, we see this is also equivalent to (3.7).

The limiting distribution of $X_n^+(\pi, 2)$ is well known to be Poisson with mean 1 (cf. Dwass 1973; Tanny 1976; Reilly and Tanny 1979; Fu 1995) and Corollary 3.14 is consistent with (Fu 1995). We also note, from Corollary 3.3, that

$$E[X_n^+(\pi,2)] = \frac{n-1}{n}, \qquad Var[X_n^+(\pi,2)] = 1 - \frac{1}{n} - \frac{1}{n^2}.$$

For the case $\ell = 3$, Riordan (1945) gives

$$\left\langle {n \atop k} \middle\uparrow 3 \right\rangle = \sum_{j=0}^{n-2} (-1)^{j+k} \binom{j}{k} \sum_{i=0}^{j-1} \binom{j-1}{i} \binom{n-j-1}{j-i} (n-2j+i)!, \tag{3.19}$$

where $\binom{j}{k} = 0$ if $j < k$; and shows that

$$E[X_n^+(\pi,3)] = \frac{n-2}{n(n-1)} \quad \text{and} \quad Var[X_n^+(\pi,3)] = \frac{n^4 - 3n^3 + n^2 - 6n + 8}{(n)_2 (n)_3},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. The expectation is consistent with Corollary 3.13 and we have

$$\sum_{j=0}^{n-2}(-1)^{j+k}\binom{j}{k}\sum_{i=0}^{j-1}\binom{j-1}{i}\binom{n-j-1}{j-i}(n-2j+i)! \\ = \sum_{p=0}^{n}\left\langle {n-p \atop 0}\middle| 2\right\rangle\binom{n-p}{p-k}\binom{p-1}{k}. \quad (3.20)$$

Abramson and Moser (1967) give the solution for the general case as

$$\begin{aligned}\left\langle {n \atop k}\middle| \ell\right\rangle = &\sum_{i=1}^{n-\ell-1}(-1)^i\binom{k+i}{i}\sum_{a_1=0}^{i-1}\sum_{a_2=0}^{a_1}\sum_{a_3=0}^{a_2} \\ &\cdots\sum_{a_{\ell-1}=0}^{a_{\ell-2}}\binom{i-1}{a_1}\binom{a_1}{a_2}\binom{a_2}{a_3}\cdots\binom{a_{\ell-2}}{a_{\ell-1}} \\ &\times\binom{n-\ell-k-i-a_1-\cdots-a_{\ell-2}+2}{a_{\ell-1}+1} \\ &\times(n-k-i-a_1-a_2-\cdots-a_{\ell-2}-\ell+2)!.\end{aligned} \quad (3.21)$$

This summation is complicated to evaluate and the number of terms is exponential in ℓ. We note that, by a change of variables, the product of the binomials can be cast to a multinomial coefficient; hence, in effect, we are summing over partitions — a method that is explored in the next section.

Jackson and Reilly (1976) and Jackson and Aleliunas (1977) provide independent proofs that

$$\left\langle {n \atop k}\middle| \ell\right\rangle = [z^n w^k]\sum_{j\geq 0} j!z^j\left(\frac{1-wz-(1-w)z^{\ell-1}}{1-wz-(1-w)z^{\ell}}\right)^j, \quad (3.22)$$

and we obtain, for $\ell \in \mathbb{N}$,

$$[z^n w^k] \sum_{j \geq 0} j! z^j \left(\frac{1 - wz - (1-w)z^{\ell-1}}{1 - wz - (1-w)z^{\ell}} \right)^j$$
$$= \sum_{p=0}^{n} \sum_{d=0}^{p} \left\langle {p \atop 0} \middle\uparrow 2 \right\rangle \binom{p}{d} \binom{k-1}{k-d} C_{n-k-d(\ell-1),p-d}(\ell - 1). \quad (3.23)$$

Jackson and Reilly (1976) further show that

$$\left\langle {n \atop k} \middle\uparrow \ell \right\rangle = [z^k] \sum_{j=0}^{n} j!\, a(n,j), \quad (3.24)$$

where $a(n,j)$ satisfies

$$\begin{aligned} a(n,j) &= z(n-1,j) + (1-z)a(n-\ell,j) + a(n-1,j-1) \\ &\quad - za(n-2,j-1) - (1-z)a(n-\ell,j-1) \end{aligned} \quad (3.25)$$

with boundary conditions $a(n,j) = [n = j]$ for $n \leq \ell - 1$ and $j \geq 0$ and $a(n,0) = [n = 0]$. They give an algorithm for calculating the generating function in $O(n^3)$ time. Since the number of summands in (3.2) of Theorem 3.1 is of order $k(n-k)$ by Corollary 3.10 and, given ℓ, the values of $C_{n,m}(\ell)$ can be pre-calculated in $O(n^2)$ time, we have a more efficient method of determining $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ than that of Jackson and Reilly (1976).

3.2 An Alternate Method

Let Ξ_n denote the set of all partitions of the integer n and let $\xi = \{\xi_i\}_1^n$ denote an arbitrary element of Ξ_n so that

$$\xi_1 + 2\xi_2 + \cdots n\xi_n = n.$$

Define the mapping $\Phi_n : \mathbb{S}_n \rightarrow \Xi_n$ as

$$\Phi_n(\pi) := \xi = \{\xi_i\}_1^n, \qquad \text{where } \xi_i = R_n^+(\pi, i) \text{ for } i = 1, \ldots, n. \tag{3.26}$$

Clearly Φ_n is defined for all $\pi \in \mathbb{S}_n$ and onto Ξ_n. It will be convenient to fix this definition of Φ_n and refer to it as the *canonical mapping* of $\mathbb{S}_n$ onto Ξ_n. We now prove

Theorem 3.15 *Let $\mathbb{S}_n$ the symmetric group on n elements, let Ξ_n the set of all partitions of n and let Φ_n the canonical mapping of $\mathbb{S}_n$ onto Ξ_n. Then, given $\xi = \{\xi_i\}_1^n \in \Xi_n$,*

$$\#\Phi_n^{-1}[\xi] = \#\{\pi \in \mathbb{S}_n \mid \Phi_n(\pi) = \xi\} = \frac{(\sum_{i=1}^n \xi_i)\,!}{\xi_1!\,\xi_2!\,\cdots\,\xi_n!} \left\langle \begin{matrix} \sum_{i=1}^n \xi_i \\ 0 \end{matrix} \middle| 2 \right\rangle. \tag{3.27}$$

Proof: Let $\mathbb{S}_n$, Ξ_n and Φ_n as given and let $\xi = \{\xi_i\}_1^n \in \Xi_n$. Then ξ corresponds to the unique multi-set $\{1 \cdots 12 \cdots 2 \cdots n \cdots n\}$ where there are ξ_1 copies of 1's, ξ_2 copies of 2's, and so on, up to ξ_n copies of n's. The possible distinct arrangements of this multi-set is clearly

$$\frac{(\sum_{i=1}^n \xi_i)\,!}{\xi_1!\,\xi_2!\,\cdots\,\xi_n!}.$$

Each possible arrangement may be used to divide the sequence $123 \cdots n$ into $\sum \xi_i$ subsequences by using the k^{th} element of the multi-set as the length of the k^{th} subsequence. For example, if $n = 8$ and $\xi = (1, 2, 1, 0, 0, 0, 0, 0)$, then ξ corresponds to the multi-set $\{1223\}$ and one possible arrangement is $\{2312\}$ which defines the subsequences (12), (345), (6) and (78). By definition, the number of ways to arrange these subsequences such that no longer subsequences are created is

$$\left\langle {\sum_{i=1}^{n} \xi_i \atop 0} \middle\uparrow 2 \right\rangle.$$

Clearly, each $\pi \in \mathbb{S}_n$ such that $\Phi_n(\pi) = \xi$ is enumerated by the above method. Also, if $\pi \in \mathbb{S}_n$ such that $\Phi_n(\pi) = \xi$ and $\pi \in \Phi_n^{-1}[\xi^*]$ then $\xi^* = \xi$, which completes the proof. $\square$

This result immediately yields the generating function

$$G(z, w : n) = \sum_{\xi \in \Xi_n} \frac{(\sum_{i=1}^{n} \xi_i)!}{\xi_1! \, \xi_2! \cdots \xi_n!} \left\langle {\sum_{i=1}^{n} \xi_i \atop 0} \middle\uparrow 2 \right\rangle \sum_{\ell=2}^{n} z^{\ell} w^{\xi_\ell + 2\xi_{\ell+1} + \cdots + (n-\ell+1)\xi_n}, \tag{3.28}$$

where the number of permutations in $\mathbb{S}_n$ with exactly k increasing ℓ sequences is $[z^{\ell} w^{k}] G(z, w : n)$.

Apart for the curious nature of the generating function, we now have a method for enumerating the numbers $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ for all $\ell = 2, \ldots, n$ and all $k = 0, \ldots, n - \ell + 1$. Figure 3.1 gives an algorithm for calculating the values by making use of the fact that, given π and $\Phi_n(\pi) = \xi = \{\xi_i\}_1^n$,

$$X_n^+(\pi, \ell) = \sum_{k=\ell}^{n} \xi_k (k - \ell + 1) = X_n^+(\pi, \ell + 1) + \sum_{k=\ell}^{n} \xi_k. \tag{3.29}$$

To analyze the running time of the algorithm we first note that the cardinality of Ξ_n is $p(n)$. Ehrlich (1973) gives an algorithm to iterate though the partitions of n such that each iteration takes constant time and the number of operations in steps 1–3 is clearly proportional to n; hence, the total running time is of order $O(np(n))$. For large n, this algorithm cannot compete with the results in the previous section

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For each $\xi \in \Xi_n$ do:

1. **calculate**

$$f(\xi) = \frac{(\sum_{i=1}^{n} \xi_i)\,!}{\xi_1!\,\xi_2!\,\cdots\,\xi_n!} \left\langle {\sum_{i=1}^{n} \xi_i \atop 0} \middle\uparrow 2 \right\rangle .$$

2. **set** $s = 0$ **and** $k = 0$. Here, s plays the role of $\sum_{i=\ell}^{n} \xi_i$ and k plays the role of $X_n^+(\pi, \ell)$ in (3.29).

3. **For** $\ell = n$ **downto** 2 **do:**

   3a. set $s = s + \xi_\ell$.

   3b. set $k = k + s$.

   3c. set $\left\langle {n \atop k} \uparrow \ell \right\rangle = \left\langle {n \atop k} \uparrow \ell \right\rangle + f(\xi)$.

Figure 3.1: Algorithm for Calculating $\left\langle {n \atop k} \uparrow \ell \right\rangle$.

or with the (Jackson and Reilly 1976) algorithm. It should be noted however that $np(n) < n^3$ for $n \leq 16$. The main advantage is that it is simple to implement and simultaneously computes all values of $\left\langle {n \atop k} \uparrow \ell \right\rangle$ given n. The (Jackson and Reilly 1976) algorithm requires n and ℓ given and calculates $\left\langle {m \atop k} \uparrow \ell \right\rangle$ for $0 \leq k < m \leq n$. Another advantage of the above algorithm is that the memory requirements are much less than that required in (Jackson and Reilly 1976).

Hardy and Ramanujan (1918) show that $p(n)$ satisfies

$$p(n) \leq g(n) = \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right),$$

and that $p(n)/g(n) \to 1$ as $n \to \infty$. As such, we see that $p(n)$ is a relatively slow exponential function of n.

3.3 A Special Case

When $\ell > \lfloor n/2 \rfloor$ a closed form for $\left\langle {n \atop k} \middle\uparrow \ell \right\rangle$ is available by direct enumeration. We have

Theorem 3.16 *Let $n \in \mathbb{N}$ and S_n the symmetric group of n elements. Let $\ell \in \mathbb{N}$ such that $\lfloor n/2 \rfloor < \ell \leq n$. Then*

$$\left\langle {n \atop 1} \middle\uparrow \ell \right\rangle = (n-\ell)!\,\left\{(n-\ell)^2+1\right\} + (n-\ell-1)(n-\ell-1)!\,[\ell < n]. \qquad (3.30)$$

Proof: Since $\ell > \lfloor n/2 \rfloor$, the remaining $n-\ell$ elements of the permutation cannot form another increasing ℓ-sequence. We have 6 cases to consider depending on what position the increasing ℓ-sequence begins in and what integer it begins with.

1. The increasing ℓ-sequence begins in $\pi(1)$ and $\pi(\ell) = n$ which can only happen one way. The remaining $n-\ell$ elements may be permuted at leisure and we have a total of $(n-\ell)!$ ways.

2. The increasing ℓ-sequence begins in $\pi(1)$ but $\pi(\ell) \neq n$ which can happen total of $n-\ell$ ways. The remaining $n-\ell$ elements must now be permuted with the restriction that $\pi(\ell+1) \neq \pi(\ell)+1$ which may be done in $(n-\ell-1)(n-\ell-1)!$ ways. The total for case 2 is therefore $(n-\ell)(n-\ell-1)(n-\ell-1)! = (n-\ell-1)(n-\ell)!$.

3. The increasing ℓ sequence begins in $\pi(n-\ell)$ and $\pi(n-\ell+1) = 1$. By symmetry with case 1, we have a total of $(n-\ell)!$ ways that this can occur.

4. The increasing ℓ-sequence begins in $\pi(n-\ell+1)$ and $\pi(n-\ell+1) \neq 1$. By symmetry with case 2, this may occur in $(n-\ell-1)(n-\ell)!$ ways.

5. The increasing ℓ-sequence starts in $\pi(k)$ for some $1 < k < n-\ell+1$ which can happen a total of $(n-\ell+1)-2 = n-\ell-1$ ways.

   (a) If $\pi(k) = 1$ or $\pi(k) = n-\ell+1$ (2 ways) then there are $(n-\ell-1)(n-\ell-1)!$ ways to permute the remaining elements so that the ℓ-sequence is not increased in length. Hence, we have a total of $2(n-\ell-1)(n-\ell-1)!$ ways.

   (b) If $\pi(k) \neq 1$ and $\pi(k) \neq n-\ell+1$ which may happen $(n-\ell-1)$ ways, then there are two restriction on the remaining $n-\ell$ elements and we have a total of $(n-\ell-1)[(n-\ell)! - 2(n-\ell-1)! + (n-\ell-2)!]$ ways.

   Adding (5a) and (5b) and multiplying by $(n-\ell-1)$ we get the total for case 5 as

$$(n-\ell-1)^2(n-\ell)! + (n-\ell-1)!.$$

Summing all cases yields

$$2\{(n-\ell)! + (n-\ell-1)(n-\ell-1)!\} + (n-\ell-1)^2(n-\ell)! + (n-\ell-1)!,$$

which simplifies to the desired result. □

It is also easy to verify

Theorem 3.17 *Let $n \in \mathbb{N}$ and $\mathbb{S}_n$ the symmetric group of n elements. Let $\ell \in \mathbb{N}$ such that $\lfloor n/2 \rfloor < \ell \leq n$. Then*

$$\left\langle \begin{matrix} n \\ k \end{matrix} \middle\uparrow \ell \right\rangle = \left\langle \begin{matrix} n \\ 1 \end{matrix} \middle\uparrow \ell + k - 1 \right\rangle .$$

Chapter 4

Applications

The numbers $\left\langle {n \atop k} \uparrow \ell \right\rangle$ have a number of applications in mathematics and statistics. In this chapter, we discuss two applications in non-parametric inference and one in graph theory.

4.1 A Test for Independence

Let $\{X_i\}$, for $i = 1, \ldots, n$, be a finite sequence of random variates hypothesized to be independent and identically distributed (i.i.d.) according to some unspecified continuous distribution function $F(x)$. Suppose that $x_1, \ldots, x_n$ is a particular realization of this sequence, listed in order of occurrence. We wish to test if the observed values are independent of the order in which they occurred.

Let $r(X_i)$ denote the rank of X_i and define $\pi \in \mathbb{S}_n$ by $\pi(i) := r(X_i)$ for $i = 1, \ldots, n$. Under the null-hypothesis, the exact distribution of $X_n^+(\pi, 2)$ and $X_n^-(\pi, 2)$ are known and thus these may be used as test statistics. In testing against the alternative hypothesis that the values are tending to get larger over time (an *upward trend*) then we would reject the null-hypothesis for large values of the statistic $X_n^+(\pi, 2)$. In testing against a *downward trend*, we reject for large values of $X_n^-(\pi, 2)$.

As an example, consider the realization

i	1	2	3	4	5	6	7	8	9	10	11
x_i	97	47	15	56	66	20	78	82	96	29	86
$r(x_i)$	11	4	1	5	6	2	7	8	9	3	10

The induced permutation is $\pi = (11, 4, 1, 5, 6, 2, 7, 8, 9, 3, 10)$. We wish to test for an *upward trend* and the appropriate test statistic is $X_n^+(\pi, 2) = 3$; corresponding to $(5, 6)$, $(7, 8)$ and $(8, 9)$. From Table A.1, we find

$$\Pr[X_{11}(\pi, 2) \geq 3] = 1 - \frac{16{,}019{,}531 + 14{,}684{,}570 + 6{,}674{,}805}{11!} \approx 0.064.$$

If we wish to test for a *downward trend*, then the appropriate statistic is $X_n^-(\pi, 2) = 0$ and, trivially, we have $\Pr[X_n^-(\pi, 2) \geq 0] = 1$.

This test is also appropriate to testing a large source of pseudo-random numbers, one may examine m consecutive blocks of n numbers each; calculating $X_n^+(\pi, 2)$ on each of the m blocks. A suitable goodness of fit test may then be performed on these $X_n^+(\pi, 2)$ against its theoretical distribution.

4.2 A Test for Independence in Paired Samples

Wolfowitz (1942) describes a non-parametric test for independence based on the distribution of $R_n(\pi) = n - X_n(\pi, 2)$. He shows that, under certain assumptions, a test based on $R_n^+(\pi)$ provides a reasonable approximation to a maximum likelihood test. In this section, we recast this test for use with a one-sided alternative hypothesis.

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ a random sample from a population specified by the joint continuous distribution function $F_{X,Y}(x, y)$ and let $F_X(x), F_Y(y)$ denote the marginal distribution functions of X and Y (which will be continuous by definition). Consider testing the null hypothesis, H_0, that the distributions of X and Y are independent (i.e. that $F_{X,Y}(x,y) = F_X(x)F_Y(y)$) against the one sided alternative, H_{a_1}, that the X_i are directly related to the Y_i. Let $r(X_i)$ denote the rank of X_i, for $1 \le i \le n$, among $X_1, \ldots, X_n$. Similarly, let $r(Y_i)$ denote the rank of Y_i among $Y_1, \ldots, Y_n$. Construct a permutation, $\pi \in \mathbb{S}_n$, as

$$\pi\left(r(X_k)\right) = r\left(Y_k\right), \qquad 1 \le k \le n,$$

As an example, consider the realization $(x_1, y_1), \ldots, (x_n, y_n)$:

i	1	2	3	4	5	6	7	8	9	10
x_i	3.9	3.6	0.3	0.7	8.0	1.5	5.3	6.3	7.5	1.4
y_i	1.8	3.9	6.9	7.7	5.7	4.5	2.4	5.1	2.6	2.5
$r(x_i)$	6	5	1	2	10	4	7	8	9	3
$r(y_i)$	1	5	9	10	8	6	2	7	4	3

The induced permutation is hence $\pi = (9, 10, 3, 6, 5, 1, 2, 7, 4, 8)$. Under the null hypothesis, we do not expect too many increasing 2-sequences and hence a test based on $X_n^+(\pi, 2)$ is appropriate. In the above example, we have $X_n^+(\pi, 2) = 2$ and

$$\Pr[X_{10}^+(\pi, 2) \ge 2] = 1 - \frac{1}{n!}\left\{ \left\langle \begin{matrix} 10 \\ 0 \end{matrix} \middle\uparrow 2 \right\rangle + \left\langle \begin{matrix} 10 \\ 1 \end{matrix} \middle\uparrow 2 \right\rangle \right\} \approx 0.22745.$$

If we consider the alternate hypothesis, H_{a_2}, that the X_i are inversely related to the Y_i, then the appropriate statistic is $X_n^-(\pi, 2)$ and the test follows as above.

Due to the asymptotic distribution of $X_n^+(\pi, 2)$, for large n, we require 4 or more increasing 2-sequences in order to reject the null-hypothesis at a significance level of $\alpha = 0.05$.

4.3 A Simple Application in Graph Theory

Consider a complete directed graph, G, consisting of n nodes labeled $1, \ldots, n$. Each $\pi \in \mathbb{S}_n$ may be though of as a simple path on G by considering $\pi(i)$ as the i^{th} node visited in the path for $i = 1, \ldots, n$. We see that $\left\langle {n \atop k} \uparrow 2 \right\rangle$ represents the number of simple paths such that we never visit nodes i and $i+1$ in order for any $i = 1, \ldots, n-1$.

Chapter 5

Conclusions

Using the theory of compositions of integers, theorem 3.1 gives a more efficient formula for the determination of $\left\langle {n \atop k} \uparrow \ell \right\rangle$. Furthermore, its corollaries easily give all known results for the special case $\ell = 2$, and provide more efficient results for $\ell = 3$. Theorem 3.11 gives a closed form for the expectation of $X_n^+(\pi, \ell)$, a result previously unknown. Its corollaries use this to easily show the limiting distribution of $X_n^+(\pi, \ell)$ is degenerate about 0 for $\ell > 2$.

In the remainder of this chapter, we discuss an extension to the numbers examined here.

5.1 Future Research

As an extension to the numbers studied in this thesis, consider the multi-set specified by $[\xi_i]$ where there are ξ_1 copies of 1's, ξ_2 copies of 2's, and so on, up to ξ_n copies of n's. One may reasonably ask how many distinct arrangements of such a set contain exactly k increasing ℓ-sequences. For rises and falls, this is known as the Simon-Newcomb problem and is related to the Eulerian Numbers (cf. Carlitz, Roselle, and Scoville 1966; Dillon and Roselle 1969; Carlitz 1972; Harris and Park 1994; Fu, Lou, and Wang 1996).

Appendix A

Table of $\left\langle {n \atop k} \uparrow \ell \right\rangle$

The following table lists the values for $\left\langle {n \atop k} \uparrow \ell \right\rangle$ for all $2 \leq n \leq 20$, $2 \leq \ell \leq n$ and $0 \leq k \leq n - \ell + 1$. The numbers were produced using the algorithm described figure 3.1 on page 40. The total CPU time was less than 2 seconds on a 486 33Mhz computer running Linux.

Table A.1: Values of $\left\langle {n \atop k} \uparrow \ell \right\rangle$.

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
2	2	0	1
		1	1
3	2	0	3
		1	2
		2	1
3	3	0	5
		1	1
4	2	0	11
		1	9
		2	3
		3	1
4	3	0	21
		1	2
		2	1
4	4	0	23
		1	1
5	2	0	53
		1	44
		2	18
		3	4
		4	1
5	3	0	106
		1	11
		2	2
		3	1
5	4	0	117
		1	2
		2	1
5	5	0	119
		1	1
6	2	0	309
		1	265
		2	110
		3	30
		4	5
		5	1
6	3	0	643
		1	62
6	3	2	12
		3	2
		4	1
6	4	0	706
		1	11
		2	2
		3	1
6	5	0	717
		1	2
		2	1
6	6	0	719
		1	1
7	2	0	2,119
		1	1,854
		2	795
		3	220
		4	45
		5	6
		6	1
7	3	0	4,547
		1	406
		2	71
		3	13
		4	2
		5	1
7	4	0	4,962
		1	64
		2	11
		3	2
		4	1
7	5	0	5,026
		1	11
		2	2
		3	1
7	6	0	5,037
		1	2
		2	1
7	7	0	5,039
8	1	1	1
8	2	0	16,687
		1	14,833
		2	6,489
		3	1,855
		4	385
		5	63
		6	7
		7	1
8	3	0	36,696
		1	3,046
		2	481
		3	80
		4	14
		5	2
		6	1
8	4	0	39,817
		1	424
		2	65
		3	11
		4	2
		5	1
8	5	0	40,242
		1	64
		2	11
		3	2
		4	1
8	6	0	40,306
		1	11
		2	2
		3	1
8	7	0	40,317
		1	2
		2	1
8	8	0	40,319
		1	1
9	2	0	148,329
		1	133,496
9	2	2	59,332
		3	17,304
		4	3,710
		5	616
		6	84
		7	8
		8	1
9	3	0	332,769
		1	25,737
		2	3,708
		3	559
		4	89
		5	15
		6	2
		7	1
9	4	0	359,171
		1	3,196
		2	433
		3	66
		4	11
		5	2
		6	1
9	5	0	362,376
		1	426
		2	64
		3	11
		4	2
		5	1
9	6	0	362,802
		1	64
		2	11
		3	2
		4	1
9	7	0	362,866
		1	11
		2	2
		3	1
9	8	0	362,877
9	8	1	2
		2	1
9	9	0	362,879
		1	1
10	2	0	1,468,457
		1	1,334,961
		2	600,732
		3	177,996
		4	38,934
		5	6,678
		6	924
		7	108
		8	9
		9	1
10	3	0	3,349,507
		1	242,094
		2	32,028
		3	4,414
		4	640
		5	98
		6	16
		7	2
		8	1
10	4	0	3,597,936
		1	27,070
		2	3,271
		3	442
		4	67
		5	11
		6	2
		7	1
10	5	0	3,625,081
		1	3,214
		2	427
		3	64
		4	11
		5	2
		6	1

Table A.1 continued...

n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$	n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$	n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$	n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$
10	6	0	3,628,296	11	4	3	3,346	12	2	3	24,474,285	12	6	5	11
		1	426			4	451			4	5,506,710			6	2
		2	64			5	68			5	978,978			7	1
		3	11			6	11			6	142,758	12	7	0	478,997,880
		4	2			7	2			7	17,490			1	3,216
		5	1			8	1			8	1,815			2	426
10	7	0	3,628,722	11	5	0	39,885,851			9	165			3	64
		1	64			1	27,220			10	11			4	11
		2	11			2	3,223			11	1			5	2
		3	2			3	428	12	3	0	446,867,351			6	1
		4	1			4	64			1	28,473,604	12	8	0	479,001,096
10	8	0	3,628,786			5	11			2	3,228,804			1	426
		1	11			6	2			3	378,592			2	64
		2	2			7	1			4	46,343			3	11
		3	1	11	6	0	39,913,080			5	5,958			4	2
10	9	0	3,628,797			1	3,216			6	811			5	1
		1	2			2	426			7	116	12	9	0	479,001,522
		2	1			3	64			8	18			1	64
10	10	0	3,628,799			4	11			9	2			2	11
		1	1			5	2			10	1			3	2
11	2	0	16,019,531			6	1	12	4	0	476,066,277			4	1
		1	14,684,570	11	7	0	39,916,296			1	2,641,713	12	10	0	479,001,586
		2	6,674,805			1	426			2	261,231			1	11
		3	2,002,440			2	64			3	28,415			2	2
		4	444,990			3	11			4	3,421			3	1
		5	77,868			4	2			5	460	12	11	0	479,001,597
		6	11,130			5	1			6	69			1	2
		7	1,320	11	8	0	39,916,722			7	11			2	1
		8	135			1	64			8	2	12	12	0	479,001,599
		9	10			2	11			9	1			1	1
		10	1			3	2	12	5	0	478,714,416	13	2	0	2,467,007,773
11	3	0	37,054,436			4	1			1	256,150			1	2,290,792,932
		1	2,510,733	11	9	0	39,916,786			2	27,295			2	1,057,289,046
		2	306,723			1	11			3	3,232			3	323,060,540
		3	38,893			2	2			4	429			4	73,422,855
		4	5,164			3	1			5	64			5	13,216,104
		5	724	11	10	0	39,916,797			6	11			6	1,957,956
		6	107			1	2			7	2			7	244,728
		7	17			2	1			8	1			8	26,235
		8	2	11	11	0	39,916,799	12	6	0	478,970,641			9	2,420
		9	1			1	1			1	27,238			10	198
11	4	0	39,630,372	12	2	0	190,899,411			2	3,217			11	12
		1	254,808			1	176,214,841			3	426			12	1
		2	27,741			2	80,765,135			4	64				

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
13	3	0	5,834,728,509
		1	350,651,588
		2	37,080,394
		3	4,040,110
		4	457,966
		5	54,389
		6	6,796
		7	901
		8	125
		9	19
		10	2
		11	1
13	4	0	6,194,080,387
		1	29,931,510
		2	2,708,064
		3	267,698
		4	29,092
		5	3,496
		6	469
		7	70
		8	11
		9	2
		10	1
13	5	0	6,224,078,292
		1	2,654,568
		2	256,821
		3	27,370
		4	3,241
		5	430
		6	64
		7	11
		8	2
		9	1
13	6	0	6,226,733,531
		1	256,300
		2	27,247
		3	3,218
		4	426
		5	64
		6	11
		7	2
		8	1
13	7	0	6,226,989,840
13	7	1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
13	8	0	6,227,017,080
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
13	9	0	6,227,020,296
		1	426
		2	64
		3	11
		4	2
		5	1
13	10	0	6,227,020,722
		1	64
		2	11
		3	2
		4	1
13	11	0	6,227,020,786
		1	11
		2	2
		3	1
13	12	0	6,227,020,797
		1	2
		2	1
13	13	0	6,227,020,799
		1	1
14	2	0	34,361,893,981
		1	32,071,101,049
		2	14,890,154,058
		3	4,581,585,866
		4	1,049,946,755
		5	190,899,423
		6	28,634,892
		7	3,636,204
		8	397,683
14	2	9	37,895
		10	3,146
		11	234
		12	13
		13	1
14	3	0	82,003,113,550
		1	4,661,105,036
		2	461,569,226
		3	46,936,856
		4	4,949,551
		5	545,110
		6	63,042
		7	7,678
		8	994
		9	134
		10	20
		11	2
		12	1
14	4	0	86,776,390,796
		1	368,145,933
		2	30,671,367
		3	2,774,989
		4	274,209
		5	29,772
		6	3,571
		7	478
		8	71
		9	11
		10	2
		11	1
14	5	0	87,145,277,160
		1	30,064,344
		2	2,661,000
		3	257,492
		4	27,445
		5	3,250
		6	431
		7	64
		8	11
		9	2
		10	1
14	6	0	87,175,347,936
		1	2,655,910
14	5	2	256,375
		3	27,256
		4	3,219
		5	426
		6	64
		7	11
		8	2
		9	1
14	7	0	87,178,003,921
		1	256,318
		2	27,241
		3	3,216
		4	426
		5	64
		6	11
		7	2
		8	1
14	8	0	87,178,260,240
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
14	9	0	87,178,287,480
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
14	10	0	87,178,290,696
		1	426
		2	64
		3	11
		4	2
		5	1
14	11	0	87,178,291,122
		1	64
		2	11
		3	2
		4	1

Table A.1 continued...

n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$
14	12	0	87,178,291,186
		1	11
		2	2
		3	1
14	13	0	87,178,291,197
		1	2
		2	1
14	14	0	87,178,291,199
		1	1
15	2	0	513,137,616,783
		1	481,066,515,734
		2	224,497,707,343
		3	69,487,385,604
		4	16,035,550,531
		5	2,939,850,914
		6	445,431,987
		7	57,269,784
		8	6,363,357
		9	618,618
		10	53,053
		11	4,004
		12	273
		13	14
		14	1
15	3	0	1,234,297,698,757
		1	66,529,260,545
		2	6,192,527,700
		3	590,070,030
		4	58,126,425
		5	5,962,080
		6	640,289
		7	72,313
		8	8,604
		9	1,090
		10	143
		11	21
		12	2
		13	1
15	4	0	1,302,376,048,620
		1	4,886,708,928
		2	377,034,018
15	4	3	31,418,508
		4	2,842,488
		5	280,764
		6	30,455
		7	3,646
		8	487
		9	72
		10	11
		11	2
		12	1
15	5	0	1,307,271,652,917
		1	369,627,369
		2	30,130,827
		3	2,667,435
		4	258,163
		5	27,520
		6	3,259
		7	432
		8	64
		9	11
		10	2
		11	1
15	6	0	1,307,641,346,772
		1	30,077,208
		2	2,656,581
		3	256,450
		4	27,265
		5	3,220
		6	426
		7	64
		8	11
		9	2
		10	1
15	7	0	1,307,671,424,651
		1	2,656,060
		2	256,327
		3	27,242
		4	3,216
		5	426
		6	64
		7	11
15	7	8	2
		9	1
15	8	0	1,307,674,080,720
		1	256,320
		2	27,240
		3	3,216
		4	426
		5	64
		6	11
		7	2
		8	1
15	9	0	1,307,674,337,040
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
15	10	0	1,307,674,364,280
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
15	11	0	1,307,674,367,496
		1	426
		2	64
		3	11
		4	2
		5	1
15	12	0	1,307,674,367,922
		1	64
		2	11
		3	2
		4	1
15	13	0	1,307,674,367,986
		1	11
		2	2
		3	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$	n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$	n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
15	14	0	1,307,674,367,997	16	4	6	287,363	16	7	9	2
		1	2			7	31,141			10	1
		2	1			8	3,721	16	8	0	20,922,786,944,641
15	15	0	1,307,674,367,999			9	496			1	2,656,078
		1	1			10	73			2	256,321
16	2	0	8,178,130,767,479			11	11			3	27,240
		1	7,697,064,251,745			12	2			4	3,216
		2	3,607,998,868,005			13	1			5	426
		3	1,122,488,536,715	16	5	0	20,917,481,850,667			6	64
		4	260,577,696,015			1	4,904,506,950			7	11
		5	48,106,651,593			2	370,368,948			8	2
		6	7,349,627,285			3	30,197,354			9	1
		7	954,497,115			4	2,673,873	16	9	0	20,922,789,600,720
		8	107,380,845			5	258,834			1	256,320
		9	10,605,595			6	27,595			2	27,240
		10	927,927			7	3,268			3	3,216
		11	72,345			8	433			4	426
		12	5,005			9	64			5	64
		13	315			10	11			6	11
		14	15			11	2			7	2
		15	1			12	1			8	1
16	3	0	19,809,901,558,841	16	6	0	20,922,387,099,240	16	10	0	20,922,789,857,040
		1	1,014,985,068,610			1	369,760,344			1	27,240
		2	89,102,492,915			2	30,083,640			2	3,216
		3	7,984,564,400			3	2,657,252			3	426
		4	737,549,220			4	256,525			4	64
		5	70,734,390			5	27,274			5	11
		6	7,082,703			6	3,221			6	2
		7	743,768			7	426			7	1
		8	82,213			8	64	16	11	0	20,922,789,884,280
		9	9,574			9	11			1	3,216
		10	1,189			10	2			2	426
		11	152			11	1			3	64
		12	22	16	7	0	20,922,756,866,016			4	11
		13	2			1	30,078,550			5	2
		14	1			2	2,656,135			6	1
16	4	0	20,847,721,870,931			3	256,336	16	12	0	20,922,789,887,496
		1	69,645,189,376			4	27,243			1	426
		2	5,001,404,982			5	3,216			2	64
		3	386,016,398			6	426			3	11
		4	32,172,944			7	64			4	2
		5	2,910,561			8	11			5	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
16	13	0	20,922,789,887,922
		1	64
		2	11
		3	2
		4	1
16	14	0	20,922,789,887,986
		1	11
		2	2
		3	1
16	15	0	20,922,789,887,997
		1	2
		2	1
16	16	0	20,922,789,887,999
		1	1
17	2	0	138,547,156,531,409
		1	130,850,092,279,664
		2	61,576,514,013,960
		3	19,242,660,629,360
		4	4,489,954,146,860
		5	833,848,627,248
		6	128,284,404,248
		7	16,799,148,080
		8	1,908,994,230
		9	190,899,280
		10	16,968,952
		11	1,349,712
		12	96,460
		13	6,160
		14	360
		15	16
		16	1
17	3	0	337,707,109,446,702
		1	16,484,495,344,135
		2	1,369,014,167,140
		3	115,748,765,205
		4	10,060,799,445
		5	905,448,369
		6	84,847,894
		7	8,316,479
		8	855,812
		9	92,753
17	3	10	10,588
		11	1,291
		12	161
		13	23
		14	2
		15	1
17	4	0	354,549,730,559,949
		1	1,060,918,965,608
		2	71,229,862,678
		3	5,117,370,308
		4	395,093,338
		5	32,934,686
		6	2,979,208
		7	294,006
		8	31,830
		9	3,796
		10	505
		11	74
		12	11
		13	2
		14	1
17	5	0	355,612,235,468,396
		1	69,874,864,413
		2	4,913,416,887
		3	371,111,101
		4	30,263,925
		5	2,680,314
		6	259,505
		7	27,670
		8	3,277
		9	434
		10	64
		11	11
		12	2
		13	1
17	6	0	355,682,119,243,320
		1	4,905,990,240
		2	369,826,836
		3	30,090,072
		4	2,657,923
		5	256,600
17	6	6	27,283
		7	3,222
		8	426
		9	64
		10	11
		11	2
		12	1
17	7	0	355,687,025,300,052
		1	369,773,208
		2	30,079,221
		3	2,656,210
		4	256,345
		5	27,244
		6	3,216
		7	426
		8	64
		9	11
		10	2
		11	1
17	8	0	355,687,395,073,931
		1	30,078,700
		2	2,656,087
		3	256,322
		4	27,240
		5	3,216
		6	426
		7	64
		8	11
		9	2
		10	1
17	9	0	355,687,425,152,640
		1	2,656,080
		2	256,320
		3	27,240
		4	3,216
		5	426
		6	64
		7	11
		8	2
		9	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
17	10	0	355,687,427,808,720
		1	256,320
		2	27,240
		3	3,216
		4	426
		5	64
		6	11
		7	2
		8	1
17	11	0	355,687,428,065,040
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
17	12	0	355,687,428,092,280
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
17	13	0	355,687,428,095,496
		1	426
		2	64
		3	11
		4	2
		5	1
17	14	0	355,687,428,095,922
		1	64
		2	11
		3	2
		4	1
17	15	0	355,687,428,095,986
		1	11
		2	2
		3	1
17	16	0	355,687,428,095,997
		1	2
		2	1
17	17	0	355,687,428,095,999
		1	1
18	2	0	2,486,151,753,313,617
		1	2,355,301,661,033,953
		2	1,112,225,784,377,144
		3	348,933,579,412,440
		4	81,781,307,674,780
		5	15,265,844,099,324
		6	2,362,571,110,536
		7	311,547,838,888
		8	35,698,189,670
		9	3,605,877,990
		10	324,528,776
		11	26,224,744
		12	1,912,092
		13	126,140
		14	7,480
		15	408
		16	17
		17	1
18	3	0	6,094,059,760,690,035
		1	283,989,434,253,186
		2	22,373,840,093,040
		3	1,790,141,293,730
		4	146,876,983,360
		5	12,445,840,614
		6	1,095,255,712
		7	100,555,934
		8	9,668,520
		9	976,686
		10	103,944
		11	11,646
		12	1,396
		13	170
		14	24
		15	2
		16	1
18	4	0	6,384,006,047,649,910
		1	17,204,867,229,956
		2	1,084,282,429,946
		3	72,832,453,416
		4	5,234,609,806
18	4	5	404,265,103
		6	33,703,745
		7	3,048,429
		8	300,693
		9	32,522
		10	3,871
		11	514
		12	75
		13	11
		14	2
		15	1
18	5	0	6,401,234,296,266,540
		1	1,064,092,121,088
		2	69,989,843,418
		3	4,922,334,108
		4	371,853,828
		5	30,330,540
		6	2,686,758
		7	260,176
		8	27,745
		9	3,286
		10	435
		11	64
		12	11
		13	2
		14	1
18	6	0	6,402,298,503,373,917
		1	69,892,686,009
		2	4,906,731,951
		3	369,893,331
		4	30,096,504
		5	2,658,594
		6	256,675
		7	27,292
		8	3,223
		9	426
		10	64
		11	11
		12	2
		13	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
18	7	0	6,402,368,396,801,640
		1	4,906,123,224
		2	369,779,640
		3	30,079,892
		4	2,656,285
		5	256,354
		6	27,245
		7	3,216
		8	426
		9	64
		10	11
		11	2
		12	1
18	8	0	6,402,373,302,931,296
		1	369,774,550
		2	30,078,775
		3	2,656,096
		4	256,323
		5	27,240
		6	3,216
		7	426
		8	64
		9	11
		10	2
		11	1
18	9	0	6,402,373,672,705,921
		1	30,078,718
		2	2,656,081
		3	256,320
		4	27,240
		5	3,216
		6	426
		7	64
		8	11
		9	2
		10	1
18	10	0	6,402,373,702,784,640
		1	2,656,080
		2	256,320
		3	27,240
		4	3,216
		5	426
		6	64
18	10	7	11
		8	2
		9	1
18	11	0	6,402,373,705,440,720
		1	256,320
		2	27,240
		3	3,216
		4	426
		5	64
		6	11
		7	2
		8	1
18	12	0	6,402,373,705,697,040
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
18	13	0	6,402,373,705,724,280
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
18	14	0	6,402,373,705,727,496
		1	426
		2	64
		3	11
		4	2
		5	1
18	15	0	6,402,373,705,727,922
		1	64
		2	11
		3	2
		4	1
18	16	0	6,402,373,705,727,986
		1	11
		2	2
		3	1
18	17	0	6,402,373,705,727,997
18	17	1	2
		2	1
18	18	0	6,402,373,705,727,999
		1	1
19	2	0	47,106,033,220,679,059
		1	44,750,731,559,645,106
		2	21,197,714,949,305,577
		3	6,673,354,706,262,864
		4	1,570,201,107,355,980
		5	294,412,707,629,208
		6	45,797,532,297,972
		7	6,075,182,855,664
		8	700,982,637,498
		9	71,396,379,340
		10	6,490,580,382
		11	531,047,088
		12	39,337,116
		13	2,647,512
		14	162,180
		15	8,976
		16	459
		17	18
		18	1
19	3	0	116,052,543,892,621,951
		1	5,173,041,992,087,562
		2	387,602,212,164,321
		3	29,427,095,480,435
		4	2,285,694,716,330
		5	182,917,590,694
		6	15,165,333,989
		7	1,308,506,455
		8	117,949,361
		9	11,143,991
		10	1,106,655
		11	115,797
		12	12,748
		13	1,504
		14	179
		15	25
		16	2
		17	1

Table A.1 continued...

n	ℓ	k	$\langle {n \atop k} \uparrow \ell \rangle$
19	4	0	121,330,369,923,079,290
		1	295,971,058,578,160
		2	17,571,257,417,630
		3	1,107,912,107,740
		4	74,453,045,290
		5	5,353,128,376
		6	413,531,958
		7	34,480,132
		8	3,118,224
		9	307,424
		10	33,217
		11	3,946
		12	523
		13	76
		14	11
		15	2
		16	1
19	5	0	121,626,707,638,142,280
		1	17,251,647,930,540
		2	1,065,680,603,796
		3	70,104,916,762
		4	4,931,258,613
		5	372,597,129
		6	30,397,199
		7	2,693,205
		8	260,847
		9	27,820
		10	3,295
		11	436
		12	64
		13	11
		14	2
		15	1
19	6	0	121,643,960,874,649,867
		1	1,064,322,100,950
		2	69,901,597,668
		3	4,907,473,706
		4	369,959,829
		5	30,102,936
		6	2,659,265
		7	256,750
		8	27,301
19	6	9	3,224
		10	426
		11	64
		12	11
		13	2
		14	1
19	7	0	121,645,025,205,662,520
		1	69,894,169,440
		2	4,906,189,716
		3	369,786,072
		4	30,080,563
		5	2,656,360
		6	256,363
		7	27,246
		8	3,216
		9	426
		10	64
		11	11
		12	2
		13	1
19	8	0	121,645,095,099,898,452
		1	4,906,136,088
		2	369,775,221
		3	30,078,850
		4	2,656,105
		5	256,324
		6	27,240
		7	3,216
		8	426
		9	64
		10	11
		11	2
		12	1
19	9	0	121,645,100,006,035,211
		1	369,774,700
		2	30,078,727
		3	2,656,082
		4	256,320
		5	27,240
		6	3,216
		7	426
		8	64
19	9	9	11
		10	2
		11	1
19	10	0	121,645,100,375,809,920
		1	30,078,720
		2	2,656,080
		3	256,320
		4	27,240
		5	3,216
		6	426
		7	64
		8	11
		9	2
		10	1
19	11	0	121,645,100,405,888,640
		1	2,656,080
		2	256,320
		3	27,240
		4	3,216
		5	426
		6	64
		7	11
		8	2
		9	1
19	12	0	121,645,100,408,544,720
		1	256,320
		2	27,240
		3	3,216
		4	426
		5	64
		6	11
		7	2
		8	1
19	13	0	121,645,100,408,801,040
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
19	14	0	121,645,100,408,828,280
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
19	15	0	121,645,100,408,831,496
		1	426
		2	64
		3	11
		4	2
		5	1
19	16	0	121,645,100,408,831,922
		1	64
		2	11
		3	2
		4	1
19	17	0	121,645,100,408,831,986
		1	11
		2	2
		3	1
19	18	0	121,645,100,408,831,997
		1	2
		2	1
19	19	0	121,645,100,408,831,999
		1	1
20	2	0	939,765,362,752,547,227
		1	895,014,631,192,902,121
		2	425,131,949,816,628,507
		3	134,252,194,678,935,321
		4	31,698,434,854,748,604
		5	5,966,764,207,952,724
		6	932,306,907,492,492
		7	124,307,587,665,924
		8	14,428,559,282,202
		9	1,479,852,234,718
		10	135,653,120,746
		11	11,211,002,478
		12	840,824,556
		13	57,492,708
		14	3,593,052
20	2	15	205,428
		16	10,659
		17	513
		18	19
		19	1
20	3	0	2,325,905,946,434,516,516
		1	99,346,991,708,245,506
		2	7,095,737,193,164,187
		3	512,444,547,794,096
		4	37,780,575,533,555
		5	2,863,450,315,286
		6	224,323,049,959
		7	18,245,978,788
		8	1,546,783,084
		9	137,120,880
		10	12,748,110
		11	1,245,984
		12	128,323
		13	13,894
		14	1,615
		15	188
		16	26
		17	2
		18	1
20	4	0	2,427,196,999,663,678,987
		1	5,383,788,700,672,945
		2	302,064,161,086,250
		3	17,941,815,445,990
		4	1,131,809,403,480
		5	76,091,722,053
		6	5,472,930,918
		7	422,894,168
		8	35,263,858
		9	3,188,593
		10	314,199
		11	33,915
		12	4,021
		13	532
		14	77
		15	11
		16	2
		17	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
20	5	0	2,432,586,885,636,105,251
		1	296,704,637,928,676
		2	17,275,065,176,586
		3	1,067,270,356,836
		4	70,220,084,456
		5	4,940,190,402
		6	373,341,004
		7	30,463,902
		8	2,699,655
		9	261,518
		10	27,895
		11	3,304
		12	437
		13	64
		14	11
		15	2
		16	1
20	6	0	2,432,883,613,692,550,316
		1	17,254,825,178,973
		2	1,064,437,101,807
		3	69,910,509,901
		4	4,908,215,505
		5	370,026,330
		6	30,109,368
		7	2,659,936
		8	256,825
		9	27,310
		10	3,225
		11	426
		12	64
		13	11
		14	2
		15	1
20	7	0	2,432,900,868,632,730,720
		1	1,064,339,924,400
		2	69,894,911,160
		3	4,906,256,208
		4	369,792,504
20	7	5	30,081,234
		6	2,656,435
		7	256,372
		8	27,247
		9	3,216
		10	426
		11	64
		12	11
		13	2
		14	1
20	8	0	2,432,901,932,973,396,840
		1	69,894,302,424
		2	4,906,142,520
		3	369,775,892
		4	30,078,925
		5	2,656,114
		6	256,325
		7	27,240
		8	3,216
		9	426
		10	64
		11	11
		12	2
		13	1
20	9	0	2,432,902,002,867,705,696
		1	4,906,137,430
		2	369,774,775
		3	30,078,736
		4	2,656,083
		5	256,320
		6	27,240
		7	3,216
		8	426
		9	64
		10	11
		11	2
		12	1

Table A.1 continued...

n	ℓ	k	$\left\langle {n \atop k} \uparrow \ell \right\rangle$
20	10	0	2,432,902,007,773,843,201
		1	369,774,718
		2	30,078,721
		3	2,656,080
		4	256,320
		5	27,240
		6	3,216
		7	426
		8	64
		9	11
		10	2
		11	1
20	11	0	2,432,902,008,143,617,920
		1	30,078,720
		2	2,656,080
		3	256,320
		4	27,240
		5	3,216
		6	426
		7	64
		8	11
		9	2
		10	1
20	12	0	2,432,902,008,173,696,640
		1	2,656,080
		2	256,320
		3	27,240
		4	3,216
		5	426
		6	64
		7	11
		8	2
		9	1
20	13	0	2,432,902,008,176,352,720
		1	256,320
		2	27,240
		3	3,216
		4	426
		5	64
20	13	6	11
		7	2
		8	1
20	14	0	2,432,902,008,176,609,040
		1	27,240
		2	3,216
		3	426
		4	64
		5	11
		6	2
		7	1
20	15	0	2,432,902,008,176,636,280
		1	3,216
		2	426
		3	64
		4	11
		5	2
		6	1
20	16	0	2,432,902,008,176,639,496
		1	426
		2	64
		3	11
		4	2
		5	1
20	17	0	2,432,902,008,176,639,922
		1	64
		2	11
		3	2
		4	1
20	18	0	2,432,902,008,176,639,986
		1	11
		2	2
		3	1
20	19	0	2,432,902,008,176,639,997
		1	2
		2	1
20	20	0	2,432,902,008,176,639,999
		1	1

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