SMALL p_n-SEQUENCES

by

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SMALL p_n- SEQUENCES

BY

DABIN WANG

A Thesis/Practicum submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Dabin Wang

ABSTRACT

A lot of works in the past dealt with the small p_n-sequences and the algebras related to them (see G. Grätzer, G. Grätzer and R. Padmanabhan, K. M. Koh, J. Dudek and others [**28**], [**29**], [**30**] and [**21**]). The p_n-sequence $\langle 0,1,1,4\rangle$ (i.e. $\langle 0,0,1,4\rangle$ in classic notation) seems complicated and no paper about it has been published yet. This sequence is related to Problem 7 and Problem 14 in [**29**].

For sequence $\langle 0,1,1,4\rangle$, we have obtained the following results:

(1) If algebra $\mathfrak{A}$ represents $\langle 0,1,1,4\rangle$, then $p_4(\mathfrak{A}) \geqslant 19$ and $\langle 0,1,1,4,19\rangle$ is representable. Thus, $p_n(\mathfrak{A}) \geqslant 2^n + 2^{n-2} - 1$, for all $n \geqslant 4$.

As the by-products, we have the following statements:

(2) There are two varieties $\mathbf{K}_i$ of type $\langle 2,3\rangle$ and two four-element algebras $Z_i \in \mathbf{K}_i$, $i = 1,2$, such that

(a) both Z_i represent $\langle 0,1,1,4,19,151\rangle$ and every algebra $\mathfrak{A} \in \mathbf{K}_i$ representing $\langle 0,1,1,4\rangle$ contains a subalgebra such that Z_i is a homomorphic image of this subalgebra;

(b) for algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ which is not in $\mathbf{K}_1 \cup \mathbf{K}_2$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ iff $\mathfrak{A}$ is a nontrivial join algebra; thus, for any algebra $\mathfrak{A}$ without ternary reduct equivalent to some algebra in $\mathbf{K}_1 \cup \mathbf{K}_2$, the p_n-sequence of $\mathfrak{A}$ identifies that of a nontrivial join algebra iff $\mathfrak{A}$ is equivalent to a nontrivial join algebra.

(3) For any algebra $\mathfrak{A} = \langle A;\cdot,f\rangle$ of type $\langle 2,3\rangle$ with the property: either $f(x,x,y) = x$ or $f(x,y,x) = x$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ iff it is a nontrivial join algebra.

(4) There exist two subvarieties $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbf{K}_1$ and $\mathbf{K}_2$ respectively such that algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ represents $\langle 0,1,1,4,19\rangle$ iff $|\mathfrak{A}| > 3$ and $\mathfrak{A} \in \mathbf{H}_1 \cup \mathbf{H}_2$.

(5) If algebra $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$, then $\mathfrak{A}$ has a ternary reduct which is a nontrivial join algebra or in $\mathbf{K}_1 \cup \mathbf{K}_2$.

We also show that the p_n-sequence $\langle 0,1,1,7\rangle$ has the minimal extension property in the class of algebras with a proper non-associative binary operation, which generalizes the result of J. Galuszka and gives a partial solution of the open problem 18 raised by G. Grätzer and J. Kisielewicz whether $\langle 0, 1, 1, 7\rangle$ has the minimal extension property. From this result, the problem is reduced to consider the class of algebras with a semilattice operation.

In Part 3, we give some other results about small p_n-sequences. For example, K. M. Koh has investigated sequence $\langle 0, 1, 1, 2\rangle$ intensively. He has proved that there exist two varieties, $\mathfrak{K}_1$ and $\mathfrak{K}_2$, of type $\langle 2, 3\rangle$, such that a ternary algebra $\mathfrak{A}$ represents the sequence $\langle 0, 1, 1, 2\rangle$ if and only if $\mathfrak{A}$ is equivalent to an algebra with more than one element belonging to $\mathfrak{K}_1$ or $\mathfrak{K}_2$. In Chapter 9 of Part 3, we give characterizations of $\mathfrak{K}_1$ and $\mathfrak{K}_2$.

Contents

Introduction and History

A *universal algebra* $\mathfrak{A}$ is an ordered pair $\langle A; F\rangle$ where A is a nonempty set and $F = \langle f_0, \cdots, f_\lambda, \cdots\rangle$, $\lambda < \alpha$, is a family of finitary operations on A. The operations in F are called the *basic operations* of $\mathfrak{A}$. The *type* of $\mathfrak{A}$ is $\tau = \langle n_0, \cdots, n_\lambda, \cdots\rangle$, $\lambda < \alpha$, where n_λ is the arity of f_λ. $\mathfrak{A}$ is called k-ary, if $n_\lambda \leqslant k$ for all $\lambda < \alpha$.

The *set of term operations (polynomials)* of $\mathfrak{A}$, denoted by Term$\mathfrak{A}$, is the set of term operations on A obtained from all the projection operations by composition with the basic operations. For each integer $n > 0$, if we start from the n-ary projection operations, we get $\mathrm{Term}_n\mathfrak{A}$, the *set of n-ary term operations* of $\mathfrak{A}$.

An n-ary term operation p over $\mathfrak{A}$ is said to *depend* on x_i if there exist $a_1, \dots, a_i$, $a_i', \dots, a_n$ in A such that

$$p(a_1, \dots, a_i, \dots, a_n) \neq p(a_1, \dots, a_i', \dots, a_n).$$

If $p \in \mathrm{Term}_n\mathfrak{A}$ depends on all n variables $x_1, \dots, x_n$, then p is called *essentially n-ary* over $\mathfrak{A}$.

For $n \geqslant 1$, let $\mathrm{Term}_n^e\mathfrak{A}$ be the set of all the essentially n-ary term operations and set

$$p_n(\mathfrak{A}) = |\mathrm{Term}_n^e\mathfrak{A}|.$$

Let $p_0(\mathfrak{A})$ denote the number of constant unary term operations. Thus, the *p_n-sequence of* $\mathfrak{A}$ is defined as

$$p(\mathfrak{A}) = \langle p_0(\mathfrak{A}), p_1(\mathfrak{A}), \dots, p_n(\mathfrak{A}), \dots\rangle.$$

For a nontrivial variety $\mathbf{V}$, we define the *p_n-sequence* of $\mathbf{V}$ as the p_n-sequence of $\mathbf{F}_\omega(\mathbf{V})$, the free algebra on ω generators in $\mathbf{V}$.

An $p \in \mathrm{Term}_n\mathfrak{A}$ is called *an n-ary polynomial* of $\mathfrak{A}$; and an $p \in \mathrm{Term}_n^e\mathfrak{A}$ is called *an essentially n-ary polynomial* of $\mathfrak{A}$.

Let $\mathbf{K}$ be a class of algebras. A sequence $\langle p_0, p_1, \dots, p_n, \dots\rangle$ of cardinals is said to be *representable* in $\mathbf{K}$, if there exists an algebra $\mathfrak{A}$ in $\mathbf{K}$ such that $p_n = p_n(\mathfrak{A})$ for every $n \geqslant 0$.

An algebra $\mathfrak{A} = \langle A; F\rangle$ is *proper* if all operations in F are pairwise distinct and every nonnullary $f \in F$ depends on all its variables. In this case, $f \in F$ is said to be *proper.*

An algebra $\mathfrak{A} = \langle A; F\rangle$ is *idempotent* if $f(x, x, \ldots, x) = x$, for every $f \in F$. Obviously, $\mathfrak{A}$ *is idempotent if and only if* $p_0(\mathfrak{A}) = 0$ *and* $p_1(\mathfrak{A}) = 1$.

Two algebras $\mathfrak{A}_1 = \langle A; F_1\rangle$ and $\mathfrak{A}_2 = \langle A; F_2\rangle$ defined on the same base A are *equivalent* iff $\mathrm{Term}\mathfrak{A}_1 = \mathrm{Term}\mathfrak{A}_2$, which is equivalent to that any operation in F_1 is a composition of term operations over $\mathfrak{A}_2$ and vice versa.

We say that a sequence $p = \langle p_0, p_1, \ldots, p_n, \ldots\rangle$ of cardinals is a *minimal extension* of the given sequence $p' = \langle p_0, p_1, \ldots, p_n\rangle$ (in some class $\mathbf{K}$ of algebras) if the p is representable (in $\mathbf{K}$) and for every algebra $\mathfrak{A}$ (in $\mathbf{K}$) which represents p' we have $p_n(\mathfrak{A}) \geqslant p_n$ for all n. In this situation, we say that p' has the *minimal extension property* (MEP) (with p as the minimal extension).

For nonempty subset $F' \subseteq \mathrm{Term}\mathfrak{A}$, the algebra $\langle A; F'\rangle$ is called a *reduct* of $\mathfrak{A}$. If $F' \subseteq \mathrm{Term}_n\mathfrak{A}$, it is called a *n-ary reduct* of $\mathfrak{A}$.

Let $\mathbf{V}$ be a nontrivial variety of algebras and $\mathbf{F}_n(\mathbf{V})$ denote the free algebra on n generators over $\mathbf{V}$, and for $n > 0$, define

$$f_n(\mathbf{V}) = |\mathbf{F}_n(\mathbf{V})|;$$

and $f_0(\mathbf{V})$ is the number of constant unary term operations. The *free spectrum of* $\mathbf{V}$ is the sequence

$$\mathbf{f}(\mathbf{V}) = \langle f_0(\mathbf{V}), f_1(\mathbf{V}), \ldots, f_n(\mathbf{V}), \ldots\rangle.$$

For an algebra $\mathfrak{A}$, let

$$f_n(\mathfrak{A}) = |\mathbf{F}_n(\mathfrak{A})|,$$

where $\mathbf{F}_n(\mathfrak{A})$ is the free algebra on n generators in the variety generated by $\mathfrak{A}$. The *free spectrum of* $\mathfrak{A}$ is defined similarly.

The basic problem is to study and characterize the representable sequences. An easy combinatorial argument shows that this problem is equivalent to that characterize the free spectrum (Problem 42 in [**27**]). Indeed, for an algebra $\mathfrak{A}$ we have the following connections:

(C1)
$$f_n(\mathfrak{A}) = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathfrak{A})$$

and

(C2)
$$p_n(\mathfrak{A}) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f_k(\mathfrak{A})$$

The development of the study of p_n sequence may briefly be divided into four stages. The initial stage can be considered as starting in 1910. Though there were no significant contributions to the theory, the idea was formed by the work of S. Sierpinski. He published a series of articles between 1918 and 1945 for the purpose of investigating the composition of functions.

The explicit formulation of the basic problem was given by E. Marczewski and his school in Wroclaw from 1963 to 1966. One of the deepest results was found by K. Urbanik [1965], who characterized the *zero set*

$$Z(\mathfrak{A}) = \{n \mid p_n(\mathfrak{A}) = 0\}.$$

A great deal of systematic and intensive work was carried out in G. Grätzer's seminar at the University of Manitoba in the academic year 1968–1969. G. Grätzer, J. Plonka and R. Padmanahan enriched and clarified this subject. The survey article by G. Grätzer [1970] has reviewed the field up to 1969. Since then, much progress has been made. Especially, A. Kisielewicz completed the solutions of the two most interesting open problems (see Theorem 3 and Theorem 4 in [**29**]).

The new survey paper by G. Grätzer and A. Kisielewicz [1992] begins a new period. In this survey, twenty nine open problems were raised. Some of them would reveal the nature of the p_n-sequence.

The purpose of this thesis is to attempt to attack Problem 7, Problem 14 and Problem 18 in the survey [**29**] and partial solutions of them are obtained. Obviously, Problem 14 can be thought of as a part of Problem 7. We list these problems in the following:

(1) Problem 7: Characterize the algebras with $p_3 < 6$.

(2) Problem 14: Consider minimal extension for small sequence such as $\langle 0,1,5\rangle$, $\langle 0,1,1,2\rangle$, $\langle 0,1,1,4\rangle$, $\langle 0,1,2,5\rangle$. What kind of algebras represent them?

(3) Problem 18: Do the sequences $\langle 0,1,2,9\rangle$, $\langle 0,1,2,10\rangle$ and $\langle 0,1,1,7\rangle$ have the minimal extension property?

For $\langle 0,1,1,4\rangle$, there are no publications about it. We have obtained the following results (see Part 1):

(1) If an algebra $\mathfrak{A}$ represents $\langle 0,1,1,4\rangle$, then $p_4(\mathfrak{A}) \geqslant 19$ and $\langle 0,1,1,4,19\rangle$ is representable. Thus, $p_n(\mathfrak{A}) \geqslant 2^n + 2^{n-2} - 1$, for all $n \geqslant 4$.

As the by-products, we have the following statements:

(2) There are two varieties $\mathbf{K}_i$ of type $\langle 2,3\rangle$ and two four-element algebras $Z_i \in \mathbf{K}_i$, $i = 1,2$, such that

(a) both Z_i represent $\langle 0,1,1,4,19,151\rangle$ and every algebra $\mathfrak{A} \in \mathbf{K}_i$ representing $\langle 0,1,1,4\rangle$ contains a subalgebra such that Z_i is a homomorphic image of this subalgebra;

(b) for an algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ that is not in $\mathbf{K}_1 \cup \mathbf{K}_2$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ iff $\mathfrak{A}$ is a nontrivial join algebra; thus, for any algebra $\mathfrak{A}$ without ternary reduct equivalent to some in $\mathbf{K}_1 \cup \mathbf{K}_2$, the p_n sequence of $\mathfrak{A}$ identifies that of a nontrivial join algebra iff $\mathfrak{A}$ is equivalent to a nontrivial join algebra.

(3) For any algebra $\mathfrak{A} = \langle A;\ \cdot,\ f\rangle$ of type $\langle 2,3\rangle$ with the property: either $f(x,x,y) = x$ or $f(x,y,x) = x$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ iff it is a nontrivial join algebra.

(4) There exist two subvarieties $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbf{K}_1$ and $\mathbf{K}_2$ respectively such that an algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ represents $\langle 0,1,1,4,19\rangle$ iff $|\mathfrak{A}| > 3$ and $\mathfrak{A} \in \mathbf{H}_1 \cup \mathbf{H}_2$.

(5) If algebra $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$, then $\mathfrak{A}$ has a ternary reduct which is a nontrivial join algebra or in $\mathbf{K}_1 \cup \mathbf{K}_2$.

For the sequence $\langle 0,1,1,7\rangle$, there is a well-known result that $\langle 0,1,1,7\rangle$ has the minimal extension property in the class of groupoids (**[25]**, **[29]**). We have proved that $\langle 0,1,1,7\rangle$ has the minimal extension property in the class of all algebras with a proper nonassociative binary operation, which generalizes the previous result (see Part 2).

K. M. Koh has investigated sequence $\langle 0,1,1,2\rangle$ intensively. He has proved that there exist two varieties, $\mathfrak{K}_1$ and $\mathfrak{K}_2$, of type $\langle 2,3\rangle$, such that a ternary algebra $\mathfrak{A}$ represents the sequence $\langle 0,1,1,2\rangle$ if and only if $\mathfrak{A}$ is equivalent to an algebra with more than one element belonging to $\mathfrak{K}_1$ or $\mathfrak{K}_2$. In Chapter 9 of Part 3, we give characterizations of $\mathfrak{K}_1$ and $\mathfrak{K}_2$.

We give additional results on p_n-sequences in Part 3.

The proof of the results on sequence $\langle 0,1,1,4\rangle$ occupies almost the whole thesis. For the notions and notations, we refer to G. Grätzer's book [**27**] and two survey papers [**28**], [**29**].

Part 1

On the p_n-sequence $\langle 0, 1, 1, 4 \rangle$

CHAPTER 1

On binary term operation

Let algebra $\mathfrak{A}$ represent sequence $\langle 0, 1, 1, 4\rangle$. Then $\mathfrak{A}$ has one and only one idempotent commutative essentially binary polynomial $x \cdot y$ and $\mathfrak{A}$ is nontrivial. In this chapter, we will prove that the binary operation " $\cdot$ " defined by this polynomial is associative, so " $\cdot$ " is a semilattice operation. To prove this, we assume the contrary, i.e., that " $\cdot$ " is nonassociative. We will find a contradiction. By the assumption, $(xy)z, (yz)x$ and $(zx)y$ are pairwise distinct essentially ternary polynomials. Let $f(x, y, z)$ be the fourth essentially ternary polynomial.

LEMMA 1-0.1. *$f(x, y, z)$ is symmetric.*

PROOF. If $f(x, y, z)$ is not symmetric, without loss of generality, we can suppose that

$$f(y, x, z) \neq f(x, y, z).$$

Since $f(y, x, z)$ is an essentially ternary polynomial, from $p_3(\mathfrak{A}) = 4$,

$$f(y, x, z) \in \{(xy)z, (yz)x, (zx)y\}.$$

But this implies $f(x, y, z) \in \{(xy)z, (yz)x, (zx)y\}$, a contradiction. $\square$

LEMMA 1-0.2. *$f(x, y, y) = xy$.*

PROOF. Obviously,

$$f(x, y, y) = \begin{cases} x, \\ xy, \\ y. \end{cases}$$

Assume that $f(x,y,y) = x$. We will prove that

$$(1\text{-}0.2^*) \qquad f(x,y,z),\ f(x,y,z)x,\ f(x,y,z)y \text{ and } f(x,y,z)z$$

are distinct essentially ternary polynomials.

Firstly, we claim that every polynomial in (1-0.2*) is essentially ternary. By symmetry, we need only check $f(x,y,z)x$. Setting $z = y$, $f(x,y,z)x = f(x,y,y)x = xx = x$ so $f(x,y,z)x$ depends on x. Notice that $f(x,y,z)x$ depends on y iff it depends on z. Thus, if $f(x,y,z)x$ is not essential, then $f(x,y,z)x = x$. Let $y = x$. By the assumption above, $x = f(x,y,z)x = f(z,x,x)x = zx$, a contradiction.

Secondly, all polynomials of (1-0.2*) are pairwise distinct. If $f(x,y,z)x = f(x,y,z)y$, then, setting $x = z$, $\quad yx = yy = y$, a contradiction. By symmetry,

$$f(x,y,z)x, \quad f(x,y,z)y \text{ and } f(x,y,z)z$$

are pairwise distinct. If $f(x,y,z) = f(x,y,z)x$, then, setting $z = x$,

$$y = f(y,x,x) = f(x,y,x) = f(x,y,x)x = f(y,x,x)x = yx,$$

a contradiction. So the polynomials in (1-0.2*) are pairwise distinct.

Thirdly, we claim that

$$(1\text{-}0.2^{**}) \qquad f(x,y,z)x = x(yz), \quad f(x,y,z)y = (xz)y, \quad f(x,y,z)z = (xy)z.$$

If $f(x,y,z)x = (xy)z$, then, exchanging y for z,

$$(xy)z = f(x,y,z)x = f(x,z,y)x = (xz)y,$$

a contradiction to the assumption of the nonassociativity in this section. If $f(x,y,z)x = (xz)y$, then exchanging y for z,

$$(xz)y = f(x,y,z)x = f(x,z,y)x = (xy)z,$$

a contradiction again. So $f(x,y,z)x = x(yz)$. By symmetry, $f(x,y,z)y = (xz)y$ and $f(x,y,z)z = (xy)z$.

Let $z = y$ in $f(x,y,z)x = x(yz)$. Then $x = xx = f(x,y,z)x = x(yy) = xy$, a contradiction. So $f(x,y,y) \neq x$.

A similar reasoning implies that $f(x,y,y)=y$ is impossible. (proof: first we have

$$f(x,y,z),\quad f(x,y,z)x,\quad f(x,y,z)y \text{ and } f(x,y,z)z$$

are distinct essentially ternary polynomials. Indeed, let $z=y$. Then $f(x,y,z)x=xy$ so $f(x,y,z)x$ depends on x. Since $f(x,y,z)x$ is symmetric in y and z, it is essentially ternary by $f(x,y,z)x \neq x$. By applying permutations, the others are essentially ternary. If $f(x,y,z)x=f(x,y,z)$, then, setting $z=y$, $yx=y$, a contradiction. If $f(x,y,z)x=f(x,y,z)y$, then, setting $z=y$, $yx=y$, a contradiction. By permutations, we can show that the above four ternary polynomials are pairwise distinct. Next, the same reason as above implies (1-0.2**); and then a contradiction will appear (perhaps, you need $(xy)y=y$ implies $x=y$)). □

LEMMA 1-0.3. $(xy)y=x$.

PROOF. Obviously, $(xy)y \in \{x,y,xy\}$. If $(xy)y=y$, then $xy=((xy)y)(xy)=y(xy)=y$, a contradiction. If $(xy)y=xy$, then consider

$$f_1=(xy)z, f_2=(yz)x, f_3=(zx)y, f_4=(xy)(yz), f_5=(yz)(zx), f_6=(zx)(xy).$$

As the stated in G. Grätzer and R. Padmanabhan ([**30**], p78), $f_i's$ are essentially ternary and pairwise distinct since " $\cdot$ " is nonassociative. Therefore, $p_3(\mathfrak{A}) \geqslant 6$, contradicting $p_3(\mathfrak{A})=4$. Thus $(xy)y=x$. □

LEMMA 1-0.4.

$$f(x,y,z)x=f(x,y,z);$$

or

$$f(x,y,z)x=(yz)x,$$

$$f(x,y,z)y=(xz)y,$$

$$f(x,y,z)z=(xy)z.$$

PROOF. Suppose that $f \cdot x \neq f$. Then $f \cdot x = (yz)x$ or $(zx)y$ or $(xy)z$ since $f \cdot x$ is essentially ternary from Lemma 1-0.2. If $fx = (xz)y$, then, by the symmetry of fx in y and z,

$$(xz)y = f(x, y, z)x = f(x, z, y)x = (xy)z$$

which implies associativity.

Similarly, $fx \neq (xy)z$. So $fx = (yz)x$.

Similarly, the other results can be proved. □

Now we are in a position to prove our main theorem in this section.

THEOREM 1-0.1. *If algebra $\mathfrak{A} = \langle A; F \rangle$ represents sequence $\langle 0, 1, 1, 4 \rangle$, then the binary operation "$\cdot$" is associative.*

PROOF. Suppose that "$\cdot$" is nonassociative. Then there are only two cases to consider by Lemma 1-0.4.

Case 1. $f(x, y, z)x = f$:

By Lemma 1-0.3, $x = (fx)f = ff = f$, a contradiction.

Case 2. $f(x, y, z)x = (yz)x$:

By Lemma 1-0.3, $f = (fx)x = ((yz)x)x = yz$, a contradiction. □

REMARK . *We can prove that if an algebra $\mathfrak{A}$ represents $\langle 0, 1, 1, 5 \rangle$ or $\langle 0, 1, 1, 6 \rangle$, then there is one and only one essentially binary semilattice term operation xy (see Part 3).*

CHAPTER 2

Basic results on ternary term operations

Let S_n be the symmetric group on n letters $\{1, 2, \ldots, n\}$. For every essentially n-ary polynomial $p = p(x_1, x_2, \ldots, x_n)$, recall that the *symmetric group* of p is

$$\mathrm{Inv}(p) = \{\alpha \in S_n \mid p = p^\alpha\},$$

where $p = p^\alpha$ means $p(x_1, x_2, \ldots, x_n) = p(x_{1\alpha}, x_{2\alpha}, \ldots, x_{n\alpha})$. Define an equivalence relation $\sim$ over the set $\mathrm{Term}_n^e(\mathfrak{A})$ of all essentially n-ary polynomials as follows:

$$p \sim q \Longleftrightarrow p = q^\alpha \quad \text{for some } \alpha \in S_n.$$

Let $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m$ be all equivalence classes containing $f_1, f_2, \ldots, f_m$ as their representatives, respectively. Then

$$p_n = |\mathrm{Term}_n^e(\mathfrak{A})| = \sum_{i=1}^{m} \frac{n!}{|\mathrm{Inv}(f_i)|}.$$

Suppose that algebra $\mathfrak{A}$ represents $\langle 0, 1, 1, 4\rangle$. Then its unique binary polynomial induces a semilattice operation " $\cdot$ " . Let $g_1(x, y, z)$, $g_2(x, y, z)$ and $g_3(x, y, z)$ (in abbreviation, g_1, g_2 and g_3) denote the other three essentially ternary polynomials different from xyz. We have

THEOREM 2-0.2. *There are three cases (up to equivalence)*

(1) all g_i's are symmetric ;

(2) g_1 is symmetric and $|\mathrm{Inv}(g_2)| = |\mathrm{Inv}(g_3)| = 3$;

(3) $|\mathrm{Inv}(g_i)| = 2$, for $i = 1, 2, 3$.

PROOF. Suppose that not all of them are symmetric. Without loss of generality, let g_1 be symmetric. If g_2 is symmetric, then $|\mathrm{Inv}(g_k)| = 6$ for $k = 1, 2$. Since

$$4 = \frac{6}{|\mathrm{Inv}(xyz)|} + \frac{6}{|\mathrm{Inv}(g_1)|} + \frac{6}{|\mathrm{Inv}(g_2)|} + \frac{6}{|\mathrm{Inv}(g_3)|},$$

$|\mathrm{Inv}(g_3)| = 6$, that is, g_3 is symmetric, a contradiction. Similarly, g_3 is not symmetric. Thus, $\tilde{g}_2 = \tilde{g}_3$ so $|\mathrm{Inv}(g_2)| = |\mathrm{Inv}(g_3)| = 3$.

If none of them is symmetric, then all $\tilde{g}_i$'s are equal, so $|\mathrm{Inv}(g_i)| = 2$ for all i. $\square$

Based on the results of sequence $\langle 0, 1, 1, 2 \rangle$, K. M. Koh gave examples of algebras representing $\langle 0, 1, 1, n \rangle$ in 1971 [45] (Proposition 2.4, p.101). He considered the class $K(1)$ of all algebras on which all basic operations are symmetric. He proved that if $\mathfrak{A} \in K(1)$ represents $\langle 0, 1, 1, n \rangle$, then $p_4(\mathfrak{A}) \geqslant 10n - 9$. Thus, *for case (1) in Theorem 2-0.2, $p_4(\mathfrak{A}) \geqslant 31$.*

In symmetric group S_3, there are only two elements of degree 3, (123) and (132). Since $(123)^2 = (132)$ and $(132)^2 = (123)$, for case (2), we can assume that

(s0) $$g_1(x, y, z) \text{ is symmetric,}$$

(s1) $$g_2(x, y, z) = g_2(z, x, y) = g_2(y, z, x) = g_3(y, x, z),$$

(s2) $$g_3(x, y, z) = g_3(z, x, y) = g_3(y, z, x) = g_2(y, x, z).$$

For case (3), we will prove that $\mathrm{Inv}(g_i)'s$ are pairwise different, $i = 1, 2, 3$. Thus, we can assume that

(s3) $$\mathrm{Inv}(g_1) = \langle (23) \rangle, \quad \mathrm{Inv}(g_2) = \langle (13) \rangle, \quad \mathrm{Inv}(g_3) = \langle (12) \rangle.$$

where $\langle (23) \rangle$ denotes the subgroup of S_3 generated by the permutation (23) and so on. If the three symmetric groups are all equal, then without loss of generality, we can assume that $g_i(x, y, z) = g_i(y, x, z)$, i.e., $\mathrm{Inv}(g_i) = \langle (12) \rangle$, for all $i = 1, 2, 3$. Thus,

$$g_1(x,z,y) = \begin{cases} g_2(x,y,z), \\ g_3(x,y,z). \end{cases}$$

If $g_1(x,z,y) = g_2(x,y,z)$, then

$$g_2(x,y,z) = g_1(x,z,y) = g_1(z,x,y) = g_2(z,y,x).$$

Therefore, $(13) \in \mathrm{Inv}(g_2)$, a contradiction. If $g_1(x,z,y) = g_3(x,y,z)$, then

$$g_3(x,y,z) = g_1(x,z,y) = g_1(z,x,y) = g_3(z,y,x),$$

so $(13) \in \mathrm{Inv}(g_3)$, a contradiction.

Suppose that two of the symmetric groups are equal, say the first two, i.e.,

$$g_1(x,y,z) = g_1(y,x,z), \quad g_2(x,y,z) = g_2(y,x,z)$$

and

$$g_3(x,y,z) = g_3(x,z,y).$$

Then

$$g_1(x,z,y) = \begin{cases} g_2(x,y,z), \\ g_3(x,y,z). \end{cases}$$

If $g_1(x,z,y) = g_2(x,y,z)$, then

$$g_2(x,y,z) = g_1(x,z,y) = g_1(z,x,y) = g_2(z,y,x).$$

Therefore, $(13) \in \mathrm{Inv}(g_2)$, a contradiction. If $g_1(x,z,y) = g_3(x,y,z)$, then

$$g_1(x,z,y) = g_3(x,y,z) = g_3(x,z,y) = g_1(x,y,z),$$

which implies $(23) \in \mathrm{Inv}(g_1)$, a contradiction.

Therefore $\mathrm{Inv}(g_i)'s$ are pairwise different, $i = 1,2,3$; and we can assume (s3).

Let $\mathbf{S_1}$ denote the class of algebras representing $\langle 0,1,1,4\rangle$ and satisfying (s0), (s1), and (s2). Let $\mathbf{S_2}$ denote the class of algebras representing $\langle 0,1,1,4\rangle$ and satisfying (s3). In the following chapters, we will consider the lower bounds of $p_4(\mathfrak{A})$ in these two classes and discuss some algebras related to those lower bounds. When we say permutation (23) acting on $f(x, y, z)$, it means that (23) does on the second and the third variables of f; and so on. **From now on, suppose that the semilattice term operation is meet semilattice operation.**

CHAPTER 3

On Class $\mathbf{S}_1$

In this chapter, we will classify all algebras representing $\langle 0,1,1,4\rangle$ in $\mathbf{S}_1$ as six subclasses. Then we will prove that $p_4(\mathfrak{A}) \geqslant 31$ for each subclass.

3-1. Classification of $\mathbf{S}_1$

Suppose that algebra $\mathfrak{A}$ represents $\langle 0,1,1,4\rangle$ and there are three essentially ternary polynomials g_i, $i=1,2,3$, satisfying that g_1 is symmetric, and conditions (s1) and (s2):

$$g_2(x,y,z) = g_2(y,z,x) = g_2(z,x,y) = g_3(y,x,z),$$

$$g_3(x,y,z) = g_3(y,z,x) = g_3(z,x,y) = g_2(y,x,z).$$

For simplification, sometimes $g_i(x,y,z)$ is simply denoted by g_i.

LEMMA 3-1.1. *$g_i(x,y,y) = xy$ for all i.*

PROOF. We only need to prove that $g_2(x,y,y) = xy$ since the others are similar. If $g_2(x,y,y) = x$, we claim that the polynomials

$$g_2,\ g_2x,\ g_2y \text{ and } g_2z$$

are pairwise distinct and essentially ternary. If $g_2 = g_2x$, then, letting $z=x$,

$$y = g_2(y,x,x) = g_2(y,x,x)x = yx,$$

a contradiction. Similarly, $g_2 \neq g_2y, g_2z$. If $g_2x = g_2y$, then, letting $z=y$,

$$x = xx = g_2(x,y,y)x = g_2(x,y,y)y = xy,$$

a contradiction. Symmetrically, $g_2x \neq g_2z$ and $g_2y \neq g_2z$. Set $z = y$, $g_2(x,y,z)x = g_2(x,y,y)x = x$, so $g_2(x,y,z)x$ depends on x. Letting $z = x$,

$$g_2(x,y,z)x = g_2(x,y,x)x = g_2(y,x,x)x = yx$$

so $g_2(x,y,z)x$ depends on y. Letting $y = x$,

$$g_2(x,y,z)x = g_2(x,x,z)x = g_2(z,x,x)x = zx,$$

so $g_2(x,y,z)x$ depends on z. Thus, $g_2(x,y,z)x$ is essentially ternary. By permutation (xyz), g_2y and g_2z are essentially ternary. So our claim is true.

We claim that g_2x, g_2y and g_2z are not symmetric. Otherwise, all of them are the same by permutation (xyz), a contradiction to the claim above. Therefore we obtain at least three nonsymmetric essentially ternary polynomials, which contradicts our assumption.

Similarly, $g_2(x,y,y) = y$ is impossible. So $g_2(x,y,y) = xy$. □

From Lemma 3-1.1, it is easy to show that

LEMMA 3-1.2. *g_ix, g_iy and g_iz are essentially ternary for all i.*

PROOF. Let $z = y$. Then $g_2(x,y,z)x = g_2(x,y,y)x = xy$, so $g_2(x,y,z)x$ depends on x. Similarly, $g_2(x,y,z)x$ depends on y and z, respectively. By permutation (xyz), g_2y and g_2z are essentially ternary. Similarly, we can prove the others. □

LEMMA 3-1.3. *One of g_1x, g_1y and g_1z is symmetric if and only if they are all symmetric so they are all equal.*

LEMMA 3-1.4. *g_1x, g_1y, g_1z are all symmetric and then either*

$$g_1x = g_1y = g_1z = xyz$$

or

$$g_1x = g_1y = g_1z = g_1.$$

PROOF. Otherwise, they are nonsymmetric and pairwise distinct by Lemma 3-1.3. But there are only two nonsymmetric essentially ternary polynomials by assumption, a contradiction. □

LEMMA 3-1.5. *$g_1(xy, x, y) = xy$ so $g_1(xyz, xy, z) = xyz$ and $g_1(xyz, xy, xz) = xyz$.*

PROOF. The last two are direct consequences of the first one. And the first follows from the statement that $g_1(xy, x, y)$ depends on x if and only if it depends on y (since it is symmetric in x and y). □

LEMMA 3-1.6. *$g_1(xy, y, z)$ is an essentially ternary polynomial.*

PROOF. Let $x = y$. Then $g_1(xy, y, z) = g_1(x, x, z) = xz$ by Lemma 3-1.1. So $g_1(xy, y, z)$ depends on z. Let $z = x$. Then $g_1(xy, y, z) = g_1(xy, y, x) = xy$ by Lemma 3-1.5. So $g_1(xy, y, z)$ depends on y. Similarly, it depends on x by Lemma 3-1.1, so it is essentially ternary. □

LEMMA 3-1.7. *$g_1(xy, y, z) \neq g_1(x, y, z)$.*

PROOF. If

$$g_1(xy, y, z) = g_1(x, y, z), \tag{3-1.7-1}$$

then

$$\begin{aligned}
g_1(x,y,z) &= g_1(y,z,xy) && (\text{ by permutation } (123)\ ) \\
&= g_1(yz,z,xy) && (\text{by } (3\text{-}1.7\text{-}1)) \\
&= g_1(z,xy,yz) && (\text{by permutation } (123)) \\
&= g_1(xyz,xy,yz) && (\text{by } (3\text{-}1.7\text{-}1)) \\
&= g_1(xy,yz,xyz) && (\text{by permutation } (123)) \\
&= g_1(xyz,yz,xyz) && (\text{by } (3\text{-}1.7\text{-}1)) \\
&= g_1(yz,xyz,xyz) && (\text{by permutation } (123))
\end{aligned}$$

$$\begin{aligned} &= g_1(xyz, xyz, xyz) \quad \text{(by (3-1.7-1))} \\ &= xyz, \qquad\qquad \text{(by idempotence)} \end{aligned}$$

a contradiction. □

COROLLARY 3-1.1. *$g_i(xy, y, z) \neq g_i(x, y, z)$ for i=2,3.*

PROOF. In the proof of Lemma 3-1.7 , we only use permutation (123) . □

LEMMA 3-1.8. *$g_1(xy, y, z) = xyz$.*

PROOF. By Lemma 3-1.6, $g_1(xy, y, z)$ is essentially ternary. By Lemma 3-1.7, $g_1(xy, y, z) \neq g_1(x, y, z)$. If

$$g_1(xy, y, z) = g_2(x, y, z),$$

then

$$\begin{aligned} g_2(x, y, z) &= g_1(xy, y, z) \qquad \text{(by assumption)} \\ &= g_1((xy)y, y, z) \\ &= g_2(xy, y, z), \qquad \text{(by assumption)} \end{aligned}$$

contradicting Corollary 3-1.1. Hence $g_1(xy, y, z) \neq g_2(x, y, z)$. Similarly, $g_1(xy, y, z) \neq g_3(x, y, z)$. So $g_1(xy, y, z) = xyz$. □

COROLLARY 3-1.2. *$g_1(xy, xz, z) = xyz$, $g_1(xy, xz, x) = xyz$, $g_1(xyz, y, z) = xyz$.*

LEMMA 3-1.9. *$g_1(xy, yz, zx) = xyz$.*

PROOF. Obviously, $g_1(xy, yz, zx)$ is symmetric . It is easy to show that it is essentially ternary by Lemma 3-1.1. So

$$g_1(xy, yz, zx) = \begin{cases} xyz, \\ g_1(x, y, z). \end{cases}$$

If $g_1(xy, yz, zx) = g_1(x, y, z)$, then

$$\begin{aligned} xyz &= g_1(xyz, xyz, xyz) \\ &= g_1(xy \cdot yz, yz \cdot zx, zx \cdot xy) \\ &= g_1(xy, yz, zx) \\ &= g_1(x, y, z), \end{aligned}$$

a contradiction. □

LEMMA 3-1.10. *If $g_i(xy, y, z)$ is essentially ternary, then $g_i(xy, y, z) = xyz$ for i=2,3.*

PROOF. If $g_2(xy, y, z) = g_3(x, y, z)$, then

$$\begin{aligned} g_3(x, y, z) &= g_2(xy, y, z) \qquad \text{(by assumption)} \\ &= g_2((xy)y, y, z) \\ &= g_3(xy, y, z), \qquad \text{(by assumption)} \end{aligned}$$

contradicting Corollary 3-1.1. So $g_2(xy, y, z) \neq g_3(x, y, z)$. Similarly, $g_2(xy, y, z) \neq g_1(x, y, z)$ by Lemma 3-1.7-1. By Corollary 3-1.1, $g_2(xy, y, z) = xyz$. Similarly, $g_3(xy, y, z) = xyz$. □

COROLLARY 3-1.3. *If $g_i(xy, y, z)$ is essentially ternary, then*

$$\begin{aligned} &g_i(xy, xz, z) = xyz, \quad g_i(xy, xz, y) = xyz, \\ &g_i(xy, xz, y) = xyz, \quad g_i(xyz, y, z) = xyz, \\ &g_i(y, xy, z) = xyz. \end{aligned}$$

We will prove in the proof of Theorem 3-1.1 that $g_i(xy, y, z)$ is essentially ternary, for $i = 2, 3$. Threrfore the conditions in Lemma 3-1.10 and Corollary 3-1.3 can be eliminated.

LEMMA 3-1.11. *$g_i(xy, x, y) = xy$ for all i.*

PROOF. The case $i = 1$ is Lemma 3-1.5. We only prove the case i=2 since the proof of the other is similar.

If $g_2(xy, x, y) \neq xy$, then

$$g_2(xy, x, y) = \begin{cases} x, \\ y. \end{cases}$$

Case 1 . $g_2(xy, x, y) = x$ i.e., $g_2(xy, y, x) = y$:

Subcase 1.1 . If $g_2(xy, y, z)$ is essentially ternary, then, by Lemma 3-1.10, $g_2(xy, y, z) = xyz$. Let $z = x$. Then $xy = g_2(xy, y, x) = y$ by assumption of case 1, a contradiction.

Subcase 1.2 . Suppose that $g_2(xy, y, z)$ is not essentially ternary. Then

$$g_2(xy, y, z) = \begin{cases} xy \\ xz \\ yz \\ x \\ y \\ z. \end{cases}$$

If $g_2(xy, y, z) = xy$, then, setting $z = x$, $xy = g_2(xy, y, x) = y$ by the assumption of case 1, a contradiction. If $g_2(xy, y, z) = xz$, then, setting $z = x$,

$$x = g_2(xy, y, x),$$

contradicting the assumption of case 1. If $g_2(xy, y, z) = yz$, then, setting $z = x$,

$$xy = g_2(xy, y, x),$$

contradicting the assumption of case 1. If $g_2(xy, y, z) = x$, then, setting $z = xy$, $x = g_2(xy, y, xy) = xy$ by Lemma 3-1.1, a contradiction. If $g_2(xy, y, z) = y$, then, setting $z = y$, $y = g_2(xy, y, y) = xy$ by Lemma 3-1.1, a contradiction. If $g_2(xy, y, z) = z$, then, setting $z = x$, $x = g_2(xy, y, x)$, contradicting the assumption of case 1.

Case 2 . $g_2(xy, x, y) = y$ i.e., $g_2(xy, y, x) = x$:

Subcase 2.1 . If $g_2(xy, y, z)$ is essentially ternary, then, by Lemma 3-1.10, $g_2(xy, y, z) = xyz$. Let $z = x$. Then $xy = g_2(xy, y, x) = x$ by the assumption of case 2, a contradiction.

Subcase 2.2 . Suppose that $g_2(xy, y, z)$ is not essentially ternary. Then

$$g_2(xy, y, z) = \begin{cases} xy \\ xz \\ yz \\ x \\ y \\ z. \end{cases}$$

If $g_2(xy, y, z) = xy$, then, setting $z = x$, $g_2(xy, y, x) = xy$, contradicting the assumption of case 2. If $g_2(xy, y, z) = yz$, then, setting $z = x$, $xy = g_2(xy, y, x) = x$ by the assumption of case 2, a contradiction. If $g_2(xy, y, z) = xz$, then, replacing y by xy,

$$xz = g_2(x(xy), xy, z) = g_2(xy, xy, z) = (xy)z = xyz$$

by Lemma 3-1.1, a contradiction. Since $g_2(xy, y, z)$ depends on z, $g_2(xy, y, z) \neq x$ or y. If $g_2(xy, y, z) = z$, then, substituting xy for y,

$$z = g_2(x(xy), xy, z) = g_2(xy, xy, z) = xyz,$$

a contradiction.

Thus, $g_2(xy, x, y) = xy$. $\square$

COROLLARY 3-1.4. *$g_i(xy, y, x) = xy$ for all i.*

THEOREM 3-1.1. *$g_i(xy, y, z) = xyz$, for all $i = 1, 2, 3$.*

PROOF. The case $i = 1$ has been already proved in Lemma 3-1.8. By Lemma 3-1.10, we only need to prove that $g_i(xy, y, z)$ is essentially ternary. By Lemma 3-1.1, it is easy to see that $g_i(xy, y, z)$ depends on z. Let $z = x$. Then $g_i(xy, y, z) =$

$g_i(xy, y, x) = xy$ by Lemma 3-1.11, so it depends on y. Let $z = x$. Then $g_i(xy, y, z) = g_i(xy, y, y) = xy$ by Lemma 3-1.1, so it depends on x. □

LEMMA 3-1.12. *$g_2(x, y, z)g_3(x, y, z) = g_1(x, y, z)$ or xyz.*

PROOF. It is easy to check that the left hand side is essentially ternary and symmetric. So we obtain the lemma. □

LEMMA 3-1.13. *$g_i(xy, xz, yz) = xyz$ for all i.*

PROOF. Lemma 3-1.9 is the case i=1. We will prove the Lemma for i=2. By Lemma 3-1.1, it is easy to show that $g_2(xy, yz, xz)$ is essentially ternary.

If

$$g_2(xy, yz, xz) = g_1(x, y, z), \tag{3-1.13-1}$$

then, by the symmetry of the right hand side,

$$\begin{aligned} g_1(x, y, z) &= g_2(xy, xz, yz)(\text{by permutation } (xy)) \\ &= g_3(xy, yz, xz)(\text{by permutation } (23)), \end{aligned}$$

so

$$\begin{aligned} g_1(x, y, z) &= g_1(x, y, z)g_1(x, y, z) \\ &= g_2(xy, yz, xz)g_3(xy, yz, xz) \\ &= \begin{cases} g_1(xy, yz, xz), \\ (xy)(yz)(xz) \end{cases} && \text{(by Lemma 3-1.12)} \\ &= xyz, && \text{(by Lemma 3-1.9)} \end{aligned}$$

a contradiction.

If

$$g_2(xy, yz, xz) = g_2(x, y, z), \tag{3-1.13-2}$$

then

$$\begin{aligned} xyz &= g_2(xyz, xyz, xyz) && \text{(by idempotence)} \\ &= g_2(xy \cdot yz, xz \cdot yz, xy \cdot xz) \\ &= g_2(xy, yz, xz) && \text{(by (3-1.13-2))} \\ &= g_2(x, y, z) && \text{(by (3-1.13-2))}, \end{aligned}$$

a contradiction.

If $g_2(xy, yz, xz) = g_3(x, y, z) = g_2(x, z, y)$, then, repeating the procedure in the paragraph above, $xyz = g_2(x, z, y) = g_3(x, y, z)$, a contradiction. Thus, $g_2(xy, yz, xz) = xyz$. The proof is complete. □

LEMMA 3-1.14. *For each fixed* $i = 1, 2, 3$, $g_i(x, y, z)x$, $g_i(x, y, z)y$ *and* $g_i(x, y, z)z$ *are equal.*

PROOF. By Lemma 3-1.1, they are essentially ternary. If they are pairwise distinct for fixed i, then there are at least three essentially ternary polynomials

$$g_i(x, y, z)x, \quad g_i(x, y, z)y, \quad \text{and } g_i(x, y, z)z,$$

which contradicts the fact that there are only two nonsymmetric essentially ternary polynomials since none of them is symmetric (otherwise, they would be equal). Without loss of generality, we suppose that $g_i(x, y, z)x = g_i(x, y, z)y$ (for fixed i). Using permutation (xyz),

$$g_i(x, y, z)z = g_i(z, x, y)z = g_i(z, x, y)x = g_i(x, y, z)x$$

by (s1) and (s2) , so

$$g_i(x, y, z)x = g_i(x, y, z)y = g_i(x, y, z)z.$$

□

For $i = 1$, the fact that the polynomials in Lemma 3-1.14 are identical implies that $g_1(x, y, z)x$ is symmetric. Therefore, we have

COROLLARY 3-1.5.

$$g_1(x,y,z)x = g_1(x,y,z)y = g_1(x,y,z)z = \begin{cases} xyz \\ g_1(x,y,z). \end{cases}$$

LEMMA 3-1.15. *We have the following statements*

(1) $g_2(x,y,z)x = xyz$ *iff* $g_3(x,y,z)x = xyz$;

(2) $g_2(x,y,z)x = g_1(x,y,z)$ *iff* $g_3(x,y,z)x = g_1(x,y,z)$;

(3) $g_2(x,y,z)x \neq g_3(x,y,z)$ *and* $g_3(x,y,z)x \neq g_2(x,y,z)$;

(4) $g_2(x,y,z)x = g_2(x,y,z)$ *and* $g_3(x,y,z)x = g_3(x,y,z)$ *implies that*

$$g_2g_3 = g_1 \text{ and } g_1(x,y,z)x = g_1(x,y,z).$$

PROOF. By permutation (yz), we have (1), (2), and $g_2(x,y,z)x = g_3(x,y,z)$ iff $g_3(x,y,z)x = g_2(x,y,z)$.

Notice that the two statements in (3) are equivalent. If $g_2(x,y,z)x = g_3(x,y,z)$, i.e., $g_3(x,y,z)x = g_2(x,y,z)$ by permutation (yz), then by Lemma 3-1.14,

$$g_2(x,y,z)xy = g_2(x,y,z)y = g_2(x,y,z)x = g_3(x,y,z);$$

$$g_3(x,y,z)xy = g_3(x,y,z)y = g_3(x,y,z)x = g_2(x,y,z).$$

So

$$g_3(x,y,z) = g_2(x,y,z)xy = g_3(x,y,z)y = g_2(x,y,z),$$

a contradiction.

Suppose that $g_2(x,y,z)x = g_2(x,y,z)$ and $g_3(x,y,z)x = g_3(x,y,z)$. Then, by permutation (xyz),

$$g_2(x,y,z)y = g_2(x,y,z), \quad g_2(x,y,z)z = g_2(x,y,z),$$

If $g_2g_3 = xyz$, then

$$g_2 = g_2x = g_2xyz = g_2g_2g_3 = g_2g_3 = xyz,$$

a contradiction. Therefore by Lemma 3-1.12, $g_2g_3 = g_1$. If $g_1x = xyz$, then

$$g_2 = g_2xyz = g_2g_1x = g_2(g_2g_3)x = g_2g_3x = g_2g_3 = g_1,$$

a contraction. So $g_1 x = g_1$ by Corollary 3-1.5. □

LEMMA 3-1.16. *For each* $i = 1, 2, 3$, $g_1 x = xyz$ *and* $g_i x = g_1$ *do not hold simultaneously.*

PROOF. If they hold at the same time, then $g_1 = g_i x = g_i xx = g_i xy = g_1 y = xyz$ by Lemma 3-1.14, a contradiction. □

Now we prove a general theorem.

THEOREM 3-1.2. *Let* $\mathfrak{A} = \langle A; F \rangle$ *be an idempotent algebra and assume that there is only one essentially binary term operation over* $\mathfrak{A}$, *denoted by "xy ". If every* $f \in F$ *satisfies*

(E) $$f(x_1 y_1, \cdots, x_n y_n) = xy$$

whenever $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\} = \{x, y\}$, *then a polynomial* p *is essentially* n*-ary if and only if* p *contains exactly* n *different variables.*

PROOF. The necessity is trivial. We will prove sufficiency. Suppose that e_i^n is ith projection, $i = 0, 1, \cdots, n-1$. When $p = e_i^n$ or $p = (e_0^2 \cdot e_1^2)$, $p = x$ or xy, so the statement is true in this case.

First, we show that p *is essentially binary iff* p *contains exactly two variables* x *and* y*; so* $p = xy$. Suppose that p contains exactly two variables x and y . We only need to show that $p = xy$. Obviously, it is true for $p = (e_0^2 \cdot e_1^2)$. If $p = f(p_1, \cdots, p_n)$, where $f \in F$ and the theorem is true for p_i , $n = 1, 2$; $i = 1, 2, \cdots, n$. Therefore, $p_i = x_i y_i$ for every i, where $x_i, y_i \in \{x, y\}$ and $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\} = \{x, y\}$. Therefore, $p = xy$ by the condition (E) of Lemma.

Next, suppose that p contains exactly n variables $x_1, x_2, \cdots, x_n$. If p is not essentially n-ary, then $p = q(x_{i_1}, \cdots, x_{i_m})$ is essentially m-ary and $m < n$. Without loss of generality, suppose that $x_{i_k} = x_k$, $k = 1, 2, \cdots, m$. Let $x_1 = x_2 = \cdots = x_m = x$ and $x_{m+1} = \cdots = x_n = y$. Then, by the previous paragraph and idempotence, $xy = x$, a contradiction. □

Thus, we have the following corollaries.

LEMMA 3-1.17. *$g_i(g_j(x,y,z),s,t)$ is essentially ternary for any i and j, where $s,t \in \{x,y,z,xy,yz,xz,xyz\}$.*

LEMMA 3-1.18. *$g_i(g_1,g_2,t), g_i(g_1,g_3,t)$, and $g_i(g_2,g_3,t)$ are essentially ternary, where $t = z, yz$, or xyz.*

LEMMA 3-1.19. *$g_i(g_1,g_2,g_3)$ is essentially ternary for any i.*

Conclusions of this section:

From Lemma 3-1.14, Lemma 3-1.15, Lemma 3-1.16, Corollary 3-1.5 and Lemma 3-1.12, we have the following six cases:

(N1) $g_1(x,y,z)x = g_1(x,y,z), \quad g_2(x,y,z)x = g_1(x,y,z), \quad g_3(x,y,z)x = g_1(x,y,z)$;

(N2) $g_1(x,y,z)x = g_1(x,y,z), \quad g_2(x,y,z)x = xyz, \quad g_3(x,y,z)x = xyz$;

(N3) $g_i(x,y,z)x = g_i(x,y,z)$ for all i;

(N4)

$$g_1(x,y,z)x = xyz, \quad g_2(x,y,z)x = xyz, \quad g_3(x,y,z)x = xyz,$$
$$g_2g_3 = xyz, \quad g_1g_2 = xyz;$$

(N5)

$$g_1(x,y,z)x = xyz, \quad g_2(x,y,z)x = xyz, \quad g_3(x,y,z)x = xyz,$$
$$g_2g_3 = xyz, \quad g_1g_2 = g_2;$$

(N6) $g_1(x,y,z)x = xyz, \quad g_2(x,y,z)x = xyz, \quad g_3(x,y,z)x = xyz, \quad g_2g_3 = g_1$.

Indeed, for $g_1(x,y,z)x = g_1(x,y,z)$, we have three cases (N1), (N2) and (N3) by Lemma 3-1.15 and Lemma 3-1.16.

Suppose that $g_1(x,y,z)x = xyz$. Then $g_ix \neq g_1$ by Lemma 3-1.16 and

$$g_2x \neq g_3, \quad g_3x \neq g_2$$

by Lemma 3-1.15. If $g_2x = g_2$, i.e., $g_3x = g_3$, then, by Lemma 3-1.15, $g_1x = g_1$, contradicting the assumption $g_1x = xyz$. Hence,

$$g_2(x,y,z)x = xyz, \quad g_3(x,y,z)x = xyz.$$

If $g_2g_3 = g_1$, then we have (N6). Otherwise, $g_2g_3 = xyz$ by Lemma 3-1.12. Since g_1g_2 is essentially ternary,

$$g_1g_2 = \begin{cases} xyz \\ g_1 \\ g_2 \\ g_3. \end{cases}$$

If $g_1g_2 = g_1$, i.e., $g_1g_3 = g_1$, then

$$g_1 = g_1g_2g_3 = g_1xyz = xyz$$

by $g_1x = xyz$, a contradiction. If $g_1g_2 = g_3$, then $g_1g_3 = g_2$, so

$$g_3 = g_1g_2 = g_1(g_1g_3) = g_1g_3 = g_2,$$

a contradiction. Thus, we have (N4) or (N5).

3-2. $p_4 \geqslant 31$ for the algebras in $\mathbf{S}_1$

In this section, we will prove that $p_4 \geqslant 31$ for the algebras in $\mathbf{S}_1$. Suppose that $g_1(x,y,z)$, $g_2(x,y,z)$, and $g_3(x,y,z)$ satisfy the conditions (s0), (s1) and (s2). By Theorem 3-1.2, the following polynomials are essentially 4-ary:

$$(A_0) \qquad x_1x_2x_3x_4;$$

$$(A_1) \qquad \begin{array}{ll} g_1(x_1,x_2,x_3)x_4, & g_1(x_2,x_3,x_4)x_1, \\ g_1(x_3,x_4,x_1)x_2, & g_1(x_4,x_1,x_2)x_3; \end{array}$$

$$(A_2) \qquad \begin{array}{ll} g_2(x_1,x_2,x_3)x_4, & g_2(x_2,x_3,x_4)x_1, \\ g_2(x_3,x_4,x_1)x_2, & g_2(x_4,x_1,x_2)x_3, \\ g_3(x_1,x_2,x_3)x_4, & g_3(x_2,x_3,x_4)x_1, \\ g_3(x_3,x_4,x_1)x_2, & g_3(x_4,x_1,x_2)x_3; \end{array}$$

$$(A_3) \qquad \begin{array}{ll} g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4), & g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4), \\ g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3), & g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1), \\ g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3), & g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2); \end{array}$$

$$(A_4) \qquad \begin{array}{ll} g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4), & g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4), \\ g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3), & g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1), \\ g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3), & g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2), \\ g_3(x_1,x_2,x_3)g_3(x_1,x_2,x_4), & g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4), \\ g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3), & g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1), \\ g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3), & g_3(x_3,x_4,x_1)g_3(x_3,x_4,x_2). \end{array}$$

Obviously, $A_0 \cap (A_3 \cup A_4) = \varnothing$. We will prove that the polynomials in $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct. Let us start with the following concepts.

DEFINITION 3-2.1. *Given a set P of essentially n-ary polynomials, we define the* symmetric group $\operatorname{Inv}(P)$ *of P as*

$$\operatorname{Inv}(P) = \{\alpha \in S_n \mid p^\alpha \in P \textit{ for all } p \in P\},$$

where S_n is the symmetric group on n letters.

DEFINITION 3-2.2. *Given a set P of essentially n-ary polynomials and a permutation $\alpha \in S_n$, we say that P is* closed *under α if*

$$P^\alpha = \{p^\alpha \mid p \in P, \alpha \in S_n\} \subseteq P.$$

DEFINITION 3-2.3. *A set P of essentially n-ary polynomials is called* invariant *under a permutation $\alpha \in S_n$ if $P^\alpha = P$. It is called invariant under S_n if $P^\alpha = P$ for every $\alpha \in S_n$.*

LEMMA 3-2.1. *Each A_i is invariant under S_4, $i = 1, 2, 3, 4$.*

PROOF. We only need to prove that A_i is invariant under each of permutations (12), (13), (14), (23), (24), (34). However, it is routine to check. □

LEMMA 3-2.2. *For every fixed i, for any $p, q \in A_i$, there is a $\alpha \in S_4$ such that $p = q^\alpha$.*

PROOF. Since every $\alpha \in S_4$ is bijective, we only need to prove that for some fixed p, for every $q \in S_4$ there exists an α such that $q = p^\alpha$.

Obviously, the lemma is true for A_0 and A_1.

For A_2, we have

$$g_2(x_2, x_3, x_4)x_1 = (g_2(x_1, x_2, x_3)x_4)^{(14)}, \quad g_2(x_3, x_4, x_1)x_2 = (g_2(x_1, x_2, x_3)x_4)^{(142)},$$
$$g_2(x_4, x_1, x_2)x_3 = (g_2(x_1, x_2, x_3)x_4)^{(34)}, \quad g_3(x_1, x_2, x_3)x_4 = (g_2(x_1, x_2, x_3)x_4)^{(12)},$$
$$g_3(x_2, x_3, x_4)x_1 = (g_2(x_1, x_2, x_3)x_4)^{(134)}, \quad g_3(x_3, x_4, x_1)x_2 = (g_2(x_1, x_2, x_3)x_4)^{(24)},$$
$$g_3(x_4, x_1, x_2)x_3 = (g_2(x_1, x_2, x_3)x_4)^{(314)}.$$

For A_3, we have

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = (g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4))^{(23)},$$
$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = (g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4))^{(24)},$$
$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = (g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4))^{(123)},$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = (g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4))^{(214)},$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = (g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4))^{(4321)}.$$

For A_4, we have

$$g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(23)},$$

$$g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(24)},$$

$$g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(123)},$$

$$g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(124)},$$

$$g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(1324)},$$

$$g_3(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(12)},$$

$$g_3(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(132)},$$

$$g_3(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(142)},$$

$$g_3(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(13)},$$

$$g_3(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(14)},$$

$$g_3(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = (g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4))^{(14)(23)}.$$

□

We will continue the discussion in six parts.

3-2.1. $p_4 \geqslant 31$ **for algebras satisfying (N1).** Suppose that $g_i's$ satisfy (N1). We will prove that $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct.

LEMMA 3-2.3. *$g_i(x,y,z)t = g_1(x,y,z)$ for every i, where $t \in \{x,y,z,xy,yz,zx,xyz\}$.*

PROOF. We only prove the case i=2. Using permutation (xyz) in $g_2(x,y,z)x = g_1(x,y,z)$, $g_1 = g_1(z,x,y) = g_2(z,x,y)z = g_2(x,y,z)z$ (by (N1)). Again using (xyz), $g_1 = g_1(y,z,x) = g_2(y,z,x)y = g_2(x,y,z)y$ (by (N1)). So

$$g_1 = g_1g_1 = g_2x \cdot g_2y = g_2xy$$

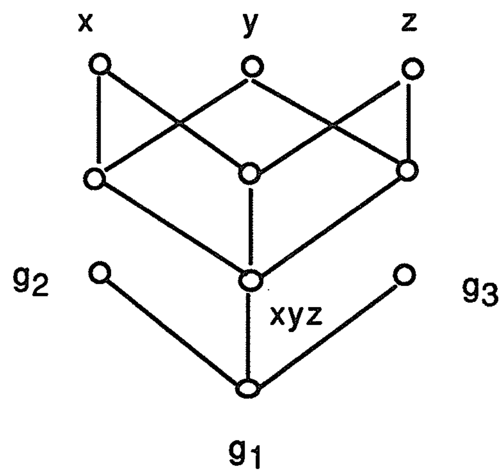

FIGURE 1. N1

$$= g_1 g_1 = g_2 x \cdot g_2 z = g_2 xz$$

$$= g_1 g_1 = g_2 z \cdot g_2 y = g_2 yz$$

$$= g_1 g_1 g_1 = g_2 x \cdot g_2 y \cdot g_2 z$$

$$= g_2 xyz.$$

□

LEMMA 3-2.4. $g_1 g_2 = g_1, \quad g_1 g_3 = g_1, \quad g_2 g_3 = g_1.$

PROOF. By $(N1)$,

$$g_1 g_2 = (g_1 x) g_2 = g_1 (x g_2) = g_1 g_1 = g_1.$$

Similarly, $g_1 g_3 = g_1$.

If $g_2 g_3 = xyz$, then $xyz = g_2 x g_3 x = g_1$ by $(N1)$, a contradiction. So by Lemma 3-1.12, $g_2 g_3 = g_1$. □

The semilattice structure of free algebra with three generators satisfying (N1) and representing $\langle 0, 1, 1, 4 \rangle$ is as in Figure 1. It is easy to see

LEMMA 3-2.5. $A_0 \cap \bigcup_{i=1}^{4} A_i = \varnothing$.

PROOF. For example, let $x_4 = x_1$,

$$x_1x_2x_3x_4 = x_1x_2x_3 \neq g_1(x_1, x_2, x_3) = g_1(x_1, x_2, x_3)x_4.$$

□

LEMMA 3-2.6. $A_1 \cap A_2 = \varnothing$.

PROOF. Let $x_4 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3),$$

$$g_2(x_3, x_4, x_1)x_2 = g_2(x_3, x_1, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = g_2(x_1, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = g_3(x_3, x_1, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = g_3(x_1, x_1, x_2)x_3 = x_1x_2x_3.$$

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3.$$

Let $x_4 = g_2(x_1, x_2, x_3)$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3) \quad \text{(by Lemma 3-2.4)},$$

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3).$$

Let $x_4 = g_3(x_1, x_2, x_3)$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3) \quad \text{(by Lemma 3-2.4)},$$

$$g_3(x_1, x_2, x_3)x_4 = g_3(x_1, x_2, x_3).$$

Thus, from the above, $g_1(x_1, x_2, x_3)x_4 \notin A_2$.

By lemmas 3-2.1 and 3-2.2, any polynomial in A_1 is not in A_2 by applying permutations in S_4. □

LEMMA 3-2.7. $A_1 \cap A_3 = \varnothing$.

PROOF. Let $x_3 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_4 \quad \text{(by Lemma 3-1.1)},$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_4) \quad \text{(by (N1) and Lemma 3-1.1)},$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = g_1(x_1, x_2, x_4),$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = g_1(x_1, x_4, x_2),$$

$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = g_1(x_2, x_4, x_1),$$

$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = g_1(x_2, x_4, x_1).$$

Let $x_1 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_2x_3x_4,$$

$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = g_1(x_2, x_3, x_4).$$

Thus, $g_1(x_1, x_2, x_3)x_4 \notin A_3$. By lemmas 3-2.1 and 3-2.2, any polynomial in A_1 is not in A_3 by applying permutations in S_4. □

LEMMA 3-2.8. $A_1 \cap A_4 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)x_4 \notin A_4$:

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)g_3(x_1, x_2, x_4) = g_3(x_1, x_2, x_3).$$

Let $x_4 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_1, x_3, x_2)g_2(x_1, x_3, x_4) = g_2(x_1, x_3, x_2),$$

$$g_3(x_1, x_3, x_2)g_3(x_1, x_3, x_4) = g_3(x_1, x_3, x_2).$$

Let $x_3 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_4,$$

$$g_2(x_1, x_4, x_2)g_2(x_1, x_4, x_3) = g_2(x_1, x_4, x_2),$$

$$g_3(x_1, x_4, x_2)g_3(x_1, x_4, x_3) = g_3(x_1, x_4, x_2).$$

Let $x_4 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_2, x_3, x_4)g_2(x_2, x_3, x_1) = g_2(x_2, x_3, x_1),$$

$$g_3(x_2, x_3, x_4)g_3(x_2, x_3, x_1) = g_3(x_2, x_3, x_1).$$

Let $x_3 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_4,$$

$$g_2(x_2, x_4, x_1)g_2(x_2, x_4, x_3) = g_2(x_2, x_4, x_1),$$

$$g_3(x_2, x_4, x_1)g_3(x_2, x_4, x_3) = g_3(x_2, x_4, x_1).$$

Let $x_2 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_3x_4,$$

$$g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2) = g_2(x_3, x_4, x_1),$$

$$g_3(x_3, x_4, x_1)g_3(x_3, x_4, x_2) = g_3(x_3, x_4, x_1).$$

Therefore, $g_1(x_1, x_2, x_3)x_4 \notin A_4$.

(2) By lemmas 3-2.1 and 3-2.2, any polynomial in A_1 is not in A_4 by applying permutations in S_4 to (1). □

LEMMA 3-2.9. $A_2 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin A_2$:

Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_2(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3,x_4,x_1)x_2 = x_1x_2x_3.$$

Let $x_3 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_4) \quad \text{(by (N1) and Lemma 3-1.1)},$$

$$g_2(x_1,x_2,x_3)x_4 = x_1x_2x_4,$$

$$g_3(x_1,x_2,x_3)x_4 = x_1x_2x_4.$$

Let $x_4 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3) \quad \text{(by (N1) and Lemma 3-1.1)},$$

$$g_2(x_4,x_1,x_2)x_3 = x_1x_2x_3,$$

$$g_2(x_4,x_1,x_2)x_3 = x_1x_2x_3.$$

Therefore, $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin A_2$.

(2) By lemmas 3-2.1 and 3-2.2, any polynomial in A_2 is not in A_3 by applying permutations in S_4 to (1). □

LEMMA 3-2.10. $A_2 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)x_4 \notin A_4$:

Let $x_4 = x_3$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3),$$

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = x_1x_2x_3,$$

$$g_3(x_1,x_2,x_3)g_3(x_1,x_2,x_4) = g_3(x_1,x_2,x_3),$$

$$g_3(x_3,x_4,x_1)g_3(x_3,x_4,x_2) = x_1x_2x_3.$$

Let $x_4 = x_1$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3),$$

$$g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = x_2x_3x_4,$$

$$g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = g_2(x_2,x_3,x_1),$$

$$g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3) = x_2x_3x_4,$$

$$g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1) = g_3(x_2,x_3,x_1).$$

Let $x_4 = x_2$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3),$$

$$g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = g_2(x_1,x_3,x_2),$$

$$g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4) = g_3(x_1,x_3,x_2).$$

Thus, $g_2(x_1,x_2,x_3)x_4 \notin A_4$.

(2) By lemmas 3-2.1 and 3-2.2, any polynomial in A_2 is not in A_4 by applying permutations in S_4 to (1). □

LEMMA 3-2.11. $A_3 \cap A_4 = \varnothing$.

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin A_4$:

Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = x_1x_2x_3,$$

$$g_3(x_1,x_2,x_3)g_3(x_1,x_2,x_4) = g_3(x_1,x_2,x_3),$$

$$g_3(x_3,x_4,x_1)g_3(x_3,x_4,x_2) = x_1x_2x_3.$$

Let $x_4 = x_1$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3) \quad \text{(by (N1) and Lemma 3-1.1)},$$

$$g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = x_2x_3x_4,$$

$$g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = g_2(x_2,x_3,x_1),$$

$$g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3) = x_2x_3x_4,$$

$$g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1) = g_3(x_2,x_3,x_1).$$

Let $x_4 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3) \quad \text{(by (N1) and Lemma 3-1.1)},$$

$$g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = g_2(x_1,x_3,x_2),$$

$$g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4) = g_3(x_1,x_3,x_2).$$

Thus, $g_2(x_1,x_2,x_3)x_4 \notin A_4$.

(2) By lemmas 3-2.1 and 3-2.2, any polynomial in A_3 is not in A_4 by applying permutations in S_4 to (1). □

LEMMA 3-2.12. *The polynomials in A_1 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)x_4$ is different from others:

Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3.$$

Let $x_4 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

Therefore, $g_1(x_1, x_2, x_3)x_4$ is different from others.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_1 are pairwise distinct by applying permutations in S_4 to (1). □

LEMMA 3-2.13. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)x_4$ is different from others: Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3.$$

Let $x_4 = x_2$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3),$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

Let $x_4 = g_2(x_1, x_2, x_3)$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)x_4 = g_3(x_1, x_2, x_3)g_2(x_1, x_2, x_3) = g_1(x_1, x_2, x_3) \quad \text{(by Lemma 3-2.4)}.$$

Therefore, $g_2(x_1, x_2, x_3)x_4$ is different from others.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_2 are pairwise distinct by applying permutations in S_4 to (1). □

LEMMA 3-2.14. *The polynomials in A_3 are pairwise distinct.*

PROOF. Let $x_1 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = x_2x_3x_4,$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = g_1(x_2,x_3,x_4),$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_2,x_3,x_4),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = g_1(x_3,x_4,x_2).$$

Therefore, $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4)$ is different from others. By lemmas 3-2.1 and 3-2.2, the polynomials in A_3 are pairwise distinct by applying permutations in S_4. □

LEMMA 3-2.15. *The polynomials in A_4 are pairwise distinct.*

PROOF. (1) $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4)$ is different from others:
Let $x_1 = x_2$. Then

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = x_2x_3x_4,$$

$$g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = g_1(x_2,x_3,x_4),$$

$$g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = g_1(x_2,x_3,x_4),$$

$$g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = g_1(x_3,x_4,x_2),$$

$$g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4) = g_1(x_2,x_3,x_4),$$

$$g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1) = g_1(x_2,x_3,x_4),$$

$$g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3) = g_1(x_2,x_4,x_3),$$

$$g_3(x_3, x_4, x_1)g_3(x_3, x_4, x_2) = g_1(x_3, x_4, x_2).$$

Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)g_3(x_1, x_2, x_4) = g_3(x_1, x_2, x_3).$$

Therefore, $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ is different from others.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_4 are pairwise distinct by applying permutations in S_4 to (1). □

From the lemmas above, $p_4 \geqslant 31$ for any algebra representing $\langle 0, 1, 1, 4\rangle$ and satisfying (N1) under conditions (s0), (s1) and (s2).

3-2.2. $p_4 \geqslant 31$ for algebras satisfying (N2). In this part, we assume that g_i's satisfy the condition (N2). We will prove that $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct.

LEMMA 3-2.16.

$$g_1(x, y, z)t = g_1(x, y, z), \quad g_i(x, y, z)t = xyz,$$

where $t \in \{x, y, z, xy, yz, zx, xyz\}$ *and* $i = 2, 3$.

PROOF. It is similar to Lemma 3-2.3. □

LEMMA 3-2.17. $g_2g_3 = xyz, g_1g_2 = g_1, g_1g_3 = g_1$.

PROOF. $g_1g_2 = g_1xg_2 = g_1(g_2x) = g_1xyz = g_1$ by Lemma 3-2.16. Similarly, $g_1g_3 = g_1$. If $g_2g_3 = g_1$, then $g_1 = g_1x = g_2g_3x = g_2xyz = xyz$ by Lemma 3-2.16, a contradiction. So $g_2g_3 = xyz$ by Lemma 3-1.12. □

The semilattice structure of free algebra with three generators representing $\langle 0, 1, 1, 4\rangle$ and satisfying (N2) is as in Figure 2. As the previous subsection, it is easy to see

LEMMA 3-2.18. $A_0 \cap \bigcup_{i=1}^{4} A_i = \varnothing$.

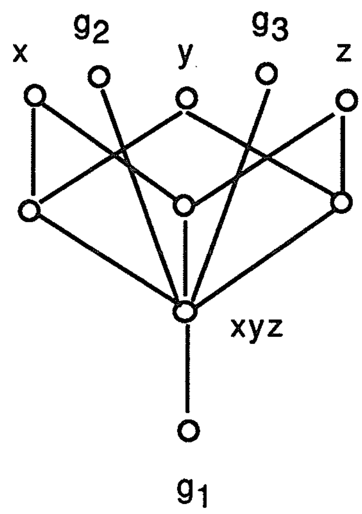

FIGURE 2. N2

PROOF. For example, let $x_4 = g_2(x_1, x_2, x_3)$. Then

$$x_1x_2x_3x_4 = x_1x_2x_3g_2(x_1, x_2, x_3) = x_1x_2x_3,$$
$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3).$$

□

LEMMA 3-2.19. $A_1 \cap A_2 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)x_4 \notin A_2$:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3) \quad \text{(by Lemma 3-2.16)},$$
$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$
$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$
$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$
$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$
$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$
$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_2 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.20. $A_1 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)x_4 \notin A_3$:
Let $x_3 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_4,$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_4) \quad \text{(by Lemma 3-2.16)},$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = g_1(x_1, x_2, x_4),$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = g_1(x_1, x_4, x_2),$$

$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = g_1(x_2, x_4, x_1),$$

$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = g_1(x_2, x_4, x_1).$$

Let $x_3 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_4,$$

$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = g_1(x_2, x_1, x_4).$$

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_3 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.21. $A_1 \cap A_4 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)x_4 \notin A_4$:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3) \quad \text{(by Lemma 3-2.16)},$$

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) = g_2(x_1,x_3,x_2)x_1x_3 = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$

$$g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) = x_1x_2x_3,$$

$$g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) = x_1x_2x_3,$$

$$g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) = x_1x_2x_3,$$

$$g_3(x_1,x_2,x_3)g_3(x_1,x_2,x_4) = g_3(x_1,x_2,x_3),$$

$$g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4) = g_3(x_1,x_3,x_2)x_1x_3 = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$

$$g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3) = x_1x_2x_3,$$

$$g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1) = x_1x_2x_3,$$

$$g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_3(x_3,x_4,x_1)g_3(x_3,x_4,x_2) = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.22. $A_2 \cap A_3 = \varnothing$.

PROOF. (2) $g_2(x_1,x_2,x_3)x_4 \notin A_3$:
Let $x_4 = x_3$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_2(x_1,x_2,x_3)x_3 = x_1x_2x_3,$$

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = g_1(x_1,x_3,x_2)x_1x_3 = g_1(x_1,x_3,x_2),$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_1,x_3,x_2),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_2,x_3,x_1),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_2,x_3,x_1).$$

(2) By lemmas 3-2.1 and 3-2.2, $A_2 \cap A_3 = \varnothing$ by applying permutations in S_4 to (1). ☐

LEMMA 3-2.23. $A_2 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_2$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3, \quad \text{(by Lemma 3-2.16)},$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3, \quad \text{(by Lemma 3-2.16)},$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

Therefore, $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_2$.

(2) By lemmas 3-2.1 and 3-2.2, $A_2 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). ☐

LEMMA 3-2.24. $A_3 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_3$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = g_1(x_1, x_3, x_2) \quad \text{(by Lemma 3-2.16)},$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_1,x_3,x_2),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_2,x_3,x_1),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_2,x_3,x_1),$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3.$$

Therefore, $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) \notin A_3$.

(2) By lemmas 3-2.1 and 3-2.2, $A_3 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.25. *The polynomials in A_1 are pairwise distinct.*

PROOF. The proof is similar to that of Lemma 3-2.12 by using $g_1x = g_1$ and $g_1(x,y,y) = xy$. □

LEMMA 3-2.26. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1,x_2,x_3)x_4$ is different from others:

If $g_2(x_1,x_2,x_3)x_4 = g_2(x_2,x_3,x_4)x_1$, then

$$\begin{aligned} g_2(x_1,x_2,x_3)x_4 &= g_2(x_2,x_3,x_4)x_1x_4 \\ &= x_1x_2x_3x_4, \end{aligned}$$

which is in A_0, a contradiction.

Similarly, $g_2(x_1,x_2,x_3)x_4$ is not identical to any of the following:

$$g_2(x_3,x_4,x_1)x_2, \quad g_2(x_4,x_1,x_2)x_3,$$

$$g_3(x_2,x_3,x_4)x_1, \quad g_3(x_3,x_4,x_1)x_2,$$

$$g_3(x_4,x_1,x_2)x_3.$$

Let $x_4 = g_2(x_1,x_2,x_3)$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_2(x_1,x_2,x_3),$$

$$g_3(x_1,x_2,x_3)x_4 = x_1x_2x_3.$$

Therefore, $g_2(x_1, x_2, x_3)x_4$ is different from others.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_2 are pairwise distinct by applying permutations in S_4 to (1). □

LEMMA 3-2.27. *The polynomials in A_3 are pairwise distinct.*

PROOF. It is similar to the proof of Lemma 3-2.14 by $g_1 x = g_1$. □

LEMMA 3-2.28. *The polynomials in A_4 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4)$ is different from others:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_2(x_1, x_3, x_2)g_2(x_1, x_3, x_4) = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$

$$g_2(x_1, x_4, x_2)g_2(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)g_2(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_2(x_2, x_4, x_1)g_2(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2) = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)g_3(x_1, x_2, x_4) = g_3(x_1, x_2, x_3),$$

$$g_3(x_1, x_3, x_2)g_3(x_1, x_3, x_4) = x_1x_2x_3 \quad \text{(by Lemma 3-2.16)},$$

$$g_3(x_1, x_4, x_2)g_3(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)g_3(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_3(x_2, x_4, x_1)g_3(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)g_3(x_3, x_4, x_2) = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_4 are pairwise distinct by applying permutations in S_4 to (1). □

From Lemmas 3-2.18 to 3-2.28, we have $p_4 \geqslant 31$ under the assumption of this subsection.

3-2.3. $p_4 \geqslant 31$ for algebras satisfying (N3). In this part, we assume that $p_i's$ satisfy the condition (N3). We will prove that $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct.

LEMMA 3-2.29. *$g_i(x,y,z)x = g_i(x,y,z)y = g_i(x,y,z)z = g_i(x,y,z)$; $g_i(x,y,z)xy = g_i(x,y,z)xz = g_i(x,y,z)yz = g_i(x,y,z)$; $g_i(x,y,z)xyz = g_i(x,y,z)$, for every i=1,2,3.*

PROOF. By the conditions of (N3), use permutation (xyz). □

LEMMA 3-2.30. *$g_2(x,y,z)g_3(x,y,z) = g_1(x,y,z)$.*

PROOF. If $g_2g_3 = xyz$, then $xyz = g_2g_3 = g_2g_3g_3 = xyzg_3 = g_3$ by Lemma 3-2.29, a contradiction. □

It is easy to see that the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (N3) is as in Figure 3.

LEMMA 3-2.31. *$A_0 \cap \bigcup_{i=1}^{4} A_i = \varnothing$.*

LEMMA 3-2.32. *$A_1 \cap A_2 = \varnothing$.*

PROOF. (1) $g_2(x_1,x_2,x_3)x_4 \notin A_1$:
Let $x_4 = x_3$. Then

$$\begin{aligned}
g_2(x_1,x_2,x_3)x_4 &= g_2(x_1,x_2,x_3) \quad \text{(by Lemma 3-2.29)},\\
g_1(x_1,x_2,x_3)x_4 &= g_1(x_1,x_2,x_3) \quad \text{(by Lemma 3-2.29)},\\
g_1(x_2,x_3,x_4)x_1 &= x_1x_2x_3,\\
g_1(x_3,x_4,x_1)x_2 &= x_1x_2x_3,\\
g_1(x_4,x_1,x_2)x_3 &= g_1(x_1,x_2,x_3).
\end{aligned}$$

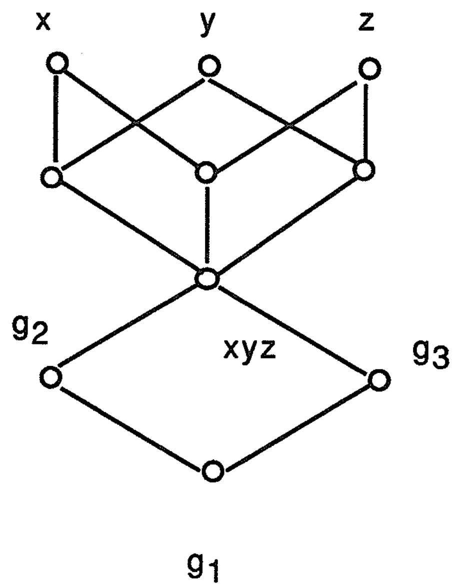

FIGURE 3. N3

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_2 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.33. $A_1 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin A_1$:
Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$
$$g_1(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$
$$g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3.$$

Let $x_4 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$
$$g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3.$$

Let $x_3 = x_2$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_4),$$

$$g_1(x_1,x_2,x_3)x_4 = x_1x_2x_4.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_3 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.34. $A_1 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) \notin A_1$:
Let $x_4 = x_3$. Then

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_1(x_4,x_1,x_2)x_3 = g_1(x_1,x_2,x_3).$$

(2) By lemmas 3-2.1 and 3-2.2, $A_1 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.35. $A_2 \cap A_3 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)x_4 \notin A_3$:
Let $x_4 = x_3$. Then, by lemma 3-2.29,

$$g_2(x_1,x_2,x_3)x_4 = g_2(x_1,x_2,x_3),$$

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_1,x_2,x_3),$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_2 \cap A_3 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.36. $A_2 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) \notin A_2$:

Let $x_3 = x_2$. Then, by Lemma 3-2.29,

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_4),$$

$$g_2(x_1,x_2,x_3)x_4 = x_1x_2x_4,$$

$$g_2(x_2,x_3,x_4)x_1 = x_1x_2x_4,$$

$$g_3(x_1,x_2,x_3)x_4 = x_1x_2x_4,$$

$$g_3(x_2,x_3,x_4)x_1 = x_1x_2x_4.$$

Let $x_4 = x_1$. Then

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_2(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4,x_1,x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4,x_1,x_2)x_3 = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_2 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.37. $A_3 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) \notin A_3$:
Let $x_4 = x_3$. Then, by Lemma 3-2.29,

$$g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) = g_2(x_1,x_2,x_3),$$

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_1,x_2,x_3),$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3.$$

(2) By lemmas 3-2.1 and 3-2.2, $A_3 \cap A_4 = \varnothing$ by applying permutations in S_4 to (1). □

LEMMA 3-2.38. *The polynomials in A_1 are pairwise distinct.*

PROOF. It is similar to the proof of Lemma 3-2.12 by $g_1x = g_1$. □

LEMMA 3-2.39. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1,x_2,x_3)x_4$ is different from others:
Let $x_4 = x_3$. Then, by Lemma 3-2.29,

$$g_2(x_1,x_2,x_3)x_4 = g_2(x_1,x_2,x_3),$$

$$g_2(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_1,x_2,x_3)x_4 = g_3(x_1,x_2,x_3),$$

$$g_3(x_2,x_3,x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3,x_4,x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = g_3(x_1, x_2, x_3).$$

Let $x_4 = x_2$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

Thus, our claim (1) is complete.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_2 are pairwise distinct by applying permutations in S_4 to (1). □

LEMMA 3-2.40. *The polynomials in A_3 are pairwise distinct.*

PROOF. The proof is the same as that of Lemma 3-2.14. □

LEMMA 3-2.41. *The polynomials in A_4 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4)$ is different from others:
Let $x_2 = x_1$. Then, by Lemma 3-2.29, we have

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = x_1x_3x_4,$$

$$g_2(x_1, x_3, x_2)g_2(x_1, x_3, x_4) = g_2(x_1, x_3, x_4),$$

$$g_2(x_1, x_4, x_2)g_2(x_1, x_4, x_3) = g_2(x_1, x_4, x_3),$$

$$g_2(x_2, x_3, x_4)g_2(x_2, x_3, x_1) = g_2(x_1, x_3, x_4),$$

$$g_2(x_2, x_4, x_1)g_2(x_2, x_4, x_3) = g_2(x_1, x_4, x_3),$$

$$g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2) = g_2(x_3, x_4, x_1),$$

$$g_3(x_1, x_3, x_2)g_3(x_1, x_3, x_4) = g_3(x_1, x_3, x_4),$$

$$g_3(x_1, x_4, x_2)g_3(x_1, x_4, x_3) = g_3(x_1, x_4, x_3),$$

$$g_3(x_2, x_3, x_4)g_3(x_2, x_3, x_1) = g_3(x_1, x_3, x_4),$$

$$g_3(x_2, x_4, x_1)g_3(x_2, x_4, x_3) = g_3(x_1, x_4, x_3),$$

$$g_3(x_3, x_4, x_1)g_3(x_3, x_4, x_2) = g_3(x_3, x_4, x_1).$$

Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)g_3(x_1, x_2, x_4) = g_3(x_1, x_2, x_3).$$

So the proof of (1) is complete.

(2) By lemmas 3-2.1 and 3-2.2, the polynomials in A_3 are pairwise distinct by applying permutations in S_4 to (1). □

From Lemmas 3-2.31 to 3-2.41, we have $p_4 \geqslant 31$ under the assumption of this subsection.

3-2.4. $p_4 \geqslant 31$ for algebras satisfying (N4). In this part, we assume that $g_i's$ satisfy the condition (N4). We will prove that the polynomials in $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct.

LEMMA 3-2.42.

$$\begin{aligned} g_i(x,y,z)x &= g_i(x,y,z)y = g_i(x,y,z)z \\ &= g_i(x,y,z)xy = g_i(x,y,z)xz = g_i(x,y,z)yz \\ &= g_i(x,y,z)xyz = xyz. \end{aligned}$$

PROOF. The proof is similar to Lemma 3-2.29. □

It is easy to see that the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (N4) is as in Figure 4.

LEMMA 3-2.43. $A_0 \cap \bigcup_{i=1}^{4} A_i = \varnothing$.

LEMMA 3-2.44. $A_1 \cap A_2 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)x_4 \notin A_1$:
Let $x_4 = g_2(x_1, x_2, x_3)$. Then, by Lemma 3-2.42 and (N4), we have

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_3.$$

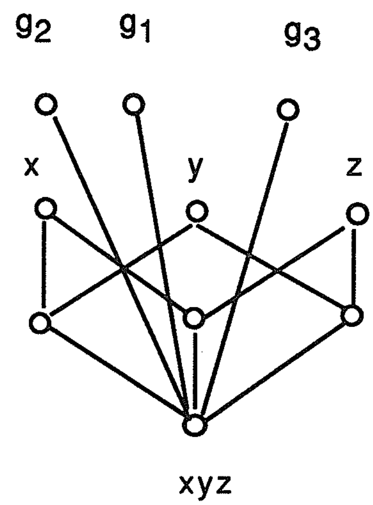

FIGURE 4. N4

Since $x_1x_2x_3 < g_2(x_1, x_2, x_3)$, there exist $a, b, c \in A$ such that

$$e_2 = g_2(a, b, c) > abc = d.$$

Let $(x_1, x_2, x_3) = (a, b, c)$ and $x_4 = e_2$. Then

$$g_2(x_1, x_2, x_3)x_4 = e_2e_2 = e_2,$$

$$g_1(x_2, x_3, x_4)x_1 = g_1(b, c, e_2)a.$$

So they are not equal; otherwise, $e_2 \leqslant a$ therefore

$$e_2 = g_2(a, b, c) = g_2(a, b, c)a = abc$$

by (N4), a contradiction. Similarly,

$$g_2(x_1, x_2, x_3)x_4 \neq g_1(x_3, x_4, x_1)x_2, \quad g_1(x_4, x_1, x_2)x_3.$$

Thus, the proof of (1) is complete.

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_2 is not in A_1 by applying the permutations in S_4 to (1). $\square$

LEMMA 3-2.45. $A_1 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin A_1$:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_3 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.46. $A_1 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_1$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.47. $A_2 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin A_2$:

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_3 is not in A_2 by applying the permutations in S_4 to (1). □

LEMMA 3-2.48. $A_2 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_2$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_2 by applying the permutations in S_4 to (1). □

LEMMA 3-2.49. $A_3 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_3$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = x_1x_2x_3,$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_3 by applying the permutations in S_4 to (1). □

LEMMA 3-2.50. *The polynomials in A_1 are pairwise distinct.*

PROOF. Since $g_1(x_1, x_2, x_3) > x_1x_2x_3$, there exist $a, b, c \in A$ such that

$$e_1 = g_1(a, b, c) > abc = d.$$

Let $(x_1, x_2, x_3) = (a, b, c)$ and $x_4 = e_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = e_1e_1 = e_1,$$

$$g_1(x_2, x_3, x_4)x_1 = g_1(b, c, e_1)a,$$

$$g_1(x_3, x_4, x_1)x_2 = g_1(c, e_1, a)b,$$

$$g_1(x_4, x_1, x_2)x_3 = g_1(e_1, a, b)c.$$

As in the proof of Lemma 3-2.44, $g_1(x_1, x_2, x_3)x_4$ is different from the others. And then, by Lemma 3-2.1 and Lemma 3-2.2, the proof of this lemma is complete by applying the permutations in S_4. □

LEMMA 3-2.51. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)x_4$ is different from the others:

Similar to the proof of Lemma 3-2.44, $g_2(x_1, x_2, x_3)x_4$ is different from any of the following polynomials

$$g_2(x_2, x_3, x_4)x_1, \quad g_2(x_3, x_4, x_1)x_2, \quad g_2(x_4, x_1, x_2)x_3,$$

$$g_3(x_2, x_3, x_4)x_1, \quad g_3(x_3, x_4, x_1)x_2, \quad g_3(x_4, x_1, x_2)x_3.$$

Let $x_4 = g_2(x_1, x_2, x_3)$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_2 are pairwise distinct. □

LEMMA 3-2.52. *The polynomials in A_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ is different from the others:

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = x_1x_2x_3,$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_3 are pairwise distinct.

□

LEMMA 3-2.53. *The polynomials in A_4 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4)$ is different from the others:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_2(x_1, x_3, x_2)g_2(x_1, x_3, x_4) = x_1x_2x_3,$$

$$g_2(x_1, x_4, x_2)g_2(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)g_2(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_2(x_2, x_4, x_1)g_2(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)g_2(x_3, x_4, x_2) = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)g_3(x_1, x_2, x_4) = g_3(x_1, x_2, x_3),$$

$$g_3(x_1, x_3, x_2)g_3(x_1, x_3, x_4) = x_1x_2x_3,$$

$$g_3(x_1, x_4, x_2)g_3(x_1, x_4, x_3) = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)g_3(x_2, x_3, x_1) = x_1x_2x_3,$$

$$g_3(x_2, x_4, x_1)g_3(x_2, x_4, x_3) = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)g_3(x_3, x_4, x_2) = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_4 are pairwise distinct.

□

By Lemmas from 3-2.43 to 3-2.53, $p_4 \geqslant 31$ under the conditions of (N4).

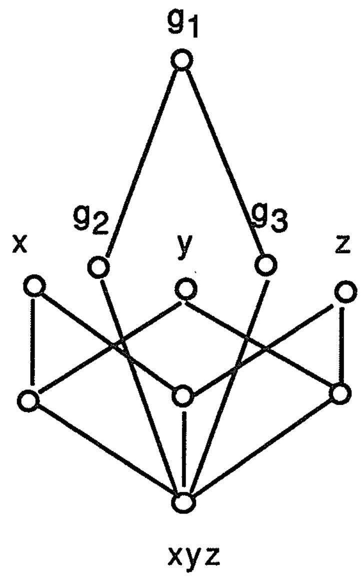

FIGURE 5. N5

3-2.5. $p_4 \geqslant 31$ **for algebras satisfying (N5).** In this part, we assume that $g_i's$ satisfy the condition (N5). We will prove that the polynomials in $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct. It is easy to see that

$$\begin{aligned} g_i(x,y,z)x = g_i(x,y,z)y &= g_i(x,y,z)z \\ &= g_i(x,y,z)xy = g_i(x,y,z)xz \\ &= g_i(x,y,z)yz = g_i(x,y,z)xyz \\ &= xyz. \end{aligned}$$

It is easy to see that the semilattice structure of algebra generated by three elements representing $\langle 0,1,1,4 \rangle$ and satisfying (N5) is as in Figure 5.

LEMMA 3-2.54. $A_0 \cap \bigcup_{i=1}^{4} A_i = \varnothing$.

LEMMA 3-2.55. $A_1 \cap A_2 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)x_4 \notin A_1$:

Let $x_4 = g_1(x_1, x_2, x_3)$. Then, by (N5), we have

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3).$$

Since $x_1x_2x_3 < g_2(x_1, x_2, x_3)$, there exist $a, b, c \in A$ such that

$$e_2 = g_2(a, b, c) > abc = d.$$

Let $(x_1, x_2, x_3) = (a, b, c)$ and $x_4 = e_2$. Then

$$g_2(x_1, x_2, x_3)x_4 = e_2e_2 = e_2,$$

$$g_1(x_2, x_3, x_4)x_1 = g_1(b, c, e_2)a.$$

So they are not equal; otherwise, $e_2 \leqslant a$ therefore

$$e_2 = g_2(a, b, c) = g_2(a, b, c)a = abc$$

by (N5), a contradiction. Similarly,

$$g_2(x_1, x_2, x_3)x_4 \neq g_1(x_3, x_4, x_1)x_2, \quad g_1(x_4, x_1, x_2)x_3.$$

Thus, the proof of (1) is complete.

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_2 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.56. $A_1 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin A_1$:

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_3 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.57. $A_1 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_1$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_1(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_1(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.58. $A_2 \cap A_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin A_2$:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_3 is not in A_2 by applying the permutations in S_4 to (1). □

LEMMA 3-2.59. $A_2 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_2$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_2(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_2(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_2(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_2(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$$g_3(x_1, x_2, x_3)x_4 = x_1x_2x_3,$$

$$g_3(x_2, x_3, x_4)x_1 = x_1x_2x_3,$$

$$g_3(x_3, x_4, x_1)x_2 = x_1x_2x_3,$$

$$g_3(x_4, x_1, x_2)x_3 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_2 by applying the permutations in S_4 to (1). □

LEMMA 3-2.60. $A_3 \cap A_4 = \varnothing$.

PROOF. (1) $g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) \notin A_3$:
Let $x_4 = x_3$. Then

$$g_2(x_1, x_2, x_3)g_2(x_1, x_2, x_4) = g_2(x_1, x_2, x_3),$$

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_1x_2x_3,$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_1x_2x_3,$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_2x_3,$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_4 is not in A_3 by applying the permutations in S_4 to (1). □

LEMMA 3-2.61. *The polynomials in A_1 are pairwise distinct.*

PROOF. Since $g_1(x_1,x_2,x_3) > x_1x_2x_3$, there exist $a,b,c \in A$ such that

$$e_1 = g_1(a,b,c) > abc = d.$$

Let $(x_1,x_2,x_3) = (a,b,c)$ and $x_4 = e_1$. Then

$$g_1(x_1,x_2,x_3)x_4 = e_1e_1 = e_1,$$

$$g_1(x_2,x_3,x_4)x_1 = g_1(b,c,e_1)a,$$

$$g_1(x_3,x_4,x_1)x_2 = g_1(c,e_1,a)b,$$

$$g_1(x_4,x_1,x_2)x_3 = g_1(e_1,a,b)c.$$

As in the proof of Lemma 3-2.55, $g_1(x_1,x_2,x_3)x_4$ is different from the others. And then, by Lemma 3-2.1 and Lemma 3-2.2, the proof of this lemma is complete by applying the permutations in S_4. □

LEMMA 3-2.62. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1,x_2,x_3)x_4$ is different from the others:

Similarly to the proof of Lemma 3-2.55, $g_2(x_1,x_2,x_3)x_4$ is different from any of the following polynomials

$$g_2(x_2,x_3,x_4)x_1,\quad g_2(x_3,x_4,x_1)x_2,\quad g_2(x_4,x_1,x_2)x_3,$$
$$g_3(x_2,x_3,x_4)x_1,\quad g_3(x_3,x_4,x_1)x_2,\quad g_3(x_4,x_1,x_2)x_3.$$

Let $x_4 = g_2(x_1,x_2,x_3)$. Then

$$g_2(x_1,x_2,x_3)x_4 = g_2(x_1,x_2,x_3),$$
$$g_3(x_1,x_2,x_3)x_4 = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_2 are pairwise distinct.

□

LEMMA 3-2.63. *The polynomials in A_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4)$ is different from the others:
Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$
$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_1x_2x_3,$$
$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_1x_2x_3,$$
$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_2x_3,$$
$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_2x_3,$$
$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3.$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_3 are pairwise distinct.

□

LEMMA 3-2.64. *The polynomials in A_4 are pairwise distinct.*

PROOF. (1) (1) $g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4)$ is different from the others:

Let $x_4 = x_3$. Then

$$
\begin{aligned}
g_2(x_1,x_2,x_3)g_2(x_1,x_2,x_4) &= g_2(x_1,x_2,x_3),\\
g_2(x_1,x_3,x_2)g_2(x_1,x_3,x_4) &= x_1x_2x_3,\\
g_2(x_1,x_4,x_2)g_2(x_1,x_4,x_3) &= x_1x_2x_3,\\
g_2(x_2,x_3,x_4)g_2(x_2,x_3,x_1) &= x_1x_2x_3,\\
g_2(x_2,x_4,x_1)g_2(x_2,x_4,x_3) &= x_1x_2x_3,\\
g_2(x_3,x_4,x_1)g_2(x_3,x_4,x_2) &= x_1x_2x_3,\\
g_3(x_1,x_2,x_3)g_3(x_1,x_2,x_4) &= g_3(x_1,x_2,x_3),\\
g_3(x_1,x_3,x_2)g_3(x_1,x_3,x_4) &= x_1x_2x_3,\\
g_3(x_1,x_4,x_2)g_3(x_1,x_4,x_3) &= x_1x_2x_3,\\
g_3(x_2,x_3,x_4)g_3(x_2,x_3,x_1) &= x_1x_2x_3,\\
g_3(x_2,x_4,x_1)g_3(x_2,x_4,x_3) &= x_1x_2x_3,\\
g_3(x_3,x_4,x_1)g_3(x_3,x_4,x_2) &= x_1x_2x_3.
\end{aligned}
$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_4 are pairwise distinct. □

By Lemmas from 3-2.54 to 3-2.64, $p_4 \geqslant 31$ under the conditions of (N5).

3-2.6. $p_4 \geqslant 31$ for algebras satisfying (N6). In this part, we assume that $g_i's$ satisfy the condition (N6). We will prove that the polynomials in $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$ are pairwise distinct. However, the proof is the same as in subsection 3-2.4 except for

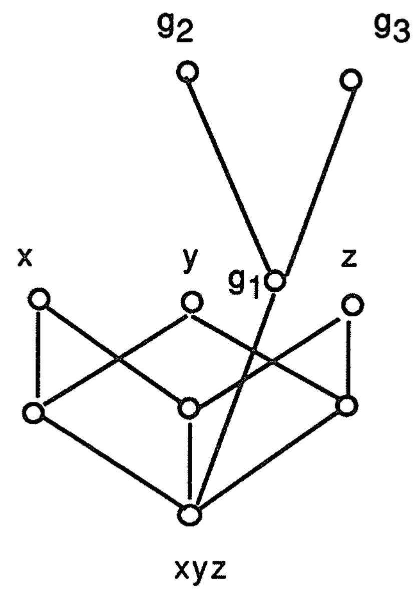

FIGURE 6. N6

Lemma 3-2.44 and Lemma 3-2.50 because we only use the following identities there:

$$\begin{aligned} g_i(x,y,z)x &= g_i(x,y,z)y = g_i(x,y,z)z \\ &= g_i(x,y,z)xy = g_i(x,y,z)xz \\ &= g_i(x,y,z)yz = g_i(x,y,z)xyz \\ &= xyz. \end{aligned}$$

and

$$g_i(x,y,z) > xyz.$$

It is easy to see that the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (N6) is as in Figure 6.

The proofs corresponding to Lemma 3-2.44 and Lemma 3-2.50 are similar to subsection 3-2.4 as the following.

LEMMA 3-2.65. $A_1 \cap A_2 = \varnothing$.

PROOF. (1) $g_2(x_1,x_2,x_3)x_4 \notin A_1$:

Let $x_4 = g_2(x_1, x_2, x_3)$. Then, by (N5), we have

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3).$$

Since $x_1x_2x_3 < g_2(x_1, x_2, x_3)$, there exist $a, b, c \in A$ such that

$$e_2 = g_2(a, b, c) > abc = d.$$

Let $(x_1, x_2, x_3) = (a, b, c)$ and $x_4 = e_2$. Then

$$g_2(x_1, x_2, x_3)x_4 = e_2e_2 = e_2,$$

$$g_1(x_2, x_3, x_4)x_1 = g_1(b, c, e_2)a.$$

So they are not equal; otherwise, $e_2 \leqslant a$ therefore

$$e_2 = g_2(a, b, c) = g_2(a, b, c)a = abc$$

by (N6), a contradiction. Similarly,

$$g_2(x_1, x_2, x_3)x_4 \neq g_1(x_3, x_4, x_1)x_2, \quad g_1(x_4, x_1, x_2)x_3.$$

Thus, the proof of (1) is complete.

(2) By Lemma 3-2.1 and Lemma 3-2.2, every polynomial in A_2 is not in A_1 by applying the permutations in S_4 to (1). □

LEMMA 3-2.66. *The polynomials in A_2 are pairwise distinct.*

PROOF. (1) $g_2(x_1, x_2, x_3)x_4$ is different from the others:

Similarly to the proof of Lemma 3-2.44, $g_2(x_1, x_2, x_3)x_4$ is different from any of the following polynomials

$$g_2(x_2, x_3, x_4)x_1, \quad g_2(x_3, x_4, x_1)x_2, \quad g_2(x_4, x_1, x_2)x_3,$$

$$g_3(x_2, x_3, x_4)x_1, \quad g_3(x_3, x_4, x_1)x_2, \quad g_3(x_4, x_1, x_2)x_3.$$

Let $x_4 = g_2(x_1, x_2, x_3)$. Then

$$g_2(x_1, x_2, x_3)x_4 = g_2(x_1, x_2, x_3),$$

$$g_3(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3).$$

(2) By Lemma 3-2.1 and Lemma 3-2.2, the polynomials in A_2 are pairwise distinct.

□

CHAPTER 4

On Class $\mathbf{S_2}$

In this chapter, we will classify all the algebras representing $\langle 0,1,1,4\rangle$ in $\mathbf{S_2}$ as six subclasses, and then find lower bounds of p_4 for each subclass and the corresponding algebras to some lower bounds. We will see that $p_4 \geqslant 19$ and the sequence $\langle 0,1,1,4,27\rangle$ is related to the join algebras, i.e., ternary reducts of distributive lattices.

4-1. Classification of the class $\mathbf{S_2}$

Suppose that algebra $\mathfrak{A}$ represents $\langle 0,1,1,4\rangle$ and $g_i's$ $(i = 1,2,3)$ are pairwise distinct essentially 4-ary polynomials other than xyz satisfying the condition (s3) and

$$\mathrm{Inv}(g_1) = \langle (23)\rangle, \mathrm{Inv}(g_2) = \langle (13)\rangle, \mathrm{Inv}(g_3) = \langle (12)\rangle.$$

For convenience, we often use g_i to denote $g_i(x,y,z)$ or $g_i(x_1,x_2,x_3)$.

LEMMA 4-1.1.

(1) $$(g_1)^{(12)} = g_2, \quad (g_1)^{(13)} = g_3, \quad (g_1)^{(23)} = g_1;$$

(2) $$(g_2)^{(12)} = g_1, \quad (g_2)^{(13)} = g_2, \quad (g_2)^{(23)} = g_3;$$

(3) $$(g_3)^{(12)} = g_3, \quad (g_3)^{(13)} = g_1, \quad (g_3)^{(23)} = g_2;$$

(4) $$g_1 g_2 g_3 = xyz.$$

PROOF. By (s3), $(g_1)^{(23)} = g_1$. By (s3),

$$g_1(z,y,x) = \begin{cases} g_2(x,y,z), \\ g_3(x,y,z). \end{cases}$$

If $g_1(z,y,x) = g_2(x,y,z)$, then

$$\begin{aligned} g_2(x,y,z) &= g_1(z,y,x) \quad \text{(by assumption)} \\ &= g_1(z,x,y) \quad \text{(by (s3))} \\ &= g_2(y,x,z) \quad \text{(by assumption)}, \end{aligned}$$

which implies $(12) \in \mathrm{Inv}(g_2)$, a contradiction. So

$$(g_1)^{(13)} = g_3.$$

Similarly, $(g_1)^{(12)} = g_2$.

By (1),

$$\begin{aligned} (g_2)^{(12)} &= (g_1)^{(12)(12)} \\ &= g_1 \end{aligned}$$

and

$$\begin{aligned} (g_2)^{(23)} &= (g_1)^{(23)(12)} \\ &= (g_1)^{(23)(12)(23)} \\ &= (g_1)^{(13)} \\ &= g_3, \end{aligned}$$

which proves (2).

Similarly, we can prove (3).

Since $g_1g_2g_3$ is symmetric and there is no constant polynomial, $g_1g_2g_3$ depends on all variables so $g_1g_2g_3 = xyz$. □

From Lemma above, we have

LEMMA 4-1.2.

$$(g_1g_2)^{(12)} = g_1g_2, \quad (g_1g_3)^{(13)} = g_1g_3, \quad (g_2g_3)^{(23)} = g_2g_3.$$

LEMMA 4-1.3.

$$g_1g_2 = \begin{cases} xy \\ xyz \end{cases} ; \quad g_1g_3 = \begin{cases} xz \\ xyz \end{cases} ; \quad g_2g_3 = \begin{cases} yz \\ xyz \end{cases} .$$

PROOF. Since these three formulas are equivalent, we will prove the first one.

If $g_1g_2 = x$, then

$$x = g_1g_2 = (g_1g_2)^{(12)} = (x)^{(xy)} = y,$$

a contradiction. Similarly, $g_1g_2 \neq y$.

If $g_1g_2 = xz$, then

$$xz = g_1g_2 = (g_1g_2)^{(12)} = (xz)^{(xy)} = yz,$$

a contradiction. Similarly, $g_1g_2 \neq yz$.

If $g_1g_2 = g_1$, then, by Lemma 4-1.1 and Lemma 4-1.2,

$$g_2 = (12)(g_1) = (g_1g_2)^{(12)} = g_1g_2 = g_1,$$

a contradiction. Similarly, $g_1g_2 \neq g_2$.

If $g_1g_2 = g_3$, then

$$g_3 = g_3g_3 = g_1g_2g_3 = xyz$$

by (4) of Lemma 4-1.1. Thus, either $g_1g_2 = xy$ or $g_1g_2 = xyz$. □

By Lemma 4-1.1, we have

$$g_1g_2 = xy \Leftrightarrow g_1g_3 = xz \Leftrightarrow g_2g_3 = yz.$$

and

$$g_1g_2 = xyz \Leftrightarrow g_1g_3 = xyz \Leftrightarrow g_2g_3 = xyz.$$

So the following lemma is immediately obtained.

LEMMA 4-1.4. *Either*

$$g_1g_2 = xy, \quad g_1g_3 = xz, \quad g_2g_3 = yz;$$

or

$$g_1g_2 = xyz, \quad g_1g_3 = xyz, \quad g_2g_3 = xyz.$$

Now let us consider $g_1(x,y,z)x$.

LEMMA 4-1.5.

$$g_1(x,y,z)x = \begin{cases} x \\ g_1 \\ xyz. \end{cases}$$

PROOF. Since

$$g_1(x,y,y) = \begin{cases} x \\ y \\ xy, \end{cases}$$

$$g_1(x,y,y)x = \begin{cases} x \\ xy \end{cases}$$

so $g_1(x,y,z)x$ depends on x. Since $g_1(x,y,z)x$ is symmetric in y and z, it depends on y iff it depends on z. Hence $g_1(x,y,z)x = x$ or $g_1(x,y,z)x$ is essentially ternary. For the latter case,

$$g_1(x,y,z)x = \begin{cases} g_1 \\ xyz \end{cases}$$

since $p_4 = 4$ and it is symmetric in y and z. $\square$

LEMMA 4-1.6. *We have the following*

(1) If $g_1(x,y,z)x = x$, *then* $g_1(x,y,y) = x$, $\quad g_1(x,y,x) = x$.

(2) If $g_1(x,y,z)x = g_1$, *then*

$$g_1(x,y,x) = \begin{cases} x \\ xy, \end{cases} \qquad g_1(x,y,y) = \begin{cases} x \\ xy. \end{cases}$$

(3) If $g_1(x,y,z)x = xyz$, *then*

$$g_1(x,y,x) = \begin{cases} y \\ xy, \end{cases} \qquad g_1(x,y,y) = \begin{cases} y \\ xy. \end{cases}$$

PROOF. First we prove (1). If $g_1(x,y,y) = y$, then $g_1(x,y,y)x = xy$ which contradicts assumption (1) (let $z = y$). If $g_1(x,y,y) = xy$, then $g_1(x,y,y)x = xy$ which also contradicts assumption (1). So $g_1(x,y,y) = x$. Similarly, $g_1(x,y,x) = x$.

Suppose that condition (2) holds. If $g_1(x,y,x) = y$, then $g_1(x,y,x)x = xy$ but by assumption of (2),

$$g_1(x,y,x)x = g_1(x,y,x) = y,$$

a contradiction. Similarly, $g_1(x,y,y) \neq y$.

Finally, assume that condition (3) holds. If $g_1(x,y,x) = x$, then $g_1(x,y,x)x = x$. On the other hand, let $z = x$ in $g_1(x,y,z)x = xyz$. Then $g_1(x,y,x)x = xy$, a contradiction. □

From Lemma 4-1.5, there are three cases.

Firstly, we consider case

(C4-1.1) $$g_1x = x.$$

LEMMA 4-1.7. *If* $g_1(x,y,z)x = x$, *then (1)* $g_1g_2 = xy$; *(2)* $g_1y = xy$.

PROOF. By Lemma 4-1.4, to prove $g_1g_2 = xy$, we only need show that $g_1g_2 = xyz$ is impossible.

By Lemma 4-1.1 and assumption, $g_2(x,y,z)y = y$. If $g_1g_2 = xyz$, then

$$xyz = g_1xg_2y = xy,$$

a contradiction. Thus, proof of Lemma 4-1.7 (1) is complete.

In the following, we will prove Lemma 4-1.7 (2). Since g_1y depends on y (consider $z = x$ and use Lemma 4-1.6 (1)),

$$g_1y = \begin{cases} g_1 \\ g_2 \\ g_3 \\ y \\ xy \\ yz \\ xyz. \end{cases}$$

If $g_1y = g_1$, then

$$xy = g_1xy = g_1yx = g_1x = x,$$

which is impossible. If $g_1y = g_2$, then

$$g_2 = g_2g_2 = g_2g_1y = xyy = xy$$

by Lemma 4-1.7 (1), which is impossible. If $g_1y = g_3$, then

$$g_3 = g_3g_3 = g_3g_1y = xzy = xyz$$

since $g_1g_3 = xz$ by Lemma 4-1.7 (1) and Lemma 4-1.2. But this is impossible. If $g_1y = y$, then, by $g_1x = x$ and Lemma 4-1.1 and Lemma 4-1.7 (1),

$$y = g_2y = g_2g_1y = xyy = xy,$$

a contradiction. If $g_1y = yz$, then, setting $y = x$, $g_1(x,x,z)x = xz$. But, from $g_1x = x$, $g_1(x,x,z)x = x$, a contradiction. If $g_1y = xyz$, then, setting $y = x$, $g_1(x,x,z)x = xz$.

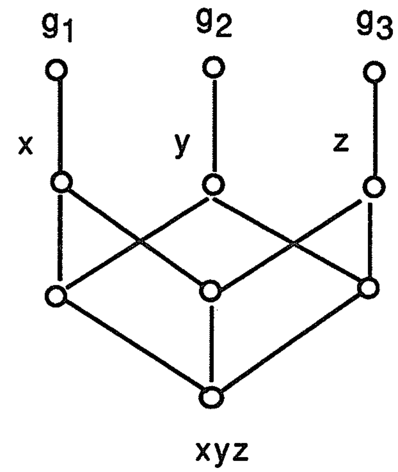

FIGURE 7. T1

But, from $g_1x = x$, $g_1(x,x,z)x = x$, a contradiction. Hence, Lemma 4-1.7 (2) is proved. □

From the above lemmas, among the algebras representing $\langle 0,1,1,4\rangle$, the semilattice structure of the free algebra generated by three elements satisfying $g_1x = x$ is as in Figure 7. Under condition $g_1x = x$, we have the following formulas

$$\text{(T1)}\qquad \begin{aligned} &g_1x = x, && g_1y = xy, && g_1z = xz,\\ &g_2x = xy, && g_2y = y, && g_2z = yz,\\ &g_3x = xz, && g_3y = yz, && g_3z = z,\\ &g_1g_2 = xy, && g_1g_3 = xz, && g_2g_3 = yz,\\ &g_1g_2g_3 = xyz. \end{aligned}$$

Secondly, consider the case

$$\textbf{(C4-1.2)}\qquad g_1x = g_1,$$

which implies $g_2y = g_2$ and $g_3z = g_3$. We have two subcases from Lemma 4-1.4

$$\textbf{(C4-1.2.1)}\qquad g_1g_2 = xy,$$

which is equivalent to either of $g_1g_3 = xz$ and $g_2g_3 = yz$;

(C4-1.2.2) $$g_1g_2 = xyz,$$

which is equivalent to either of $g_1g_3 = xyz$ and $g_2g_3 = xyz$.

LEMMA 4-1.8. *Let $g_1x = g_1$. If $g_1g_2 = xy$, then*

$$g_1y = xy, \quad g_1z = xz.$$

PROOF.

$$g_1y = g_1xy = g_1g_1g_2 = g_1g_2 = xy.$$

Applying permutation (yz) to this formula, $g_1z = xz$. □

Thus, under conditions (s3) and $g_1x = g_1$ and (**C4-1.2.1**), the semilattice structure of the free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ is as in Figure 8. For this case, we have

$$\text{(T2)} \qquad \begin{array}{lll} g_1x = g_1, & g_1y = xy, & g_1z = xz, \\ g_2x = xy, & g_2y = g_2, & g_2z = yz, \\ g_3x = xz, & g_3y = yz, & g_3z = g_3, \\ g_1g_2 = xy, & g_1g_3 = xz, & g_2g_3 = yz, \\ g_1g_2g_3 = xyz. & & \end{array}$$

LEMMA 4-1.9. *Let $g_1x = g_1$. If $g_1g_2 = xyz$, then*

$$g_1y = xyz, \quad g_1z = xyz.$$

PROOF. Since $g_1y = xyz$ and $g_1z = xyz$ are equivalent, we only need prove $g_1y = xyz$. Since g_1y depends on x and y (set $z = y$ and $z = x$, resp., and apply

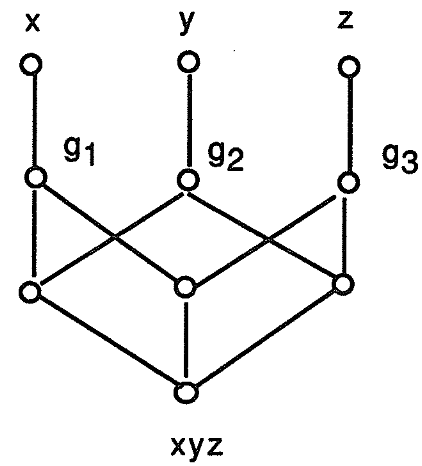

FIGURE 8. T2

Lemma 4-1.6 (2)),

$$g_1 y = \begin{cases} g_1 \\ g_2 \\ g_3 \\ xy \\ xyz. \end{cases}$$

If $g_1 y = g_2$, then $g_2 = g_2 g_2 = g_2 g_1 y = xyzy = xyz$, a contradiction.

If $g_1 y = g_3$, then $g_3 = g_3 g_3 = g_3 g_1 y = xyz$, a contradiction.

If $g_1 y = g_1$ i.e., $g_1 z = g_1$, then

$$g_1 = g_1 x = g_1 yx = g_1 zyx = g_1 g_1 g_2 g_3 = xyz,$$

a contradiction.

If $g_1 y = xy$, then $g_2 x = xy$. Notice that $g_2 y = g_2$ from $g_1 x = g_1$. So

$$xyz = g_1 g_2 = g_1 x g_2 y = g_1 y g_2 x = xy,$$

a contradiction. Therefore $g_1 y = xyz$. $\square$

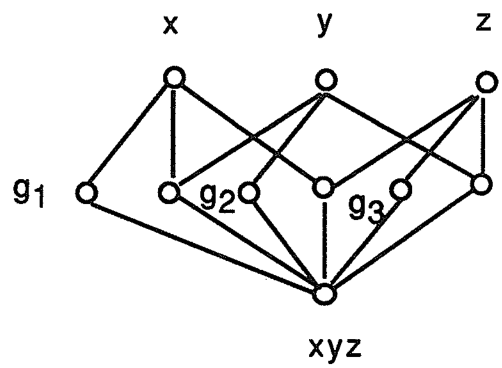

FIGURE 9. T3

Thus, under conditions (s3) and $g_1x = g_1$ and **(C4-1.2.2**) , the semilattice structure of the free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ is as in Figure 9. For this case, we have

$$\text{(T3)}\qquad \begin{aligned} &g_1x = g_1, && g_1y = xyz, && g_1z = xyz,\\ &g_2x = xyz, && g_2y = g_2, && g_2z = xyz,\\ &g_3x = xyz, && g_3y = xyz, && g_3z = g_3,\\ &g_1g_2 = xyz, && g_1g_3 = xyz, && g_2g_3 = xyz,\\ &g_1g_2g_3 = xyz. \end{aligned}$$

Thirdly, consider case

(C4-1.3) $$g_1x = xyz,$$

which implies $g_2y = xyz$ and $g_3z = xyz$.

LEMMA 4-1.10. *Let $g_1x = xyz$. Then $g_1g_2 = xyz$, so $g_1g_3 = xyz$ and $g_2g_3 = xyz$.*

PROOF. If $g_1g_2 \neq xyz$, then $g_1g_2 = xy$ by Lemma 4-1.3. So

$$xy = xyx = g_1g_2x = (g_1x)g_2 = xyzg_2 = xyz,$$

a contradiction. □

LEMMA 4-1.11. *Let $g_1x = xyz$. Then*

$$g_1y = \begin{cases} g_1 \\ yz \\ xyz. \end{cases}$$

PROOF. As in the proof of previous lemma, g_1y depends on y (by Lemma 4-1.6 (3), setting $z = x$, $g_1y = y$). So

$$g_1y = \begin{cases} g_1 \\ g_2 \\ g_3 \\ y \\ xy \\ yz \\ xyz. \end{cases}$$

If $g_1y = g_2$, then

$$g_2 = g_2g_2 = g_2g_1y = g_1(g_2y) = g_1(xyz) = xyz,$$

a contradiction. If $g_1y = g_3$, then, by Lemma 4-1.10,

$$g_3 = g_3g_3 = g_3g_1y = (xyz)y = xyz,$$

a contradiction. If $g_1y = y$, then

$$xy = x(g_1y) = (xg_1)y = xyzy = xyz,$$

a contradiction. If $g_1y = xy$, then

$$xy = x(xy) = x(g_1y) = (xg_1)y = xyzy = xyz,$$

a contradiction.

Thus, from the above, our proof is complete. □

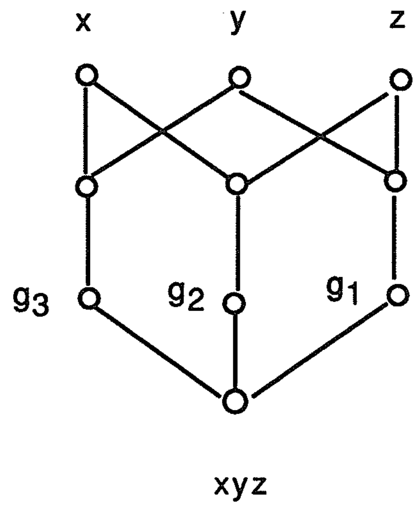

FIGURE 10. T4

By Lemma 4-1.11, for case $g_1x = xyz$, we have three subcases:

$$g_1y = g_1, \tag{C4-1.3.1}$$

$$g_1y = yz, \tag{C4-1.3.2}$$

$$g_1y = xyz. \tag{C4-1.3.3}$$

From Lemma 4-1.1, Lemma 4-1.2 and Lemma 4-1.10, we have the following lemmas immediately.

LEMMA 4-1.12. *Let $g_1x = xyz$ and $g_1y = g_1$. Then*

$$\begin{array}{lll} g_1x = xyz, & g_1y = g_1, & g_1z = g_1; \\ g_2x = g_2, & g_2y = xyz, & g_2z = g_2; \\ g_3x = g_3, & g_3y = g_3, & g_3z = xyz; \\ g_1yz = g_1, & g_2xz = g_2, & g_3xy = g_3; \\ g_1g_2 = xyz, & g_1g_3 = xyz, & g_2g_3 = xyz. \end{array} \tag{T4}$$

*Thus, the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (**C4-1.3**) and (**C4-1.3.1**) is as in Figure 10.*

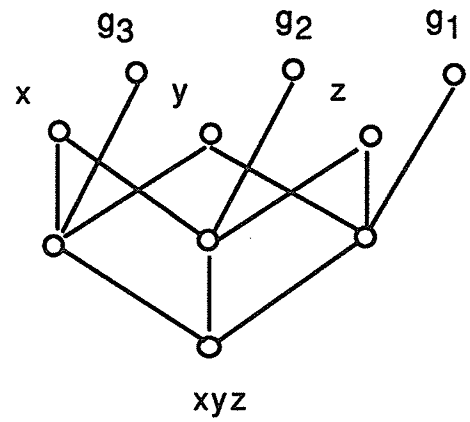

FIGURE 11. T5

LEMMA 4-1.13. *Let $g_1x = xyz$ and $g_1y = yz$. Then*

$$\begin{aligned} &g_1x = xyz, \quad &g_1y = yz, \quad &g_1z = yz;\\ &g_2x = xz, \quad &g_2y = xyz, \quad &g_2z = xz;\\ &g_3x = xy, \quad &g_3y = xy, \quad &g_3z = xyz;\\ &g_1g_2 = xyz, \quad &g_1g_3 = xyz, \quad &g_2g_3 = xyz. \end{aligned} \tag{T5}$$

*So the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (**C4-1.3**) and (**C4-1.3.2**) is as in Figure 11.*

LEMMA 4-1.14. *Let $g_1x = xyz$ and $g_1y = xyz$. Then*

$$\begin{aligned} &g_1x = xyz, \quad &g_1y = xyz, \quad &g_1z = xyz;\\ &g_2x = xyz, \quad &g_2y = xyz, \quad &g_2z = xyz;\\ &g_3x = xyz, \quad &g_3y = xyz, \quad &g_3z = xyz;\\ &g_1g_2 = xyz, \quad &g_1g_3 = xyz, \quad &g_2g_3 = xyz. \end{aligned} \tag{T6}$$

*So the semilattice structure of free algebra generated by three elements representing $\langle 0,1,1,4\rangle$ and satisfying (**C4-1.3**) and (**C4-1.3.3**) is as in Figure 12.*

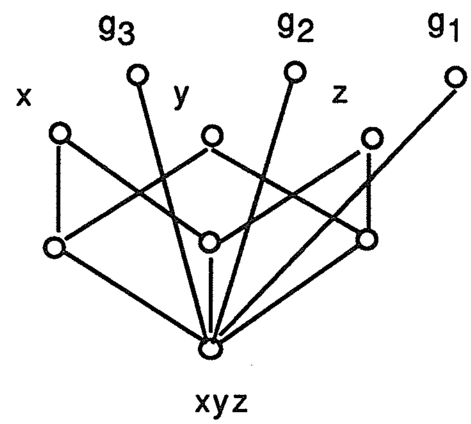

FIGURE 12. T6

For convenience, we call the class of algebras representing $\langle 0,1,1,4\rangle$ and satisfying (Ti) *the class* $\mathbf{T}_i$.

In the following section, we consider the value of $p_4(\mathfrak{A})$. Similarly to the previous chapter, the following sets of polynomials are very useful.

$$(B_0) \qquad x_1x_2x_3x_4;$$

$$(B_1) \qquad \begin{array}{lll} g_1(x_1,x_2,x_3)x_4, & g_1(x_2,x_3,x_1)x_4, & g_1(x_3,x_1,x_2)x_4, \\ g_1(x_2,x_3,x_4)x_1, & g_1(x_3,x_4,x_2)x_1, & g_1(x_4,x_2,x_3)x_1, \\ g_1(x_3,x_4,x_1)x_2, & g_1(x_4,x_1,x_3)x_2, & g_1(x_1,x_3,x_4)x_2, \\ g_1(x_4,x_1,x_2)x_3, & g_1(x_1,x_2,x_4)x_3, & g_1(x_2,x_4,x_1)x_3. \end{array}$$

$$
(B_2) \qquad
\begin{array}{ll}
g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4), & g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4),\\
g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3), & g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1),\\
g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3), & g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4),\\
g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4), & g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4),\\
g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2), & g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3),\\
g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1), & g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2).
\end{array}
$$

$$
(B_3) \qquad
\begin{array}{ll}
g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4), & g_1(x_2,x_4,x_1)g_1(x_3,x_4,x_1),\\
g_1(x_1,x_2,x_4)g_1(x_3,x_2,x_4), & g_1(x_1,x_2,x_3)g_1(x_4,x_2,x_3),\\
g_1(x_2,x_3,x_1)g_1(x_4,x_3,x_1), & g_1(x_3,x_1,x_2)g_1(x_4,x_1,x_2),
\end{array}
$$

$$
(B_4) \qquad
\begin{array}{lll}
g_1(x_1x_2,x_3,x_4), & g_1(x_1x_3,x_2,x_4), & g_1(x_1x_4,x_2,x_3),\\
g_1(x_2x_3,x_1,x_4), & g_1(x_2x_4,x_1,x_3), & g_1(x_3x_4,x_1,x_2).
\end{array}
$$

$$
(B_5) \qquad
\begin{array}{lll}
g_1(x_1,x_2x_3,x_4), & g_1(x_1,x_2x_4,x_3), & g_1(x_1,x_3x_4,x_2),\\
g_1(x_2,x_1x_3,x_4), & g_1(x_2,x_1x_4,x_3), & g_1(x_2,x_3x_4,x_1),\\
g_1(x_3,x_1x_2,x_4), & g_1(x_3,x_1x_4,x_2), & g_1(x_3,x_2x_4,x_1),\\
g_1(x_4,x_1x_2,x_3), & g_1(x_4,x_1x_3,x_2), & g_1(x_4,x_2x_3,x_1).
\end{array}
$$

We need the following subset of B_1 for later use:

$$
(B_{11}) \qquad
\begin{array}{lll}
g_1(x_1,x_2,x_3)x_4, & g_1(x_2,x_1,x_3)x_4, & g_1(x_3,x_1,x_2)x_4,\\
g_1(x_2,x_3,x_4)x_1, & g_1(x_3,x_4,x_2)x_1, & g_1(x_3,x_4,x_1)x_2.
\end{array}
$$

Obviously, $B_0 \cap (B_2 \cup B_3 \cup B_4 \cup B_5) = \varnothing)$ by identifying two variables.

THEOREM 4-1.1. *The polynomials in $B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ are essentially 4-ary.*

PROOF. By G.Grätzer ([**28**] pp.44-45), the polynomials in $B_0 \cup B_1 \cup B_4 \cup B_5$ are essentially 4-ary.

To prove the others, we only need prove that $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ and $g_1(x_1, x_3, x_4)g_1(x$ are essentially 4- ary. By similarity of the proof, we only need prove the first one. Firstly, $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ depends on x_3 and x_4; otherwise, by its symmetry in x_3 and x_4, it does not depend on x_3 and x_4. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = \begin{cases} x_1x_2 \\ x_1 \\ x_2. \end{cases}$$

Let $x_4 = x_3$,

$$g_1(x_1, x_2, x_3) = \begin{cases} x_1x_2 \\ x_1 \\ x_2. \end{cases}$$

But the left hand side is essentially ternary, a contradiction. Secondly,

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$$

depends on x_2; otherwise, without loss of generality, suppose that it is essentially ternary. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = \begin{cases} x_1x_3x_4 \\ g_i(x_1, x_3, x_4) \quad \text{for some i.} \end{cases}$$

Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3) = \begin{cases} x_1x_3 \\ g_i(x_1, x_3, x_3). \end{cases}$$

Thus, the left hand side is essentially ternary but the right hand side depends on at most two variables, a contradiction. □

As in the previous chapter, for formal polynomials satisfying (s3), we have the following lemmas.

LEMMA 4-1.15. *Each B_i is invariant under S_4, $i = 1, 2, 3, 4, 5$.*

PROOF. We only need prove that B_i is invariant under each of permutations (12), (13), (14), (23), (24), (34). However, it is routine to check them. □

LEMMA 4-1.16. *For every fixed i, for any $p, q \in B_i$ there is an $\alpha \in S_4$ such that $p = q^\alpha$.*

PROOF. Since every $\alpha \in S_4$ is bijective, we only need to prove that for some fixed p, for every $q \in S_4$ there exists α such that $q = p^\alpha$.

(1) On B_1:

$$g_1(x_2, x_3, x_1)x_4 = (g_1(x_1, x_2, x_3)x_4)^{(12)},$$

$$g_1(x_3, x_1, x_2)x_4 = (g_1(x_1, x_2, x_3)x_4)^{(13)},$$

$$g_1(x_2, x_3, x_4)x_1 = (g_1(x_1, x_2, x_3)x_4)^{(214)},$$

$$g_1(x_3, x_4, x_2)x_1 = (g_1(x_1, x_2, x_3)x_4)^{(23)(214)},$$

$$g_1(x_4, x_2, x_3)x_1 = (g_1(x_1, x_2, x_3)x_4)^{(24)(214)},$$

$$g_1(x_3, x_4, x_1)x_2 = (g_1(x_1, x_2, x_3)x_4)^{(13)(24)},$$

$$g_1(x_4, x_1, x_3)x_2 = (g_1(x_3, x_4, x_1)x_2)^{(34)} = (g_1(x_1, x_2, x_3)x_4)^{(34)(13)(24)},$$

$$g_1(x_1, x_3, x_4)x_2 = (g_1(x_1, x_2, x_3)x_4)^{(24)},$$

$$g_1(x_4, x_1, x_2)x_3 = (g_1(x_1, x_2, x_3)x_4)^{(134)},$$

$$g_1(x_1, x_2, x_4)x_3 = (g_1(x_4, x_1, x_2)x_3)^{(14)} = (g_1(x_1, x_2, x_3)x_4)^{(14)(134)},$$

$$g_1(x_2, x_4, x_1)x_3 = (g_1(x_4, x_1, x_2)x_3)^{(24)} = (g_1(x_1, x_2, x_3)x_4)^{(24)(134)}.$$

(2) On B_2: take $p = g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$. Since we have the following identities, our claim on B_2 is true.

$$g_1(x_2, x_1, x_3)g_1(x_2, x_1, x_4) = (g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4))^{(12)},$$
$$g_1(x_3, x_1, x_2)g_1(x_3, x_1, x_4) = (g_1(x_2, x_1, x_3)g_1(x_2, x_1, x_4))^{(23)},$$
$$g_1(x_3, x_2, x_1)g_1(x_3, x_2, x_4) = (g_1(x_3, x_1, x_2)g_1(x_3, x_1, x_4))^{(12)},$$
$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = (g_1(x_3, x_2, x_1)g_1(x_3, x_2, x_4))^{(24)},$$
$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = (g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3))^{(13)},$$
$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = (g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3))^{(34)},$$
$$g_1(x_4, x_3, x_1)g_1(x_4, x_3, x_2) = (g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4))^{(14)},$$
$$g_1(x_4, x_2, x_3)g_1(x_4, x_2, x_1) = (g_1(x_4, x_3, x_1)g_1(x_4, x_3, x_2))^{(23)},$$
$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = (g_1(x_4, x_2, x_3)g_1(x_4, x_2, x_1))^{(24)},$$
$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = (g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3))^{(34)},$$
$$g_1(x_4, x_1, x_2)g_1(x_4, x_1, x_3) = (g_1(x_3, x_1, x_2)g_1(x_3, x_1, x_4))^{(34)}.$$

(3) On B_3: take $p = g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4)$. And we have

$$g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3) = (g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4))^{(24)},$$
$$g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1) = (g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3))^{(12)},$$
$$g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1) = (g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1))^{(34)},$$
$$g_1(x_1, x_2, x_4)g_1(x_3, x_2, x_4) = (g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1))^{(12)},$$
$$g_1(x_3, x_1, x_2)g_1(x_4, x_1, x_2) = (g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1))^{(23)}.$$

(4) On B_4: take $p = g_1(x_1x_2, x_3, x_4)$.

$$g_1(x_1x_3, x_2, x_4) = (g_1(x_1x_2, x_3, x_4))^{(23)},$$

$$g_1(x_1x_4, x_2, x_3) = (g_1(x_1x_3, x_2, x_4))^{(34)},$$

$$g_1(x_3x_4, x_1, x_2) = (g_1(x_1x_4, x_2, x_3))^{(13)},$$

$$g_1(x_2x_4, x_1, x_3) = (g_1(x_3x_4, x_1, x_2))^{(23)},$$

$$g_1(x_2x_3, x_1, x_4) = (g_1(x_2x_4, x_1, x_3))^{(34)}.$$

(5) On B_5:

$$g_1(x_1, x_2x_4, x_3) = (g_1(x_1, x_2x_3, x_4))^{(34)}, \quad g_1(x_1, x_3x_4, x_2) = (g_1(x_1, x_2x_4, x_3))^{(23)},$$

$$g_1(x_2, x_3x_4, x_1) = (g_1(x_1, x_3x_4, x_2))^{(12)}, \quad g_1(x_3, x_2x_4, x_1) = (g_1(x_2, x_3x_4, x_1))^{(23)},$$

$$g_1(x_4, x_2x_3, x_1) = (g_1(x_3, x_2x_4, x_1))^{(34)}, \quad g_1(x_4, x_1x_3, x_2) = (g_1(x_4, x_2x_3, x_1))^{(12)},$$

$$g_1(x_3, x_1x_4, x_2) = (g_1(x_4, x_1x_3, x_2))^{(34)}, \quad g_1(x_2, x_1x_4, x_3) = (g_1(x_3, x_1x_4, x_2))^{(23)},$$

$$g_1(x_2, x_1x_3, x_4) = (g_1(x_2, x_1x_4, x_3))^{(34)}, \quad g_1(x_3, x_1x_2, x_4) = (g_1(x_2, x_1x_3, x_4))^{(23)},$$

$$g_1(x_4, x_1x_2, x_3) = (g_1(x_3, x_1x_2, x_4))^{(34)}.$$

□

In the following, we will discuss the value of $p_4(\mathfrak{A})$ in six parts, respectively under conditions $(Ti), i = 1, 2, 3, 4, 5, 6$.

4-2. Lower bounds of p_4 and some algebras in the class $\mathbf{S_2}$

In the following subsections, we will find that the greatest lower bound for $p_4(\mathfrak{A})$ is 19 and $\langle 0, 1, 1, 4, 19\rangle$ is representable. There are two varieties $\mathbf{K}_i$ of type $\langle 2, 3\rangle$ and two four-element algebras $Z_i \in \mathbf{K}_i$, $i = 1, 2$, such that both Z_i represent $\langle 0, 1, 1, 4, 19, 251\rangle$ and every algebra $\mathfrak{A} \in \mathbf{K}_i$ representing $\langle 0, 1, 1, 4\rangle$ contains a subalgebra such that Z_i

is a homomorphic image of this subalgebra. The sequence $\langle 0,1,1,4,27\rangle$ is related to the variety of join algebras.

4-2.1. $p_4 \geqslant 31$ under condition (T1). In this subsection, we suppose that (T1) holds. We will prove that $p_4(\mathfrak{A}) \geqslant 31$.

LEMMA 4-2.1. $g_1(x,y,x) = g_1(x,x,y) = x, \quad g_1(x,y,y) = x.$

PROOF. If $g_1(x,y,y) = y$, then $g_1(x,y,y)x = xy$ but $g_1(x,y,y)x = x$ by (T1), a contradiction. If $g_1(x,y,y) = xy$, then $g_1(x,y,y)x = xy$ but $g_1(x,y,y)x = x$ by (T1), a contradiction. So $g_1(x,y,y) = x$.

Similarly to above, $g_1(x,y,x) \neq y$ or xy. So $g_1(x,y,x) = x$. □

In the following, we will show that the polynomials in $B_0 \cup B_1 \cup B_2 \cup B_3$ are pairwise distinct and then $p_4 \geqslant 31$.

LEMMA 4-2.2. $B_0 \cap (B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5) = \varnothing$.

PROOF. It is trivial. □

LEMMA 4-2.3. $B_1 \cap B_2 = \varnothing$.

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin B_1$:

Let $x_4 = x_3$. Then $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3)$ and by (T1) and Lemma 4-2.1, we have

$$
\begin{array}{lll}
g_1(x_1,x_2,x_3)x_4 = x_1x_3, & g_1(x_2,x_3,x_1)x_4 = x_2x_3, & g_1(x_3,x_1,x_2)x_4 = x_3, \\
g_1(x_2,x_3,x_4)x_1 = x_2x_1, & g_1(x_3,x_4,x_2)x_1 = x_3x_1, & g_1(x_4,x_2,x_3)x_1 = x_3x_1, \\
g_1(x_3,x_4,x_1)x_2 = x_3x_2, & g_1(x_4,x_1,x_3)x_2 = x_3x_2, & g_1(x_1,x_3,x_4)x_2 = x_1x_2, \\
g_1(x_4,x_1,x_2)x_3 = x_3, & g_1(x_1,x_2,x_4)x_3 = x_1x_3, & g_1(x_2,x_4,x_1)x_3 = x_2x_3.
\end{array}
$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.4. $B_1 \cap B_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)x_4 \notin B_3$:
Let $x_2 = x_1$. Then $g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = g_1(x_1, x_3, x_4)$ and $g_1(x_1, x_2, x_3)x_4 = x_1x_4$. Similarly, $g_1(x_1, x_2, x_3)x_4$ is not equal to any other in B_3.

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.5. $B_2 \cap B_3 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin B_3$:
Let $x_4 = x_3$. Then $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3)$ and

$$g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = x_1x_2,$$

$$g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1) = g_1(x_2, x_3, x_1)x_3 = x_2x_3,$$

$$g_1(x_1, x_2, x_4)g_1(x_3, x_2, x_4) = g_1(x_1, x_2, x_3)x_3 = x_1x_3,$$

$$g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3) = g_1(x_1, x_2, x_3)x_3 = x_1x_3,$$

$$g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1) = g_1(x_2, x_3, x_1)x_3 = x_2x_3,$$

$$g_1(x_3, x_1, x_2)g_1(x_4, x_1, x_2) = g_1(x_3, x_1, x_2).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.6. *The polynomials in B_1 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2, x_3)x_4$ is different from others:
Let $x_4 = x_3$. $g_1(x_1, x_2, x_3)x_4 = x_1x_3$, and by (T1) and Lemma 4-2.1

$$g_1(x_2, x_3, x_1)x_4 = x_2x_3, \quad g_1(x_3, x_1, x_2)x_4 = x_3, \quad g_1(x_2, x_3, x_4)x_1 = x_2x_1,$$

$$g_1(x_3, x_4, x_2)x_1 = x_3x_1, \quad g_1(x_4, x_2, x_3)x_1 = x_3x_1, \quad g_1(x_3, x_4, x_1)x_2 = x_2x_3,$$

$$g_1(x_4, x_1, x_3)x_2 = x_2x_3, \quad g_1(x_1, x_3, x_4)x_2 = x_1x_2, \quad g_1(x_4, x_1, x_2)x_3 = x_3,$$

$g_1(x_1, x_2, x_4)x_3 = x_1x_3, \quad g_1(x_2, x_4, x_1)x_3 = x_2x_3.$

Let $x_4 = x_1$. Then

$$g_1(x_1, x_2, x_3)x_4 = x_1, \quad g_1(x_3, x_4, x_2)x_1 = x_1x_3,$$

$$g_1(x_4, x_2, x_3)x_1 = x_1, \quad g_1(x_1, x_2, x_4)x_3 = x_1x_3.$$

If $g_1(x_1, x_2, x_3)x_4 = g_1(x_4, x_2, x_3)x_1$, then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_4, x_2, x_3)x_4x_1 = x_4x_1,$$

contradicting the fact that $g_1(x_1, x_2, x_3)x_4$ is essentially 4-ary. (2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.7. *The polynomials in B_2 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ is different from others:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3),$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = g_1(x_1, x_3, x_2)x_1 = x_1,$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = g_1(x_1, x_3, x_2)x_1 = x_1,$$

$$g_1(x_2, x_3, x_4)g_1(x_2, x_3, x_1) = g_1(x_2, x_3, x_1)x_2 = x_2,$$

$$g_1(x_2, x_4, x_1)g_1(x_2, x_4, x_3) = g_1(x_2, x_3, x_1)x_2 = x_2,$$

$$g_1(x_2, x_1, x_3)g_1(x_2, x_1, x_4) = g_1(x_2, x_1, x_3) = g_2(x_1, x_2, x_3),$$

$$g_1(x_3, x_1, x_2)g_1(x_3, x_1, x_4) = g_1(x_3, x_1, x_2)x_3 = x_3,$$

$$g_1(x_3, x_2, x_1)g_1(x_3, x_2, x_4) = g_1(x_3, x_2, x_1)x_3 = x_3,$$

$$g_1(x_3, x_4, x_1)g_1(x_3, x_4, x_2) = x_3,$$

$$g_1(x_4, x_1, x_2)g_1(x_4, x_1, x_3) = g_1(x_3, x_1, x_2)x_3 = x_3,$$

$$g_1(x_4, x_2, x_3)g_1(x_4, x_2, x_1) = g_1(x_3, x_2, x_1)x_3 = x_3,$$

$$g_1(x_4, x_3, x_1)g_1(x_4, x_3, x_2) = x_3.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.8. *The polynomials in B_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4)$ is different from others:
Let $x_2 = x_1$. Then

$$g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = g_1(x_1, x_3, x_4),$$

$$g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1) = g_1(x_3, x_4, x_1)x_1 = x_3x_1,$$

$$g_1(x_1, x_2, x_4)g_1(x_3, x_2, x_4) = g_1(x_3, x_1, x_4)x_1 = x_1x_3,$$

$$g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3) = g_1(x_4, x_1, x_3)x_1 = x_4x_1,$$

$$g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1) = g_1(x_4, x_3, x_1)x_1 = x_4x_1,$$

$$g_1(x_3, x_1, x_2)g_1(x_4, x_1, x_2) = x_3x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

4-2.2. Lower bound of p_4 under condition (T2) and join algebras. In this subsection, we suppose that (T2) holds. We will prove that $p_4(\mathfrak{A}) \geqslant 27$ and $\langle 0, 1, 1, 4, 27\rangle$ is related to join algebras.

LEMMA 4-2.9. $g_1(x, x, y) = g_1(x, y, x) = x$ *and*

$$g_1(x, y, y) = \begin{cases} x \\ xy. \end{cases}$$

PROOF. (1) $g_1(x, x, y) \neq xy$:

If $g_1(x, x, y) = xy$, then $g_1(x, x, z) = xz$ so $g_1(x, x, z)x = xz$. On the other hand, letting $y = x$ in $g_1(x, y, z)y = xy$, $g_1(x, x, z)x = x$, a contradiction.

(2) $g_1(x, x, y) \neq y, \quad g_1(x, y, y) \neq y$:

If $g_1(x,x,y) = y$, then $g_1(x,x,y)x = xy$. On the other hand, setting $z = x$ in $g_1(x,z,y)x = g_1(x,z,y)$,

$$g_1(x,x,y)x = g_1(x,x,y) = y,$$

a contradiction.

If $g_1(x,y,y) = y$, then $g_1(x,y,y)x = xy$. On the other hand, letting $z = y$ in $g_1(x,y,z)x = g_1(x,y,z)$,

$$g_1(x,y,y)x = g_1(x,y,y) = y,$$

a contradiction. □

Thus, we have two cases:

(C4-2.2.1) $$g_1(x,y,y) = x,$$

and

(C4-2.2.2) $$g_1(x,y,y) = xy.$$

4-2.2.1. $p_4 \geqslant 31$ *under condition* (**C4-2.2.1**). We suppose that the condition (**C4-2.2.1**) holds. Then we will prove that the polynomials of $B_0 \cup B_2 \cup B_3 \cup B_5$ are pairwise distinct, so $p_4(\mathfrak{A}) \geqslant 31$.

LEMMA 4-2.10. *If $g_1(x,y,y) = x$, then $g_1(x,y,xy) = x$.*

PROOF. If $g_1(x,y,xy) = y$, then $y = g_1(x,y,xy)y = xy$, a contradiction.

If $g_1(x,y,xy) = xy$, then, replacing y by xy,

$$xy = g_1(x,xy,x(xy)) = g_1(x,xy,xy) = x$$

by (**C4-2.2.1**), a contradiction. □

LEMMA 4-2.11. *If $g_1(x,y,y) = x$, then $g_1(xy,x,y) = xy$.*

PROOF. If $g_1(xy, x, y) = y$, then

$$y = g_1(xy, x, y)y = (xy)y = xy$$

by (T2), a contradiction.

If $g_1(xy, x, y) = x$, then

$$x = g_1(xy, x, y)x = (xy)x = xy,$$

a contradiction. □

LEMMA 4-2.12. *If $g_1(x, y, y) = x$, then $g_1(xy, y, z) = xy$.*

PROOF. If $g_1(xy, y, z) = x$, then, setting $z = y$, $x = g_1(xy, y, y) = xy$, a contradiction.

If $g_1(xy, y, z) = y$, then, setting $z = y$, $y = g_1(xy, y, y) = xy$, a contradiction.

If $g_1(xy, y, z) = z$, then, setting $z = y$, $z = g_1(xy, y, y) = xy$, a contradiction.

If $g_1(xy, y, z) = xz$, then, setting $z = x$, $x = g_1(xy, y, x) = xy$ by Lemma 4-2.11, a contradiction.

If $g_1(xy, y, z) = yz$, then, setting $z = y$, $y = g_1(xy, y, y) = xy$ by **(C4-2.2.1)**, a contradiction.

If $g_1(xy, y, z) = g_1(x, y, z)$, then, setting $z = y$, $xy = x$ by **(C4-2.2.1)**, a contradiction.

If $g_1(xy, y, z) = g_2(x, y, z) = g_1(y, x, z)$, then, setting $z = x$, $xy = y$ by **(C4-2.2.1)** and Lemma 4-2.11, a contradiction.

If $g_1(xy, y, z) = g_3(x, y, z) = g_1(z, y, x)$, then, setting $z = y$, $xy = y$ by **(C4-2.2.1)** and Lemma 4-2.9, a contradiction.

If $g_1(xy, y, z) = xyz$, then, replacing y by xy,

$$xy = g_1(xy, xy, z) = xyz,$$

a contradiction. □

LEMMA 4-2.13. *If $g_1(x,y,y)=x$, then*

$$g_1(x,xy,z)=\begin{cases} x, \\ g_1(x,y,z). \end{cases}$$

PROOF. If $g_1(x,xy,z) = y$, then, Letting $z = x$, $y = x$ by Lemma 4-2.9, a contradiction.

If $g_1(x,xy,z)=z$, then, letting $z=xy$,

$$xy=g_1(x,xy,xy)=x$$

by (**C4-2.2.1**), a contradiction.

If $g_1(x,xy,z)=xy$, then, setting $z=x$,

$$x=g_1(x,xy,x)=xy$$

by Lemma 4-2.9, a contradiction.

If $g_1(x,xy,z)=xz$, then, setting $z=xy$,

$$xy=g_1(x,xy,xy)=x$$

by (**C4-2.2.1**), a contradiction.

If $g_1(x,xy,z)=yz$, then, setting $z=xy$,

$$xy=g_1(x,xy,xy)=x$$

by (**C4-2.2.1**), a contradiction.

If $g_1(x,xy,z)=xyz$, then, setting $z=xy$,

$$xy=g_1(x,xy,xy)=x$$

by (**C4-2.2.1**), a contradiction.

If $g_1(x,xy,z)=g_2(x,y,z)=g_1(y,x,z)$, then, letting $z=x$,

$$x=g_1(x,xy,x)=g_1(y,x,x)=y$$

by (**C4-2.2.1**) and Lemma 4-2.9, a contradiction.

If $g_1(x, xy, z) = g_3(x, y, z) = g_1(z, y, x)$, then, setting $y = x$,

$$x = g_1(x, x, z) = g_1(z, x, x) = z$$

by **(C4-2.2.1)** and Lemma 4-2.9, a contradiction. □

LEMMA 4-2.14. *If* $g_1(x, y, y) = x$, *then* $g_1(x, yz, z) = x$.

PROOF. If $g_1(x, yz, z) = y$, then, letting $z = x$,

$$y = g_1(x, yx, x) = x$$

by Lemma 4-2.9, a contradiction.

If $g_1(x, yz, z) = z$, then, replacing z by yz,

$$yz = g_1(x, yz, yz) = x$$

by **(C4-2.2.1)**, a contradiction.

If $g_1(x, yz, z) = xy$, then, setting $z = x$,

$$xy = g_1(x, yx, x) = x$$

by Lemma 4-2.9, a contradiction.

If $g_1(x, yz, z) = xz$, then, replacing z by yz,

$$xyz = g_1(x, yz, yz) = x$$

by Lemma 4-2.9, a contradiction.

If $g_1(x, yz, z) = yz$, then replacing z by yz,

$$yz = g_1(x, yz, yz) = x$$

by Lemma 4-2.9, a contradiction. Similarly,

$$g_1(x, yz, z) \neq xyz.$$

If $g_1(x, yz, z) = g_2(x, y, z) = g_1(y, x, z)$, then, letting $z = x$,

$$x = g_1(x, yx, x) = g_1(y, x, x) = y$$

by Lemma 4-2.9 and **(C4-2.2.1)**, a contradiction.

If $g_1(x, yz, z) = g_3(x, y, z) = g_1(z, y, x)$, then

$$g_1(z, y, x) = g_1(z, y, x)z = g_1(x, yz, z)z = xz$$

by (T2), a contradiction.

If $g_1(x, yz, z) = g_1(x, y, z)$, then

$$\begin{aligned} g_1(x, y, z) &= g_1(x, yz, z) \\ &= g_1(x, z, yz) \\ &= g_1(x, z(yz), yz) \quad \text{(by assumption)} \\ &= g_1(x, yz, yz) \\ &= x, \qquad\qquad\qquad \text{(by } \mathbf{(C4\text{-}2.2.1)}\text{)} \end{aligned}$$

a contradiction. □

Now we are in a position to prove that the polynomials in $B_0 \cup B_2 \cup B_3 \cup B_5$ are pairwise distinct under condition (**C4-2.2.1**).

LEMMA 4-2.15. $B_0 \cap (B_2 \cup B_3 \cup B_5) = \varnothing$.

LEMMA 4-2.16. $B_2 \cap B_3 = \varnothing$.

PROOF. It is the same as the proof of Lemma 4-2.5 because we only used Lemma 4-2.1 there. □

LEMMA 4-2.17. $B_2 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin B_5$:

Let $x_4 = x_3$. Then $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3)$ and by the Lemmas 4-2.13 and 4-2.14 we have

$$g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2x_3, x_3) = x_1,$$

$$g_1(x_1, x_2x_4, x_3) = g_1(x_1, x_2x_3, x_3) = x_1,$$

$$g_1(x_1, x_3x_4, x_2) = g_1(x_1, x_2, x_3),$$

$$g_1(x_2, x_1x_3, x_4) = g_1(x_2, x_1x_3, x_3) = x_2,$$

$$g_1(x_2, x_1x_4, x_3) = g_1(x_2, x_1x_3, x_3) = x_2,$$

$$g_1(x_2, x_3x_4, x_1) = g_2(x_1, x_2, x_3),$$

$$g_1(x_3, x_1x_2, x_4) = g_1(x_3, x_1x_2, x_3) = x_3,$$

$$g_1(x_3, x_1x_4, x_2) = g_1(x_3, x_1x_3, x_2) = \begin{cases} x_3 \\ g_1(x_3, x_1, x_2) \end{cases} = \begin{cases} x_3 \\ g_3(x_1, x_2, x_3) \end{cases}$$

$$g_1(x_3, x_2x_4, x_1) = g_1(x_3, x_2x_3, x_1) = \begin{cases} x_3 \\ g_3(x_1, x_2, x_3) \end{cases}$$

$$g_1(x_4, x_1x_2, x_3) = g_1(x_3, x_1x_2, x_3) = x_3,$$

$$g_1(x_4, x_1x_3, x_2) = g_1(x_3, x_1x_3, x_2) = \begin{cases} x_3 \\ g_3(x_1, x_2, x_3) \end{cases}$$

$$g_1(x_4, x_2x_3, x_1) = g_1(x_3, x_2x_3, x_1) = \begin{cases} x_3 \\ g_3(x_1, x_2, x_3). \end{cases}$$

Let $x_4 = x_2$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3)x_1 = g_1(x_1, x_2, x_3) \quad \text{(by } \mathbf{C4\text{-}2.2.1}\text{)}$$

$$g_1(x_1, x_3x_4, x_2) = g_1(x_1, x_3x_2, x_2) = x_1. \quad \text{(by Lemma 4-2.14)}$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.18. $B_3 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2x_3, x_4) \notin B_3$:
Let $x_3 = x_2$. Then $g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4)$ and

$$g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = g_1(x_1, x_2, x_4)x_2 = x_1x_2,$$

$$g_1(x_2,x_4,x_1)g_1(x_3,x_4,x_1) = g_1(x_2,x_4,x_1) = g_2(x_1,x_2,x_4),$$

$$g_1(x_1,x_2,x_4)g_1(x_3,x_2,x_4) = g_1(x_1,x_2,x_4)x_2 = x_1x_2,$$

$$g_1(x_1,x_2,x_3)g_1(x_4,x_2,x_3) = x_1x_4,$$

$$g_1(x_2,x_3,x_1)g_1(x_4,x_3,x_1) = g_1(421)x_2 = x_2x_4,$$

$$g_1(x_3,x_1,x_2)g_1(x_4,x_1,x_2) = g_1(412)x_2 = x_2x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.19. *The polynomials in B_2 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4)$ is different from others:
Let $x_4 = x_3$. Then by (**C4-2.2.1**) and (T2) and Lemma 4-2.9

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = g_1(x_1,x_3,x_2)x_1 = g_1(x_1,x_2,x_3),$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = g_1(x_1,x_3,x_2)x_1 = g_1(x_1,x_2,x_3),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = g_1(x_2,x_3,x_1)x_2 = g_2(x_1,x_2,x_3),$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = g_1(x_2,x_3,x_1)x_2 = g_2(x_1,x_2,x_3),$$

$$g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4) = g_2(x_1,x_2,x_3),$$

$$g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4) = g_1(x_3,x_1,x_2)x_3 = g_3(x_1,x_2,x_3),$$

$$g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4) = g_1(x_3,x_2,x_1)x_3 = g_3(x_1,x_2,x_3),$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_3,$$

$$g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3) = g_1(x_3,x_1,x_2)x_3 = g_3(x_1,x_2,x_3),$$

$$g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1) = g_1(x_3,x_2,x_1)x_3 = g_3(x_1,x_2,x_3),$$

$$g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2) = x_3.$$

Let $x_2 = x_1$. Then

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = x_1,$$

$$g_1(x_1, x_3, x_2)g_1(x_1, x_3, x_4) = x_1 g_1(x_1, x_3, x_4) = g_1(x_1, x_3, x_4),$$

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = x_1 g_1(x_1, x_3, x_4) = g_1(x_1, x_3, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.20. *The polynomials in B_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4)$ is different from others:
Let $x_2 = x_1$. Then

$$g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = g_1(x_1, x_3, x_4),$$

$$g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1) = g_1(x_3, x_4, x_1)x_1 = x_3 x_1,$$

$$g_1(x_1, x_2, x_4)g_1(x_3, x_2, x_4) = g_1(x_3, x_1, x_4)x_1 = x_1 x_3,$$

$$g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3) = g_1(x_4, x_1, x_3)x_1 = x_4 x_1,$$

$$g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1) = g_1(x_4, x_3, x_1)x_1 = x_4 x_1,$$

$$g_1(x_3, x_1, x_2)g_1(x_4, x_1, x_2) = x_3 x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.21. *The polynomials in B_5 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2 x_3, x_4)$ is different from others:
Let $x_3 = x_2$. Then, by lemmas 4-2.13 and 4-2.14,

$$g_1(x_1, x_2 x_3, x_4) = g_1(x_1, x_2, x_4),$$

$$g_1(x_1, x_2 x_4, x_3) = g_1(x_1, x_2 x_4, x_2) = x_1,$$

$$g_1(x_1, x_3x_4, x_2) = g_1(x_1, x_2x_4, x_2) = x_1,$$

$$g_1(x_2, x_1x_3, x_4) = g_1(x_2, x_1x_2, x_4) = \begin{cases} x_2 \\ g_2(x_1, x_2, x_4) \end{cases}$$

$$g_1(x_2, x_1x_4, x_3) = g_1(x_2, x_1x_4, x_2) = x_2,$$

$$g_1(x_2, x_3x_4, x_1) = g_1(x_2, x_2x_4, x_1) = \begin{cases} x_2 \\ g_2(x_1, x_2, x_4) \end{cases}$$

$$g_1(x_3, x_1x_2, x_4) = g_1(x_2, x_1x_2, x_4) = \begin{cases} x_2 \\ g_2(x_1, x_2, x_4), \end{cases}$$

$$g_1(x_3, x_1x_4, x_2) = g_1(x_2, x_1x_4, x_2) = x_2,$$

$$g_1(x_3, x_2x_4, x_1) = g_1(x_2, x_2x_4, x_1) = \begin{cases} x_2 \\ g_2(x_1, x_2, x_4), \end{cases}$$

$$g_1(x_4, x_1x_2, x_3) = g_1(x_4, x_1x_2, x_2) = x_4,$$

$$g_1(x_4, x_1x_3, x_2) = g_1(x_4, x_1x_2, x_2) = x_4,$$

$$g_1(x_4, x_2x_3, x_1) = g_3(x_1, x_2, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

From lemmas 4-2.15 to 4-2.21, the polynomials of $B_0 \cup B_2 \cup B_3 \cup B_5$ are pairwise distinct, so $p_4(\mathfrak{A}) \geqslant 31$ under (**C4-2.2.1**).

4-2.2.2. *Condition* (**C4-2.2.2**) *and join algebras.* We suppose that the condition (**C4-2.2.2**) holds. We will see that this case is related to the join algebra, i.e., a ternary reduct of distributive lattice. Recall that *a join algebra* is an algebra $J =$

s	x	y	z
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

TABLE 1. J_2

$\langle J,\ s\rangle$ of type $\langle 3\rangle$ satisfying the following identities [**5**]:

$$s(x,x,y)=x, \tag{J1}$$

$$s(x,y,y)=s(y,x,x), \tag{J2}$$

$$s(x,y,z)=s(x,z,y), \tag{J3}$$

$$w\cdot s(x,y,z)=s(w\cdot x,y,z), \tag{J4}$$

$$w\cdot s(x,y,z)=s(x,w\cdot y,w\cdot z), \tag{J5}$$

where " $\cdot$ " is derived operation given by

$$x\cdot y=s(x,y,y). \tag{D1}$$

In other words, it is an algebra of type $\langle 2,3\rangle$ satisfying (J1)-(J5) and (D1).

Let $\mathfrak{J}$ be the variety of join algebras. Then $\mathfrak{J}$ is generated by two element chain join algebra $J_2=\langle\{0,1\},\ s\rangle$, where s is defined by

$$s(x,y,z)=x\wedge(y\vee z)$$

and $\langle\{0,1\};\vee,\wedge\rangle$ is two-element lattice (ref. [**48**], [**5**] and [**36**]). Notice that J_2 has the operation Table 1.

g_1	x	y	z
d	d	d	d
d	d	d	e
d	d	e	d
d	d	e	e
d	e	d	d
e	e	d	e
e	e	e	d
e	e	e	e

TABLE 2. C_2

Since $g_1 > xyz$, there exist $a, b, c \in A$ such that $d = abc < g_1(a, b, c) = e$. Consider two-element algebra $C_2 = \langle\{d, e\}; g_1, \cdot\rangle$. By Lemma 4-2.9 and the condition (**C4-2.2.2**), C_2 has the operation Table 2. It is the same as that of J_2. Therefore, every algebra which represents $\langle 0, 1, 1, 4\rangle$ and satisfies (T2) and (**C4-2.2.2**) contains a ternary reduct which contains J_2 as a subalgebra. Since $p_4(J_2) = 27$, we have the following lemma.

LEMMA 4-2.22. *Let algebra $\mathfrak{A}$ in the class $\mathbf{T}_2$ and represent $\langle 0, 1, 1, 4\rangle$. Then $p_4(\mathfrak{A}) \geqslant 27$ and $\langle 0, 1, 1, 4\rangle$ has the minimal extension property in the class $\mathbf{T}_{22}$, where $\mathbf{T}_{22}$ is the subclass of $\mathbf{T}_2$ in which algebra satisfies (***C4-2.2.2***).*

THEOREM 4-2.1. *Let algebra $\mathfrak{A}$ of type $\langle 2, 3\rangle$ satisfy (T2). Then $\mathfrak{A}$ represents $\langle 0, 1, 1, 4, 27\rangle$ if and only if $\mathfrak{A}$ is a nontrivial join algebra.*

PROOF. For a nontrivial join algebra $\mathfrak{A}$, it is obvious that $p_0(\mathfrak{A}) = 0$, $p_1(\mathfrak{A}) = p_2(\mathfrak{A}) = 1$ and $p_3(\mathfrak{A}) = 4$. Consider the following 53 4-ary polynomials over $\mathfrak{A}$:

$$(E_0) \qquad x_i, \quad x_i \wedge x_j, \quad x_i \wedge x_j \wedge x_m, \quad x_i \wedge (x_j \vee x_m),$$

$$(E_1) \qquad x_1 \wedge x_2 \wedge x_3 \wedge x_4,$$

$$(E_2) \qquad x_i \wedge x_j \wedge (x_m \vee x_n),$$

$$(E_3) \qquad x_i \wedge (x_j \vee x_m) \wedge (x_j \vee x_n) = x_i \wedge (x_j \vee x_m \vee x_n),$$

$$(E_4) \qquad x_i \wedge [(x_j \wedge x_m) \vee (x_j \wedge x_n) \vee (x_m \wedge x_n)],$$

$$(E_5) \qquad x_i(x_j \vee x_m \vee x_n),$$

where $\{i, j, m, n\} = \{1, 2, 3, 4\}$.

We can prove that the free join algebra with four generators consists of $\bigcup_{k=0}^{5} E_k$ (see Figure 13). Indeed, it is easy to check

(1) the polynomials in $\bigcup_{k=0}^{5} E_k$ are pairwise distinct (since they are pairwise different in C_2);

(2) $\bigcup_{k=0}^{5} E_k$ is closed under $\wedge$;

(3) $\bigcup_{k=0}^{5} E_k$ is closed under $\vee$ for any two polynomials with a common upper bound.

Thus, we have exactly 27 essentially 4-ary polynomials $\bigcup_{k=1}^{5} E_k$.

Conversely, suppose that algebra $\mathfrak{A} = \langle A; \cdot, g_1 \rangle$ of type $\langle 2, 3 \rangle$ satisfies (T2) and represents $\langle 0, 1, 1, 4, 27 \rangle$. Then it is nontrivial. Since $p_4(\mathfrak{A}) = 27$, it is only possibility that g_1 satisfy the condition (s3). Without loss of generality, we can suppose that $\mathrm{Inv}(g_1) = \langle (23) \rangle$. Since it satisfies (T2), $\mathfrak{A}$ contains J_2 as a subalgebra by the statement before Lemma 4-2.22. The 27 essentially 4-ary polynomials correspondent to $\bigcup_{k=1}^{5} E_k$ are as follows:

$$(F_1) \qquad x_1 x_2 x_3 x_4;$$

$$(F_2) \qquad g_1(x_1, x_2, x_3)x_4, \quad g_1(x_2, x_3, x_4)x_1, \quad g_1(x_3, x_4, x_1)x_2, \quad g_1(x_4, x_1, x_2)x_3,$$
$$g_1(x_2, x_1, x_3)x_4, \quad g_1(x_3, x_2, x_4)x_1;$$

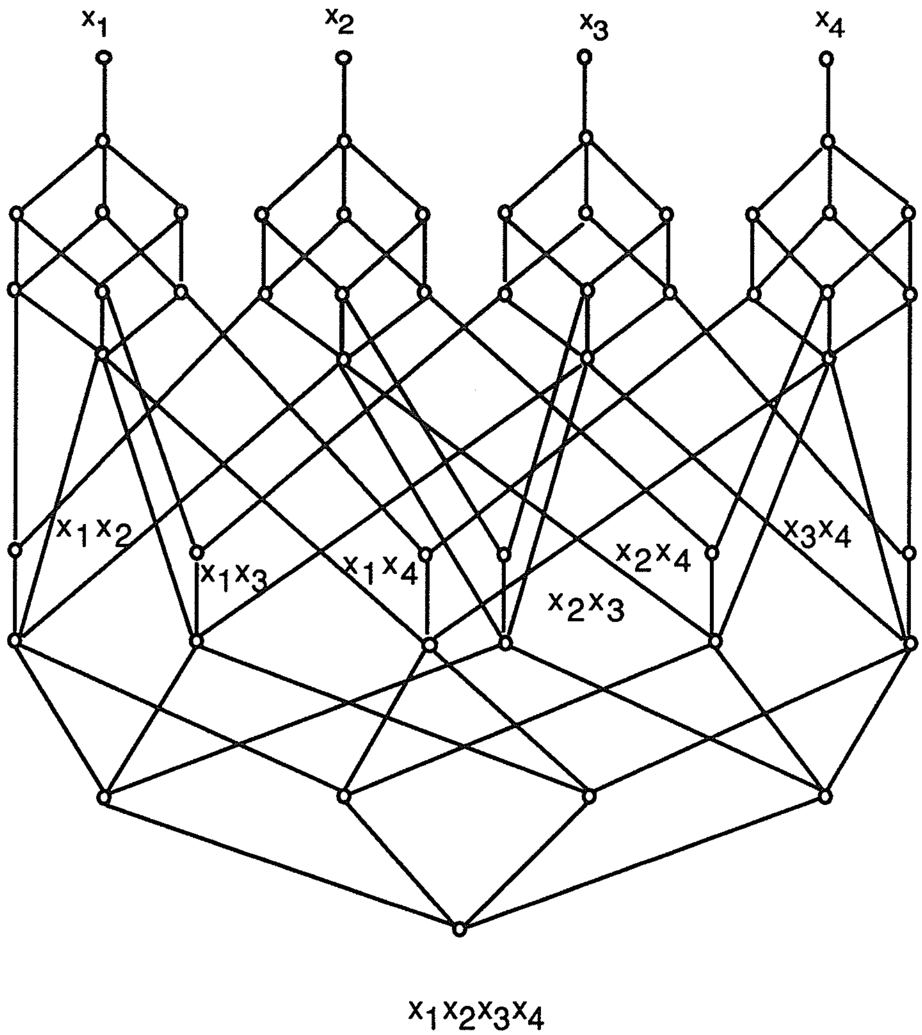

FIGURE 13. The free join algebra with four generators

(F_3)

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4),\quad g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4),\quad g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3),$$

$$g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1),\quad g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3),\quad g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2),$$

$$g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4),\quad g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4),\quad g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3),$$

$$g_1(x_3,x_2,x_4)g_1(x_3,x_2,x_1),\quad g_1(x_4,x_2,x_1)g_1(x_4,x_2,x_3),\quad g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2);$$

$$(F_4) \qquad g_1(x_1,x_2,g_1(x_1,x_3,x_4)), \quad g_1(x_2,x_3,g_1(x_2,x_1,x_4)),$$
$$g_1(x_3,x_4,g_1(x_3,x_1,x_2)), \quad g_1(x_4,x_1,g_1(x_4,x_2,x_3));$$
$$(F_5) \qquad g_1(x_1,x_3x_4,g_1(x_2,x_3,x_4)), \quad g_1(x_2,x_1x_4,g_1(x_3,x_1,x_4)),$$
$$g_1(x_3,x_1x_2,g_1(x_4,x_1,x_2)), \quad g_1(x_4,x_2x_3,g_1(x_1,x_2,x_3)).$$

First, $g_1(x_4x_1,x_2,x_3)$ and $g_1(x_1,x_4x_2,x_4x_3)$ are essentially 4-ary. Indeed, let $x_3 = x_2$. Then

$$g_1(x_1,x_4x_2,x_4x_3) = g_1(x_1,x_4x_2,x_4x_2) = x_1x_2x_4$$

depends on x_1, x_4, x_2 (or x_3). Since $g_1(x_1,x_4x_2,x_4x_3)$ is symmetric in x_2 and x_3, it depends on x_2 and x_3. So it is essentially 4-ary. Similarly, $g_1(x_4x_1,x_2,x_3)$ is essentially 4-ary.

Second, we have

$$g_1(x_1,x_2,x_3)x_4 = g_1(x_4x_1,x_2,x_3), \tag{4}$$
$$g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_4x_2,x_4x_3). \tag{5}$$

Since (4) and (5) hold in J_2 and J_2 is a subalgebra of $\mathfrak{A}$,

$$g_1(x_4x_1,x_2,x_3), \quad g_1(x_1,x_4x_2,x_4x_3) \notin \bigcup_{k=1}^{5} F_k - \{g_1(x_1,x_2,x_3)x_4\}$$

because this is true for J_2. Since the polynomials in $\bigcup_{k=1}^5 F_k$ are pairwise distinct over J_2 and $p_4(\mathfrak{A}) = 27$, the polynomials in $\bigcup_{k=1}^5 F_k$ are pairwise distinct over $\mathfrak{A}$. Therefore, we have identities (4) and (5) over $\mathfrak{A}$. By Lemma 4-2.9 and (T2) and the definition of join algebra, $\mathfrak{A}$ is a join algebra. $\square$

4-2.3. Lower bound under condition (T3) and the varieties $\mathbf{K}_1$ and $\mathbf{K}_2$. In this subsection, suppose that all algebras satisfy (T3) and represent $\langle 0,1,1,4\rangle$. We will prove that $p_4 \geqslant 19$ and $\langle 0,1,1,4,19\rangle$ has the minimal extension property in some class.

LEMMA 4-2.23.

$$g_1(x,y,x) = g_1(x,x,y) = xy, \quad g_1(x,y,y) = \begin{cases} x, \\ xy. \end{cases}$$

PROOF. If $g_1(x,y,x) = x$, then $g_1(x,y,x)x = x$. On the other hand, let $z = x$ in $g_1 z = xyz$. Then $g_1(x,y,x)x = xy$, a contradiction.

If $g_1(x,y,x) = y$, then $g_1(x,x,z) = z$ so $g_1(x,x,z)z = z$. On the other hand, let $y = x$ in $g_1 z = xyz$. Then $g_1(x,x,z)z = xz$, a contradiction.

If $g_1(x,y,y) = y$, then $g_1(x,y,y)y = y$. On the other hand, let $z = y$ in $g_1 z = xyz$. Then $g_1(x,y,y)y = xy$, a contradiction. □

Thus, we have two cases:

(4-2.3.A) $$g_1(x,x,y) = g_1(x,y,x) = xy, \quad g_1(x,y,y) = x;$$

and

(4-2.3.B) $$g_1(x,x,y) = g_1(x,y,x) = xy, \quad g_1(x,y,y) = xy,$$

First, we prove that case (4-2.3.A) is impossible.

LEMMA 4-2.24. *Case (4-2.3.A) is impossible.*

PROOF. Suppose that (4-2.3.A) holds. We will find it impossible. Consider $g_1(x,yz,z)$. Let $z = y$. Then $g_1(x,yz,z) = x$ by (4-2.3.A). Setting $z = x$, $g_1(x,yz,z) = xy$ by Lemma 4-2.23. So $g_1(x,yz,z)$ depends on x and y.

If $g_1(x,yz,z)$ is not essentially ternary, then $g_1(x,yz,z) = xy$. Substituting yz for z,

$$xy = g_1(x,yz,z) = g_1(x,yz,yz) = x$$

by (4-2.3.A), a contradiction.

Assume that $g_1(x,yz,z)$ is essentially ternary. If $g_1(x,yz,z) = xyz$, then, substituting yz for z,

$$xyz = g_1(x,yz,yz) = x$$

by (4-2.3.A), a contradiction. If $g_1(x, yz, z) = g_1(x, y, z)$, then

$$\begin{aligned} g_1(x,y,z) &= g_1(x,z,yz) \\ &= g_1(x, z(yz), yz) && \text{(by assumption)} \\ &= g_1(x, yz, yz) \\ &= x, && \text{(by (4-2.3.A))} \end{aligned}$$

a contradiction. If $g_1(x, yz, z) = g_2(x, y, z)$, then

$$\begin{aligned} g_2(x,y,z) &= g_1(x,yz,z) \\ &= g_1(x,yz,z)x && \text{(by (T3))} \\ &= g_2 x && \text{(by assumption)} \\ &= xyz, && \text{(by (T3))} \end{aligned}$$

a contradiction. If $g_1(x, yz, z) = g_3(x, y, z)$, then

$$\begin{aligned} g_3(x,y,z) &= g_1(x,yz,z) \\ &= g_1(x,yz,z)x && \text{(by (T3))} \\ &= g_3 x && \text{(by assumption)} \\ &= xyz, && \text{(by (T3))} \end{aligned}$$

a contradiction. □

Thus, we have

COROLLARY 4-2.1. *Under (T3), (4-2.3.B) holds.*

LEMMA 4-2.25. *Under (T3),*

$$g_1(x, xy, y) = g_1(x, y, xy) = xy, \quad g_1(xy, x, y) = xy.$$

PROOF. (1) $g_1(xy, x, y) = xy$:
If $g_1(xy, x, y) = x$, then $x = g_1(xy, x, y)x = (xy)xy = xy$ by (T3), a contradiction.
If $g_1(xy, x, y) = y$, then $y = g_1(xy, x, y)y = (xy)xy = xy$ by (T3), a contradiction.

(2) $g_1(x, xy, y) = xy$:

If $g_1(x, xy, y) = x$, then, substituting xy for y,

$$x = g_1(x, x(xy), xy) = g_1(x, xy, xy) = xy$$

by (4-2.3.B), a contradiction. If $g_1(x, xy, y) = y$, then, substituting xy for x,

$$y = g_1(xy, (xy)y, y) = g_1(xy, xy, x) = xy$$

by (4-2.3.B), a contradiction. □

LEMMA 4-2.26.

$$g_1(xy, y, z) = xyz, \quad g_1(x, xy, z) = g_1(x, z, xy) = \begin{cases} xyz \\ g_1(x, y, z). \end{cases}$$

PROOF. By Lemma 4-2.25 and (4-2.3.B), it is easy to check that both $g_1(x, xy, z)$ and $g_1(xy, y, z)$ are essentially ternary.

If $g_1(x, xy, z) = g_2(x, y, z)$, then

$$g_2(x, y, z) = g_1(x, xy, z) = g_1(x, xy, z)x = g_2(x, y, z)x = xyz,$$

a contradiction. If $g_1(x, xy, z) = g_3(x, y, z)$, then

$$g_3(x, y, z) = g_1(x, xy, z) = g_1(x, xy, z)x = g_3(x, y, z)x = xyz,$$

a contradiction. If $g_1(xy, y, z) = g_1(x, y, z)$, then

$$g_1(x, y, z) = g_1(xy, y, z) = g_1(xy, y, z)(xy) = g_1(x, y, z)(xy) = xyz$$

by (T3), a contradiction. If $g_1(xy, y, z) = g_2(x, y, z)$, then

$$g_2(x, y, z) = g_2(x, y, z)y = g_1(xy, y, z)y = (xy)yz = xyz$$

by (T3), a contradiction. If $g_1(xy, y, z) = g_3(x, y, z)$, then

$$g_3(x, y, z) = g_3(x, y, z)z = g_1(xy, y, z)z = (xy)yz = xyz$$

by (T3), a contradiction. □

LEMMA 4-2.27. $g_1(x, yz, z) = xyz$.

PROOF. By Lemma 4-2.25, it is easy to see that $g_1(x, yz, z)$ is essentially ternary. If $g_1(x, yz, z) = g_1(x, y, z)$, then

$$\begin{aligned} g_1(x,y,z) &= g_1(x,yz,z) \\ &= g_1(x,z,yz) \\ &= g_1(x,z(yz),yz) \quad &\text{(by assumption)} \\ &= g_1(x,yz,yz) \\ &= xyz, &\text{(by (4-2.3.B))} \end{aligned}$$

a contradiction. If $g_1(x, yz, z) = g_3(x, y, z)$, then

$$g_3 = g_3 z = g_1(x,yz,z)z = x(yz)z = xyz$$

by (T3), a contradiction. If $g_1(x, yz, z) = g_2(x, y, z) = g_1(y, x, z)$, then

$$\begin{aligned} g_1(y,x,z) &= g_1(x,yz,z) \\ &= g_1(yz,xz,z) \quad &\text{(by assumption)} \\ &= g_1(yz,z,xz) \\ &= g_1(z,(yz)(xz),xz) &\text{(by assumption)} \\ &= g_1(z,xyz,xz) \\ &= g_1(xyz,z(xz),xz) &\text{(by assumption)} \\ &= g_1(xyz,xz,xz) \\ &= xyz, &\text{(by Lemma 4-2.25)} \end{aligned}$$

a contradiction. (or we have

$$\begin{aligned} g_1(y,x,z) &= g_1(x,yz,z) \\ &= g_1(x,yz,z)x \\ &= g_1(y,x,z)x \quad &\text{(by assumption)} \\ &= xyz, &\text{(by (T3)} \end{aligned}$$

a contradiction.) □

In the following, we will consider two cases respectively satisfying the conditions

$$g_1(x_1, x_2x_3, x_4) \neq g_1(x_1, x_2, x_4)g_1(x_1, x_3, x_4) \tag{T3.1}$$

and

$$g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4)g_1(x_1, x_3, x_4). \tag{T3.2}$$

We will prove that $p_4 \geqslant 31$ under (T3.1) and $p_4 \geqslant 19$ under (T3.2).

4-2.3.1. $p_4 \geqslant 31$ *under* (T3.1). **Suppose that (T3.1) is satisfied.** We have the following lemmas 4-2.28 to 4-2.34

LEMMA 4-2.28. $B_0 \cap (B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5) = \varnothing$.

PROOF. It is trivial. □

LEMMA 4-2.29. $B_1 \cap B_2 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) \notin B_1$:
Let $x_4 = x_3$. Then $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3)$ and any one in B_1 is $x_1x_2x_3$ except that $g_1(x_3, x_1, x_2)x_4 = g_1(x_3, x_1, x_2) = g_3(x_1, x_2, x_3)$ and $g_1(x_4, x_1, x_2)x_3 = g_1(x_3, x_1, x_2) = g_3(x_1, x_2, x_3)$.

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.30. $B_1 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2x_3, x_4) \notin B_1$:
Let $x_3 = x_2$. Then $g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4)$ and any one in B_1 is $x_1x_2x_4$ except that

$$g_1(x_3, x_4, x_1)x_2 = g_1(x_2, x_4, x_1) = g_2(x_1, x_2, x_4)$$

and

$$g_1(x_2, x_4, x_1)x_3 = g_1(x_2, x_4, x_1) = g_2(x_1, x_2, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.31. $B_2 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2x_3, x_4) \notin B_2$:

Let $x_3 = x_2$. Then $g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4)$ and any one in B_2 is $x_1x_2x_4$ except that

$$g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3) = g_1(x_1, x_4, x_2) = g_1(x_1, x_2, x_4)$$

and

$$g_1(x_4, x_1, x_2)g_1(x_4, x_1, x_3) = g_1(x_4, x_1, x_2) = g_3(x_1, x_2, x_4).$$

But by assumption (T3.1), $g_1(x_1, x_2x_3, x_4) \neq g_1(x_1, x_4, x_2)g_1(x_1, x_4, x_3)$.

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.32. *The polynomials in B_2 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)$ is different from others:

Let $x_4 = x_3$. $g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4) = g_1(x_1, x_2, x_3)$ and others are equal to $x_1x_2x_3$ except that

$$g_1(x_2, x_1, x_3)g_1(x_2, x_1, x_4) = g_1(x_2, x_1, x_3) = g_2(x_1, x_2, x_3).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.33. *The following polynomials in B_{11} are distinct:*

$$g_1(x_1, x_2, x_3)x_4, \quad g_1(x_2, x_1, x_3)x_4, \quad g_1(x_3, x_1, x_2)x_4,$$

$$g_1(x_2, x_3, x_4)x_1, \quad g_1(x_3, x_4, x_2)x_1, \quad g_1(x_3, x_4, x_1)x_2.$$

PROOF. Let $x_4 = x_1$. $g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3)$ and the others are equal to $x_1x_2x_3$. Similarly, the rest can be easily checked. □

LEMMA 4-2.34. *The polynomials in B_5 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2x_3, x_4)$ is different from the others:
Let $x_3 = x_2$. We have

$$g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4);$$

by Lemma 4-2.27 or Lemma 4-2.25 or (4-2.3.B),

$$g_1(x_1, x_2x_4, x_3) = x_1x_2x_4,$$

$$g_1(x_1, x_3x_4, x_2) = x_1x_2x_4,$$

$$g_1(x_2, x_1x_4, x_3) = x_1x_2x_4,$$

$$g_1(x_3, x_1x_4, x_2) = x_1x_2x_4,$$

$$g_1(x_4, x_1x_2, x_3) = x_1x_2x_4,$$

$$g_1(x_4, x_1x_3, x_2) = x_1x_2x_4;$$

by Lemma 4-2.26,

$$g_1(x_2, x_1x_3, x_4) = \begin{cases} x_1x_2x_4 \\ g_1(x_2, x_1, x_4) = g_2(x_1, x_2, x_4), \end{cases}$$

$$g_1(x_2, x_3x_4, x_1) = \begin{cases} x_1x_2x_4 \\ g_1(x_2, x_4, x_1) = g_2(x_1, x_2, x_4), \end{cases}$$

$$g_1(x_3, x_1x_2, x_4) = \begin{cases} x_1x_2x_4 \\ g_1(x_2, x_1, x_4) = g_2(x_1, x_2, x_4); \end{cases}$$

$$g_1(x_3, x_2x_4, x_1) = \begin{cases} x_1x_2x_4 \\ g_1(x_2, x_4, x_1) = g_2(x_1, x_2, x_4); \end{cases}$$

and

$$g_1(x_4, x_2x_3, x_1) = g_1(x_4, x_2, x_1) = g_3(x_1, x_2, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

By Lemmas from 4-2.28 to 4-2.34, the polynomials in $B_0 \cup B_{11} \cup B_2 \cup B_3$ are pairwise distinct so $p_4(\mathfrak{A}) \geqslant 31$ under (T3) and (T3.1).

4-2.3.2. $p_4 \geqslant 19$ *under* (T3.2) *and two varieties.* Before continuing discussion, we give two general lemmas.

LEMMA 4-2.35. *Under (T3),*

$$g_1(g_1(x,y,z),y,z) = \begin{cases} xyz \\ g_1(x,y,z). \end{cases}$$

PROOF. By Lemma 4-2.25, it is easy to see that $g_1(g_1, y, z)$ is essentially ternary. If $g_1(g_1, y, z) = g_2$, then

$$g_2 = g_2y = g_1(g_1, y, z)y = g_1yz = xyz,$$

a contradiction. If $g_1(g_1, y, z) = g_3$, then

$$g_3 = g_3z = g_1(g_1, y, z)z = g_1yz = xyz,$$

a contradiction. □

LEMMA 4-2.36. *If* $g_1(g_1(x,y,z),y,z) = xyz$, *then*

$$g_1(x_1x_2, x_3, x_4) \neq g_1(x_1, x_3, x_4)x_2,$$

$$g_1(x_1x_2, x_3, x_4) \neq g_1(x_2, x_3, x_4)x_1.$$

PROOF. We only need prove the first one. To do this, assume that

$$g_1(x_1x_2, x_3, x_4) = g_1(x_1, x_3, x_4)x_2.$$

Then

$$g_1(g_1(x,y,z),y,z) = g_1(xg_1(x,y,z),y,z) = g_1(x,y,z)g_1(x,y,z) = g_1(x,y,z),$$

contradicting the assumption; so completing the proof by Lemma 4-2.35. □

Now suppose that condition (T3.2) is satisfied in the following.

LEMMA 4-2.37. *The condition* (T3.2) *implies* $g_1(x, xy, z) = xyz$.

PROOF. By Lemma 4-2.26, we need show that $g_1(x,xy,z) \neq g_1(x,y,z)$. If $g_1(x,xy,z) = g_1(x,y,z)$, then, letting $x_2 = x_1$ in (T3.2),

$$g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_1x_3, x_4) = g_1(x_1, x_3, x_4)$$

and

$$g_1(x_1, x_2, x_4)g_1(x_1, x_3, x_4) = x_1x_4g_1(x_1, x_3, x_4) = x_1x_3x_4,$$

so, by(T3),

$$g_1(x_1, x_3, x_4) = x_1x_3x_4$$

by assumption of Lemma 4-2.37, a contradiction. □

Note that there are two subcases $g_1(g_1, y, z) = xyz$ and $g_1(g_1, y, z) = g_1$ from Lemma 4-2.35. For the first case, we have $p_4 \geqslant 31$.

LEMMA 4-2.38. *Suppose that* $g_1(g_1(x,y,z),y,z) = xyz$ *and (T3.2). Then the polynomials in* $B_0 \cup B_1 \cup B_2 \cup B_4$ *are pairwise distinct, so* $p_4 \geqslant 31$ *in this case.*

PROOF. From the proof of Lemmas 4-2.28 to 4-2.34, we need show that the polynomials in $B_1 \cup B_4$ are pairwise distinct and

$$B_4 \bigcap (B_1 \cup B_2) = \varnothing.$$

(1) The polynomials in B_1 are pairwise distinct:

Let $x_4 = x_1$. Then

$$\begin{aligned}
&g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3), && g_1(x_2,x_3,x_1)x_4 = x_1x_2x_3,\\
&g_1(x_3,x_1,x_2)x_4 = x_1x_2x_3, && g_1(x_2,x_3,x_4)x_1 = x_1x_2x_3,\\
&g_1(x_3,x_4,x_2)x_1 = x_1x_2x_3, && g_1(x_4,x_2,x_3)x_1 = g_1(x_1,x_2,x_3),\\
&g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3, && g_1(x_4,x_1,x_3)x_2 = x_1x_2x_3,\\
&g_1(x_1,x_3,x_4)x_2 = x_1x_2x_3, && g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3,\\
&g_1(x_1,x_2,x_4)x_3 = x_1x_2x_3, && g_1(x_2,x_4,x_1)x_3 = x_1x_2x_3.
\end{aligned}$$

So $g_1(x_1,x_2,x_3)x_4$ is different from others except for $g_1(x_4,x_2,x_3)x_1$. Now we will show that $g_1(x_1,x_2,x_3)x_4 \neq g_1(x_4,x_2,x_3)x_1$. Since

$$x \geqslant g_1(x,y,z) > xyz,$$

there exist $a,b,c \in A$ such that

$$a \geqslant e = g_1(a,b,c) > abc.$$

Let $x_4 = e$ and $(x_1,x_2,x_3) = (a,b,c)$. Then

$$g_1(x_1,x_2,x_3)x_4 = g_1(a,b,c)$$

and by assumption,

$$g_1(x_4,x_2,x_3)x_1 = g_1(e,b,c)a = g_1(g_1(a,b,c),b,c)a = (abc)a = abc.$$

So $g_1(x_1,x_2,x_3)x_4 \neq g_1(x_4,x_2,x_3)x_1$.

Applying permutations in S_4 to the above, (1) is complete by Lemma 4-1.15 and Lemma 4-1.16.

(2) The polynomials in B_4 are pairwise distinct:

Let $x_2 = x_1$. Then, by Lemma 4-2.26,

$$\begin{aligned}
&g_1(x_1x_2,x_3,x_4) = g_1(x_1,x_3,x_4), && g_1(x_1x_3,x_2,x_4) = x_1x_3x_4,\\
&g_1(x_1x_4,x_2,x_3) = x_1x_3x_4, && g_1(x_2x_3,x_1,x_4) = x_1x_3x_4,\\
&g_1(x_2x_4,x_1,x_3) = x_1x_3x_4, && g_1(x_3x_4,x_1,x_2) = x_1x_3x_4.
\end{aligned}$$

So $g_1(x_1x_2, x_3, x_4)$ is distinct from others. Applying permutations in S_4, (2) is complete.

(3) $B_1 \cap B_4 = \varnothing$:

First, $g_1(x_1x_2, x_3, x_4) \notin B_1$. Let $x_2 = x_1$. It is easy to check that $g_1(x_1x_2, x_3, x_4)$ is different from the polynomials in B_1 except for $g_1(x_2, x_3, x_4)x_1$ and $g_1(x_1, x_3, x_4)x_2$. But by assumption and Lemma 4-2.36, $g_1(x_1x_2, x_3, x_4)$ is different from $g_1(x_2, x_3, x_4)x_1$ and $g_1(x_1, x_3, x_4)x_2$. Applying permutations in S_4, (3) is complete.

(4) $B_2 \cap B_4 = \varnothing$:

Let $x_2 = x_1$. It is easy to check that $g_1(x_1x_2, x_3, x_4)$ is different from the polynomials in B_2. Applying permutations in S_4, (4) is complete. □

We suppose that (T3.2) and the following condition (T3.3) are satisfied in the remainder of this subsection

$$g_1(g_1(x, y, z), y, z) = g_1(x, y, z). \tag{T3.3}$$

We will see that there exist only two varieties of ternary reducts in this case.

LEMMA 4-2.39. *$g_1(g_1(x, y, z), s, t) = xyz$, where $s, t \in \{x, y, xy, xz, yz, xyz\}$ or only one of s and t is x.*

PROOF. $g_1(g_1, x, y) = g_1(xg_1, x, y) = g_1xy = xyz$ by Lemma 4-2.37 and (T3). Similarly, by Lemmas 4-2.26-4-2.27-4-2.37, we have

$$g_1(g_1, x, z) = xyz, \quad g_1(g_1, x, xy) = xyz, \quad g_1(g_1, x, xz) = xyz,$$
$$g_1(g_1, x, yz) = xyz, \quad g_1(g_1, y, xy) = xyz, \quad g_1(g_1, y, yz) = xyz,$$
$$g_1(g_1, z, xz) = xyz, \quad g_1(g_1, z, yz) = xyz, \quad g_1(g_1, t, xyz) = xyz,$$

where $t \in \{x, y, z, xy, xz, yz, xyz\}$. By the above and (T3.2),

$$g_1(g_1, y, xz) = g_1(g_1, y, x)g_1(g_1, y, z) = (xyz)g_1 = xyz.$$

Similarly,

$$g_1(g_1, z, xy) = xyz, \quad g_1(g_1, xy, xz) = xyz,$$

$$g_1(g_1, xy, yz) = xyz, \quad g_1(g_1, xz, yz) = xyz.$$

The others are trivial. □

LEMMA 4-2.40.

$$g_1(g_1, g_2, z) = \begin{cases} xyz \\ g_1 \end{cases}$$

and

$$g_1(g_1,\ g_2, s) = xyz,$$

where $s \in \{x, y, xy, xz, yz, xyz, g_1, g_2\}$.

PROOF. If $g_1(g_1, g_2, z) = g_2$, then

$$g_2 = g_1(g_1, g_2, z)g_2 = g_1g_2z = xyz,$$

a contradiction. If $g_1(g_1, g_2, z) = g_3$, then

$$g_3 = g_1(g_1, g_2, z)g_3 = g_1(g_1, g_2, z)g_3z = g_1g_2g_3z = xyz,$$

a contradiction. So

$$g_1(g_1, g_2, z) = \begin{cases} xyz \\ g_1 \end{cases}$$

since it is essentially ternary by Theorem 3-1.2.

$$g_1(g_1, g_2, x) = g_1(xg_1, x, g_2) = xg_1g_2 = xyz.$$

$$g_1(g_1, g_2, y) = g_1(g_1, yg_2, y) = g_1g_2y = xyz.$$

Similarly, we can prove the others by (T3.2) or (4-2.3.B). □

LEMMA 4-2.41.

$$g_1(g_1, g_2, g_3) = \begin{cases} xyz \\ g_1 \end{cases}$$

PROOF. If $g_1(g_1, g_2, g_3) = g_3$, then

$$g_3 = g_1(g_1, g_2, g_3)g_3 = g_1g_2g_3 = xyz,$$

a contradiction. Similarly, $g_1(g_1, g_2, g_3) \neq g_2$. By Theorem 3-1.2, it is essentially ternary. So the proof is complete. □

By Lemma 4-2.40 and permutation (yz), we have

LEMMA 4-2.42.

$$g_1(g_1, y, g_3) = g_1(g_1, g_3, y) = \begin{cases} xyz \\ g_1 \end{cases}$$

and

$$g_1(g_1, s, g_3) = xyz, \quad \textit{where } s \in \{x, z, xy, xz, yz, xyz, g_1, g_3\}.$$

LEMMA 4-2.43.

$$g_1(y, g_1, z) = \begin{cases} xyz \\ g_2, \end{cases} \qquad g_1(z, g_1, y) = \begin{cases} xyz \\ g_3, \end{cases}$$

$$g_1(x, g_1, x) = g_1, \qquad g_1(s, g_1, t) = xyz,$$

where $s, t \in \{x, y, xy, xz, yz, xyz\}$, $\{s, t\} \neq \{y, z\}$ *and* $\{s, t\} \neq \{x\}$.

PROOF. By (T3.2),

$$g_1(y, g_1, z) = g_1(y, xg_1, z) = g_1(y, x, z)g_1(y, g_1, z) = g_2(x, y, z)g_1(y, g_1, z)$$

which implies

$$g_1(y, g_1, z) = \begin{cases} xyz \\ g_2. \end{cases}$$

By permutation (yz),

$$g_1(z, g_1, y) = \begin{cases} xyz \\ g_3. \end{cases}$$

The proof of others is similar to that of Lemma 4-2.39. □

LEMMA 4-2.44.

$$g_1(z, g_1, g_2) = \begin{cases} xyz \\ g_3, \end{cases} \qquad g_1(s, g_1, g_2) = xyz,$$

where $s \in \{x, y, xy, xz, yz, xyz\}$.

PROOF. If $g_1(z, g_1, g_2) = g_1$, then

$$g_1 = g_1(z, g_1, g_2)g_1 = zg_1g_2 = xyz,$$

a contradiction. Similarly, $g_1(z, g_1, g_2) \neq g_2$. So

$$g_1(z, g_1, g_2) = \begin{cases} xyz \\ g_3. \end{cases}$$

The proof of others is similar to that of Lemma 4-2.39. □

By applying permutations (yz) and (xz) , respectively, we have

LEMMA 4-2.45.

$$g_1(y, g_1, g_3) = \begin{cases} xyz \\ g_2, \end{cases} \qquad g_1(s, g_1, g_3) = xyz,$$

where $s \in \{x, z, xy, xz, yz, xyz\}$;

$$g_1(x, g_2, g_3) = \begin{cases} xyz \\ g_1, \end{cases} \qquad g_1(s, g_2, g_3) = xyz,$$

where $s \in \{y, z, xy, xz, yz, xyz\}$.

LEMMA 4-2.46. *If* $g_1(x, g_2, z) = g_1$, *then*

$$g_1(x, y, g_3) = g_1, \quad g_1(x, g_2, g_3) = g_1, \quad g_1(g_1, y, g_3) = g_1,$$
$$g_1(g_1, g_2, z) = g_1, \quad g_1(g_1, g_2, g_3) = g_1.$$

PROOF. By permutation (yz), $g_1(x, g_2, z) = g_1$ is equivalent to $g_1(x, y, g_3) = g_1$. By substituting g_3 for z in $g_1(g_1, y, z) = g_1$,

$$\begin{aligned} g_1 &= g_1(x, y, g_3) && \text{(by the equivalent condition)} \\ &= g_1(g_1(x, y, g_3), y, g_3) \\ &= g_1(g_1, y, g_3). && \text{(substitution by the equivalent condition)} \end{aligned}$$

By permutation (yz), $g_1(g_1, g_2, z) = g_1$.

From $g_1(g_1, y, g_3) = g_1$,

$$\begin{aligned} g_1 &= g_1(x, g_2, z) && \text{(by assumption of Lemma 4-2.46)} \\ &= g_1(g_1(x, g_2, z), g_2, g_3(x, g_2, z)) && \text{(substituting } g_2 \text{ for } y \text{ in } g_1(g_1, y, g_3) = g_1) \\ &= g_1(g_1, g_2, g_3) && \text{(by assumption of Lemma 4-2.46)} \end{aligned}$$

since $g_3(x, g_2, z) = g_3$ by permutation (xz) in $g_1(x, g_2, z) = g_1$.

At the beginning of our proof, we have $g_1(x, y, g_3) = g_1$, that is,

$$g_2(y, x, g_3(x, y, z)) = g_2(y, x, z).$$

Exchanging x for y, $g_2(x, y, g_3) = g_2$. Now from $g_1(x, g_2, z) = g_1$,

$$\begin{aligned} g_1 &= g_1(x, y, g_3) && \text{(by the equivalent condition above)} \\ &= g_1(x, g_2(x, y, g_3), g_3) && \text{(substituting } g_3 \text{ for } z \text{ in } g_1(x, g_2, z) = g_1) \\ &= g_1(x, g_2, g_3) && \text{(by the previous result)} \end{aligned}$$

□

LEMMA 4-2.47. *If* $g_1(x, g_2, z) = xyz$, *then*

$$g_1(x, y, g_3) = xyz, \quad g_1(x, g_2, g_3) = xyz, \quad g_1(g_1, y, g_3) = xyz,$$
$$g_1(g_1, g_2, z) = xyz, \quad g_1(g_1, g_2, g_3) = xyz.$$

PROOF. By permutation (yz), $g_1(x, g_2, z) = xyz$ is equivalent to $g_1(x, y, g_3) = xyz$.

If $g_1(g_1, g_2, z) = g_1$, then

$$\begin{aligned} g_1 &= g_1(g_1, y, z) && \text{(by (T3.3))} \\ &= g_1(g_1(g_1, y, z), g_2(g_1, y, z), z) && \text{(substituting } g_1 \text{ for } x \text{ in } g_1(g_1, g_2, z) = g_1) \\ &= g_1(g_1, g_2(g_1, y, z), z). && \text{(by the first identity or (T3.3))} \end{aligned}$$

By permutation (xy) in $g_1(x, g_2, z) = xyz$, $g_1(y, g_1, z) = xyz$, that is,

$$g_2(g_1, y, z) = xyz. \tag{G1}$$

Then by permutation (yz),

$$g_3(g_1, y, z) = xyz. \tag{G2}$$

So $g_1 = g_1(g_1, xyz, z) = xyz$ by the above , a contradiction. Therefore,

$$g_1(g_1, g_2, z) = xyz$$

from Lemma 4-2.42. By permutation (yz), $g_1(g_1, y, g_3) = xyz$.

If $g_1(x, g_2, g_3) = g_1$, then

$$\begin{aligned} g_1 &= g_1(g_1, y, z) && \text{(by (T3.3))} \\ &= g_1(g_1, g_2(g_1, y, z), g_3(g_1, y, z)) && \text{(substituting } g_1 \text{ for } x \text{ in } g_1(x, g_2, g_3) = g_1) \\ &= g_1(g_1, xyz, xyz) && \text{(by (G1) and (G2))} \\ &= xyz, \end{aligned}$$

a contradiction. So $g_1(x, g_2, g_3) = xyz$.

If $g_1(g_1, g_2, g_3) = g_1$, then, similarly to the previous paragraphs,

$$\begin{aligned} g_1 &= g_1(g_1, y, z) \\ &= g_1(g_1(g_1, y, z), g_2(g_1, y, z), g_3(g_1, y, z)) \\ &= g_1(g_1, xyz, xyz) \\ &= xyz, \end{aligned}$$

a contradiction. So $g_1(g_1, g_2, g_3) = xyz$.

NOTE: for later use, we give a direct proof of the following identities

$$g_1(x, g_2, g_3) = xyz$$

and

$$g_1(g_1, g_2, g_3) = xyz.$$

Indeed,

$$\begin{aligned} g_1(x, g_2, g_3) &= g_1(x, g_2y, g_3) && \text{(by (T3))} \\ &= g_1(x, g_2, g_3)g_1(x, y, g_3) && \text{(by (T3.2))} \\ &= g_1(x, g_2, g_3)xyz && \text{(by the first identity in the lemma)} \\ &= g_1(x, g_2, g_3)g_1g_2 && \text{(by (T3))} \\ &= xg_2g_3g_1 && \text{(by (T3))} \\ &= xyz. && \text{(by (T3))} \end{aligned}$$

and

$$\begin{aligned} g_1(g_1, g_2, g_3) &= g_1(g_1, g_2y, g_3) \\ &= g_1(g_1, g_2, g_3)g_1(g_1, y, g_3) && \text{(by (T3.2))} \\ &= g_1(g_1, g_2, g_3)xyz && \text{(by the third identity in the lemma)} \\ &= g_1g_2g_1(g_1, g_2, g_3) && \text{(by (T3))} \\ &= xg_2g_3g_1 && \text{(by (T3))} \end{aligned}$$

$$= xyz. \qquad \text{(by (T3))}$$

□

Thus, from the conditions (T3.2) and (T3.3), we consider two varieties $\mathbf{K}_1$ and $\mathbf{K}_2$ of algebras of type $\langle 2,3\rangle$ respectively satisfying the following identities

(K1)
$$\begin{aligned}
xx &= x, & xy &= yx,\\
(xy)z &= x(yz), & g_1(x,y,z) &= g_1(x,z,y),\\
g_1(x,y,y) &= xy, & g_1(x,x,y) &= xy,\\
g_1(x,y,z)x &= g_1(x,y,z), & g_1(x,y,z)y &= xyz,\\
g_1(x,y,z)g_2(x,y,z) &= xyz, & g_1(xy,y,z) &= xyz,\\
g_1(x,yz,z) &= xyz, & g_1(g_1(x,y,z),y,z) &= g_1(x,y,z),\\
g_1(x,g_2(x,y,z),z) &= g_1(x,y,z), & &\\
g_1(x_1,x_2x_3,x_4) &= g_1(x_1,x_2,x_4)g_1(x_1,x_3,x_4). & &
\end{aligned}$$

and

(K2)
$$\begin{aligned}
xx &= x, & xy &= yx,\\
(xy)z &= x(yz), & g_1(x,y,z) &= g_1(x,z,y),\\
g_1(x,y,y) &= xy, & g_1(x,x,y) &= xy,\\
g_1(x,y,z)x &= g_1(x,y,z), & g_1(x,y,z)y &= xyz,\\
g_1(x,y,z)g_2(x,y,z) &= xyz, & g_1(xy,y,z) &= xyz,\\
g_1(x,yz,z) &= xyz, & g_1(g_1(x,y,z),y,z) &= g_1(x,y,z),\\
g_1(x,g_2(x,y,z),z) &= xyz, & g_1(g_1,g_2,z) &= xyz,\\
g_1(x_1,x_2x_3,x_4) &= g_1(x_1,x_2,x_4)g_1(x_1,x_3,x_4). & &
\end{aligned}$$

where $g_2(x,y,z) = g_1(y,x,z)$, and $g_3(x,y,z) = g_1(z,y,x)$. The free algebras $F_{K_i}(3)$ generated by three elements in $\mathbf{K}_i$ is as in Figure 9 and the values of the operations are the same as in lemmas from 4-2.39 to 4-2.47. By the proofs of the lemmas from

4-2.39 to 4-2.47, (K1) and (K2) are closed, respectively, and we have the results of those lemmas.

Since $g_1(x,y,z) > xyz$, there exist $a,b,c \in A$ such that

$$a \geqslant e_1 = g_1(a,b,c) > abc,$$

which implies

$$a \parallel b \parallel c$$

by Lemmas 4-2.37 and 4-2.27, so

$$e_1 \ngeqslant b, c.$$

Otherwise, for example, if

$$e_1 \geqslant b,$$

then, by (T3.3),

$$\begin{aligned} e_1 &= g_1(a,b,c) \\ &= g_1(e_1,b,c) \\ &= g_1(e_1,e_1b,c) \\ &= e_1bc \\ &= abc, \end{aligned}$$

a contradiction. Let

$$e_2 = g_1(b,a,c) = g_2(a,b,c)$$

and

$$e_3 = g_1(c,a,b) = g_3(a,b,c).$$

Similarly,

$$e_2 \ngeqslant a, c; \quad e_3 \ngeqslant b, c.$$

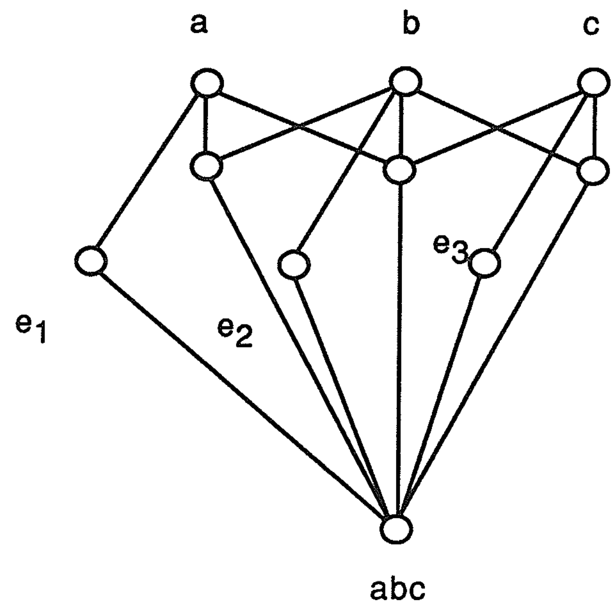

FIGURE 14. The semilattice structure of A(a,b,c)

Consider the subalgebra $A(a,b,c)$ generated by $\{a,b,c\}$. Since

$$b \geqslant e_2 \geqslant abc \quad \text{and} \quad c \geqslant e_3 \geqslant abc,$$

$$abe_3 = ae_2c = ae_2e_3 = e_1bc = e_1be_3 = e_1e_2c = e_1e_2e_3 = abc.$$

By (T3), it is easy to check that

$$e_1b = e_1c = abc, \quad e_2a = e_2c = abc, \quad e_3a = e_3b = abc.$$

Therefore, the semilattice structure of

$$A(a,b,c) = \{a,b,c,e_1,e_2,e_3,ab,ac,bc,abc\}$$

is shown as in Figure 14.

In $\mathbf{K}_1$, by Lemma 4-2.46, we have

$$\begin{aligned} g_1(e_1,b,c) &= g_1(e_1,e_2,c) \\ &= g_1(e_1,b,e_3) \\ &= g_1(a,e_2,c) \end{aligned}$$

$$\begin{aligned}&= g_1(a, e_2, e_3)\\ &= g_1(a, b, e_3)\\ &= g_1(e_1, e_2, e_3)\\ &= g_1(a, b, c)\\ &= e_1\\ &> abc.\end{aligned}$$

Similarly to the previous paragraph,

$$e_1 \parallel e_2 \parallel e_3$$

and

$$e_i > abc,$$

where $i = 1, 2, 3$. (otherwise, for example, if $e_2 = abc$, then $g_1(e_1, e_2, e_3) = abc$, a contradiction.) Consider the partition over $A(a, b, c)$:

$$\{\{a, e_1\}, \{b, e_2\}, \{c, e_3\}, \{ab, ac, bc, abc\}\}.$$

It is easy to see that it forms a congruence relation θ_1 over $A(a, b, c)$. Thus, $A(a, b, c)/\theta_1$ is isomorphic to a four-element algebra $Z_1 = \langle\{0, a_1, b_1, c_1\}; \cdot, f_1\rangle$ in $\mathbf{K}_1$ (see Figure 15), where " $\cdot$ " is semilattice operation such that a_1, b_1, c_1 are pairwise incomparable and 0 is the smallest element, and f_1 is a ternary operation defined by

$$f_1(x, y, z) = \begin{cases} a_1, & \text{if } (x, y, z) = (a_1, b_1, c_1) \text{ or } (a_1, c_1, b_1);\\ b_1, & \text{if } (x, y, z) = (b_1, a_1, c_1) \text{ or } (b_1, c_1, a_1);\\ c_1, & \text{if } (x, y, z) = (c_1, a_1, b_1) \text{ or } (c_1, b_1, a_1);\\ xyz, & \text{otherwise.}\end{cases}$$

Therefore, we have

LEMMA 4-2.48. *For every algebra $\mathfrak{A}$ in $\mathbf{K}_1$, if it represents $\langle 0, 1, 1, 4\rangle$, then $\mathfrak{A}$ contains a subalgebra such that Z_1 is a homomorphic image of this subalgebra.*

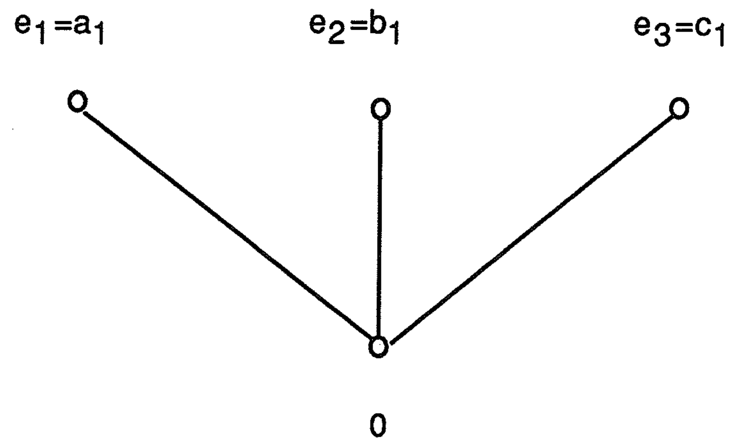

FIGURE 15. Z_1

In $\mathbf{K}_2$,

$$g_1(e_1, b, c) = g_1(a, b, c) = e_1 > abc$$

and by Lemma 4-2.47,

$$\begin{aligned} g_1(e_1, e_2, c) &= g_1(e_1, b, e_3) \\ &= g_1(a, e_2, c) \\ &= g_1(a, e_2, e_3) \\ &= g_1(a, b, e_3) \\ &= g_1(e_1, e_2, e_3) \\ &= abc. \end{aligned}$$

So

$$b > e_2, \quad \text{and} \quad c > e_3.$$

For example, if $b = e_2$, then

$$g_1(a, e_2, c) = g_1(a, b, c) > abc,$$

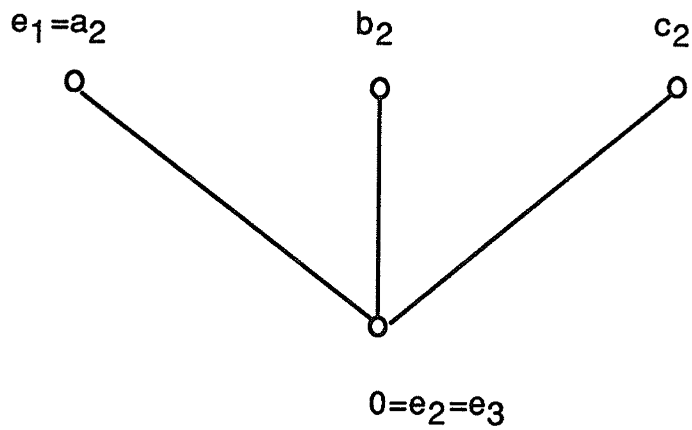

FIGURE 16. Z_2

a contradiction. Consider the partition on $A(a,b,c)$:

$$\{\{a,e_1\},\{b\},\{c\},\{e_2,e_3,ab,ac,bc,abc\}\}.$$

It forms a congruence relation θ_2 over $A(a,b,c)$. Thus, $A(a,b,c)/\theta_2$ is isomorphic to a four-element algebra $Z_2 = \langle\{0,a_2,b_2,c_2\};\cdot,f_2\rangle$ in $\mathbf{K}_2$ (see Figure 16), where " · " is semilattice operation such that a_2,b_2,c_2 are pairwise incomparable and 0 is the smallest, and f_2 is a ternary operation defined by

$$f_2(x,y,z) = \begin{cases} a_2, & \text{if } (x,y,z) = (a_2,b_2,c_2) \text{ or } (a_2,c_2,b_2); \\ xyz, & \text{otherwise.} \end{cases}$$

Therefore, we have

LEMMA 4-2.49. *For every algebra $\mathfrak{A}$ in $\mathbf{K}_2$, if it represents $\langle 0,1,1,4\rangle$, then $\mathfrak{A}$ contains a subalgebra such that Z_2 is a homomorphic image of this subalgebra.*

In the following, we will prove that $p_4(Z_i) = 19$, $i = 1,2$.

As before, it is easy to check

LEMMA 4-2.50. *The polynomials in $B_0 \cup B_{11} \cup B_2$ are pairwise distinct over Z_i and $F_{K_i}(3)$, $i = 1,2$.*

Obviously, Z_i is a homomorphic image of $F_{K_i}(3)$. The following lemmas show that there is no other essentially 4-ary polynomials over $F_{K_i}(3)$, so

$$p_4(Z_i) = p_4(F_{K_i}(3)) = 19, i = 1, 2.$$

It is routine to check the following lemmas.

LEMMA 4-2.51. *Over $F_{K_1}(3)$, we have*

$$g_1(x_1x_2, x_3, x_4) = g_1(x_1, x_3, x_4)x_2 = g_1(x_2, x_3, x_4)x_1,$$

$$g_1(x_2, g_1(x_1, x_2, x_3), x_4) = g_1(x_2, x_1, x_3)g_1(x_2, x_1, x_4),$$

$$g_1(x_4, g_1(x_1, x_2, x_3), x_2) = x_4g_1(x_3, x_1, x_2),$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)g_1(x_1, x_3, x_4) = x_1x_2x_3x_4.$$

From this Lemma and the previous identities which defined $\mathbf{K}_1$, we have

LEMMA 4-2.52. *There are only 19 essentially 4-ary polynomials $B_0 \cup B_{11} \cup B_2$ over $F_{K_1}(3)$, so*

$$p_4(Z_1) = p_4(F_{K_1}(3)) = 19.$$

Similarly, we have

LEMMA 4-2.53. *Over $F_{K_2}(3)$, we have*

$$g_1(x_1x_2, x_3, x_4) = g_1(x_1, x_3, x_4)x_2 = g_1(x_2, x_3, x_4)x_1,$$

$$g_1(x_2, g_1(x_1, x_2, x_3), x_4) = x_1x_2x_3x_4,$$

$$g_1(x_4, g_1(x_1, x_2, x_3), x_2) = x_1x_2x_3x_4,$$

$$g_1(x_1, x_2, x_3)g_1(x_1, x_2, x_4)g_1(x_1, x_3, x_4) = x_1x_2x_3x_4.$$

LEMMA 4-2.54. *There are only 19 essentially 4-ary polynomials $B_0 \cup B_{11} \cup B_2$ over $F_{K_2}(3)$, so*

$$p_4(Z_2) = p_4(F_{K_2}(3)) = 19.$$

REMARK . *By using the program written by J. Berman, we also have*

$$p_4(Z_i) = p_4(F_{K_i}(3)) = 19, i = 1, 2$$

because

$$|F_{Z_i}(4)| = |F_{F_{K_i}(3)}(4)| = 45.$$

and

$$p_5(Z_i) = p_5(F_{K_i}(3)) = 151, i = 1, 2$$

because

$$|F_{Z_i}(5)| = |F_{F_{K_i}(3)}(5)| = 301.$$

From Lemma 4-2.48 and Lemma 4-2.49,

LEMMA 4-2.55. *Under the assumption of this subsection, $p_4 \geqslant 19$.*

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the subvarieties of $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively, such that they satisfy the identities in Lemma 4-2.51 and Lemma 4-2.53, respectively. Then, from the previous results and their proofs, we have the following result.

THEOREM 4-2.2. *For every algebra $\mathfrak{A}$ with more than three elements in $\mathbf{H}_1$ or $\mathbf{H}_2$, $\mathfrak{A}$ represents $\langle 0, 1, 1, 4, 19 \rangle$.*

PROOF. The reason is that $B_0 \cup B_{11} \cup B_2$ is closed. □

4-2.4. $p_4 \geqslant 31$ under condition (T4). In this subsection, suppose that all g_i's satisfy (T4). We will prove that $p_4 \geqslant 31$.

LEMMA 4-2.56.

$$g_1(x,y,x) = g_1(x,x,y) = xy, \quad g_1(x,y,y) = \begin{cases} y, \\ xy. \end{cases}$$

PROOF. Let $z = x$ in $g_1(x,y,z)x = xyz$. We have $g_1(x,y,x)x = xy$.

If $g_1(x,y,x) = x$, then $x = g_1(x,y,x)x = xy$, a contradiction. If $g_1(x,y,x) = y$, then, letting $z = x$ in $g_1(x,y,z)z = g_1(x,y,z)$,

$$g_1(x,y,x)x = g_1(x,y,x)$$

so $yx = y$, a contradiction.

If $g_1(x,y,y) = x$, then, letting $z = y$ in $g_1(x,y,z)z = g_1(x,y,z)$,

$$xy = g_1(x,y,y)y = g_1(x,y,y) = x,$$

a contradiction. □

LEMMA 4-2.57. $g_1(x,xy,y) = xy$ *(i.e.,* $g_1(x,y,xy) = xy$*).*

PROOF. If $g_1(x,xy,y) = x$, then $g_1(x,xy,y)x = x$. On the other hand, by $g_1(x,y,z)x = xyz$,

$$g_1(x,xy,y)x = x(xy)y = xy,$$

a contradiction.

If $g_1(x,xy,y) = y$, then, by $g_1(x,y,z)y = g_1(x,y,z)$,

$$xy = y(xy) = g_1(x,xy,y)(xy) = g_1(x,xy,y) = y,$$

a contradiction. □

LEMMA 4-2.58. $g_1(x,xy,z) = xyz$.

PROOF. By lemmas 4-2.56 and 4-2.57, it is easy to check that $g_1(x, xy, z)$ is essentially ternary.

If $g_1(x, xy, z) = g_1(x, y, z)$, then

$$g_1(x, y, z) = g_1(x, xy, z) = g_1(x, xy, z)(xy) = g_1(x, y, z)xy = xyz,$$

a contradiction.

If $g_1(x, xy, z) = g_2(x, y, z)$, then

$$g_2(x, y, z) = g_1(x, xy, z) = g_1(x, xy, z)xy = g_2(x, y, z)yx = (xyz)x = xyz,$$

a contradiction.

If $g_1(x, xy, z) = g_3(x, y, z)$, then

$$g_3(x, y, z) = g_1(x, xy, z) = g_1(x, xy, z)z = g_3(x, y, z)z = xyz,$$

a contradiction. □

LEMMA 4-2.59. $g_1(xy, x, y) = xy$.

PROOF. If $g_1(xy, x, y) = x$, then, substituting xy for y,

$$x = g_1(xy, x, xy) = xy$$

by Lemma 4-2.56, a contradiction.

If $g_1(xy, x, y) = y$, then, substituting xy for x,

$$y = g_1(xy, xy, y) = xy$$

by Lemma 4-2.56, a contradiction. □

LEMMA 4-2.60.

$$g_1(x, y, yz) = \begin{cases} yz \\ xyz. \end{cases}$$

PROOF. By Lemma 4-2.56 and Lemma 4-2.57, it is easy to check that $g_1(x,y,yz)$ depends on y and z. If it does not depend on x, then

$$g_1(x,y,yz) = yz.$$

Suppose that $g_1(x,y,yz)$ is essentially ternary. If $g_1(x,y,yz) = g_1(x,y,z)$, then

$$\begin{aligned} g_1(x,y,z) &= g_1(x,yz,y) \\ &= g_1(x,yz,(yz)y) \quad \text{(by assumption)} \\ &= g_1(x,yz,yz) \\ &= \begin{cases} yz \\ xyz, \end{cases} \quad \text{(by Lemma 4-2.56)} \end{aligned}$$

a contradiction. If $g_1(x,y,yz) = g_2(x,y,z)$, then

$$xyz = g_2(x,y,z)y = g_1(x,y,yz)y = g_1(x,y,yz) = g_2(x,y,z),$$

a contradiction. If $g_1(x,y,yz) = g_3(x,y,z)$, then

$$xyz = g_3(x,y,z)yz = g_1(x,y,yz)yz = g_1(x,y,yz) = g_3(x,y,z),$$

a contradiction. □

LEMMA 4-2.61. *We have the following two equivalence conditions:*
(1) $g_1(x,y,y) = y \Longleftrightarrow g_1(x,y,yz) = yz$.
(2) $g_1(x,y,y) = xy \Longleftrightarrow g_1(x,y,yz) = xyz$.

PROOF. Suppose that $g_1(x,y,yz) \neq yz$. Then, by Lemma 4-2.60, $g_1(x,y,yz) = xyz$. Let $z = y$. Then $g_1(x,y,y) = xy$. So the necessity of (1) is true. However, the inverse is trivial.

To prove the necessity of (2), assume that $g_1(x,y,y) = xy$. By Lemma 4-2.60, we only need show that $g_1(x,y,yz)$ depends on x. Indeed, letting $z = y$,

$$g_1(x,y,yz) = g_1(x,y,y) = xy$$

so it depends on x. □

LEMMA 4-2.62.

$$g_1(xy, y, z) = \begin{cases} yz \\ xyz \\ g_1(x, y, z) \end{cases}$$

and if $g_1(x, y, y) = xy$, *then*

$$g_1(xy, y, z) = \begin{cases} xyz \\ g_1(x, y, z). \end{cases}$$

PROOF. By Lemma 4-2.56 and Lemma 4-2.57, it is easy to show that $g_1(xy, y, z)$ depends on y and z. If it does not depend on x, then $g_1(xy, y, z) = yz$. Suppose that it is essentially ternary. If $g_1(xy, y, z) = g_2(x, y, z)$, then

$$g_2(x, y, z) = g_1(xy, y, z) = g_1(xy, y, z)y = g_2(x, y, z)y = xyz,$$

a contradiction. If $g_1(xy, y, z) = g_3(x, y, z)$, then

$$g_3(x, y, z) = g_1(xy, y, z) = g_1(xy, y, z)z = g_3(x, y, z)z = xyz,$$

a contradiction. If $g_1(x, y, y) = xy$, then, letting $z = y$,

$$g_1(xy, y, z) = g_1(xy, y, y) = xy$$

so $g_1(xy, y, z)$ depends on x. □

We will prove that the polynomials in $B_0 \cup B_1 \cup B_4 \cup B_5$ are pairwise distinct. So $p_4 \geqslant 31$.

LEMMA 4-2.63. $B_0 \bigcap (B_1 \cup B_4 \cup B_5) = \varnothing$.

PROOF. It is trivial. □

LEMMA 4-2.64. $B_1 \cap B_4 = \varnothing$.

PROOF. (1) $g_1(x_1x_2, x_3, x_4) \notin B_1$:
Let $x_2 = x_1$. Then $g_1(x_1x_2, x_3, x_4) = g_1(x_1, x_3, x_4)$ and, by the previous lemmas,

$$g_1(x_1, x_2, x_3)x_4 = x_1x_3x_4, \qquad g_1(x_2, x_1, x_3)x_4 = x_1x_3x_4,$$

$$g_1(x_3, x_1, x_2)x_4 = \begin{cases} x_1x_4, \\ x_1x_3x_4, \end{cases} \qquad g_1(x_2, x_3, x_4)x_1 = g_1(x_1, x_3, x_4)x_1 = x_1x_3x_4,$$

$$g_1(x_3, x_4, x_2)x_1 = g_1(x_3, x_4, x_1) = g_2(x_1, x_3, x_4),$$
$$g_1(x_4, x_2, x_3)x_1 = g_1(x_4, x_1, x_3) = g_3(x_1, x_3, x_4),$$
$$g_1(x_3, x_4, x_1)x_2 = g_1(x_3, x_4, x_1) = g_2(x_1, x_3, x_4),$$
$$g_1(x_4, x_1, x_3)x_2 = g_1(x_4, x_1, x_3) = g_3(x_1, x_3, x_4),$$

$$g_1(x_1, x_3, x_4)x_2 = x_1x_3x_4, \quad g_1(x_4, x_1, x_2)x_3 = \begin{cases} x_1x_3, \\ x_1x_3x_4, \end{cases}$$
$$g_1(x_1, x_2, x_4)x_3 = x_1x_3x_4, \quad g_1(x_2, x_4, x_1)x_3 = x_1x_3x_4.$$

(2) By lemmas 4-1.15 and 4-1.16, our claim is complete by applying permutations in S_4 to (1). □

LEMMA 4-2.65. $B_1 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1, x_2x_3, x_4) \notin B_1$:
Let $x_3 = x_2$. Then $g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4)$ and

$$g_1(x_1, x_2, x_3)x_4 = \begin{cases} x_2x_4, \\ x_1x_2x_4, \end{cases} \qquad g_1(x_2, x_3, x_1)x_4 = x_1x_2x_4,$$

$$g_1(x_3, x_1, x_2)x_4 = x_1x_2x_4, \qquad g_1(x_2, x_3, x_4)x_1 = x_1x_2x_4,$$

$$g_1(x_3,x_4,x_2)x_1 = x_1x_2x_4, \qquad g_1(x_4,x_2,x_3)x_1 = \begin{cases} x_1x_2, \\ x_1x_2x_4, \end{cases}$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_2x_4, \qquad g_1(x_4,x_1,x_3)x_2 = g_3(x_1,x_2,x_4),$$

$$g_1(x_4,x_1,x_2)x_3 = g_3(x_1,x_2,x_4), \quad g_1(x_2,x_4,x_1)x_3 = x_1x_2x_4.$$

Let $x_4 = x_2$. Then

$$g_1(x_1,x_3,x_4)x_2 = g_1(x_1,x_3,x_2),$$

and by Lemma 4-2.60

$$g_1(x_1,x_2x_3,x_4) = g_1(x_1,x_2x_3,x_2) = \begin{cases} x_2x_3 \\ x_1x_2x_3. \end{cases}$$

Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_4)x_3 = g_1(x_1,x_2,x_3)$$

and by Lemma 4-2.60

$$g_1(x_1,x_2x_3,x_4) = g_1(x_1,x_2x_3,x_3) = \begin{cases} x_2x_3 \\ x_1x_2x_3. \end{cases}$$

(2) By lemmas 4-1.15 and 4-1.16, our claim is complete by applying permutations in S_4 to (1). □

LEMMA 4-2.66. $B_4 \cap B_5 = \varnothing$.

PROOF. (1) $g_1(x_1x_2,x_3,x_4) \notin B_5$:

Let $x_2 = x_1$. Then $g_1(x_1x_2,x_3,x_4) = g_1(x_1,x_3,x_4)$ and by Lemma 4-2.56, Lemma 4-2.58 and Lemma 4-2.60,

$$g_1(x_1,x_2x_3,x_4) = g_1(x_1,x_1x_3,x_4) = x_1x_3x_4,$$

$$g_1(x_1,x_2x_4,x_3) = g_1(x_1,x_1x_4,x_3) = x_1x_3x_4,$$

$$g_1(x_1,x_3x_4,x_2) = g_1(x_1,x_3x_4,x_1) = x_1x_3x_4,$$

$$g_1(x_2, x_1x_3, x_4) = g_1(x_1, x_1x_3, x_4) = x_1x_3x_4,$$

$$g_1(x_2, x_1x_4, x_3) = g_1(x_1, x_1x_4, x_3) = x_1x_3x_4,$$

$$g_1(x_2, x_3x_4, x_1) = g_1(x_1, x_3x_4, x_1) = x_1x_3x_4,$$

$$g_1(x_3, x_1x_2, x_4) = g_1(x_3, x_1, x_4) = g_2(x_1, x_3, x_4),$$

$$g_1(x_3, x_1x_4, x_2) = g_1(x_3, x_1x_4, x_1) = \begin{cases} x_1x_4 \\ x_1x_3x_4, \end{cases}$$

$$g_1(x_3, x_2x_4, x_1) = g_1(x_3, x_1x_4, x_1) = \begin{cases} x_1x_4 \\ x_1x_3x_4, \end{cases}$$

$$g_1(x_4, x_1x_2, x_3) = g_1(x_4, x_1, x_3) = g_3(x_1, x_3, x_4),$$

$$g_1(x_4, x_1x_3, x_2) = g_1(x_4, x_1x_3, x_1) = \begin{cases} x_1x_3 \\ x_1x_3x_4, \end{cases}$$

$$g_1(x_4, x_2x_3, x_1) = g_1(x_4, x_1x_3, x_1) = \begin{cases} x_1x_3 \\ x_1x_3x_4. \end{cases}$$

(2) By lemmas 4-1.15 and 4-1.16, our claim is complete by applying permutations in S_4 to (1). □

LEMMA 4-2.67. *The polynomials in B_1 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2, x_3)x_4$ is different from others:
Let $x_4 = x_3$. Then

$$g_1(x_1, x_2, x_3)x_4 = g_1(x_1, x_2, x_3), \qquad g_1(x_2, x_3, x_1)x_4 = g_2(x_1, x_2, x_3),$$

$$g_1(x_3, x_1, x_2)x_4 = x_1x_2x_3, \qquad g_1(x_2, x_3, x_4)x_1 = \begin{cases} x_1x_3 \\ x_1x_2x_3, \end{cases}$$

$$g_1(x_3, x_4, x_2)x_1 = x_1x_2x_3, \qquad g_1(x_4, x_2, x_3)x_1 = x_1x_2x_3,$$

$g_1(x_3, x_4, x_1)x_2 = x_1x_2x_3,$ $\qquad$ $g_1(x_4, x_1, x_3)x_2 = x_1x_2x_3,$

$$g_1(x_1, x_3, x_4)x_2 = \begin{cases} x_2x_3 \\ x_1x_2x_3, \end{cases} \qquad g_1(x_4, x_1, x_2)x_3 = x_1x_2x_3,$$

$g_1(x_2, x_4, x_1)x_3 = g_1(x_2, x_3, x_1) = g_2(x_1, x_2, x_3).$

Let $x_3 = x_2$. Then

$$g_1(x_1, x_2, x_3)x_4 = \begin{cases} x_2x_4 \\ x_1x_2x_4, \end{cases}$$

and

$$g_1(x_1, x_2, x_4)x_3 = g_1(x_1, x_2, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.68. *The polynomials in B_4 are pairwise distinct.*

PROOF. (1) $g_1(x_1x_2, x_3, x_4)$ is different from others:
Let $x_2 = x_1$. Then, by Lemma 4-2.62,

$$g_1(x_1x_2, x_3, x_4) = g_1(x_1, x_3, x_4),$$

$$g_1(x_1x_3, x_2, x_4) = g_1(x_1x_3, x_1, x_4) = \begin{cases} x_1x_4 \\ x_1x_3x_4 \\ g_2(x_1, x_3, x_4), \end{cases}$$

$$g_1(x_1x_4, x_2, x_3) = g_1(x_1x_4, x_1, x_3) = \begin{cases} x_1x_3 \\ x_1x_3x_4 \\ g_3(x_1, x_3, x_4), \end{cases}$$

$$g_1(x_2x_3, x_1, x_4) = g_1(x_1x_3, x_1, x_4) = \begin{cases} x_1x_4 \\ x_1x_3x_4 \\ g_2(x_1, x_3, x_4), \end{cases}$$

$$g_1(x_2x_4, x_1, x_3) = g_1(x_1x_4, x_1, x_3) = \begin{cases} x_1x_3 \\ x_1x_3x_4 \\ g_3(x_1, x_3, x_4), \end{cases}$$

$$g_1(x_3x_4, x_1, x_2) = g_1(x_3x_4, x_1, x_1) = \begin{cases} x_1 \\ x_1x_3x_4. \end{cases}$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.69. *The polynomials in B_5 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_2x_3, x_4)$ is different from others:
Let $x_3 = x_2$. Then, by Lemma 4-2.56, Lemma 4-2.58 and Lemma 4-2.60,

$$g_1(x_1, x_2x_3, x_4) = g_1(x_1, x_2, x_4),$$

$$g_1(x_1, x_2x_4, x_3) = g_1(x_1, x_2x_4, x_2) = \begin{cases} x_2x_4 \\ x_1x_2x_4, \end{cases}$$

$$g_1(x_1, x_3x_4, x_2) = g_1(x_1, x_2x_4, x_2) = \begin{cases} x_2x_4 \\ x_1x_2x_4, \end{cases}$$

$$g_1(x_2, x_1x_3, x_4) = g_1(x_2, x_1x_2, x_4) = x_1x_2x_4,$$

$$g_1(x_2, x_1x_4, x_3) = g_1(x_2, x_1x_4, x_2) = x_1x_2x_4,$$

$$g_1(x_2, x_3x_4, x_1) = g_1(x_2, x_2x_4, x_1) = x_1x_2x_4,$$

$$g_1(x_3, x_1x_2, x_4) = g_1(x_2, x_1x_2, x_4) = x_1x_2x_4,$$

$$g_1(x_3, x_1x_4, x_2) = g_1(x_2, x_1x_4, x_2) = x_1x_2x_4,$$

$$g_1(x_3, x_2x_4, x_1) = g_1(x_2, x_2x_4, x_1) = x_1x_2x_4,$$

$$g_1(x_4, x_1x_2, x_3) = g_1(x_4, x_1x_2, x_2) = \begin{cases} x_1x_2 \\ x_1x_2x_4, \end{cases}$$

$$g_1(x_4, x_1x_3, x_2) = g_1(x_4, x_1x_2, x_2) = \begin{cases} x_1x_2 \\ x_1x_2x_4, \end{cases}$$

$$g_1(x_4, x_2x_3, x_1) = g_1(x_4, x_2, x_1) = g_3(x_1, x_2, x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

4-2.5. $p_4 \geqslant 31$ under condition (T5). In this subsection, suppose that all g_i's satisfy (T5) and represent $\langle 0, 1, 1, 4\rangle$. We will prove that $p_4(\mathfrak{A}) \geqslant 31$.

LEMMA 4-2.70. $g_1(x, x, y) = g_1(x, y, x) = xy, \quad g_1(x, y, y) = y.$

PROOF. If $g_1(x, y, x) = x$, then, by (T5),

$$x = xx = g_1(x, y, x)x = xy,$$

a contradiction. If $g_1(x, y, x) = y$, then, by (T5),

$$y = yy = g_1(x, y, x)y = xy,$$

a contradiction. Therefore, $g_1(x, y, x) = xy$.

If $g_1(x, y, y) = x$, then, by (T5),

$$x = xx = g_1(x, y, y)x = xy,$$

a contradiction. If $g_1(x, y, y) = xy$, then, by (T5),

$$xy = g_1(x, y, y)y = yy = y,$$

a contradiction. Hence $g_1(x, y, y) = y$. □

LEMMA 4-2.71. $g_1(xy, x, y) = g_1(xy, y, x) = xy, \quad g_1(x, xy, y) = xy$.

PROOF. If $g_1(xy, x, y) = x$, then, by (T5),

$$x = g_1(xy, x, y)x = xy,$$

a contradiction. If $g_1(xy, x, y) = y$, then, by (T5),

$$y = g_1(xy, x, y)y = xy,$$

a contradiction. Therefore, $g_1(xy, x, y) = xy$.

If $g_1(x, xy, y) = x$, then, by (T5),

$$x = xx = g_1(x, xy, y)x = x(xy)y = xy,$$

a contradiction. If $g_1(x, xy, y) = y$, then, by (T5),

$$y = g_1(x, xy, y)y = (xy)y = xy,$$

a contradiction. Therefore, $g_1(x, xy, y) = xy$. □

LEMMA 4-2.72. $g_1(x, xy, z) = xyz$.

PROOF. It is easy to check that $g_1(x, xy, z)$ is essentially ternary.

If $g_1(x, xy, z) = g_1(x, y, z)$, then, by (T5),

$$xyz = g_1(x, xy, z)z = g_1(x, y, z)z = yz,$$

a contradiction. If $g_1(x, xy, z) = g_2(x, y, z)$, then

$$xyz = x(xy)z = g_1(x, xy, z)x = g_2(x, y, z)x = xz,$$

a contradiction. If $g_1(x, xy, z) = g_3(x, y, z)$, then

$$x(xy)z = g_1(x, xy, z)x = g_3(x, y, z)x = xy,$$

a contradiction. Hence $g_1(x, xy, z) = xyz$. □

LEMMA 4-2.73.

$$g_1(xy, y, z) = \begin{cases} yz \\ g_1(x, y, z). \end{cases}$$

PROOF. It is easy to check that $g_1(x, xy, z)$ depends on y and z. If it is not essentially ternary, then $g_1(xy, y, z) = yz$.

Suppose that $g_1(xy, y, z)$ is essentially ternary. If $g_1(xy, y, z) = g_2(x, y, z)$, then, by (T5),

$$yz = g_1(xy, y, z)y = g_2(x, y, z)y = xyz,$$

a contradiction. If $g_1(xy, y, z) = g_3(x, y, z)$, then

$$yz = g_1(xy, y, z)z = g_3(x, y, z)z = xyz,$$

a contradiction. If $g_1(xy, y, z) = xyz$, then

$$xyz = g_1(xy, y, z)y = yz,$$

a contradiction. Thus, $g_1(xy, y, z) = g_1(x, y, z)$. □

LEMMA 4-2.74. $g_1(x, y, yz) = yz$.

PROOF. It is easy to check that $g_1(x, y, yz)$ depends on y and z. We only need show that it is impossible that $g_1(x, y, yz)$ be essentially ternary.

Assume that $g_1(x, y, yz)$ is essentially ternary. If $g_1(x, y, yz) = g_1(x, y, z)$, then

$$\begin{aligned} g_1(x, y, z) &= g_1(x, y, yz) \\ &= g_1(x, yz, y) \\ &= g_1(x, yz, (yz)y) \\ &= g_1(x, yz, yz) \\ &= yz \end{aligned}$$

by Lemma 4-2.70, a contradiction. If $g_1(x, y, yz) = g_2(x, y, z)$, then

$$xyz = g_1(x, y, yz)x = g_2(x, y, z)x = xz,$$

a contradiction. If $g_1(x,y,yz) = g_3(x,y,z)$, then

$$xyz = g_1(x,y,yz)x = g_3(x,y,z)x = xy,$$

a contradiction. If $g_1(x,y,yz) = xyz$, then

$$xyz = g_1(x,y,yz)y = yz,$$

a contradiction. $\square$

We will prove that the polynomials in $B_0 \cup B_1 \cup B_2 \cup B_3$ are pairwise distinct. So $p_4 \geqslant 31$.

LEMMA 4-2.75. $B_0 \bigcap (B_1 \cup B_2 \cup B_3) = \varnothing$.

PROOF. It is trivial. $\square$

LEMMA 4-2.76. $B_1 \cap B_2 = \varnothing$.

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin B_1$:

Let $x_4 = x_3$. Then $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3)$ and by (T5) and previous lemmas,

$$\begin{array}{lll}
g_1(x_1,x_2,x_3)x_4 = x_2x_3, & g_1(x_2,x_3,x_1)x_4 = x_1x_3, & g_1(x_3,x_1,x_2)x_4 = x_1x_2x_3, \\
g_1(x_2,x_3,x_4)x_1 = x_3x_1, & g_1(x_3,x_4,x_2)x_1 = x_1x_2x_3, & g_1(x_4,x_2,x_3)x_1 = x_1x_2x_3, \\
g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3, & g_1(x_4,x_1,x_3)x_2 = x_1x_2x_3, & g_1(x_1,x_3,x_4)x_2 = x_2x_3, \\
g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3, & g_1(x_1,x_2,x_4)x_3 = x_2x_3, & g_1(x_2,x_4,x_1)x_3 = x_1x_3.
\end{array}$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.77. $B_1 \cap B_3 = \varnothing$.

PROOF. (1) $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) \notin B_1$:

Let $x_2 = x_1$. Then $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) = g_1(x_1,x_2,x_3)$ and

$$g_1(x_1,x_2,x_3)x_4 = x_1x_3x_4, \quad g_1(x_2,x_3,x_1)x_4 = x_1x_3x_4, \quad g_1(x_3,x_1,x_2)x_4 = x_1x_4,$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_3x_4, \quad g_1(x_3,x_4,x_2)x_1 = x_1x_4, \quad g_1(x_4,x_2,x_3)x_1 = x_1x_3,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_4, \quad g_1(x_4,x_1,x_3)x_2 = x_1x_3, \quad g_1(x_1,x_3,x_4)x_2 = x_1x_3x_4,$$

$$g_1(x_4,x_1,x_2)x_3 = x_1x_3, \quad g_1(x_1,x_2,x_4)x_3 = x_1x_3x_4, \quad g_1(x_2,x_4,x_1)x_3 = x_1x_3x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.78. *$B_2 \cap B_3 = \varnothing$.*

PROOF. (1) $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) \notin B_2$:

Let $x_2 = x_1$. Then $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) = g_1(x_1,x_3,x_4)$ and

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = x_1x_3x_4, \quad g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_1x_3x_4,$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_1x_3x_4, \quad g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_3x_4,$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_3x_4, \quad g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4) = x_1x_3x_4,$$

$$g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4) = x_1x_4, \quad g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4) = x_1x_4,$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = g_2(x_1,x_3,x_4), \quad g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3) = x_1x_3,$$

$$g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1) = x_1x_3, \quad g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2) = g_3(x_1,x_3,x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.79. *The polynomials in B_1 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)x_4$ is different from others:

Let $x_4 = x_1$. Then

$$g_1(x_1,x_2,x_3)x_4 = x_1x_2x_3, \quad g_1(x_2,x_3,x_1)x_4 = x_1x_3, \quad g_1(x_3,x_1,x_2)x_4 = x_1x_2,$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_3, \quad g_1(x_3,x_4,x_2)x_1 = x_1x_2, \quad g_1(x_4,x_2,x_3)x_1 = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_2, \quad g_1(x_4,x_1,x_3)x_2 = x_1x_2x_3, \quad g_1(x_1,x_3,x_4)x_2 = x_1x_2x_3,$$

$$g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3, \quad g_1(x_1,x_2,x_4)x_3 = x_1x_2x_3, \quad g_1(x_2,x_4,x_1)x_3 = x_1x_3.$$

Let $x_4 = x_2$. Then

$$g_1(x_4,x_1,x_3)x_2 = x_1x_2x_3, \quad g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3,$$

$$g_1(x_4,x_2,x_3)x_1 = x_1x_2x_3, \quad g_1(x_1,x_2,x_3)x_4 = x_2x_3.$$

Let $x_4 = g_1(x_1,x_2,x_3)$. Then $g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_2,x_3)$. If

$$g_1(x_1,x_2,x_4)x_3 = g_1(x_1,x_2,g_1(x_1,x_2,x_3))x_3 = g_1(x_1,x_2,x_3),$$

then

$$g_1(x_1,x_2,x_3) = g_1(x_1,x_2,g_1(x_1,x_2,x_3))g_1(x_1,x_2,x_3)x_3 = (x_2g_1(x_1,x_2,x_3))x_3 = x_2x_3,$$

a contradiction. If

$$g_1(x_1,x_3,x_4)x_2 = g_1(x_1,x_3,g_1(x_1,x_2,x_3))x_2 = g_1(x_1,x_2,x_3),$$

then

$$g_1(x_1,x_2,x_3) = g_1(x_1,x_3,g_1(x_1,x_2,x_3))g_1(x_1,x_2,x_3)x_2 = (3g_1(x_1,x_2,x_3))x_2 = x_2x_3,$$

a contradiction.

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.80. *The polynomials in B_2 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4)$ is different from others:

Let $x_4 = x_3$. Then

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3), \quad g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_2x_3,$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_2x_3, \quad g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_3,$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_3, \quad g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4) = g_2(x_1,x_2,x_3),$$

$$g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4) = x_1x_2x_3, \qquad g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4) = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3, \qquad g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3) = x_1x_2x_3,$$

$$g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1) = x_1x_2x_3, \qquad g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2) = x_1x_2x_3.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.81. *The polynomials in B_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4)$ is different from others:
Let $x_2 = x_1$. Then

$$g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) = g_1(x_1,x_3,x_4),$$

$$g_1(x_2,x_4,x_1)g_1(x_3,x_4,x_1) = g_1(x_3,x_4,x_1)x_1x_4 = x_1x_4,$$

$$g_1(x_1,x_2,x_4)g_1(x_3,x_2,x_4) = g_1(x_3,x_1,x_4)x_1x_4 = x_1x_4,$$

$$g_1(x_1,x_2,x_3)g_1(x_4,x_2,x_3) = g_1(x_4,x_1,x_3)x_1x_3 = x_1x_3,$$

$$g_1(x_2,x_3,x_1)g_1(x_4,x_3,x_1) = g_1(x_4,x_3,x_1)x_1x_3 = x_1x_3,$$

$$g_1(x_3,x_1,x_2)g_1(x_4,x_1,x_2) = x_1.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

4-2.6. $p_4 \geqslant 31$ under condition (T6). In this subsection, suppose that all g_i's satisfy (T6). We will prove that $p_4(\mathfrak{A}) \geqslant 31$.

LEMMA 4-2.82. $g_1(x,x,y) = g_1(x,y,x) = xy, \quad g_1(x,y,y) = xy.$

PROOF. If $g_1(x,y,x) = x$, then $g_1(x,y,x)x = x$. On the other hand,

$$g_1(x,y,x)x = xy$$

from $g_1(x,y,z)x = xyz$ by letting $z = x$, a contradiction. If $g_1(x,y,x) = y$, then, similarly,

$$y = g_1(x,y,x)y = xy,$$

a contradiction.

If $g_1(x,y,y) = x$, then

$$x = g_1(x,y,y)x = xy,$$

a contradiction. If $g_1(x,y,y) = y$, then

$$y = g_1(x,y,y)y = xy,$$

a contradiction. □

LEMMA 4-2.83. $g_1(x,xy,y) = xy, \quad g_1(xy,y,x) = xy.$

PROOF. If $g_1(x,xy,y) = x$, then

$$x = xx = g_1(x,xy,y)x = x(xy)y = xy,$$

a contradiction. If $g_1(x,xy,y) = y$, then

$$y = yy = g_1(x,xy,y)y = x(xy)y = xy,$$

a contradiction.

If $g_1(xy,x,y) = x$, then

$$x = xx = g_1(xy,x,y)x = (xy)xy = xy,$$

a contradiction. If $g_1(xy,x,y) = y$, then

$$y = yy = g_1(xy,x,y)y = (xy)xy = xy,$$

a contradiction. □

We will prove that the polynomials in $B_0 \cup B_1 \cup B_2 \cup B_3$ are pairwise distinct. So $p_4(\mathfrak{A}) \geqslant 31$.

LEMMA 4-2.84. $B_0 \bigcap (B_1 \cup B_2 \cup B_3) = \varnothing$.

PROOF. It is trivial. $\square$

LEMMA 4-2.85. $B_1 \cap B_2 = \varnothing$.

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) \notin B_1$:

Let $x_4 = x_3$. Then $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3)$ and by (T6) and previous lemmas,

$$g_1(x_1,x_2,x_3)x_4 = x_1x_2x_3, \quad g_1(x_2,x_3,x_1)x_4 = x_1x_2x_3, \quad g_1(x_3,x_1,x_2)x_4 = x_1x_2x_3,$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_2x_3, \quad g_1(x_3,x_4,x_2)x_1 = x_1x_2x_3, \quad g_1(x_4,x_2,x_3)x_1 = x_1x_2x_3,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_2x_3, \quad g_1(x_4,x_1,x_3)x_2 = x_1x_2x_3, \quad g_1(x_1,x_3,x_4)x_2 = x_1x_2x_3,$$

$$g_1(x_4,x_1,x_2)x_3 = x_1x_2x_3, \quad g_1(x_1,x_2,x_4)x_3 = x_1x_2x_3, \quad g_1(x_2,x_4,x_1)x_3 = x_1x_2x_3.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.86. $B_1 \cap B_3 = \varnothing$.

PROOF. (1) $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) \notin B_1$:

Let $x_2 = x_1$. Then $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) = g_1(x_1,x_2,x_3)$ and

$$g_1(x_1,x_2,x_3)x_4 = x_1x_3x_4, \quad g_1(x_2,x_3,x_1)x_4 = x_1x_3x_4, \quad g_1(x_3,x_1,x_2)x_4 = x_1x_3x_4,$$

$$g_1(x_2,x_3,x_4)x_1 = x_1x_3x_4, \quad g_1(x_3,x_4,x_2)x_1 = x_1x_3x_4, \quad g_1(x_4,x_2,x_3)x_1 = x_1x_3x_4,$$

$$g_1(x_3,x_4,x_1)x_2 = x_1x_3x_4, \quad g_1(x_4,x_1,x_3)x_2 = x_1x_3x_4, \quad g_1(x_1,x_3,x_4)x_2 = x_1x_3x_4,$$

$$g_1(x_4,x_1,x_2)x_3 = x_1x_3x_4, \quad g_1(x_1,x_2,x_4)x_3 = x_1x_3x_4, \quad g_1(x_2,x_4,x_1)x_3 = x_1x_3x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. $\square$

LEMMA 4-2.87. $B_2 \cap B_3 = \varnothing$.

PROOF. (1) $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) \notin B_2$:

Let $x_2 = x_1$. Then $g_1(x_1,x_3,x_4)g_1(x_2,x_3,x_4) = g_1(x_1,x_3,x_4)$ and

$$g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = x_1x_3x_4, \qquad g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_1x_3x_4,$$

$$g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_1x_3x_4, \qquad g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_3x_4,$$

$$g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_3x_4, \qquad g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4) = x_1x_3x_4,$$

$$g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4) = x_1x_3x_4, \qquad g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4) = x_1x_3x_4,$$

$$g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = g_2(x_1,x_3,x_4), \quad g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3) = x_1x_3x_4,$$

$$g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1) = x_1x_3x_4, \qquad g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2) = g_3(x_1,x_3,x_4).$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.88. *The polynomials in B_1 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)x_4$ is different from others:

Let $x_4 = g_1(x_1,x_2,x_3)$. Then

$$g_1(x_1,x_2,x_3)x_4 = g_1(x_1,x_3,x_4), \quad g_1(x_2,x_3,x_1)x_4 = x_1x_2x_3,$$

$$g_1(x_3,x_1,x_2)x_4 = x_1x_2x_3.$$

If $g_1(x_2,x_3,g_1(x_1,x_2,x_3))x_1 = g_1(x_2,x_3,x_4)x_1 = g_1(x_1,x_2,x_3)x_4 = g_1(x_1x_2x_3)$, then

$$\begin{aligned} g_1(x_1,x_2,x_3) &= g_1(x_2,x_3,g_1(x_1,x_2,x_3))g_1(x_1,x_2,x_3)x_1 \\ &= x_2x_3g_1(x_1,x_2,x_3)x_1 \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_3,x_4,x_2)x_1 = g_1(x_1,x_2,x_3)x_4$, then

$$\begin{aligned} g_1(x_1,x_2,x_3) &= g_1(x_3,g_1(x_1,x_2,x_3),x_2)x_1g_1(x_1,x_2,x_3) \\ &= x_1x_2x_3g_1(x_1,x_2,x_3) \end{aligned}$$

$$= x_1x_2x_3,$$

a contradiction. If $g_1(x_4, x_2, x_3)x_1 = g_1(x_1, x_2, x_3)x_4$, then

$$\begin{aligned} g_1(x_1, x_2, x_3) &= g_1(g_1(x_1, x_2, x_3), x_2, x_3)x_1g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_3, x_4, x_1)x_2 = g_1(x_1, x_2, x_3)x_4$, then

$$\begin{aligned} g_1(x_1, x_2, x_3) &= g_1(x_3, g_1(x_1, x_2, x_3), x_1)x_2g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_4, x_1, x_3)x_2 = g_1(x_1, x_2, x_3)x_4$, then

$$\begin{aligned} g_1(x_1, x_2, x_3) &= g_1(g_1(x_1, x_2, x_3), x_1, x_3)x_2g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_1, x_3, x_4)x_2 = g_1(x_1, x_2, x_3)x_4$, then

$$\begin{aligned} g_1(x_1, x_2, x_3) &= g_1(x_1, x_3, g_1(x_1, x_2, x_3))x_2g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_4, x_1, x_2)x_3 = g_1(x_1, x_2, x_3)x_4$, then

$$\begin{aligned} g_1(x_1, x_2, x_3) &= g_1(g_1(x_1, x_2, x_3), x_1, x_2)x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3g_1(x_1, x_2, x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_1,x_2,x_4)x_3 = g_1(x_1,x_2,x_3)x_4$, then

$$\begin{aligned} g_1(x_1,x_2,x_3) &= g_1(x_1,x_2,g_1(x_1,x_2,x_3))x_3g_1(x_1,x_2,x_3) \\ &= x_1x_2x_3g_1(x_1,x_2,x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction. If $g_1(x_2,x_4,x_1)x_3 = g_1(x_1,x_2,x_3)x_4$, then

$$\begin{aligned} g_1(x_1,x_2,x_3) &= g_1(x_2,g_1(x_1,x_2,x_3),x_1)x_3g_1(x_1,x_2,x_3) \\ &= x_1x_2x_3g_1(x_1,x_2,x_3) \\ &= x_1x_2x_3, \end{aligned}$$

a contradiction.

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.89. *The polynomials in B_2 are pairwise distinct.*

PROOF. (1) $g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4)$ is different from others:
Let $x_4 = x_3$. Then

$$\begin{aligned} &g_1(x_1,x_2,x_3)g_1(x_1,x_2,x_4) = g_1(x_1,x_2,x_3), && g_1(x_1,x_3,x_2)g_1(x_1,x_3,x_4) = x_1x_2x_3, \\ &g_1(x_1,x_4,x_2)g_1(x_1,x_4,x_3) = x_1x_2x_3, && g_1(x_2,x_3,x_4)g_1(x_2,x_3,x_1) = x_1x_2x_3, \\ &g_1(x_2,x_4,x_1)g_1(x_2,x_4,x_3) = x_1x_2x_3, && g_1(x_2,x_1,x_3)g_1(x_2,x_1,x_4) = g_2(x_1,x_2,x_3), \\ &g_1(x_3,x_1,x_2)g_1(x_3,x_1,x_4) = x_1x_2x_3, && g_1(x_3,x_2,x_1)g_1(x_3,x_2,x_4) = x_1x_2x_3, \\ &g_1(x_3,x_4,x_1)g_1(x_3,x_4,x_2) = x_1x_2x_3, && g_1(x_4,x_1,x_2)g_1(x_4,x_1,x_3) = x_1x_2x_3, \\ &g_1(x_4,x_2,x_3)g_1(x_4,x_2,x_1) = x_1x_2x_3, && g_1(x_4,x_3,x_1)g_1(x_4,x_3,x_2) = x_1x_2x_3. \end{aligned}$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

LEMMA 4-2.90. *The polynomials in B_3 are pairwise distinct.*

PROOF. (1) $g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4)$ is different from others:
Let $x_2 = x_1$. Then

$$g_1(x_1, x_3, x_4)g_1(x_2, x_3, x_4) = g_1(x_1, x_3, x_4),$$

$$g_1(x_2, x_4, x_1)g_1(x_3, x_4, x_1) = g_1(x_3, x_4, x_1)x_1x_4 = x_1x_3x_4,$$

$$g_1(x_1, x_2, x_4)g_1(x_3, x_2, x_4) = g_1(x_3, x_1, x_4)x_1x_4 = x_1x_3x_4,$$

$$g_1(x_1, x_2, x_3)g_1(x_4, x_2, x_3) = g_1(x_4, x_1, x_3)x_1x_3 = x_1x_3x_4,$$

$$g_1(x_2, x_3, x_1)g_1(x_4, x_3, x_1) = g_1(x_4, x_3, x_1)x_1x_3 = x_1x_3x_4,$$

$$g_1(x_3, x_1, x_2)g_1(x_4, x_1, x_2) = x_1x_3x_4.$$

(2) Applying permutations in S_4 to (1), our proof is complete by Lemma 4-1.15 and Lemma 4-1.16. □

CHAPTER 5

Conclusions and Conjectures

5-1. Conclusions

From the previous chapters, we have the following results.

THEOREM 5-1.1. *If an algebra* $\mathfrak{A}$ *represents* $\langle 0,1,1,4\rangle$, *then* $p_4(\mathfrak{A}) \geqslant 19$. *Thus,*

$$p_n(\mathfrak{A}) \geqslant 2^n + 2^{n-2} - 1,$$

for all $n \geqslant 4$.

The last statement of Theorem 5-1.1 is directly from Theorem 10 in [**28**] by mathematical induction.

THEOREM 5-1.2. *There are two varieties* $\mathbf{K}_i$ *of type* $\langle 2,3\rangle$ *and two four-element algebras* $Z_i \in \mathbf{K}_i$, $i = 1,2$, *such that*
(1) for any algebra $\mathfrak{A}$ *of type* $\langle 2,3\rangle$ *not in* $\mathbf{K}_i$ *for* $i = 1,2$, $\mathfrak{A}$ *represents* $\langle 0,1,1,4,27\rangle$ *if and only if* $\mathfrak{A}$ *is a nontrivial join algebra.*
(2) both Z_i *represents* $\langle 0,1,1,4,19,151\rangle$ *and every algebra* $\mathfrak{A} \in \mathbf{K}_i$ *representing* $\langle 0,1,1,4\rangle$ *contains a subalgebra such that* Z_i *is a homomorphic image of this subalgebra.*

PROOF. The necessity of (1) is trivial. Conversely, suppose that $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$, not in $\mathbf{K}_1 \cup \mathbf{K}_2$. Then the essentially ternary polynomials over $\mathfrak{A}$ satisfy (T2) and then the proof is complete by Theorem 4-2.1.

(2) is Lemma 4-2.48 and Lemma 4-2.49. □

THEOREM 5-1.3. *For an algebra* $\mathfrak{A} = \langle A;\ \cdot,\ g\rangle$ *of type* $\langle 2,3\rangle$, $\mathfrak{A}$ *represents* $\langle 0,1,1,4,19\rangle$ *iff* $\mathfrak{A}$ *has more than three elements and belongs to* $\mathbf{H}_1 \cup \mathbf{H}_2$.

PROOF. The sufficiency is Theorem 4-2.2. For the necessity, suppose that $\mathfrak{A}$ represents $\langle 0,1,1,4,19\rangle$. Then the ternary operation g is essential and $|\mathrm{Inv}(g)| = 2$ and $\mathfrak{A} \in \mathbf{K}_1 \cup \mathbf{K}_2$ by the previous sections. Without loss of generality, assume that $\mathrm{Inv}(g) = \langle (23)\rangle$, i.e., $g = g_1$. If $\mathfrak{A}$ has less than or equal three elements, then, by condition (4-2.3.B), Lemma 4-2.23, Lemma 4-2.25, Lemma 4-2.26 and Lemma 4-2.27, it is impossible that $\mathfrak{A}$ represents $\langle 0,1,1,4,19\rangle$.

Since the essentially 4-ary polynomials in $B_0 \cup B_{11} \cup B_2$ are pairwise distinct, there are only 19 essentially 4-ary polynomials $B_0 \cup B_{11} \cup B_2$. We will prove that either the identities of Lemma 4-2.51 or the identities of Lemma 4-2.53 hold.

(1) $g(x_1x_2,x_3,x_4) = x_1g(x_2,x_3,x_4)$:
Obviously, $g(x_1x_2,x_3,x_4) \notin B_0$. Let $x_2 = x_1$. Then $g(x_1x_2,x_3,x_4) = g(x_1,x_3,x_4)$ and

$$\begin{aligned}
&g(x_1,x_2,x_3)g(x_1,x_2,x_4) = x_1x_3x_4, && g(x_1,x_3,x_2)g(x_1,x_3,x_4) = x_1x_3x_4,\\
&g(x_1,x_4,x_2)g(x_1,x_4,x_3) = x_1x_3x_4, && g(x_2,x_3,x_4)g(x_2,x_3,x_1) = x_1x_3x_4,\\
&g(x_2,x_4,x_1)g(x_2,x_4,x_3) = x_1x_3x_4, && g(x_2,x_1,x_3)g(x_2,x_1,x_4) = x_1x_3x_4,\\
&g(x_3,x_1,x_2)g(x_3,x_1,x_4) = x_1x_3x_4, && g(x_3,x_2,x_1)g(x_3,x_2,x_4) = x_1x_3x_4,\\
&g(x_3,x_4,x_1)g(x_3,x_4,x_2) = g_2(x_1,x_3,x_4), && g(x_4,x_1,x_2)g(x_4,x_1,x_3) = x_1x_3x_4,\\
&g(x_4,x_2,x_3)g(x_4,x_2,x_1) = x_1x_3x_4, && g(x_4,x_3,x_1)g(x_4,x_3,x_2) = g_3(x_1,x_3,x_4).
\end{aligned}$$

Therefore, $g(x_1x_2,x_3,x_4) \notin B_2$. Similarly, $g(x_1x_2,x_3,x_4) \notin B_{11} - \{x_1 \cdot g(x_2,x_3,x_4)\}$. Thus, $g(x_1x_2,x_3,x_4) = x_1g(x_2,x_3,x_4)$.

(2) $g(x_1,x_2,x_3)g(x_1,x_2,x_4)g(x_1,x_3,x_4) = x_1x_2x_3x_4$:
Let $x_4 = x_1$. Then

$$g(x_1,x_2,x_3)g(x_1,x_2,x_4)g(x_1,x_3,x_4) = x_1x_2x_3, \quad g(x_1,x_2,x_3)x_4 = g(x_1,x_2,x_3).$$

Similarly, it is easy to check that

$$g(x_1,x_2,x_3)g(x_1,x_2,x_4)g(x_1,x_3,x_4) \notin B_{11} \cup B_2.$$

(3) $\mathfrak{A} \in \mathbf{K}_1 \Longrightarrow \mathfrak{A} \in \mathbf{H}_1$:

We want to prove

$$\text{(h1)} \qquad g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, x_1, x_3)g(x_2, x_1, x_4),$$

$$\text{(h2)} \qquad g(x_4, g(x_1, x_2, x_3), x_2) = x_4 g(x_3, x_1, x_2).$$

By Theorem 3-1.2, $g(x_2, g(x_1, x_2, x_3), x_4)$ and $g(x_4, g(x_1, x_2, x_3), x_2)$ are essentially 4-ary.

Let $x_4 = x_3$,

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, g(x_1, x_2, x_3), x_3) = g_2(x_1, x_2, x_3)$$

because $g_1(y, g_1(x, y, z), z) = g_2(x, y, z)$ from $g_1(x, g_2, z) = g_1$ in (K1) and every one in B_{11} is $x_1x_2x_3$ except for $g_1(x_3, x_1, x_2)x_4 = g_3(x_1, x_2, x_3)$, and any one in B_2 is $x_1x_2x_3$ except for

$$g(x_2, x_1, x_3)g(x_2, x_1, x_4) = g_2(x_1, x_2, x_3).$$

So (h1) holds since $p_4 = 19$.

Let $x_4 = x_3$,

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_3, g(x_1, x_2, x_3), x_2) = g(x_3, x_1, x_2) = g_3(x_1, x_2, x_3)$$

since $g_1(z, g_1(x, y, z), y) = g_3(x, y, z)$ from $g_1(x, g_2, z) = g_1$ in (K1), and anyone in B_{11} is $x_1x_2x_3$ except for

$$x_4 g(x_3, x_1, x_2) = g_3(x_1, x_2, x_3),$$

and each in B_2 is $x_1x_2x_3$ except for

$$g(x_2, x_1, x_3)g(x_2, x_1, x_4) = g_2(x_1, x_2, x_3).$$

So (h2) holds.

(4) $\mathfrak{A} \in \mathbf{K}_2 \Longrightarrow \mathfrak{A} \in \mathbf{H}_2$:

We want to prove

$$\text{(k1)} \qquad g(x_2, g(x_1, x_2, x_3), x_4) = x_1x_2x_3x_4,$$

$$\text{(k2)} \qquad g(x_4, g(x_1, x_2, x_3), x_2) = x_1x_2x_3x_4.$$

Let $x_4 = x_1$. Then

$$g(x_1, x_2, x_3)x_4 = g(x_1, x_2, x_3),$$

$$g(x_2, x_3, x_1)g(x_2, x_3, x_4) = g(x_2, x_3, x_1),$$

$$g(x_3, x_2, x_1)g(x_3, x_2, x_4) = g(x_3, x_2, x_1),$$

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, x_1 g(x_1, x_2, x_3), x_1) = g(x_1, x_2, x_3)x_1x_2 = x_1x_2x_3,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_1, x_1 g(x_1, x_2, x_3), x_2) = x_1x_2x_3.$$

Let $x_4 = x_2$. Then

$$g(x_2, x_3, x_1)x_4 = g(x_2, x_3, x_1),$$

$$g(x_1, x_3, x_2)g(x_1x_3x_4) = g(x_1, x_3, x_2),$$

$$g(x_3, x_1, x_2)g(x_3, x_1, x_4) = g(x_3, x_1, x_2),$$

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, g(x_1, x_2, x_3), x_2) = g(x_1, x_2, x_3)x_2 = x_1x_2x_3,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_2, g(x_1, x_2, x_3), x_2) = g(x_1, x_2, x_3)x_2 = x_1x_2x_3,$$

Let $x_4 = x_3$. Then, from $g_1(x, g_2, z) = xyz$ in (K2),

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, g(x_1, x_2, x_3), x_3) = x_1x_2x_3,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_3, g(x_1, x_2, x_3), x_2) = x_1x_2x_3,$$

$$g(x_3, x_1, x_2)x_4 = g(x_3, x_1, x_2),$$

$$g(x_1, x_2, x_3)g(x_1x_2x_4) = g(x_1, x_2, x_3),$$

$$g(x_2, x_1, x_3)g(x_2, x_1, x_4) = g(x_1, x_2, x_3).$$

Let $x_2 = x_1$. Then

$$g(x_2, x_3, x_4)x_1 = g(x_1, x_3, x_4),$$

$$g(x_3, x_4, x_1)g(x_3, x_4, x_2) = g(x_3, x_4, x_1),$$

$$g(x_4, x_3, x_1)g(x_4, x_3, x_2) = g(x_4, x_3, x_1),$$

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_1, x_1x_3, x_4) = x_1x_3x_4,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_4, x_1x_3, x_1) = x_1x_3x_4.$$

Let $x_3 = x_1$. Then

$$g(x_3, x_4, x_2)x_1 = g(x_1, x_4, x_2),$$

$$g(x_2, x_4, x_1)g(x_2, x_4, x_3) = g(x_2, x_4, x_1),$$

$$g(x_4, x_2, x_1)g(x_4, x_2, x_3) = g(x_4, x_2, x_1),$$

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, x_1x_2, x_4) = x_1x_2x_4,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_4, x_1x_2, x_2) = x_1x_2x_4.$$

Let $x_3 = x_2$. Then

$$g(x_3, x_4, x_1)x_2 = g(x_2, x_4, x_1),$$

$$g(x_1, x_4, x_2)g(x_1, x_4, x_3) = g(x_1, x_4, x_2),$$

$$g(x_4, x_1, x_2)g(x_4, x_1, x_3) = g(x_4, x_1, x_2),$$

$$g(x_2, g(x_1, x_2, x_3), x_4) = g(x_2, x_1x_2, x_4) = x_1x_2x_4,$$

$$g(x_4, g(x_1, x_2, x_3), x_2) = g(x_4, x_1x_2, x_2) = x_1x_2x_4.$$

Therefore, (k1) and (k2) hold. □

THEOREM 5-1.4. *For any algebra $\mathfrak{A} = \langle A; \cdot, f\rangle$ of type $\langle 2,3\rangle$ with property: either $f(x,x,y) = x$ or $f(x,y,x) = x$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ iff it is a nontrivial join algebra.*

PROOF. The reason is that only (T2) satisfy the property: $f(x,x,y) = f(x,y,x) = x$ and $p_4 \leqslant 30$, and then apply Theorem 4-2.1. However, the condition either $f(x,x,y) = x$ or $f(x,y,x) = x$ is sufficient to get our result. □

THEOREM 5-1.5. *If nontrivial algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ represents $\langle 0,1,1,4,27\rangle$, then $\mathfrak{A}$ is a nontrivial join algebra or $\mathfrak{A} \in \mathbf{K}_1 \cup \mathbf{K}_2$. Thus, if algebra $\mathfrak{A}$ represents*

$\langle 0,1,1,4,27\rangle$, *then* $\mathfrak{A}$ *has a ternary reduct which is a nontrivial join algebra or in* $\mathbf{K}_1 \cup \mathbf{K}_2$.

THEOREM 5-1.6. *Let* $\mathfrak{J}$ *be the variety of join algebras and algebra* $\mathfrak{A}$ *has no ternary reduct equivalent to some algebra in* $\mathbf{K}_i$ *for* $i = 1,2$. *Then* $\mathfrak{A}$ *is equivalent to a nontrivial join algebra iff* $\mathfrak{A}$ *represents the* p_n*-sequence* $\langle 0,1,1,4,27,\ldots,p_n(\mathfrak{J}),\ldots\rangle$.

PROOF. Since J_2 generates $\mathfrak{J}$, we have the sufficiency. Conversely, by assumption, $p_4(\mathfrak{A}) \geqslant 27$. Since $\mathfrak{A}$ represents the $p_n(\mathfrak{J})$, $p_4(\mathfrak{A}) = 27$. By Theorem 5-1.2 (2), $\mathfrak{A}$ contains a ternary reduct equivalent to a nontrivial join algebra; therefore, $\mathfrak{A}$ is equivalent to this nontrivial join algebra. □

5-2. Conjectures

Naturally, we have the following conjectures:

(Q1) $\langle 0,1,1,4,19\rangle$ has the minimal extension property.

(Q2) For any algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$, $\mathfrak{A}$ represents $\langle 0,1,1,4,27\rangle$ if and only if $\mathfrak{A}$ is a nontrivial join algebra.

(Q3) For any algebra $\mathfrak{A}$, $\mathfrak{A}$ represents the p_n-sequence of nontrivial join algebra if and only if $\mathfrak{A}$ is equivalent to a nontrivial join algebra, that is, the free spectra decides the variety of join algebras.

Part 2

On the p_n-sequence $\langle 0, 1, 1, 7 \rangle$

CHAPTER 6

On the minimal extension of the p_n-sequence $\langle 0, 1, 1, 7\rangle$

In this chapter, we show that the p_n-sequence $\langle 0,1,1,7\rangle$ has the minimal extension property in the class of algebras with a proper non-associative binary operation. This generalizes the result of J. Galuszka and gives a partial solution of problem 18 raised by G. Grätzer and J. Kisielewicz whether $\langle 0,1,1,7\rangle$ has the minimal extension property. This result reduces the problem to the class of algebras with a semilattice operation.

6-1. Main result

It is well-known that ([**29**], [**25**]):

The sequence $\langle 0,1,1,7,\dots,p_n(\mathfrak{B}),\dots\rangle$ is the minimal extension of the sequence $\langle 0,1,1,7\rangle$ in the class of groupoids, where $\mathfrak{B}$ is the four-element Park algebra.

The question was raised in G. Grätzer and J. Kisielewicz [**29**] (Problem 18): Does $\langle 0,1,1,7\rangle$ (in classic notation, $\langle 0,0,1,7\rangle$) have the minimal extension property? In this part, we will show that

THEOREM 6-1.1. *$\langle 0,1,1,7\rangle$ has the minimal extension property in the class of all algebras with a proper non-associative binary operation.*

COROLLARY 6-1.1. *$\langle 0,1,1,7\rangle$ has the minimal extension property in the class of algebras with a commutative non-associative binary operation.*

Thus, our result generalizes the case of groupoids. Note that the sequence $\langle p_n(\mathfrak{B})\rangle$ is not the minimal extension of the sequence $\langle 0,1,1,7\rangle$ (see the section *Comments*).

Let S_n be the symmetric group on n letters $\{1,2,\dots,n\}$. For every essentially n-ary polynomial $p = p(x_1,x_2,\dots,x_n)$, recall that the *symmetric group* of p is

$$\mathrm{Inv}(p) = \{\alpha \in S_n \mid p = p^{\alpha}\},$$

where $p = p^\alpha$ means that $p(x_1, x_2, \ldots, x_n) = p(x_{1\alpha}, x_{2\alpha}, \ldots, x_{n\alpha})$. Define an equivalence relation $\sim$ over the set $\mathrm{Term}_n^e(\mathfrak{A})$ of all essentially n-ary polynomials as follows:

$$p \sim q \Longleftrightarrow p = q^\alpha \quad \text{for some } \alpha \in S_n.$$

Let $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m$ be all equivalence classes containing $f_1, f_2, \ldots, f_m$ as their representatives, respectively. Then

$$p_n(\mathfrak{A}) = |\mathrm{Term}_n^e(\mathfrak{A})| = \sum_{i=1}^{m} \frac{n!}{|\mathrm{Inv}(f_i)|}.$$

6-2. Proof of main result

Assume that algebra $\mathfrak{A} = \langle A; F\rangle$ represents $\langle 0, 1, 1, 7\rangle$ and the essentially binary term operation (polynomial) $x \cdot y$ is nonassociative. Then $x(yz)$, $y(zx)$, and $z(xy)$ are pairwise distinct essentially ternary polynomials. By J. Galuszka **[25]** or J. Dudek **[14]**, either the idempotent groupoid $\langle A; \cdot\rangle$ is *distributive Steiner*, i.e., it satisfies

$$(xy)y = x \text{ and } (xy)z = (xz)(yz), \tag{6-2.1}$$

or it is a *near semilattice* (i.e., it satisfies $(xy)y = xy$) and the following identities hold

$$((xy)z)x = (xy)z, \tag{6-2.2}$$

$$((xy)(yz))(zx) = ((yz)(zx))(xy), \tag{6-2.3}$$

$$(((xy)z)((yz)x))((zx)y) = (((yz)x)((zx)y))((xy)z), \tag{6-2.4}$$

$$((xy)z)((yz)x) = (xy)(yz) \tag{6-2.5}$$

since $p_3(\mathfrak{A}) < 9$. Furthermore, if $\langle A; \cdot\rangle$ is distributive Steiner, then

$$\mathrm{Term}_3^e(A; \cdot) = \{x(yz), y(zx), z(xy)\}. \tag{6-2.6}$$

If $\langle A; \cdot\rangle$ is a near semilattice, then

$$\begin{aligned}\mathrm{Term}_3^e(A; \cdot) = \{&x(yz), y(zx), z(xy), (xy)(xz),\\ &(yz)(yx), (zx)(zy), ((xy)(yz))(zx)\}\end{aligned} \tag{6-2.7}$$

Notice that for a non-distributive Steiner groupoid, we have at least nine essentially ternary polynomials

$$\begin{array}{ccc} (xy)z & (yz)x & (zx)y \\ (xy)(xz) & (yz)(yx) & (zx)(zy) \\ ((xy)(yz))(zx) & ((yz)(zx))(xy) & ((zx)(xy))(yz). \end{array}$$

The cancellation law holds for distributive Steiner groupoids.

Let $g_1(x,y,z)$, $g_2(x,y,z)$, $g_3(x,y,z)$, and $g_4(x,y,z)$ be the other four essentially ternary polynomials different from $x(yz)$, $y(zx)$, $z(xy)$ over $\mathfrak{A}$. We have five cases:

(1) all $\tilde{g}_i{}'s$ are equal;

(2) exactly three of $\tilde{g}_i{}'s$ are equal;

(3) two of $\tilde{g}_i{}'s$ are equal and the other two are equal too;

(4) two of $\tilde{g}_i{}'s$ are equal but the other two are not equal;

(5) none of the $\tilde{g}_i{}'s$ are equal.

The proof of Main Theorem will be completed in several steps. We will eliminate the cases (1), (3), (4), and (5).

6-2.1. Case (1) **is impossible.** If case (1) holds, then

$$7 = \frac{6}{|\mathrm{Inv}(x(yz))|} + \frac{6}{|\mathrm{Inv}(g_i)|} = 3 + \frac{6}{|\mathrm{Inv}(g_i)|},$$

$$|\mathrm{Inv}(g_i)| = \frac{6}{4},$$

which is impossible.

6-2.2. Case (2) **implies that** $\langle A; \cdot \rangle$ **is non-Steiner.** By case (2), we can assume that g_4 is symmetric and $|\mathrm{Inv}(g_i)| = 2$ for $i = 1, 2, 3$. Since

$$\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3,$$

$$g_i^{(jk)} \notin \{x(yz), y(xz), z(xy)\}$$

for any permutation (jk); so

$$g_i^{(jk)} \in \{g_1, g_2, g_3\}.$$

Just like in class $\mathbf{S_2}$, we have

LEMMA 6-2.1.

$$\mathrm{Inv}(g_1),\ \mathrm{Inv}(g_2)\ \textit{and}\ \mathrm{Inv}(g_3)$$

are pairwise distinct.

Thus, we can assume that

$$\mathrm{Inv}(g_1) = \langle(23)\rangle,\ \mathrm{Inv}(g_2) = \langle(13)\rangle,\ \mathrm{Inv}(g_3) = \langle(12)\rangle. \tag{6*}$$

Similarly to Lemma 4-1.1, we have

LEMMA 6-2.2.

$$\begin{aligned} g_1^{(12)} &= g_2, \quad g_1^{(13)} = g_3,\\ g_2^{(12)} &= g_1, \quad g_2^{(23)} = g_3,\\ g_3^{(13)} &= g_1, \quad g_3^{(23)} = g_2. \end{aligned} \tag{6-2.8}$$

PROOF. Obviously

$$g_1^{(12)} \notin \{x(yz), y(zx), z(xy), g_1, g_4\}.$$

If $g_1^{(12)} = g_3$, then

$$g_3(x,y,z) = g_1(y,x,z) = g_1(y,z,x) = g_3(z,y,x),$$

which implies $(13) \in \mathrm{Inv}(g_3)$, a contradiction. Therefore,

$$g_1^{(12)} = g_2.$$

By applying permutations to $g_1^{(12)} = g_2$, we obtain (6-2.8) immediately. □

If $\langle A;\cdot\rangle$ is Steiner, then it must be distributive (otherwise, $p_3(\mathfrak{A}) \geqslant 9$). Now we assume that $\langle A;\cdot\rangle$ is distributive Steiner. Consider $g_4(x,y,z)x$. We will find a contradiction.

If $g_4(x,y,z)x = x$, then by Steiner property

$$x = xx = (g_4x)x = g_4,$$

a contradiction. Similarly, by multiplying x,

$$g_4(x,y,z)x \notin \{y, z, xy, xz, yz, x(yz), y(xz), z(xy)\}.$$

If $g_4(x,y,z)x = g_2$, then

$$x = (xg_4)g_4 = g_2g_4,$$

but the left hand side is symmetric in x and z, a contradiction. Similarly, $g_4(x,y,z)x \neq g_3$.

Now let us prove that $g_4(x,y,z)x \neq g_1$. Assume that

$$g_4(x,y,z)x = g_1. \tag{6-2.9}$$

Then

$$x = (g_4x)g_4 = g_1g_4. \tag{6-2.10}$$

By applying permutations (xy) and (xz) to (6-2.9) and (6-2.10),

$$g_4(x,y,z)y = g_2, \quad g_4(x,y,z)z = g_3, \tag{6-2.11}$$

$$g_2g_4 = y, \quad g_3g_4 = z. \tag{6-2.12}$$

From (6-2.10),

$$xg_1 = (g_1g_4)g_1 = g_4. \tag{6-2.13}$$

From (6-2.11),

$$g_2(x,y,z)y = g_4, g_3(x,y,z)z = g_4. \tag{6-2.14}$$

Consider the ternary polynomials

$$g_1(g_2g_3),\ g_2(g_3g_1),\ \text{and}\ g_3(g_1g_2). \tag{6-2.15}$$

We will prove that the polynomials in (6-2.15) are essentially ternary. First of all, we have

$$g_ig_j \neq g_4 \text{ for all } i \neq j. \tag{6-2.16}$$

Obviously, $i \neq 4$ and $j \neq 4$ by the cancellation law. If $g_1g_2 = g_4$, then

$$g_2 = (g_1g_2)g_1 = g_1g_4.$$

Since $(g_1g_4)^{(23)} = g_1g_4$, $(23) \in \mathrm{Inv}(g_2)$, contradicting the assumption that

$$\mathrm{Inv}(g_2) = \langle(13)\rangle.$$

Similarly,

$$g_1g_3 \neq g_4 \text{ and } g_2g_3 \neq g_4.$$

If $g_1(g_2g_3) = x$, then

$$g_2g_3 = (g_1(g_2g_3))g_1 = xg_1 = g_4$$

by (6-2.13), contradicting (6-2.16). If $g_1(g_2g_3) = y$, then

$$g_2g_3 = (g_1(g_2g_3))g_1 = yg_1$$

so

$$yg_1 = g_2g_3 = zg_1$$

by permutation (yz). By the cancellation, $y = z$, a contradiction. Similarly,

$$g_1(g_2g_3) \notin \{z, xy, xz\}.$$

If $g_1(g_2g_3) = yz$, then

$$\begin{aligned}
(g_2g_3)(g_2g_3) &= g_2g_3 \\
&= (g_1(g_2g_3))g_1 && \text{(by Steiner property)} \\
&= (yz)g_1 \\
&= (yz)(xg_4) && \text{(by (6-2.9))} \\
&= ((yz)x)((yz)g_4) && \text{(by distributivity)} \\
&= (x(yz))((yg_4)(zg_4)) \\
&= (x(yz))(g_2g_3), && \text{(by (6-2.11))}
\end{aligned}$$

so by the cancellation law

$$g_2g_3 = x(yz).$$

Therefore

$$\begin{aligned} g_1 &= (g_1(x(yz)))(x(yz)) \\ &= (g_1(g_2g_3))(x(yz)) \\ &= (yz)(x(yz)) \\ &= x, \end{aligned}$$

a contradiction. Thus, $g_1(g_2g_3)$ is essentially ternary and then so are $g_2(g_3g_1)$ and $g_3(g_1g_2)$.

Note that $g_1(g_2g_3) = g_4$ is impossible. Indeed, If

$$g_1(g_2g_3) = g_4.$$

Then by the symmetry of g_4

$$g_1(g_2g_3) = g_2(g_3g_1) = g_4,$$

so

$$\begin{aligned} g_1g_1 &= g_1 \\ &= (g_1(g_2g_3))(g_2g_3) \\ &= (g_2g_3)g_4 \\ &= (g_2g_3)(g_2(g_1g_3)) \\ &= ((g_2g_3)g_2)((g_2g_3)(g_1g_3)) \\ &= g_3(g_3(g_1g_2)) && \text{(by Steiner property and distributivity)} \\ &= g_1g_2, \end{aligned}$$

and then

$$g_1 = g_2,$$

a contradiction.

Now we claim that the polynomials in (6-2.15) are not in $\{x(yz), y(zx), z(xy)\}$. If $g_1(g_2g_3) = x(yz)$ then by permutations

$$g_2(g_3g_1) = y(zx),\ g_3(g_1g_2) = z(xy).$$

Thus,

$$\begin{aligned}
(g_2g_3)(g_2g_3) &= g_2g_3 \\
&= (g_1(g_2g_3))g_1 && \text{(by Steiner property)} \\
&= g_1(x(yz)) \\
&= (xg_1)((g_1y)(g_1z)) \\
&= g_4((g_1y)(g_1z)) && \text{(by (6-2.13))} \\
&= ((g_4g_1)(g_4y))((g_4g_1)(g_4z)) && \text{(by distributivity)} \\
&= ((g_4g_1)g_2)((g_4g_1)g_3) && \text{(by (6-2.11))} \\
&= (g_1g_4)(g_2g_3), && \text{(by distributivity)}
\end{aligned}$$

so, by the cancellation law,

$$g_2g_3 = g_1g_4,$$

then

$$x(yz) = g_1(g_2g_3) = g_1(g_1g_4) = g_4,$$

a contradiction. If $g_1(g_2g_3) = z(xy)$, then by permutation (xy)

$$g_1(g_2g_3) = z(xy) = g_2(g_1g_3),$$

so by permutation (yz)

$$g_1(g_2g_3) = g_3(g_1g_2).$$

Therefore, $g_1(g_2g_3)$ is symmetric and then by $p_3(\mathfrak{A}) = 7$

$$x(yz) = g_1(g_2g_3) = g_4,$$

a contradiction. If $g_1(g_2g_3) = y(zx)$, then

$$y(xz) = g_1(g_2g_3) = z(xy)$$

by permutation (yz), which is impossible by non-associativity. Thus, $g_1(g_2g_3)$ is not in $\{x(yz), y(zx), z(xy)\}$. This implies that $g_2(g_3g_2)$ and $g_3(g_1g_2)$ are not in it either.

From the arguments above, if $g_4x = g_1$, then $p_3(\mathfrak{A}) \geqslant 10$, a contradiction.

Thus, g_4x must be a constant polynomial, contradicting the fact that $p_0(\mathfrak{A}) = 0$.

6-2.3. Case (3) **is impossible.** Without loss of generality, assume that $\tilde{g}_1 = \tilde{g}_2$ and $\tilde{g}_3 = \tilde{g}_4$. Then

$$|\mathrm{Inv}(g_1)| = |\mathrm{Inv}(g_2)| = 3 \text{ and } |\mathrm{Inv}(g_3)| = |\mathrm{Inv}(g_4)| = 3,$$

$$\mathrm{Inv}(g_1) = \mathrm{Inv}(g_2) = \mathrm{Inv}(g_3) = \mathrm{Inv}(g_4) = \langle(123)\rangle,$$

and

$$g_1(x,z,y) = g_2(x,y,z),\ g_3(x,z,y) = g_4(x,y,z).$$

Consider $h_1(x,y,z) = g_1(x,y,z)g_2(x,y,z)$ and $h_2(x,y,z) = g_3(x,y,z)g_4(x,y,z)$. Then they are essentially symmetric ternary, so

$$p_3(\mathfrak{A}) \geqslant 8,$$

a contradiction. Indeed, If $\langle A;\cdot\rangle$ is distributive Steiner, then, by (6-2.6),

$$p_3(\mathfrak{A}) \geqslant 8.$$

If $\langle A;\cdot\rangle$ is near semilattice, then, by (6-2.7), $p_3(\mathfrak{A}) \geqslant 11$.

6-2.4. Cases (4) **and** (5) **are impossible.** We only prove that the case (4) is impossible. The proof of case (5) is similar. By condition, we can assume that $|\mathrm{Inv}(g_1)| = |\mathrm{Inv}(g_2)| = 3$ and $|\mathrm{Inv}(g_3)| = |\mathrm{Inv}(g_4)| = 6$. Consider $\langle A;\cdot\rangle$. If it is near semilattice, then there are seven essentially ternary polynomials as in (6-2.7) and the last one $((xy)(yz))(zx)$ is symmetric (since $p_3(A;\cdot) < 9$). Now we have at least two essentially symmetric ternary polynomials by assumption and then $p_3(\mathfrak{A}) \geqslant 8$, contradicting the assumption that $p_3(\mathfrak{A}) = 7$.

Thus, $\langle A; \cdot \rangle$ must be distributive Steiner. Then the set of ternary polynomials over $\langle A; \cdot \rangle$ is

$$\mathrm{P}^{(3)}(A; \cdot) = \{x, y, z, xy, xz, yz, x(yz), y(zx), z(xy)\}.$$

Consider g_3x. If $g_3x = x$, then

$$g_3 = (g_3x)x = xx = x,$$

a contradiction. The same proof implies that

$$g_3x \notin \mathrm{P}^{(3)}(A; \cdot).$$

If $g_3x = g_1$, then

$$x = (g_3x)g_3 = g_1g_3,$$

but the right hand side satisfies that $(g_1g_3)^{(123)} = g_1g_3$, a contradiction. Similarly,

$$g_3x \notin \{g_2, g_3, g_4\}.$$

By the above arguments, g_3x is a constant, contradicting the assumption that $p_0(\mathfrak{A}) = 0$.

From the above results, only case (2) is true. Therefore, $\langle A; \cdot \rangle$ is a near semilattice. Since $p_3(A; \cdot) < 9$, there are only seven essentially ternary polynomials over $\langle A; \cdot \rangle$, i.e., (6-2.7). Now, since $p_3(\mathfrak{A}) = 7$, every g_i is some one in (6-2.7) for $i = 1, 2, 3, 4$. Thus, by the result of J. Galuszka ([**25**]),

$$p_n(\mathfrak{B}) \leqslant p_n(A; \cdot) \leqslant p_n(\mathfrak{A}),$$

for $n \geqslant 0$. Since Park algebra $\mathfrak{B}$ is in the algebraic class considered, our proof is complete.

6-3. Comments

The p_n-sequence $\langle p_n(\mathfrak{B})\rangle$ of Park four-element algebra $\mathfrak{B}$ is not the minimal extension of the sequence $\langle 0,1,1,7\rangle$ though we have proved that it is the minimal extension of the sequence $\langle 0,1,1,7\rangle$ in the class of all algebras with a proper non-associative binary operation. In K. M. Koh [45], there is an algebra with a semilattice binary operation that represents $(0,1,1,7,61)$. However, $p_4(\mathfrak{B}) = 232$ by using the program written by J. Berman. Thus, the open problem is reduced to considering the class of algebras with a semilattice operation.

Part 3

Other results

CHAPTER 7

Binary term operation for the sequence $\langle 0,1,1,5\rangle$

Suppose that algebra $\mathfrak{A} = \langle A; F\rangle$ represents p_n-sequence $\langle 0,1,1,5\rangle$. Then there is one and only one essentially binary term operation $x \cdot y$. We will prove that " $\cdot$ " is associative. **Assuming the contrary, $(xy)z, (yz)x$ and $(zx)y$ are pairwise essentially ternary polynomials.** Suppose that h_1 and h_2 are the other two essentially ternary polynomials. We have the following two cases:

$$h_1 \notin \mathrm{Inv}(h_2), \tag{7.1}$$

$$h_1 \in \mathrm{Inv}(h_2). \tag{7.2}$$

LEMMA 7-0.1. *The case (7.2) is impossible.*

PROOF. If $h_1 \in \mathrm{Inv}(h_2)$, then

$$\frac{6}{|\mathrm{Inv}((xy)z)|} + \frac{6}{|\mathrm{Inv}(h_i)|} = 5,$$

so

$$|\mathrm{Inv}(h_i)| = 3.$$

Since S_3 has only one subgroup of degree 3, $\langle (123)\rangle$,

$$\mathrm{Inv}(h_i) = \langle (123)\rangle.$$

Also, since $|\mathrm{Inv}((xy)z)| = 2$,

$$h_1(x,y,z) = h_2(x,z,y).$$

Notice that

$$h(x,y,z) = h_1(x,y,z)h_2(x,y,z)$$

is essentially symmetric ternary polynomial. Therefore, $p_3 \geqslant 6$, contradicting the assumption that $p_3 = 5$. $\square$

Thus, case (7.1) holds.

LEMMA 7-0.2. *If "$\cdot$" is nonassociative, then both of h_1 and h_2 are symmetric.*

PROOF. By (7.1),

$$\frac{6}{|\mathrm{Inv}((xy)z)|}+\frac{6}{|\mathrm{Inv}(h_1)|}+\frac{6}{|\mathrm{Inv}(h_2)|}=5,$$

so

$$\frac{6}{|\mathrm{Inv}(h_1)|}+\frac{6}{|\mathrm{Inv}(h_2)|}=2.$$

Since

$$1 \leqslant |\mathrm{Inv}(h_i)| \leqslant 6,$$

$$|\mathrm{Inv}(h_i)| = 6$$

so h_i are symmetric. $\square$

LEMMA 7-0.3. *$h_i(x,y,y) = xy,$, for $i = 1,2$.*

PROOF. If $h_1(x,y,y) = x$ or y, then

$$h_1(x,y,z), \quad h_1(x,y,z)x, \quad h_1(x,y,z)y, \quad h_1(x,y,z)z \tag{7-0.3.1}$$

are pairwise essentially ternary polynomials (the proof is the same as the case $\langle 0,1,1,4\rangle$. We only use the symmetry there). So they are not totally symmetric. Next, we claim that

$$\begin{aligned} h_1(x,y,z)x &= x(yz), \\ h_1(x,y,z)y &= y(xz), \\ h_1(x,y,z)z &= z(xy). \end{aligned} \tag{7-0.3.2}$$

Since $h_i(x,y,z)$ are symmetric, they are different from

$$h_1(x,y,z)x, \quad h_1(x,y,z)y, \quad h_1(x,y,z)z.$$

Since both $h_1(x,y,z)x$ and $x(yz)$ are symmetric in y and z, $h_1(x,y,z)x = x(yz)$. The same reason implies that

$$h_1(x,y,z)y = (zx)y, \quad h_1(x,y,z)z = (xy)z.$$

That is, we have (7-0.3.2).

However, if $h_1(x,y,y) = x$, then, letting $z = y$ in $h_1(x,y,z)x = x(yz)$,

$$xy = x(yy) = h_1(x,y,y)x = xx = x,$$

a contradiction; if $h_1(x,y,y) = y$, then, by (7-0.3.2),

$$y = yy = h_1(x,y,y)y = h_1(y,y,x)y = (yx)y = (xy)y,$$

so

$$xy = ((xy)y)(xy) = y(xy) = y,$$

a contradiction.

Similarly, $h_2(x,y,y) = xy$. □

Since $p_3 < 7$, $\langle A; \cdot\rangle$ is distributive Steiner by Dudek's result [**14**], i.e., we have

LEMMA 7-0.4. $(xy)y = x, \quad (xy)z = (xz)(yz)$.

Notice that any one in (7-0.3.3) implies the other two and $h_i(x,y,z)x$ are essentially ternary by Lemma 7-0.3. We have

LEMMA 7-0.5. *For fixed* $i = 1,2$, *we have that*

$$h_i(x,y,z)x = h_1(x,y,z)$$

or that

$$h_i(x,y,z)x = h_2(x,y,z)$$

or that

$$h_i(x,y,z)x = x(yz), \quad h_i(x,y,z)y = (zx)y, \quad h_i(x,y,z)z = (xy)z.$$

LEMMA 7-0.6. *$h_1(x,y,z)x = h_2(x,y,z)$ and $h_2(x,y,z)x = h_1(x,y,z)$.*

PROOF. If $h_1(x,y,z)x = h_2(x,y,z)$, then

$$h_2(x,y,z)x = (h_1(x,y,z)x)x = h_1(x,y,z)$$

by Lemma 7-0.4. These two identities are equivalent.

If $h_1(x,y,z)x = h_1(x,y,z)$, then

$$x = (h_1(x,y,z)x)h_1(x,y,z) = h_1(x,y,z)h_1(x,y,z) = h_1(x,y,z)$$

by Lemma 7-0.4, a contradiction. Similarly,

$$h_2(x,y,z)x \neq h_2(x,y,z).$$

If $h_1(x,y,z)x = x(yz)$, then

$$h_1(x,y,z) = (h_1(x,y,z)x)x = (x(yz))x = yz$$

by Lemma 7-0.4, a contradiction. Similarly,

$$h_1(x,y,z)x \neq y(zx), \quad z(xy).$$

□

Now we prove our main theorem in this chapter.

THEOREM 7-0.1. *If algebra $\mathfrak{A}$ represents $\langle 0,1,1,5\rangle$, then the essentially binary term is associative.*

PROOF. Suppose the contrary. Consider $h(x,y,z) = h_1(x,y,z)h_2(x,y,z)$. Then it is essentially ternary. Thus, by previous lemmas and assumption,

$$h(x,y,z) = h_1(x,y,z) \text{ or } h_2(x,y,z).$$

If $h(x,y,z) = h_1(x,y,z)$, then

$$h_2(x,y,z) = (h_2(x,y,z)h_1(x,y,z))h_1(x,y,z) = h_1(x,y,z)h_1(x,y,z) = h_1(x,y,z)$$

by Lemma 7-0.4, a contradiction. Similarly,

$$h(x,y,z) \neq h_2(x,y,z).$$

□

CHAPTER 8

Binary term operation for the sequence $\langle 0,1,1,6\rangle$

The main result of this chapter is

THEOREM 8-0.2. *If algebra $\mathfrak{A}$ represents $\langle 0,1,1,6\rangle$, then the essentially binary term operation is associative.*

The following lemmas will complete the proof of the theorem. Suppose the contrary. Then $x(yz)$, $y(zx)$ and $z(xy)$ are pairwise different essentially ternary. By J. Galuszka [**25**], we can assume that " $\cdot$ " is distributive Steiner, i.e., it satisfies

$$(xy)y = x \quad \text{and } (xy)z = (xz)(yz). \tag{8.1}$$

Let $g_1(x,y,z)$, $g_2(x,y,z)$ and $g_3(x,y,z)$ be the other three essentially ternary term operations different from the above. We have three cases:

(1) all $\tilde{g}_i's$ are equal;

(2) exactly two of $\tilde{g}_i's$ are equal;

(3) none of the $\tilde{g}_i's$ are equal.

The following lemmas show that these three cases are impossible under the assumption that " $\cdot$ " is non associative. Therefore, the proof of the theorem is complete.

LEMMA 8-0.7. *Case (2) is impossible.*

PROOF. By case (2), assume that $\tilde{g}_2 = \tilde{g}_3$. Then

$$|\mathrm{Inv}(g_2)| = |\mathrm{Inv}(g_3)| = 3 \quad \text{and } |\mathrm{Inv}(g_1)| = 6.$$

Since g_2g_3 is symmetric, $g_2g_3 = g_1$. By (8.1),

$$g_2 = (g_2g_3)g_3 = g_1g_3$$

and

$$g_3 = (g_2g_3)g_2 = g_1g_2.$$

First, we claim that

$$g_1(x,y,y) = xy. \tag{8.2}$$

Otherwise, similarly to case of sequence $\langle 0,1,1,4\rangle$, the following

$$g_1(x,y,z), \quad g_1(x,y,z)x,$$
$$g_1(x,y,z)y, \quad g_1(x,y,z)z$$

are pairwise distinct and essentially ternary.

Consider $g_1(x,y,z)x$. If

$$g_1(x,y,z)x = g_2(x,y,z),$$

then, by permutation (xyz),

$$g_1(x,y,z)z = g_1(y,z,x)z = g_2(x,y,z) = g_1(x,y,z)x,$$

a contradiction to the previous result. Similarly,

$$g_1(x,y,z)x \neq g_3(x,y,z).$$

So, similarly to case on $\langle 0,1,1,4\rangle$,

$$g_1(x,y,z)x = x(yz).$$

Therefore, by (8.1),

$$g_1(x,y,z) = (g_1(x,y,z)x)x = (x(yz))x = yz,$$

a contradiction. So (8.2) holds.

Thus, $g_1(x,y,z)x$, $g_1(x,y,z)y$ and $g_1(x,y,z)z$ are essentially ternary. If $g_1(x,y,z)x = g_1(x,y,z)y$, then $g_1(x,y,z)$ is symmetric. Therefore, $g_1(x,y,z)x = g_1(x,y,z)$ since $p_3 = 6$. It follows that

$$x = (g_1(x,y,z)x)g_1(x,y,z) = g_1(x,y,z)g_1(x,y,z) = g_1(x,y,z),$$

a contradiction. If $g_1(x,y,z)x \neq g_1(x,y,z)y$, then

$$g_1(x,y,z), \quad g_1(x,y,z)x, \quad g_1(x,y,z)y, \quad g_1(x,y,z)z \tag{8.3}$$

are pairwise distinct. Similarly to the case of the sequence $\langle 0,1,1,4\rangle$,

$$g_1(x,y,z)x = x(yz), \quad g_1(x,y,z)y = y(zx), \quad g_1(x,y,z)z = z(xy).$$

By (8.1),

$$g_1(x,y,z) = (g_1(x,y,z)x)x = (x(yz))x = yz,$$

a contradiction. □

LEMMA 8-0.8. *Case (3) is impossible.*

PROOF. By case (3), all $g_i's$ are symmetric. Similar to Lemma 8-0.7, the following polynomials

$$g_1(x,y,z), \quad g_1(x,y,z)x,$$
$$g_1(x,y,z)y, \quad g_1(x,y,z)z$$

are pairwise distinct and essentially ternary, so $g_1(x,y,z)x$ is only symmetric in y and z. Thus,

$$g_1(x,y,z)x = x(yz)$$

which implies $g_1(x,y,z) = yz$, a contradiction. □

LEMMA 8-0.9. *Case (1) is impossible.*

PROOF. By (1), $|\mathrm{Inv}(g_i)| = 2$ for $i = 1,2,3$. First, for any permutation (ij),

$$g_k^{(ij)} \notin \{x(yz), y(xz), z(xy)\}.$$

For example, suppose that $\mathrm{Inv}(g_1) = \langle (23)\rangle$. If $g_1(y,x,z) = x(yz)$, then interchanging x and z,

$$z(xy) = g_1(y,z,x) = g_1(y,x,z) = x(yz),$$

a contradiction. Similarly, $g_1(y,x,z) \neq z(xy)$. If $g_1(y,x,z) = y(xz)$, then

$$g_1(x,y,z) = x(yz),$$

a contradiction.

Similarly to the case of the sequence $\langle 0,1,1,4\rangle$, we can assume that

$$\mathrm{Inv}(g_1) = \langle(23)\rangle, \quad \mathrm{Inv}(g_2) = \langle(13)\rangle, \quad \mathrm{Inv}(g_3) = \langle(12)\rangle.$$

Thus, interchanging x and y,

$$g_1(y,x,z) = g_2(x,y,z) \text{ or } g_3(x,y,z).$$

Without loss of generality, assume that $g_1(y,x,z) = g_2(x,y,z)$. Then we have (1), (2) and (3) of Lemma 4-1.1.

Consider $g_1(x,y,z)x$. If $g_1(x,y,z)x = g_2(x,y,z)$, then

$$g_2(x,z,y) = g_1(x,z,y)x = g_1(x,y,z)x = g_2(x,y,z)$$

which implies $(23) \in \mathrm{Inv}(g_2)$, a contradiction. Similarly,

$$g_1(x,y,z)x \neq g_3(x,y,z).$$

If $g_1(x,y,z)x = g_1(x,y,z)$, then, by (8.1),

$$x = (g_1x)g_1 = g_1g_1 = g_1,$$

a contradiction. Similarly,

$$g_1(x,y,z)x \notin \{y,z,xy,yz,xz\}.$$

If $g_1(x,y,z)x = x$, then

$$g_1 = (g_1x)x = xx = x,$$

a contradiction. If $g_1(x,y,z)x = x(yz)$, then

$$g_1 = (g_1x)x = (x(yz))x = yz,$$

a contradiction. If $g_1(x,y,z)x = y(xz)$, then

$$y(xz) = g_1(x,y,z)x = g_1(x,z,y)x = z(xy),$$

contradicting the non-associativity. Similarly, $g_1(x,y,z)x \neq z(xy)$. $\square$

CHAPTER 9

Two results on Koh's and Dudek's

Here we find some result about the p_n-sequence $\langle 0,1,1,2\rangle$ and a simple proof of Dudek's result.

9-1. Koh's result on $\langle 0,1,1,2\rangle$

In [**45**], K. M. Koh proved that *there exist two varieties $\mathfrak{K}_1$ and $\mathfrak{K}_2$ of type $\langle 2,3\rangle$ such that a ternary algebra $\mathfrak{A}$ represents sequence $\langle 0,1,1,2\rangle$ if and only if $\mathfrak{A}$ is equivalent to an algebra with more than one element belonging to $\mathfrak{K}_1$ or $\mathfrak{K}_2$.* The version here can be found in G. Grätzer and A. Kisielewicz [**29**]. In Koh's paper, eight basic examples were given, four in $\mathfrak{K}_1$ and the other four in $\mathfrak{K}_2$, denoted by $I(j)$ and $II(j)$, $j = 1,2,3,4$, respectively (pp. 22–25 K. M. Koh in [**45**]). He proved the following result:

Let algebra $\mathfrak{A}$ of type $\langle 2,3\rangle$ represent $\langle 0,1,1,2\rangle$. If $\mathfrak{A} \in \mathfrak{K}_1$, then $\mathfrak{A}$ contains one of the $I(j)'s$ ($j = 1,2,3,4$) as a subalgebra. If $\mathfrak{A} \in \mathfrak{K}_2$, then $\mathfrak{A}$ contains one of the $II(j)'s$ ($j = 1,2,3,4$) as a subalgebra.

Now we conclude that the converse of the result above is also true.

LEMMA 9-1.1. *$I(j) \notin \mathfrak{K}_2$ and $II(j) \notin \mathfrak{K}_1$, for every $j = 1,2,3,4$.*

THEOREM 9-1.1. *Let $\mathfrak{A} = \langle A, \cdot, f\rangle$ be an algebra of type $\langle 2,3\rangle$ which represents $\langle 0,1,1,2\rangle$. Then $\mathfrak{A} \in \mathfrak{K}_1$ if and only if $\mathfrak{A}$ contains one of the $I(j)'s$ ($j = 1,2,3,4$) as a subalgebra. $\mathfrak{A} \in \mathfrak{K}_2$ if and only if $\mathfrak{A}$ contains one of the $II(j)'s$ ($j = 1,2,3,4$) as a subalgebra.*

PROOF. Let $\mathfrak{A}$ contain one of the $I(j)'s$. We consider two cases.

case 1. xy is not essentially binary:

This is impossible. For instance, if $xy = x$, then $xy = x$ hold in $I(j)$ for some j, which implies $d = ab = a$.

case 2. xy is essentially binary.

Since $p_2 = 1$, $xy = yx$. Since $p_3 = 2$, $f(x,y,z)$ is essentially ternary and $f(x,y,z) \neq xyz$. By the argument of Koh's paper from page 4 to page 15, $f(x,y,z)x$ is essentially ternary and

$$f(x,y,z)x = \begin{cases} xyz, \\ f(x,y,z). \end{cases}$$

Therefore, $\mathfrak{A} \in \mathfrak{K}_1$ or $\mathfrak{A} \in \mathfrak{K}_2$. If $\mathfrak{A} \in \mathfrak{K}_2$, then $I(j) \in \mathfrak{K}_2$ for some j, which contradicts to the Lemma 9-1.1. So $\mathfrak{A} \in \mathfrak{K}_1$.

Similarly, if $\mathfrak{A}$ contains one of the $II(j)'s$, then $\mathfrak{A} \in \mathfrak{K}_2$. □

9-2. A simple proof of Dudek's result

In [7], Dudek has proved that *for every algebra* $\mathfrak{A} = \langle A; F \cup \{\cdot\}\rangle$ *having a binary communicatative idempotent and non-associative operation, for every positive integer* n*,*

$$p_{n+1}(\mathfrak{A}) \geqslant p_n(\mathfrak{A}) + \frac{2}{3}(2^{n-1} - (-1)^{n-1}).$$

Now we give a simple proof of this result.

As in the first comment of Dudek's proof, it is very easy to show that the sequence a_n, being the number of all essentially n-ary term operations (i.e., polynomials) on $\mathfrak{A}$ which are not term operations on the algebra $\langle A; \cdot\rangle$, is non-decreasing. Consequently, it is enough to prove that the inequality

$$p_{n+1}(A;\cdot) \geqslant p_n(A;\cdot) + \frac{2}{3}(2^{n-1} - (-1)^{n-1}).$$

holds for every n. Obviously, this is true for $n = 1$. For $n \geqslant 2$, by Theorem 1 in G. Grätzer and R. Padmanabhan [**30**],

$$p_n(A;\cdot) \geqslant q_n,$$

where $q_n = \frac{1}{3}(2^n - (-1)^n)$. Also, by (13) in G. Grätzer and R. Padmanabhan [**30**] [**30**],

$$p_{n+1}(A;\cdot) \geqslant p_n(A;\cdot) + 2p_{n-1}(A;\cdot)$$

so

$$\begin{aligned} p_{n+1}(A;\cdot) &\geqslant p_n(A;\cdot) + 2q_{n-1} \\ &= p_n(A;\cdot) + \frac{2}{3}(2^{n-1} - (-1)^{n-1}), \end{aligned}$$

completing the proof.

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