DELTA FUNCTIONS, INTEGRATION BY PARTS, AND CALCULUS OF VARIATIONS

by

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ABSTRACT

Integration by parts is analyzed by means of delta function methods. Differentiation is done by convolution against the derivative of the delta function. Single and double integration by parts between finite and infinite limits are done with delta functions. Fundamental Theorem of Calculus is discussed.

Calculus of variations is done using delta functions instead of integration by parts. Taylor's theorem is written out with delta functions, to illustrate a duality with moment expansions.

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CHAPTER 1.

Introduction

In Professor Finlayson's course on differential equations, he points out various situations in which an operation on a continuous function can be compared to a simple vector or matrix calculation. The most basic example is the vector dot product. If $\vec{a} \bullet \vec{b} = 0$, then two vectors $\vec{a}$ and $\vec{b}$ are orthogonal. Similarly, if $\int_{-\infty}^{\infty} f(x)g(x)dx = 0$, then f and g can be said to be orthogonal. It therefore makes sense to call the above integral expression the "dot product of f and g". (We will sometimes drop the integral sign and write "$f \bullet g$".)

Under the Fourier Transformation, the dot product of two functions is an invariant. Power calculations are a good example. If the total power delivered to a load is given by $\int_{-\infty}^{\infty} v(t)i(t)dt$, do we not expect to get the same -∞ value when we calculate power in the frequency domain, i.e., $\int_{-\infty}^{\infty} V(s)I^*(s)ds$? We should therefore be free to write "$v \bullet i$", without specifying time domain or frequency domain.

Quantum mechanics provides another example. In Dirac's representation, we find the scalar quantity $<\alpha | \beta>$, indicating a calculation to be done on two state vectors. If we represent the particle in position space, then $\alpha(x)$ and $\beta(x)$ are Schroedinger wave functions, and $<\alpha | \beta>$ is the dot product of α and β. If we take Fourier Transforms of α and β, we get $A(p)$ and $B(p)$, which correspond to representation of the same particle in momentum space. The dot product of A and B should evaluate the same physical quantity as the dot product of α and β, because the only thing different is the set of basis functions with which we choose to describe the particle. Therefore, $\alpha \bullet \beta$ should be invariant under Fourier Transforms.

Why is the dot product important for vectors? It evaluates a quantity which is independent of the basis used to represent the vectors. For example, work is (force) $\times$ (displacement). Without dot products, we can express this mathematically only by choosing a coordinate system and writing $W = f_x dx + f_y dy + f_z dz$. The significance of work is much more clearly shown by writing it as $w = \vec{f} \bullet d\vec{s}$, where the calculation does not seem to depend on choice of coordinates. This example suggests that those properties of functions which have the deepest physical significance should be those which do not depend on the basis we use to describe the functions.

Under this criterion, is the derivative of a function physically significant? We can indeed evaluate the derivative of a function $f(t)$ at the origin by taking $f(t) \bullet \delta'(t)$. This same dot product can be evaluated for different representations of f. For example, if we take Fourier transforms, then the calculation becomes $F(s) \bullet s$. In this case, we can identify the first moment of the transformed function with the derivative of the original function.

This example does not prove that the derivative is a physically significant quantity. It does, however, show that a particular quantity, which we all believe is significant, can in fact be expressed in a form, i.e., the dot product, which does not require us to explicitly state the system of representation, or basis, with which we choose to describe the function.

In Chapter 4 of this thesis, we pursue the connection between derivatives and moments, and show that the description of a function in terms of its Taylor Series is related to the description of the Fourier Transform in terms of its moment expansion.

Chapter 3 of this thesis describes a typical problem in variational calculus. The technique of solution calls for the conversion of an integral equation into a differential equation. Ordinarily, this step is accomplished via integration by parts. The author first solved the problem in question by a self-taught method which seemed to avoid integration by parts, using instead a delta function interpretation of the perturbing function as suggested by Feynman. The success of this method on this particular problem led the author to work out solutions to several other classical variational problems, notably, the catenary and brachistochrone, using the same technique. Finally, it seemed that it would be worthwhile to investigate the connection between delta functions and integration by parts. This became the subject of Chapter 2 of the thesis. The result is to present a mathematical development of integration by parts, which justifies the results shown in Chapters 3 and 4.

In fact, Chapter 2 goes into considerably more depth than is required for the subsequent sections, in order to present a reasonably complete picture of what we might call the "operational" view of integration by parts. Section 2.1

outlines the connection between delta functions and differentiation, as stated in various standard works. Section 2.2 uses this machinery to illustrate the "weak form" of integration by parts,i.e., when the calculation is between infinite limits. Here we point out conceptual linkages with quantum mechanics and linear algebra.

Section 2.3 expands the scope of Section 2.2 by introducing the characteristic function on an interval to allow us to handle cases of integration by parts between finite limits. Section 2.4 shows the generalization to multiple execution of integration by parts.

Section 2.5 goes back to the fundamental Theorem of Calculus. In showing how to do this with delta functions, an extension is clearly suggested to functions of several variables; in particular, with regard to the divergence theorem, which is the 2-d analog of the Fundamental Theorem.

Throughout this thesis, we will sometimes want to refer to first or higher order derivatives of the delta function. To facilitate readability, we will make use of the following convention. "The delta function" will indicate the delta function itself, while "delta functions" (without the definite article) will encompass the delta function and its derivatives. In some cases, we will use "the delta function" to indicate a specific derivative, where the meaning is clear from context.

CHAPTER 2.

Integration by Parts

2.1. Differentiation

The remarks in this section correspond to the ideas developed in Carrier and Pearson (1976) and other standard works. Since we will be taking derivatives of the delta function, we will define it in terms of a normalized Gaussian whose variance approaches zero, as follows:

$$\delta(x) \triangleq \lim_{k \to \infty} \frac{k}{\sqrt{2\pi}} e^{-(kx)^2/2} \tag{1}$$

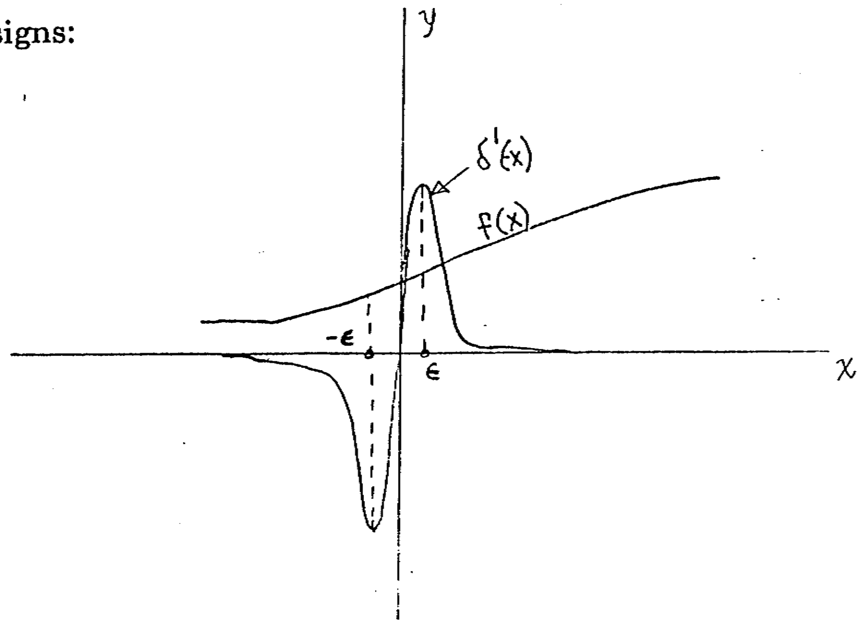
We are familiar with the fact that when a function is multiplied by the delta function and integrated, the result is the value of that function at the origin, i.e.:

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0) \tag{2}$$

It is also true, but less widely recognized, that a similar result arises when the derivative of the delta function is used.

$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx = -f'(0) \tag{3}$$

This can be proven for suitable f via integration by parts. But it is more easily remembered by thinking of $\delta'(x)$ as two large spikes very close together with opposite signs:



When we take the dot product of these two functions, we can think of the first spike as evaluating f at $-\varepsilon$, the second spike evaluating f at $+\varepsilon$. Eq. (3) is plausible if we recall, from the definition of the derivative, that:

$$f'(0) \cong \frac{f(\varepsilon) - f(-\varepsilon)}{2\varepsilon} \tag{4}$$

It is natural to generalize these ideas to higher derivatives, i.e.:

$$f^n(0) = \int_{-\infty}^{\infty} \delta^n(-x) f(x)dx \tag{5}$$

Furthermore, we are not constrained to evaluate derivatives only at the origin. If we replace the argument of the delta function by c-x, we evaluate the derivative at c. In this case, we need only integrate between finite limits a and b such that $a < c < b$:

$$f^n(c) = \int_a^b \delta^n(c-x)f(x)dx \tag{6}$$

If we call our displacement ξ instead of c and integrate over all x, then we will think of the result as a function rather than a single value. In general, then, the act of differentiation can be viewed as an integral operation, as follows:

$$f^n(\xi) = \int_{-\infty}^{\infty} \delta^n(\xi-x)f(x)dx \tag{7}$$

We will also make use of the equivalent formula with the variables reversed:

$$f^n(x) = \int_{-\infty}^{\infty} \delta^n(x-\xi)f(\xi)d\xi \tag{8}$$

2.2 Integration by Parts (between infinite limits)

For integrable functions f and g which approach zero sufficiently smoothly as $x \to \pm\infty$, integration by parts tells us that we can differentiate whichever function we please when taking dot products:

$$\int_{-\infty}^{\infty} f(x)g'(x)dx = -\int_{-\infty}^{\infty} f'(x)g(x)dx \tag{9}$$

It is amusing to see how this calculation looks when using identities (7) and (8). We can write $g'(x)$ as follows:

$$g'(x) = \int_{-\infty}^{\infty} \delta'(x-\xi)g(\xi)d\xi \tag{10}$$

Then we can write $f \bullet g'$ as follows:

$$\int_{-\infty}^{\infty} f(x)g'(x)dx = \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \delta'(x-\xi)g(\xi)d\xi dx \tag{11}$$

If the functions are reasonably well behaved, we can execute the double integration in whichever order we please. Consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\delta'(x-\xi)g(\xi)dxd\xi \tag{12}$$

If we operate toward the right with the delta function, we get back to where we started. If we operate toward the left, using identity (7), we get:

$$\int_{-\infty}^{\infty} - f'(\xi)g(\xi)d\xi \tag{13}$$

We have moved the operation of differentiation from g to f. (Notice that the need for the sign reversal follows from the parity of the delta operator.)

At this point, the reader should note the similarity between (12) and the following examples:

(i) integration by parts: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\delta'(x-\xi)g(\xi)dxd\xi$ (12)

(ii) vector products: $(f_1, f_2, f_3 \dots f_n) \begin{pmatrix} D_{11} & D_{12} & \cdot\ \cdot \\ D_{21} & \cdot\ \cdot\ \cdot & \\ \cdot\ \cdot\ \cdot & & \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ g_n \end{pmatrix}$ (14)

(iii) Dirac's bra-ket notation for quantum mechanics: $<f|D|g>$ (15)

Notice that in all cases the "operator" can work either to the right or left. Dirac's system was, in fact, developed to describe situations which might be evaluated either as matrices (14) or integrals (12), depending on the choice of how the physical system is described. The bra-ket notation does not require the user to specify whether he is referring to vectors (finite/discrete basis) or functions (infinite/continuous basis).

2.3 Integration by Parts between Finite Limits

We have seen how integration by parts can be done between infinite limits. How can we handle the case of finite limits?

$$\int_a^b f(x)g'(x)dx \tag{16}$$

We can think of this as being an integral of type (12) if we make use of the so-called "characteristic function", char(x,a,b), defined as:

$$\text{char}\,(x,a,b,) = \begin{cases} 0 \text{ outside } (a,b) \\ 1 \text{ inside } (a,b) \end{cases} \tag{17}$$

We will find it useful to designate this function in the following suggestive way:

$$\text{char}(x,a,b) = \underset{a \qquad b}{\_\!\sqcap\!\_}$$

(We will leave out the limit markers a and b when they are clear from context. In this notation, (16) can be written as:

$$\int_{-\infty}^{\infty} \underset{a \qquad b}{\_\!\sqcap\!\_} \cdot f(x)g'(x)dx \tag{19}$$

We can now write this out like Eq. (12) as a double integral:

$$\int_{-\infty}^{\infty}\int \sqcap \cdot f(x)\delta'(x-\xi)g(\xi)dxd\xi \tag{20}$$

As before, we will operate to the left, evaluating the integral on dx first. How can we do this? Using identity (7), we think of it as differentiation, and make use of the product rule. To indicate derivatives of the step functions, we suggest the use of the following pictorial notations. It is hoped that the reader will find them to be self-explanatory.

$$-\int_{-\infty}^{\infty} (\sqcap' \cdot f(\xi) + \sqcap \cdot f'(\xi))\, g(\xi)d\xi \tag{21}$$

This consists of two parts:

$$-\int_{-\infty}^{\infty} \sqcap' \cdot f \cdot g - \int_{-\infty}^{\infty} \sqcap \cdot f'g \tag{22}$$

The delta functions simply evaluate the product $f \bullet g$ at the points a and b; so we get:

$$f \cdot g \Big|_{b}^{a} - \int_{a}^{b} f'g \tag{23}$$

This is the result we would ordinarily get from integration by parts.

2.4 Repeated Integration by Parts

Consider the integral:

$$\int_a^b f(x)g''(x)dx \tag{24}$$

This can be done by two steps of integration by parts. Let us see how it works with delta functions. As before, modify the problem so the limits become infinite:

$$\int_{-\infty}^{\infty}\int_{a\;b} \cdot f(x)\delta''(x-\xi)g(\xi)dxd\xi \tag{25}$$

Now operate toward the left with the differentiator, using the binomial theorem for derivatives:

$$\int_{-\infty}^{\infty} [\;\cdot f(\xi) + 2 \cdot \;\cdot f'(\xi) + \;\cdot f''(\xi)]\, g(\xi)d\xi \tag{26}$$

There are three pieces to Eq. (26). Evaluate them separately, and add them up:

$$\int_{-\infty}^{\infty} \cdot f \cdot g = (f'g + g'f)\Big|_a^b$$

(because the delta functions evaluate the derivative of $f{\cdot}g$ at points a and b) (28)

$$2\int \cdot f'g = -2f'g\Big|_a^b \tag{29}$$

$$\int \cdot f''g = \int_a^b f''g \tag{30}$$

$$\int_a^b fg'' = \{g'f - f'g\}\Big|_a^b + \int_a^b f''g$$

2.5 Fundamental Theorem of Calculus

The fundamental theorem of calculus says that integration is the same as anti-differentiation. It can be stated simply as:

$$\int_a^b f'(x)dx = f(x)\Big|_a^b \tag{31}$$

With delta functions, it looks like this:

$$\int_{-\infty}^{\infty} \int \delta'(x-\xi)f(\xi)d\xi dx \tag{32}$$

Operate to the left:

$$-\int_{-\infty}^{\infty} f(\xi)d(\xi) = f(\xi)\Big|_a^b \tag{33}$$

This is a "round about" way to arrive at an easy result. It is presented in order to suggest an extension to two dimensions. The two-dimensional analog of the Fundamental Theorem of Calculus is the Divergence Theorem, i.e., the divergence over a region is equal to the flux evaluated at the boundary.

This analogy suggests that the use of two-dimensional integral operators based on taking the divergence of the two-dimensional delta function would be an interesting area of study. However, in order to limit the scope of this thesis, we will refrain from pursuing it further.

Chapter 3

CALCULUS OF VARIATIONS

There are different ways in which mathematical ideas can arise. Certainly, the mixing together of ideas from different sources can be a source of stimulation toward the development of new ideas. The author first developed these ideas on delta functions while learning calculus of variations. At first it seemed that these ideas related specifically to that branch of mathematics. Later, it became apparent that the technique in question dealt specifically with integration by parts, and that the variational problem was merely a particular application. In any case, the reader may be interested in the problem which led to these ideas in the first place. What follows is an account of how the author solved the problem of elastic strain in an infinite solid medium with a spherical cavity.

Consider an infinite medium containing a spherical cavity of radius 100 mm. If we push outward on the walls of the cavity, how can we describe the distortion of the medium which will result?

We will assume a spherically symmetrical displacement, meaning that our description should entail a function of r alone. Now imagine that instead of a solid medium like steel we have a still liquid. If we expand the cavity from 100 to 101 mm, then we can easily see that a particle of water 200 mm from the origin will be pushed back ¼ mm. Let us use this as the basis of our functional description:

$f(100) = 1$ (boundary condition)

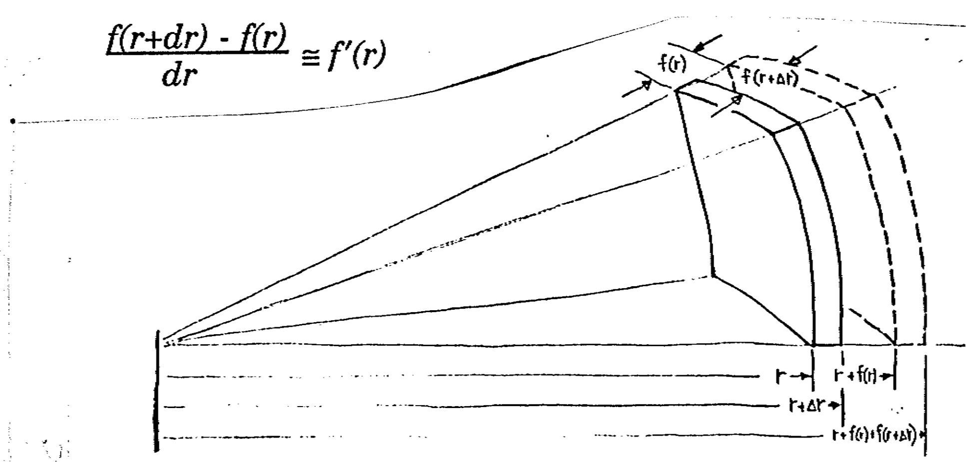
$f(200) = 1/4$

$f(300) = 1/9$ etc.

This function ought to be consistent with the idea that a volume element of water remains the same size under the action of this function. Let us apply the condition of constant volume to a moving spherical shell, and see if in fact we can derive the inverse square law.

A typical shell, say, which starts at 200 mm, gets bigger in radius by a proportion $f(r)/r$ (= ¼/200 in this case). The area goes as the square of the linear dimension, so the total increase of area is $2f(r)/r$, or 0.25%. This should be accompanied by a reduction in the thickness of the shell by an equal percentage in order that the volume remain constant. The shell initially occupies the interval $(r, r{+}dr)$. After expansion, the interval becomes $(r{+}f(r), r{+}dr{+}f(r{+}dr))$. The proportional change in shell thickness is clearly

$$\frac{f(r+dr) - f(r)}{dr} \cong f'(r) \qquad (34)$$



Spherical shell being pushed outwards.

The condition that the volume of the shell remains constant is thus:

$$\frac{2f(r)}{r} + f'(r) = 0 \tag{35}$$

This differential equation can be integrated.

$$\frac{2f}{r} = \frac{-df}{dr} \rightarrow \frac{2dr}{r} = \frac{-df}{f}$$

$$2log(r) = -log(f) \quad \text{(except for constants)}$$

$$log\ (r^2) = log\ (\frac{1}{f})$$

$$f \sim \frac{1}{r^2} \tag{36}$$

This is how water goes. Water is very easy to distort and very hard to compress. Can we imagine a medium with the opposite characteristics? In other words, it should accommodate an internal stress by pure change in volume, with no distortional effects.

This is a somewhat nonphysical case, corresponding to a Poisson's ratio of $\gamma = -1$, as opposed to $.1 < \gamma < .5$ for most ordinary materials. Orientations are preserved as scale changes. We have seen that the diameter of a spherical shell changes by $f(r)\,/\,r$ under the mapping f, and that the thickness of the shell changes by $f'(r)$. These quantities must be equal for distortionless change, so we obtain the D.E:

$$\frac{f(r)}{r} = \frac{df}{dr} \tag{37}$$

This solves as $df/f = dr/r \rightarrow f(r) \sim r$ (38)

which is the equation of distortionless change, as required.

We are now ready to return to the case of an infinite solid medium with a spherical cavity. Suppose we wish to evaluate the total elastic strain energy of the solid under the distortion given by $f(r)$. For a simple spring, the energy of distortion is given by $\frac{1}{2}kx^2$, where x is the quantity of distortion. For linear elastic media, it is reasonable to expect equations of this type to govern strain energy. We must simply remember to account for both kinds of strain: volumetric and distortional. For volumetric change, we have seen that the quantity in Eq. (35) gives a measure of the strain; therefore, for some constant of stiffness k_1 we would expect the volumetric strain energy to be given by:

$$dE_{vol} = \tfrac{1}{2}K_1 \left(\frac{2f(r)}{r} + f'(r)\right)^2 4\pi r^2 dr \tag{39}$$

The distortional energy should be given by a similar formula based on Eq. (37):

$$dE_{dist} = \tfrac{1}{2}K_2 \left(\frac{f(r)}{r} + f'(r)\right)^2 4\pi r^2 dr \tag{40}$$

(Note the stiffness constant is in general different.) Now we can state the grand variational principle which we will try to satisfy -- if it is not already obvious: For a given displacement of the boundary of the cavity, e.g., for $f(100) = 1$, we will require that $f(r)$ assumes the shape which will minimize the total strain energy, both volumetric and distortional, over the entire solid medium.

A few remarks are in order. If we were only concerned with volumetric strain, we would minimize the integral of Eq. (39) simply by setting $f(r) \sim 1/r^2$. Since this takes no account of the distortional strains we are creating, it seems entirely plausible that in order to minimize the total energy we might well modify the $1/r^2$ relation somewhat, either by increasing the exponent in the denominator, or perhaps by multiplying by an exponential envelope. The effect would be to shield the outer reaches of the medium from the effects of the swelling cavity. Since the $1/r^2$ function is so "far-reaching", it might well be worthwhile to incur some extra energy expenses in the near zone in order to confine the region of distortion somewhat.

Let us write the integral of total strain energy. Combining Eqs. (39) and (40), we get:

$$E_{tot} = \tfrac{1}{2} \int_{100}^{\infty} [K_1 (\frac{2f}{r} + f')^2 + K_2 (\frac{f}{r} - f')^2] 4\pi r^2 dr \tag{41}$$

We can obviously ignore the factor of $\tfrac{1}{2}\cdot 4\pi$. Collecting terms, we get:

$$E_{tot} \sim \int_{100}^{\infty} [(4K_1 + K_2)f^2 + (4K_1 - 2K_2)\dot{f}fr + (K_1 + K_2)\dot{f}^2 r^2]\, dr \tag{42}$$

We are using $\dot{f}$ for $f'(r)$ in order to abbreviate expressions like $[f'(r)]^2$. Also, for the time being, let us refer to the 3 constants in (42) as A, B, and C:

$$4K_1 + K_2 = A \tag{43a}$$

$$4K_2 - 2K_2 = B \tag{43b}$$

$$K_1 + K_2 = C \tag{43c}$$

So our goal is to minimize the value of the following integral:

$$E = \int_{100}^{\infty} [Af^2 + B\dot{f}fr + C\dot{f}2r^2]dr \tag{44}$$

The technique of variational calculus is described well in Feynmann (1964). We assume that f is the correct choice. Then any deviation from f must lead to an increase in E. If f were not a true minimizing function, then a given deviation from f might either tend to increase or decrease the total energy (first-order change).

In particular, we try to modify f by adding a small "blip" to it somewhere, i.e.,

$$f(r) \rightarrow f(r) + \varepsilon\delta(r\text{-}r_o),\ \varepsilon << 1 \tag{45}$$

We then write out the condition that the change in the total energy should be zero to a first-order approximation. The modified energy equation looks like this:

$$E + dE \cong \int_{100}^{\infty} A(f^2 + 2f\delta) + B(\dot{f}fr + \dot{\delta}fr + \delta fr) + C(\dot{f}^2r^2 + 2\dot{f}\delta r^2\ )dr \tag{46}$$

In Eq. (44), the notation "δ" is intended to represent the blip we invoked earlier, i.e.,:

$$\delta \leftrightarrow \varepsilon\delta(r\text{-}r_o) \tag{47a}$$

$$\dot{\delta} \leftrightarrow \varepsilon\dot{\delta}(r\text{-}r_o) \tag{47a}$$

We have omitted all second-order terms, which is to say, terms with an (implied) ε^2.

Now we compare Eq. (42), for E, with Eq. (46), for $E + dE$. By subtracting these two equations, we get the formula for the variation of the energy under the influence of δ:

$$dE = \int_{100}^{\infty} \varepsilon\,[(2Af + Bf\dot{r})\delta(r_o\text{-}r) + (Bfr + 2C\dot{f}r^2)\dot{\delta}(r\text{-}r_o)]dr \tag{48}$$

We will now convert Eq. (48) into two pieces corresponding to the form of Eq. (6). We need only note, by parity considerations, that $\delta(r\text{-}r_o) = \delta(r_o\text{-}r)$, while $\dot{\delta}\,(r\text{-}r_o) = \text{-}\dot{\delta}(r\text{-}r_o)$. Therefore,

$$dE = \int_{100}^{\infty} \varepsilon\,(2Af + B\dot{f}r)\delta(r_o\text{-}r)dr - \int_{100}^{\infty} \varepsilon\,(Bfr + 2C\dot{f}r^2)\dot{\delta}(r_o\text{-}r)]dr \tag{49}$$

Since $100 < r_o < \infty$, the delta functions in Eq. (49) evaluate appropriate derivatives on their respective target functions, in accordance with Eq. (6). If f is a true minimizing function the variation in energy dE should be zero no matter where we locate the blip, i.e., for arbitrary value of r_o.

We can therefore set $dE = 0$ and write directly:

$$2Af(\mathrm{r}_o) + Br_o\dot{f}(r_o) = \frac{d}{dr}[Brf(r) + 2Cr^2\dot{f}(r)]\Big|_{r_o} \tag{50}$$

Notice that since r_o was arbitrary, Eq. (50) must be true for all $r > 100$; so we can replace r_o with r. Then we can execute the differentiation on the R.H.S., yielding:

$$2Af + Br\dot{f} = Bf + Br\dot{f} + 4Cr\dot{f} + 2Cr^2\ddot{f} \tag{51}$$

In this way, we have converted our variational integral problem to an ordinary differential equation. To solve it, we begin by collecting terms:

$$\ddot{f}r^2 + 2\dot{f}r + (\frac{B\text{-}2A}{2C})\, f = 0 \tag{52}$$

Something surprising happens here. If we go back to Eqs. (41) defining A, B, and C, we find that the expression $(B\text{-}2A)/2C$ reduces to -2. The solution to our problem is going to be independent of its Poisson's ratio, because the terms in the equation which told us whether it is harder to distort or compress have disappeared. We should start to suspect that steel will behave the same way water does. If we execute the solution, this is in fact borne out:

$$\ddot{f}r^2 + 2\dot{f}r - 2f = 0$$

assume $f = \dfrac{1}{r^n}$

then $\dot{f} = \dfrac{-n}{r^{n+1}}$

$$\ddot{f} = \frac{n(n+1)}{r^{n+2}}$$

Substitute f, $\dot{f}$, and $\ddot{f}$ back into *D.E.*:

$$\frac{n^2+n}{r^n} - \frac{2n}{r^n} - \frac{2}{r^n} = 0$$

$$n^2 - n - 2 = 0 \quad \rightarrow \quad n{=}2 \text{ or } n{=}{-1}$$

Note: It is worthwhile to compare the logical steps leading from Eqs. (48) to (51) with the traditional method of variational calculus. There we also have a δ, but it stands for an arbitrary deviation from the minimizing function, not restricted to a deviation in the form of a blip. We begin by taking Eq. (48) and breaking it into two pieces:

$$dE = \int_{100}^{\infty} \varepsilon\,(2Af + B\dot{f}r)\delta(r)dr + \int_{100}^{\infty} \varepsilon\,(Bfr + 2C\dot{f}r^2)\dot{\delta}(r)dr \tag{49a}$$

Note the integral on $\dot{\delta}$ is positive (as contrasted with the negative sign in Eq. (49)). Now we use integration by parts to get rid of the $\dot{\delta}$:

$$\int_{100}^{\infty} (Bfr + 2C\dot{f}r^2)\dot{\delta}(r)dr = (Bfr + 2C\dot{f}r^2)\delta(r)\Big|_{100}^{\infty}$$

$$- \int_{100}^{\infty} \frac{d}{dr}(Bfr + 2C\dot{f}r^2)\delta(r)dr \tag{49b}$$

The integrated portion is zero because we require $\delta(r) = 0$ at $r = 100$ and $r = \infty$ in order to match the boundary conditions of the problem. The remaining piece is replaced into Eq. (49a), giving

$$dE = \int_{100}^{\infty} \varepsilon\,[2Af + B\dot{f}r - \frac{d}{dr}(Bfr + 2C\dot{f}r^2)]\delta(r)dr \tag{49c}$$

Since we require dE to be zero for any choice of $\delta(r)$, we conclude that the term within the square brackets must be zero everywhere. This gives us our differential equation, the same as we got from Eq. (50).

The "expected" solution, c/r^2, is verified. The other solution, $f(r) \sim r$, is the distortionless case. A complete solution to a given problem uses a superposition of the distortionless and compressionless cases to match the stated boundary conditions. For our problem, only the $1/r^2$ term is used. Interestingly enough, for the converse problem, i.e., the hydrostatically compressed bowling ball, it is the distortionless term which gives the physical answer.

3.1 An Original Duality

Taylor's Theorem shows how a suitable (i.e., analytic) function can be expressed in terms of its derivatives evaluated at the origin. Using Eq. (4), it is apparent that Taylor's Expansion Theorem can be expressed as follows:

$$\text{Let } C_n = \int \delta^n(-x) f(x) dx \tag{55}$$

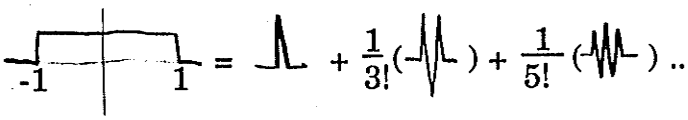
$$\text{Then } f(x) = C_o + C_1 x + C_2 \frac{x^2}{2!} + C_3 \frac{x^3}{3!} \ldots \tag{56}$$

A surprising duality arises if we interchange the roles of delta functions with powers of x:

$$\text{Let } d_n = \int (-x)^n g(x) dx \tag{57}$$

$$\text{Then } g(x) = d_o \delta(x) + d_1 \delta'(x) + d_2 \frac{\delta''(x)}{2!} \cdots \tag{58}$$

This unusual formula suggests that, for example, a square pulse can be constructed from a series of delta functions like so:



(59)

(If we take Fourier Transforms of both sides of Eq. (59), we get the well-known Taylor expansion for *(sinx)/x.)*

The author proposes that expressions like Eq. (59) should be called "Dirac Expansions". While there is certainly a widespread "de facto" recognition that a function can be uniquely specified by its moment expansion (Korner, 1988), it is quite uncommon to see expressions such as Eq. (59), where the method of reconstructing the original function is explicitly set forth.

One example of a similar expression was provided to me by Paul Chernoff of the University of California at Berkeley. In Gelfond, Vol. 1, p. 160, we find (paraphrasing):

$$\delta(x\text{-}1) = \delta(x) - \frac{\delta'(x)}{1!} + \frac{\delta''(x)}{2!} - \frac{\delta'''(x)}{3!} \dots \tag{60}$$

It is not absolutely clear that Gelfond intends to synthesize an impulse at x=1 by summation of a series of delta functions centred at the origin. Many authorities continue to say, as did Lighthill (on p. 26 of "Generalized Functions"), that any number of delta functions added up together would be zero anywhere except at the origin. A different interpretation of Eq. (60) is that it is simply an expression of Taylor's Theorem, stripped of a "target" function to act upon. Taylor's theorem comes from taking the "dot product" of both sides of Eq. (60) with an arbitrary function f(x):

$$\int_{-\infty}^{\infty} f(x)\delta(x\text{-}1)dx = \int_{-\infty}^{\infty} f(x)[\delta(x) - \frac{\delta'(x)}{1!} \quad + \frac{\delta''(x)}{2!} \quad ...]dx \tag{61}$$

$$f(1) = f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} ... \tag{62}$$

Are we allowed to make a more graphical interpretation of expressions such as (60)? In the following series of graphs, we show the summation of terms in Eq. (59), where we replace the delta function by a sequence of progressively narrower Gaussians. These graphs suggest that as the Gaussian approaches the limit of the delta function, the sum of the series does indeed approach the square pulse. The number of terms needed for convergence, however, increases rapidly.

One might ask why we would every want to use such an unusual system to represent a function. A good example is provided by electrostatic theory. If we wish to calculate the potential due to a finite line of charge, it is done most easily by expanding the charge distribution into its Dirac expansion. The field due to each component of the expansion series assumes an explicit analytic form (i.e., monopole + quadrupole + higher order terms) which converges rapidly if we are far from the source. The nearer we wish to approach the source, the more terms are needed.

In this sequence of graphs, the line indicated by crosses is a Gaussian of constant area which we will call $\Delta(x)$ to indicate that it replaces a delta function.

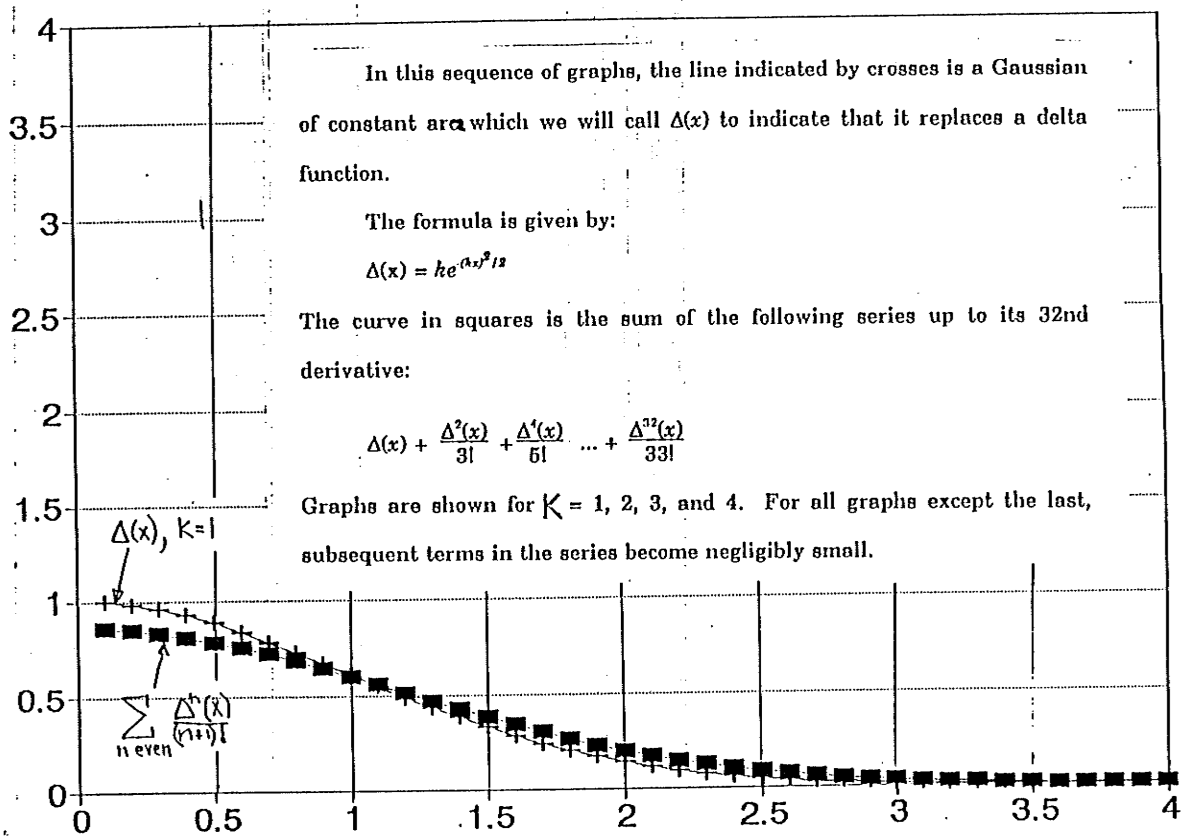
The formula is given by:

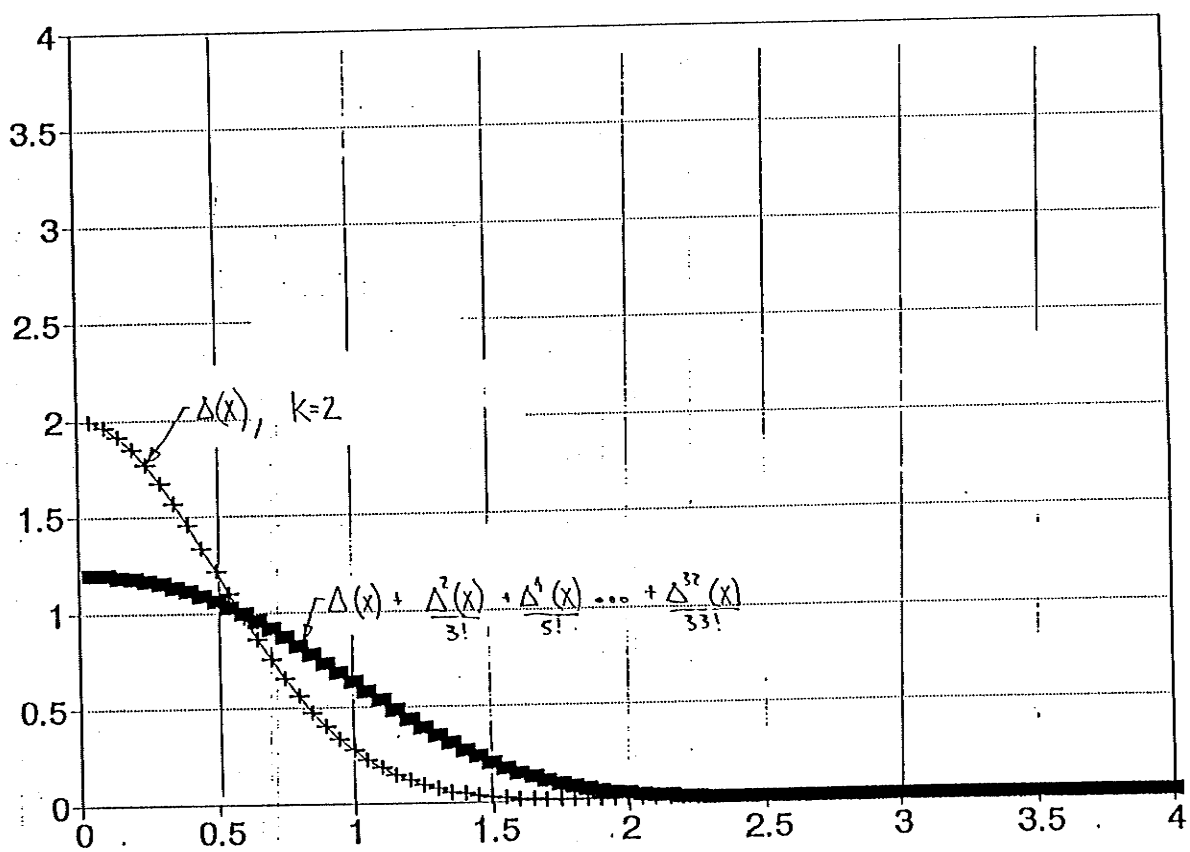
$\Delta(x) = ke^{-(kx)^2/2}$

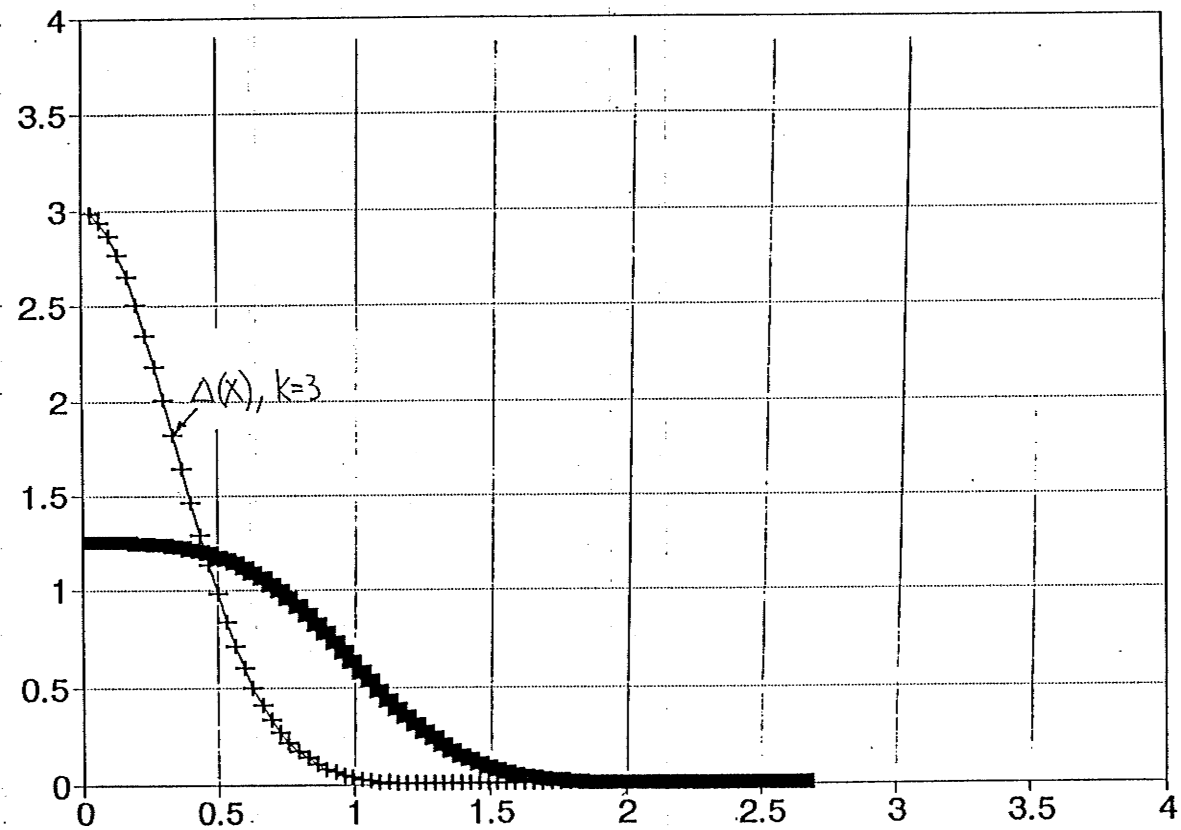
The curve in squares is the sum of the following series up to its 32nd derivative:

$$\Delta(x) + \frac{\Delta^2(x)}{3!} + \frac{\Delta^4(x)}{5!} \dots + \frac{\Delta^{32}(x)}{33!}$$

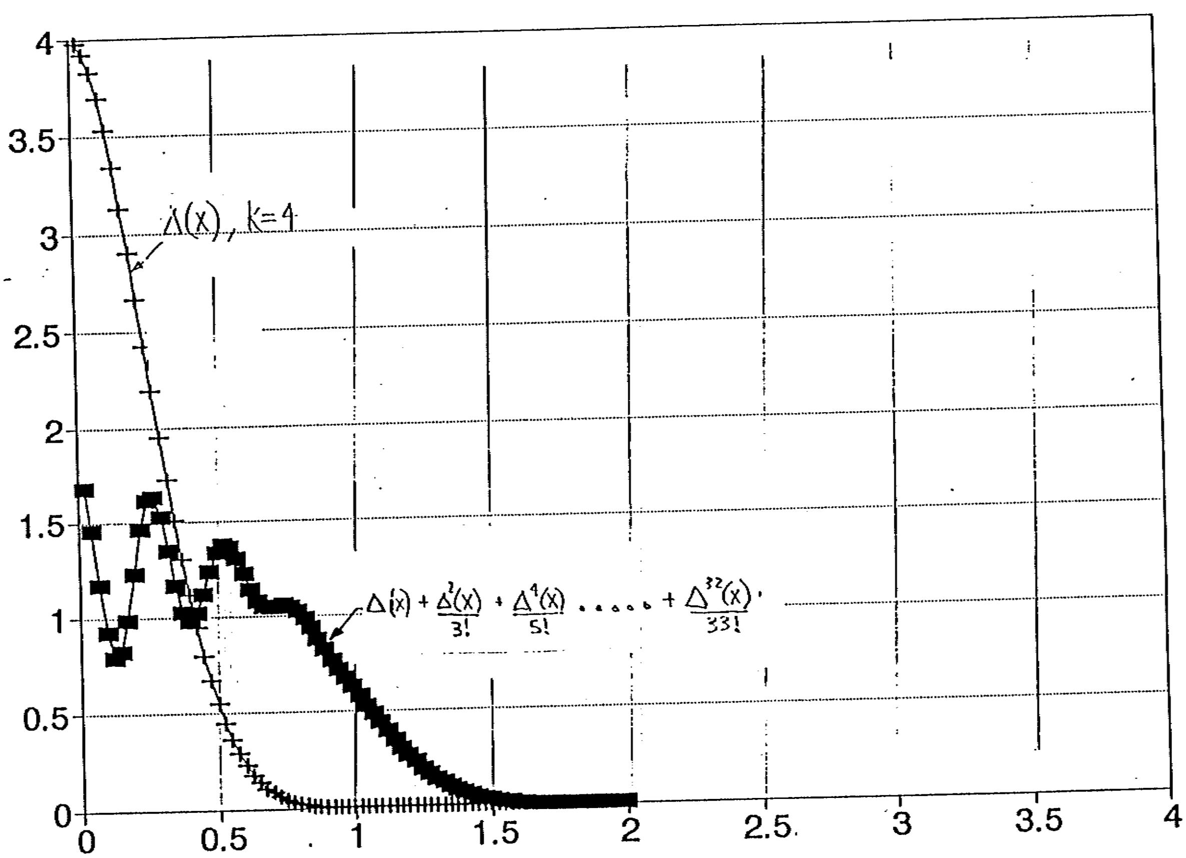
Graphs are shown for K = 1, 2, 3, and 4. For all graphs except the last, subsequent terms in the series become negligibly small.







4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
$\Delta(X)$, $k=3$



4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
$\Delta(X)$, $k=4$
$\Delta(x) + \frac{\Delta^2(X)}{3!} + \frac{\Delta^4(X)}{5!} \ldots\ldots + \frac{\Delta^{32}(X)}{33!}$

CHAPTER 5.

Conclusion

In this thesis, we discuss several calculations which ordinarily make use of integration by parts. Here we present them using delta function methods.

In Chapter 2, we discuss integration by parts in detail, giving several examples of how it would be done using delta functions.

In Chapter 3, we look at calculus of variations. The process of converting an integral condition into a differential equation normally requires integration by parts. The same result can be achieved from a different point of view by using delta functions.

In Section 3.1, we take the Fourier Transform of a Taylor Series and create a moment expansion in the frequency domain, which we have called a Dirac series. In this case, there is an unexpected benefit from writing out the mathematical expressions using delta functions. It becomes possible to express the relation between delta functions and powers of x as a duality. In other words, a true mathematical statement is converted into another true statement simply by reversing the roles of $\delta^n(x)$ with $x^n/n!$.

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APPENDIX A

SUBSEQUENT REMARKS

The sequence of graphs presented in this thesis suggests that when the delta function is defined as the limiting case of a Gaussian, we can use a sequence of delta functions to approximate an arbitrary function in the vicinity of the origin. It has been asked whether the same kind of sequence can be created based on other definitions of the delta function, and if so, whether the coefficients of the terms in the series would depend on how the delta function was defined.

A propos to this discussion, Professor Finlayson has suggested an example of how the delta function could be defined in a way which would make all its derivatives well-defined, but which could not possibly be used to build up functions away from the origin. His example is based on the function $e^{-(1/x^2)}$.

Consider the function defined as follows:

$$f(x) = \begin{cases} 0 & \text{for } -\infty < x \leq 0 \\ e^{-(k/x)^2} & \text{for } 0 < x < \infty \end{cases}$$

This function is everywhere differentiable an infinite number of times. It is similar to a unit step function; so as $k \to 0$, its first derivative behaves like a delta function. But, since it is perfectly flat to the left of the origin, no matter how often it is differentiated, it will never build up to any value on the negative x-axis.

This example shows that the existence of all higher-order derivates does not guarantee that the delta functions can be used as a basis to generate other functions.