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Dirac's Constrained Systems: Two-Dimensional Gravity and Spinning Relativistic Particle

by

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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

Department of Physics
Winnipeg, Manitoba

ISBN 0-612-13316-8

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DIRAC'S CONSTRAINED SYSTEMS:

TWO-DIMENSIONAL GRAVITY AND SPINNING RELATIVISTIC PARTICLE

BY

DOMINGO JESÚS LOUIS MARTÍNEZ

A Thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

A mis padres

Contents

Abstract

Classical and quantum aspects of the constrained dynamical systems are considered. The thesis is divided into three separate but related sections. First, a straightforward method for obtaining the generators of Lagrangian gauge transformations within the Lagrangian formalism is presented. This procedure can be carried out completely without the need of developing the Dirac-Hamilton formalism. It is shown that, in the generic case of systems having chains of Lagrangian constraints up to a level k ($k \geq 2$), the variations δq under which the action is invariant depend on q, $\frac{dq}{dt}$ and higher time derivatives up to $\frac{d^k q}{dt^k}$. The relation between this Lagrangian approach and the standard Dirac-Hamilton approach is discussed. Secondly, the most general dilaton gravity theory in two space-time dimensions is studied. It is proven that, up to space-time diffeomorphisms, the general solutions of the field equations form a one parameter family of distinct solutions. A Hamiltonian analysis is performed and the reduced phase space, which is two dimensional, is explicitly constructed in a suitable parametrization of the fields. The theory is then quantized via the Dirac method in a functional Schrödinger representation. The quantum constraints are solved exactly to yield the (spatial) diffeomorphism invariant physical wave functionals for all theories considered. These wave functionals depend explicitly on the single configuration space coordinate as well as on the embedding of space into space-time (i.e. on the choice of time). Finally, Hamiltonian models are proposed for the description of the motion of a spinning relativistic particle with gyromagnetic ratio $g = 2$ in an external uniform and static electromagnetic field in 3+1 and 2+1 dimensions. The antisymmetric spin matrix is written in terms of some new generalized coordinates and their conjugate momenta by analogy with the orbital angular momentum matrix so that the Poincare algebra is satisfied. The correct equations of motion are obtained in both cases. The Dirac canonical quantization of the 2+1 dimensional model is also considered.

Acknowledgments

I am very grateful to Gabor Kunstatter for his patience, support and guidance along all my Ph.D. studies. I would also like to thank him for his careful reading of the manuscript, detailed criticisms and comments, and his many suggestions for improvements.

I wish to express my gratitude to Masud Chaichian for his hospitality at the University of Helsinki where part of this work was done and for the encouragement and constant support I have received from him in all these years.

I am grateful to Jack Gegenberg and Ricardo Gonzalez Felipe for fruitful collaboration and discussions. My interest in the subjects of this thesis emerged also from many useful discussions with Alejandro Cabo, Hugo Pérez Rojas, Luca Lusanna, Hugo Morales, Igor Komarov and Richard Epp. I thank all of them. I would like to thank Patrick Sutton for useful comments.

Special thanks to José L. Martínez Cuéllar for being a true friend in an hour of need.

1 Gauge Invariance of Systems with Constraints

1.1 Introduction

Gauge theories play a very important role in modern theoretical physics and in particular in elementary particle physics. It is well known that gauge theories belong to the class of the so-called singular Lagrangian theories. A systematic approach for studying singular Lagrangian theories was developed by Dirac [1]. The Dirac-Hamilton canonical formalism [1] stays at the root of modern Quantum Field Theory [2, 3].

The standard quantization methods cannot be applied to the singular Lagrangian theories directly due to the lack of a well-defined theory of physical observables. Some problems concerning the theory of physical observables in the Lagrangian and Hamiltonian formalisms have been studied in [1, 4, 5, 6, 7]. The Dirac-Hamilton formalism is at the foundation of the most general methods of quantization of unified interaction theories [8, 9, 10, 11, 12, 13]. Nowadays the so-called BRST (Becchi-Rouet-Stora-Tyutin)[14, 15] and BFV (Batalin-Fradkin-Vilkovilsky) [9, 10, 11, 12] procedures are of special importance. The modern approach to the quantization of gauge theories makes use of the BRST and BFV methods in order to save the explicit covariance while remaining within the scope of renormalizable field theories. However, the BRST-BFV approach has been criticized from several points of view [16, 17].

We think that a better understanding of the classical theory of singular Lagrangians in the finite-dimensional case would clarify many unsolved problems in the modern quantization methods.

It is recognized that at the classical formulation of the singular Lagrangian theories there remain some problems which need clarification. In [18] the relation between the Hamiltonian and Lagrangian constraints has been studied. In [3, 19, 21] the Hamiltonian and Lagrangian formalisms for constrained systems were developed and several relations between the two were obtained. In [7] a unification of the two descriptions was proposed.

The question of gauge invariance for systems with constraints has been investigated by many authors. Finding the Lagrangian gauge transformations under which a given action remains invariant is, in general, a non trivial task. A way of solving this problem, based on the Dirac-Hamilton formalism [1], has been developed in [20, 21, 22, 23, 24, 25, 26, 27]. The method consists of finding the generating function of infinitesimal canonical gauge transformations Ψ as a proper linear combination of the Hamiltonian constraints. In general $\Psi(t, q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, ...)$ depends on the coordinates q, the conjugate momenta p and the Lagrange multipliers λ and their time derivatives $\dot{\lambda}, \ddot{\lambda}$, etc. The Lagrangian gauge transformations derived in this Dirac-Hamilton approach take the form:

$$\delta q_i = \frac{\partial \Psi}{\partial p_i}(t, q, \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \lambda(q, \dot{q}), \dot{\lambda}(q, \dot{q}, \ddot{q}), ...).$$

Our purpose is to develop a straightforward method of finding the Lagrangian gauge transformations of a given action within the Lagrangian formalism. This method is a generalization of the one used previously in [5] for systems having at most Lagrangian constraints of the first level. Here we consider the generic case of systems having Lagrangian constraints of first, second and higher order levels. It is assumed that the Lagrangian constraints are projectable or weakly projectable into the momentum phase space. The Lagrangian generators of gauge transformations are obtained from the consistency conditions obeyed by the Lagrangian constraints. Following the method presented here it is not necessary to assume a priori that the action is invariant under certain gauge transformations. In fact, the gauge transformations are simply derived from the constraint structure. This is an important difference between our approach and the one used in [19, 28, 29, 30, 31, 32] which assumes the action to be invariant under a certain type of gauge transformations. In that approach it is usually assumed [19, 29, 30, 31, 32] that the gauge transformations $q \to q + \delta q$ are such that δq depends only on the coordinates q and the velocities $\dot{q} = \frac{dq}{dt}$. We show that this assumption, although valid for some particular cases, may be not valid in the most generic case. Indeed, we prove that in the generic case of systems having chains of Lagrangian constraints up to a level k ($k \geq 2$), the variations

δq under which the action is invariant depend on q, $\frac{dq}{dt}$ and higher time derivatives up to $\frac{d^k q}{dt^k}$.

In Section 1.2 the Dirac-Hamilton method for finding the gauge transformations is reviewed. It is assumed that the system has only first-class constraints. In Section 1.3 the conditions obeyed by the physical quantities in the Dirac-Hamilton approach are derived. In Section 1.4 the Lagrangian generators of gauge transformations are obtained explicitly for the case of systems having projectable Lagrangian constraints. It is also shown how a similar procedure can be carried out when the Lagrangian constraints are weakly projectable. The relation between the Lagrangian approach proposed here and the standard Dirac-Hamilton approach of Section 1.2 [20, 21, 22, 23, 24, 25, 26, 27] is discussed in Section 1.5.

Sections 1.2 and 1.3 contain well known material and are intended only as a review. The original results that we have obtained are presented in Sections 1.4 and 1.5 and were preprinted in [33].

1.2 The Dirac-Hamilton approach

Let us consider systems with only first-class Hamiltonian constraints[1]. According to Dirac [1] a first-class function is one that has vanishing (in the constrained manifold) Poisson brackets with all the constraints. The system is described by the action:

$$S = \int dtL \tag{1}$$

L is the Lagrangian of the system which is defined in the velocity phase space TQ (TQ is the tangent bundle of the n-dimensional configuration space Q). The q_i $(i = 1, 2, ..., n)$ are the generalized coordinates and $\dot{q}_i \equiv \frac{dq_i}{dt}$ are the generalized velocities.

The Euler-Lagrange equations of motion may be written in the following form:

$$L_i \equiv W_{ij}\ddot{q}_j - \alpha_i = 0 \ , \qquad (i, j = 1, 2, ..., n) \tag{2}$$

where,

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \tag{3}$$

$$\alpha_i \equiv \frac{\partial L}{\partial q_i} - \dot{q}_\ell \frac{\partial^2 L}{\partial q_\ell \partial \dot{q}_i} \tag{4}$$

We assume that the dimension of the kernel of the Hessian matrix W is constant in TQ and equal to m:

$$rank||W_{ij}|| = n - m \tag{5}$$

We denote as $\gamma^{(\mu)}(q, \dot{q})$ $(\mu = 1, 2, ..., m)$ a set of linearly independent null eigenvectors of the Hessian matrix:

$$\gamma_i^{(\mu)} W_{ij} \equiv 0, \qquad (\mu = 1, 2, ..., m) \tag{6}$$

Following Dirac's method the primary Hamiltonian constraints are obtained from the relations defining the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{7}$$

The number of primary Hamiltonian constraints coincides with the dimension of the kernel of the Hessian matrix.

The total Hamiltonian equations (which are equivalent to the Euler-Lagrange equations of motion (2)) can be written as follows:

$$\dot{q}_i = \frac{\partial H_c}{\partial p_i} + \lambda_\mu \frac{\partial \phi_\mu^{(0)}}{\partial p_i} \tag{8}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \lambda_\mu \frac{\partial \phi_\mu^{(0)}}{\partial q_i} \tag{9}$$

$$\phi_\mu^{(0)}(q,p) = 0 \tag{10}$$

where $\phi_\mu^{(0)}$ are the primary Hamiltonian constraints ($\mu = 1, 2, ..., m$). The corresponding Lagrange multipliers are denoted by λ_μ. The canonical Hamiltonian $H_c(q,p)$ is any function of (q,p) satisfying

$$H_c\left[q, \frac{\partial L}{\partial \dot{q}}\right] = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q, \dot{q}) \tag{11}$$

The total Hamiltonian is defined as

$$H_T^{(\lambda)}(q,p) = H_c(q,p) + \lambda_\mu \phi_\mu^{(0)}(q,p) \tag{12}$$

All the primary Hamiltonian constraints $\phi_\mu^{(0)}$ are assumed to be first-class. This means that the Poisson brackets among them are equal to a linear combination of themselves:

$$\{\phi_\mu^{(0)}, \phi_\nu^{(0)}\} = D_{\mu\nu\eta}^{(0,0,0)} \phi_\eta^{(0)} \stackrel{M_0}{=} 0 \tag{13}$$

The label above the equality indicates in which subspace of the momentum phase space T^*Q this equality holds. M_0 is the subset of the momentum phase space T^*Q defined by the primary Hamiltonian constraints $\phi_\mu^{(0)}(q,p) = 0$ ($\mu = 1, 2, ..., m$).

The secondary constraints $\phi_\mu^{(1)}$ are obtained from the consistency conditions on the primary constraints:

$$\dot{\phi}_\mu^{(0)} = \{\phi_\mu^{(0)}, H_c\} + \lambda_\nu \{\phi_\mu^{(0)}, \phi_\nu^{(0)}\} = 0 \qquad (\mu = 1, 2, ..., m) \tag{14}$$

Equations (14) should be satisfied for all the solutions of the total Hamiltonian equations of motion (8-10). From (13) and (14) it follows that we can define the secondary Hamiltonian constraints $\phi_\mu^{(1)}$ as $\phi_\mu^{(1)} = \{\phi_\mu^{(0)}, H_c\}$.

The tertiary Hamiltonian constraints result from the consistency conditions on the secondary ones, and so on. All the Hamiltonian constraints of the system can be obtained following this algorithm (it is the so-called Dirac algorithm).

We may write all the Hamiltonian constraints of the system in the following form:

$$\begin{aligned}
&\phi_\mu^{(0)} && \text{- primary Hamiltonian constraints,} \\
&\phi_\mu^{(1)} = \{\phi_\mu^{(0)}, H_c\} && \text{- secondary Hamiltonian constraints,} \\
&\phi_\mu^{(2)} = \{\phi_\mu^{(1)}, H_c\} && \text{- tertiary Hamiltonian constraints,} \\
&\quad \dots \\
&\phi_\mu^{(k)} = \{\phi_\mu^{(k-1)}, H_c\} && \text{- } (k+1)\text{-ary Hamiltonian constraints.}
\end{aligned} \tag{15}$$

For simplicity, we assume that k is independent of μ. The primary Hamiltonian constraints define a subset of the momentum phase space T^*Q denoted as M_0; the secondary Hamiltonian constraints define a subset of M_0 which we denote as M_1, and so on, the $(k+1)$-ary Hamiltonian constraints define a subset of M_{k-1} denoted as M_k. Therefore, M_k is defined by the vanishing of all the Hamiltonian constraints of the system. It is assumed that Dirac's algorithm stops at some level denoted by k, which means that

$$\{\phi_\mu^{(k)}, H_c\} = \sum_{s=0}^{k} F_{\mu\nu}^{(s)}(q,p)\phi_\nu^{(s)}(q,p) \tag{16}$$

All the Hamiltonian constraints are assumed to be first-class, therefore, the Poisson brackets among the Hamiltonian constraints should vanish in the constraint surface:

$$\{\phi_\mu^{(\ell)}, \phi_\nu^{(r)}\} = \sum_s D_{\mu\nu\eta}^{(\ell,r,s)} \phi_\eta^{(s)} \stackrel{M_k}{=} 0 \tag{17}$$

Let us consider the infinitesimal canonical transformations mapping solutions into solutions of the total Hamiltonian equations of motion (8-10). Under such canonical transformations the total Hamiltonian transforms as follows:

$$H_T^{(\lambda)}(q,p) \longrightarrow H_T^{(\lambda')}(q,p) \tag{18}$$

The generating function Ψ of these infinitesimal canonical transformations should satisfy the following two conditions [20, 21, 22, 23, 24, 25, 26, 27]:

$$\Psi \qquad \text{- is a first-class function} \tag{19}$$

$$\frac{D\Psi}{Dt} + \{\Psi, H_c\} + \lambda_\mu \{\Psi, \phi_\mu^{(0)}\} = \omega_\mu \phi_\mu^{(0)} \tag{20}$$

where ω_μ are some functions of t, q, p and λ, $\dot{\lambda}$, etc. The derivative $\frac{D}{Dt}$ denotes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{\lambda}_\mu \frac{\partial}{\partial \lambda_\mu} + \ddot{\lambda}_\mu \frac{\partial}{\partial \dot{\lambda}_\mu} + \cdots \tag{21}$$

The infinitesimal canonical transformations generated by Ψ map the solution $(q(t), p(t), \lambda(t))$ into another trajectory $(q'(t), p'(t), \lambda'(t))$:

$$\begin{aligned} q_i'(t) &= q_i(t) + \varepsilon \frac{\partial \Psi}{\partial p_i}(t, q(t), p(t), \lambda(t), \dot{\lambda}(t), \ldots) && (22) \\ p_i'(t) &= p_i(t) - \varepsilon \frac{\partial \Psi}{\partial q_i}(t, q(t), p(t), \lambda(t), \dot{\lambda}(t), \ldots) && (23) \\ \lambda_\mu'(t) &= \lambda_\mu(t) + \varepsilon \omega_\mu(t, q(t), p(t), \lambda(t), \dot{\lambda}(t), \ldots) && (24) \end{aligned}$$

which satisfies the equations of motion (8-10) with the total Hamiltonian $H_T^{(\lambda')}$:

$$H_T^{(\lambda')} = H_T^{(\lambda)} + \varepsilon \omega_\mu \phi_\mu^{(0)} = H_c + \lambda_\mu' \phi_\mu^{(0)} \tag{25}$$

The condition (19) of Ψ being a first-class function guarantees that Ψ generates transformations from points (q,p) of M_k into points (q',p') of the same subset M_k. The condition (20) furnishes that the canonical transformations of the total Hamiltonian equations of motion bring about the same total Hamitonian equations of motion in which only the Lagrange multipliers are changed.

The existence of a function Ψ satisfying (19,20) can be proven by expressing it as a linear combination of all the first-class Hamiltonian constraints of the system in the following way [20, 21, 22, 23, 24, 25, 26, 27]:

$$\Psi = \sum_{s=0}^{k} \sum_{\mu=1}^{m} C_{\mu}^{(s)}(t, q, p, \lambda, \dot{\lambda}, ...)\phi_{\mu}^{(s)}(q, p) \tag{26}$$

The condition (19) on Ψ is thus automatically satisfied. The coefficients $C_{\mu}^{(s)}$ $(s = 0, 1, ..., k)$ $(\mu = 1, 2, ..., m)$ can always be selected in order to satisfy (20). Indeed, the coefficients $C_{\mu}^{(k)}$ $(\mu = 1, 2, ..., m)$ can be chosen to be some arbitrary functions of t, q and p, while the other coefficients $C_{\mu}^{(s)}$ $(s = 0, 1, ..., k-1)$ $(\mu = 1, 2, ..., m)$ are determined by the recursive relations:

$$\begin{aligned} C_{\mu}^{(s)} &= -\frac{DC_{\mu}^{(s+1)}}{Dt} - \{C_{\mu}^{(s+1)}, H_c\} - C_{\mu}^{(k)} F_{\mu\nu}^{(s+1)} - \\ &\quad - \sum_{\nu} \lambda_{\nu} \left[\{C_{\mu}^{(s+1)}, \phi_{\nu}^{(0)}\} + \sum_{r=s+1}^{k} \sum_{\eta} C_{\eta}^{(r)} D_{\eta\nu\mu}^{(r,0,s+1)} \right] \end{aligned} \tag{27}$$

$s = 0, 1, ..., k-1$

Notice that

$$\begin{aligned} \omega_{\mu} &= \frac{DC_{\mu}^{(0)}}{Dt} + \{C_{\mu}^{(0)}, H_c\} + C_{\mu}^{(k)} F_{\mu\nu}^{(0)} + \\ &\quad + \lambda_{\nu} \left[\{C_{\mu}^{(0)}, \phi_{\nu}^{(0)}\} + \sum_{r=0}^{k} \sum_{\eta} C_{\eta}^{(r)} D_{\eta\nu\mu}^{(r,0,0)} \right] \end{aligned} \tag{28}$$

As we can see from (27) in general the coefficients $C_{\mu}^{(k-1)}$ depend on the t, q, p and λ; the coefficients $C_{\mu}^{(k-2)}$ depend on the t, q, p, λ and $\dot{\lambda}$; and so on. The coefficients $C_{\mu}^{(0)}$ in general depend on the t, q, p, λ, $\dot{\lambda}$, $\ddot{\lambda}$, ..., $\frac{d^{k-1}\lambda}{dt^{k-1}}$. Therefore, the generating function Ψ (26) of the canonical transformations (22-24) depends in general on t, q, p, λ, $\dot{\lambda}$, $\ddot{\lambda}$,..., $\frac{d^{k-1}\lambda}{dt^{k-1}}$. Remember that k is the step at which the Dirac constraint algorithm (15) stops. It is interesting to notice that in general the functions ω_{μ} (28) depend on t, q, p, λ, $\dot{\lambda}$, $\ddot{\lambda}$,..., $\frac{d^{k}\lambda}{dt^{k}}$.

Following [18] we consider the application fiber derivative of the Lagrangian (FL) of the tangent bundle TQ on the cotangent bundle T^*Q

$$\mathrm{FL}:\ TQ \longrightarrow T^*Q$$

given by $\mathrm{FL}(q,\dot{q}) = (q,p)$, where

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q,\dot{q}) \equiv \mathcal{P}_i(q,\dot{q}) \qquad (i = 1,2,...,n).$$

For the primary Hamiltonian constraints we have at the Lagrangian level that

$$\mathrm{FL}^*\phi_\mu^{(0)} = \phi_\mu^{(0)}\left(q,\mathcal{P}(q,\dot{q})\right) \equiv 0 \tag{29}$$

where FL^* is the pullback application. From equation (29) we can deduce the following relations:

$$\mathrm{FL}^*\frac{\partial \phi_\mu^{(0)}}{\partial p_i} = \gamma_i^{(\mu)}(q,\dot{q}), \tag{30}$$

$$\mathrm{FL}^*\frac{\partial \phi_\mu^{(0)}}{\partial q_i} = -\gamma_j^{(\mu)}(q,\dot{q})\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}. \tag{31}$$

Indeed, by differentiating (29) with respect to $\dot{q}_j$ we obtain

$$\mathrm{FL}^*\frac{\partial \phi_\mu^{(0)}}{\partial p_i}\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0. \tag{32}$$

Equation (32) allows us to identify $\mathrm{FL}^*\frac{\partial \phi_\mu^{(0)}}{\partial p_i}$ with the linearly independent eigenvectors $\gamma_i^{(\mu)}$ (6).

Equation (31) is obtained by differentiating (29) with respect to q_j. Notice that the relations (30,31) are valid for any $(q,\dot{q}) \in TQ$.

From the definition of the canonical Hamiltonian (11):

$$\mathrm{FL}^* H_c = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q,\dot{q}) \tag{33}$$

it follows that

$$\mathrm{FL}^*\frac{\partial H_c}{\partial p_i} = \dot{q}_i - \lambda_\mu(q,\dot{q})\gamma_i^{(\mu)}(q,\dot{q}), \tag{34}$$

$$\mathrm{FL}^*\frac{\partial H_c}{\partial q_i} = -\frac{\partial L}{\partial q_i} + \lambda_\mu(q,\dot{q})\gamma_j^{(\mu)}(q,\dot{q})\frac{\partial^2 L}{\partial \dot{q}_j \partial q_i} \tag{35}$$

for any $(q, \dot{q}) \in TQ$.

Indeed, differentiating (33) by $\dot{q}_j$ we get

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \left[\dot{q}_i - \mathrm{FL}^* \frac{\partial H_c}{\partial p_i} \right] = 0 \qquad \forall (q, \dot{q}) \in TQ \tag{36}$$

This means that $\dot{q}_i - \mathrm{FL}^* \frac{\partial H_c}{\partial p_i}$ is a null eigenvector of the Hessian matrix W_{ij} (3), and therefore it may be expressed as a linear combination of the basis vectors $\gamma_i^{(\mu)}$ (6):

$$\dot{q}_i - \mathrm{FL}^* \frac{\partial H_c}{\partial p_i} = \lambda_\mu(q, \dot{q}) \gamma_i^{(\mu)}(q, \dot{q}). \tag{37}$$

This is actually the *definition* of the functions $\lambda_\mu(q, \dot{q})$ which will appear in the Dirac-Hamilton equations (8-10).

On the other hand, differentiating (33) by q_j we obtain

$$\mathrm{FL}^* \frac{\partial H_c}{\partial q_j} = -\frac{\partial L}{\partial q_j} + \left[\dot{q}_i - \mathrm{FL}^* \frac{\partial H_c}{\partial p_i} \right] \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \tag{38}$$

Using (37) in (38) we finally obtain that

$$\mathrm{FL}^* \frac{\partial H_c}{\partial q_j} = -\frac{\partial L}{\partial q_j} + \lambda_\mu(q, \dot{q}) \gamma_i^{(\mu)}(q, \dot{q}) \frac{\partial^2 L}{\partial \dot{q}_i \partial q_i} \tag{39}$$

Theorem. The existence of infinitesimal canonical transformations

$$\delta^H q_i = \varepsilon \frac{\partial \Psi}{\partial p_i} , \tag{40}$$

$$\delta^H p_i = -\varepsilon \frac{\partial \Psi}{\partial q_i} \tag{41}$$

generated by a function Ψ, which satisfies the conditions

$$\Psi \qquad \text{- is a first-class function} \tag{42}$$

$$\frac{D\Psi}{Dt} + \{\Psi, H_c\} + \lambda_\mu \{\Psi, \phi_\mu^{(0)}\} = \omega_\mu \phi_\mu^{(0)} \tag{43}$$

guarantees, at the Lagrangian level, the off-shell gauge invariance of the action $S = \int L\, dt$ under the induced gauge transformations

$$\delta q_i = \mathrm{FL}^* \delta^H q_i , \tag{44}$$

for *any* trajectory $q(t)$.

The variation of the Lagrange function L is equal to:

$$\delta L = \frac{d}{dt}\left[\varepsilon\frac{\partial L}{\partial \dot{q}_i}\mathrm{FL}^*\frac{\partial \Psi}{\partial p_i} - \varepsilon \mathrm{FL}^*\Psi\right] \tag{45}$$

Proof. The induced transformations for *any* trajectory $q(t)$ in the configuration space Q may be written in the following way:

$$\delta q_i(t) = \varepsilon \mathrm{FL}^*\frac{\partial \Psi}{\partial p_i}. \tag{46}$$

The variation $\delta \dot{q}_i$ is defined as

$$\delta \dot{q}_i(t) = \frac{d}{dt}\delta q_i(t)\ . \tag{47}$$

Using the definition of the canonical Hamiltonian

$$H_c(q, \mathcal{P}(q,\dot{q})) = \mathcal{P}_i(q,\dot{q})\cdot \dot{q}_i - L(q,\dot{q})\ , \tag{48}$$

we may write the following expression for the variation of L:

$$\begin{aligned} \delta L &= \delta\mathcal{P}_i\dot{q}_i + \mathcal{P}_i\cdot\delta\dot{q}_i - \delta H_c \\ &= \frac{d}{dt}(\mathcal{P}_i\delta q_i) + \delta\mathcal{P}_i\dot{q}_i - \delta q_i\dot{\mathcal{P}}_i - \frac{\partial H_c}{\partial q_i}\delta q_i - \frac{\partial H_c}{\partial p_i}\delta\mathcal{P}_i\ . \end{aligned} \tag{49}$$

Substituting into Eq.(49) the following relations

$$\delta q_i(t) = \varepsilon \mathrm{FL}^*\frac{\partial \Psi}{\partial p_i}. \tag{50}$$

$$\begin{aligned} \delta\mathcal{P}_i &= \mathrm{FL}^*\delta^H p_i + (\delta\mathcal{P}_i - \mathrm{FL}^*\delta^H p_i) \\ &= -\varepsilon\mathrm{FL}^*\frac{\partial\Psi}{\partial q_i} + (\delta\mathcal{P}_i - \mathrm{FL}^*\delta^H p_i) \end{aligned} \tag{51}$$

$$\frac{d}{dt}[\mathrm{FL}^*\Psi] = \mathrm{FL}^*\frac{\partial\Psi}{\partial q_j}\dot{q}_j + \mathrm{FL}^*\frac{\partial\Psi}{\partial p_i}\dot{\mathcal{P}}_j + \mathrm{FL}^*\frac{D\Psi}{Dt} \tag{52}$$

we obtain that

$$\begin{aligned} \delta L &= \varepsilon\frac{d}{dt}\left[\mathcal{P}_i(q,\dot{q})\mathrm{FL}^*\frac{\partial\Psi}{\partial p_i} - \mathrm{FL}^*\Psi\right] + \varepsilon\mathrm{FL}^*\left[\frac{D\Psi}{Dt} + \{\Psi, H_c\}\right] + \\ &+ \left(\dot{q}_i - \frac{\partial H_c}{\partial p_i}(q,\mathcal{P}(q,\dot{q}))\right)(\delta\mathcal{P}_i - \mathrm{FL}^*\delta^H p_i) \end{aligned} \tag{53}$$

As a result of (29) and the condition (43) the second term in (53) can be rewritten as

$$\varepsilon \mathrm{FL}^* \left[\frac{D\Psi}{Dt} + \{\Psi, H_c\}\right] = -\varepsilon \mathrm{FL}^* \sum_\mu \lambda_\mu \{\Psi, \phi_\mu^{(0)}\} \tag{54}$$

Considering the relations (30,31) we can write

$$\begin{aligned}
\mathrm{FL}^*\varepsilon \cdot \{\Psi, \phi_\mu^{(0)}\} &= \varepsilon \cdot \left(\mathrm{FL}^* \frac{\partial \Psi}{\partial q_i}\right) \cdot \gamma_i^{(\mu)} + \varepsilon \left(\mathrm{FL}^* \frac{\partial \Psi}{\partial p_i}\right) \gamma_j^{(\mu)} \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \\
&= \gamma_i^{(\mu)} \left[-\mathrm{FL}^* \delta^H p_i + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} (\mathrm{FL}^* \delta^H q_j)\right] \\
&= \gamma_i^{(\mu)} \left[-\delta \mathcal{P}_i + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \cdot \delta q_j\right] + \gamma_i^{(\mu)} (\delta \mathcal{P}_i - \mathrm{FL}^* \delta^H p_i)
\end{aligned} \tag{55}$$

From the definition of $\mathcal{P}_i$ it follows

$$\delta \mathcal{P}_i = \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \delta q_j + \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \delta \dot{q}_j \tag{56}$$

Substituting (56) into (55) and using (6) we get

$$\mathrm{FL}^*\varepsilon \cdot \{\Psi, \phi_\mu^{(0)}\} = \gamma_i^{(\mu)} (\delta \mathcal{P}_i - \mathrm{FL}^* \delta^H p_i) \tag{57}$$

From (54), (57) and (34) it follows that

$$\varepsilon \mathrm{FL}^* \left[\frac{D\Psi}{Dt} + \{\Psi, H_c\}\right] + \left(\dot{q}_i - \frac{\partial H_c}{\partial p_i}(q, \mathcal{P}(q, \dot{q}))\right) (\delta \mathcal{P}_i - \mathrm{FL}^* \delta^H p_i) = 0 \tag{58}$$

In other words, the second and third terms in (53) exactly cancel each other. We finally obtain the following expression for the variation of the Lagrange function

$$\delta L = \frac{d}{dt} \left[\varepsilon \frac{\partial L}{\partial \dot{q}_i} \mathrm{FL}^* \frac{\partial \Psi}{\partial p_i} - \varepsilon \mathrm{FL}^* \Psi\right] \tag{59}$$

for *any* trajectory $q(t)$ in the configuration space.

Therefore, the action $S = \int L\, dt$ remains invariant under the induced gauge transformations (44) when the variations δq vanish in the initial and final instants of time. Thus the theorem is proven.

In this Section we have proven the existence of a generating function Ψ of infinitesimal canonical transformations satisfying Eqs(42,43) for systems with first-class constraints. Indeed, the function Ψ can be constructed in the form (26) with as many arbitrary functions as primary first-class constraints. The arbitrary functions are the coefficients $C_{\mu}^{(k)}$, $(\mu = 1, 2, ..., m)$ (k is the final step of Dirac's algorithm). The rest of the coefficients $C_{\mu}^{(s)}$ $(s = 0, 1, ..., k-1)$ are uniquely determined by the recursive relations (27). Thus, (26) and (27) provide an explicit way of constructing the generating functions Ψ. The above enunciated theorem guarantees that the induced Lagrangian transformations $\delta q_i(t) = \varepsilon \mathrm{FL}^* \frac{\partial \Psi}{\partial p_i}$ are the Lagrangian gauge transformations under which the action of the system remains invariant. This is the Dirac-Hamilton approach [20, 21, 22, 23, 24, 25, 26, 27] for finding the Lagrangian gauge transformations. As it can be seen, it requires developing the Hamiltonian formulation, constructing infinitesimal canonical gauge transformations in the momentum phase space, and then going back again to the Lagrangian formulation.

1.3 Physical observables in the Dirac-Hamilton formalism

Due to the arbitrariness of the Lagrange multipliers λ_μ in (8-10), the Cauchy problem for the canonical equations of motion does not have a unique solution. This means that we are faced with a degenerate theory. In Section 1.2 we studied the infinitesimal canonical transformations generated by Ψ (26,27) mapping solutions of the total Hamiltonian equations of motion into solutions of the same equations. We define [3] as an instantaneous state of the solution $x(t) = (q(t), p(t), \lambda(t))$ of (8-10) the set of values of the variables x and their derivatives $\dot{x}, \ddot{x}, \dddot{x}$, etc. at the given instant of time. We say that two solutions $x(t)$ and $x'(t)$ intersect if at some given instant of time their instantaneous states coincide. It is clear that in the so-called degenerate theories there exist intersecting solutions.

The physical interpretation of the systems with first-class constraints is based on the following two statements [1, 4, 3]:

(a) the physical state of the system, and therefore all the physical quantities, are uniquely determined by the instantaneous state of the solutions of the corresponding theory:

$$\mathcal{A}^H(x, \dot{x}, \ddot{x}, \dddot{x}, ...) \tag{60}$$

(b) the values of the physical quantities coincide for equal-time points of the solutions which intersect:

$$\mathcal{A}^H(x, \dot{x}, \ddot{x}, \dddot{x}, ...) = \mathcal{A}^H(x', \dot{x}, \ddot{x}', \dddot{x}', ...), \qquad \forall t \tag{61}$$

Naturally, we will consider that any function which satisfies (a) and (b) corresponds to a certain physical quantity.

The generating function Ψ maps a solution $x(t)$ of the total Hamiltonian equations (8-10) into another solution $x'(t)$ (22-24) of the same equations of motion. The explicit time dependence of Ψ is determined by the time dependence of the arbitrarily chosen coefficients $C^{(k)}_\mu$ $(\mu = 1, 2, ..., m)$. Indeed, if for example we choose $C^{(k)}_\mu$ to be some functions which depend only on time but not on q, p or λ, such that we denote

$\nu_\mu(t) \equiv \varepsilon C_\mu^{(k)}(t)$, then we see from (26,27) that Ψ will necessarily depend on $\nu_\mu(t)$, $\dot{\nu}_\mu(t)$, $\ddot{\nu}_\mu(t)$, ..., $\frac{d^{k-1}\nu_\mu}{dt^{k-1}}(t)$. On the other hand, as follows from (28), the functions ω_μ depend necessarily on $\nu_\mu(t)$, $\dot{\nu}_\mu(t)$, $\ddot{\nu}_\mu(t)$, ..., $\frac{d^{k-1}\nu_\mu}{dt^{k-1}}(t)$, $\frac{d^k\nu_\mu}{dt^k}(t)$.

Using the arbitrariness of the functions $\nu_\mu(t)$ we can demand that at a certain instant of time $t = t_0$:

$$\nu_\mu(t_0) = \dot{\nu}_\mu(t_0) = \ddot{\nu}_\mu(t_0) = \ldots = 0 \tag{62}$$

This means that the solutions $x(t)$ and $x'(t)$ intersect at $t = t_0$. Therefore, these two solutions will describe the same physical evolution of the system. The values of the physical quantities must coincide for equal-time points, and we can write this in the following way:

$$\mathcal{A}^H(x, \dot{x}, \ddot{x}, \dddot{x}, \ldots) = \mathcal{A}^H(x', \dot{x}', \ddot{x}', \dddot{x}', \ldots) \tag{63}$$

From the total Hamiltonian equations of motion (8-10) it follows that the values of $\dot{q}$ and $\dot{p}$ are uniquely determined by the values of q, p and λ, at a given instant of time. Taking the time derivatives of these equations we see that the variables $\ddot{q}$ and $\ddot{p}$ can be expressed as functions of the variables q, p, λ and $\dot{\lambda}$; and so on. In other words, the instantaneous state of a solution $x(t)$ may be well defined by knowing only the values of the variables q, p, λ, $\dot{\lambda}$, $\ddot{\lambda}$, $\dddot{\lambda}$, etc. at the given instant of time. The conditions (63) may therefore be rewritten as follows:

$$\mathcal{A}^H(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, \ldots) = \mathcal{A}^H(q', p', \lambda', \dot{\lambda}', \ddot{\lambda}', \ldots) \qquad \forall t \tag{64}$$

From (22-24) we can see that due to the arbitrariness of $\nu_\mu(t)$ and to the fact that the variations of λ_μ depend on $\nu_\mu(t)$, $\dot{\nu}_\mu(t)$, $\ddot{\nu}_\mu(t)$, ..., $\frac{d^{k-1}\nu_\mu}{dt^{k-1}}(t)$, $\frac{d^k\nu_\mu}{dt^k}(t)$, while the variations of the coordinates q and the momenta p depend on $\nu_\mu(t)$, $\dot{\nu}_\mu(t)$, $\ddot{\nu}_\mu(t)$, ..., $\frac{d^{k-1}\nu_\mu}{dt^{k-1}}(t)$, we can choose arbitrarily the variations $\delta\lambda$, $\delta\dot{\lambda}$, $\delta\ddot{\lambda}$, etc., at an instant of time $t = t_1$ $(t_1 \neq t_0)$ maintaining the values of q and p fixed $(\delta q(t_1) = \delta p(t_1) = 0)$:

$$\mathcal{A}^H(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, \ldots) = \mathcal{A}^H(q, p, \lambda + \delta\lambda, \dot{\lambda} + \delta\dot{\lambda}, \ddot{\lambda} + \delta\ddot{\lambda}, \ldots) \tag{65}$$

Since $\delta\lambda$, $\delta\dot{\lambda}$, $\delta\ddot{\lambda}$, etc., are completely arbitrary, we conclude that

$$\frac{\partial \mathcal{A}^H}{\partial \lambda} = \frac{\partial \mathcal{A}^H}{\partial \dot{\lambda}} = \frac{\partial \mathcal{A}^H}{\partial \ddot{\lambda}} = \ldots = 0 \tag{66}$$

Therefore, the physical quantities in the Dirac-Hamilton formalism depend only on the variables q and p for the solutions of the total Hamiltonian equations (8-10). The condition (64) can be rewritten as

$$\mathcal{A}^H(q,p) = \mathcal{A}^H(q',p') \qquad\qquad \forall t \tag{67}$$

where for any given time $(q,p) \in M_k$, $(q',p') \in M_k$ and

$$\begin{aligned} q_i' &= q_i + \varepsilon \frac{\partial \Psi}{\partial p_i} && (68)\\ p_i' &= p_i - \varepsilon \frac{\partial \Psi}{\partial q_i} && (69) \end{aligned}$$

It was conjectured by Dirac [1] that the first-class constraints are always generators of canonical transformations mapping points (q,p) of M_k into points (q',p') of M_k which describe the same physical state of the system. In [34, 35, 23] Dirac's conjecture was proven for systems having only first-class constraints. A more general proof of the conjecture suitable for systems having first- and second-class constraints was given in [26].

We will not present the details of the proof here. The proof relies on the fact that all the first-class constraints enter into the expression (26) for the generating function Ψ of canonical gauge transformations and that at any two given instants of time the values of the functions $\nu_\mu(t)$ and their derivatives can be chosen arbitrarily. This guarantees that any two points (q,p) and (q',p') related by a canonical transformation

$$\begin{aligned} q_i' &= q_i + \varepsilon \frac{\partial \phi_\mu^{(s)}}{\partial p_i}(q,p) && (70)\\ p_i' &= p_i - \varepsilon \frac{\partial \phi_\mu^{(s)}}{\partial q_i}(q,p) && (71) \end{aligned}$$

belong to two different solutions of the total Hamiltonian equations (8-10) which can be transformed into one another by a canonical gauge transformation generated by Ψ.

We can then write the conditions (67) and (70,71) as

$$\mathcal{A}^H(q,p) = \mathcal{A}^H\left(q_i + \varepsilon\frac{\partial\phi_\mu^{(s)}}{\partial p_i}(q,p), p_i - \varepsilon\frac{\partial\phi_\mu^{(s)}}{\partial q_i}(q,p)\right) \qquad \begin{array}{l} s = 0,1,\ldots,k \\ \mu = 1,2,\ldots,m \end{array} \tag{72}$$

Since through each point $(q,p) \in M_k$ there passes a solution of (8-10) we conclude that the condition (72) is valid for any point belonging to the subset M_k.

From (72) we obtain finally the conditions required for a function to be considered as a physical quantity in the Hamiltonian formalism, namely [1, 4, 3]:

$$\{\mathcal{A}^H, \phi_\mu^{(s)}\} \overset{M_k}{=} 0, \qquad \begin{array}{l} s = 0,1,\ldots,k \\ \mu = 1,2,\ldots,m \end{array} \tag{73}$$

Therefore, the physical quantities in the Dirac-Hamilton formalism may be expressed as functions of the coordinates and the momenta. Their Poisson brackets with all the first-class constraints vanish in the subset M_k defined by all the constraints.

1.4 The Lagrangian approach

Let us consider a system described by the action (1). The Euler-Lagrange equations of motion may be written in the form (2-4). Assume the dimension of the kernel of the Hessian matrix W is constant in TQ and given by (5). We denote as $\gamma^{(\mu)}(q,\dot{q})$ ($\mu = 1, ..., m$) a set of linearly independent null eigenvectors of the Hessian matrix (6).

Using the Euler-Lagrange equations (2) and the identities (6) we can obtain the Lagrangian constraints of the first-level as [19, 21, 18]:

$$\chi_\mu^{(1)} = \gamma_i^{(\mu)} \alpha_i \qquad \mu = 1, 2, ..., m \tag{74}$$

The set of constraints $\chi_\mu^{(1)} = 0$ defines a subspace of the velocity phase space TQ which we denote as S_1. The consistency conditions require that the constraints are preserved in time due to the equations of motion. From the consistency conditions satisfied by the Lagrangian constraints of the first-level (74) the corresponding Lagrangian constraints of the second level $\chi_\mu^{(2)}$ [19, 18] can be obtained. These new constraints define a subspace of S_1 that can be denoted as S_2. From the consistency conditions on the Lagrangian constraints of the second-level we can obtain Lagrangian constraints of the third-level $\chi_\mu^{(3)}$, and so on. For a finite dimensional system this algorithm stops at some level k (no more new constraints appear from the consistency conditions on $\chi_\mu^{(k)}$). We assume that our Lagrangian is such that the consistency conditions on the Lagrangian constraints do not generate any new equations involving the accelerations $\ddot{q}$. This is equivalent to assuming that all the Lagrangian constraints are projectable or at least weakly projectable into the momentum phase space T^*Q.

First, let us consider the case when all the Lagrangian constraints are projectable into T^*Q. The necessary and sufficient conditions for a constraint $\chi_\mu^{(\ell)}$ to be projectable into T^*Q are [30, 18, 5]:

$$\gamma_i^{(\nu)} \frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_i} = 0 \ , \qquad \nu = 1, 2, ..., m \tag{75}$$

where (75) are satisfied in TQ.

The fact that the consistency conditions on $\chi_\mu^{(\ell)}$ do not generate new equations involving the accelerations is a direct consequence of (75). Indeed,

$$\frac{d\chi_\mu^{(\ell)}}{dt} = \frac{\partial \chi_\mu^{(\ell)}}{\partial q_j}\dot{q}_j + \frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_j}\ddot{q}_j \,, \tag{76}$$

and, from the identities (6) and (75) it follows that $\frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_j}$ can be written as a linear combination of W_{ij} (i=1,...,n):

$$\frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_j} = b_{\mu i}^{(\ell)}(q,\dot{q})W_{ij}(q,\dot{q}) \tag{77}$$

Substituting (77) into (76) we get:

$$\frac{d\chi_\mu^{(\ell)}}{dt} = \frac{\partial \chi_\mu^{(\ell)}}{\partial q_j}\dot{q}_j + b_{\mu i}^{(\ell)}\alpha_i + b_{\mu i}^{(\ell)}L_i \tag{78}$$

Notice that in (78) the only term involving the accelarations is the last one which is a linear combination of the left-hand-sides of the originally derived Euler-Lagrange equations (2). Therefore, the consistency conditions $\frac{d\chi_\mu^{(\ell)}}{dt} = 0$ do not generate any new equations involving $\ddot{q}$.

It is then clear that, when all the Lagrangian constraints are projectable, the consistency conditions take the form:

$$\begin{aligned}
\frac{d\chi_\mu^{(1)}}{dt} &= \chi_\mu^{(2)}(q,\dot{q}) + b_{\mu i}^{(1)}(q,\dot{q})L_i(q,\dot{q},\ddot{q}) \\
\frac{d\chi_\mu^{(2)}}{dt} &= \chi_\mu^{(3)}(q,\dot{q}) + b_{\mu i}^{(2)}(q,\dot{q})L_i(q,\dot{q},\ddot{q}) \\
&\dots \\
\frac{d\chi_\mu^{(k-1)}}{dt} &= \chi_\mu^{(k)}(q,\dot{q}) + b_{\mu i}^{(k-1)}(q,\dot{q})L_i(q,\dot{q},\ddot{q}) \\
\frac{d\chi_\mu^{(k)}}{dt} &= \sum_{\ell=1}^{k} a_{\mu\nu}^{(\ell)}(q,\dot{q})\,\chi_\nu^{(\ell)}(q,\dot{q}) + b_{\mu i}^{(k)}(q,\dot{q})L_i(q,\dot{q},\ddot{q})
\end{aligned} \tag{79}$$

Notice that, since the algorithm stops at level k, the consistency conditions on the constraints $\chi_\mu^{(k)}$ involve a linear combination of all the constraints. In general, the functions $a_{\mu\nu}^{(\ell)}$ depend on q and $\dot{q}$. Here, for simplicity, we assume that k is independent of μ. In general, of course, the chains of Lagrangian constraints can be of different length.

Our purpose in this Section is to obtain from (79) the Noether identities. First, it can be noticed that from (79) it follows:

$$\frac{d^k\chi_\mu^{(1)}}{dt^k} = \sum_{\ell=1}^{k} a_{\mu\nu}^{(\ell)}\chi_\nu^{(\ell)} + \sum_{\ell=0}^{k-1}\frac{d^\ell}{dt^\ell}(b_{\mu i}^{(k-\ell)}L_i) \qquad \mu = 1,2,...,m \tag{80}$$

From the definition (74) of the Lagrangian constraints of the first-level and the identities (6) we see that we can rewrite $\chi_\mu^{(1)}$ as follows:

$$\chi_\mu^{(1)} = -b_{\mu i}^{(0)}L_i \tag{81}$$

where,

$$b_{\mu i}^{(0)} = \gamma_i^{(\mu)} \tag{82}$$

On the other hand, from (79) it follows that:

$$\chi_\mu^{(\ell)} = -\sum_{s=0}^{\ell-1}\frac{d^s}{dt^s}(b_{\mu i}^{(\ell-1-s)}L_i) \qquad \ell = 1,2,...,k \tag{83}$$

Substituting (83) into (80), and redefining the summations, we get:

$$\sum_{\ell=0}^{k}\frac{d^\ell}{dt^\ell}(b_{\mu i}^{(k-\ell)}L_i) - \sum_{\ell=0}^{k-1}\sum_{s=\ell+1}^{k} a_{\mu\nu}^{(s)}\frac{d^\ell}{dt^\ell}(b_{\nu i}^{(s-\ell-1)}L_i) = 0 \tag{84}$$

It is possible to prove that for any arbitrary functions of time $A(t)$ and $B(t)$ the following relation holds:

$$A\frac{d^\ell B}{dt^\ell} = \frac{d^\ell}{dt^\ell}(AB) + \sum_{q=0}^{\ell-1}\frac{d^q}{dt^q}(v_{\ell-q}^{(\ell)}\frac{d^{\ell-q}A}{dt^{\ell-q}}B) \qquad \ell \geq 1 \tag{85}$$

where,

$$v_{\ell-q}^{(\ell)} = \sum_{\alpha=1}^{\ell-q} V_{\ell-q}^{(\alpha,\ell)} \tag{86}$$

and,

$$\begin{aligned} V_\sigma^{(1,\ell)} &= -\frac{\ell!}{\sigma!(\ell-\sigma)!} \\ V_\sigma^{(\alpha,\ell)} &= (-1)^\alpha \sum_{r_1=1}^{\sigma-\alpha+1}\ \sum_{r_2=1}^{\sigma-\alpha+2-r_1}\ \sum_{r_3=1}^{\sigma-\alpha+3-r_1-r_2} \cdots \sum_{r_{\alpha-1}=1}^{\sigma-1-r_1-r_2-\cdots-r_{\alpha-2}} \\ &\quad \frac{\ell!}{r_1!r_2!\cdots r_{\alpha-1}!(\sigma-r_1-r_2-\cdots-r_{\alpha-1})!(\ell-\sigma)!} \end{aligned} \tag{87}$$

$$\alpha \geq 2$$

Notice that $v^{(\ell)}_{\ell-q}$ are just some integer numbers.

It is convenient to make use of the relations (85-87) in (84). Indeed (85-87) allow us to rewrite the second term in (84) in a suitable form. As a result of this substitution we obtain that:

$$\sum_{\ell=0}^{k} \frac{d^\ell}{dt^\ell}(b^{(k-\ell)}_{\mu i} L_i) - \sum_{\ell=0}^{k-1} \frac{d^\ell}{dt^\ell}(\sum_{s=\ell+1}^{k} a^{(s)}_{\mu\nu} b^{(s-\ell-1)}_{\nu i} L_i) -$$
$$- \sum_{\ell=1}^{k-1} \sum_{q=0}^{\ell-1} \frac{d^q}{dt^q}(v^{(\ell)}_{\ell-q} \sum_{s=\ell+1}^{k} \frac{d^{\ell-q}}{dt^{\ell-q}} a^{(s)}_{\mu\nu}\, b^{(s-\ell-1)}_{\nu i} L_i) = 0 \tag{88}$$

By performing a proper redefinition of the summations in the last term in (88), we can rewrite (88) as follows:

$$\sum_{\ell=0}^{k} \frac{d^\ell}{dt^\ell}(b^{(k-\ell)}_{\mu i} L_i) - \sum_{\ell=0}^{(k-1)} \frac{d^\ell}{dt^\ell}(\sum_{s=\ell+1}^{k} a^{(s)}_{\mu\nu}\, b^{(s-\ell-1)}_{\nu i} L_i) -$$
$$- \sum_{\ell=0}^{k-2} \frac{d^\ell}{dt^\ell} \left[\sum_{q=\ell+1}^{k-1} \sum_{s=q+1}^{k} v^{(q)}_{q-\ell} \frac{d^{q-\ell} a^{(s)}_{\mu\nu}}{dt^{q-\ell}} b^{(s-q-1)}_{\nu i} L_i \right] = 0 \tag{89}$$

or equivalently:

$$\frac{d^k}{dt^k}(b^{(0)}_{\mu i} L_i) + \frac{d^{k-1}}{dt^{k-1}} \left[b^{(1)}_{\mu i} - a^{(k)}_{\mu\nu} b^{(0)}_{\nu i}) L_i \right] + \sum_{\ell=0}^{k-2} \frac{d^\ell}{dt^\ell} \left[(b^{(k-\ell)}_{\mu i} - \sum_{s=\ell+1}^{k} a^{(s)}_{\mu\nu} b^{(s-\ell-1)}_{\nu i} - \right.$$
$$\left. - \sum_{q=\ell+1}^{k-1} \sum_{s=q+1}^{k} v^{(q)}_{q-\ell} \frac{d^{q-\ell} a^{(s)}_{\mu\nu}}{dt^{q-\ell}} b^{(s-q-1)}_{\nu i}) L_i \right] = 0 \tag{90}$$

$$\mu = 1, 2, ..., m$$

The relations (90) are the Noether identities satisfied by the left-hand-sides $L_i = W_{ij}\ddot{q}_j - \alpha_i$ of the Euler-Lagrange equations. The Noether identities are valid for any trajectory $q(t)$ in configuration space Q. The second Noether theorem states that the Noether identities are necessary and sufficient conditions for the action (1) to be invariant under gauge transformations. The number of independent Noether identities is equal to the number of arbitrary functions of time involved in the gauge transformations. If the L_i satisfy the identities:

$$\sum_{\ell=0}^{k} (-1)^\ell \frac{d^\ell}{dt^\ell}(P^{(\ell)}_{\mu i} L_i) = 0 \qquad \mu = 1, 2, ..., m \tag{91}$$

where $P^{(\ell)}_{\mu i}$ are certain functions of q, $\dot{q}$, etc, then the action (1) is invariant under the gauge transformations:

$$q_i \rightarrow q_i + \delta q_i \ ,$$

where,

$$\delta q_i = \sum_{\ell=0}^{k} \frac{d^\ell \nu_\mu(t)}{dt^\ell} P^{(\ell)}_{\mu i} \tag{92}$$

and, vice versa, if (1) is invariant under (92) then the left-hand-sides L_i of the Euler-Lagrange equations satisfy the identities (91). Notice that $\nu_\mu(t)$ ($\mu = 1, ..., n$) are arbitrary functions of time.

In this Section, from the consistency conditions satisfied by the projectable Lagrangian constraints (79), we have obtained the Noether identities (91) in the form (90). The functions $P^{(\ell)}_{\mu i}$ are called Lagrangian generators of the gauge transformations (92). We see that when the Lagrangian constraints satisfy consistency conditions of the form (79) the Lagrangian generators of gauge transformations can be expressed in terms of the functions $b^{(\ell)}_{\mu i}$ and $a^{(\ell)}_{\mu\nu}$ defined in (79) and of the time derivatives of $a^{(\ell)}_{\mu\nu}$, as follows:

$$\begin{aligned}
P^{(k)}_{\mu i} &= (-1)^k b^{(0)}_{\mu i} \\
P^{(k-1)}_{\mu i} &= (-1)^{(k-1)} (b^{(1)}_{\mu i} - a^{(k)}_{\mu\nu} b^{(0)}_{\nu i}) \\
P^{(\ell)}_{\mu i} &= (-1)^\ell \left[b^{(k-\ell)}_{\mu i} - \sum_{s=\ell+1}^{k} a^{(s)}_{\mu\nu} b^{(s-\ell-1)}_{\nu i} - \right. \\
&\qquad \left. \sum_{q=\ell+1}^{k-1} \sum_{s=q+1}^{k} v^{(q)}_{q-\ell} \frac{d^{q-\ell} a^{(s)}_{\mu\nu}}{dt^{q-\ell}} b^{(s-q-1)}_{\nu i} \right]
\end{aligned} \tag{93}$$

$$\ell = 0, 1, ..., k-2$$

where $v^{(\ell)}_{\ell-q}$ are integer numbers defined in (86-87).

Notice that $P^{(k)}_{\mu i}$ and $P^{(k-1)}_{\mu i}$ depend only on q and $\dot{q}$, while $P^{(\ell)}_{\mu i}$ ($\ell = 0, 1, ..., k-2$) depend on $q, \dot{q}, \ddot{q}$ and higher derivatives up to $\frac{d^{k-\ell}q}{dt^{k-\ell}}$. This is due to the fact that in the expression for $P^{(\ell)}_{\mu i}$ the time derivatives of $a^{(k)}_{\mu\nu}$ appear up to order $k-1-\ell$ and, in general, $a^{(k)}_{\mu\nu}$ depend on q and $\dot{q}$ for $k \geq 2$. The gauge transformations δq_i, therefore, in general depend on $q, \dot{q}, \ddot{q}$ and higher time derivatives up to $\frac{d^k q}{dt^k}$ for $k \geq 2$. As was stated before, k denotes the level at which the constraint algorithm (79) stops.

Let us now consider the more general case when the Lagrangian constraints of the second, third and higher levels are weakly projectable into the momentum phase space T^*Q. The Lagrangian constraints of the first level are assumed to be strongly projectable. In other words, we are considering now the case when the Lagrangian constraints satisfy the following conditions [18]:

$$\gamma_i^{(\nu)} \frac{\partial \chi_\mu^{(1)}}{\partial \dot{q}_i} = 0 \tag{94}$$

$$\gamma_i^{(\nu)} \frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_i} \overset{S_{\ell-1}}{=} 0 \qquad \ell = 2, 3, \ldots k \tag{95}$$

The label above the equality in (95) indicates in which subspace of TQ the equality holds.

Notice that from (95) and the identities (6) it follows that for $\ell \geq 2$ *each* $\frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_j}$ should be equal to a linear combination of W_{ij} and the constraints (up to level $\ell - 1$):

$$\frac{\partial \chi_\mu^{(\ell)}}{\partial \dot{q}_j} = b_{\mu i}^{(\ell)}(q, \dot{q}) W_{ij}(q, \dot{q}) + \sum_{s=1}^{\ell-1} h_{\mu\nu j}^{(s,\ell)}(q, \dot{q}) \chi_\nu^{(s)} \tag{96}$$

For $\ell = 1$ we have as before:

$$\frac{\partial \chi_\mu^{(1)}}{\partial \dot{q}_j} = b_{\mu i}^{(1)}(q, \dot{q}) W_{ij}(q, \dot{q}) \tag{97}$$

The time derivatives of the Lagrangian constraints can be therefore expressed as follows:

$$\begin{aligned} \frac{d\chi_\mu^{(1)}}{dt} &= \frac{\partial \chi_\mu^{(1)}}{\partial q_i} \dot{q}_i + b_{\mu i}^{(1)} \alpha_i + b_{\mu i}^{(1)} L_i \\ \frac{d\chi_\mu^{(\ell)}}{dt} &= \frac{\partial \chi_\mu^{(\ell)}}{\partial q_i} \dot{q}_i + b_{\mu i}^{(\ell)} \alpha_i + b_{\mu i}^{(\ell)} L_i + \sum_{s=1}^{\ell-1} h_{\mu\nu i}^{(s,\ell)} \ddot{q}_i \chi_\nu^{(s)} \end{aligned} \tag{98}$$

$\ell = 2, 3, \ldots, k$

The consistency conditions satisfied by the Lagrangian constraints take the following more general form:

$$
\begin{aligned}
\frac{d\chi_\mu^{(1)}}{dt} &= \chi_\mu^{(2)}(q,\dot q) + b_{\mu i}^{(1)}(q,\dot q)L_i(q,\dot q,\ddot q) \\
\frac{d\chi_\mu^{(2)}}{dt} &= \chi_\mu^{(3)}(q,\dot q) + b_{\mu i}^{(2)}(q,\dot q)L_i(q,\dot q,\ddot q) + h_{\mu\nu j}^{(1,2)}(q,\dot q)\ddot q_j \chi_\nu^{(1)}(q,\dot q) \\
& \dots \\
\frac{d\chi_\mu^{(k-1)}}{dt} &= \chi_\mu^{(k)}(q,\dot q) + b_{\mu i}^{(k-1)}(q,\dot q)L_i(q,\dot q,\ddot q) + \sum_{s=1}^{k-2} h_{\mu\nu j}^{(s,k-1)}(q,\dot q)\ddot q_j \chi_\nu^{(s)}(q,\dot q) \\
\frac{d\chi_\mu^{(k)}}{dt} &= \sum_{\ell=1}^{k} a_{\mu\nu}^{(\ell)}(q,\dot q)\chi_\nu^{(\ell)}(q,\dot q) + b_{\mu i}^{(k)}(q,\dot q)L_i(q,\dot q,\ddot q) + \\
&\qquad + \sum_{s=1}^{k-1} h_{\mu\nu j}^{(s,k)}(q,\dot q)\ddot q_j \chi_\nu^{(s)}(q,\dot q)
\end{aligned}
\tag{99}
$$

This is the more general form that the consistency conditions can have, assuming that no new independent equations involving the accelerations are generated. Comparing the relations (99) with the relations (79) of the previously considered case, we can see that although (99) look rather unwieldy the same procedure that was used to obtain the Noether identities applies here. Indeed, from (99) it follows that:

$$
\frac{d^k\chi_\mu^{(1)}}{dt^k} = \sum_{\ell=1}^{k} a_{\mu\nu}^{(\ell)}\chi_\nu^{(\ell)} + \sum_{\ell=0}^{k-1} \frac{d^\ell}{dt^\ell}(b_{\mu i}^{(k-\ell)}L_i) + \sum_{\ell=0}^{k-1} \frac{d^\ell}{dt^\ell}\Big(\sum_{s=1}^{k-\ell-1} h_{\mu\nu j}^{(s,k-\ell)}\ddot q_j \chi_\nu^{(s)}\Big) \tag{100}
$$

$\mu = 1, 2, ..., m$

It is convenient to substitute (81) into (100), so that (100) can be rewritten in the form:

$$
\sum_{\ell=0}^{k} \frac{d^\ell}{dt^\ell}(b_{\mu i}^{(k-\ell)}L_i) + \sum_{\ell=1}^{k} a_{\mu\nu}^{(\ell)}\chi_\nu^{(\ell)} + \sum_{\ell=0}^{k-1} \frac{d^\ell}{dt^\ell}\Big(\sum_{s=1}^{k-\ell-1} h_{\mu\nu j}^{(s,k-\ell)}\ddot q_j \chi_\nu^{(s)}\Big) = 0 \tag{101}
$$

$\mu = 1, 2, ..., m$

On the other hand, from (99) it also follows that the constraints can be expressed in the following way:

$$
\chi_\mu^{(\ell)} = -\sum_{s=0}^{\ell-1} \frac{d^s}{dt^s}(B_{\mu i}^{(\ell-1-s)}L_i) \qquad \begin{aligned} &\ell = 1, 2, ..., k \\ &\mu = 1, 2, ..., m \end{aligned} \tag{102}
$$

The functions $B^{(q)}_{\mu i}$ depend on the b's and may also depend on the h's and the time derivatives of the h's. For example:

$$\begin{aligned} B^{(0)}_{\mu i} &= b^{(0)}_{\mu i} \\ B^{(1)}_{\mu i} &= b^{(1)}_{\mu i} \\ B^{(2)}_{\mu i} &= b^{(2)}_{\mu i} - h^{(1,2)}_{\mu\nu j}\ddot{q}_j b^{(0)}_{\nu i} \\ & \dots \end{aligned} \tag{103}$$

Substituting (102) into (101) we get:

$$\begin{aligned} &\sum_{\ell=0}^{k} \frac{d^\ell}{dt^\ell}(b^{(k-\ell)}_{\mu i} L_i) - \sum_{\ell=1}^{k}\sum_{s=0}^{\ell-1} a^{(\ell)}_{\mu\nu}\frac{d^s}{dt^s}(B^{(\ell-1-s)}_{\nu i} L_i) - \\ &- \sum_{\ell=0}^{k-1}\sum_{s=1}^{k-\ell-1}\sum_{r=0}^{s-1}\frac{d^\ell}{dt^\ell}(h^{(s,k-\ell)}_{\mu\nu j}\ddot{q}_j \frac{d^r}{dt^r}(B^{(\ell-1-r)}_{\nu i} L_i)) = 0 \end{aligned} \tag{104}$$

using (85-87) in (104) and properly redefining the summations, the Noether identities for the generic case can be obtained.

It is clear that once we have the Noether identities we can obtain the Lagrangian generators of gauge transformations.

When the Lagrangian is singular but the system has no Lagrangian constraints ((74) are identically satisfied in TQ), then the gauge transformations depend only on the coordinates q and the velocities $\dot{q}$, and do not depend on the accelerations or higher derivatives. The same is valid for singular Lagrangians generating only Lagrangian constraints of the first level. However, if the system has Lagrangian constraints of the second level (or higher order levels) then, in general, the gauge transformations depend on $q, \dot{q}$ and the accelerations $\ddot{q}$ (or higher order derivatives). For systems having Lagrangian constraints up to the level $k \geq 2$ the gauge transformations δq, under which the action (1) is invariant, depend on q, $\dot{q}$ and higher time derivatives up to $\frac{d^k q}{dt^k}$.

It may be worth remarking that the case of weakly projectable Lagrangian constraints is not really different from the case of strongly projectable constraints. In fact, in general it is always possible to redefine the weakly projectable Lagrangian constraints in order to make them strongly projectable into the momentum phase space T^*Q. This can be achieved by defining a new equivalent set of Lagrangian constraints, each new constraint being constructed as a proper linear combination of the old ones. For example, a weakly projectable Lagrangian constraint of the second-level can be transformed into a projectable one by adding to it a proper linear combination of the Lagrangian constraints of the first level. Analogously a weakly projectable Lagrangian constraint of the third-level can be made strongly projectable by adding to it a proper linear combination of the Lagrangian constraints of the first- and second-levels. So, it could be argued that there is no need for considering weakly projectable constraints. However, in practice the procedure described above for transforming weakly projectable constraints into projectable ones can be very complicated. In fact, it involves solving certain first order partial differential equations. This is why we decided to show in this Section a method suitable for systems with weakly projectable Lagrangian constraints.

1.5 Relation between the Dirac-Hamilton and the Lagrange methods

In Section 1.4 we have shown how to derive explicitly the Lagrangian generators of gauge transformations for singular Lagrangians that may generate projectable or weakly projectable constraints. The method developed here is carried out completely within the Lagrangian formalism.

The usual approach to this problem [20, 22, 23, 24, 26, 27] is based on the Dirac-Hamilton formalism and it was presented in Section 1.2. The method consists of constructing the generating function of infinitesimal canonical gauge transformations Ψ as a proper linear combination of the Hamiltonian constraints. Ψ should be a first class function [1] and should obey certain conditions (see Section 1.2). The Lagrangian gauge transformations derived from the Dirac-Hamilton formalism take the form:

$$\delta q_i = \frac{\partial \Psi}{\partial p_i}(t, q, \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \lambda(q, \dot{q}), \dot{\lambda}(q, \dot{q}, \ddot{q}), ...) \,. \tag{105}$$

It is interesting to discuss the relationship between our Lagrangian approach and the Dirac-Hamilton approach of Section 1.2. We have assumed in Section 1.4 that the consistency conditions on the Lagrangian constraints do not generate new equations involving the accelerations (or, equivalently, that the Lagrangian constraints are projectable or weakly projectable). As was proven in [18], new equations involving the accelerations in the Lagrangian formalism correspond to determining some of the Lagrange multipliers in the Dirac-Hamilton formalism. The latter can happen if and only if the system has second class constraints [1]. Therefore, our assumption that no new equations for $\ddot{q}$ are generated is equivalent to assuming that for the considered systems all the Hamiltonian constraints are first-class. This is precisely what was assumed in [23, 27] and in Section 1.2.

Our approach is in fact the Lagrangian counterpart of the approach used in [27]. In order to illustrate this, let us consider the example proposed in [27] of a particular

Lagrangian having the following form:

$$L = \frac{1}{2}[(\dot{q}_2 - e^{q_1})^2 + (\dot{q}_3 - q_2)^2] \tag{106}$$

The Euler-Lagrange equations derived from (106) are the following:

$$L_1 \equiv e^{q_1}(\dot{q}_2 - e^{q_1}) = 0 \tag{107}$$

$$L_2 \equiv \ddot{q}_2 - \dot{q}_1 e^{q_1} + \dot{q}_3 - q_2 = 0 \tag{108}$$

$$L_3 \equiv \ddot{q}_3 - \dot{q}_2 = 0 \tag{109}$$

From the point of view of the Lagrangian formalism the system described by (106) has the following Lagrangian constraints:

$$\chi^{(1)} = -e^{q_1}(\dot{q}_2 - e^{q_1}) \tag{110}$$

$$\chi^{(2)} = e^{q_1}(\dot{q}_3 - q_2) \tag{111}$$

The constraints (110) and (111) are of the first- and second-levels respectively. Both $\chi^{(1)}$ and $\chi^{(2)}$ are projectable into the momentum phase space T^*Q.

The consistency conditions on (110-111) take the form:

$$\frac{d\chi^{(1)}}{dt} = \chi^{(2)} - \dot{q}_1 L_1 - e^{q_1} L_2 \tag{112}$$

$$\frac{d\chi^{(2)}}{dt} = \dot{q}_1 \chi^{(2)} + e^{q_1} L_3 \tag{113}$$

Notice that (112-113) are of the form (79). Therefore we have,

$$\begin{aligned} &b_1^{(0)} = 1 \quad , \quad b_2^{(0)} = b_3^{(0)} = 0 \\ &b_1^{(1)} = -\dot{q}_1 \quad , \quad b_2^{(1)} = -e^{q_1}, \quad b_3^{(1)} = 0 \\ &b_1^{(2)} = b_2^{(2)} = 0 \quad , \quad b_3^{(2)} = e^{q_1} \end{aligned} \tag{114}$$

and,

$$a^{(1)} = 0 \quad , \quad a^{(2)} = \dot{q}_1 \tag{115}$$

Substituting (114) into (92-93) we finally get the gauge transformations:

$$\begin{aligned}
\delta q_1 &= \ddot{\nu} + 2\dot{\nu}\dot{q}_1 + \nu((\dot{q}_1)^2 + \ddot{q}_1) \\
\delta q_2 &= \dot{\nu} e^{q_1} + \nu \dot{q}_1 e^{q_1} \\
\delta q_3 &= \nu e^{q_1}
\end{aligned} \tag{116}$$

This result (116) is in perfect agreement with the result obtained in [27] for a system described by (106).

As stressed in [27], in general the canonical gauge transformations depend not only on the coordinates q and the momenta p but also on the Lagrangian multipliers and their time derivatives up to a certain order. In the approach proposed in Section 1.4, this corresponds to the fact that in general the Lagrangian gauge transformations depend not only on the coordinates q and the velocities $\dot{q}$ but also on the accelerations $\ddot{q}$ and higher time derivatives. In [36, 37] the problem of the existence of canonical gauge transformations independent of the Lagrange multipliers λ_μ and their time derivatives was considered. It was suggested in [37] that for systems having tertiary or higher order Hamiltonian constraints it seems in general impossible to eliminate the dependence on the λ's and their time derivatives. This fact is corroborated in the Lagrangian approach developed here. Indeed, tertiary or higher order Hamiltonian constraints correspond to Lagrangian constraints of the second-level or higher levels. As was shown above, for these systems ($k \geq 2$) the gauge transformations may depend on the accelerations or higher order derivatives. Nevertheless, it seems to us that this problem is still open to further study.

In [24, 26] the generating function of infinitesimal canonical gauge transformations Ψ was obtained for the more general case of systems having first- and second-class constraints. A classification of systems with second-class constraints was presented in [38]. It would be interesting to generalize the Lagrangian approach proposed here to that case.

2 Two Dimensional Gravity

2.1 Introduction

Two dimensional (2d) gravity is a subject that has attracted much attention in recent years because of its potential usefulness as a theoretical toy model for studying the end-point of Hawking radiation and because of its relation to string theory in non critical dimensions. Two dimensional theories of gravity could provide as well some insight into the difficulties that appear in the quantization of gravity in 3+1 dimensions.

The number of independent components of the Riemann curvature tensor in an N dimensional space-time is equal to [39]: $C_N = \frac{1}{12}N^2(N^2-1)$. For a two dimensional space-time (N=2) $C_2 = 1$. So, the 2d Riemann curvature tensor has only one independent component. Indeed, in 2 dimensions the Riemann curvature tensor can be expressed in terms of the scalar curvature as [39] $R_{\lambda\mu\eta\nu} = \frac{1}{2}R(g_{\lambda\eta}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\eta})$. Since by definition the Ricci tensor $R_{\mu\nu}$ is equal to $R^{\lambda}_{\mu\lambda\nu}$, then in two space-time dimensions we have

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}R(\delta^{\lambda}_{\lambda}g_{\mu\nu} - \delta^{\lambda}_{\mu}g_{\lambda\nu}) \\ &= \frac{1}{2}g_{\mu\nu}R \end{aligned} \tag{117}$$

In Einstein's General Theory of Relativity the gravitational dynamics is described by the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{118}$$

where R is the scalar curvature and Λ is the cosmological constant.

Since in 2 dimensions the Ricci tensor $R_{\mu\nu}$ is identically equal to $\frac{1}{2}g_{\mu\nu}R$, it is clear that 2d gravity cannot be described by Einstein's theory of General Relativity [40, 41]. Indeed, in two dimensions the equations (118) reduce to the following:

$$\Lambda g_{\mu\nu} = 0 \tag{119}$$

When the cosmological constant Λ is different from zero, (119) give the unacceptable solution $g_{\mu\nu} = 0$. If Λ vanishes then $g_{\mu\nu}$ is undetermined [41]. Therefore, it is necessary to go beyond Einstein's theory in order to construct a meaningful 2d theory of gravity [40, 41].

Several 2d models of gravity have been proposed. We will mention here some of the models that have received considerable attention.

Jackiw and Teitelboim (J-T)[40, 42, 43] developed a 2d geometrical model of gravity in which it is assumed that the gravitational field satisfies the following constant curvature equation:

$$R - \Lambda = 0 \tag{120}$$

In (120) R is the scalar curvature and Λ is a constant. Three cases can be considered: $\Lambda = 0$ (flat space-time), $\Lambda > 0$ (deSitter case) and $\Lambda < 0$ (anti-deSitter case).

Equation (120) is invariant under arbitrary general coordinate transformations $x^\mu \rightarrow x'^\mu = f^\mu(x)$. It is desirable to derive (120) from a local action principle such that the action is also invariant under general coordinate transformations. Equation (120) can be derived from the following diffeomorphism invariant action [40, 41]:

$$S = \int d^2x\sqrt{-g}\phi(R - \Lambda) \tag{121}$$

The local action (121) is not entirely geometrical because it involves a scalar auxiliary field ϕ[40, 41]. This seems to be a general feature of all 2d diffeomorphism invariant theories of gravity [44]. A purely geometrical diffeomorphism invariant action describing (120) should be non-local[44]. A local covariant action can be defined only if an auxiliary field is introduced into the theory[40, 41]. Notice that in two space-time dimensions the Hilbert-Einstein action $S = \int d^2x\sqrt{-g}R$ is simply a surface term [40, 41] (see section 2.3).

The auxiliary scalar field ϕ can be determined from the field equations:

$$\nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu\nu} \Lambda \phi = 0, \tag{122}$$

where ∇_μ is the covariant derivative with respect to x^μ. As it is shown in [45] the

Jackiw-Teitelboim model does have black hole solutions (space-time configurations with an event horizon) without any curvature singularity.

The exact Dirac quantization of the open J-T model was carried out by M. Henneaux in [46]. The covariant phase space quantization of Jackiw-Teitelboim gravity in the compact case was presented in [47].

Another 2d model of gravity derived from string theories [48, 49] was developed by Callan, Giddings, Harvey and Strominger (CGHS)[50] and other authors[51, 52]. The classical action of CGHS model (without matter fields) can be written as follows [50]:

$$S = \int d^2x\sqrt{-g}e^{-2\xi}(R + 4g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\xi + 4\lambda^2) \tag{123}$$

One of the peculiarities of this model is that its field equations have black hole solutions of the Schwarzschid type. In [53] the symplectic structure of the CGHS model for compact spatial manifolds was given. When scalar matter fields are incorporated into the model, the CGHS functional action takes the form [50]:

$$S = \int d^2x\sqrt{-g}\left[e^{-2\xi}(R + 4g^{\mu\nu}\nabla_\mu\xi\nabla_\nu\xi + 4\lambda^2) - \frac{1}{2}\sum_{i=1}^{N} g^{\mu\nu}\nabla_\mu f_i\nabla_\nu f_i\right] \tag{124}$$

In (124) the scalar fields f_i are minimally coupled to the gravitational field. Based on (124) the Hawking radiation of the 2d Schwarzschild black holes has been studied semiclassically [50, 51, 52]. In [54] the backreaction of the Hawking radiation on the geometry of an evaporating black hole has been studied also in the semiclassical approximation. As it is argued in [54] a full analysis of the problem requires going beyond the semiclassical approximation, i.e, considering quantum gravity effects. This is expected to provide an explanation for the so-called information loss paradox.

In [55] the Dirac quantization of the CGHS model minimally coupled to an even number of matter fields has been developed up to the point of finding explicitly the physical quantum wave functional.

A 2d model of gravity that seems to be directly connected with the 3+1 Einstein's General Relativity has been developed by Berger, Hajicek, Gegenberg, Kunstatter, Unruh and other authors[56, 57, 58, 59, 60, 61, 62].

This model can be obtained from the 3+1 Einstein's gravitational theory in the spherically symmetric case by dimensional reduction. A careful study of this 2d model may provide a clue for the quantum evolution of black holes in 3+1 dimensions.

The spherically symmetric model in two dimensions is described by the following action functional:

$$S = \int d^2x \sqrt{-g} e^{-2\xi} (R + 2g^{\mu\nu} \nabla_\mu \xi \nabla_\nu \xi + 2e^{2\xi}) \tag{125}$$

The field equations derived from (125) have black hole solutions of Schwarzschild type. The Hawking radiation of these black holes was studied in [60] using the semiclassical approximation. The Dirac quantization of the spherically symmetric model without matter fields has been carried out exactly in [61, 63, 64]. The theory is solvable classically and quantum mechanically. Some results concerning the Hawking radiation based on the exact form of the quantum wave functional were also presented in [61]. In [62] an analysis of the global physical observables for spherically symmetric gravity was given.

The most general action functional depending on the metric tensor $g_{\mu\nu}$ and a scalar field ϕ in two spacetime dimensions, such that it contains at most second derivatives of the fields can be written [65]:

$$S[g,\phi] = \int d^2x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + D(\phi) R \right) \tag{126}$$

where R is the Ricci scalar associated with $g_{\mu\nu}$ and $V(\phi)$ is an arbitrary function of ϕ.

Our purpose in the following sections is to study some properties of the space of solutions of the field equations for two dimensional models of gravity and also to develop the Hamiltonian analysis and exact Dirac quantization for such theories.

We propose a convenient parametrization for the functional action describing all two dimensional dilaton gravity theories. In this parametrization we obtain the field equations and prove that for any solution of these equations there always exists a coordinate frame in which the solution is independent either of the time coordinate or the spatial coordinate. We also prove that the solutions form a one-parameter family. This parameter is shown to be a physical observable invariant under arbitrary diffeomorphisms. These results remind us of the so-called Birkhoff's theorem in 3+1 General Relativity which states that all the spherically symmetric solutions of the Einstein's equations are static and parametrized by a single parameter. In other words, Birkhoff's theorem states that the Schwarzschild solutions are the only solutions of the Einstein's equations in the spherically symmetric case. Since the Berger-Hajicek model can be obtained by dimensional reduction from spherically symmetric 3+1 Einstein's General Relativity, it is clear that Birkhoff's theorem should be valid for this theory. Here we extend this result to all two dimensional dilaton gravity theories. Another proof of Birkhoff's theorem in 2d dilaton gravity was proposed recently by Kiem [66]. However, our analysis is considerably simpler than that of Kiem and allows us to make direct contact with a phase space analysis. Moreover, Kiem did not prove that the one parameter in his solutions was diffeomorphism invariant.

We also obtain the Noether identities and the generators of Lagrangian gauge transformations using the Lagrangian approach of Part I.

We carry out the Hamiltonian analysis of all 2d gravity theories. We identify the constraints of the system and obtain their Poisson brackets. The generating functional of the canonical gauge transformations is constructed as a suitable linear combination of the first class constraints. We prove that although the systems have no propagating local degrees of freedom, they do have two global degrees of freedom. It is shown that one of these global degrees of freedom coincides with the parameter of the solutions that is obtained from the study of the configuration space. It is invariant under arbitrary coordinate transformations. The other global degree of freedom is proven to be invariant under gauge transformations but not invariant under global diffeomorphisms. Based on this fact we explain the apparent discrepancy between the size of the covariant space of solutions and the size of the reduced phase space.

We develop the Dirac quantization for all 2d dilaton gravity models. By solving the constraints for the momenta and imposing these (so-called quantum) constraints on the physical wave functional we obtain the analog of the Wheeler-deWitt equations in two dimensions. We solve these equations and obtain an expression for the quantum wave functional. Some discussions about its properties is also presented.

It should be noticed that the results presented here are very general. They are valid for all two dimensional dilaton gravity theories. However, to some extent our analysis is carried out at a formal level. We do not discuss the properties of these models that are related to the presence of surface terms in the functional action or in the Hamiltonian. These properties are very important for a complete understanding of the theories but it is not possible to study them in such a general approach because the specific form of the surface terms is determined in part by the boundary conditions on the fields, which in principle vary from one specific model to another.

Most of the results presented in this part of the thesis have been published in [67, 68].

2.2 Functional action for 2d gravity. The field equations

In 2d theories of gravity the space-time is assumed to be a 2d manifold on which a Lorentz metric $g_{\mu\nu}$ having signature $-+$ is defined.

The models are generally assumed to be invariant under general coordinate transformations (diffeomorphisms): $x^\mu \to x'^\mu = f^\mu(x)$. The gravitational field is described by the metric tensor $g_{\mu\nu}$.

As usual, the Christoffel symbols are defined as follows [69]:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right) \tag{127}$$

where the contravariant second-rank tensor $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$:

$$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda \tag{128}$$

The Riemann curvature tensor is defined as [69]:

$$R^\lambda_{\mu\eta\nu} = -\frac{\partial \Gamma^\lambda_{\mu\eta}}{\partial x^\nu} + \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\eta} - \Gamma^\sigma_{\mu\eta}\Gamma^\lambda_{\nu\sigma} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\eta\sigma} \tag{129}$$

The Ricci tensor and the scalar curvature are [69]:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = -\frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} + \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\lambda\eta} \tag{130}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{131}$$

Let us consider the functional action

$$S[g,\phi] = \int d^2x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + D(\phi) R \right) \tag{132}$$

where $D(\phi)$ must be a differentiable function of ϕ, such that $D(\phi) \neq 0$, and $\frac{dD}{d\phi} \neq 0$ for any admissible value of ϕ. Note that if $D(\phi) = \mathit{constant}$, the curvature term decouples and becomes a total divergence, leaving a single, propagating scalar degree of freedom.

Notice that by performing a reparametrization of the scalar field $\phi = 2^{3/2} e^{-\xi}$, the CGHS action functional (123) takes the form

$$S = \int d^2x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{\lambda^2}{2}\phi^2 + \frac{\phi^2}{8} R \right) \tag{133}$$

By redefining the scalar field $\phi = 2e^{-\xi}$, the action (125) for spherically symmetric gravity can be written as

$$S = \int d^2x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2 + \frac{\phi^2}{4}R\right) \tag{134}$$

As we can see from (133) and (134) the CGHS and spherically symmetric gravity models are particular cases of (132). It can also be shown that the J-T action can be reparametrized to the form (132).

The action (132) contains only one independent function of ϕ. This can be readily seen as follows. Let us redefine the metric $g_{\mu\nu}$ and the scalar field ϕ as:

$$\overline{g}_{\mu\nu}(x) = \Omega^2(\phi(x))g_{\mu\nu}(x) \tag{135}$$

$$\overline{\phi}(x) = D(\phi(x)), \tag{136}$$

where $\Omega(\phi)$ is yet to be determined. Notice that (135) is a conformal transformation of the metric and (136) is a reparametrization of the scalar field. From (135), it follows that $g_{\mu\nu} = \Omega^{-2}\overline{g}_{\mu\nu}$ and therefore:

$$\sqrt{-g} = \Omega^{-2}\sqrt{-\overline{g}} \tag{137}$$

The Christoffel symbols (127), the Riemann curvature tensor (129), the Ricci tensor and the scalar curvature (130-131) can be defined for either $g_{\mu\nu}$ or $\overline{g}_{\mu\nu}$. The scalar curvature R associated with $g_{\mu\nu}$ can be expressed in terms of the scalar curvature $\overline{R}$ associated with $\overline{g}_{\mu\nu}$ [69] as follows:

$$R = \Omega^2\overline{R} - 2\Omega^3\overline{g}^{\mu\nu}\overline{\nabla}_\mu\overline{\nabla}_\nu(\Omega^{-1}) + 2\Omega^4\overline{g}^{\mu\nu}\overline{\nabla}_\mu(\Omega^{-1})\overline{\nabla}_\nu(\Omega^{-1}) \tag{138}$$

In (138) $\overline{\nabla}_\mu$ denotes the covariant derivative associated with $\overline{g}_{\mu\nu}$. It is not difficult to prove that:

$$\begin{aligned}\overline{\nabla}_\mu(\Omega^{-1}) &= -\frac{\overline{\nabla}_\mu\Omega}{\Omega^2}\\ \overline{\nabla}_\mu\overline{\nabla}_\nu(\Omega^{-1}) &= \frac{2}{\Omega^3}\overline{\nabla}_\mu\Omega\overline{\nabla}_\nu\Omega - \frac{1}{\Omega^2}\overline{\nabla}_\mu\overline{\nabla}_\nu\Omega\end{aligned} \tag{139}$$

By substituting (139) into (138) we get the following more convenient expression for R

$$R = \Omega^2 \overline{R} + 2\Omega \overline{g}^{\mu\nu} \overline{\nabla}_\mu \overline{\nabla}_\nu \Omega - 2\overline{g}^{\mu\nu} \overline{\nabla}_\mu \Omega \overline{\nabla}_\nu \Omega \tag{140}$$

Notice that in (140) Ω is defined as a function of ϕ. In order to express ϕ as a function of $\overline{\phi}$ we use (136). Our assumption $\frac{dD}{d\phi} \neq 0$ is crucial at this point because it guarantees that the inverse function $\phi = \phi(\overline{\phi})$ is well defined. It is not difficult to see that

$$\overline{\nabla}_\mu \phi = (\frac{dD}{d\phi})^{-1} \overline{\nabla}_\mu \overline{\phi}$$

The covariant derivative $\overline{\nabla}_\mu \Omega$ can therefore be expressed as:

$$\overline{\nabla}_\mu \Omega = \frac{d\Omega}{d\phi} (\frac{dD}{d\phi})^{-1} \overline{\nabla}_\mu \overline{\phi} \tag{141}$$

We also can write the second covariant derivative of Ω as follows:

$$\begin{aligned} \overline{\nabla}_\mu \overline{\nabla}_\nu \Omega \; &= \; \left(\frac{d^2\Omega}{d\phi^2} - \frac{d\Omega}{d\phi} (\frac{dD}{d\phi})^{-1} \frac{d^2 D}{d\phi^2} \right) (\frac{dD}{d\phi})^{-2} \overline{\nabla}_\mu \overline{\phi} \overline{\nabla}_\nu \overline{\phi} + \\ &\quad + \frac{d\Omega}{d\phi} (\frac{dD}{d\phi})^{-1} \overline{\nabla}_\mu \overline{\nabla}_\nu \overline{\phi} \end{aligned} \tag{142}$$

Substituting (135-136), (140) and (141-142) into (132) we obtain, using integration by parts, the following expression for the functional action:

$$\begin{aligned} S \; &= \; \int d^2x \sqrt{-\overline{g}} \left[\left(\frac{1}{2} - \frac{2}{\Omega} \frac{d\Omega}{d\phi} \frac{dD}{d\phi} \right) (\frac{dD}{d\phi})^{-2} \overline{g}^{\mu\nu} \overline{\nabla}_\mu \overline{\phi} \overline{\nabla}_\nu \overline{\phi} + \right. \\ &\quad \left. + \overline{\phi} \overline{R} - \frac{V\left(\phi(\overline{\phi})\right)}{\Omega^2 \left(\phi(\overline{\phi})\right)} \right] \end{aligned} \tag{143}$$

We see that the kinetic term for the scalar field can be eliminated if $\Omega(\phi)$ satisfies the differential equation:

$$\frac{1}{2} - 2\frac{dD}{d\phi} \frac{d \ln \Omega}{d\phi} = 0 \tag{144}$$

Therefore, by means of the field redefinitions (135-136) with Ω being a solution of (144), the action functional (132) takes the simpler form

$$S = \int d^2x \sqrt{-\overline{g}} (\overline{\phi}\, \overline{R} - \overline{V}(\overline{\phi})) \tag{145}$$

Notice that

$$\overline{V}(\overline{\phi}) \equiv \frac{V(\phi(\overline{\phi}))}{\Omega^2(\phi(\overline{\phi}))} \tag{146}$$

The conditions on $D(\phi)$ given above are sufficient, formally, to guarantee the existence of $\overline{V}$

In Jackiw-Teitelboim model of gravity the action (121) is already in the form (145) where $\overline{V} = \Lambda\overline{\phi}$. For the CGHS model the action (133) can be reparametrized to the form (145) with $\overline{V} = -\frac{\lambda^2}{2}$. In the case of spherically symmetric gravity (134) can be rewritten in the form (145) with $\overline{V} = -\frac{1}{\sqrt{\overline{\phi}}}$.

The parametrization of the functional action in the form (145) proves to be extremely useful. As shown in the following sections, in this form it is easier to study the space of solutions. It also simplifies the Hamiltonian analysis because the Hamiltonian constraints associated with the functional action (145) turn out to be solvable for the momenta.

Therefore, we study the action functional in the parametrization

$$S = \int d^2x\sqrt{-g}(\phi R - V(\phi)) \tag{147}$$

where no kinetic term for the scalar field is present. Notice that for convenience all the bars over the fields have been dropped.

Let us derive the field equations from (147) using the minimal action principle. The solutions of the field equations are the extremal configurations for which the variation of the action (147) vanishes:

$$\delta S = 0 \tag{148}$$

The variation (148) can be written

$$\begin{aligned} \delta S &= \int d^2x\delta\left(\sqrt{-g}(\phi R - V(\phi))\right) \\ &= \int d^2x\left[\delta\phi\sqrt{-g}(R - \frac{dV}{d\phi}) + \delta(\sqrt{-g}R)\phi - \delta(\sqrt{-g})V(\phi)\right] \end{aligned} \tag{149}$$

It is not difficult to prove that [39]:

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{150}$$

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} \tag{151}$$

On the other hand,

$$\delta(\sqrt{-g}R) = \delta(\sqrt{-g})R + \sqrt{-g}\delta g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} \tag{152}$$

Substituting (150-151) into (152) we obtain

$$\delta(\sqrt{-g}R) = -\delta g_{\mu\nu}\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} \tag{153}$$

The first term in (153) vanishes identically in two space-time dimensions (117), and therefore

$$\delta(\sqrt{-g}R) = \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} \tag{154}$$

In order to evaluate the variations of the components of the Ricci tensor we make use of the Palatini identities [39]:

$$\delta R_{\alpha\beta} = \nabla_\beta(\delta\Gamma^\lambda_{\alpha\lambda}) - \nabla_\lambda(\delta\Gamma^\lambda_{\alpha\beta}) \tag{155}$$

The covariant derivatives in (155) are well defined because $\delta\Gamma^\lambda_{\mu\nu}$ is a tensor.

Since the covariant derivative of the metric tensor is equal to zero we can write

$$\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= \frac{\partial}{\partial x^\eta}\left(g^{\alpha\beta}\delta\Gamma^\eta_{\alpha\beta} - g^{\alpha\eta}\delta\Gamma^\rho_{\alpha\rho}\right) \\ &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\eta}\left(\sqrt{-g}(g^{\alpha\beta}\delta\Gamma^\eta_{\alpha\beta} - g^{\alpha\eta}\delta\Gamma^\rho_{\alpha\rho})\right) \end{aligned} \tag{156}$$

From the definition of the Christoffel symbols (127) it follows

$$\frac{\partial g_{\alpha\beta}}{\partial x^\sigma} = \Gamma^\rho_{\sigma\alpha}g_{\rho\beta} + \Gamma^\rho_{\sigma\beta}g_{\rho\alpha} \tag{157}$$

The variations of the Christoffel symbols can be expressed as functions of the components of the metric tensor, their first partial derivatives, the variations of the components of the metric tensor and their derivatives. Substituting the first partial derivatives of the components of the metric tensor by (157) we obtain

$$
\begin{aligned}
g^{\alpha\beta}\delta\Gamma^{\eta}_{\alpha\beta} - g^{\alpha\eta}\delta\Gamma^{\rho}_{\alpha\rho} &= \left(g^{\eta\rho}g^{\sigma\mu}\Gamma^{\nu}_{\rho\sigma} - g^{\eta\mu}g^{\sigma\rho}\Gamma^{\nu}_{\sigma\rho}\right)\delta g_{\mu\nu} + \\
&\quad + \left(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}\right)\frac{\partial(\delta g_{\mu\nu})}{\partial x^{\sigma}} \qquad (158)
\end{aligned}
$$

So, finally, for the variation $\delta(\sqrt{-g}R)$ (154) we have

$$
\begin{aligned}
\delta(\sqrt{-g}R) &= \frac{\partial}{\partial x^{\eta}}\Big[\sqrt{-g}\left(g^{\eta\rho}g^{\sigma\mu}\Gamma^{\nu}_{\rho\sigma} - g^{\eta\mu}g^{\sigma\rho}\Gamma^{\nu}_{\sigma\rho}\right)\delta g_{\mu\nu} + \\
&\quad + \sqrt{-g}\left(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}\right)\frac{\partial(\delta g_{\mu\nu})}{\partial x^{\sigma}}\Big] \qquad (159)
\end{aligned}
$$

Substituting (150) and (159) into (149) we obtain (up to surface terms):

$$
\begin{aligned}
\delta S &= \int d^2x \Big[\delta\phi\sqrt{-g}(R - \frac{dV}{d\phi}) + \delta g_{\mu\nu}\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}V(\phi) - \\
&\quad - \frac{\partial\phi}{\partial x^{\eta}}\left(g^{\eta\rho}g^{\sigma\mu}\Gamma^{\nu}_{\rho\sigma} - g^{\eta\mu}g^{\sigma\rho}\Gamma^{\nu}_{\sigma\rho}\right) + \\
&\quad + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial\phi}{\partial x^{\eta}}\sqrt{-g}\left(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}\right)\right)\Big)\Big] \qquad (160)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\sigma}}\left(\frac{\partial\phi}{\partial x^{\eta}}\sqrt{-g}\left(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}\right)\right) = \\
&= \frac{\partial^2\phi}{\partial x^{\sigma}\partial x^{\eta}}(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}) + \\
&\quad + \frac{\partial\phi}{\partial x^{\eta}}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\sigma}}(\sqrt{-g})(g^{\mu\sigma}g^{\nu\eta} - g^{\mu\nu}g^{\sigma\eta}) + \\
&\quad + \frac{\partial\phi}{\partial x^{\eta}}\left(\frac{\partial g^{\mu\sigma}}{\partial x^{\sigma}}g^{\nu\eta} + g^{\mu\sigma}\frac{\partial g^{\nu\eta}}{\partial x^{\sigma}} - \frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}g^{\sigma\eta} - g^{\mu\nu}\frac{\partial g^{\sigma\eta}}{\partial x^{\sigma}}\right) \qquad (161)
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
\frac{\partial}{\partial x^{\sigma}}(\sqrt{-g}) &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^{\sigma}} \qquad (162) \\
\frac{\partial g^{\alpha\beta}}{\partial x^{\sigma}} &= -g^{\alpha\mu}g^{\beta\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \qquad (163)
\end{aligned}
$$

and the definition of the second covariant derivative of scalar fields

$$
\nabla_{\mu}\nabla_{\nu}\phi = \frac{\partial^2\phi}{\partial x^{\mu}\partial x^{\nu}} - \Gamma^{\rho}_{\mu\nu}\frac{\partial\phi}{\partial x^{\rho}} \qquad (164)
$$

we finally obtain the following expression for the variation of the action functional:

$$\begin{aligned} \delta S &= \int d^2x \left[\delta\phi\sqrt{-g}(R - \frac{dV}{d\phi}) + \delta g_{\mu\nu}\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}V(\phi) + \right.\right. \\ &\left.\left. + \nabla^\mu\nabla^\nu\phi - g^{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi \right)\right] \end{aligned} \tag{165}$$

It is important to notice that a surface term should be added to the action functional (147) for consistency. The variation of such a surface term must cancel with the surface term obtained for δS. Provided that this cancellation takes place, the extremals of the action (147) are those configurations that satisfy the following field equations:

$$R - \frac{dV}{d\phi} = 0 \tag{166}$$

$$-\frac{1}{2}g_{\mu\nu}V(\phi) + \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0 \tag{167}$$

Equations (167) can be written in a more compact form. Indeed, multiplying (167) by $g^{\mu\nu}$ we obtain

$$-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}V(\phi) + g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - g^{\mu\nu}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0 \tag{168}$$

Since $g^{\mu\nu}g_{\mu\nu} = 2$, then

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = -V(\phi) \tag{169}$$

Substituting (169) into (167) we get

$$\nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) = 0 \tag{170}$$

Therefore, the field equations (166-167) can be rewritten in the form

$$R - \frac{dV}{d\phi} = 0 \tag{171}$$

$$\nabla_\mu\nabla_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) = 0 \tag{172}$$

As it can be noticed the scalar curvature R appears in the action (147) and also, of course, in the field equations. Sometimes it is convenient to express $\sqrt{-g}R$ in the form [3]:

$$\sqrt{-g}R = -G - \partial_\mu\omega^\mu \tag{173}$$

where,

$$G = \sqrt{-g}g^{\mu\nu}(-\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}) \tag{174}$$

$$\omega^{\mu} = \sqrt{-g}(g^{\mu\nu}\Gamma^{\lambda}_{\nu\lambda} - g^{\nu\lambda}\Gamma^{\mu}_{\nu\lambda}) \tag{175}$$

The formulae (173-175) are valid for any N-dimensional space-time. They are particularly useful in the Lagrangian and Hamiltonian formulations of 3+1 Einstein's Theory of Relativity because they allow us to rewrite the Hilbert-Einstein action functional $\int d^2x\sqrt{-g}R$ (which contains second derivatives of the metric tensor) as an action depending only on the metric and its first derivatives. In two-dimensional gravity we use (173-175) as a convenient way to compute the scalar curvature. As we will see below, in two space-time dimensions $\sqrt{-g}R$ is a total divergence.

2.3 Field equations in the ADM parametrization

The two-dimensional space-time is assumed to be locally a direct product $R \times \Sigma$, where the spatial manifold Σ can at this stage be either open or closed. In any coordinate frame the metric tensor can be parametrized as follows [44]:

$$g_{\mu\nu} = \begin{pmatrix} -\sigma^2 + M^2 & M \\ M & 1 \end{pmatrix} e^{2\rho} \tag{176}$$

where M and σ are the shift and lapse functions.

From (176) it immediately follows that $\sqrt{-g} = \sigma e^{2\rho}$. In this parametrization (176) the contravariant metric tensor (128) takes the form

$$g^{\mu\nu} = \begin{bmatrix} -1 & M \\ M & \sigma^2 - M^2 \end{bmatrix} \frac{e^{-2\rho}}{\sigma^2} \tag{177}$$

A straightforward calculation of the Christoffel symbols (127) gives:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{\sigma^2}\{\dot{\sigma}\sigma + \sigma^2\dot{\rho} + M^2\dot{\rho} + M\sigma^2\rho' - M^3\rho' + M\sigma'\sigma - M'M^2\} \\
\Gamma^0_{10} &= \frac{1}{\sigma^2}\{\sigma'\sigma - M'M + \sigma^2\rho' - M^2\rho' + M\dot{\rho}\} \\
\Gamma^0_{11} &= \frac{1}{\sigma^2}\{\dot{\rho} - M' - M\rho'\} \\
\Gamma^1_{00} &= \frac{1}{\sigma^2}\{-M\dot{\sigma}\sigma + \sigma^2\dot{M} + (\sigma^2 - M^2)(\sigma\sigma' - MM' + M\dot{\rho} + \sigma^2\rho' - M^2\rho')\} \\
\Gamma^1_{10} &= \frac{1}{\sigma^2}\{-M\sigma'\sigma + M^2M' - M\sigma^2\rho' + M^3\rho' + \sigma^2\dot{\rho} - M^2\dot{\rho}\} \\
\Gamma^1_{11} &= \frac{1}{\sigma^2}\{-M\dot{\rho} + MM' + \sigma^2\rho' + M^2\rho'\}
\end{aligned} \tag{178}$$

Dots and primes denote differentiation with respect to time and space respectively. Substituting (178) into (174-175) yields

$$\begin{aligned}
G &= \frac{1}{\sigma^2}\{\sigma'\dot{M} - \dot{\sigma}M'\} \\
&= -\frac{\partial}{\partial x}(\frac{\dot{M}}{\sigma}) + \frac{\partial}{\partial t}(\frac{M'}{\sigma})
\end{aligned} \tag{179}$$

$$\omega^0 = \frac{1}{\sigma}\{2M\rho' + M' - 2\dot{\rho}\} \tag{180}$$

$$\omega^1 = \frac{1}{\sigma}\{\dot{M} + 2\sigma\sigma' - 2MM' + 2M\dot{\rho} + 2\sigma^2\rho' - 2M^2\rho'\} \tag{181}$$

And, therefore, substituting (179-181) into (173) we obtain

$$\begin{aligned}\sqrt{-g}R &= -\frac{\partial}{\partial t}\left[\frac{1}{\sigma}(2M\rho' + 2M' - 2\dot{\rho})\right] - \\ &\quad -\frac{\partial}{\partial x}\left[\frac{1}{\sigma}(2\sigma\sigma' - 2MM' + 2M\dot{\rho} + 2\sigma^2\rho' - 2M^2\rho')\right] \end{aligned} \tag{182}$$

We see that the combination $\sqrt{-g}R$ in two space-time dimensions is a total divergence. Since $\sqrt{-g} = \sigma e^{2\rho}$, the scalar curvature is given by

$$\begin{aligned} R &= -\frac{e^{-2\rho}}{\sigma}\frac{\partial}{\partial t}\left[\frac{1}{\sigma}(2M\rho' + 2M' - 2\dot{\rho})\right] - \\ &\quad -\frac{e^{-2\rho}}{\sigma}\frac{\partial}{\partial x}\left[\frac{1}{\sigma}(2\sigma\sigma' - 2MM' + 2M\dot{\rho} + 2\sigma^2\rho' - 2M^2\rho')\right] \end{aligned} \tag{183}$$

Let us rewrite the field equations (171-172) in the parametrization (176). Notice that by definition of the covariant derivative:

$$\nabla_\mu\nabla_\nu\phi = \frac{\partial^2\phi}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial\phi}{\partial x^\lambda} \tag{184}$$

Therefore, substituting (183) into (171) and (178,184) into (172) we obtain the field equations in the form

$$\begin{aligned} A &\equiv -2\frac{e^{-2\rho}}{\sigma^2}\Big[-\ddot{\rho} + \sigma^2\rho'' + \dot{M}\rho' + 2M\dot{\rho}' + \dot{M}' + \sigma\sigma'' - (M')^2 - \\ &\quad -MM'' + M'\dot{\rho} + +2\sigma\sigma'\rho' - 2MM'\rho' - M^2\rho'' - \\ &\quad -\frac{\dot{\sigma}}{\sigma}(M\rho' + M' - \dot{\rho}) - \frac{\sigma'}{\sigma}(-MM' + M\dot{\rho} + \sigma^2\rho' - M^2\rho')\Big] - \frac{dV}{d\phi} \\ &= 0 \end{aligned} \tag{185}$$

$$\begin{aligned} B_{00} &\equiv \ddot{\phi} - \frac{\dot{\phi}}{\sigma^2}(\sigma^2\dot{\rho} + \sigma\dot{\sigma} + M^2\dot{\rho} + M\sigma^2\rho' - M^3\rho' + M\sigma\sigma' - M^2M') - \\ &\quad -\frac{\phi'}{\sigma^2}\Big((\sigma^2 - M^2)(\sigma^2\rho' - M^2\rho' + \sigma\sigma' - MM' + M\dot{\rho}) - M\sigma\dot{\sigma} + \sigma^2\dot{M}\Big) - \\ &\quad -\frac{1}{2}(\sigma^2 - M^2)e^{2\rho}V(\phi) \\ &= 0 \end{aligned} \tag{186}$$

$$\begin{aligned} B_{01} &\equiv \dot{\phi}' - \frac{\dot{\phi}}{\sigma^2}(\sigma^2\rho' + \sigma\sigma' - MM' - M^2\rho' + M\dot{\rho}) - \\ &\quad -\frac{\phi'}{\sigma^2}(\sigma^2\dot{\rho} - M\sigma\sigma' + M^2M' - M\sigma^2\rho' + M^3\rho' - M^2\dot{\rho}) + \frac{1}{2}Me^{2\rho}V(\phi) \end{aligned}$$

$$
\begin{aligned}
&= 0 && (187)\\
B_{11} &\equiv \phi'' - \frac{\dot\phi}{\sigma^2}(\dot\rho - M' - M\rho') - \frac{\phi'}{\sigma^2}(\sigma^2\rho' - M\dot\rho + MM' + M^2\rho') + \\
&+\frac{1}{2}e^{2\rho}V(\phi) \\
&= 0 && (188)
\end{aligned}
$$

Clearly (187-188) do not contain second time derivatives of the fields. They are the Lagrangian constraints of the first level in this theory.

Another equivalent way of obtaining the field equations is to rewrite the functional action (147) in the ADM parametrization (176) and make use of the minimal action principle. Indeed, substituting (182) into (147) we obtain (up to surface terms):

$$
\begin{aligned}
S &= \int dtdx \left[\frac{\dot\phi}{\sigma}(2M\rho' + 2M' - 2\dot\rho) + \frac{\phi'}{\sigma}(2\sigma\sigma' - 2MM' + 2M\dot\rho \right.\\
&\qquad \left. + 2\sigma^2\rho' - 2M^2\rho') - \sigma e^{2\rho}V(\phi)\right] && (189)
\end{aligned}
$$

Computing the variation of (189) with respect to the fields ϕ, ρ, σ and M one can obtain the field equations in the form:

$$
\begin{aligned}
L_\phi &\equiv -\frac{\partial}{\partial t}\left(\frac{2}{\sigma}(M\rho' + M' - \dot\rho)\right) - 2\sigma'' - 2(\sigma\rho')' - \sigma e^{2\rho}\frac{dV}{d\phi} + \\
&+\frac{\partial}{\partial x}\left(\frac{2M}{\sigma}(M\rho' + M' - \dot\rho)\right) = 0 && (190)\\
L_\rho &\equiv -\frac{\partial}{\partial t}\left(\frac{2}{\sigma}(-\dot\phi + M\phi')\right) + \frac{\partial}{\partial x}\left(\frac{2M}{\sigma}(-\dot\phi + M\phi')\right) - \\
&-2(\sigma\phi')' - 2\sigma e^{2\rho}V(\phi) = 0 && (191)\\
L_\sigma &\equiv -2\phi'' + 2\phi'\rho' + \frac{2}{\sigma^2}(M\rho' + M' - \dot\rho)(-\dot\phi + M\phi') - \\
&-e^{2\rho}V(\phi) = 0 && (192)\\
L_M &\equiv -\frac{2\rho'}{\sigma}(-\dot\phi + M\phi') - \frac{2\phi'}{\sigma}(M\rho' + M' - \dot\rho) + \\
&+\frac{\partial}{\partial x}\left(\frac{2}{\sigma}(-\dot\phi + M\phi')\right) = 0 && (193)
\end{aligned}
$$

It is not difficult to prove that (190-193) are equivalent to (185-188). Indeed,

$$
L_\phi = \sigma e^{2\rho}A \qquad (194)
$$

$$
\begin{aligned}
L_\rho &= \frac{2}{\sigma}(B_{00} - MB_{01} - \sigma^2 B_{11}) && (195)\\
L_\sigma &= \frac{1}{2}B_{11} && (196)\\
L_M &= \frac{2}{\sigma}(B_{01} - \frac{M}{4}B_{11}) && (197)\\
& && (198)
\end{aligned}
$$

Obviously, L_σ and L_M are the Lagrangian constraints of the first level since they do not involve second time derivatives.

2.4 Lagrangian gauge transformations

The functional action (147) of two dimensional gravity is invariant under general coordinate transformations (diffeomorphisms) $x^\mu \to x'^\mu = x^\mu + f^\mu(x)$, where f^μ ($\mu = 0, 1$) are differentiable functions of x^μ. Under diffeomorphisms the components of the metric tensor and the scalar field transform as follows:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu} g_{\alpha\beta}(x) \tag{199}$$

$$\phi'(x') = \phi(x) \tag{200}$$

The invariance of the functional action (147) under diffeomorphisms can be interpreted as an invariance under gauge transformations of the form:

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \tag{201}$$

$$\phi'(x) = \phi(x) + \delta\phi(x) \tag{202}$$

where, for infinitesimal gauge transformations, we can write

$$\delta g_{\mu\nu}(x) = -\frac{\partial g_{\mu\nu}}{\partial x^\lambda} f^\lambda - g_{\mu\lambda}\frac{\partial f^\lambda}{\partial x^\nu} - g_{\lambda\nu}\frac{\partial f^\lambda}{\partial x^\mu} \tag{203}$$

$$\delta\phi(x) = -\frac{\partial \phi}{\partial x^\lambda} f^\lambda \tag{204}$$

Here it is important to notice that, contrary to the case of Electrodynamics or Yang-Mills theories, the Lagrangian $\sqrt{-g}(\phi R - V(\phi))$ is not invariant under the gauge transformations (203-204). However, it is not difficult to prove that its variation is equal to a total divergence:

$$\delta\left(\sqrt{-g}(\phi R - V(\phi))\right) = -\partial_\mu\left(f^\mu \sqrt{-g}(\phi R - V(\phi))\right) \tag{205}$$

This guarantees the invariance of the action (147) under gauge transformations when the functions f^μ vanish at the boundaries. This is the price that one has to pay for dealing with gravity as gauge field theory. It is convenient to use the gauge field theory approach because in it we can carry out the Hamiltonian analysis and the Dirac quantization. However we need to keep in mind that the gauge transformations (being defined for test functions which vanish at spatial infinity) are more restricted than the generic diffeomorphism transformations (199-200).

In the ADM parametrization (176):

$$\begin{aligned} g_{00} &= (-\sigma^2 + M^2)e^{2\rho} \\ g_{01} &= g_{10} = Me^{2\rho} \\ g_{11} &= e^{2\rho} \end{aligned} \tag{206}$$

Using (203-204) and the relations (206) we obtain (after some algebraic calculations) the variations of the fields ϕ, ρ, σ and M to be

$$\delta\phi = \xi\phi' + \frac{\eta}{\sigma}(\dot{\phi} - M\phi') \tag{207}$$

$$\delta\rho = \xi\rho' + \xi' + \frac{\eta}{\sigma}(\dot{\rho} - M\rho' - M') \tag{208}$$

$$\delta\sigma = \dot{\eta} + \xi\sigma' - \xi'\sigma + \eta M' - \eta' M \tag{209}$$

$$\delta M = \dot{\xi} + \xi M' - \xi' M + \eta\sigma' - \eta'\sigma \tag{210}$$

where

$$\begin{aligned} \eta &= -\sigma f^0 \\ \xi &= -f^1 - Mf^0 \end{aligned} \tag{211}$$

In the standard generally covariant description of gravity two configurations $(g'_{\mu\nu}(x'), \phi'(x'))$ and $(g_{\mu\nu}(x), \phi(x))$ are regarded as physically indistinguishable if there exists a diffeomorphism $x^\mu \to x'^\mu = x^\mu + f^\mu(x)$, such that (199,200). In the gauge field theory approach two configurations $(g'_{\mu\nu}(x), \phi'(x))$ and $(g_{\mu\nu}(x), \phi(x))$ are regarded as physically indistinguishable if there exists a gauge transformation (203,204) such that (201,202) are valid. Therefore, some configurations, which according to the standard generally covariant description are physically equivalent, can be regarded as physically inequivalent from the point of view of gauge field theory.

As was shown in Section 1.4 it is possible to obtain the Lagrangian gauge transformations by finding all the Lagrangian constraints and the consistency equations that all these constraints satisfy. Let us now show how this method can be applied to 2d gravity. In Section 2.3 we found the Lagrangian constraints of the first level for 2d gravity to be L_σ and L_M (192,193).

The consistency conditions on the Lagrangian constraints are:

$$\begin{aligned}
\dot{L}_M &= (ML_M)' + M'L_M + (\sigma L_\sigma)' + \sigma' L_\sigma + \phi' L_\phi - L_\rho' + \rho' L_\rho && (212)\\
\dot{L}_\sigma &= (ML_\sigma)' + M'L_\sigma + (\sigma L_M)' + \sigma' L_M + \frac{1}{\sigma}(\dot{\phi} - M\phi')L_\phi + \\
&\quad + \frac{1}{\sigma}(\dot{\rho} - M\rho' - M')L_\rho && (213)
\end{aligned}$$

As it can be seen from (212,213) the time derivatives of the constraints L_σ and L_M are equal to a linear combination of L_σ, L_M and their spatial derivatives plus a linear combination of the left hand sides of the remaining Euler-Lagrange field equations (190,191) and their spatial derivatives. Therefore, no new constraints are generated. L_σ and L_M are the only Lagrangian constraints of the theory. It is easy recognize that the consistency equations (212,213) are of the form (79). The Lagrangian constraints L_σ and L_M are strongly projectable into momentum phase space (see section 2.7).

We can rewrite (212,213) in the form:

$$\begin{aligned}
&\int dt'dx' \, (R_{\xi M}(t,x;t',x')L_M(t',x') + R_{\xi\sigma}(t,x;t',x')L_\sigma(t',x') + \\
&\quad + R_{\xi\phi}(t,x;t',x')L_\phi(t',x') + R_{\xi\rho}(t,x;t',x')L_\rho(t',x')) = 0 \qquad (214)
\end{aligned}$$

$$\begin{aligned}
&\int dt'dx' \, (R_{\eta M}(t,x;t',x')L_M(t',x') + R_{\eta\sigma}(t,x;t',x')L_\sigma(t',x') + \\
&\quad + R_{\eta\phi}(t,x;t',x')L_\phi(t',x') + R_{\eta\rho}(t,x;t',x')L_\rho(t',x')) = 0 \qquad (215)
\end{aligned}$$

where

$$\begin{aligned}
R_{\xi M}(t,x;t',x') &= R_{\eta\sigma}(t,x;t',x') \\
&= \delta(x-x')\frac{\partial}{\partial t'}\delta(t-t') - \delta(t-t')M(t',x')\frac{\partial}{\partial x'}\delta(x-x') + \\
&\qquad + \frac{\partial M}{\partial x'}(t',x')\delta(t-t')\delta(x-x') && (216)\\
R_{\xi\sigma}(t,x;t',x') &= R_{\eta M}(t,x;t',x')
\end{aligned}$$

$$
\begin{aligned}
&= -\delta(t-t')\sigma(t',x')\frac{\partial}{\partial x'}\delta(x-x') + \\
&\qquad +\frac{\partial \sigma}{\partial x'}(t',x')\delta(t-t')\delta(x-x') \qquad (217)
\end{aligned}
$$

$$
\begin{aligned}
R_{\xi\phi}(t,x;t',x') &= \frac{\partial \phi}{\partial x'}(t',x')\delta(t-t')\delta(x-x') && (218)\\
R_{\xi\rho}(t,x;t',x') &= \delta(t-t')\frac{\partial}{\partial x'}\delta(x-x') + \frac{\partial \rho}{\partial x'}(t',x')\delta(t-t')\delta(x-x') && (219)\\
R_{\eta\phi}(t,x;t',x') &= -\frac{1}{\sigma}(-\dot{\phi} + M\phi')\delta(t-t')\delta(x-x') && (220)\\
R_{\eta\rho}(t,x;t',x') &= -\frac{1}{\sigma}(M\rho' + M' - \dot{\rho})\delta(t-t')\delta(x-x') && (221)
\end{aligned}
$$

The relations (214,215) are the Noether identities in two dimensional gravity. As expected we have two identities which coincides with the number of Lagrangian constraints of the first level.

We can now write the gauge transformations for the fields as follows:

$$
\begin{aligned}
\delta\phi(t,x) &= \int dt'dx'\,(\xi(t',x')R_{\xi\phi}(t',x';t,x) + \eta(t',x')R_{\eta\phi}(t',x';t,x)) && (222)\\
\delta\rho(t,x) &= \int dt'dx'\,(\xi(t',x')R_{\xi\rho}(t',x';t,x) + \eta(t',x')R_{\eta\rho}(t',x';t,x)) && (223)\\
\delta\sigma(t,x) &= \int dt'dx'\,(\xi(t',x')R_{\xi\sigma}(t',x';t,x) + \eta(t',x')R_{\eta\sigma}(t',x';t,x)) && (224)\\
\delta M(t,x) &= \int dt'dx'\,(\xi(t',x')R_{\xi M}(t',x';t,x) + \eta(t',x')R_{\eta M}(t',x';t,x)) && (225)
\end{aligned}
$$

It is not difficult to verify that by substituting the expressions (216-221) for the Lagrangian gauge generators into (222-225) we obtain exactly (207-210).

2.5 The conformal gauge

It is well known that one can always choose a coordinate frame in which the metric is conformally flat ($\sigma^2 = 1, M = 0$) [41]:

$$g_{\mu\nu}(t,x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^{2\rho(t,x)} \tag{226}$$

The field equations (185-188) can be written in the conformal gauge as follows:

$$2e^{-2\rho}(\ddot{\rho} - \rho'') - \frac{dV}{d\phi} = 0 \tag{227}$$

$$\ddot{\phi} - \dot{\phi}\dot{\rho} - \phi'\rho' - \frac{1}{2}e^{2\rho}V(\phi) = 0 \tag{228}$$

$$\dot{\phi}' - \dot{\phi}\rho' - \phi'\dot{\rho} = 0 \tag{229}$$

$$\phi'' - \dot{\phi}\dot{\rho} - \phi'\rho' + \frac{1}{2}e^{2\rho}V(\phi) = 0 \tag{230}$$

Let us introduce the light cone variables

$$\begin{aligned} z_+ &= x + t \\ z_- &= x - t \end{aligned} \tag{231}$$

Equations (231) define a coordinate transformation. The components of the metric tensor can be written in the frame associated with the light cone coordinates. Indeed, under any coordinate transformation $x \to x'$:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{232}$$

In our case (231) $t = \frac{1}{2}(z_+ - z_-)$, $x = \frac{1}{2}(z_+ + z_-)$, and therefore

$$\begin{aligned} g_{++}(z_+, z_-) &= 0 \\ g_{+-}(z_+, z_-) &= g_{-+}(z_+, z_-) = \frac{1}{2}e^{2\rho(z_+,z_-)} \\ g_{--}(z_+, z_-) &= 0 \end{aligned} \tag{233}$$

So, the metric tensor in light cone coordinates has the form

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{2\rho(z_+,z_-)} \tag{234}$$

There exists a class of residual diffeomorphism transformations $z \rightarrow \tilde{z}$ that do not change the form of the metric (234). In other words, the metric tensor in the frame associated with the new coordinates $\tilde{z}$ remains in the form

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{2\tilde{\rho}(\tilde{z}_+,\tilde{z}_-)} \tag{235}$$

This means that the residual diffeomorphism transformations must obey the following equations:

$$\begin{aligned}
\tilde{g}_{++} &= \frac{\partial z_+}{\partial \tilde{z}_+}\frac{\partial z_-}{\partial \tilde{z}_+}e^{2\rho} = 0 \\
\tilde{g}_{+-} &= \tilde{g}_{-+} = \frac{1}{2}\left(\frac{\partial z_+}{\partial \tilde{z}_+}\frac{\partial z_-}{\partial \tilde{z}_-} + \frac{\partial z_-}{\partial \tilde{z}_+}\frac{\partial z_+}{\partial \tilde{z}_-}\right)e^{2\rho} = \frac{1}{2}e^{2\tilde{\rho}} \\
\tilde{g}_{--} &= \frac{\partial z_+}{\partial \tilde{z}_-}\frac{\partial z_-}{\partial \tilde{z}_-}e^{2\rho} = 0
\end{aligned} \tag{236}$$

Equations (236) have two classes of solutions: either

$$\begin{aligned}
\tilde{z}_+ &= \tilde{z}_+(z_+) \\
\tilde{z}_- &= \tilde{z}_-(z_-)
\end{aligned} \tag{237}$$

where $z_+(\tilde{z}_+)$ and $z_-(\tilde{z}_-)$ are arbitrary differentiable invertible functions of their arguments; or

$$\begin{aligned}
\tilde{z}_+ &= \tilde{z}_+(z_-) \\
\tilde{z}_- &= \tilde{z}_-(z_+)
\end{aligned} \tag{238}$$

So, we can say that the most generic residual diffeomorphism transformations that leave the metric tensor in the form (234) can be always expressed as (237) or as a product of two transformations, one being (237) and the other $z_+ \rightarrow \tilde{z}_+ = z_-$, $z_- \rightarrow \tilde{z}_- = z_+$. Notice that the diffeomorphism transformation $z_+ \rightarrow \tilde{z}_+ = z_-$, $z_- \rightarrow \tilde{z}_- = z_+$ is simply a time inversion ($\tilde{x} = x, \tilde{t} = -t$). Under time inversion we clearly have

$$\begin{aligned}
\tilde{\phi}(\tilde{z}_+,\tilde{z}_-) &= \phi(z_+,z_-) \\
\tilde{\rho}(\tilde{z}_+,\tilde{z}_-) &= \rho(z_+,z_-)
\end{aligned} \tag{239}$$

Therefore, under the most generic residual diffeomorphisms, the scalar field ϕ and the field ρ transform as follows:

$$\tilde{\phi}(\tilde{z}_+, \tilde{z}_-) = \phi(z_+, z_-) \tag{240}$$

$$2\tilde{\rho}(\tilde{z}_+, \tilde{z}_-) = 2\rho(z_+, z_-) + \ln(\alpha(z_+)\beta(z-)) \tag{241}$$

where $\alpha(z_+) \equiv \frac{dz_+}{d\tilde{z}_+}$ and $\beta(z_-) \equiv \frac{dz_-}{d\tilde{z}_-}$. Note that (241) is only well defined if the derivatives of the gauge functions satisfy:

$$\alpha(z_+)\beta(z_-) > 0 \tag{242}$$

Using the definition of the light cone variables (231), it is straightforward to obtain the field equations in the conformal gauge (227-230) as follows:

$$8e^{-2\rho}\frac{\partial^2 \rho}{\partial z_+ \partial z_-} + \frac{dV}{d\phi} = 0 \tag{243}$$

$$\frac{\partial^2 \phi}{\partial {z_+}^2} + \frac{\partial^2 \phi}{\partial {z_-}^2} - 2\frac{\partial^2 \phi}{\partial z_+ \partial z_-} - 2\frac{\partial \rho}{\partial z_+}\frac{\partial \phi}{\partial z_+} - 2\frac{\partial \rho}{\partial z_-}\frac{\partial \phi}{\partial z_-} - \frac{1}{2}e^{2\rho}V(\phi) = 0 \tag{244}$$

$$\frac{\partial^2 \phi}{\partial {z_+}^2} - \frac{\partial^2 \phi}{\partial {z_-}^2} - 2\frac{\partial \rho}{\partial z_+}\frac{\partial \phi}{\partial z_+} + 2\frac{\partial \rho}{\partial z_-}\frac{\partial \phi}{\partial z_-} = 0 \tag{245}$$

$$\frac{\partial^2 \phi}{\partial {z_+}^2} + \frac{\partial^2 \phi}{\partial {z_-}^2} + 2\frac{\partial^2 \phi}{\partial z_+ \partial z_-} - 2\frac{\partial \rho}{\partial z_+}\frac{\partial \phi}{\partial z_+} - 2\frac{\partial \rho}{\partial z_-}\frac{\partial \phi}{\partial z_-} + \frac{1}{2}e^{2\rho}V(\phi) = 0 \tag{246}$$

We first note that equations (243-246) are equivalent to the following:

$$\frac{\partial^2 \rho}{\partial z_+ \partial z_-} + \frac{1}{8}e^{2\rho}\frac{dV}{d\phi} = 0 \tag{247}$$

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} + \frac{1}{4}e^{2\rho}V(\phi) = 0 \tag{248}$$

$$\frac{\partial^2 \phi}{\partial {z_+}^2} - 2\frac{\partial \rho}{\partial z_+}\frac{\partial \phi}{\partial z_+} = e^{2\rho}\frac{\partial}{\partial z_+}\left(e^{-2\rho}\frac{\partial \phi}{\partial z_+}\right) = 0 \tag{249}$$

$$\frac{\partial^2 \phi}{\partial {z_-}^2} - 2\frac{\partial \rho}{\partial z_-}\frac{\partial \phi}{\partial z_-} = e^{2\rho}\frac{\partial}{\partial z_-}\left(e^{-2\rho}\frac{\partial \phi}{\partial z_-}\right) = 0 \tag{250}$$

In particular (248) is the difference between (244) and (246), while Eqs.(249) and (250) are obtained by taking appropriate linear combinations of (244,246) and (245).

Equations (249-250) imply that

$$e^{-2\rho}\frac{\partial\phi}{\partial z_+} = g(z_-) \tag{251}$$

$$e^{-2\rho}\frac{\partial\phi}{\partial z_-} = f(z_+) \tag{252}$$

Clearly neither $f(z_+)$ or $g(z_-)$ can be identically zero unless $V(\phi)$ is zero (cf. Eq (248)). Note also that as a direct consequence of (251) and (252):

$$f(z_+)\frac{\partial\phi}{\partial z_+} = g(z_-)\frac{\partial\phi}{\partial z_-} \tag{253}$$

We assume f and g to be smooth functions.

Substituting

$$e^{2\rho} = \frac{1}{g(z_-)}\frac{\partial\phi}{\partial z_+} \tag{254}$$

into (248) we obtain

$$\frac{\partial^2\phi}{\partial z_+\partial z_-} + \frac{1}{4}e^{2\rho}V(\phi) = \frac{\partial^2\phi}{\partial z_+\partial z_-} + \frac{1}{4}\frac{1}{g(z_-)}\frac{\partial\phi}{\partial z_+}V(\phi) \tag{255}$$

Defining the function j to be a solution of

$$\frac{dj}{d\phi} = V(\phi) \tag{256}$$

we have

$$\frac{\partial\phi}{\partial z_+}V(\phi) = \frac{\partial\phi}{\partial z_+}\frac{dj}{d\phi} = \frac{\partial j}{\partial z_+} \tag{257}$$

Substituting (257) into (255), we see that

$$\begin{aligned}\frac{\partial^2\phi}{\partial z_+\partial z_-} + \frac{1}{4}e^{2\rho}V(\phi) &= \frac{\partial^2\phi}{\partial z_+\partial z_-} + \frac{1}{4}\frac{1}{g(z_-)}\frac{\partial j}{\partial z_+} \\ &= \frac{\partial}{\partial z_+}\left(\frac{\partial\phi}{\partial z_-} + \frac{1}{4}\frac{1}{g(z_-)}j\right)\end{aligned} \tag{258}$$

Analogously, since

$$e^{2\rho} = \frac{1}{f(z_+)}\frac{\partial\phi}{\partial z_-} \tag{259}$$

we can also write

$$\frac{\partial^2\phi}{\partial z_+\partial z_-} + \frac{1}{4}e^{2\rho}V(\phi) = \frac{\partial}{\partial z_-}\left(\frac{\partial\phi}{\partial z_+} + \frac{1}{4}\frac{1}{f(z_+)}j\right) \tag{260}$$

and therefore, from (248) it follows that

$$\begin{aligned}\frac{\partial}{\partial z_+}\left(\frac{\partial\phi}{\partial z_-}+\frac{1}{4}\frac{1}{g(z_-)}j(\phi)\right) &= 0\\ \frac{\partial}{\partial z_-}\left(\frac{\partial\phi}{\partial z_+}+\frac{1}{4}\frac{1}{f(z_+)}j(\phi)\right) &= 0\end{aligned} \tag{261}$$

Then, we can write

$$\begin{aligned}\frac{\partial\phi}{\partial z_-}+\frac{1}{4}\frac{1}{g(z_-)}j(\phi) &= C^{(-)}(z_-)\\ \frac{\partial\phi}{\partial z_+}+\frac{1}{4}\frac{1}{f(z_+)}j(\phi) &= C^{(+)}(z_+)\end{aligned} \tag{262}$$

In (262) $C^{(-)}(z_-)$ and $C^{(+)}(z_+)$ are some functions of their arguments. Multiplying the above equations by $g(z_-)$ and $f(z_+)$, respectively, and using (253) leads to the result that

$$C^{(+)}(z_+)f(z_+) = C^{(-)}(z_-)g(z_-) = constant \equiv -\frac{1}{4}C \tag{263}$$

This in turn leads immediately to the following equation:

$$f(z_+)\frac{\partial\phi}{\partial z_+}+\frac{1}{4}j(\phi) = -\frac{1}{4}C \tag{264}$$

Therefore, we have proven that the field equations in the conformal gauge lead to the following equations

$$f(z_+)\frac{\partial\phi}{\partial z_+}+\frac{1}{4}j(\phi) = -\frac{1}{4}C \tag{265}$$

$$e^{-2\rho}\frac{\partial\phi}{\partial z_-} = f(z_+) \tag{266}$$

where,

$$\frac{dj}{d\phi} = V(\phi) \tag{267}$$

On the other hand, it is not difficult to prove that (247-250) can be derived from (265-266). Indeed, equation (250) is a direct consequence of (266). Differentiating (265) with respect z_+ we get:

$$\frac{df}{dz_+}\frac{\partial\phi}{\partial z_+}+f\frac{\partial^2\phi}{\partial {z_+}^2}+\frac{1}{4}V(\phi)\frac{\partial\phi}{\partial z_+} = 0 \tag{268}$$

Differentiating (265) with respect z_- and (266) with respect to z_+ gives

$$f(z_+)\frac{\partial^2\phi}{\partial z_+\partial z_-}+\frac{1}{4}V(\phi)\frac{\partial\phi}{\partial z_-}=0 \tag{269}$$

$$\frac{df}{dz_+}+2\frac{\partial\rho}{\partial z_+}\frac{\partial\phi}{\partial z_-}e^{-2\rho}-\frac{\partial^2\phi}{\partial z_+\partial z_-}e^{-2\rho}=0 \tag{270}$$

From (269-270) and (266) it follows that

$$\frac{df}{dz_+}+2\frac{\partial\rho}{\partial z_+}f+\frac{1}{4}V(\phi)=0 \tag{271}$$

Multiplying (271) by $\frac{\partial\phi}{\partial z_+}$ and taking the difference between this equation and (268) we obtain (249). Equation (248) is obtained by differentiating (265) with respect to z_- and substituting $f(z_+)$ from (266).

Finally, it is not difficult to prove that (247) is not an independent equation, but can be derived from (248-250). Indeed, differentiating (249) with respect to z_- and (250) with respect to z_+ we get:

$$\frac{\partial^3\phi}{\partial z_+^2\partial z_-}-2\frac{\partial^2\rho}{\partial z_+\partial z_-}\frac{\partial\phi}{\partial z_+}-2\frac{\partial\rho}{\partial z_+}\frac{\partial^2\phi}{\partial z_+\partial z_-} = 0 \tag{272}$$

$$\frac{\partial^3\phi}{\partial z_-^2\partial z_+}-2\frac{\partial^2\rho}{\partial z_+\partial z_-}\frac{\partial\phi}{\partial z_-}-2\frac{\partial\rho}{\partial z_-}\frac{\partial^2\phi}{\partial z_+\partial z_-} = 0 \tag{273}$$

Using (248) we can express the second partial derivative of ϕ as

$$\frac{\partial^2\phi}{\partial z_+\partial z_-}=-\frac{1}{4}e^{2\rho}V(\phi) \tag{274}$$

Differentiating (274) with respect to z_+ and z_-, we have

$$\frac{\partial^3\phi}{\partial z_+^2\partial z_-} = -\frac{1}{4}e^{2\rho}\frac{dV}{d\phi}\frac{\partial\phi}{\partial z_+}-\frac{1}{2}\frac{\partial\rho}{\partial z_+}V(\phi) \tag{275}$$

$$\frac{\partial^3\phi}{\partial z_-^2\partial z_+} = -\frac{1}{4}e^{2\rho}\frac{dV}{d\phi}\frac{\partial\phi}{\partial z_-}-\frac{1}{2}\frac{\partial\rho}{\partial z_-}V(\phi) \tag{276}$$

Substituting (275-276) into (272-273) we get

$$\frac{\partial\phi}{\partial z_+}\left(-\frac{1}{4}e^{2\rho}\frac{dV}{d\phi}-2\frac{\partial^2\rho}{\partial z_+\partial z_-}\right) = 0 \tag{277}$$

$$\frac{\partial\phi}{\partial z_-}\left(-\frac{1}{4}e^{2\rho}\frac{dV}{d\phi}-2\frac{\partial^2\rho}{\partial z_+\partial z_-}\right) = 0 \tag{278}$$

We are considering the case when the function $V(\phi)$ is not identically equal to zero, and therefore, $\phi \neq const$. This of course means that either $\frac{\partial \phi}{\partial z_+}$ or $\frac{\partial \phi}{\partial z_-}$ is different from zero and then equation (247) follows as direct consequence of (277-278).

2.6 Birkhoff theorem in 2d gravity

The field equations are invariant under spacetime diffeomorphisms. In particular, if we transform the fields according to (240,241) then the transformed fields $\tilde{\phi}$ and $\tilde{\rho}$ obey:

$$\tilde{f}(\tilde{z}_+)\frac{\partial\tilde{\phi}}{\partial\tilde{z}_+} + \frac{1}{4}j(\tilde{\phi}) = -\frac{1}{4}\tilde{C} \tag{279}$$

$$e^{-2\tilde{\rho}}\frac{\partial\tilde{\phi}}{\partial\tilde{z}_-} = \tilde{f}(\tilde{z}_+) \tag{280}$$

and:

$$e^{-2\tilde{\rho}}\frac{\partial\tilde{\phi}}{\partial\tilde{z}_+} = \tilde{g}(\tilde{z}_-) \tag{281}$$

where the functions $\tilde{f}$ and $\tilde{g}$ are related to f and g as follows:

$$\tilde{f}(\tilde{z}_+) = \frac{1}{\alpha(z_+)}f(z_+) \tag{282}$$

$$\tilde{g}(\tilde{z}_-) = \frac{1}{\beta(z_-)}g(z_-) \tag{283}$$

Moreover, it can easily be verified that $\tilde{C} = C$. Thus the parameter C is invariant under the transformations (240,241). Indeed, the parameter C is invariant under arbitrary spacetime diffeomorphisms since it can be written in the following covariant form:

$$C = -g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - j(\phi) \tag{284}$$

In the conformally flat gauge (226):

$$C = -4e^{-2\rho}\frac{\partial\phi}{\partial z_+}\frac{\partial\phi}{\partial z_-} - j(\phi) \tag{285}$$

as in (265).

On the other hand, $f(z_+)$ and $g(z_-)$ are coordinate functions. As long as

$$f(z_+)g(z_-) > 0$$

it is always possible to choose a local coordinate system in which $\tilde{f}(\tilde{z}_+) = 1$ and $\tilde{g}(\tilde{z}_-) = 1$. (Recall the condition (242).) This can be achieved by performing a diffeomorphism of the type (282,283) such that $\alpha = f(z_+)$ and $\beta = g(z_-)$. In these coordinates (we henceforth drop the tildes over the variables):

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial z_+} - \frac{\partial\phi}{\partial z_-} = 0 \tag{286}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z_+} + \frac{\partial\phi}{\partial z_-} = 2e^{2\rho} \tag{287}$$

so that we can rewrite the equations (265,266) as follows:

$$\frac{1}{2}\frac{d\phi}{dx} + \frac{1}{4}j(\phi) = -\frac{1}{4}C \tag{288}$$

$$e^{2\rho} = \frac{1}{2}\frac{d\phi}{dx} \tag{289}$$

Notice that since ϕ is independent of the time coordinate, so is ρ. We have therefore proven that for any 2d gravity model all the solutions of the field equations (extremals of the action) are essentially static.

Equations (288,289) are easy to solve. Eq(288) is equivalent to

$$\int \frac{d\phi}{(j(\phi) + C)} = -\frac{1}{2}x + const \tag{290}$$

The constant in (290) is of no physical significance and can be made equal to zero. It is related to the fact that after fixing the gauge $\alpha = f(z_+)$, $\beta = g(z_-)$, there still remains a residual symmetry under the transformations $z_+ \rightarrow z_+ + const$, $z_- \rightarrow z_- + const$.

The solution $\phi = \phi(x)$, $\rho = \rho(x)$ of (288,289) is the following: $\phi = \phi(x)$ is the inverse of the function $x = H(\phi)$ where,

$$H(\phi) = -2\int \frac{d\phi}{(j(\phi) + C)} \tag{291}$$

and from (289):

$$e^{2\rho} = -\frac{C + j(\phi)}{4} \tag{292}$$

It is worth noting that in the case $f(z_+)g(z_-) < 0$, one can repeat the argument above to show that there always exists a local coordinate system in which the solution is independent of the spatial coordinate. The solution is then identical to the one above, with the x dependence replaced everywhere by t dependence. This situation arises for the interior Schwarzschild solution: below the event horizon, the coordinate r on which the metric depends becomes timelike.

We have shown that the analogue of Birkhoff's theorem is valid for all the two-dimensional models of gravity described by the action (147) (or equivalently by the action (132) , since the extremals of (147) and (132) are in a one to one correspondence through a conformal transformation of the metric and a redefinition of the scalar field). That is, for any solution of the field equations generated by the action (147)(or (132)) there exits a local coordinate frame in which the solution is time independent. Up to spacetime diffeomorphisms, the solutions are parametrized by the value of a single physical observable.

2.7 The Hamiltonian formulation of 2d gravity

In terms of the parametrization (176) the action functional (147) can be written (up to surface terms) as:

$$
\begin{aligned}
S &= \int dt dx \left[\frac{\dot{\phi}}{\sigma}(2M\rho' + 2M' - 2\dot{\rho}) + \frac{\phi'}{\sigma}(2\sigma\sigma' - 2MM' + 2M\dot{\rho} \right. \\
&\qquad \left. + 2\sigma^2\rho' - 2M^2\rho') - \sigma e^{2\rho} V(\phi) \right] \qquad (293)
\end{aligned}
$$

In the above, dots and primes denote differentiation with respect to time and space, respectively. The canonical momenta associated with the fields $\{\phi, \rho, \sigma, M\}$ are:

$$
\begin{aligned}
\Pi_\phi &= \frac{2}{\sigma}(M\rho' + M' - \dot{\rho}) \qquad (294) \\
\Pi_\rho &= \frac{2}{\sigma}(-\dot{\phi} + M\phi') \qquad (295) \\
\Pi_\sigma &= 0 \qquad (296) \\
\Pi_M &= 0 \qquad (297)
\end{aligned}
$$

Clearly (296) and (297) are primary constraints. By definition (see (11)), the canonical Hamiltonian has the form

$$
H_c = \int dx \left[\dot{\phi}\Pi_\phi + \dot{\rho}\Pi_\rho + \dot{\sigma}\Pi_\sigma + \dot{M}\Pi_M - \mathcal{L} \right] \qquad (298)
$$

where $\mathcal{L}$ is the integrand of (293).

The canonical Hamiltonian (298) can be expressed (up to surface terms) as:

$$
H_c = \int dx \, [M\mathcal{F} + \sigma\mathcal{G}] \qquad (299)
$$

where

$$
\begin{aligned}
\mathcal{F} &\equiv \rho'\Pi_\rho + \phi'\Pi_\phi - \Pi_\rho' \qquad (300) \\
\mathcal{G} &\equiv 2\phi'' - 2\phi'\rho' - \frac{1}{2}\Pi_\phi\Pi_\rho + e^{2\rho}V(\phi) \qquad (301)
\end{aligned}
$$

We define the total Hamiltonian as

$$
H_T = H_c + \int dx (\Lambda_\sigma \Pi_\sigma + \Lambda_M \Pi_M) \qquad (302)
$$

where Λ_σ and Λ_M are the Lagrange multipliers associated with the primary constraints $\Pi_\sigma \approx 0$ and $\Pi_M \approx 0$.

Equations (294-297) define the Legendre transformations from the so called velocity phase space into the momentum phase space. In the momentum phase space we can define the Poisson brackets. For any functions A and B the Poisson bracket is

$$\begin{aligned}
\{A(t,x), B(t,y)\} &= \int dz \left(\frac{\delta A(t,x)}{\delta\phi(t,z)} \frac{\delta B(t,y)}{\delta\Pi_\phi(t,z)} - \frac{\delta A(t,x)}{\delta\Pi_\phi(t,z)} \frac{\delta B(t,y)}{\delta\phi(t,z)} + \right. \\
&\quad + \frac{\delta A(t,x)}{\delta\rho(t,z)} \frac{\delta B(t,y)}{\delta\Pi_\rho(t,z)} - \frac{\delta A(t,x)}{\delta\Pi_\rho(t,z)} \frac{\delta B(t,y)}{\delta\rho(t,z)} + \\
&\quad + \frac{\delta A(t,x)}{\delta\sigma(t,z)} \frac{\delta B(t,y)}{\delta\Pi_\sigma(t,z)} - \frac{\delta A(t,x)}{\delta\Pi_\sigma(t,z)} \frac{\delta B(t,y)}{\delta\sigma(t,z)} + \\
&\quad \left. + \frac{\delta A(t,x)}{\delta M(t,z)} \frac{\delta B(t,y)}{\delta\Pi_M(t,z)} - \frac{\delta A(t,x)}{\delta\Pi_M(t,z)} \frac{\delta B(t,y)}{\delta M(t,z)} \right) \qquad (303)
\end{aligned}$$

Clearly the Poisson brackets between the primary constraints (296,297) vanish. In the Dirac-Hamilton formalism the time derivative of any function A is equal to

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H_T\} \qquad (304)$$

The Dirac-Hamilton field equations can therefore be written in the form:

$$\begin{aligned}
\dot{\sigma} &= \Lambda_\sigma & (305) \\
\dot{M} &= \Lambda_M & (306) \\
\dot{\phi} &= -\frac{\sigma}{2}\Pi_\rho + M\phi' & (307) \\
\dot{\rho} &= -\frac{\sigma}{2}\Pi_\phi + M\rho' + M' & (308) \\
\dot{\Pi}_\phi &= -2\sigma'' - 2(\sigma\rho')' - \sigma e^{2\rho}\frac{dV}{d\phi} + (M\Pi_\phi)' & (309) \\
\dot{\Pi}_\rho &= (M\Pi_\rho)' - 2(\sigma\phi')' - 2\sigma e^{2\rho}V(\phi) & (310) \\
\Pi_\sigma &= 0 & (311) \\
\Pi_M &= 0 & (312)
\end{aligned}$$

In order to obtain (305-312) we made use of (304) and of the following expressions for the functional derivatives of $\mathcal{F}$ and $\mathcal{G}$:

$$\frac{\delta F(x)}{\delta\phi(z)} = -\frac{\partial}{\partial z}\left(\Pi_\phi(z)\delta(x-z)\right)$$

$$
\begin{aligned}
\frac{\delta F(x)}{\delta \Pi_\phi(z)} &= \phi'(z)\delta(x-z) \\
\frac{\delta F(x)}{\delta \rho(z)} &= -\frac{\partial}{\partial z}\left(\Pi_\rho(z)\delta(x-z)\right) \\
\frac{\delta F(x)}{\delta \Pi_\rho(z)} &= \rho'(z)\delta(x-z) + \frac{\partial}{\partial z}\delta(x-z) \qquad (313) \\
\frac{\delta G(x)}{\delta \phi(z)} &= 2\frac{\partial^2}{\partial z^2}\delta(x-z) + 2\frac{\partial}{\partial z}\left(\rho'(z)\delta(x-z)\right) + e^{2\rho}\frac{dV(\phi)}{d\phi}\delta(x-z) \\
\frac{\delta G(x)}{\delta \Pi_\phi(z)} &= -\frac{1}{2}\Pi_\rho(z)\delta(x-z) \\
\frac{\delta G(x)}{\delta \rho(z)} &= 2\frac{\partial}{\partial z}\left(\phi'(z)\delta(x-z)\right) + 2e^{2\rho}V(\phi)\delta(x-z) \\
\frac{\delta G(x)}{\delta \Pi_\rho(z)} &= -\frac{1}{2}\Pi_\phi(z)\delta(x-z) \qquad (314)
\end{aligned}
$$

The primary constraints (296,297) should satisfy the Dirac consistency conditions which require that if a constraint is satisfied for an initial field configuration then it should be satisfied for any field configuration obtained from the initial one by the Dirac-Hamilton field equations (305-312). The Dirac consistency conditions on the primary constraints (296,297) mean that for the solutions of the field equations (305-312):

$$
\begin{aligned}
\dot{\Pi}_\sigma &= 0 \\
\dot{\Pi}_M &= 0 \qquad (315)
\end{aligned}
$$

On the other hand, from (304) and (302) it follows that

$$
\begin{aligned}
\dot{\Pi}_\sigma &= \{\Pi_\sigma, H_T\} = -\mathcal{G} \\
\dot{\Pi}_M &= \{\Pi_M, H_T\} = -\mathcal{F} \qquad (316)
\end{aligned}
$$

In this way we see that the system has the following secondary constraints:

$$
\mathcal{F} \equiv \rho'\Pi_\rho + \phi'\Pi_\phi - \Pi_\rho' \approx 0 \qquad (317)
$$

$$
\mathcal{G} \equiv 2\phi'' - 2\phi'\rho' - \frac{1}{2}\Pi_\phi\Pi_\rho + e^{2\rho}V(\phi) \approx 0 \qquad (318)
$$

Notice that when in the Hamiltonian constraints (317,318) one substitutes the expressions ((294,295) for the momenta, one obtains exactly (192,193).

The Dirac consistency conditions should also be applied on the secondary constraints (317,318). We therefore require that $\dot{\mathcal{F}}$ and $\dot{\mathcal{G}}$ vanish for all the field configurations that are solutions of the Dirac-Hamilton field equations (305-312). From (304) and (302) it follows that:

$$\begin{aligned}
\dot{\mathcal{F}} &= \{\mathcal{F}(t,x), H_T(t)\} \\
&= \int dy\,(M(t,y)\{\mathcal{F}(t,x),\mathcal{F}(t,y)\} + \sigma(t,y)\{\mathcal{F}(t,x),\mathcal{G}(t,y)\}) \\
\dot{\mathcal{G}} &= \{\mathcal{G}(t,x), H_T(t)\} \\
&= \int dy\,(M(t,y)\{\mathcal{G}(t,x),\mathcal{F}(t,y)\} + \sigma(t,y)\{\mathcal{G}(t,x),\mathcal{G}(t,y)\})
\end{aligned} \tag{319}$$

Using the definition of the Poisson brackets (303) and the expressions for the functional derivatives (313,314) we find that:

$$\begin{aligned}
\{\mathcal{F}(t,x),\mathcal{F}(t,y)\} &= -\frac{\partial\mathcal{F}}{\partial y}\delta(x-y) - 2\mathcal{F}\frac{\partial}{\partial y}\delta(x-y) \\
\{\mathcal{G}(t,x),\mathcal{G}(t,y)\} &= -\frac{\partial\mathcal{F}}{\partial y}\delta(x-y) - 2\mathcal{F}\frac{\partial}{\partial y}\delta(x-y) \\
\{\mathcal{F}(t,x),\mathcal{G}(t,y)\} &= -\frac{\partial\mathcal{G}}{\partial y}\delta(x-y) - 2\mathcal{G}\frac{\partial}{\partial y}\delta(x-y)
\end{aligned} \tag{320}$$

and therefore:

$$\begin{aligned}
\dot{\mathcal{F}} = -M\mathcal{F}' + 2(M\mathcal{F})' - \sigma\mathcal{G}' + 2(\sigma\mathcal{G})' &\approx 0 \\
\dot{\mathcal{G}} = -M\mathcal{G}' + 2(M\mathcal{G})' - \sigma\mathcal{F}' + 2(\sigma\mathcal{F})' &\approx 0
\end{aligned} \tag{321}$$

No further constraints appear in the Dirac algorithm. It is straightforward to verify that all the constraints of the theory are first-class, indeed:

$$\begin{aligned}
\{\Pi_\sigma(t,x),\Pi_\sigma(t,y)\} &= 0 \\
\{\Pi_M(t,x),\Pi_M(t,y)\} &= 0 \\
\{\Pi_\sigma(t,x),\Pi_M(t,y)\} &= 0
\end{aligned}$$

$$
\begin{aligned}
\{\Pi_\sigma(t,x),\mathcal{F}(t,y)\} &= 0 \\
\{\Pi_M(t,x),\mathcal{F}(t,y)\} &= 0 \\
\{\Pi_\sigma(t,x),\mathcal{G}(t,y)\} &= 0 \\
\{\Pi_M(t,x),\mathcal{G}(t,y)\} &= 0 \\
\{\mathcal{F}(t,x),\mathcal{F}(t,y)\} &= -\frac{\partial\mathcal{F}}{\partial y}\delta(x-y) - 2\mathcal{F}\frac{\partial}{\partial y}\delta(x-y) \approx 0 \\
\{\mathcal{G}(t,x),\mathcal{G}(t,y)\} &= -\frac{\partial\mathcal{F}}{\partial y}\delta(x-y) - 2\mathcal{F}\frac{\partial}{\partial y}\delta(x-y) \approx 0 \\
\{\mathcal{F}(t,x),\mathcal{G}(t,y)\} &= -\frac{\partial\mathcal{G}}{\partial y}\delta(x-y) - 2\mathcal{G}\frac{\partial}{\partial y}\delta(x-y) \approx 0
\end{aligned}
\tag{322}
$$

2.8 Canonical gauge transformations

In Section 2.4 we obtained the infinitesimal gauge transformations for the fields σ, M, ρ and ϕ in the form:

$$\begin{aligned} \delta\sigma &= \dot{\eta} + \xi\sigma' - \xi'\sigma + \eta M' - \eta' M && (323) \\ \delta M &= \dot{\xi} + \xi M' - \xi' M + \eta\sigma' - \eta'\sigma && (324) \\ \delta\rho &= \xi\rho' + \xi' + \frac{\eta}{\sigma}(\dot{\rho} - M\rho' - M') && (325) \\ \delta\phi &= \xi\phi' + \frac{\eta}{\sigma}(\dot{\phi} - M\phi') && (326) \end{aligned}$$

where ξ and η are two arbitrary infinitesimal functions of t and x. The same result can be obtained from the Dirac-Hamilton analysis. Indeed, according to the general theory presented in Part I the generating functional of infinitesimal gauge transformations is a linear combination of the first class constraints that satisfies the conditions (19,20). For 2d dilaton gravity theories:

$$G = \int dx [C_\sigma \Pi_\sigma + C_M \Pi_M + C_{\mathcal{F}} \mathcal{F} + C_{\mathcal{G}} \mathcal{G}] \qquad (327)$$

$$\frac{\partial G}{\partial t} + \{G, H_T\} = \int dx [\Omega_\sigma \Pi_\sigma + \Omega_M \Pi_M] \qquad (328)$$

In (327) the coefficients C_σ, C_M, $C_{\mathcal{F}}$, $C_{\mathcal{G}}$ are functions of t and x. They are not completely independent, indeed from (328) it follows that C_σ and C_M are related to $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$ as follows:

$$\begin{aligned} C_M &= \dot{C}_{\mathcal{F}} + C_{\mathcal{F}} M' - C'_{\mathcal{F}} M + C_{\mathcal{G}} \sigma' - C'_{\mathcal{G}} \sigma \\ C_\sigma &= \dot{C}_{\mathcal{G}} + C_{\mathcal{F}} \sigma' - C'_{\mathcal{F}} \sigma + C_{\mathcal{G}} M' - C'_{\mathcal{G}} M && (329) \end{aligned}$$

We assume $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$ to be any arbitrary functions explicitly depending on the variables t and x but independent of the fields and the canonical momenta. Then renaming $C_{\mathcal{F}} \equiv \xi(t,x)$ and $C_{\mathcal{G}} \equiv \eta(t,x)$ we obtain the generating functional of the canonical gauge transformations (327) as follows:

$$\begin{aligned} G = \int dx \, [&(\dot{\eta} + \xi\sigma' - \xi'\sigma + \eta M' - \eta' M)\Pi_\sigma + \\ &+(\dot{\xi} + \xi M' - \xi' M + \eta\sigma' - \eta'\sigma)\Pi_M + +\xi\mathcal{F} + \eta\mathcal{G}] && (330) \end{aligned}$$

The canonical gauge transformations of the fields and their conjugate canonical momenta generated by G (330) are:

$$
\begin{aligned}
\delta\sigma &= \frac{\delta G}{\delta \Pi_\sigma} = \dot{\eta} + \xi\sigma' - \xi'\sigma + \eta M' - \eta' M \\
\delta M &= \frac{\delta G}{\delta \Pi_M} = \dot{\xi} + \xi M' - \xi' M + \eta\sigma' - \eta'\sigma \\
\delta\rho &= \frac{\delta G}{\delta \Pi_\rho} = \xi\rho' + \xi' - \frac{\eta}{2}\Pi_\phi \\
\delta\phi &= \frac{\delta G}{\delta \Pi_\phi} = \xi\phi' - \frac{\eta}{2}\Pi_\rho
\end{aligned}
\qquad (331)
$$

$$
\begin{aligned}
\delta\Pi_\sigma &= -\frac{\delta G}{\delta \sigma} = 0 \\
\delta\Pi_M &= -\frac{\delta G}{\delta M} = 0 \\
\delta\Pi_\rho &= -\frac{\delta G}{\delta \rho} = (\xi\Pi_\rho)' - 2(\eta\phi')' - 2\eta e^{2\rho}V(\phi) \\
\delta\Pi_\phi &= -\frac{\delta G}{\delta \phi} = (\xi\Pi_\phi)' - 2\eta'' - 2(\eta\rho')' - \eta e^{2\rho}\frac{dV}{d\phi}
\end{aligned}
\qquad (332)
$$

Using the definitions for the momenta (294,295) we see that (331) exactly coincide with (323-326).

2.9 Global physical observables in 2d gravity

Given the existence of four first class constraints, there are no propagating modes in the theory. The reduced phase space is finite dimensional. We will now show that there are two phase space degrees of freedom. It is useful to define the following linear combination of the constraints:

$$\begin{aligned} \tilde{\mathcal{G}} &\equiv -e^{-2\rho}(\phi'\mathcal{G} + \frac{1}{2}\Pi_\rho\mathcal{F}) \\ &= (C[\rho, \Pi_\rho, \phi, \Pi_\phi])' \end{aligned} \tag{333}$$

where we have defined the functional C by

$$C[\rho, \Pi_\rho, \phi, \Pi_\phi] \equiv e^{-2\rho}\left(\frac{\Pi_\rho^2}{4} - (\phi')^2\right) - j(\phi) \tag{334}$$

with $j(\phi)$ a solution to the equation:

$$\frac{dj(\phi)}{d\phi} = V(\phi) \tag{335}$$

As long as $e^{-2\rho}\phi' \neq 0$, the set of constraints $(\tilde{\mathcal{G}}, \mathcal{F})$ are equivalent to the original set $(\mathcal{G}, \mathcal{F})$. C commutes with both $\tilde{\mathcal{G}}$ and $\mathcal{F}$ and (333) implies that the constant mode of the functional C is unconstrained. Thus C is a physical observable. The momentum canonically conjugated to C must also be physical in the Dirac sense, and is found to be:

$$P = -\int dx \frac{2e^{2\rho}\Pi_\rho}{(\Pi_\rho^2 - 4(\phi')^2)} \tag{336}$$

Notice that to (336) we can add any function of C.

P also commutes with the constraints, and has a Poisson bracket with C of:

$$\{C, P\} = 1 \tag{337}$$

As expected, both C and P are global variables: C is constant on Σ while its conjugate is defined as a spatial integral.

Under the gauge transformations (331,332) the observable C remains invariant for arbitrary test functions ξ and η:

$$\delta C = 0 \tag{338}$$

For static spherically symmetric gravity the variable C is equal to the ADM energy of the system[61, 62].

For the observable P we have

$$\delta P = -2 \int dx \left(\frac{e^{2\rho}(\xi\Pi_\rho - 2\eta\phi')}{\Pi_\rho^2 - 4(\phi')^2} \right)' \tag{339}$$

From (339) we see that $\delta P = 0$ only if the test functions ξ and η vanish sufficiently rapidly at the spatial infinity for the open case. In the compact case the variation δP is always equal to zero. This means that in the compact case both variables C and P are invariant under diffeomorphisms and therefore are truly physical observables. In the open case C is a true physical observable but the status of P is less clear.

The momentum variable P is associated with time diffeomorphisms

$$\begin{aligned} t &\rightarrow t' = t + f^0(t,x) \\ x &\rightarrow x' = x \end{aligned} \tag{340}$$

such that $f^0(t,x)$ does not vanish at the spatial infinity.

Let us study how these time diffeomorphisms (340) can be generated canonically. For this we use our expression (330) for the generating functional of infinitesimal canonical gauge transformations and the relations (211) which tell us how the test functions ξ and η are related to the functions f^0 and f^1 of the generic diffeomorphism transformations. Clearly for time diffeomorphisms (340) $f^1 = 0$, and therefore we have

$$\begin{aligned} \eta &= -\sigma f^0(t,x) \\ \xi &= -M f^0(t,x) \end{aligned} \tag{341}$$

Substituting (341) into (330) we get

$$\begin{aligned} G_t = \int dx \Big[&-f^0(t,x)\left(\dot{M}\Pi_M + \dot{\sigma}\Pi_\sigma + M\mathcal{F} + \sigma\mathcal{G}\right) - \\ &-\dot{f}^0(t,x)\left(\sigma\Pi_\sigma + M\Pi_M\right) + f^{0\prime}(t,x)\left(2M\sigma\Pi_\sigma + (M^2+\sigma^2)\Pi_M\right)\Big] \end{aligned} \tag{342}$$

G_t is the generator of time diffeomorphism transformations.

Substituting (341) into (339) we obtain:

$$\delta P = -2\int dx \left(\frac{e^{2\rho}(-Mf^0\Pi_\rho + 2\sigma f^0\phi')}{\Pi_\rho^2 - 4(\phi')^2}\right)' \tag{343}$$

For stationary configurations, since $\Pi_\rho = \frac{2}{\sigma}M\phi'$ (295), we have

$$P = -\int dx \frac{e^{2\rho}\sigma}{\phi'}\frac{M}{M^2 - \sigma^2} \tag{344}$$

It is not difficult to prove that $\frac{e^{2\rho}\sigma}{\phi'} = v = const$ for stationary solutions. Indeed, for stationary configurations the momenta (294,295) take the form

$$\Pi_\phi = \frac{2}{\sigma}(M\rho' + M') \tag{345}$$

$$\Pi_\rho = \frac{2}{\sigma}M\phi' \tag{346}$$

Substituting (345,346) into the expressions for the constraints $\mathcal{F} \approx 0$ (317), $\mathcal{G} \approx 0$ (318), and into the equation (310) we get

$$2\frac{M}{\sigma}\rho'\phi' + \frac{\sigma'}{\sigma^2}M\phi' - \frac{M}{\sigma}\phi'' = 0 \tag{347}$$

$$2\phi'' - 2\phi'\rho' + e^{2\rho}V(\phi) - \frac{2}{\sigma^2}M\phi'(M\rho' + M') = 0 \tag{348}$$

$$(\sigma\phi')' + \sigma e^{2\rho}V(\phi) - (\frac{M^2}{\sigma}\phi')' = 0 \tag{349}$$

From (349) and (347) follows that

$$(\sigma\phi')' + \sigma e^{2\rho}V(\phi) - 2\frac{M}{\sigma}\phi'(M\rho' + M') = 0 \tag{350}$$

If we multiply (348) by σ and take the difference between this equation and Eq(350) we obtain

$$2\sigma\phi'' - 2\sigma\phi'\rho' - (\sigma\phi')' = 0 \tag{351}$$

On the other hand, it is not dificult to show that:

$$\left(\frac{e^{2\rho}\sigma}{\phi'}\right)' = -\frac{e^{2\rho}}{(\phi')^2}\left(2\sigma\phi'' - 2\sigma\phi'\rho' - (\sigma\phi')'\right) \tag{352}$$

Therefore, from (351) and (352) it follows that for stationary solutions

$$\frac{e^{2\rho}\sigma}{\phi'} = v = const \tag{353}$$

Then we can write the variable P as

$$P = -v \int dx \frac{M}{M^2 - \sigma^2} \tag{354}$$

or equivalently (since we are using the ADM parametrization of the metric):

$$P = -v \int dx \frac{g_{01}}{g_{00}} \tag{355}$$

For the variation of P under time diffeomorphisms (343) we have

$$\begin{aligned} \delta P &= -2 \int dx \left(\frac{f^0 e^{2\rho}(-\frac{2}{\sigma} M^2 \phi' + 2\sigma\phi')}{\frac{4}{\sigma^2} M^2 \phi'^2 - 4\phi'^2} \right)' \\ &= \int dx \left(\frac{f^0 e^{2\rho}\sigma}{\phi'} \right)' \end{aligned} \tag{356}$$

Again, since $\frac{e^{2\rho}\sigma}{\phi'} = v = const$, we obtain that for stationary solutions the variation of the variable P is equal to:

$$\delta P = v \int dx f^{0'} \tag{357}$$

Therefore, we can give the following interpretation to the variable P: for stationary configurations the generator P is associated with global time diffeomorphisms of the form (340) such that $f^{0'}(t,x) \neq 0$ and f^0 does not vanish at the spatial infinity. Since gauge transformations are defined in terms of test functions that vanish at the boundaries, P is gauge invariant in this sense and commutes with the constraints as claimed above. However, the observable P can nonetheless be changed by a "non-canonical" diffeomorphism; one for which the test functions ξ and η do not obey the usual boundary conditions.

Notice that for static configurations $P = 0$ and the variation δP is also given by the formula (357). Static are those stationary configurations for which $M = 0$. Under time diffeomorphisms

$$\delta M = \sigma^2 f^{0'}(t,x) \tag{358}$$

Equation (358) can be easily obtained by substituting (341) into (324) and taking into account that for static configurations $M = 0$.

Then, another possible interpretation for P is that it is associated with global time diffeomorphisms that transform static solutions into stationary solutions.

Notice that if one is interested only in the transformations of the fields ρ and ϕ and their canonical conjugated momenta Π_ρ and Π_ϕ under time diffeomorphisms, then one can write the generating functional of these transformations as follows:

$$G_t = -\int dx f^0 (M\mathcal{F} + \sigma\mathcal{G}) \tag{359}$$

This is a direct consequence of (342). For stationary solutions we can write

$$\begin{aligned} G_t &= -\int dx f^0 (\frac{1}{2}\frac{\sigma\Pi_\rho}{\phi'}\mathcal{F} + \sigma\mathcal{G}) \\ &= -\int dx f^0 \frac{\sigma e^{2\rho}}{\phi'} e^{-2\rho} (\phi'\mathcal{G} + \frac{1}{2}\Pi_\rho \mathcal{F}) \\ &= \int dx f^0 v \tilde{\mathcal{G}} \\ &= v \int dx f^0 \tilde{\mathcal{G}} \end{aligned} \tag{360}$$

We see that $\tilde{\mathcal{G}}$ acts as the generator of time diffeomorphisms for stationary configurations. An infinitesimal change in P is generated by the constraint $\tilde{\mathcal{G}}$ as follows:

$$\begin{aligned} \delta P &= \{P, v \int dx f^0 \tilde{\mathcal{G}}\} \\ &= v \int dx f^{0\prime} \end{aligned} \tag{361}$$

As expected Eq(341) and Eq(357) are exactly the same.

This explains the apparent discrepancy between the size of the covariant solution space and the size of the reduced phase space. In the case of spherically symmetric gravity this has been mentioned in [70, 71].

For the solutions of the field equations in the conformal gauge (265,266), since

$$\Pi_\rho = 2(\frac{\partial\phi}{\partial z_-} - \frac{\partial\phi}{\partial z_+}),$$

the phase space variable P takes the form:

$$
\begin{aligned}
P &= -\int dx \frac{2e^{2\rho}\Pi_\rho}{(\Pi_\rho^2 - 4(\phi')^2)} \\
&= -\frac{1}{4}\int dx e^{2\rho} \frac{\left(\frac{\partial\phi}{\partial z_+} - \frac{\partial\phi}{\partial z_-}\right)}{\frac{\partial\phi}{\partial z_+}\frac{\partial\phi}{\partial z_-}} \\
&= -\frac{1}{4}\int dx \left[\frac{1}{f(z_+)} - \frac{1}{g(z_-)}\right]
\end{aligned}
\tag{362}
$$

Clearly P is zero in the coordinate frame in which the solution is time independent ($f(z_+) = g(z_-)$ =constant), and can be made different from zero by performing an appropriate diffeomorphism of the form (237). This is a direct consequence of (282) and (283). Thus, P is not invariant under space time diffeomorphisms, despite the fact that it commutes with the constraints.

2.10 Dirac quantization

We now proceed with the Dirac quantization of the theory in the functional Schrödinger representation in which states are given by functionals of the fields, namely

$$\Psi = \Psi[\sigma, M, \rho, \phi] \tag{363}$$

As usual in the Schrödinger representation, we define the momentum operators as functional derivatives:

$$\begin{aligned}
\hat{\Pi}_\sigma &= -i\hbar\frac{\delta}{\delta\sigma} &(364)\\
\hat{\Pi}_M &= -i\hbar\frac{\delta}{\delta M} &(365)\\
\hat{\Pi}_\rho &= -i\hbar\frac{\delta}{\delta\rho} &(366)\\
\hat{\Pi}_\phi &= -i\hbar\frac{\delta}{\delta\phi} &(367)
\end{aligned}$$

Following the Dirac quantization scheme, we define the physical states $\Psi_{\mathbf{phys}}$ as those that are annihilated by the constraints:

$$\begin{aligned}
\hat{\Pi}_\sigma\Psi_{\mathbf{phys}} &= 0 &(368)\\
\hat{\Pi}_M\Psi_{\mathbf{phys}} &= 0 &(369)\\
\hat{\mathcal{F}}\Psi_{\mathbf{phys}} &= 0 &(370)\\
\hat{\mathcal{G}}\Psi_{\mathbf{phys}} &= 0 &(371)
\end{aligned}$$

The conditions (368,369) will require the wave functional $\Psi_{\mathbf{phys}}$ to be independent of the fields M and σ. We therefore drop them from the following discussion.

$$\Psi_{\mathbf{phys}} = \Psi_{\mathbf{phys}}[\rho, \phi] \tag{372}$$

The physical wave functional (372) is required to satisfy (370,371). As it is usual in the quantum theory, there exists a factor ordering ambiguity in the definition of the operators $\mathcal{F}$ and $\mathcal{G}$ because these constraints involve products of noncommuting variables. In order to circumvent this problem we follow the method used by Henneaux for the quantization of the J-T model in [46] and by Gegenberg and Kunstatter for 2d spherically symmetric gravity in [61]. We replace the Hamiltonian constraints $\mathcal{F}$ and $\mathcal{G}$ by equivalent classical constraints which are solved for the momenta Π_ρ and Π_ϕ. First, use Eq (318) to express Π_ϕ as a function of Π_ρ:

$$\Pi_\phi \approx \frac{g[\rho,\phi]}{\Pi_\rho} \tag{373}$$

where we have defined

$$g[\rho,\phi] \equiv 4\phi'' - 4\phi'\rho' + 2e^{2\rho}V(\phi) \tag{374}$$

After substituting this expression into $\mathcal{F} \approx 0$ (317) and doing some algebra one finds

$$2\rho'\Pi_\rho^2 - (\Pi_\rho^2)' = -2\phi' g[\rho,\phi] \tag{375}$$

The left hand side of the above equation can be related to a total divergence:

$$2\rho'\Pi_\rho^2 - (\Pi_\rho^2)' = -e^{2\rho}\left(e^{-2\rho}\Pi_\rho^2\right)' \tag{376}$$

so that:

$$e^{-2\rho}\Pi_\rho^2 = 2\int dx e^{-2\rho}\phi' g[\rho,\phi] \tag{377}$$

It is straightforward to verify that the resulting solution is:

$$\begin{aligned} \Pi_\rho &\approx Q[C;\rho,\phi] && (378)\\ \Pi_\phi &\approx \frac{g[\rho,\phi]}{Q[C;\rho,\phi]} && (379) \end{aligned}$$

where we have defined:

$$\begin{aligned} Q[C;\rho,\phi] &\equiv 2\sqrt{(\phi')^2 + (C + j(\phi))e^{2\rho}} && (380)\\ g[\rho,\phi] &\equiv 4\phi'' - 4\phi'\rho' + 2e^{2\rho}V(\phi) && (381) \end{aligned}$$

and C is a constant of integration that corresponds precisely to the observable defined in (284).

Note that values of the fields for which $Q^2 < 0$ represent "classically forbidden" regions of the configuration space since they yield imaginary momenta.

It is straightforward to prove that

$$\frac{\delta}{\delta\rho}\left[\frac{g[\rho,\phi]}{Q[C;\rho,\phi]}\right] = \frac{\delta}{\delta\phi}Q[C;\rho,\phi] \tag{382}$$

Eq (382) guarantees that the constraints (378,379) have vanishing Poisson brackets. We have accomplished the abelianization of the constraints.

We now define the physical states $\Psi_{\mathbf{phys}}[C;\rho,\phi]$ as those that are annihilated by the constraints:

$$\left(\hat{\Pi}_\phi - \frac{g[\rho,\phi]}{Q[C;\rho,\phi]}\right)\Psi_{\mathbf{phys}} = 0 \tag{383}$$

$$\left(\hat{\Pi}_\rho - Q[C;\rho,\phi]\right)\Psi_{\mathbf{phys}} = 0 \tag{384}$$

2.11 The quantum wave functional for 2d gravity

We look for a solution of (383,384) in the form

$$\Psi_{\rm phys}[C;\rho,\phi] = \exp\left(\frac{i}{\hbar}S[C;\rho,\phi]\right) \qquad (385)$$

where S satisfies the linear, coupled functional differential equations:

$$\begin{aligned} \frac{\delta}{\delta\phi}S[C;\rho,\phi] &= \frac{g[\rho,\phi]}{Q[C;\rho,\phi]} \\ \frac{\delta}{\delta\rho}S[C;\rho,\phi] &= Q[C;\rho,\phi] \end{aligned} \qquad (386)$$

The integrability condition for these equations is satisfied, it is precisely (382).

The second of these equations can be directly integrated, since it does not involve spatial derivatives, to yield:

$$S[C;\rho,\phi] = \int dx\left[Q + \phi'\ln\left(\frac{2\phi' - Q}{2\phi' + Q}\right)\right] + F[C;\phi] \qquad (387)$$

where $F[C;\phi]$ as an arbitrary functional independent of ρ. Remarkably, the remaining operator constraint, applied to (387) yields the simple result that $F[C;\phi] = F[C] = constant$(independent of ϕ). Therefore,

$$\Psi_{\rm phys}[C;\rho,\phi] = \chi[C]\exp\frac{i}{\hbar}\int dx\left[Q + \phi'\ln\left(\frac{2\phi' - Q}{2\phi' + Q}\right)\right] \qquad (388)$$

where $\chi[C] = \exp i/\hbar F[C]$ is a completely arbitrary function of the configuration space coordinate C.

Notice that from (334),(380) and (384) it follows

$$\hat{C}\Psi_{\rm phys}[C;\rho,\phi] = C\Psi_{\rm phys}[C;\rho,\phi] \qquad (389)$$

or in other words, $\Psi_{\rm phys}[C;\rho,\phi]$ is an eigenstate of the physical observable $\hat{C}$.

The wave functional in (388) is invariant under the quantum constraint:

$$\hat{\mathcal{F}} = \rho'\hat{\Pi}_\rho + \phi'\hat{\Pi}_\phi - \frac{\partial}{\partial x}\hat{\Pi}_\rho \qquad (390)$$

It is also annihilated by the constraint $\hat{\mathcal{G}}$, with factor ordering:

$$\hat{\mathcal{G}} = \frac{1}{2}g[\phi, \rho] - \frac{1}{2}Q\hat{\Pi}_\phi Q^{-1}\hat{\Pi}_\rho \tag{391}$$

We also note that the phase of the wave functional becomes complex when Q is imaginary. This is consistent with traditional quantum mechanics: $Q^2 < 0$ corresponds to classically forbidden regions in which the classical momenta are imaginary (378,379). The phase can also pick up an imaginary part when the argument of the logarithm is negative, i.e. when

$$4(\phi')^2 - Q^2 < 0 \tag{392}$$

The physical significance of this contribution is less clear at this stage. However, it is interesting to note that in the case of spherically symmetric gravity, (392) is satisfied for static solutions expressed in Kruskal coordinates when $r < 2m$.

Finally, we remark that the wave functional Ψ_{phys} yields, as expected, a consistent representation for the physical phase space in the Schrödinger representation. That is, one can explicitly verify the relation:

$$-i\hbar\frac{\partial}{\partial C}\Psi_{\text{phys}} = -P\Psi_{\text{phys}} \tag{393}$$

where P is equal to the expression in (336) evaluated on the constraint surface.

The solution (388) has very interesting features that highlight several important issues in quantum gravity. As the notation indicates, the wave functional is an explicit function of the physical configuration space coordinate C as well a functional of the embedding variables ρ and ϕ.

The wave functional (388) is invariant under spatial diffeomorphisms

$$\begin{aligned} t &\rightarrow t' = t \\ x &\rightarrow x' = x + f^1(t,x) \end{aligned} \tag{394}$$

Notice that from (211) it follows that for spatial diffeomorphisms

$$\begin{aligned} \eta &= 0 \\ \xi &= -f^1(t,x) \end{aligned} \tag{395}$$

The fields ρ and ϕ transform as follows:

$$\begin{aligned} \delta\rho &= -f^1\rho' - f^{1'} \\ \delta\phi &= -f^1\phi' \end{aligned} \tag{396}$$

and, of course, $\delta C = 0$.

From (396) it follows that

$$\begin{aligned} \delta Q &= -f^1 Q' - f^{1'} Q \\ &= -(f^1 Q)' \end{aligned} \tag{397}$$

Substituting (396) and (397) into $\delta\Psi_{\mathbf{phys}}$ we get

$$\delta\Psi_{\mathbf{phys}} = -\frac{i}{\hbar}\int dx \left(f^1 \left(Q + \phi' \ln \left(\frac{2\phi' - Q}{2\phi' + Q} \right) \right) \right)' \Psi_{\mathbf{phys}} \tag{398}$$

Therefore, $\delta\Psi_{\mathbf{phys}} = 0$ under the spatial diffeomorphism transformations (394) for test functions f^1 sufficiently rapidly vanishing at spatial infinity.

Invariance of the wave functional under spatial diffeomorphisms guarantees that one of the two functions ρ or ϕ is redundant: it can be trivially eliminated by choosing an appropriate spatial coordinate. The remaining function is essentially the time variable in the problem. Different choices correspond to different time slicings. The fact that the wave functional depends in a non- trivial way on this choice of time slice (as opposed to the choice of spatial coordinate) is sometimes referred to as the many-fingered time problem in quantum gravity[72]. However, in the present context it can also be interpreted as a consequence of the fact that the solution which solves the constraint $\hat{\mathcal{G}}$ is analogous to a time dependent Schrödinger state.

We have so far avoided the question of the correct Hilbert space measure. One can perhaps ask whether there exists a functional measure on the space of ϕ and ρ that makes this state normalizable. Alternatively, one can first choose a spatial coordinate and time slicing and then define the measure only on the (one-dimensional) space of physical observables. This question can perhaps best be studied in the context of particular models, such as spherically symmetric gravity or the CGHS model. In any case, it is hoped that the existence of the above exact solution in the most general theory of two dimensional dilaton gravity provides a useful laboratory for the study of this, and other fundamental questions in quantum gravity.

Finally we remark that in [61, 62], the solution (388) was applied to spherically symmetric gravity in order to find the quantum wave functional for an isolated black hole. This wave functional was shown to have interesting properties consistent with the presence of a quantum mechanical instability (i.e. Hawking radiation.)

3 Spinning Relativistic Particle

3.1 Introduction.

The description of the motion of a spinning relativistic particle in 3+1 space-time dimensions has been a subject of research for many years. Equations describing the evolution of the classical spin in a uniform and static external electromagnetic field have been proposed by several authors [73, 74]. Modifications of the Lorentz force law for spinning particles moving in a slowly varying external electromagnetic field together with some generalizations for the equations describing the spin precession in this case have also been suggested [75, 76, 77].

It is interesting to note that so far all the Lagrangian or Hamiltonian formulations which allow one to obtain the equations of motion for a spinning relativistic particle in an external electromagnetic field from an action principle rely either on the use of Grassmann variables [78, 79, 80, 81] or on the analogy with the relativistic quantum equations [82]. There exist however some Lagrangian and Hamiltonian models [83, 84, 85] that describe the motion of a free relativistic spinning particle using classical commuting variables, but, to our knowledge, they have proven to be too complicated to allow the introduction of an external field.

Recently, Yee and Bander [86] have proposed a Routhian function from which the equations of motion for a spinning relativistic particle coupled to an external electromagnetic and gravitational field can be derived. We think that to date this Routhian formulation provides the most complete description. However, puzzled by the absence of a classical Lagrangian or Hamiltonian model in 3+1 space-time dimensions which would at least describe the motion of a spinning relativistic particle in a homogeneous electromagnetic field, we decided to see how such a model could be built. The results of our investigation are presented in Section 3.2.

We propose a model that uses an idea suggested by Plyushchay in [84, 85] of expressing the antisymmetric spin matrix in terms of some new coordinates and their conjugate momenta as in the case of the orbital angular momentum matrix. We also introduce some of the constraints proposed by Plyushchay in his model for a free relativistic particle and we extend them to include interactions with an external field. As mentioned before, Plyushchay's model describes a free particle only; no interaction with external fields is considered in [84, 85]. Our model describes the interaction of a spinning relativistic particle with gyromagnetic ratio $g = 2$ in an external static uniform electromagnetic field in 3+1 space-time dimensions. When the external field vanishes it can be considered as a modification of the model presented in [84, 85].

The existence of anyons or particles with arbitrary spin and statistics in 2+1 dimensions [88] has been attracting a great deal of attention due to the applications to different planar physical phenomena such as the fractional quantum Hall effect and possibly high-Tc superconductivity, and to the description of physical processes in the presence of cosmic strings. Several field-theoretic models have been proposed in which anyons appear as topological solitons [89, 90] or electrically charged vortices [91, 92]. In another approach point particles, described by scalar or spinor fields, are coupled minimally to a $U(1)$ gauge field, the so-called statistical gauge field whose dynamics is governed by the Chern-Simons action [93]. However, none of the above models gives a description for a free particle with arbitrary spin.

In a field-theoretical context, it was first pointed out in [94] that the angular momentum of single-particle states can have fractional values in 2+1 dimensions. Recently, in Refs. [95, 96] the field equations for a free particle with fractional spin were proposed and it was shown that their solutions realize the one-particle states as the appropriate induced representation of the Poincaré group in 2+1 dimensions. In [95] the corresponding classical action for the fields was constructed by analogy with the action of the massive vector field. Ref. [97] dealt with the same problem, but starting from the description of the classical action for a relativistic particle with fixed mass and fixed arbitrary spin. There the set of Hamiltonian constraints was found and two different schemes of the canonical quantization of the model were considered. However, no interaction of anyons with the electromagnetic field has been considered in the above mentioned works.

In the last few years the properties of anyons in an external electromagnetic field have been studied [98, 99, 100, 101, 102, 103]. In particular, in [98], using the approach of coupling fermions or bosons to a statistical Chern-Simons field, it was shown that one-anyon states acquire an induced magnetic moment consistent with a value of $g = 2$ for the gyromagnetic ratio. In [101], based on heuristical arguments an equation for an anyon in a constant magnetic field was assumed for the description of the relativistic fractional quantum Hall effect. In a recent paper [103] the electromagnetic interaction of anyons has been considered on the basis of classical analogy with the behaviour of a spin in an electromagnetic field and intuitive arguments.

In Sections 3.3 and 3.4 we propose a model for a relativistic particle with arbitrary spin and gyromagnetic ratio $g = 2$ in an electromagnetic field in 2+1 space-time dimensions. The model shares some similarities with the one of Section 3.2, but it also has differences due to the fact that in 2+1 dimensions the spin should be parallel to the momentum and therefore has only one physical degree of freedom associated with it. It is achieved by extending the original phase space and then imposing additional constraints so that the dimension of the reduced phase space be the correct one. Most of the results presented in these last two Sections were published in [87].

3.2 Spinning relativistic particle in an electromagnetic field: 3+1 space-time dimensions.

Let the position of the relativistic particle be described by the four-vector x^μ ($\mu = 0,1,2,3$), and the spin by the antisymmetric spin matrix $S_{\mu\nu}$. We introduce the auxiliary 'coordinates' n^μ to describe the spin. In phase space (T^*Q) we work with the coordinates x^μ, n^μ and their conjugate momenta p_μ and $p^{(n)}_\mu$. Therefore we can write the following Poisson bracket (PB) relations:

$$\{x_\mu, p_\nu\} = -g_{\mu\nu} \tag{399}$$

$$\{x_\mu, x_\nu\} = \{p_\mu, p_\nu\} = 0 \tag{400}$$

$$\{n_\mu, p^{(n)}_\nu\} = -g_{\mu\nu} \tag{401}$$

$$\{n_\mu, n_\nu\} = \{p^{(n)}_\mu, p^{(n)}_\nu\} = 0 \tag{402}$$

$$\{x_\mu, n_\nu\} = \{x_\mu, p^{(n)}_\nu\} = \{p_\mu, n_\nu\} = \{p_\mu, p^{(n)}_\nu\} = 0 \tag{403}$$

where the metric tensor $g_{\mu\nu} = diag(+1,-1,-1,-1)$.

Let us define the antisymmetric spin matrix $S_{\mu\nu}$ as follows:

$$S_{\mu\nu} = n_\mu p^{(n)}_\nu - n_\nu p^{(n)}_\mu \tag{404}$$

The total angular momentum matrix $M_{\mu\nu}$ can therefore be written as

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu} \tag{405}$$

It is not difficult to verify that the momentum four-vector p_μ and the total angular momentum tensor $M_{\mu\nu}$ satisfy the Poincare algebra. Indeed,

$$\{p_\mu, p_\nu\} = 0 \tag{406}$$

$$\{M_{\mu\nu}, p_\lambda\} = -g_{\mu\lambda} p_\nu + g_{\nu\lambda} p_\mu \tag{407}$$

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = -g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma} + g_{\alpha\delta} M_{\beta\gamma} + g_{\beta\gamma} M_{\alpha\delta} \tag{408}$$

We denote by $F_{\mu\nu}$ a uniform static external electromagnetic field. The vector potential describing such a uniform static field can be expressed as

$$A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu \tag{409}$$

From the definition (404) it follows

$$F^{\mu\nu} n_\mu p_\nu^{(n)} = \frac{1}{2} F_{\mu\nu} S^{\mu\nu} \tag{410}$$

For the description of the spinning relativistic particle with gyromagnetic ratio $g = 2$ in an electromagnetic field $F_{\mu\nu}$ we propose the canonical Hamiltonian

$$H_c = -\frac{1}{2} n^2 \tag{411}$$

and the primary constraints

$$\Phi \equiv \pi^2 - e S^{\mu\nu} F_{\mu\nu} - m^2 \tag{412}$$

$$\phi \equiv (\pi p^{(n)}) \tag{413}$$

where by π_μ we mean

$$\pi_\mu = p_\mu - e A_\mu \tag{414}$$

The constraints (412,413) define a subspace of T^*Q which we call M_0. Notice that for the free case ($F = 0$) the constraint (412) becomes $p^2 - m^2 = 0$.

It is not difficult to prove that

$$\{\pi_\mu, \pi_\nu\} = -e F_{\mu\nu} \tag{415}$$

The primary constraints (412,413) PB commute, indeed,

$$\{\Phi, \phi\} = 2\pi^\mu p^{(n)\nu} \{\pi_\mu, \pi_\nu\} - 2e F^{\mu\nu} p_\nu^{(n)} \{n_\mu, p_\alpha^{(n)}\} \pi^\alpha \tag{416}$$

and substituting (401) and (415) into (416) we obtain

$$\{\Phi, \phi\} = -2e F_{\mu\nu} \pi^\mu p^{(n)\nu} + 2e F_{\mu\nu} \pi^\mu p^{(n)\nu} = 0 \tag{417}$$

The total Hamiltonian is defined to be equal to the canonical Hamiltonian plus a linear combination of the primary constraints:

$$H_T = -\frac{1}{2} n^2 + \Lambda \Phi + \lambda \phi \tag{418}$$

where Λ and λ are the Lagrange multipliers.

The Dirac-Hamilton equations of motion can therefore be written as

$$\begin{aligned}
\dot{x}^\mu &= \frac{\partial H_T}{\partial p_\mu} = 2\Lambda \pi^\mu + \lambda p^{(n)\mu} && (419)\\
\dot{n}^\mu &= \frac{\partial H_T}{\partial p^{(n)}_\mu} = 2\Lambda e F^{\mu\nu} n_\nu + \lambda \pi^\mu && (420)\\
\dot{p}_\mu &= -\frac{\partial H_T}{\partial x^\mu} = \Lambda e F_{\mu\nu}\pi^\nu + \frac{1}{2}\lambda e F_{\mu\nu} p^{(n)\nu} && (421)\\
\dot{p}^{(n)}_\mu &= -\frac{\partial H_T}{\partial n^\mu} = n_\mu + 2\Lambda e F_{\mu\nu} p^{(n)\nu} && (422)
\end{aligned}$$

together with the constraints (412,413).

The consistency conditions on the primary constraints are :

$$\dot{\Phi} \doteq 0 \qquad (423)$$

$$\dot{\phi} \doteq 0 \qquad (424)$$

where $\doteq$ means that the equality holds for the solutions of the equations of motion (419-422).

On the other hand

$$\dot{\Phi} \doteq \{\Phi, H_c\} + \Lambda\{\Phi,\Phi\} + \Lambda\{\Phi,\phi\} \qquad (425)$$

$$\dot{\phi} \doteq \{\phi, H_c\} + \Lambda\{\phi,\Phi\} + \Lambda\{\phi,\phi\} \qquad (426)$$

Obviously $\{\Phi,\Phi\} = \{\phi,\phi\} = 0$. Substituting (411) and (417) into (425,426) we obtain the secondary constraint

$$\theta \equiv (\pi n) \qquad (427)$$

The constraints (412,413) and (427) define the subspace M_1.

Notice that

$$\begin{aligned}
\{\Phi,\theta\} &= 0 && (428)\\
\{\phi,\theta\} &= \pi^2 + eF_{\mu\nu} n^\mu p^{(n)\nu} \overset{M_0}{=} m^2 + \frac{3}{2} e F_{\mu\nu} S^{\mu\nu} \neq 0 && (429)
\end{aligned}$$

The consistency condition on the secondary constraint θ is:

$$\dot{\theta} \doteq 0 \tag{430}$$

On the other hand,

$$\begin{aligned} \dot{\theta} &\doteq \{\theta, H_c\} + \Lambda\{\theta, \Phi\} + \Lambda\{\theta, \phi\} \\ &\doteq \lambda(-m^2 - \frac{3}{2}eF_{\mu\nu}S^{\mu\nu}) \end{aligned} \tag{431}$$

Therefore,

$$\lambda \overset{M_1}{=} 0 \tag{432}$$

and no more constraints appear in the Dirac algorithm.

The constraint Φ is of the first-class because it PB commutes with ϕ and θ. The constraints ϕ and θ form a second-class pair. The Lagrangian multiplier Λ remains undetermined.

The number of degrees of freedom in the phase space is equal to the dimension of the phase space T^*Q minus twice the number of first-class constraints minus the number of second-class constraints: $16 - 2 - 2 = 12$.

Therefore, there are 12 degrees of freedom in phase space. In the configuration space, thus, there are 6 degrees of freedom. It is natural to think that 3 of these correspond to the position of the relativistic particle and the other 3 to the components of the spin.

Notice that the constraints ϕ and θ guarantee that the Frenkel conditions [77, 86]:

$$S_{\mu\nu}\dot{x}^{\nu} \overset{M_1}{=} 0 \tag{433}$$

are satisfied. Indeed, from (404), (419) and (432) it follows:

$$\begin{aligned} S_{\mu\nu}\dot{x}^{\nu} &= 2n(n_{\mu}p_{\nu}^{(n)} - n_{\nu}p_{\mu}^{(n)})\pi^{\mu} \\ &= 2n(n\pi)p_{\nu}^{(n)} - 2n(\pi p^{(n)})n_{\nu} \\ &= 2n\theta p_{\nu}^{(n)} - 2n\phi n_{\nu} \\ &\overset{M_1}{=} 0 \end{aligned} \tag{434}$$

Taking into account (419) and (432) we see that we can rewrite the Frenkel conditions (433) in the equivalent form:

$$S_{\mu\nu}\pi^{\nu} \overset{M_1}{=} 0 \tag{435}$$

These conditions guarantee that only three components of the antisymmetric spin matrix $S_{\mu\nu}$ are truly independent, as required by the fact that only three degrees of freedom are associated with the spin.

It is useful to define the spin four-vector as

$$S_{\mu} = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\frac{\pi^{\alpha}}{\sqrt{\pi^2}}S^{\beta\gamma} \tag{436}$$

Notice that in the absence of an external electromagnetic field the expression (436) becomes equal to the Pauli-Lubanski four-vector [104].

From the definition (436) it is clear that the spin four-vector is perpendicular to the momentum π_{μ}:

$$S_{\mu}\pi^{\mu} = 0 \tag{437}$$

Therefore, as expected, the spin four-vector S_{μ} has only three independent components. It provides an equivalent and convenient way of describing the spin. It is not difficult to prove that from (435,436) the spin four-vector can be written on the constrained surface M_1 as follows:

$$S_{\mu\nu} \overset{M_1}{=} \varepsilon_{\mu\nu\alpha\beta}S^{\alpha}\frac{\pi^{\beta}}{\sqrt{\pi^2}} \tag{438}$$

Indeed, from the definition of the spin four-vector (436) it follows

$$\varepsilon_{\mu\nu\alpha\beta}S^{\alpha}\frac{\pi^{\beta}}{\sqrt{\pi^2}} = \frac{1}{2\pi^2}\varepsilon_{\alpha\mu\nu\beta}\varepsilon^{\alpha\rho\sigma\eta}\pi_{\rho}S_{\sigma\eta}\pi^{\beta} \tag{439}$$

On the other hand,

$$\varepsilon_{\alpha\mu\nu\beta}\,\varepsilon^{\alpha\rho\sigma\eta} = \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\delta^{\eta}_{\beta} + \delta^{\eta}_{\mu}\delta^{\rho}_{\nu}\delta^{\sigma}_{\beta} + \delta^{\sigma}_{\mu}\delta^{\eta}_{\nu}\delta^{\rho}_{\beta} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}\delta^{\eta}_{\beta} - \delta^{\eta}_{\mu}\delta^{\sigma}_{\nu}\delta^{\rho}_{\beta} - \delta^{\rho}_{\mu}\delta^{\eta}_{\nu}\delta^{\sigma}_{\beta} \tag{440}$$

Substituting (440) into (439) we obtain

$$\varepsilon_{\mu\nu\alpha\beta}S^{\alpha}\frac{\pi^{\beta}}{\sqrt{\pi^2}} = \frac{1}{\pi^2}(\pi^2 S_{\mu\nu} + \pi_{\mu}S_{\nu\beta}\pi^{\beta} - \pi_{\nu}S_{\mu\beta}\pi^{\beta}) \tag{441}$$

Using the Frenkel conditions (435) we immediately see that the second and third terms in (441) vanish in M_1 and therefore we obtain (438).

From the definition of the spin four-vector (436) it follows that

$$S^2 \equiv S_\mu S^\mu = -\frac{1}{2}(S_{\mu\nu}S^{\mu\nu}) + \frac{1}{\pi^2}(S_{\mu\alpha}\pi^\alpha)(S^{\mu\beta}\pi_\beta) \tag{442}$$

Substituting the Frenkel conditions (435) into (442) we obtain

$$S^2 \stackrel{M_1}{=} -\frac{1}{2}(S_{\mu\nu}S^{\mu\nu}) \tag{443}$$

Let us rewrite the equations of motion (419-422) in a more standard way. First notice that from (419-422) and (432) it follows

$$\dot{x}^\mu = 2\Lambda\pi^\mu \tag{444}$$

$$\dot{n}^\mu = 2\Lambda e F^{\mu\nu} n_\nu \tag{445}$$

$$\dot{p}_\mu = \Lambda e F_{\mu\nu}\pi^\nu \tag{446}$$

$$\dot{p}^{(n)}_\mu = n_\mu + 2\Lambda e F_{\mu\nu} p^{(n)\nu} \tag{447}$$

On the other hand, from (414) and (409) we have

$$\begin{aligned}
\dot{\pi}_\mu &= \dot{p}_\mu - e\dot{A}_\mu \\
&= \dot{p}_\mu - e\frac{\partial A_\mu}{\partial x^\nu}\dot{x}^\nu \\
&= \dot{p}_\mu + \frac{e}{2}F_{\mu\nu}\dot{x}^\nu
\end{aligned} \tag{448}$$

Substituting (444) and (446) into (448) we obtain

$$\dot{\pi}_\mu = 2\Lambda e F_{\mu\nu}\pi^\nu \tag{449}$$

or equivalently

$$\ddot{x}_\mu = 2\Lambda e F_{\mu\nu}\dot{x}^\nu \tag{450}$$

The derivative of the antisymmetric spin matrix $S_{\mu\nu}$ with respect to proper time τ is

$$\dot{S}_{\mu\nu} = \dot{n}_\mu p^{(n)}_\nu + n_\mu \dot{p}^{(n)}_\nu - \dot{n}_\nu p^{(n)}_\mu - n_\nu \dot{p}^{(n)}_\mu \tag{451}$$

Substituting (445) and (447) into (451) and taking into account (410) we obtain

$$\dot{S}_{\mu\nu} = 2\Lambda e F_{\mu}{}^{\alpha} S_{\alpha\nu} - 2\Lambda e F_{\nu}{}^{\alpha} S_{\alpha\mu} \tag{452}$$

Comparing (450) with the Lorentz force law we see that it is appropriate to choose $\Lambda = \frac{1}{2m}$. With this choice the equations (450) and (452) read

$$\ddot{x}_{\mu} = \frac{e}{m} F_{\mu\nu} \dot{x}^{\nu} \tag{453}$$

$$\dot{S}_{\mu\nu} = \frac{e}{m} F_{\mu}{}^{\alpha} S_{\alpha\nu} - \frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu} \tag{454}$$

Equations (453,454) coincide exactly with those previously proposed in the literature [74, 86] for a relativistic spinning particle with $g = 2$ in a uniform static electromagnetic field.

Notice that the Frenkel conditions (433) are preserved in time (see the equations (453,454)); indeed,

$$\begin{aligned}
\frac{d}{d\tau}(S_{\mu\nu}\dot{x}^{\nu}) &= \dot{S}_{\mu\nu}\dot{x}^{\nu} + S_{\mu\nu}\ddot{x}^{\nu} \\
&= \frac{e}{m} F_{\mu}{}^{\alpha} S_{\alpha\nu}\dot{x}^{\nu} - \frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu}\dot{x}^{\nu} + \frac{e}{m} S_{\mu\nu} F^{\nu\alpha}\dot{x}_{\alpha} \\
&= -\frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu}\dot{x}^{\nu} + \frac{e}{m} S_{\mu\nu} F^{\nu\alpha}\dot{x}_{\alpha} \\
&= -\frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu}\dot{x}^{\nu} + \frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu}\dot{x}^{\nu} \\
&= 0
\end{aligned} \tag{455}$$

From (454) it follows also that the total spin (443) is a constant of motion.

$$\begin{aligned}
\frac{dS^2}{d\tau} &= -\dot{S}_{\mu\nu} S^{\mu\nu} \\
&= -\frac{e}{m} F_{\mu}{}^{\alpha} S_{\alpha\nu} S^{\mu\nu} + \frac{e}{m} F_{\nu}{}^{\alpha} S_{\alpha\mu} S^{\mu\nu} \\
&= -\frac{2e}{m} F_{\mu\alpha} g^{\alpha\beta} S_{\beta\nu} S^{\mu\nu} \\
&= 0
\end{aligned} \tag{456}$$

For the spin four-vector S_{μ} we can write the following evolution equation:

$$\dot{S}_{\mu} = \frac{e}{m} F_{\mu\nu} S^{\nu} \tag{457}$$

Indeed, from (436) it follows that

$$\dot{S}_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\frac{\dot{\pi}^\alpha}{\sqrt{\pi^2}}S^{\beta\gamma} - \frac{1}{4}\varepsilon_{\mu\alpha\beta\gamma}\pi^\alpha S^{\beta\gamma}\frac{(\pi\dot{\pi})}{\pi^2} + \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\frac{\pi^\alpha}{\sqrt{\pi^2}}\dot{S}^{\beta\gamma} \tag{458}$$

From (449) it is clear that $(\pi\dot{\pi}) = 0$. Substituting (449) and (454) into (458) and doing some algebra we obtain

$$\dot{S}_\mu = \frac{e}{2m}\varepsilon_{\mu\alpha\beta\gamma}F^{\alpha\sigma}\frac{\pi_\sigma}{\sqrt{\pi^2}}S^{\beta\gamma} + \frac{e}{m}\varepsilon_{\mu\alpha\beta\gamma}\frac{\pi^\alpha}{\sqrt{\pi^2}}F^{\beta\sigma}S_\sigma{}^\gamma \tag{459}$$

If we substitute (438) into (459) and take into account that

$$\varepsilon_{\mu\nu\alpha\beta}\,\varepsilon^{\alpha\beta\gamma\eta} = 2(\delta^\gamma_\mu\delta^\eta_\nu - \delta^\eta_\mu\delta^\gamma_\nu) \tag{460}$$

we obtain straightforwardly the equation (457).

Therefore, our model seems to describe well the motion of a spinning relativistic particle with gyromagnetic ratio $g = 2$ in an external uniform and static electromagnetic field. The main ingredients of our model are the following:

- The phase space of the system is given by the coordinates x^μ and n^μ and their conjugate momenta p_μ and $p^{(n)}_\mu$.
- The antisymmetric spin matrix $S_{\mu\nu}$ is defined in terms of the four-vectors n_μ and $p^{(n)}_\mu$ as in (404).
- The canonical Hamiltonian is given by (411) and the primary constraints are (412,413).

Out of these starting points we have obtained the following results:

- The system has six degrees of freedom; three of them are related to the position of the particle and the other three to the spin.
- The Frenkel conditions (433) are satisfied.
- The correct equations of motion (453,454) are obtained and they preserve the Frenkel condition. The total spin is a constant of the motion.

It is useful to develop the Lagrangian formulation for our model. The Lagrange function is defined in the velocity phase space TQ which is described by the generalized coordinates x^μ and n^μ and the generalized velocities $\dot{x}^\mu$ and $\dot{n}^\mu$. As shown in Part 1, Section 1.2, the velocity phase space is the image of the subspace M_0 when the pullback application FL^* is realized (see (29)). M_0 is a subset of T^*Q which is defined by the primary constraints (412,413). Notice also that, as shown in Part 1, Section 1.2, the relations (419,420) are valid for any point in TQ (see Eqs.(37)) and therefore are satisfied for any arbitrary trajectory, regardless of whether it is a solution of (421,422) or not. In fact, Eqs. (419,420) are just the definition of the Lagrange multipliers. In other words, the Lagrange multipliers are essentially those 'velocities' that cannot be expresed in terms of the coordinates and the momenta alone.

Using the relations (419,420) and the primary constraints (412,413) we can express the Lagrange multipliers Λ and λ in terms of the coordinates x^μ and n^μ and the velocities $\dot{x}^\mu$ and $\dot{n}^\mu$. In order to achieve that, let us notice first that Eq. (420) can easily be solved for the momenta π^μ as

$$\lambda\pi^\mu = \dot{n}^\mu - 2\Lambda e F^{\mu\nu} n_\nu \tag{461}$$

Multiplying (419) by λ and substituting into it (461) we obtain

$$\lambda^2 p^{(n)\mu} = \lambda\dot{x}^\mu - 2\Lambda\dot{n}^\mu + 4\Lambda^2 e F^{\mu\nu} n_\nu \tag{462}$$

Therefore, multiplying (461) by (462) and using the primary constraint (413) we obtain the following equation:

$$\lambda(\dot{n}\dot{x}) - 2\Lambda\dot{n}^2 - 2\Lambda\lambda e F^{\mu\nu}\dot{x}_\mu n_\nu + 8\Lambda^2 e F^{\mu\nu}\dot{n}_\mu n_\nu - 8\Lambda^3 e^2 F_{\mu\alpha} g^{\mu\nu} F_{\nu\beta} n^\alpha n^\beta = 0 \tag{463}$$

On the other hand, from the definition of the antisymmetric spin matrix (404) and from (462) it follows that

$$\lambda^2 S^{\mu\nu} = n^\mu(\lambda\dot{x}^\nu - 2\Lambda\dot{n}^\nu + 4\Lambda^2 e F^{\nu\alpha} n_\alpha) - (\lambda\dot{x}^\mu - 2\Lambda\dot{n}^\mu + 4\Lambda^2 e F^{\mu\alpha} n_\alpha) n^\nu \tag{464}$$

From (461) it also follows

$$\lambda^2\pi^2 = \dot{n}^2 + 4\Lambda^2 e^2 F_{\mu\alpha} g^{\mu\nu} F_{\nu\beta} n^\alpha n^\beta - 4\Lambda e F^{\mu\nu} \dot{n}_\mu n_\nu \tag{465}$$

Let us use now the other primary constraint, namely (412). We can write

$$\lambda^2(\pi^2 - eF_{\mu\nu}S^{\mu\nu} - m^2) = 0 \tag{466}$$

Substituting Eqs.(464) and (465) into (466) we get the following equation:

$$\dot{n}^2 + 2\lambda e F^{\mu\nu} \dot{x}_\mu n_\nu - 8\Lambda e F^{\mu\nu} \dot{n}_\mu n_\nu + 12\Lambda^2 e^2 F_{\mu\alpha} g^{\mu\nu} F_{\nu\beta} n^\alpha n^\beta - \lambda^2 m^2 = 0 \tag{467}$$

Therefore, the Lagrange multipliers are the solutions of the system of equations (463) and (467). We have not been able to solve this system of equations exactly. We use an iterative procedure to find approximate solutions when the strength of the external field $F_{\mu\nu}$ is not large. We assume that the solutions for Λ and λ can be written as a series expansion in powers of F:

$$\Lambda = \Lambda_0 + \Lambda_1 + ... \tag{468}$$

$$\lambda = \lambda_0 + \lambda_1 + ... \tag{469}$$

where Λ_0 and λ_0 are the zeroth order terms, Λ_1 and λ_1 are the linear terms in the field F, etc.

In the zeroth approximation the equations (463) and (467) become:

$$\lambda_0(\dot{n}\dot{x}) - 2\Lambda_0\dot{n}^2 = 0 \tag{470}$$

$$\dot{n}^2 - \lambda_0^2 m^2 = 0 \tag{471}$$

From (470,471) we immediately find that

$$\lambda_0 = \frac{\sqrt{\dot{n}^2}}{m} \tag{472}$$

$$\Lambda_0 = \frac{(\dot{n}\dot{x})}{2m\sqrt{\dot{n}^2}} \tag{473}$$

In the first order approximation (463) and (467) read

$$\lambda_1(\dot{n}\dot{x}) - 2\Lambda_1\dot{n}^2 - 2\Lambda_0\lambda_0 eF^{\mu\nu}\dot{x}_\mu n_\nu + 8\Lambda_0^2 eF^{\mu\nu}\dot{n}_\mu n_\nu = 0 \tag{474}$$

$$2\lambda_0 eF^{\mu\nu}\dot{x}_\mu n_\nu - 8\Lambda_0 eF^{\mu\nu}\dot{n}_\mu n_\nu - 2\lambda_0\lambda_1 m^2 = 0 \tag{475}$$

and from them we find that

$$\lambda_1 = \frac{e}{m^2}F^{\mu\nu}\dot{x}_\mu n_\nu - 4\frac{e}{m^2}\frac{\Lambda_0}{\lambda_0}F^{\mu\nu}\dot{n}_\mu n_\nu \tag{476}$$

$$\Lambda_1 = \frac{(\dot{n}\dot{x})}{2\dot{n}^2}\lambda_1 - \frac{e}{\dot{n}^2}\Lambda_0\lambda_0 F^{\mu\nu}\dot{x}_\mu n_\nu + 4\frac{e}{\dot{n}^2}\Lambda_0^2 F^{\mu\nu}\dot{n}_\mu n_\nu \tag{477}$$

Substituting (472,473) and (476,477) into (468,469) we finally obtain the following expressions for the Lagrange multipliers Λ and λ in terms of the coordinates and velocities up to terms linear in the electromagnetic field:

$$\lambda = \frac{\sqrt{\dot{n}^2}}{m} + \frac{e}{m^2}F^{\mu\nu}\dot{x}_\mu n_\nu - 2\frac{e}{m^2}\frac{(\dot{n}\dot{x})}{\dot{n}^2}F^{\mu\nu}\dot{n}_\mu n_\nu \tag{478}$$

$$\Lambda = \frac{(\dot{n}\dot{x})}{2m\sqrt{\dot{n}^2}} \tag{479}$$

The Lagrange function is equal to

$$L = \frac{1}{2}n^2 + \dot{x}^\mu p_\mu + \dot{n}^\mu p_\mu^{(n)} \tag{480}$$

where p_μ and $p_\mu^{(n)}$ should be expressed in terms of the generalized coordinates and velocities.

Since $p_\mu = \pi_\mu + eA_\mu$ we can rewrite (480) as

$$L = \frac{1}{2}n^2 + \dot{x}\pi + \dot{n}p^{(n)} + eA\dot{x} \tag{481}$$

From (419) and (420) it follows

$$\dot{x}\pi = 2\Lambda\pi^2 + \lambda\pi p^{(n)} \tag{482}$$

$$\dot{n}p^{(n)} = 2\Lambda eF^{\mu\nu}p_\mu^{(n)}n_\nu + \lambda\pi p^{(n)} \tag{483}$$

and therefore,

$$L = \frac{1}{2}n^2 + 2\Lambda(\pi^2 - eF^{\mu\nu}n_\mu p_\nu^{(n)}) + 2\lambda\pi p^{(n)} + eA\dot{x} \tag{484}$$

Since L is a function defined in the velocity phase space TQ and since the primary constraints (412,413) identically vanish in TQ, we can write

$$L = \frac{1}{2}n^2 + 2\Lambda(m^2 + eF^{\mu\nu}n_\mu p_\nu^{(n)}) + eA\dot{x} \tag{485}$$

Substituting (479) into (485) we obtain up to the linear approximation in the field strength:

$$L = \frac{1}{2}n^2 + \frac{m(\dot{n}\dot{x})}{\sqrt{\dot{n}^2}}\left(1 + \frac{e}{m^2}F^{\mu\nu}n_\mu p_\nu^{(n)}\right) + eA\dot{x} \tag{486}$$

In the zeroth approximation,

$$\begin{aligned} p_\mu^{(n)} &= \frac{1}{\lambda_0}\left(\dot{x}_\mu - \frac{2\Lambda_0}{\lambda_0}\dot{n}_\mu\right) \\ &= \frac{m}{\sqrt{\dot{n}^2}}\left(\dot{x}_\mu - \frac{(\dot{n}\dot{x})}{\dot{n}^2}\dot{n}_\mu\right) \end{aligned} \tag{487}$$

Using (410) for the Lagrange function in the linear approximation we can finally write

$$L = \frac{1}{2}n^2 + \frac{m(\dot{n}\dot{x})}{\sqrt{\dot{n}^2}}\left(1 + \frac{e}{2m^2}F_{\mu\nu}S^{\mu\nu}\right) + eA\dot{x} \tag{488}$$

where,

$$S^{\mu\nu} = \frac{m}{\sqrt{\dot{n}^2}}(n^\mu\dot{x}^\nu - n^\nu\dot{x}^\mu) - \frac{m(\dot{n}\dot{x})}{(\dot{n}^2)^{\frac{3}{2}}}(n^\mu\dot{n}^\nu - n^\nu\dot{n}^\mu) \tag{489}$$

With this we conclude our presentation of the Hamiltonian and Lagrangian formulations for a spinning relativistic particle with gyromagnetic ratio $g = 2$ in a uniform and static external electromagnetic field in 3+1 dimensions.

A question that immediately arises is whether the model proposed here for a spinning relativistic particle in 3+1 dimensions is valid in other space-time dimensions. A very simple argument shows that the answer is negative. Indeed, if we use this model for a particle in an N-dimensional space-time we would have that the phase space T^*Q would be of dimension $4N$. Since the model has one first-class and two second-class constraints, then the number of physical degrees of freedom in phase space is: $4N - 2(1) - 1(2) = 4(N-1)$. In the configuration space Q we therefore have $2(N-1)$ physical degrees of freedom. Of these, we need $N-1$ degrees of freedom for describing the position of the particle in the N-dimensional space-time. The remaining $N-1$ degrees of freedom would be associated with the spin. On the other hand, we know that the antisymmetric spin matrix in N space-time dimensions has $\frac{1}{2}(N-1)(N-2)$ independent components. This is because from the property of antisymmetry only $\frac{1}{2}N(N-1)$ components are independent, and since the antisymmetric spin matrix satisfies the Frenkel conditions (433), it has only

$$\frac{1}{2}N(N-1)-(N-1)=\frac{1}{2}(N-1)(N-2)$$

independent components. Therefore, our model is valid only when

$$N-1=\frac{1}{2}(N-1)(N-2).$$

Obviously, this condition is satisfied only for $N=4$.

The model proposed here could be modified. If one starts with a configuration space described by the variables x^μ and n^μ in which there are F first-class and S second-class Hamiltonian constraints then the model would have $2N-F-\frac{1}{2}S$ physical degrees of freedom. $N-1$ of these degrees of freedom would describe the position of the particle and the remaining $N-F-\frac{1}{2}S+1$ should be associated with the spin. Such a model could be valid only when

$$N-F-\frac{1}{2}S+1=\frac{1}{2}(N-1)(N-2).$$

This condition can be satisfied only for $N \leq 4$ because the system must have at least one first-class constraint, which in the free case becomes $p^2-m^2=0$.

In Section 3.3 we propose a model suitable for the description of a spinning relativistic particle in 2+1 dimensions.

3.3 Spinning relativistic particle in an electromagnetic field: 2+1 space-time dimensions.

The position of the relativistic particle in 2+1 dimensions is described by the vector x^μ ($\mu = 0, 1, 2$). The auxiliary generalized coordinates n^μ ($\mu = 0, 1, 2$) and σ are introduced to describe the spin. In phase space (T^*Q) we use the generalized coordinates x^μ, n^μ and σ and their generalized conjugate momenta p_μ, $p^{(n)}_\mu$ and $p^{(\sigma)}$. We can write the following Poisson brackets (PB):

$$\begin{aligned}
\{x_\mu, p_\nu\} &= -g_{\mu\nu} && (490)\\
\{x_\mu, x_\nu\} &= \{p_\mu, p_\nu\} = 0 && (491)\\
\{n_\mu, p^{(n)}_\nu\} &= -g_{\mu\nu} && (492)\\
\{n_\mu, n_\nu\} &= \{p^{(n)}_\mu, p^{(n)}_\nu\} = 0 && (493)\\
\{x_\mu, n_\nu\} &= \{x_\mu, p^{(n)}_\nu\} = \{p_\mu, n_\nu\} = \{p_\mu, p^{(n)}_\nu\} = 0 && (494)\\
\{\sigma, p^{(\sigma)}\} &= 1 && (495)\\
\{\sigma, x_\mu\} &= \{\sigma, p_\mu\} = \{\sigma, n_\mu\} = \{\sigma, p^{(n)}_\mu\} = 0 && (496)\\
\{p^{(\sigma)}, x_\mu\} &= \{p^{(\sigma)}, p_\mu\} = \{p^{(\sigma)}, n_\mu\} = \{p^{(\sigma)}, p^{(n)}_\mu\} = 0 && (497)
\end{aligned}$$

The metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = diag(+1, -1, -1) \qquad (498)$$

We define the spin vector S_μ by analogy with the orbital angular momentum in 2+1 dimensions:

$$S_\mu = -\varepsilon_{\mu\alpha\beta}\, n^\alpha p^{(n)\beta} \qquad (499)$$

The total angular momentum vector is given by

$$J_\mu = -\varepsilon_{\mu\alpha\beta}\, x^\alpha p^\beta + S_\mu \qquad (500)$$

From the PB relations (490-494) and the definitions (499) and (500) it follows that p_μ and J_μ satisfy Poincare algebra in 2+1 dimensions [95]:

$$\{p_\mu, p_\nu\} = 0 \qquad (501)$$

$$\{J_\mu, p_\nu\} = \varepsilon_{\mu\nu\lambda}p^\lambda \tag{502}$$

$$\{J_\mu, J_\nu\} = \varepsilon_{\mu\nu\lambda}J^\lambda \tag{503}$$

Let $F_{\mu\nu}$ be a uniform static external electromagnetic field. A vector potential describing such a field can be written as

$$A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu \tag{504}$$

Let us define the vector $\tilde{F}$ as follows:

$$\tilde{F}_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta}F^{\alpha\beta} \tag{505}$$

From the definition of the spin vector (499) and from (505) it follows that

$$(\tilde{F}S) \equiv \tilde{F}_\mu S^\mu = -F^{\mu\nu}n_\mu p^{(n)}_\nu \tag{506}$$

For the description of the motion of a spinning relativistic particle in 2+1 dimensions we propose the canonical Hamiltonian

$$H_c = -\frac{\sigma}{2}(n^2+1) \tag{507}$$

and the primary constraints

$$\Phi \equiv \pi^2 + 2e(\tilde{F}S) - m^2 \tag{508}$$

$$\phi \equiv (\pi p^{(n)}) \tag{509}$$

$$B \equiv p^{(\sigma)} \tag{510}$$

where,

$$\pi_\mu = p_\mu - eA_\mu \tag{511}$$

The primary constraints (508-510) define the subspace M_0 of T^*Q. Notice that

$$\{\pi_\mu, \pi_\nu\} = -eF_{\mu\nu} \tag{512}$$

From (490-494) and (512) it follows that the primary constraints (508) and (509) PB commute:

$$\{\Phi, \phi\} = 0 \tag{513}$$

It is obvious from (508-510) that the primary constraint (510) PB commutes with Φ and ϕ:

$$\{B, \Phi\} = \{B, \phi\} = 0 \tag{514}$$

We can write the total Hamiltonian as

$$H_T = -\frac{\sigma}{2}(n^2 + 1) + \Lambda\Phi + \lambda\phi + \eta B \tag{515}$$

where Λ, λ and η are the Lagrange multipliers.

From the total Hamiltonian (515) and the primary constraints (508-510) we derive the Dirac-Hamilton equations of motion

$$\dot{x}^\mu = \frac{\partial H_T}{\partial p_\mu} = 2\Lambda\pi^\mu + \lambda p^{(n)\mu} \tag{516}$$

$$\dot{n}^\mu = \frac{\partial H_T}{\partial p^{(n)}_\mu} = 2\Lambda e F^{\mu\nu} n_\nu + \lambda\pi^\mu \tag{517}$$

$$\dot{\sigma} = \frac{\partial H_T}{\partial p^{(\sigma)}} = \eta \tag{518}$$

$$\dot{p}_\mu = -\frac{\partial H_T}{\partial x^\mu} = \Lambda e F_{\mu\nu}\pi^\nu + \frac{1}{2}\lambda e F_{\mu\nu} p^{(n)\nu} \tag{519}$$

$$\dot{p}^{(n)}_\mu = -\frac{\partial H_T}{\partial n^\mu} = \sigma n_\mu + 2\Lambda e F_{\mu\nu} p^{(n)\nu} \tag{520}$$

$$\dot{p}^{(\sigma)} = -\frac{\partial H_T}{\partial \sigma} = \frac{1}{2}(n^2 + 1) \tag{521}$$

$$\pi^2 + 2e(\tilde{F}S) - m^2 = 0 \tag{522}$$

$$(\pi p^{(n)}) = 0 \tag{523}$$

$$p^{(\sigma)} = 0 \tag{524}$$

From the consistency conditions on the primary constraints (508-510) we obtain the following secondary constraints:

$$\tilde{\theta} \equiv \sigma(\pi n) \tag{525}$$

$$\chi \equiv n^2 + 1 \tag{526}$$

Notice that the constraint $\tilde{\theta} = 0$ (525) is satisfied either when $\sigma = 0$ or $(\pi n) = 0$. The subspace M_1 defined by the constraints (508-510) and (525,526) is therefore the union of two subspaces

$$M_1 = M_1^* \cup M_1^{(0)}$$

The subspace $M_1^{(0)}$ is defined by the constraints (508-510), (526) and

$$\sigma = 0$$

The subspace M_1^* is defined by the constraints (508-510), (526) and

$$\theta \equiv (\pi n) = 0 \tag{527}$$

From the definition of the spin vector (499) and the constraints ϕ (509) and θ (527) it follows that

$$\varepsilon_{\mu\alpha\beta}\, S^\alpha \pi^\beta \overset{M_1^*}{=} 0 \tag{528}$$

In other words, in M_1^* the spin vector S_μ is parallel to the momentum π_μ. This is the analog in 2+1 dimensions of the Frenkel conditions [77, 86]. Indeed, if we define the antisymmetric spin matrix $S_{\mu\nu}$ as in Section 3.2:

$$S_{\mu\nu} = n_\mu p_\nu^{(n)} - n_\nu p_\mu^{(n)}, \tag{529}$$

we see that from this definition and from (499) it follows that

$$S_\mu = -\frac{1}{2}\varepsilon_{\mu\alpha\beta}\, S^{\alpha\beta} \tag{530}$$

Therefore, the l.h.s. of (528) can be rewritten as

$$\begin{aligned}
\varepsilon_{\mu\alpha\beta}\, S^\alpha \pi^\beta &= -\frac{1}{2}\varepsilon_{\mu\alpha\beta}\, \varepsilon^{\alpha\gamma\eta} S_{\gamma\eta}\pi^\beta \\
&= \frac{1}{2}\varepsilon_{\alpha\mu\beta}\, \varepsilon^{\alpha\gamma\eta} S_{\gamma\eta}\pi^\beta \\
&= \frac{1}{2}(\delta_\mu^\gamma \delta_\beta^\eta - \delta_\mu^\eta \delta_\beta^\gamma) S_{\gamma\eta}\pi^\beta \\
&= \frac{1}{2}S_{\mu\nu}\pi_\nu - \frac{1}{2}S_{\nu\mu}\pi^\nu \\
&= S_{\mu\nu}\pi^\nu
\end{aligned} \tag{531}$$

and the relations (528) are equivalent to the Frenkel conditions

$$S_{\mu\nu}\pi^{\nu} \stackrel{M_1^*}{=} 0 \tag{532}$$

Since we are interested in the description of particles satisfying the Frenkel conditions (528), we will study the subspace M_1^*

The constraints (508-510) and (526,527) which define M_1^* satisfy the PB relations (513,514) and

$$\{\Phi, \theta\} = 0 \tag{533}$$

$$\{\Phi, \chi\} = 0 \tag{534}$$

$$\{\phi, \theta\} = \pi^2 + eF_{\mu\nu}n^{\mu}p^{(n)\nu} \stackrel{M_0}{=} m^2 - 3e(\tilde{F}S) \neq 0 \tag{535}$$

$$\{\phi, \chi\} = 2\theta \stackrel{M_1^*}{=} 0 \tag{536}$$

$$\{B, \theta\} = 0 \tag{537}$$

$$\{B, \chi\} = 0 \tag{538}$$

$$\{\theta, \chi\} = 0 \tag{539}$$

The consistency conditions on the secondary constraints (526,527) lead to

$$\lambda \stackrel{M_1^*}{=} 0 \tag{540}$$

No more constraints appear in the Dirac algorithm. The Lagrange multipliers Λ and η remain undetermined. Notice that the constraints Φ, B and χ are of the first-class (their PB's with all the constraints vanish in M_1^*). The constraints ϕ and θ form a second-class pair.

Since the dimension of the phase space T^*Q is equal to 14 and since there are 3 first-class and 2 second-class constraints, we find that the number of physical degrees of freedom (i.e. dimension of the reduced phase space M_1^*) is: 14 - 2 (3) - 1 (2) = 6.

Therefore, there are 3 degrees of freedom in the configuration space Q. Two of these degrees of freedom are related to the position of the relativistic particle in the 2+1 space-time; the remaining degree of freedom is associated with the spin.

This makes sense because from the Frenkel conditions (528) we know that in 2+1 dimensions the spin is parallel to the momentum $\pi_m u$ and therefore, it has only one degree of freedom. We can write

$$S_\mu \stackrel{M_1^*}{=} -\alpha \frac{\pi^\mu}{\sqrt{\pi^2}} \tag{541}$$

where α is an arbitrary parameter.

We can now rewrite the equations (516-521) in a more convenient way. From (511), (516), (519) and (540) it follows that

$$\dot{\pi}_\mu = 2\Lambda e F_{\mu\nu}\pi^\nu \tag{542}$$

or equivalently,

$$\ddot{x}^\mu = 2\Lambda e F^{\mu\nu}\dot{x}_\nu \tag{543}$$

From (517), (520), (540) and (529) it follows:

$$\dot{S}_\mu = 2\Lambda e F_{\mu\nu}S^\nu \tag{544}$$

Noticing that

$$\begin{aligned} -\varepsilon_{\mu\nu\lambda}\, S^\lambda &= \frac{1}{2}\varepsilon_{\mu\nu\lambda}\, \varepsilon^{\lambda\alpha\beta} S_{\alpha\beta} \\ &= \frac{1}{2}(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha) S_{\alpha\beta} \\ &= S_{\mu\nu} \end{aligned} \tag{545}$$

and using (530), (544) and (545) we have

$$\begin{aligned} \dot{S}_\mu &= -\frac{1}{2}\varepsilon_{\mu\alpha\beta}\, \dot{S}^{\alpha\beta} \\ &= -\frac{1}{2}\varepsilon_{\mu\alpha\beta}(2\Lambda e F^\alpha{}_\gamma S^{\gamma\beta} - 2\Lambda e F^\beta{}_\gamma S^{\gamma\alpha}) \\ &= -\frac{1}{2}\varepsilon_{\mu\alpha\beta}(-2\Lambda e \varepsilon^{\gamma\beta\nu} F^\alpha{}_\gamma S_\nu + 2\Lambda e \varepsilon^{\gamma\alpha\nu} F^\beta{}_\gamma S_\nu) \\ &= \Lambda e(-\delta_\mu^\gamma \delta_\alpha^\nu + \delta_\mu^\nu \delta_\alpha^\gamma) F^\alpha{}_\gamma S_\nu - \Lambda e(\delta_\mu^\gamma \delta_\beta^\nu - \delta_\mu^\nu \delta_\beta^\gamma) F^\beta{}_\gamma S_\nu \\ &= 2\Lambda e F_{\mu\nu} S^\nu \end{aligned} \tag{546}$$

From (541) and (542) it follows that

$$\dot{S}_\mu = -\dot{\alpha}\frac{\pi_\mu}{\sqrt{\pi^2}} + 2\Lambda e F_{\mu\nu} S^\nu \tag{547}$$

Comparing (546) with (547) we conclude that for the solutions the parameter α is a constant of motion (independent of the proper time τ). Notice also that from (546) it follows that the total spin S^2 is a constant of the motion:

$$\frac{dS^2}{d\tau} = 2\dot{S}_\mu S^\mu = 2\Lambda e F_{\mu\nu} S^\mu S^\nu = 0 \tag{548}$$

In fact, the total spin

$$S^2 = \alpha^2 \tag{549}$$

as follows immediately from (541).

The Frenkel conditions are also preserved by the equations of motion, indeed,

$$\begin{aligned}
\frac{d}{d\tau}(S_{\mu\nu}\pi^\nu) &= \dot{S}_{\mu\nu}\pi^\nu + S_{\mu\nu}\dot{\pi}^\nu \\
&= 2\Lambda e F_\mu{}^\alpha S_{\alpha\nu}\pi^\nu - 2\Lambda e F_\nu{}^\alpha S_{\alpha\mu}\pi^\nu + 2\Lambda e S_{\mu\nu} F^{\nu\alpha}\pi_\alpha \\
&= -2\Lambda e F_\nu{}^\alpha S_{\alpha\mu}\pi^\nu + 2\Lambda e S_{\mu\nu} F^{\nu\alpha}\pi_\alpha \\
&= -2\Lambda e F_\nu{}^\alpha S_{\alpha\mu}\pi^\nu + 2\Lambda e F_\nu{}^\alpha S_{\alpha\mu}\pi^\nu \\
&= 0
\end{aligned} \tag{550}$$

Comparing (543) with the Lorentz force law we see that it is convenient to choose $\Lambda = \frac{1}{2m}$. Therefore, we can rewrite the equations of motion (543) and (546) as

$$\ddot{x}^\mu = \frac{e}{m} F^{\mu\nu}\dot{x}_\nu \tag{551}$$

$$\dot{S}_\mu = \frac{e}{m} F_{\mu\nu} S^\nu \tag{552}$$

Equations (551) and (552) describe the motion of a relativistic spinning particle with gyromagnetic ratio $g = 2$ in a uniform static electromagnetic field in 2+1 dimensions.

With this we conclude the presentation of the Hamiltonian formulation of our model. For completeness, let us derive the Lagrange function corresponding to the total Hamiltonian (515). The derivation is very similar to the one shown in Section 3.2 for a particle in 3+1 dimensions. The first step is to express the Lagrange multipliers Λ, λ and η in terms of the generalized coordinates x^μ, n^μ and σ and the generalized velocities $\dot{x}^\mu$, $\dot{n}^\mu$ and $\dot{\sigma}$. This can be achieved by using the relations (516-518) and the primary constraints (508-510). Indeed, from (516-518) it follows that

$$\lambda\pi^\mu = \dot{n}^\mu - 2\Lambda e F^{\mu\nu} n_\nu \tag{553}$$

$$\lambda^2 p^{(n)\mu} = \lambda\dot{x}^\mu - 2\Lambda\dot{n}^\mu + 4\Lambda^2 e F^{\mu\nu} n_\nu \tag{554}$$

$$\eta = \dot{\sigma} \tag{555}$$

Using now the primary constraints (508,509) we obtain the following equations for Λ and λ:

$$\lambda(\dot{n}\dot{x}) - 2\Lambda\dot{n}^2 - 2\Lambda\lambda e F^{\mu\nu}\dot{x}_\mu n_\nu + 8\Lambda^2 e F^{\mu\nu}\dot{n}_\mu n_\nu - 8\Lambda^3 e^2 F_{\mu\alpha} g^{\mu\nu} F_{\nu\beta} n^\alpha n^\beta = 0 \tag{556}$$

$$\dot{n}^2 + 2\lambda e F^{\mu\nu}\dot{x}_\mu n_\nu - 8\Lambda e F^{\mu\nu}\dot{n}_\mu n_\nu + 12\Lambda^2 e^2 F_{\mu\alpha} g^{\mu\nu} F_{\nu\beta} n^\alpha n^\beta - \lambda^2 m^2 = 0 \tag{557}$$

Assuming that Λ and λ can be written as a series expansion in powers of the field strength F we find that, up to terms quadratic in F:

$$\Lambda = \frac{(\dot{n}\dot{x})}{2m\sqrt{\dot{n}^2}} \tag{558}$$

$$\lambda = \frac{\sqrt{\dot{n}^2}}{m} + \frac{e}{m^2} F^{\mu\nu}\dot{x}_\mu n_\nu - 2\frac{e}{m^2}\frac{(\dot{n}\dot{x})}{\dot{n}^2} F^{\mu\nu}\dot{n}_\mu n_\nu \tag{559}$$

The Lagrange function should be equal to

$$L = \frac{\sigma}{2}(n^2+1) + \dot{x}^\mu p_\mu + \dot{n}^\mu p^{(n)}_\mu + \dot{\sigma} p^{(\sigma)} \tag{560}$$

where p_μ and $p^{(n)}_\mu$ should be expressed in terms of the generalized coordinates and velocities. We know that $p^{(\sigma)} \equiv 0$ in TQ because it is a primary constraint. From (511), (516) and (517) it follows that

$$L = \frac{\sigma}{2}(n^2+1) + 2\Lambda(\pi^2 - eF^{\mu\nu} n_\mu p^{(n)}_\nu) + 2\lambda(\pi p^{(n)}) + \dot{\sigma} p^{(\sigma)} + eA\dot{x} \tag{561}$$

Since L is a function defined in the velocity phase space TQ and since the primary constraints (508-510) identically vanish in TQ, we can write

$$L = \frac{\sigma}{2}(n^2 + 1) + 2\Lambda(m^2 + eF^{\mu\nu}n_\mu p_\nu^{(n)}) + eA\dot{x} \tag{562}$$

Finally, using (506) and (558) we can write the Lagrange function in the linear approximation as follows:

$$L = \frac{\sigma}{2}(n^2 + 1) + \frac{m(\dot{n}\dot{x})}{\sqrt{\dot{n}^2}}\left(1 - \frac{e}{m^2}(\tilde{F}S)\right) + eA\dot{x} \tag{563}$$

where

$$S_\mu = -\frac{m}{\sqrt{\dot{n}^2}}\varepsilon_{\mu\alpha\beta}\, n^\alpha \left(\dot{x}^\beta - \frac{(\dot{n}\dot{x})}{\dot{n}^2}\dot{n}^\beta\right) \tag{564}$$

3.4 Dirac quantization.

Let us consider the Dirac quantization of the model proposed in Section 3.3 for a spinning relativistic particle with gyromagnetic ratio $g = 2$ in an external electromagnetic field in 2+1 dimensions. The Dirac method consists of deriving the Dirac brackets [1] among the generalized coordinates and momenta and then writing the corresponding commutation rules among the operators in a Hilbert space. The physical quantum states should be annihilated by the first-class constraint operators.

By definition the Dirac bracket between any two variables A and B is

$$\{A, B\}_D = \{A, B\} - \{A, \varphi_i\} C^{-1}_{ij} \{\varphi_j, B\} \tag{565}$$

where φ_i are the second-class constraints and C is an antisymmetric matrix defined as

$$C_{ij} = \{\varphi_i, \varphi_j\} \tag{566}$$

The Dirac rule for obtaining the commutation relations between the quantum operators A and B is

$$[A, B] = i\{A, B\}_D \tag{567}$$

In the model proposed in Section 3.3 only two second-class constraints are present. Notice that the Poisson bracket between these second-class constraints is (535)

$$\{\phi, \theta\} \overset{M_1^*}{=} m^2 - 3e(\tilde{F}S) \tag{568}$$

Therefore, the antisymmetric matrix C is equal to

$$C = \left(m^2 - 3e(\tilde{F}S)\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{569}$$

and for the inverse C^{-1} we can write

$$C^{-1} = -\frac{1}{\left(m^2 - 3e(\tilde{F}S)\right)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{570}$$

From the PB relations (490-497) and from (570) we obtain the Dirac brackets among all the variables that define the phase space T^*Q. Then, from the Dirac rule (567) it follows that the quantum operators obey the commutation relations:

$$
\begin{aligned}
[x_\mu, \pi_\nu] &= -ig_{\mu\nu} + i\frac{e(g_{\mu\nu}(\tilde{F}S) - \tilde{F}_\mu S_\nu)}{(m^2 - 3e(\tilde{F}S))} \\
[x_\mu, x_\nu] &= i\varepsilon_{\mu\nu\lambda}\frac{S^\lambda}{(m^2 - 3e(\tilde{F}S))} \\
[\pi_\mu, \pi_\nu] &= -ieF_{\mu\nu} - i\frac{e^2 F_{\mu\alpha}F_{\nu\beta}S^{\alpha\beta}}{(m^2 - 3e(\tilde{F}S))} \\
\left[n_\mu, p_\nu^{(n)}\right] &= -ig_{\mu\nu} - i\frac{\pi_\mu \pi_\nu}{(m^2 - 3e(\tilde{F}S))} \\
[n_\mu, n_\nu] &= \left[p_\mu^{(n)}, p_\nu^{(n)}\right] = 0 \\
[x_\mu, n_\nu] &= -i\frac{n_\mu \pi_\nu}{(m^2 - 3e(\tilde{F}S))} \\
\left[x_\mu, p_\nu^{(n)}\right] &= -i\frac{p_\mu^{(n)}\pi_\nu}{(m^2 - 3e(\tilde{F}S))} \\
[\pi_\mu, n_\nu] &= -i\frac{eF_{\mu\lambda}n^\lambda \pi_\nu}{(m^2 - 3e(\tilde{F}S))} \\
\left[\pi_\mu, p_\nu^{(n)}\right] &= -i\frac{eF_{\mu\lambda}p^{(n)\lambda}\pi_\nu}{(m^2 - 3e(\tilde{F}S))} \\
\left[\sigma, p^{(\sigma)}\right] &= i \\
[\sigma, x_\mu] &= [\sigma, p_\mu] = [\sigma, n_\mu] = \left[\sigma, p_\mu^{(n)}\right] = 0 \\
\left[p^{(\sigma)}, x_\mu\right] &= \left[p^{(\sigma)}, p_\mu\right] = \left[p^{(\sigma)}, n_\mu\right] = \left[p^{(\sigma)}, p_\mu^{(n)}\right] = 0
\end{aligned}
$$

The quantum states of the system Ψ should obey the equations

$$\left(\pi^2 + 2e(\tilde{F}S) - m^2\right)\Psi = 0 \tag{571}$$

$$p^{(\sigma)}\Psi = 0 \tag{572}$$

$$(n^2 + 1)\Psi = 0 \tag{573}$$

where

$$S_\mu = -\varepsilon_{\mu\alpha\beta}n^\alpha p^{(n)\beta} \tag{574}$$

Equation (571) was heuristically assumed in [101] for an anyon in a constant magnetic field in the context of the relativistic fractional quantum Hall effect and recently it was also obtained in [103] by introducing the minimal coupling with the electromagnetic field in the symplectic structure. It is remarkable that our Dirac-Hamilton formulation leads to the same equation. The remaining conditions (572,573) are just related with the introduction of the additional variables n^μ and σ. As was shown in [103], the nonrelativistic limit of (571) gives for the magnetic moment of the anyon $\mu = -e\alpha/m$ and we see again that the gyromagnetic ratio $g = 2$.

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