CONTRIBUTIONS TO STATISTICAL METHODS FOR QUALITY IMPROVEMENT

By

HONGJIAN XIAO

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Contents

Abstract

Statistical techniques are effective tools for controlling and reducing variation in the performance of products and processes. Parameter design is used to design a product/ process that will be robust against fluctuations of operating conditions as well as variation in raw materials. Control charts are very useful for studying variation in and maintaining the stability of processes.

Data analysis strategies for parameter design under a general location-variance model are proposed. These strategies are applied to experiments based on a combined array design. An empirical procedure is proposed for selecting a transformation that separates the control factors into two groups: adjustment factors and the rest. The adjustment factors facilitate tremendously the task of finding the optimal setting of the control factors. A unifying formulation of parameter design and tolerance design is proposed. The new formulation provides a way for further reducing performance variation.

Weighted attribute control charts for variable sample size are devised. Their control limits are always constant in spite of variable sample size. Several standardized control charts for short-run processes are proposed. These short-run charts allow real-time process control. A cumulative score control scheme, that can be supplemented to the $\bar{x}$ chart, is proposed to increase the ability of detecting a change in the process mean.

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1. Introduction

1.1. Overview

In the current era of a global economy, industrialized countries as well as developing countries are competing fiercely for customers worldwide. With the success of Japanese and some American and Canadian companies in the world market, quality has become one of the most critical factors that determine the very survival of businesses, large or small. Quality is no longer considered to adversely affect productivity; on the contrary, it is now recognized to lead to higher productivity.

Statistical techniques are effective tools for reducing variation in and improving quality of products and processes. In this thesis we study two important components of statistical techniques for quality improvement: parameter design (also known as robust design) and control charts. Parameter design can be employed to build quality into the product/process in the design stage. Control charts are graphical techniques for studying and maintaining the stability of processes. If it is to meet the customer's requirements, a product should be carefully designed so that it performs well under various operating conditions, and it should also be produced by a process that is stable and capable.

In this chapter we review some of the important ideas, issues and techniques in parameter design and control charting. Section 1.2 provides a review of parameter design and recent development in the area. In section 1.3 some basic concepts of control charting are described. An outline of the thesis is provided in section 1.4.

1.2. Parameter Design

In recent years, there has been great interest in the application of statistical techniques, especially the design of experiments techniques, to product/process parameter design, inspired by Taguchi's work on quality engineering (see, e.g., Taguchi, 1986; Taguchi and Wu, 1985; Taguchi and Phadke, 1984). It has been increasingly recognized

that the quality of a product/process is in large degree determined by its design, and that the economically effective approach to quality competitiveness is to design quality into the product/process in the first place.

The performance of a product/process is influenced by two kinds of parameters or factors. One is that of control factors whose levels are chosen by design engineers. The other kind is that of hard-to-control factors, called noise factors, that fluctuate in an application setting. The response variable that characterizes the output of a product/process always fluctuates due to the existence of noise factors. There are mainly three types of noise factors:

(1) fluctuation of operating condition,

(2) raw-material variation, and

(3) component deterioration.

Well aware of the cost and difficulties of controlling the above three types of noise factors, Taguchi (Taguchi, 1986; Taguchi and Wu, 1985) presented instead an approach, called parameter design, for finding the optimal setting of the control factors that makes a product/process insensitive to these noise factors. The levels of the control factors are so chosen that the product/process performs satisfactorily despite the variation in the noise factors. Taguchi's work has shown that significant reduction of variation can be achieved through designing products and processes robust against or insensitive to noise factors.

Let $\mathbf{x}$ be the vector of the control factors and $\mathbf{e}$ the vector of the noise factors. The noise factors are assumed to be independent random variables in the actual operating environment. The response variable y considered is characterized as a function of both the control and the noise factors as follows

$$y = f(\mathbf{x}, \mathbf{e}).$$

Because of the existence of the random noise factors, the response variable is also a random variable in the actual operating environment. Taguchi introduced quality loss as a

monetary measure of the deviation of the response variable y from its target value T. Let the general form of quality loss function be

$$L(y) = L(y, T).$$

Since the noise factors fluctuate, the expected quality loss

$$R(\mathbf{x}) = E_{\mathbf{e}}\{L(y, T)\}$$

represents the average measure of quality loss for a product/process.

Because it is difficult to determine the exact form of the quality loss function, Taguchi used a quadratic loss function defined as follows

$$L(y) = k(y - T)^2,$$

where k is some constant which can be determined on the basis of the information about monetary cost caused by exceeding the customer's tolerance (see Taguchi, 1986). Under this quadratic loss function, the expected quality loss is

$$\begin{aligned} R(\mathbf{x}) &= kE[(y - T)^2] \\ &= k\{\sigma^2(\mathbf{x}) + [\mu(\mathbf{x}) - T]^2\}, \end{aligned}$$

where

$$\mu(\mathbf{x}) = E_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})],$$

and

$$\sigma^2(\mathbf{x}) = Var_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})].$$

Obviously, the expected quality loss only differs from the mean square error of y from T, $MSE(\mathbf{x})=E[(y - T)^2]$, by a coefficient k.

Taguchi distinguishes among three different cases of parameter design, each having a different objective criterion called signal-to-noise (SN) ratio. The three cases are

(a) Closeness to the Target. The response characteristic is desired to be on the target value with minimal variation.

(b) Smaller-Is-Better. The response characteristic is desired to be as small as possible.

(c) Larger-Is-Better. The response characteristic is desired to be as large as possible.

For case (a), instead of minimizing directly the expected quality loss or the mean square error, Taguchi seeks to minimize a SN ratio defined by

$$SN_T(\mathbf{x}) = \ln[\mu^2(\mathbf{x})/\sigma^2(\mathbf{x})]$$

(his original definition is slightly different but equivalent). Taguchi's method of parameter design for this case can be summarized in the following

Procedure 1:

(1) Maximize the SN ratio $SN_T(\mathbf{x})$.

(2) Fine-tune the adjustment factors that affect the mean but not the SN ratio, to bring the mean response to target.

He essentially assumed that the control factors can be partitioned into two subsets, $\mathbf{x}=(\mathbf{d}, \mathbf{a})$, such that $\mathbf{a}$ is a subset of fine-tuning adjustment factors that are independent of the signal-to-noise ratio $SN_T(\mathbf{x})$. The setting $\mathbf{d}=\mathbf{d}^*$ is found by maximizing the SN ratio, and the mean response is then brought to target by adjusting $\mathbf{a}$ to $\mathbf{a}^*$ so that $\mu(\mathbf{d}^*, \mathbf{a}^*)=T$. Then $(\mathbf{d}^*, \mathbf{a}^*)$ will be the optimal setting of the control factors with the mean response on target.

For case (b), the response variable is desired as small as possible. Taguchi introduced a new SN ratio defined by

$$SN_S(\mathbf{x}) = -10 \log\{E(y^2)\}$$
$$= -10 \log\{\sigma^2(\mathbf{x}) + \mu^2(\mathbf{x})\}.$$

For case (c), the response variable is desired as large as possible. Taguchi introduced another SN ratio defined by

$$SN_L(\mathbf{x}) = -10 \log\{E(1/y^2)\}.$$

For the latter two cases, Taguchi seeks to find the optimal setting of the control factors by maximizing the corresponding SN ratio. Box (1988) argued that these two cases should be dealt with by analyzing location effects and dispersion effects separately, and that

they are mainly about location. However, Shoemaker et al. (1988) pointed out that the dispersion as well as the location could be important in some applications.

Taguchi's approach to maximization of the SN ratio is an application of two orthogonal arrays, as illustrated in Figure 1.1. The first array is called the outer array or the control array according to which the control factors are varied. For each row in the control array, a second array, called the inner array or the noise array, is used to simulate the noise factors. The whole setup is called a product-array design. The total number of runs, M, in the product-array is the number of runs in the control array, N, times the number of runs in the noise array, J, i.e., M=NxJ.

Let y_{ij} (i=1, ..., N, j=1, ..., J) be the responses from an experiment with an N-run control array and a J-run noise array. The data are assumed to follow the model

$$y_{ij} = \mu(\mathbf{x}_i) + \sigma(\mathbf{x}_i)\xi_{ij},$$

with $E(\xi_{ij})=0$ and $Var(\xi_{ij})=1$, i=1, ..., N, j=1, ..., J.

The average $\bar{y}_i$ and variance s_i^2 of the responses for each control setting $\mathbf{x}_i$, are computed as follows

$$\bar{y}_i = \frac{1}{J}\sum_{j=1}^{J} y_{ij} ,$$

and

$$s_i^2 = \frac{1}{J-1}\sum_{j=1}^{J}(y_{ij} - \bar{y}_i)^2.$$

The SN ratios SN_T, SN_S, and SN_L for $\mathbf{x}_i$ are estimated by, respectively,

$$SN_T(\mathbf{x}_i) = \ln(\bar{y}_i^2/s_i^2),$$

$$SN_S(\mathbf{x}_i) = -10 \log\{\frac{1}{J}\sum_{j=1}^{J} y_{ij}^2\},$$

and

$$SN_L(\mathbf{x}_i) = -10 \log\{\frac{1}{J}\sum_{j=1}^{J}(1/y_{ij})^2\}.$$

Treating the SN ratio as the dependent variable, Taguchi uses the main-effect plots and the analysis of variance to find the optimal setting of the control factors that maximizes the SN ratio.

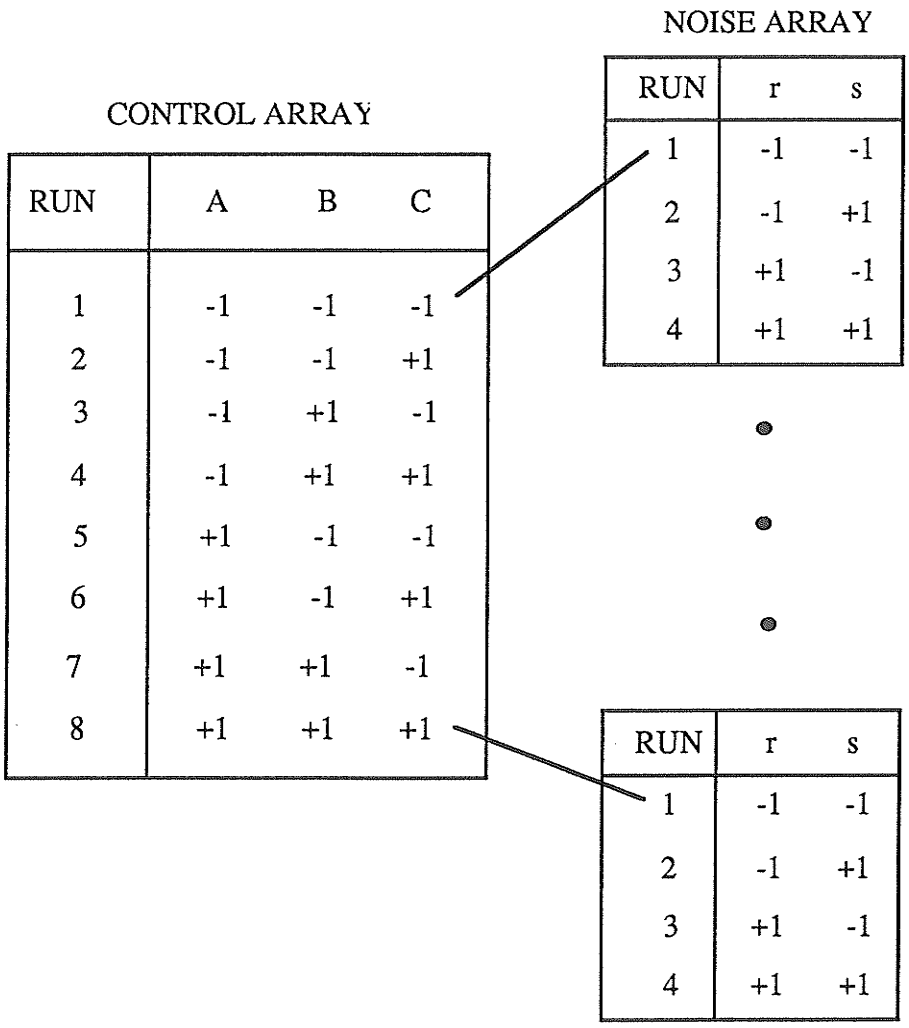

Figure 1.1. A Product-Array Strategy for Parameter Design.

For the closeness-to-target case, although he introduces the expected quality loss as the measure of the performance of a product/process, Taguchi optimizes the SN ratio instead of the expected quality loss itself. He argued that empirically the SN ratio is a better measure to be optimized, but he has never theoretically justified his change of the objective function. Phadke (1982) explained that in many applications the two-step optimization procedure involving the SN ratio indeed led to the optimal setting of the control factors.

Leon, Shoemaker, and Kacker (1987) further examined the connection between the expected quality loss and the SN ratio. When the control-factor set **x** can be divided into two subsets (**d**, **a**) such that **a** affects the mean but is independent of the SN ratio, they showed that Taguchi's two-step procedure is equivalent to the minimization of the expected quality loss. The control factors in **a** are called the adjustment factors. Leon et al. (1987) first showed that for any division of **x** into (**d**, **a**), the minimization of $R(\mathbf{d}, \mathbf{a})$ is equivalent to the following

Procedure 2:

(1) Find $\mathbf{d}^*$ that minimizes $P(\mathbf{d})=\min_{\mathbf{a}} R(\mathbf{d}, \mathbf{a})$.

(2) Find $\mathbf{a}^*$ that minimizes $R(\mathbf{d}^*, \mathbf{a})$. Then $(\mathbf{d}^*, \mathbf{a}^*)$ minimizes $R(\mathbf{d}, \mathbf{a})$.

To prove that the above procedure is equivalent to minimizing $R(\mathbf{d}, \mathbf{a})$ is simple (see Leon et al., 1987). Now assume that **a** is independent of $SN_T(\mathbf{x})$. Then $SN_T(\mathbf{x})$ can be written as $SN_T(\mathbf{d})$. The expected quality loss becomes

$$R(\mathbf{d}, \mathbf{a}) = k\{\sigma^2(\mathbf{d}, \mathbf{a}) + [\mu(\mathbf{d}, \mathbf{a}) - T]^2\}$$
$$= k\mu^2(\mathbf{d}, \mathbf{a})\{\exp[-SN_T(\mathbf{d})]\} + k[\mu(\mathbf{d}, \mathbf{a}) - T]^2.$$

To find $P(\mathbf{d})$, set

$$\frac{\partial R(\mathbf{d}, \mathbf{a})}{\partial \mathbf{a}} = 2k\frac{\partial \mu(\mathbf{d}, \mathbf{a})}{\partial \mathbf{a}}\{\mu(\mathbf{d}, \mathbf{a})[1 + \exp(-SN_T(\mathbf{d}))] - T\}=0.$$

Then

$$\mu(\mathbf{d}, \mathbf{a}^*(\mathbf{d})) = T/\{1 + \exp[-SN_T(\mathbf{d})]\}$$

and

$$P(\mathbf{d}) = T^2\exp[-SN_T(\mathbf{d})]/\{1 + \exp[-SN_T(\mathbf{d})]\}$$

$$= T^2/\{1 + \exp[SN_T(\mathbf{d})]\}.$$

Since $P(\mathbf{d})$ is a decreasing function of $SN_T(\mathbf{d})$, Procedure 2 is equivalent to the following

Procedure 3:

(1) Find $\mathbf{d}^*$ that maximizes $SN_T(\mathbf{d})$;

(2) Find $\mathbf{a}^*$ such that $\mu(\mathbf{d}^*, \mathbf{a}^*)=T/\{1+ \exp[-SN_T(\mathbf{d}^*)]\}$.

If the minimization is constrained by the requirement that the mean response of y be equal to T, Procedure 3 will be modified to the following

Procedure 3':

(1) Find $\mathbf{d}^*$ that maximizes $SN_T(\mathbf{d})$.

(2) Find $\mathbf{a}^*$ such that $\mu(\mathbf{d}^*, \mathbf{a}^*)=T$.

Procedure 3' is identical to Taguchi's procedure. The advantage of two-step procedures 3 and 3' is that they examine the location effect and the dispersion effect separately, and hence, traditional statistical techniques can be used.

The major drawback of these two-step procedures is the requirement to have the adjustment factors independent of the SN ratio. There could be, unfortunately, no adjustment factor in many parameter design problems. Leon et al. (1987) extended the concept of the SN ratio to the PerMIA or the performance measure independent of adjustment. A PerMIA is a measure of dispersion independent of a subset of the control factors. This subset is taken as the set of adjustment factors. For example, $P(\mathbf{d})$ in procedure 2 is a PerMIA.

Nair and Pregibon (1986) and Box (1988) studied the case in which the mean $\mu(\mathbf{x})$ and variance $\sigma^2(\mathbf{x})$ of the response variable are linked in a manner such that a function g(.) can be found for which $\sigma^2(\mathbf{x})/\{g[\mu(\mathbf{x})]\}^2$ only depends on a subset $\mathbf{d}$ of $\mathbf{x}=(\mathbf{d}, \mathbf{a})$. Then

$$P(\mathbf{d}) = \sigma^2(\mathbf{d}, \mathbf{a})/\{g[\mu(\mathbf{d}, \mathbf{a})]\}^2$$

is a PerMIA and $\mathbf{a}$ is the subset of the adjustment factors that can be changed without changing $P(\mathbf{d})$. Under these assumptions, the expected quality loss is

$$R(\mathbf{d}, \mathbf{a}) = k\{\sigma^2(\mathbf{d}, \mathbf{a}) + [\mu(\mathbf{d}, \mathbf{a}) - T]^2\}$$
$$= k\{g[\mu(\mathbf{d}, \mathbf{a})]\}^2 P(\mathbf{d}) + k[\mu(\mathbf{d}, \mathbf{a}) - T]^2.$$

Box (1988) provided the following procedure

Procedure 4:

(1) Find $\mathbf{d}^*$ that minimizes $P(\mathbf{d})$ or some monotonic function of $P(\mathbf{d})$.

(2) Adjust $\mathbf{a}$ to $\mathbf{a}^*$ so that $\mu(\mathbf{d}^*, \mathbf{a}^*)$ satisfies the equation

$$\mu(\mathbf{d}^*, \mathbf{a}^*) = T - g[\mu(\mathbf{d}^*, \mathbf{a}^*)]g'[\mu(\mathbf{d}^*, \mathbf{a}^*)]P(\mathbf{d}^*),$$

where $g'(.)$ is the derivative of $g(.)$.

If the minimization of the expected quality loss is constrained by the requirement that the mean response of y be equal to T, Procedure 4 becomes

Procedure 4':

(1) Find $\mathbf{d}^*$ that minimizes $P(\mathbf{d})$ or some monotonic function of $P(\mathbf{d})$.

(2) Adjust $\mathbf{a}$ to $\mathbf{a}^*$ so that $\mu(\mathbf{d}^*, \mathbf{a}^*)=T$.

Consider a special case in which $g(\mu)=\mu^{\alpha}$ and the assumptions made previously are satisfied. In this important case,

$$P(\mathbf{d}) = \sigma^2(\mathbf{d}, \mathbf{a})/[\mu(\mathbf{d}, \mathbf{a})]^{2\alpha}.$$

For $\alpha=1$,

$$P(\mathbf{d}) = \sigma^2(\mathbf{d}, \mathbf{a})/\mu^2(\mathbf{d}, \mathbf{a})$$
$$= \exp[-SN_T(\mathbf{d})].$$

Thus, minimizing $P(\mathbf{d})$ is equivalent to maximizing $SN_T(\mathbf{d})$.

An alternative approach for analyzing the dispersion effect is to transform y into $z=h(y)$ where $h'(y)=[1/g(y)]^{1/2}$. Then it can be shown by a Taylor series argument that, approximately,

$$\sigma_z^2 = \{h'[\mu(\mathbf{d}, \mathbf{a})]\}^2 \sigma_y^2$$
$$= \{h'[\mu(\mathbf{d}, \mathbf{a})]\}^2 \{g[\mu(\mathbf{d}, \mathbf{a})]\}^2 P(\mathbf{d})$$
$$= P(\mathbf{d}).$$

In practice, σ_z^2 will be replaced by its estimate s_z^2 and modeled as a function of the control factors. Such transformations are known as variance-stabilizing transformations (Bartlett and Kendall, 1946). Nair and Pregibon (1986) pointed out that empirical experiences suggest that often these transformations also have other benefits such as enhancing the symmetry of the underlying distribution and the additivity of the mean as a function of independent variables.

The real difficulty lies in diagnosing the form of the function g(.) or the transformation function h(.) from the data when neither is known. A technique of finding, when it exists, a transformation was illustrated by Box (1988). Suppose that a class of transformations $y^{(\lambda)}$ is indexed by a parameter λ. Then the dispersion effects and location effects are plotted against the value of λ, to help select a transformation that provides adjustment factors independent of the variance of the transformed response variable. The most commonly considered class of transformations is the family of power transformations defined as follows

$$y^{(\lambda)} = \log(y), \qquad \lambda=0,$$
$$= (y^{\lambda}-1)/\lambda, \qquad \text{otherwise.}$$

Shoemaker, Tsui, and Leon (1988) provided the following

Procedure 5:

(1) Find a transformation, z=h(y), to make σ_z^2 independent of **a**, a subset of control factors (**d**, **a**).

(2) Choose **d*** to minimize $\sigma_z^2(\mathbf{d})$.

(3) Adjust **a** to **a*** so that $\mu(\mathbf{d}^*, \mathbf{a}^*)=T$.

The procedures discussed above are based on modeling a measure of dispersion such as the SN ratio and the PerMIA as a function of the control factors, with a product-array design. This approach is referred to by Shoemaker et al. (1991) as the loss-model approach because it is based on modeling the expected quality loss (or other measures such as SN ratios and PerMIAs) directly as a function of the control factors.

Welch, Yu, Kang, and Sacks (1990) proposed to model the response variable instead as a function of both the control factors and the noise factors, and use the response model to discover control-factor settings that will help reduce variation in the performance of a product/process. Combining the control and noise factors in a single design matrix, called combined array, they modeled the response variable as a function of both the control and noise factors. They use the response model to obtain estimates of the expected quality loss, and then to find the optimal setting of the control factors that minimizes the expected quality loss.

The response model considered is

$$y = f(\mathbf{x}, \mathbf{e}) + \xi,$$

where ξ is the model error term with

$$E(\xi|\mathbf{x}, \mathbf{e}) = 0 \quad \text{and} \quad Var(\xi|\mathbf{x}, \mathbf{e}) = \sigma^2,$$

where σ^2 is unknown. Since the noise factors are random variables, the mean and variance of the response variable in the application setting are, respectively,

$$E(y) = E_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})],$$

and

$$Var(y) = Var_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})] + \sigma^2,$$

noting that $Cov\{f(\mathbf{x}, \mathbf{e}), \xi\}=0$. Therefore, the mean square error of y from T is

$$\begin{aligned} E(y - T)^2 &= \sigma_y^2 + (\mu_y - T)^2 \\ &= Var_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})] + \sigma^2 + \{E_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e})] - T\}^2 \\ &= E_{\mathbf{e}}[f(\mathbf{x}, \mathbf{e}) - T]^2 + \sigma^2. \end{aligned}$$

Thus, to minimize the mean square error of y from T, we only need to minimize the mean square error of $f(\mathbf{x}, \mathbf{e})$ from T. When $f(\mathbf{x}, \mathbf{e})$ is estimated by $\hat{f}(\mathbf{x}, \mathbf{e})$, its mean square error is estimated by

$$E_{\mathbf{e}}[\hat{f}(\mathbf{x}, \mathbf{e}) - T]^2.$$

Thus, to minimize $E(y - T)^2$, we seek to minimize $E_{\mathbf{e}}[\hat{f}(\mathbf{x}, \mathbf{e}) - T]^2$.

Shoemaker, Tsui and Wu (1991) further examined this response-model approach. They recommended that control-by-noise interaction plots as well as the estimated quality loss function be employed to find the optimal control-factor setting. They also discussed the advantages and disadvantages of the response-model approach. First, the combined array design provides significant reduction of run size of experiments. Second, it focuses on modeling the response variable rather than a measure of dispersion. The experimenter is more likely to have intuition or knowledge about the relationship between the response variable and the control and noise factors than about the relationship between the expected quality loss and the control factors. Finally, the drawback of this approach is that it depends more critically than the loss-model approach on how well the response model fits.

1.3. Control Charting for Statistical Process Control

The control chart technique was invented by Dr. Shewhart in the 1930's (see Shewhart, 1931). A control chart is a graphical display of a sequence of sample points taken periodically from a process. The control chart contains a center line (CL) that represents the average value of the sample statistic. There are also two other lines, called upper control limit (UCL) and lower control limit (LCL). These control limits are chosen in a way that when the process is in statistical control, nearly all the sample points will fall between them. However, a point that plots outside of the control limits, is taken as a signal that the process could be out of control. When it occurs, an investigation should be conducted to find the assignable cause or causes responsible for the problem.

The most commonly used control limits are 3-standard-deviation control limits. Suppose w is the sample statistic that is plotted on the control chart. Let $E(w)$ and $\sigma(w)$ represent its mean and standard deviation, respectively. Then the control limits and center line are set at

$$UCL = E(w) + 3\sigma(w),$$

$$CL = E(w),$$

$$LCL = E(w) - 3\sigma(w).$$

Since they are usually unknown, E(w) and $\sigma(w)$ are estimated from the collected samples. It is conventionally recommended that 20 to 30 samples are required to compute accurately the control limits. In this section we review the commonly used control charts.

For many cases, we count each item as either conforming or nonconforming to the specifications on the quality characteristic. The fraction nonconforming p is defined as the ratio of the number of nonconforming items to the total number of items in a population. The control chart for the fraction nonconforming is based on the binomial distribution. If samples of size n are taken from a process that is operating stably, then x_k, the number of nonconforming items in sample k, has a binomial distribution with parameters n and p for k=1, ..., m. The sample fraction is defined as

$$p_k = x_k/n.$$

Then the control limits and center line for the sample fraction nonconforming are set at

$$UCL = p + 3\sqrt{\frac{p(1-p)}{n}},$$

$$CL = p,$$

$$LCL = p - 3\sqrt{\frac{p(1-p)}{n}}.$$

This chart is called the p chart. Since the true fraction nonconforming p is often unknown, the usual practice is to estimate it by

$$\bar{p} = \frac{p_1 + p_2 + \ldots + p_m}{m}.$$

Thus, the control limits and center line are computed as follows

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}},$$

$$CL = \bar{p},$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}.$$

For many other cases, we may deal with the number of defects or nonconformities on a unit of product rather than the fraction nonconforming. The occurrence of nonconformities in a unit is usually well modeled by a Poisson distribution. Let c_k be the number of nonconformities in the kth independent sample, k=1, 2, ..., m. Let c be the expected number of nonconformities in a sample when the process is operating stably. Sample nonconformities are plotted the c chart with

$$UCL = c + 3\sqrt{c},$$

$$CL = c,$$

$$LCL = c - 3\sqrt{c}.$$

When it is unknown, c will be estimated by

$$\bar{c} = \frac{c_1 + c_2 + \ldots + c_m}{m}.$$

In the c chart we only consider the number of nonconformities in one single unit, however, for some applications u, the average number of nonconformities per unit, is a more convenient basis for setting up a control chart. This chart is called the u chart. If we find x_k, total nonconformities in the kth sample of n units, then the sample average number of nonconformities per unit is

$$u_k = x_k/n.$$

The control limits and center line for the current case are set at

$$UCL = u + 3\sqrt{\frac{u}{n}},$$

$$CL = u,$$

$$LCL = u - 3\sqrt{\frac{u}{n}}.$$

When it is unknown, u will be estimated by

$$\bar{u} = \frac{u_1 + u_2 + \ldots + u_m}{m}.$$

For a continuous quality characteristic, two control charts (commonly the $\bar{x}$ and R charts) are constructed together for controlling the average and variability, respectively.

These charts are called variable control charts. Suppose m samples of size n are taken from a process with mean μ and standard deviation σ. Let $\bar{x}_k$ and R_k be the mean and range of the kth sample for k=1, ..., m. Then the control limits and center line for the $\bar{x}$ chart are set at

$$UCL = \mu + A_2R',$$

$$CL = \mu,$$

$$LCL = \mu - A_2R',$$

where $R'=d_2\sigma$, $A_2=3/(\sqrt{n}d_2)$, and d_2 is a constant dependent of the sample n. The control limits and center line for the R chart are set at

$$UCL = D_4R',$$

$$CL = R',$$

$$LCL = D_3R',$$

where D_3 and D_4 are appropriate control limit factors dependent of n. In application, μ and R' are usually unknown and to be estimated from the samples obtained from the process. In fact, they are estimated by, respectively,

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_m}{m},$$

and

$$\bar{R} = \frac{R_1 + R_2 + \ldots + R_m}{m}.$$

The control limits and center line can be computed by replacing μ and R' with $\bar{\bar{x}}$ and $\bar{R}$. Thus, for the $\bar{x}$ chart,

$$UCL = \bar{\bar{x}} + A_2\bar{R},$$

$$CL = \bar{\bar{x}},$$

$$LCL = \bar{\bar{x}} - A_2\bar{R}.$$

And for the R chart,

$$UCL = D_4\bar{R},$$

$$CL = \bar{R},$$

$$LCL = D_3\bar{R}.$$

1.4. Outline of the Thesis

In chapter 2, we propose two data analysis strategies for parameter design from experiments with a combined array design, under a general location-variance model. In chapter 3, an empirical procedure is proposed to help diagnose a transformation that provides adjustment factors. The adjustment factors can be used in two-step optimization procedures for determining the optimal setting of the control factors. In chapter 4, a unifying formulation that integrates parameter design and tolerance design is presented for the case in which the theoretical model of the product/process is known to the design engineers. On the basis of the new formulation, the concept of combined robust design is introduced

In chapter 5, weighted p and u control charts for variable sample size are devised. The control limits of these weighted control charts are always constant, although the sample size varies. In chapters 6 and 7, standardized control charts for short-run processes are proposed. They can be used for statistical process control in real time. In chapter 8, we provide a cumulative score control scheme, that can be supplemented to the $\bar{x}$ chart, for increasing the ability of detecting a change in the process mean.

2. Parameter Design for a Location-Variance Model

2.1. Motivation

In chapter 1 we reviewed various techniques of parameter design developed in the last few years. The main objective of parameter design is to minimize the expected quality loss or, equivalently, the mean square error. The basic approach was to devise measures of dispersion such as the signal-to-noise ratios and PerMIAs. The optimization of the expected quality loss is then done in a two-step procedure. The product array design was proposed by Dr. Taguchi for experimentation. This strategy is easier for engineers to understand, but it is expensive to apply. The cost for such an experiment is usually very high because a large number of runs with replication is required.

A recent alternative to the product-array design is the combined array design. Combining control and noise factors in a single design array, we seek to model the response variable as a function of both control and noise factors. The response model is then used to obtain an estimate of the expected quality loss and to find the optimal control-factor setting that minimizes this loss.

The main advantage of the combined array design over the product-array design is the substantial reduction in the number of experimental runs. It was demonstrated by Shoemaker et al. (1991) that the number of experimental runs can often be substantially reduced without sacrificing important information. In addition, the combined array design allows greater flexibility in the selection of estimable effects.

A potential drawback of the response-model approach reviewed in chapter 1 is that it implicitly assumes that the variance of the model error term is constant over all the design points. Thus, it depends more critically on how well the model fits than the loss-model approach reviewed in the same chapter . For example, if some of the noise factors are omitted from the model, this approach may lead to a control-factor setting that actually increases variability (see Shoemaker et al., 1991).

In this chapter we propose two data analysis strategies for analyzing experiments from a combined array design under more general assumptions than the one discussed by Shoemaker et al. (1991). The proposed approach will correct the drawback mentioned above. We will show that the combined array approach can deal with more situations than previously recognized.

In section 2.2 we present a general location-variance model. In section 2.3 we provide an analysis strategy for dealing with experiments from a combined array design with replication, and in section 2.4 we extend the strategy to experiments with one single replicate at each design point.

2.2. The Location-Variance Model

In this chapter we consider a general model, called location-variance model, under which the variance of the model error term is also a function of both control and noise factors. That is, the model considered would be

$$Y = f(\mathbf{x}, \mathbf{e}) + \xi, \tag{2.1}$$

with

$$E_\xi(\xi|\mathbf{x}, \mathbf{e}) = 0 \quad \text{and} \quad Var_\xi(\xi|\mathbf{x}, \mathbf{e}) = \sigma^2(\mathbf{x}, \mathbf{e}). \tag{2.2}$$

When the exact distribution of the model error term ξ is needed, we will assume that it has a normal distribution. We will refer to $f(\mathbf{x}, \mathbf{e})$ and $\sigma^2(\mathbf{x}, \mathbf{e})$ as the location function and variance function, respectively.

To proceed further we need to explain the different roles of the noise factors in an experiment and in an actual application setting. In an actual operating environment, noise factors are considered to be random variables because their fluctuations are either uncontrollable or too expensive to control. To study their effects on the response variable in an experiment, however, the noise factors, along with the control factors, are considered as design factors whose levels vary according to the design array. Although they are

controlled in the experiment in the same way as the control factors are, the noise factors are ultimately random variables whose variability causes variation in the response variable.

Considering ξ and $\mathbf{e}$ as random variables,

$$\begin{aligned}\mathrm{Cov}_{\xi,\,e}\{f(\mathbf{x},\mathbf{e}),\,\xi\} &= E_{\xi,\,e}\{f(\mathbf{x},\mathbf{e})\xi\} - E_e\{f(\mathbf{x},\mathbf{e})\}E_{\xi,\,e}(\xi)\\ &= E_e\{E_\xi[f(\mathbf{x},\mathbf{e})\xi|\mathbf{x},\mathbf{e}]\} - E_e\{f(\mathbf{x},\mathbf{e})\}E_e\{E_\xi(\xi|\mathbf{x},\mathbf{e})\}\\ &= E_e\{f(\mathbf{x},\mathbf{e})E_\xi[\xi|\mathbf{x},\mathbf{e}]\} - E_e\{f(\mathbf{x},\mathbf{e})\}E_e\{E_\xi(\xi|\mathbf{x},\mathbf{e})\}\\ &= 0,\end{aligned}$$

because of the model assumption $E_\xi(\xi|\mathbf{x},\mathbf{e})=0$. Thus, the variance of the response variable y in the operating situation is

$$\begin{aligned}\mathrm{Var}_{\xi,\,e}(y) &= \mathrm{Var}_e\{f(\mathbf{x},\mathbf{e})\} + \mathrm{Var}_{\xi,\,e}(\xi)\\ &= \mathrm{Var}_e\{f(\mathbf{x},\mathbf{e})\} + E_e\{E_\xi(\xi^2|\mathbf{x},\mathbf{e})\} - \{E_e[E_\xi(\xi|\mathbf{x},\mathbf{e})]\}^2\\ &= \mathrm{Var}_e\{f(\mathbf{x},\mathbf{e})\} + E_e\{E_\xi(\xi^2|\mathbf{x},\mathbf{e})\}\\ &= \mathrm{Var}_e\{f(\mathbf{x},\mathbf{e})\} + E_e\{\sigma^2(\mathbf{x},\mathbf{e})\}.\end{aligned}$$

The location function $f(\mathbf{x},\mathbf{e})$ and variance function $\sigma^2(\mathbf{x},\mathbf{e})$ are unknown and will be estimated from a carefully planned experiment. In the rest of this chapter we only consider 2-level factorial (or fractional factorial) experiments that are often used in the screening stage. In this stage the main objective is to identify significant factors in the model with the lowest cost.

There are two major reasons for choosing a general location-variance model rather than the simple location model considered by Welch et al. (1990) and Shoemaker et al. (1991), whose model were reviewed in section 1.2 of the previous chapter. One is that the variance of the response variable may change with its mean; in fact, the basic argument for a SN ratio or PerMIA is exactly that. The other reason of more concern to us, is that some noise factors that could cause heterogeneity may not be included in the experiment. For example, suppose the true model is

$$y = f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) + \xi, \tag{2.3}$$

with

$$E_\xi(\xi|\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) = 0 \quad \text{and} \quad Var_\xi(\xi|\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) = \sigma^2, \tag{2.4}$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is the set of noise factors. If all the noise factors are included in the design layout, then the model is homogeneous because the variance of the response variable, given $\mathbf{x}$, $\mathbf{e}_1$ and $\mathbf{e}_2$, is constant. On the other hand, if only part of the noise factors is included in the design layout, the model becomes heterogeneous. To made this point more clear, suppose only $\mathbf{e}_1$ is included in the design and $\mathbf{e}_2$ is not. Then the assumed model would be like

$$y = f_1(\mathbf{x}, \mathbf{e}_1) + \xi^*, \tag{2.5}$$

with

$$E_{\xi^*}(\xi^*|\mathbf{x}, \mathbf{e}_1) = 0 \quad \text{and} \quad Var_{\xi^*}(\xi^*|\mathbf{x}, \mathbf{e}_1) = \kappa^2(\mathbf{x}, \mathbf{e}_1). \tag{2.6}$$

In fact, it can be shown that (see the Appendix to this chapter)

$$\xi^* = f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) + \xi - f_1(\mathbf{x}, \mathbf{e}_1),$$

$$f_1(\mathbf{x}, \mathbf{e}_1) = E_{\mathbf{e}_2}\{f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2)\},$$

$$\kappa^2(\mathbf{x}, \mathbf{e}_1) = Var_{\mathbf{e}_2}\{f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2)|\mathbf{x}, \mathbf{e}_1\} + \sigma^2.$$

Clearly, the original homogeneous model (2.3) and (2.4) has become a location-variance model.

In the next several sections, we propose two data analysis strategies for parameter design from replicated and unreplicated 2-level factorial (or fractional factorial) experiments. Factors in $\mathbf{x}$ and $\mathbf{e}$ will be coded to only take values -1 and 1 in the data analysis stage. Without loss of generality, we also assume that in the actual application environment the noise factors are independent random variables that are coded so that $E(e_i)=0$ and $Var(e_i)=1$, i=1, 2, ..., m, where m is the number of the noise factors that are considered.

2.3. Analysis Strategy for Replicated Experiments

In the previous section we suggested that the expected quality loss be estimated through estimating $f(\mathbf{x}, \mathbf{e})$ and $\sigma^2(\mathbf{x}, \mathbf{e})$, the location and variance functions of the response

model. In this section we consider a data analysis strategy for estimating them from two-level factorial (or fractional factorial) experiments with replicates. Let y_{ij} (i=1, ..., N, j=1, ..., J) be responses from an N-run factorial experiment with J replicates. Let $\mathbf{z}=(\mathbf{x}, \mathbf{e})$. Suppose the data follow

$$y_{ij} = f(\mathbf{z}_i) + \xi_{ij},$$

with $E(\xi_{ij})=0$ and $Var(\xi_{ij})=\sigma^2(\mathbf{z}_i)$, i=1, ..., N, j=1, ..., J. The sample mean $\bar{y}_i$ and variance s_i^2 of the responses for run $\mathbf{z}_i$ are computed through

$$\bar{y}_i = \frac{1}{J}\sum_{j=1}^{J} y_{ij}$$

and

$$s_i^2 = \frac{1}{J-1}\sum_{j=1}^{J}(y_{ij} - \bar{y}_i)^2.$$

2.3.1. Data Analysis Strategy

To estimate the location and variance functions $f(\mathbf{x}, \mathbf{e})$ and $\sigma^2(\mathbf{x}, \mathbf{e})$, we first need to identify the significant effects for each function. The effects in location function are referred to as location effects, and the effects in the variance function as dispersion effects. We now propose an analysis strategy for determining the optimal setting of the control factors. The strategy includes the following 5 steps:

Step 1. Identify significant dispersion effects.

Step 2. Identify significant location effects.

Step 3. Estimate the location and variance functions.

Step 4. Estimate the expected quality loss.

Step 5. Find the setting of the control factors that minimizes this loss.

2.3.2. Dispersion Effects

To follow the proposed strategy, we first need to identify the significant control and noise factors in the variance function $\sigma^2(\mathbf{z})$ from the sample variances s_1^2, ..., s_N^2.

Because both the mean and variance of each s_i^2 depend on $\sigma^2(\mathbf{z}_i)$, an appropriate transformation of s_i^2 is often needed. Bartlett and Kendall (1946) suggest a log transformation $\ln s_i^2$, because, in addition to other advantages, the distribution of $\ln s_i^2$ only depends on $\sigma^2(\mathbf{z}_i)$ through the term $\ln\sigma^2(\mathbf{z}_i)$ in its mean value. Many researchers and practitioners have widely used the log transformation (e.g., see Bechhofer, 1960; Box and Meyer,1986; Nair and Pregibon, 1988). However, in their article, Bartlett and Kendall do not recommend using this transformation for J<5 because of its low efficiency. Thus, for J<5, any reasonable variance stabilizing transformation may be appropriate. In this chapter we use square root transformation that would use s_i as the dependent variable, as is suggested by Vining and Myers (1990). The reason for choosing the standard deviation over the log and other transformations is mainly two-fold. One is that zero variance can be dealt with very easily, in contrast to the log transformation. The other reason is that the variance, or squared standard deviation, is in fact the characteristic we are estimating.

Let $\mathbf{Z}$ represent the NxN orthogonal array associated with the N-run (fractional) factorial design involving both control and noise factors. Thus, the first column of $\mathbf{Z}$ is a column of 1's, whereas the remaining columns consist of 1's and -1's representing the main effects and interactions. Let $\mathbf{w}_s$ represent the Nx1 vector of sample standard deviations. Consider now the following linear model

$$\mathbf{w}_s = \mathbf{Z}\beta + \varepsilon_s,$$

where $\beta=(\beta_0, ..., \beta_{N-1})^T$ represents the vector of unknown coefficients, and ε_s is the vector of random errors. $2\beta_1, ..., 2\beta_{N-1}$ are the N-1 estimable main effects and interactions of dispersion. The ordinary least squares estimator of β is

$$\hat{\beta} = (1/N)\mathbf{Z}^T\mathbf{w}_s,$$

where $\mathbf{Z}^T$ is the transpose of $\mathbf{Z}$.

We can plot the ordered effects against the expected values of the standard normal order statistics. This plot is called a normal plot. These points should fall along a straight line that can be drawn visually, when there is not a single significant effect. Otherwise,

points that lie far away from the straight line are taken as significant effects. The normal plot is a widely used and is an effective tool for identifying significant effects.

2.3.3. Location Effects

The next step is to identify the location effects. Let $\mathbf{w}_y$ represent the Nx1 vector of the sample means. Consider now the following linear model

$$\mathbf{w}_y = \mathbf{Z}\alpha + \varepsilon_y,$$

where $\alpha=(\alpha_0, ..., \alpha_{N-1})^T$ represents the vector of unknown coefficients, ε_y is the vector of random errors, and $\mathbf{Z}$ represents the same NxN orthogonal array mentioned earlier. $2\alpha_1$, ..., $2\alpha_{N-1}$ are the N-1 estimable main effects and interactions of location. The least squares estimator of α is

$$\hat{\alpha}_o = (1/N)\mathbf{Z}^T\mathbf{w}_y.$$

However, if we wish to take into account the heterogeneity of the variances of the sample mean values, we should pursue weighted least squares to estimate α. In this case we let $\mathbf{V}$ represent the variance-covariance matrix of the responses in the design. The weighted least squares estimate of α then is

$$\hat{\alpha}_w = (\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{w}_y.$$

Since $\mathbf{Z}$ is an NxN orthogonal matrix with $\mathbf{Z}^T\mathbf{Z}=N\mathbf{I}$, where $\mathbf{I}$ is the NxN unit matrix, the estimated vector of coefficients is

$$\begin{aligned}\hat{\alpha}_w &= (\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{w}_y \\ &= \mathbf{Z}^{-1}\mathbf{V}(\mathbf{Z}^T)^{-1}\mathbf{Z}^T\mathbf{V}^{-1}\mathbf{w}_y \\ &= \mathbf{Z}^{-1}\mathbf{w}_y \\ &= (1/N)\mathbf{Z}^T\mathbf{w}_y = \hat{\alpha}_o.\end{aligned}$$

This fact indicates that the weighted least squares estimate is identical to the ordinary least squares solution when the design matrix is an NxN orthogonal matrix.

Similarly, potentially significant location effects can be identified by using a normal plot.

2.3.4. Maximum Likelihood Estimation of the Location and Variance Functions

Once the significant location and dispersion effects are identified, it is possible and desirable to compute more precise estimates of both location and variance functions by using maximum likelihood. First we discuss how to compute the maximum likelihood estimate (MLE) of the variance function. As mentioned earlier, the exact distribution of ξ_{ij} (i=1, ..., N, j=1, ..., J) is assumed to be normal. Let

$$v_i = (J-1)s_i^2, \qquad i=1, ..., N.$$

Under the normality assumption, $v_i/[\sigma^2(\mathbf{z}_i)]$ has a chi-square distribution with J-1 degrees of freedom. We denote $\sigma^2(\mathbf{z}_i)$ by σ_i^2. Since v_1, v_2, ..., and v_N are independent, their log likelihood function is

$$\ln L = C - (\frac{J-1}{2})\sum_{i=1}^{N}\ln\sigma_i^2 - \frac{1}{2}\sum_{i=1}^{N}v_i/(\sigma_i^2) ,$$

where $C= -N\ln[\Gamma(\frac{J-1}{2})2^{(J-1)/2}]+(\frac{J-3}{2})\sum_{i=1}^{N}\ln v_i$, and $\Gamma(.)$ is the gamma function.

Since the dispersion effects were identified by using the sample standard deviation as the dependent variable, the function that we can estimate is the standard deviation function $\sigma(\mathbf{z})$. Let ($2\beta_1^*$, ..., $2\beta_p^*$) represent the p dispersion effects that are significant, excluding the mean. Then we can write the standard deviation function as follows

$$\sigma(\mathbf{z}^*) = \beta_0^* + \beta_1^* z_1^* + ... + \beta_p^* z_p^*.$$

where $\mathbf{z}^*=(z_1^*, ..., z_p^*)^T$ denotes the p independent variables that vary according to the columns associated with those p significant effects in the design array. Naturally, $\mathbf{z}_i^*$ is the subset of the ith run $\mathbf{z}_i$ for i=1, 2, ..., N. To estimate these effects, we replace σ_i in the log likelihood function by $\sigma(\mathbf{z}_i^*)$, and find $\hat{\beta}_0^*$, $\hat{\beta}_1^*$, ..., and $\hat{\beta}_p^*$ that maximize the log likelihood function. Then the MLE of the standard deviation function would be

$$\hat{\sigma}(\mathbf{z}^*) = \hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + ... + \hat{\beta}_p^* z_p^*.$$

It is a known fact that the MLE of the variance function is the squared MLE of the standard deviation function. Thus, the MLE of $\sigma^2(\mathbf{z}^*)$ is

$$\hat{\sigma}^2(z^*) = (\hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + \ldots + \hat{\beta}_p^* z_p^*)^2.$$

The detailed derivation for estimating the standard deviation function is provided in the Appendix to this chapter.

To estimate the location function by maximum likelihood, we recall that the weighted least squares method will produce its MLE (e.g., see Wetherill, 1981). Suppose r location effects, excluding the mean, are identified to be significant. Denote the location function by

$$y = \alpha_0^{**} + \alpha_1^{**} z_1^{**} + \ldots + \alpha_r^{**} z_r^{**},$$

where $\mathbf{z}^{**}=(z_1^{**}, \ldots, z_r^{**})^T$ denotes the r independent variables that vary according to the columns, corresponding to the r significant location effects, in the design array. Let $\mathbf{Z}^{**}$ represent the Nx(r+1) matrix consisting of r columns associated with the r effects as well as the first column associated with the mean. Further suppose that $\mathbf{V}$ is the NxN estimated covariance matrix of the N sample means. Since the random errors were assumed to be independent, $\mathbf{V}$ is diagonal. We can use the estimated variance function to compute its diagonal elements. Denote $\mathbf{w}_y$ the Nx1 vector consisting of the sample means. The weighted least squares estimate of the coefficients $\alpha^{**}=(\alpha_0^{**}, \alpha_1^{**}, \ldots, \alpha_r^{**})^T$ of the location function is

$$\hat{\alpha}^{**} = (\mathbf{Z}^{**T}\mathbf{V}^{-1}\mathbf{Z}^{**})^{-1}\mathbf{Z}^{**T}\mathbf{V}^{-1}\mathbf{w}_y.$$

Then the MLE of the location function would be

$$\hat{y} = \hat{\alpha}^{**T}\mathbf{z}^{**}$$
$$= \hat{\alpha}_0^{**} + \hat{\alpha}_1^{**} z_1^{**} + \ldots + \hat{\alpha}_r^{**} z_r^{**}.$$

2.3.5. Example 2.1

The analysis strategy is illustrated through an example that was reported by Piggnatiello and Ramberg (1985). The experiment was conducted to determine what factors affect the free height of a leaf spring used in an automobile suspension system. The factors considered are high heat temperature (B), heating time (C), transfer time (D), hold down time (E), and quench oil temperature (O). These five factors and their levels are listed in

Table 2.1. The first four factors B, C, D, E are considered to be control factors, and the fifth factor O is taken as a noise factor. The original design used by the experimenters is shown in Table 2.2. Notice that a fractional factorial 2^{4-1} design is used as an inner array for the four control factors, with the noise factor being placed in an outer-array layout.

Table 2.1. Factors in the Leaf Spring Experiment

Factor	Level -1	Level 1
Control Factor:		
B: High heat temperature (°F)	1840	1880
C: Heating time (seconds)	25	23
D: Transfer time (seconds)	12	10
E: Hold down time (seconds)	2	3
Noise Factor:		
O: Quench oil temperature (°F)	130-150	150-170

Table 2.2. The Design Array and the Data Collected

B	C	BC	D	BD	CD	E	O (-1)			O (1)		
-1	-1	1	-1	1	1	-1	7.78	7.78	7.81	7.50	7.25	7.12
1	-1	-1	-1	-1	1	1	8.15	8.18	7.88	7.88	7.88	7.44
-1	1	-1	-1	1	-1	1	7.50	7.56	7.50	7.50	7.56	7.50
1	1	1	-1	-1	-1	-1	7.59	7.56	7.50	7.63	7.75	7.56
-1	-1	1	1	-1	-1	1	7.94	8.00	7.88	7.32	7.44	7.44
1	-1	-1	1	1	-1	-1	7.69	8.09	8.06	7.56	7.69	7.62
-1	1	-1	1	-1	1	-1	7.56	7.62	7.44	7.18	7.18	7.25
1	1	1	1	1	1	1	7.56	7.81	7.69	7.81	7.50	7.59

Table 2.3. 16-Run Design Array with Summary Statistics

Run #	B	C	D	E	O	$\bar{y}$	s
1	-1	-1	-1	-1	-1	7.790	.0173
2	1	-1	-1	1	-1	8.070	.1652
3	-1	1	-1	1	-1	7.520	.0346
4	1	1	-1	-1	-1	7.633	.1021
5	-1	-1	1	1	-1	7.940	.0600
6	1	-1	1	-1	-1	7.947	.2228
7	-1	1	1	-1	-1	7.540	.0917
8	1	1	1	1	-1	7.687	.1250
9	-1	-1	-1	-1	1	7.290	.1931
10	1	-1	-1	1	1	7.733	.2540
11	-1	1	-1	1	1	7.520	.0346
12	1	1	-1	-1	1	7.647	.0961
13	-1	-1	1	1	1	7.400	.0693
14	1	-1	1	-1	1	7.623	.0651
15	-1	1	1	-1	1	7.203	.0404
16	1	1	1	1	1	7.633	.1595

To illustrate the proposed strategy, we change the above product-array design to a combined array design. Now the layout is a 16-run fractional factorial experiment with three replicates. The new array, with summary statistics, is shown in Table 2.3.

We first examine the dispersion effects. The estimated coefficients and effects based on the sample standard deviations are given in Table 2.4.

Table 2.4. Dispersion Coefficients and Effects from Standard Deviations

	Coefficient	Effect
mean	0.108	
B	0.041	0.082
C	-0.023	-0.046
BC	-0.005	-0.010
D	-0.004	-0.008
BD	-0.002	-0.004
CD	0.023	0.046
E	0.005	0.010
O	0.006	0.012
BO	-0.011	-0.022
CO	-0.009	-0.018
BCO	0.021	0.042
DO	-0.026	-0.052
BDO	0.001	0.002
CDO	0.025	0.050
EO	0.011	0.022

We plot the dispersion effects against the expected values of the standard normal order statistics of size 15. The normal plot is displayed in Figure 2.1. If there are no significant effects, the points will lie closely to a straight line. From this plot we identify that B is a significant dispersion effect. Once dispersion effect B is identified, more precise

estimation of it is possible using maximum likelihood. In fact, the MLEs of the standard deviations for B at levels -1 and 1 are

$$\hat{\sigma}(B-) = \sqrt{.0173^2+.0346^2+.0600^2+...+..0404^2} = .0837$$

and

$$\hat{\sigma}(B+) = \sqrt{.1652^2+.1021^2+.2228^2+...+.1595^2} = .1612,$$

respectively. Thus, the MLE of $\sigma(x, e)$ is, using the theoretical result in the Appendix to this chapter,

$$\hat{\sigma}(x, e) = .1225 + .0388x_B.$$

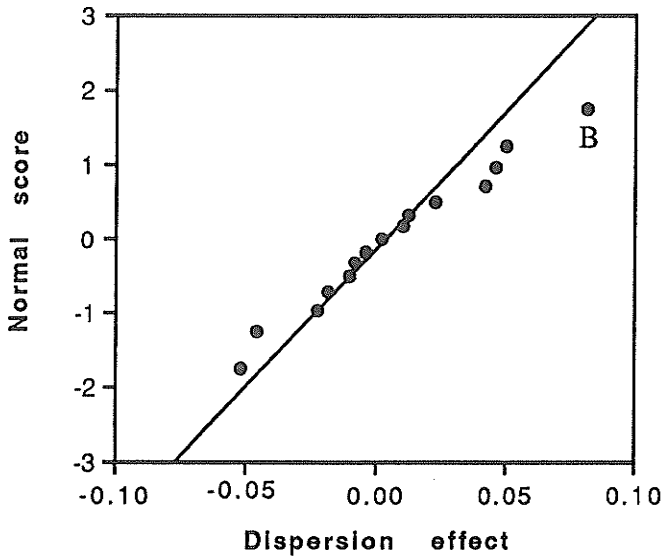

Figure 2.1. Normal plot of dispersion effects.

We next examine the location effects. Table 2.5 shows the estimated coefficients and effects of location. Again we plot the effects on the normal plot shown in Figure 2.2. Effects B, C, O, and CO are taken as significant effects.

Table 2.5. Location Coefficients and Effects

	Coefficient	Effect
mean	7.636	
B	0.111	0.221
C	-0.088	-0.176
BC	-0.009	-0.017
D	-0.014	-0.029
BD	-0.010	-0.020
CD	-0.018	-0.035
E	0.052	0.104
O	-0.130	-0.260
BO	0.042	0.084
CO	0.083	0.165
BCO	0.010	0.020
DO	-0.054	-0.108
BDO	0.020	0.040
CDO	-0.024	-0.047
EO	0.014	0.027

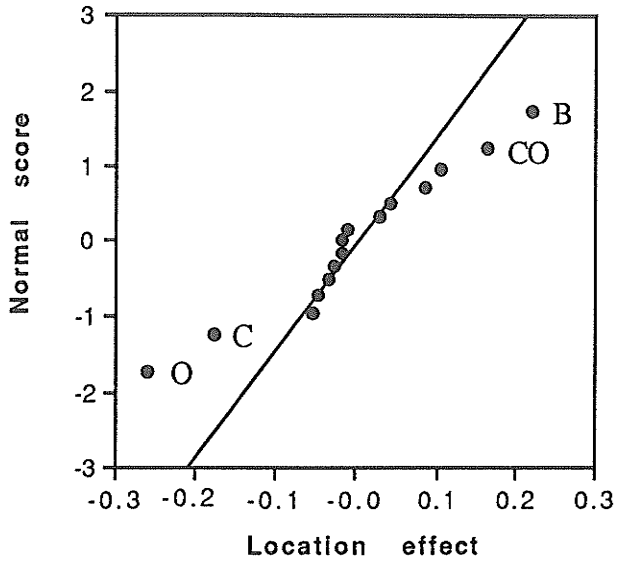

Figure 2.2. Normal plot of location effects.

The location function f(**x**, **e**) is estimated by, using weighted least squares,

$$\hat{f}(\mathbf{x}, \mathbf{e}) = 7.636 + .111x_B - .083x_C - .154x_O + .086x_Cx_O.$$

Then we can compute the mean and variance of the location function,

$$E_e\{\hat{f}(\mathbf{x}, \mathbf{e})\} = 7.636 + .111x_B - .083x_C,$$

$$\begin{aligned} Var_e\{\hat{f}(\mathbf{x}, \mathbf{e})\} &= (-.154 + .086x_C)^2 Var(x_O) \\ &= (-.154 + .086x_C)^2 \\ &= .0311 - .0265x_C, \end{aligned}$$

noting that $E(x_O)=0$ and $Var(x_O)=1$, and $x_C^2=1$.

This example is a closeness-to-target case, and the expected quality loss is

$$\begin{aligned} E_{\xi,e}(y - T)^2 &= Var_{\xi,e}(y) + \{E_{\xi,e}(y) - T\}^2 \\ &= Var_e[f(\mathbf{x}, \mathbf{e})] + E_e[\sigma^2(\mathbf{x}, \mathbf{e})] + \{E_e[f(\mathbf{x}, \mathbf{e})]^2 - T\}^2. \end{aligned}$$

The expected loss can be estimated by

$$\hat{E}(y - T)^2 = Var_e\{\hat{f}(x, e)\}^2 + E_e[\hat{\sigma}^2(x, e)] + \{E_e\hat{f}(x, e) - T\}^2 .$$

Thus, we estimate the quality loss function with

$$\begin{aligned}\hat{E}(y - T)^2 &= \{E_e\hat{f}(x, e) - T\}^2 + Var_e\{\hat{f}(x, e)\} + E_e\{\hat{\sigma}^2(x, e)\} \\ &= (7.636 + .111x_B - .083x_C - T)^2 + .0311 - .0265x_C \\ &\quad + (.1225 + .0388x_B)^2 \\ &= (7.636 + .111x_B - .083x_C - T)^2 + .0476 + .0095x_B - .0265x_C.\end{aligned}$$

Since the target value was not reported in the original article, we here assume that T=7.65. Then

$$\begin{aligned}\hat{E}(y - T)^2 &= (7.636 + .111x_B - .083x_C - 7.65)^2 + .0476 + .095x_B - .0265x_C \\ &= .0670 + .0064x_B - .0242x_C - .0184x_Bx_C,\end{aligned}$$

because $x_B^2=x_C^2=1$. If we restrict each control factor to take only the two levels -1 and 1, the estimated loss would be

$$\begin{aligned}\hat{E}(y - T)^2 &= .0664, \quad \text{when } x_B=-1, x_C=-1, \\ &= .0548, \quad \text{when } x_B=-1, x_C=1, \\ &= .1160, \quad \text{when } x_B=1, x_C=-1, \\ &= .0308, \quad \text{when } x_B=1, x_C=1.\end{aligned}$$

Thus, when $x_B=1$ and $x_C=1$, the estimated loss is at a minimum of .0308. The mean response at this point is

$$\begin{aligned}E_e\hat{f}(x, e) &= 7.636 + .111x_B - .083x_C \\ &= 7.664.\end{aligned}$$

We will later discuss exploring the control-factor region that allows the estimated loss to be further reduced.

Note that the variance function $\hat{\sigma}^2(x, e)$ takes a lower value when $x_B=-1$, instead of $x_B=1$ in the overall solution. The reason for this inconsistency is that factor B affects both the mean and variance of the response variable. The response variable has a larger bias at $x_B=-1$ than at $x_B=1$. In fact, The biases are given by

$$E_e\hat{f}(x, e) - 7.65 = 7.664 - 7.65 = .014, \quad \text{when } x_B=1 \text{ and } x_C=1,$$

$$= 7.442 - 7.65 = -.208, \quad \text{when } x_B=-1 \text{ and } x_C=1.$$

2.4. Analysis Strategy for Unreplicated Experiments

In this section, we extend the analysis strategy to two-level factorial (or fractional factorial) experiments with a single replicate. Let y_i (i=1, ..., N) be responses from an N-run experiment with single replication. Suppose the data follow

$$y_i = f(\mathbf{z}_i) + \xi_i,$$

with $E(\xi_i)=0$ and $Var(\xi_i)=\sigma^2(\mathbf{z}_i)$, i=1, ..., N.

2.4.1. Data Analysis Strategy

In the case of single replication, we first estimate the location function, then use the residuals to estimate the variance function. The strategy includes the following 5 steps:

Step 1. Identify significant location effects.

Step 2. Identify significant dispersion effects using the residuals.

Step 3. Estimate the location and variance functions.

Step 4. Estimate the expected quality loss.

Step 5. Find the optimal setting of the control factors that minimizes this loss.

2.4.2. Location Effects

The first step is to identify the location effects. Let **y** represent the Nx1 vector of the N observations. **Z** still represents the design matrix. Consider now the following linear model

$$\mathbf{y} = \mathbf{Z}\alpha + \xi,$$

where $\alpha=(\alpha_0, ..., \alpha_{N-1})^T$ represents the vector of unknown coefficients, and ξ is the vector of random errors. Thus, $2\alpha_1, ..., 2\alpha_{N-1}$ are the N-1 estimable main effects and interactions of location. The ordinary least squares estimator of α is

$$\hat{\alpha} = (1/N)\mathbf{Z}^T\mathbf{y}.$$

2.4.3. Dispersion Effects

To identify significant dispersion effects, we use a technique suggested by Box and Meyer (1986). Since there is only a single replicate, we first need to identify the significant

effects of location. Suppose p effects, including the mean, are considered significant. Let $\mathbf{Z}_1$ represent the Nxp matrix consisting of p columns associated with those p significant effects. Then the vector of the predicted responses is

$$\hat{y} = (1/N)\mathbf{Z}_1\mathbf{Z}_1^T\mathbf{y}.$$

Thus, the residual vector is obtained as follows

$$\begin{aligned}\hat{\xi} &= \mathbf{y} - \hat{y} \\ &= [\mathbf{I} - (1/N)\mathbf{Z}_1\mathbf{Z}_1^T]\mathbf{y}.\end{aligned}$$

We calculate N-1 variance ratios using the residuals. For example, for the ith column in the design matrix $\mathbf{Z}$ we could compute the $S^2(i-)$, the sum of squares from the N/2 residuals associated with a minus sign and compare it with $S^2(i+)$, the sum of squares from N/2 residuals associated with a plus sign to provide the ratio $F^*=S^2(i+)/S^2(i-)$. Note that this F* ratio does not have an F distribution because the appropriate assumptions are not met. However, we will compare the ratio with a critical value from the F distribution for a rough indication of significance. If the expected values of two variances at level -1 and 1 are the same, then lnF* will be close to 0; on the other hand, if the two expected values are quite different, the logged ratio will be far away from 0. Box and Meyer (1986) provide a more detailed examination of the F*-ratios.

It is very important to note that these N-1 ratios are not independent, thus their usefulness is only in detecting a single significant effect of N-1 effects. In fact the test can stated as

H_0: one effect is significant

vs

H_1: no effect is significant.

In application, no test could effectively detect more than one significant dispersion effect because the number of runs is usually small.

If, indeed, at least two logged variance ratios are far away from 0, careful consideration is required. Box and Meyer (1986) also briefly examined this case. The basic

idea is to project the variances into a two-way table for two factors or a three-way table for three factors.

2.4.4. MLEs of the Location and Variance Functions

To compute the maximum likelihood estimates of both the location and variance functions, we use the method suggested originally by Hartley and Jayatillake (1973), and recently by Box and Meyer (1986). In this method, conditional on the dispersion effects, location effects are obtained by weighted least squares. The dispersion effects are then recomputed from the residuals and the iteration continues until convergence is achieved. The initial estimates of the location effects can be obtained by ordinary least squares. Hartley and Jayatillake showed that the method will converge to a stationary point of the likelihood function.

2.4.5. Example 2.2

This example was originally reported in Montgomery (1991, p.524). Parts manufactured in an injection-molding process were experiencing excessive shrinkage. An experiment was conducted to study seven factors of the process. These seven factors are given in Table 2.6, and each of them was examined at two levels.

Table 2.6. Factors Under Study

A: Mold Temperature
B: Screw speed
C: Holding time
D: Cycle time
E: Moisture content
F: Gate size
G: Holding pressure

The experimenters decided to use a 16-run two-level fractional factorial design. Table 2.7 indicates that it is a 2_{IV}^{7-3} design, with generators I=ABCE, I=BCDF, and I=ACDG. The last column of Table 2.7 gives the observed shrinkage x10 (in percentage) for the test part produced by each of the 16 runs.

Table 2.7. The Design for the Injection-Molding Experiment

Run #	D	C	B	A	E	F	G	y (x10)
1	-1	-1	-1	-1	-1	-1	-1	6
2	-1	-1	-1	1	1	-1	1	10
3	-1	-1	1	-1	1	1	-1	32
4	-1	-1	1	1	-1	1	1	60
5	-1	1	-1	-1	1	1	1	4
6	-1	1	-1	1	-1	1	-1	15
7	-1	1	1	-1	-1	-1	1	26
8	-1	1	1	1	1	-1	-1	60
9	1	-1	-1	-1	-1	1	1	8
10	1	-1	-1	1	1	1	-1	12
11	1	-1	1	-1	1	-1	1	34
12	1	-1	1	1	-1	-1	-1	60
13	1	1	-1	-1	1	-1	-1	16
14	1	1	-1	1	-1	-1	1	5
15	1	1	1	-1	-1	1	-1	37
16	1	1	1	1	1	1	1	52

We first examine the location function. The coefficients and effects of location are computed and showed in Table 2.8. We then plot the effects on the normal plot shown in Figure 2.3.

Table 2.8. Location Coefficients and Effects

	Coefficient	Effect
mean	27.312	
D	0.688	1.376
C	-0.438	-0.876
CD	-0.063	-0.126
B	17.813	35.626
BD	-0.063	-0.126
BC	-0.938	-1.876
F	0.188	0.376
A	6.938	13.876
AD	-2.688	-5.376
AC	-0.813	-1.626
G	-2.438	-4.876
AB	5.938	11.876
ABD	0.063	0.126
E	0.188	0.376
DE	0.313	0.626

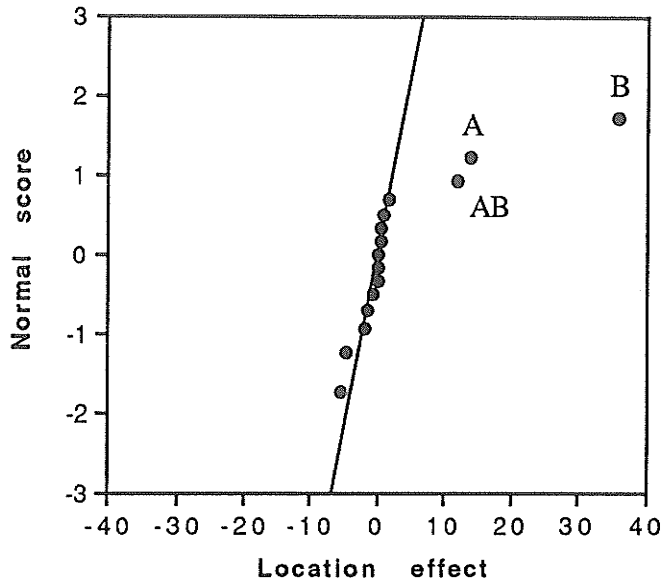

Figure 2.3. Normal plot of location effects. The effects are plotted against the expected values of the normal order statistics.

We take A, B and AB as significant location effects. The location function are, therefore, estimated by

$$\hat{y} = 27.312 + 6.938x_A + 17.813x_B + 5.938x_Ax_B.$$

In the original analysis reported in Montgomery (1991), both A and B were considered to be control factors. It is easy to see from Figure 2.4 that both A (mold temperature) and B (screw speed) should be set at the low level. More importantly, we discover from this plot that the process is very insensitive to the mold temperature when the screw speed is set at the low level. Therefore, it is not necessary to control the mold temperature in the actual manufacturing setting. We consider it to be a noise factor in the current analysis.

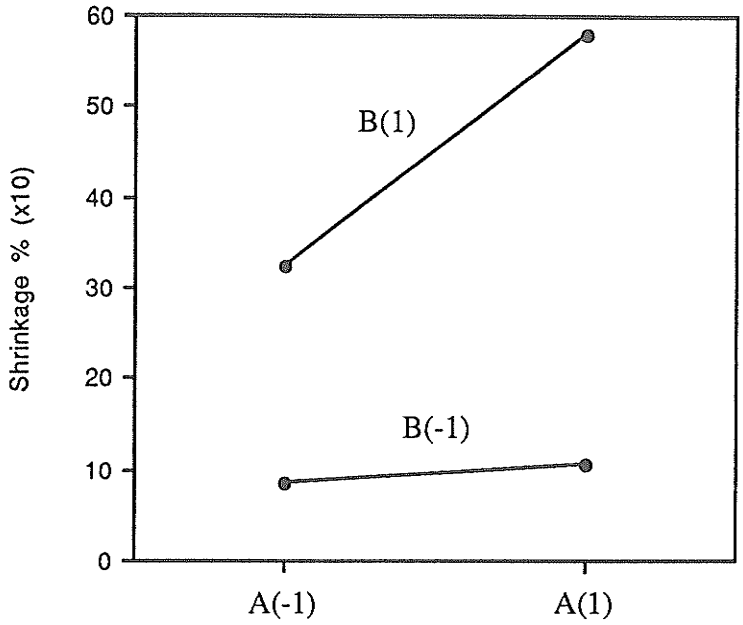

Figure 2.4. AxB, or mold temperature and screw speed interaction plot.

To examine dispersion effects, we first calculate the residuals by computing

$$\xi_i = \hat{y}_i - y_i, \qquad i=1, 2, ..., 16.$$

We then can proceed to calculate the 15 F*-ratios. For example, to compute the F*-ratio for effect D, we add up the last 8 squared residuals associated with the high level 1, and divide it by the sum of the first 8 squared residuals associated with the low level -1, this is,

$$F_D^* = \frac{\xi_9^2+\xi_{10}^2+\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2+\xi_{15}^2+\xi_{16}^2}{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2+\xi_5^2+\xi_6^2+\xi_7^2+\xi_8^2} = 1.646.$$

All the other F*-ratios are calculated in the similar way and given in Table 2.9 where the logged F*-ratios are computed as well. We compare the ratios with $\ln F_{8,\,8,\,0.025}=\ln 4.43=$ 1.488 for indication of significant dispersion effects. Evidently, C is a significant dispersion effect.

Table 2.9. The F*-Ratios and Logged F*-Ratios

Effect	F*	lnF*
D	1.646	0.498
C	11.341	2.428
CD	1.674	0.515
B	0.829	-0.188
BD	0.829	-0.188
BC	0.801	-0.222
F	0.739	-0.302
A	0.684	-0.380
AD	1.246	0.220
AC	0.674	-0.395
G	1.15	0.140
AB	1.113	0.107
ABD	1.682	0.520
E	0.965	-0.036
DE	2.044	0.715

Using the iteration method suggested in the previous section, the location function can be re-estimated by, using weighted least squares,

$$\hat{y} = 27.729 + 7.712x_A + 18.706x_B + 5.759x_Ax_B.$$

We now estimate the variance function. First we obtain the following estimates, using the residuals,

$$\hat{\sigma}(C-) = 2.456$$

and

$$\hat{\sigma}(C+) = 16.382.$$

Then $\sigma(\mathbf{x}, \mathbf{e})$ is estimated by

$$\hat{\sigma}(\mathbf{x}, \mathbf{e}) = 9.419 + 6.963x_C.$$

Since in the current example we desire the shrinkage to be as small as possible, this is a Smaller-Is-Better case. The expected quality loss is

$$E_{\xi, e}(y^2) = Var_{\xi, e}(y) + \{E_{\xi, e}(y)\}^2$$

$$= Var_e[f(\mathbf{x}, \mathbf{e})] + E_e[\sigma^2(\mathbf{x}, \mathbf{e})] + \{E_e[f(\mathbf{x}, \mathbf{e})]\}^2.$$

This expected quality loss is estimated by

$$\hat{E}(y^2) = Var_e\{\hat{f}(\mathbf{x}, \mathbf{e})\}^2 + E_e[\hat{\sigma}^2(\mathbf{x}, \mathbf{e})] + \{E_e[\hat{f}(\mathbf{x}, \mathbf{e})]\}^2.$$

Thus, we have

$$\hat{E}(y^2) = Var_A[\hat{f}(\mathbf{x}, \mathbf{e})] + E_A[\hat{\sigma}^2(\mathbf{x}, \mathbf{e})] + \{E_A[\hat{f}(\mathbf{x}, \mathbf{e})]\}^2$$

$$= (7.712+5.759x_B)^2 Var(x_A)+(9.419+6.963x_C)^2+(27.729+18.706x_B)^2$$

$$= 1348.654 + 1126.224x_B + 131.169x_C,$$

because $x_B^2=x_C^2=1$, and $E(x_A)=0$ and $E(x_A^2)=1$. Thus, the optimal setting of the control factors B and C are $x_B=-1$ and $x_C=-1$, with the minimal loss 91.261. The estimated minimal shrinkage is

$$E_A[\hat{f}(\mathbf{x}, \mathbf{e})] = 27.729 + 18.706x_B = 9.023,$$

with variance equal to

$$Var_A[\hat{f}(\mathbf{x}, \mathbf{e})] + E_A[\hat{\sigma}^2(\mathbf{x}, \mathbf{e})] = (7.712+5.759x_B)^2 + (9.419+6.963x_C)^2$$

$$= 9.846.$$

2.5. Exploring the Control-Factor Region

In the two examples discussed earlier, we optimized the expected quality loss under the constraint that the control factors only take two levels -1 and 1. When the control factors are quantitative factors, it is possible to further reduce the expected quality loss through exploring the continuous control-factor region. However, the critical requirement for this to work is that the fitted model must be adequate in the region being explored.

For illustration, consider the leaf spring example. The expected quality loss is estimated by

$$\hat{E}(y - T)^2 = Var_e\{\hat{f}(x, e)\} + E_e\{\hat{\sigma}^2(x, e)\} + \{E_e\hat{f}(x, e) - T\}^2$$

$$= (-.154 + .086x_C)^2 + (.1225 + .0388x_B)^2$$

$$+ (7.636 + .111x_B - .083x_C - 7.65)^2.$$

For the purpose of searching the optimal setting, we need to keep the squared terms. Thus, the estimated quality loss, denoting it by L, becomes

$$L = .0389 + .0064x_B - .0242x_C - .0184x_Bx_C + .0138x_B^2 + .0143x_C^2.$$

We compute partial derivatives and set them equal to 0,

$$\frac{\partial L}{\partial x_B} = .0064 - .0184x_C + .0276x_B = 0,$$

$$\frac{\partial L}{\partial x_C} = -.0242 - .0184x_B + .0286x_C = 0.$$

Solving the above two equations, we find the following optimal solution

$$x_B = .582 \quad \text{and} \quad x_C = 1.221,$$

with the estimated loss reduced from L=.0308 at x_B=1 and x_C=1, to L=.0260 at the current point.

Whether x_B=.582 and x_C=1.221 is indeed a better setting than x_B=1 and x_C=1 depends on the adequacy of both the fitted response model and the estimated variance function over the expanded control-factor region. Without this information, we can do a few more runs of test at x_B=.582 and x_C=1.221 for confirmation.

2.6. Concluding Remark

In this chapter we have proposed two data analysis strategies for parameter design for a very general location-variance model. For replicated experiments, we use the repeated observations at each design point to calculate the sample standard deviation. The dispersion effects based on these standard deviations are plotted on a normal plot to identify significant dispersion effects. To find significant location effects, the estimated effects based on the

sample means are plotted on a normal plot. Then both location and variance functions are estimated by maximum likelihood.

For unreplicated experiments, the significant location effects have to be identified and estimated first. We then use the residuals to identify significant dispersion effects. Once the significant effects of both the location and dispersion are determined, it is possible to compute more accurate estimates by maximum likelihood. For both replicated and unreplicated experiments, the expected quality loss can be estimated and minimized in the same way.

Appendix

1. Location-Variance Model

Suppose the true model is

$$y = f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) + \xi, \tag{2.3}$$

with

$$E_\xi(\xi|\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) = 0 \quad \text{and} \quad Var_\xi(\xi|\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) = \sigma^2, \tag{2.4}$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is the set of the noise factors. Further suppose that $\mathbf{e}_2$ was not included in the design array. Then the assumed model would be like

$$y = f_1(\mathbf{x}, \mathbf{e}_1) + \xi^*,$$

where $f_1(\mathbf{x}, \mathbf{e}_1)=E_{\mathbf{e}_2}\{f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2)\}$. The mean of ξ^* given $\mathbf{x}$ and $\mathbf{e}_1$, considering $\mathbf{e}_2$ and ξ as random variables, is

$$\begin{aligned} E_{\xi, \mathbf{e}_2}(\xi^*|\mathbf{x}, \mathbf{e}_1) &= E_{\xi, \mathbf{e}_2}\{y - f_1(\mathbf{x}, \mathbf{e}_1)|\mathbf{x}, \mathbf{e}_1\} \\ &= E_{\xi, \mathbf{e}_2}(y|\mathbf{x}, \mathbf{e}_1) - f_1(\mathbf{x}, \mathbf{e}_1) \\ &= E_{\xi, \mathbf{e}_2}\{f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) + \xi|\mathbf{x}, \mathbf{e}_1) - f_1(\mathbf{x}, \mathbf{e}_1) \\ &= E_{\xi, \mathbf{e}_2}(\xi|\mathbf{x}, \mathbf{e}_1) \\ &= E_{\mathbf{e}_2}\{E_\xi(\xi|\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2)\} \\ &= 0. \end{aligned}$$

Since $\xi^*=f(\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2) + \xi - f_1(\mathbf{x}, \mathbf{e}_1)$, it is easy to see that

$$E_{\xi^*}(\xi^*|x, e_1) = E_{\xi, e_2}(\xi^*|x, e_1) = 0.$$

The variance of ξ^* given x and e_1, considering e_2 and ξ as random variables, is

$$\begin{aligned}
Var_{\xi, e_2}(\xi^*|x, e_1) &= E_{\xi, e_2}\{[f(x, e_1, e_2) + \xi - f_1(x, e_1)]^2|x, e_1\} \\
&= E_{e_2}\{[f(x, e_1, e_2) - f_1(x, e_1)]^2|x, e_1\} + E_{\xi, e_2}(\xi^2|x, e_1) \\
&\quad + 2\, E_{\xi, e_2}\{[f(x, e_1, e_2) - f_1(x, e_1)]\xi|x, e_1\} \\
&= Var_{e_2}\{f(x, e_1, e_2)|x, e_1\} + E_{e_2}\{E_\xi(\xi^2|x, e_1, e_2)\} \\
&\quad + 2\, E_{e_2}(E_\xi\{[f(x, e_1, e_2) - f_1(x, e_1)]\xi|x, e_1, e_2\}) \\
&= Var_{e_2}\{f(x, e_1, e_2)|x, e_1\} + \sigma^2 \\
&\quad + 2\, E_{e_2}\{[f(x, e_1, e_2) - f_1(x, e_1)]E_\xi[\xi|x, e_1, e_2]\} \\
&= Var_{e_2}\{f(x, e_1, e_2)|x, e_1\} + \sigma^2 \\
&= \kappa^2(x, e_1),
\end{aligned}$$

where $\kappa^2(x, e_1)$ denotes $Var_{e_2}\{f(x, e_1, e_2)|x, e_1\} + \sigma^2$. Again we can easily see that

$$Var_{\xi^*}(\xi^*|x, e_1) = Var_{\xi, e_2}(\xi^*|x, e_1) = \kappa^2(x, e_1).$$

Clearly, the original homogeneous model (2.1) and (2.2) has become a location-variance model

$$y = f_1(x, e_1) + \xi^*, \tag{2.5}$$

with

$$E_{\xi^*}(\xi^*|x, e_1) = 0 \quad \text{and} \quad Var_{\xi^*}(\xi^*|x, e_1) = \kappa^2(x, e_1). \tag{2.6}$$

2. Maximum Likelihood Estimate of the Variance Function

We now discuss how to estimate the variance function. As mentioned earlier, the exact distribution of ξ_{ij} (i=1, ..., N, j=1, ..., J) is assumed to be normal. Let

$$v_i = (J-1)s_i^2, \qquad i=1, ..., N.$$

Then $v_i/[\sigma^2(z_i)]$ has a chi-square distribution with J-1 degrees of freedom. We denote $\sigma^2(z_i)$ by σ_i^2. Since $v_1, v_2, ..., v_N$ are independent, their log likelihood function is

$$\ln L = C - (\frac{J-1}{2})\sum_{i=1}^{N}\ln\sigma_i^2 - \frac{1}{2}\sum_{i=1}^{N}v_i/(\sigma_i^2)\ ,$$

where $C= -N\ln[\Gamma(\frac{J-1}{2})2^{(J-1)/2}] + (\frac{J-3}{2})\sum_{i=1}^{N}\ln v_i$.

Since the dispersion effects were identified by using the sample standard deviation as dependent variable, the standard deviation function can be written as

$$\sigma(\mathbf{z}^*) = \beta_0^* + \beta_1^* z_1^* + \ldots + \beta_p^* z_p^*,$$

where $\mathbf{z}^*=(z_1^*, \ldots, z_p^*)^T$ denotes the p independent variables that vary according to the columns, corresponding to those identified p effects, in the design array. Naturally, $\mathbf{z}_i^*$ is the subset of the ith run $\mathbf{z}_i$ for i=1, 2, ..., N. To estimate these effects, we replace σ_i in the log likelihood function by $\sigma(\mathbf{z}_i^*)$, and find $\hat{\beta}_0^*$, $\hat{\beta}_1^*$, ..., and $\hat{\beta}_p^*$ that maximize the log likelihood function. Then the MLE of the standard deviation function is

$$\hat{\sigma}(\mathbf{z}^*) = \hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + \ldots + \hat{\beta}_p^* z_p^*.$$

Consequently, the MLE of the variance function is

$$\hat{\sigma}^2(\mathbf{z}^*) = (\hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + \ldots + \hat{\beta}_p^* z_p^*)^2.$$

Now we consider some special cases.

Case 1: p=0 and $\sigma(\mathbf{z}^*)=\beta_0^*$

In this case there is no dispersion effect, this is, $\sigma_i=\sigma=\beta_0^*$, i=1, ..., N. The log likelihood function becomes

$$\ln L = C - (\frac{J-1}{2})N\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N} v_i .$$

Set

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{(J-1)N}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{N} v_i = 0.$$

The MLE of σ is obtained as follows

$$\hat{\sigma}(\mathbf{z}^*) = \hat{\sigma} = \sqrt{\frac{1}{(J-1)N}\sum_{i=1}^{N} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=1}^{N} s_i^2} .$$

Case 2: p=1 and $\sigma(z^*)=\beta_0^*+\beta_1^* z_1^*$

In this case there is one significant effect. Suppose it is effect A. Without loss of generality, we further suppose that effect A was set at level -1 for the first N/2 observations with a standard deviation $\sigma(A-)$ and at level 1 for the last N/2 observations with a standard deviation $\sigma(A+)$. Thus, $\sigma_i=\sigma(A-)$, i=1, ..., N/2, and $\sigma_i=\sigma(A+)$, i=N/2+1, ..., N. It is easy to see that β_0^* and β_1^* relate to $\sigma(A-)$ and $\sigma(A+)$ as follows

$$\beta_0^* = \{\sigma(A-) + \sigma(A+)\}/2,$$

and

$$\beta_1^* = \{\sigma(A+) - \sigma(A-)\}/2.$$

In the current case the log likelihood function becomes

$$\ln L = C - \frac{(J-1)N}{4}\ln\sigma(A-)^2 - \frac{(J-1)N}{4}\ln\sigma(A+)^2$$

$$- \frac{1}{2\sigma(A-)^2}\sum_{i=1}^{N/2} v_i - \frac{1}{2\sigma(A+)^2}\sum_{i=N/2+1}^{N} v_i .$$

Let

$$\frac{\partial \ln L}{\partial \sigma(A-)} = -\frac{(J-1)N}{2\sigma(A-)} + \frac{1}{\sigma^3(A-)}\sum_{i=1}^{N/2} v_i = 0,$$

and

$$\frac{\partial \ln L}{\partial \sigma(A+)} = -\frac{(J-1)N}{2\sigma(A+)} + \frac{1}{\sigma^3(A+)}\sum_{i=N/2+1}^{N} v_i = 0.$$

Then the MLEs of (A-) and (A+) are obtained by

$$\hat{\sigma}(A-) = \sqrt{\frac{2}{(J-1)N}\sum_{i=1}^{N/2} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=1}^{N/2} s_i^2},$$

$$\hat{\sigma}(A+) = \sqrt{\frac{2}{(J-1)N}\sum_{i=N/2+1}^{N} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=N/2+1}^{N} s_i^2}.$$

Thus, the MLEs of β_0^* and β_1^* are computed as follows

$$\hat{\beta}_0^* = \{\hat{\sigma}(A-) + \hat{\sigma}(A+)\}/2,$$

and

$$\hat{\beta}_1^* = \{\hat{\sigma}(A+) - \hat{\sigma}(A-)\}/2.$$

Case 3: p=2 and $\sigma(z^*)=\beta_0^* + \beta_1^* z_1^* + \beta_2^* z_2^*$

In this case two of the effects are significant. Suppose the two effects are A and B. Without loss of generality, further suppose that effects A and B were set at levels -1 and -1 for the first N/4 observations with a standard deviation $\sigma(A-, B-)$, at levels -1 and 1 for the second N/4 observations with a standard deviation $\sigma(A-, B+)$, at levels 1 and -1 for the third N/4 observations with a standard deviation $\sigma(A+, B-)$, and at levels 1 and 1 for the last N/4 observations with a standard deviation $\sigma(A+, B+)$. That is

$$\sigma_i = \sigma(A-, B-), \qquad i=1, ..., N/4,$$

$$\sigma_i = \sigma(A-, B+), \qquad i=N/4+1, ..., N/2,$$

$$\sigma_i = \sigma(A+, B-), \qquad i=N/2+1, ..., 3N/4,$$

$$\sigma_i = \sigma(A+, B+), \qquad i=3N/4+1, ..., N.$$

Employing similar argument, we obtain the following MLEs

$$\hat{\sigma}(A-, B-) = \sqrt{\frac{4}{(J-1)N}\sum_{i=1}^{N/4} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=1}^{N/4} s_i^2},$$

$$\hat{\sigma}(A-, B+) = \sqrt{\frac{4}{(J-1)N}\sum_{i=N/4+1}^{N/2} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=N/4+1}^{N/2} s_i^2},$$

$$\hat{\sigma}(A+, B-) = \sqrt{\frac{4}{(J-1)N}\sum_{i=N/2+1}^{3N/4} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=N/2+1}^{3N/4} s_i^2},$$

$$\hat{\sigma}(A+, B+) = \sqrt{\frac{4}{(J-1)N}\sum_{i=3N/4+1}^{N} v_i} = \sqrt{\frac{1}{J-1}\sum_{i=3N/4+1}^{N} s_i^2}.$$

The MLEs of are β_0^*, β_1^* and β_2^* computed by

$$\hat{\beta}_0^* = \{\hat{\sigma}(A-, B-) + \hat{\sigma}(A-, B+) + \hat{\sigma}(A+, B-) + \hat{\sigma}(A+, B+)\}/4,$$

$$\hat{\beta}_1^* = \{\hat{\sigma}(A+, B-) + \hat{\sigma}(A+, B+) - \hat{\sigma}(A-, B-) - \hat{\sigma}(A-, B+)\}/4,$$

$$\hat{\beta}_2^* = \{\hat{\sigma}(A-, B+) + \hat{\sigma}(A+, B+) - \hat{\sigma}(A-, B-) - \hat{\sigma}(A+, B-)\}/4.$$

Hence the MLE of the standard deviation function is

$$\hat{\sigma}(\mathbf{z}^*) = \hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + \hat{\beta}_2^* z_2^*.$$

If it is significant, we can also compute the interaction of A and B as follows

$$\hat{\beta}_{12}^* = \{\hat{\sigma}(A+, B+) + \hat{\sigma}(A-, B-) - \hat{\sigma}(A-, B+) - \hat{\sigma}(A+, B-)\}/4.$$

The standard deviation function becomes

$$\hat{\sigma}(\mathbf{z}^*) = \hat{\beta}_0^* + \hat{\beta}_1^* z_1^* + \hat{\beta}_2^* z_2^* + \hat{\beta}_{12}^* z_1 z_2^*.$$

For the other cases in which there are more than two significant dispersion effects, the MLE of the standard deviation function can be computed similarly, although the computation is certainly more tedious.

3. Transforming the Response Variable

3.1. Motivation

In chapter 2 the expected quality loss is taken as an overall measure to be minimized. Although it is considered to be an appropriate measure, the expected quality loss has its drawbacks. One that has been pointed out by some researchers is that the estimated quality loss has intractable statistical properties, unlike the sample mean and the sample variance that have very nice distributional properties (e.g., see Box, 1988; Shoemaker et al., 1991). The other main drawback is that the optimal setting depends on the target value. This dependency restricts the usefulness of the knowledge obtained regarding the process or product that is being investigated, because the optimal control-factor setting will be no longer optimal when the target value is changed to a new level in the future. From an engineer's point of view, this is undoubtedly a major disadvantage. In theory we can re-optimize the estimated quality loss of the response variable around the new target value, however, this approach may not work when the new optimal control-factor setting is out of the design region.

On the other hand, the two-step procedures reviewed in chapter 1 are tremendously advantageous, if there exist adjustment factors. In this case, all optimal levels of the control factors except for that of the adjustment factors are independent of the target value. When the target value changes, we only need to fine-tune the adjustment factors to bring the mean response to the newer target value. Product and process designs that have this feature provide flexibility in the overall engineering design as well as in process control. Simply by manipulating adjustment factors, engineers will be able to shift easily the target value of the response variable to meet the changed requirements of customers.

Some products and processes have obvious adjustment factors that are well known to design engineers. Many other products and processes may not have any known adjustment factor. When there do not exist any obvious adjustment factors, there is,

fortunately, a technique in statistics that allows us to identify them from the rest of the control factors. The technique is to transform the response variable so that the variance of the transformed response variable depends only on some of the control factors in the design. The other control factors are, therefore, chosen as adjustment factors if they can affect the mean response. This idea of data transformation is called *separation*. Besides separation, another reason for transformation is that the analysis and the interpretation of the results are often improved if an appropriate transformation results in a simpler model. This second concept is called *parsimony*. Box (1988) originally introduced the concepts of parsimony and separation in the context of parameter design. He also proposed an effective empirical procedure, called lambda plot, to help in diagnosing a transformation that would achieve these two aims. However, it only applies to product-array designs with replicates. His procedure consists of plotting the t values of the effects against the values of λ, an index parameter for a family of transformations. The transformation that yields maximum simplification and separation is selected.

In this chapter we propose an empirical procedure to help identify a suitable transformation from experiments based on a combined array design. In the combined array design both control and noise factors are accommodated in a single design layout. We plot the standardized values of the relevant statistics versus the values of λ to find a transformation that separates the control factors into adjustment factors and non-adjustment factors. The optimization of the expected loss can then be done sequentially with a two-step procedure.

3.2. Use of Transformations

Transforming the response variable is a very useful technique in statistics (e.g., see Box and Cox, 1964; and Box, 1988). In general, transformation is used to achieve a simpler model, stabilize the variance of the response variable, and normalize the error distribution. In parameter design, however, the purpose is quite different. In fact, the main

objective of transformation is to separate the control factors into two subsets, adjustment factors and the rest. The existence of adjustment factors facilitates the optimization of the expected quality loss. A secondary objective is the hope that the transformation also provides a simpler model.

Consider the following response model

$$y = f(\mathbf{x}, \mathbf{e}) + \xi,$$

where ξ is the error term, $\mathbf{x}$ is the vector of the control factors and $\mathbf{e}$ is the vector of the noise factors. Assume that z=h(y) is an appropriate transformation that separates the control vector into two subsets $\mathbf{x}=(\mathbf{d}, \mathbf{a})$ such that subset $\mathbf{a}$ affect the mean μ_z of z but not the variance σ_z^2 of z. We write

$$z = h(y) = g(\mathbf{d}, \mathbf{a}, \mathbf{e}) + \eta,$$

where η is the model error term with $E(\eta|\mathbf{d}, \mathbf{a}, \mathbf{e})=0$ and $Var(\eta|\mathbf{d}, \mathbf{a}, \mathbf{e})=\sigma^2$.

The main reason that we assume a constant variance is that our main focus is first to find a transformation that satisfies the separation requirement. Another reason is that the transformation often stabilizes the variance. Even if the variance is not stabilized, we still can use the data analysis strategies proposed in chapter 2 to investigate the variance function $\sigma^2=\sigma^2(\mathbf{d}, \mathbf{a}, \mathbf{e})$.

Consider η and $\mathbf{e}$ as random variables. The mean and variance of z in the application setting are

$$\mu_z = E_{\eta, e}(z) = E_e[g(\mathbf{d}, \mathbf{a}, \mathbf{e})] = \mu_z(\mathbf{d}, \mathbf{a}),$$

$$\sigma_z^2 = Var_{\eta, e}(z) = Var_e[g(\mathbf{d}, \mathbf{a}, \mathbf{e})] + \sigma^2 = \sigma_z^2(\mathbf{d})$$

noting that $Cov_{\eta, e}[g(\mathbf{d}, \mathbf{a}, \mathbf{e}), \eta]=0$ (see the Appendix to chapter 2). Then the modified version of Procedure 5 of chapter 1 can be applied as follows

Procedure 5':

(1) Find a transformation, z=h(y), to make $var(z)=\sigma_z^2(\mathbf{d})$, independent of

$\mathbf{a}$, the subset of adjustment factors.

(2) Choose $\mathbf{d}^*$ to minimize $\sigma_z^2(\mathbf{d})$.

(3) Adjust **a** to **a*** so that $\mu_z(\mathbf{d}^*, \mathbf{a}^*)=h(T)$, where T is the target value for y.

Note that in the above discussion we do not require either a product-array design or an experiment to be replicated. As matter of fact, neither of them is essential. The only task is to find a way for estimating $E_e[g(\mathbf{d}, \mathbf{a}, \mathbf{e})]$ and $Var_e[g(\mathbf{d}, \mathbf{a}, \mathbf{e})]$. For a product-array design, Box (1988) proposed plotting the suitable relevant statistics such as t values of both dispersion effects and location effects against the value of λ, to help identify a transformation.

We now extend the transformation technique to experiments with a combined array design. We first estimate the transformed response model as a function of both control and noise factors. Then the mean and variance functions of the transformed response variable are computed analytically from the fitted response model, considering the noise factors as random variables. The coefficients in the variance function as well as the coefficients in the mean function will be plotted against the values of λ, to help select a transformation that provides adjustment factors independent of the variance of the transformed response variable.

Suppose that a family of transformations $y^{(\lambda)}$ is indexed by a parameter λ. The most commonly considered family of transformations is that of power transformations defined as follows (e.g., see Box and Cox, 1964; Box and Fung, 1983; Grize, 1991)

$$y^{(\lambda)} = \log(y), \qquad \lambda=0,$$
$$= (y^{\lambda}-1)/\lambda, \qquad \lambda\neq 0.$$

We will use this family of transformations in this chapter.

3.3. Rescale the Effects

When the response variable is transformed, its scale will be transformed as well. For example, if the transformation selected is z=ln(y), 2 meters in the original scale will be transformed to 0.69 log-meters in the transformed scale. A lambda plot of these effects is useless since these effects are on completely different scales for the different values of λ.

We are required to rescale the effects to make them comparable. For experiments with replicates, the task is relatively simple. Under a transformation $z = h(y)$, we first identify and estimate the location effects. Then the error variance σ^2 will be estimated by the within sample mean squares s_z^2. All the effects are divided by s_z, and these rescaled effects are commonly called studentized effects.

For unreplicated experiments, the above technique is not applicable, because there is no estimate of the error variance. What we have got from such an experiment of size N are (N-1) single-degree-of-freedom main effects and interactions. Grize (1991) suggested three alternative ways to rescale the effects of interest. The first approach is to scale the effects using the standard error obtained from those N-1 effects. Its major drawback is that the estimate of the standard error will be inflated since the significant effects are included. Thus, the second approach is to trim a certain percentage (20% was suggested) of the effects before calculating the standard deviation. A third choice is to compute the pseudo standard error (PSE) of Lenth (1989) defined as

$$\text{PSE} = 1.5 \times \text{median } \{|c_i|: \ |c_i|<2.5\sigma_0, \ i=1, \ldots, N-1\},$$

where the c_i are the effects and σ_0 is defined by

$$\sigma_0 = 1.5 \times \text{median } \{|c_i|: \ i=1, \ldots, N-1\}.$$

Grize (1991) showed that the second approach is the most effective, and that the PSE-based version did not perform any better.

In this chapter we rescale the effects through using the well-known family of normalized power transformations recommended by Box and Cox (1964). Denote $\dot{y}$ the geometric mean of the data, computed from $\dot{y}=\exp(n^{-1}\sum \ln y_i)$. The family of normalized power transformations is defined as follows

$$\begin{aligned} y^{(\lambda)} &= \dot{y}\log(y), && \lambda=0, \\ &= (y^{\lambda}-1)/(\lambda \dot{y}^{\lambda-1}), && \lambda \neq 0. \end{aligned}$$

This normalization makes it possible to compare the effects of the transformed response variable. In addition, the use of the normalized power transformations is easier than the three techniques suggested by Grize (1991).

3.4. Example

We use a simulated experiment to illustrate the transformation technique described above. In this example, there are five factors, each with two levels. Factors A, B, C and D are control factors, and factor E is a noise factor. A 32-run full factorial design is chosen. The design and data are given in Table 3.1.

We compute the main effects and interactions of the response model at the original scale, i.e., $\lambda=1$. The normal plot of Figure 3.1 shows main effects A, C, D and E, along with interactions AD, AE, ADE, CD, CE and DE, are significant.

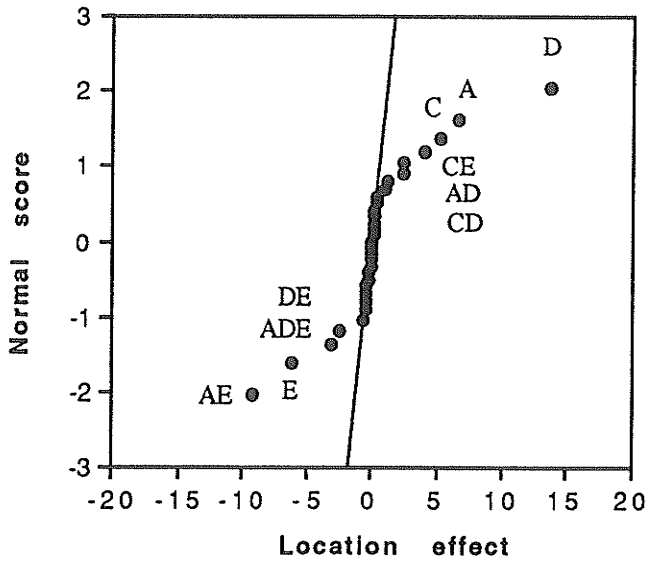

Figure 3.1. Normal plot of location effects on the original scale.

Table 3.1. Design for the Simulated Example

Run #	A	B	C	D	E	y
1	-1	-1	-1	-1	-1	11.759
2	-1	-1	-1	-1	1	10.620
3	-1	-1	-1	1	-1	22.067
4	-1	-1	-1	1	1	18.700
5	-1	-1	1	-1	-1	11.114
6	-1	-1	1	-1	1	15.425
7	-1	-1	1	1	-1	23.851
8	-1	-1	1	1	1	32.907
9	-1	1	-1	-1	-1	10.038
10	-1	1	-1	-1	1	9.335
11	-1	1	-1	1	-1	19.535
12	-1	1	-1	1	1	18.292
13	-1	1	1	-1	-1	10.838
14	-1	1	1	-1	1	18.085
15	-1	1	1	1	-1	21.958
16	-1	1	1	1	1	30.971
17	1	-1	-1	-1	-1	20.537
18	1	-1	-1	-1	1	8.841
19	1	-1	-1	1	-1	40.855
20	1	-1	-1	1	1	15.766
21	1	-1	1	-1	-1	20.280
22	1	-1	1	-1	1	13.370
23	1	-1	1	1	-1	43.772
24	1	-1	1	1	1	28.113
25	1	1	-1	-1	-1	22.106
26	1	1	-1	-1	1	9.185
27	1	1	-1	1	-1	42.619
28	1	1	-1	1	1	16.546
29	1	1	1	-1	-1	21.787
30	1	1	1	-1	1	14.173
31	1	1	1	1	-1	44.901
32	1	1	1	1	1	27.162

The response model is estimated by using least squares

$$\hat{y} = 20.110 + 3.266x_A + 2.560x_C + 6.891x_D - 3.141x_E + 1.120x_Ax_D$$
$$- 4.590x_Ax_E + 1.144x_Cx_D + 1.998x_Cx_E - 1.302x_Dx_E - 1.536x_Ax_Dx_E.$$

Since E is a noise factor, the mean and variance of the fitted response are

$$E_E(\hat{y}) = 20.110 + 3.266x_A + 2.560x_C + 6.891x_D + 1.120x_Ax_D + 1.144x_Cx_D,$$

and

$$Var_E(\hat{y}) = (- 3.141 - 4.590x_A + 1.998x_C - 1.302x_D - 1.536x_Ax_D)^2Var(x_E)$$
$$= 38.980 + 32.834x_A - 12.551x_C + 22.280x_D - 18.342x_Ax_C$$
$$+ 21.602x_Ax_D - 5.203x_Cx_D - 6.138x_Ax_Cx_D,$$

where it is assumed that $E(x_E)=0$ and $Var(x_E)=1$. From the these results we see that control factors A, C, and D affect the mean function as well as the variance function. Clearly, there is no adjustment factor on the original scale.

We now use the lambda plots to find an appropriate transformation that will separate the control factors into two groups: adjustment factors and non-adjustment factors. For λ=-2 to 2 (.5), we obtain the mean and variance functions in the same way as we did earlier for λ=1. Then the coefficients of the mean and variance are plotted in the corresponding lambda plots as shown in Figures 3.2 and 3.3.

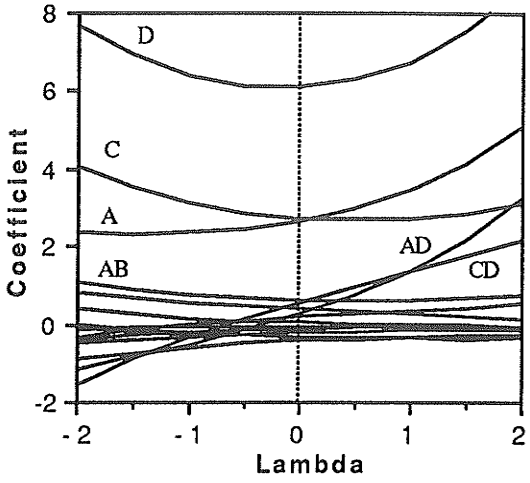

Figure 3.2. Lambda plot for the coefficients of the mean function.

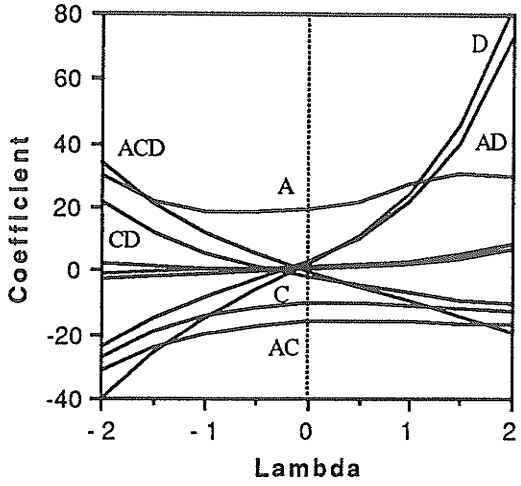

Figure 3.3. Lambda plot for the coefficients of the variance function.

From the above two plots, we find that the log transformation, or $\lambda=0$, produces an adjustment factor. Clearly, factor D affects the mean function in Figure 3.2, but not the variance function in Figure 3.3. Furthermore, on the new scale the model has a simpler form. In fact, we discover from Figure 3.4 that effects A, C, D, E, AE, and CE are significant effects of the new response variable under the log transformation.

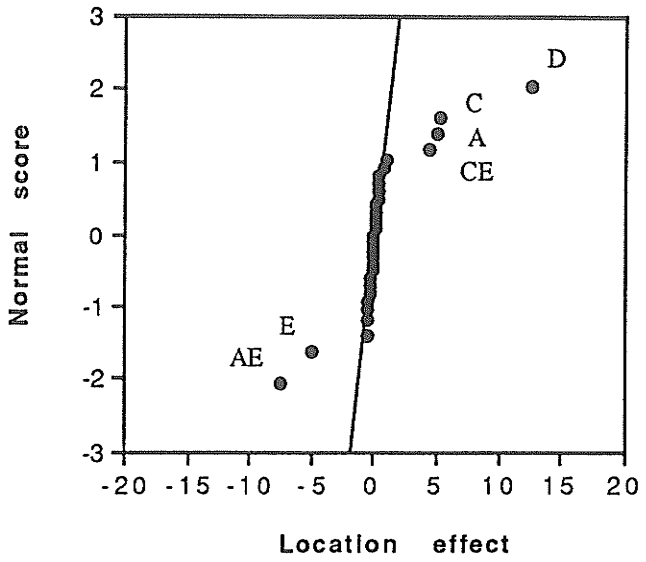

Figure 3.4. Normal plot of the location effects of the logged response variable.

From the two lambda plots, we decide to select the log transformation $z=\dot{y}\log(y)$, since it yielded greatest separation and simplification. Using least squares, we estimate the response model as follows

$$\hat{z} = 54.918 + 6.251x_D + 2.466x_A + 2.546x_C - 2.520x_E$$
$$- 3.805x_Ax_E + 2.185x_Cx_E.$$

Since E is a noise factor, we have

$$E_E(\hat{z}) = 54.918 + 6.251x_D + 2.466x_A + 2.546x_C,$$

and

$$Var_E(\hat{z}) = (- 2.520 - 3.805x_A + 2.185x_C)^2 Var(x_E)$$
$$= 25.603 + 19.177x_A - 11.012x_C - 16.628x_Ax_C.$$

From the above mean and variance functions, we see that factor D affects the mean response but not the variance, while factors A and C affect both the mean and variance of the transformed response variable z. Thus, factor D is an adjustment factor that can be fine-tuned to bring the mean response to the target value.

Table 3.2. F*-Ratios and Logged F*-Ratios

Effect	F*	lnF*
A	0.472	-0.750
B	0.962	-0.039
AB	0.803	-0.219
C	0.803	-0.219
AC	1.203	0.184
BC	0.752	-0.285
ABC	0.768	-0.264
D	0.473	-0.748
AD	1.101	0.097
BD	0.964	-0.037
ABD	1.269	0.239
CD	1.042	0.041
ACD	0.891	-0.116
BCD	0.839	-0.176
ABCD	3.291	1.191
E	0.971	-0.029
AE	1.057	0.055
BE	1.250	0.223
ABE	0.863	-0.147
CE	1.124	0.117
ACE	0.500	-0.692
BCE	1.618	0.481
ABCE	0.688	-0.375
DE	0.639	-0.448
ADE	1.553	0.440
BDE	0.738	-0.304
ABDE	0.747	-0.292
CDE	0.753	-0.283
ACDE	1.644	0.497
BCDE	1.069	0.067
ABCDE	1.022	0.022

Although we assumed earlier that the variance of the error term η is constant, it is possible to investigate the heterogeneity of the variance using the techniques of chapter 2. Choosing the log transformation, we use the fitted model to compute the residuals. We then compute the F*-ratios and logged F*-ratios that are given in Table 3.2. Comparing lnF* with $\ln F_{16,16,.01}=\ln 3.381=1.218$, we conclude that there is no potentially significant F* ratio. Thus, the variance of the error term can be considered constant.

3.5. Concluding Remark

As we have seen in the example of section 3.4, transforming the response variable is very useful technique for parameter design. It enables us to identify adjustment factors from the control factors, and consequently, parameter design can be done with a two-step procedure. The advantage of a two-step procedure over the direct optimization of the expected loss is its convenience and flexibility.

The conceptual simplicity and empirical nature of the lambda plots make them useful tools in parameter design. However, it should be recognized they are exploratory techniques and should only be used to suggest ways of analyzing the data. The experimenter should be aware of the risks inherent in the analysis of unreplicated experiments because no precise estimate of the error variance is available.

4. Combined Robust Design

4.1. Introduction

In chapters 2 and 3, we proposed several data analysis procedures for parameter design when the underlying model of the product/process is unknown. Actual physical experiments have to be done to collect data for modeling the quality characteristic of the product/process. Because industrial experimentation is often very costly, careful design and efficient analysis of experiments are of great importance. The recently developed combined array design provides an effective way of reducing the size of an experiment, and increases the flexibility in selection of estimable effects.

Although in most industrial research the exact mathematical form of the fundamental model is often unknown, there are still many situations in which the exact functional relationship of the model is known to the design engineers. For example, in the electronics industry many products and processes are assembled together with electronic components whose mechanism and behaviors are well known. There are many of assembled consumer products such as TV's, VCR's, radios and temperature controllers, among others. We call the known model that relates the response characteristic to the assembled components, a transfer function.

Knowledge of the transfer function is a tremendous advantage (see Taguchi, 1986; Taguchi and Wu, 1985; Box and Fung, 1986; Welch et al, 1990). Principally, costly physical experiments are now not necessary; experimentation is replaced by computation with the known model. Numerical computation is undoubtedly much cheaper than physical experimentation. An added advantage is that the optimization of the expected quality loss could be more accurate since the exact transfer function is given and no estimation is needed.

In general, Taguchi (Taguchi, 1986; Taguchi and Wu, 1985; Taguchi and Phadke, 1984) divides the product/process design into three stages: system design, parameter

design and tolerance design. In the system design stage, design engineers produce a basic functional prototype design. Parameter design, also called robust design, is used to determine the optimal setting of the control factors that makes the product/process insensitive to the noise factors. During this stage, the tolerances of the control factors are chosen to be large so that the raw-material and manufacturing costs are kept low. Finally, tolerance design is carried out, if necessary, to tighten the tolerances of the control factors and thus, further reduce the variation on the basis of their cost-effectiveness. Tighter tolerances will certainly reduce the variation of the response variable, while they increase the component costs.

When the transfer function is unknown, the main emphasis of quality improvement is placed on parameter design, and tolerance design is seldom dealt with. This is because parameter design can reduce variation without increasing cost. In addition, tolerance design is difficult to do without knowing the transfer function. On the other hand, when the transfer function is known, tolerance design can be easily done to further reduce the variation taking into account of the cost-effectiveness.

Although parameter design and tolerance design have always been done separately (see Taguchi, 1986; Taguchi and Wu, 1985; Box and Fung, 1986; Welch et al., 1990), they are actually inter-connected. In this chapter, we provide a unifying formulation of robust product/process design that integrates parameter design and tolerance design into one single optimization problem. This integrated problem involving parameter design and tolerance design is called *combined robust design*. The new formulation presents potential for further significant reduction of the response variation over the existing three-stage formulation.

4.2. Formulation of the Problem

Consider a product/process that is assembled together with n component parts with nominal values x_1, x_2, ..., and x_n. Suppose the response quality characteristic, y, is linked

to x_1, x_2, ..., and x_n through a known transfer function $y=F(x_1, x_2, ..., x_n)$. Generally, the nominal values of x_1, x_2, ..., and x_n are chosen by design engineers, and thus, they can be taken as control factors. Let $\mathbf{x}=(x_1, x_2, ..., x_n)$, called a control-factor setting. The transfer function can be written as

$$y = F(\mathbf{x}).$$

The target value of the quality characteristic y is assumed to be a fixed value T. Ideally, if there is no variation in the control factors, any of the infinite control-factor settings $\mathbf{x}$ satisfying $T=F(\mathbf{x})$, will be an optimal setting. In reality, there is always variation in the control factors because of the existence of hard-to-control noise factors. There are mainly three types of noise factors:

(1) unit-to-unit variation between the components.

(2) component deterioration.

(3) fluctuation of operating conditions.

We assume that the noise factors are random variables causing small variations around the nominal setting of the control factors. Thus, for control factor x_i, the input variable becomes

$$X_i = x_i + e_i, \qquad i=1, 2, ..., n,$$

where e_i is the noise factor associated with control factor x_i. Let $\mathbf{X}=(X_1, X_2, ..., X_n)$ and $\mathbf{e}=(e_1, e_2, ..., e_n)$. Then

$$\mathbf{X} = \mathbf{x} + \mathbf{e}.$$

Since the noise factors will inevitably transmit variation into the assembled product, the response quality characteristic becomes a random variable given by

$$Y = F(\mathbf{X}).$$

It is assumed in this chapter that the e_i's are uncorrelated random variables with

$$E(e_i) = 0,$$

$$E(e_i^2) = [a_i(x_i)v_i]^2,$$

$$E(e_ie_j) = 0, \qquad i\neq j,$$

for i, j =1, ..., n. v_i is referred to in the rest of this chapter as the tolerance coefficient for component x_i. $\mathbf{v}=(v_1, v_2, ..., v_n)$ will be called a tolerance-coefficient setting. The $a_i(x_i)$ are assumed to be known functions, and in practice it is often assumed $a_i(x_i)=x_i$.

4.2.1. Expected Quality Loss

When it deviates from the target value, the response variable incurs a quality loss. As in the previous chapters, we use the quadratic loss function. Then the expected quality loss (EQL) is

$$R(\mathbf{x}, \mathbf{v}) = E\{k(Y - T)^2\}$$
$$= k\{\sigma_Y^2 + (\mu_Y - T)^2\},$$

where k is some constant that can be determined on the basis of information about monetary costs caused by exceeding the customer's tolerance..

4.2.2. Cost Structure for Components

It is easy to see that the expected quality loss depends not only on the control-factor setting **x**, but also on the tolerance-coefficient setting **v**. The narrower the tolerances, the less variable the response Y is and consequently, the lower the expected quality loss is. However, we are not at freedom to tighten the tolerances of the components because of the higher cost associated with the tighter tolerances. Obviously, we need an approach to balance higher component cost and lower quality loss.

Generally, the cost of a component part is related to its tolerance. The less variable a component is, the higher the cost to produce or purchase it. We assume that the cost of a component x_i is related to its tolerance coefficient v_i but not to its nominal value x_i. Let $C(\mathbf{v})$ denote the total cost of the component parts used in assembling the product/process. Thus, the total cost $C(\mathbf{v})$ can be written into the sum of individual costs of components, this is,

$$C(\mathbf{v}) = C_1(v_1) + C_2(v_2) + ... + C_n(v_n).$$

Each $C_i(v_i)$ is assumed known to design engineers.

4.2.3. Taguchi's Formulation

Taguchi's strategy of robust product/process design is a two-stage one involving parameter design followed by tolerance design. His approach can be summarized in the following procedure:

(1) Parameter design stage: Given an initial tolerance-coefficient setting $\mathbf{v}_0$, find the control-factor setting $\mathbf{x}^*$ that optimizes $SN_T(\mathbf{x}, \mathbf{v}_0)$. Then fine-tune the adjustment factors to bring the mean response to the target value.

(2) Tolerance design stage: Find $\mathbf{v}^*$ that minimizes $R(\mathbf{x}^*, \mathbf{v}) + C(\mathbf{v})$. Hence $(\mathbf{x}^*, \mathbf{v}^*)$ is taken as the overall solution.

Instead of directly minimizing the EQL in the parameter design stage, Taguchi (see Taguchi, 1986; Taguchi and Wu, 1985; Taguchi and Phadke, 1984) seeks to maximize the following signal-to-noise ratio

$$SN_T(\mathbf{x}, \mathbf{v}) = \ln(\mu_Y/\sigma_Y)^2.$$

His procedure of parameter design can be summarized in the following two steps:

(1) Given a wide tolerance setting $\mathbf{v}_0$, find the control-factor setting $\mathbf{x}^*$ that maximizes the SN ratio.

(2) Fine-tune the adjustment factors that affect the mean but not the SN ratio, to bring the mean response to the target value T.

To make this two-step procedure work, Taguchi essentially assumed that the control factors can be divided into two subsets, $\mathbf{x}=(\mathbf{d}, \mathbf{a})$, where $\mathbf{a}$ are fine-tuning adjustment factors that are independent of the signal-to-noise ratio. The setting $\mathbf{d}^*$ is found by maximizing the SN ratio, and the setting $\mathbf{a}^*$ is found by adjusting the mean response to the target. Leon et al. (1987) showed that when adjustment factors exist, Taguchi's two-step procedure is equivalent to the minimization of the average quality loss. The major drawback of this procedure is that it requires the existence of adjustment factors. Unfortunately, there is no adjustment factor in many problems.

4.2.4. New Formulation

The objective of robust design is clearly to assemble the product/process with the highest quality at lowest cost. To balance the quality loss and the component cost, the total loss

$$M(\mathbf{x}, \mathbf{v}) = R(\mathbf{x}, \mathbf{v}) + C(\mathbf{v})$$

is an appropriate measure for optimization. Thus, instead of using Taguchi's two-stage procedure, we propose to minimize the total loss simultaneously with respect to $\mathbf{x}$ and $\mathbf{v}$. Accordingly, the problem of finding a robust product/process is to solve the following optimization problem:

$$\min_{\mathbf{x}, \mathbf{v}} M(\mathbf{x}, \mathbf{v}).$$

The minimization of the total loss will be referred to as *combined robust design* in the rest of this chapter. We will show later that the minimization of $M(\mathbf{x}, \mathbf{v})$ results in a design solution with a smaller total loss than Taguchi's two-stage solution.

With a known transfer function, the above optimization problem becomes an non-linear optimization problem. However, the $C_i(v_i)$'s are usually defined only at discrete points over which the total loss is minimized. This discreteness of the cost function adds difficulties in determining the precise optimum, because most of the existing optimization techniques and computing softwares have been devised for continuous functions. Modified and even new techniques are required to deal with the discrete cost function.

4.3. Optimization Strategies

In the last section we proposed to minimize the total loss that includes the quality loss and the component cost. In this section we propose two strategies for minimizing this total loss. They are based on the relationship

$$\min_{\mathbf{x}, \mathbf{v}} M(\mathbf{x}, \mathbf{v}) = \min_{\mathbf{v}} \min_{\mathbf{x}} M(\mathbf{x}, \mathbf{v}).$$

In fact, since, for any $(\mathbf{x}, \mathbf{v})$,

$$\min_{\mathbf{v}} \min_{\mathbf{x}} M(\mathbf{x}, \mathbf{v}) \le \min_{\mathbf{x}} M(\mathbf{x}, \mathbf{v}) \le M(\mathbf{x}, \mathbf{v}),$$

$$\min_{v} \min_{x} M(x, v) \leq \min_{x, v} M(x, v).$$

Conversely, $\min_{x, v} M(x, v) \leq \min_{v} \min_{x} M(x, v)$ because $\min_{x, v} M(x, v)$ is the global minimum.

Since the tolerances are assumed to be discrete, denote all the combinations of tolerance-coefficient settings by set G. The above equation provides, in theory, the following

Exhaustive Search Procedure:

(1) For each v in G, find $x^*(v)$ through solving $\min_{x} M(x, v)$;

(2) Select v^* in G so that $M(x^*(v), v)$ is minimized. Then $(x^*(v^*), v^*)$ is the overall solution.

Theoretically, the global minimum can be located by employing the above procedure. However, it is impractical for designs with moderate to large number of control factors, because it needs first to solve a large number of sub-optimization problems $\min_{x} M(x, v)$. The computation will be too huge even for a high-speed computer to handle; for example, if each of the n component tolerance-coefficients has two levels, the number of sub-optimizations is 2^n=64 when n=6, and 2^n=1024 when n=10.

We propose an alternative procedure that can result in a solution with a smaller total loss than Taguchi's solution. This procedure requires performing parameter design and tolerance design iterately. It starts the first round parameter design using an initial tolerance-coefficient setting v_0 to obtain $x_0=x(v_0)$, that is,

$$R(x_0, v_0) = \min_{x} R(x, v_0). \tag{4.3.1}$$

Then (x_0, v_0) is taken to be the initial round solution. Note that Taguchi recommends that the cheapest tolerance-coefficient setting be used as the initial setting. Tolerance design is conducted to determine an appropriate tolerance-coefficient setting by finding the new tolerance-coefficient setting v_1 such that

$$M(x_0, v_1) = \min_{v} M((x_0, v). \tag{4.3.2}$$

With Taguchi's procedure, $(\mathbf{x}_0, \mathbf{v}_1)$ will be taken to be the optimal solution. With the combined robust design procedure we have proposed, it is necessary and desirable to perform another round parameter design to further reduce the expected quality loss as well as the total loss. The second round parameter design will be done using updated tolerance-coefficient setting $\mathbf{v}_1$. The next round parameter design is to find $\mathbf{x}_1=\mathbf{x}(\mathbf{v}_1)$ such that

$$R(\mathbf{x}_1, \mathbf{v}_1) = \min_{\mathbf{x}} R(\mathbf{x}, \mathbf{v}_1). \tag{4.3.3}$$

Consequently, we have

$$M(\mathbf{x}_1, \mathbf{v}_1) \leq M(\mathbf{x}_0, \mathbf{v}_0)$$

(see the Appendix for the proof). Thus, $(\mathbf{x}_1, \mathbf{v}_1)$ is taken to be the second round solution. This process continues until further reduction of the total loss is impossible. In this way, we have obtained a series of solutions, $(\mathbf{x}_i, \mathbf{v}_i)$, i=0, 1, ..., with the updated solution having a smaller total loss than the former one (see the Appendix). This means that the iteration of parameter design and tolerance design can be continued until further reduction of the total loss is not possible. The above results suggest the following

Iterative Procedure:

(1) Choose an initial tolerance-coefficient setting $\mathbf{v}_0$.

(2) Perform parameter design and find $\mathbf{x}_0$. Then $(\mathbf{x}_0, \mathbf{v}_0)$ is the current round solution.

(3) Perform tolerance design to find the updated tolerance-coefficient setting $\mathbf{v}_1$.

(4) Update the tolerance-coefficient setting from $\mathbf{v}_0$ to $\mathbf{v}_1$, and return to (2).

The process continues until further reduction of the total loss is not possible.

4.4. Approximations of the Expected Quality Loss

To optimize $M(\mathbf{x}, \mathbf{v})$ or $R(\mathbf{x}, \mathbf{v})$, we first need to evaluate $R(\mathbf{x}, \mathbf{v})$ at any point of the region over which $M(\mathbf{x}, \mathbf{v})$ is being minimized. The difficulty of direct evaluation is apparent since the transfer function F is usually a very complicated function of multiple

variables, although its analytical form is known. In this section we consider approximations to R(x, v) and M(x, v).

As we described earlier, the expected quality loss is

$$R(\mathbf{x}, \mathbf{v}) = k\{\sigma_Y^2 + (\mu_Y - T)^2\}.$$

Since the response function F(X) is usually very complicated, it is almost impossible to calculate analytically the above expected quality loss. To compute this loss numerically, we first need to obtain approximations of μ_Y and σ_Y. We expand F(X) into a Taylor series around the nominal setting x and obtain

$$F(\mathbf{X}) = F(\mathbf{x}) + \sum_{i=1}^{n} f_i(\mathbf{x})e_i + \ldots,$$

where

$$f_i(\mathbf{x}) = \frac{\partial F(\mathbf{x})}{\partial x_i}.$$

Compared to the control-factor setting x, the variations in the noise factors e_i's are in general very small, and the first order approximation of F(X) is very close to the exact form. Box and Fung (1986) demonstrated that this approximation is very accurate in practice. Thus, μ_Y and σ_Y^2 are well approximated by

$$\mu_Y = F(\mathbf{x}),$$

and

$$\sigma_Y^2 = \sum_{i=1}^{n} [f_i(\mathbf{x})]^2 [a_i(x_i)v_i]^2.$$

The expected quality loss is then well approximated by

$$R(\mathbf{x}, \mathbf{v}) = k\{\sum_{i=1}^{n} [f_i(\mathbf{x})]^2 [a_i(x_i)v_i]^2 + [F(\mathbf{x}) - T]^2\}.$$

The derivatives $f_i(\mathbf{x})$ could be determined theoretically by differentiation, but in practice they are more easily computed by the following numerical approximations

$$f_i(\mathbf{x}) = \frac{Y_i - Y_0}{h a_i(x_i)v_i}$$

where

$$Y_0 = F(x),$$

$$Y_i = F(x_1, \ldots, x_i + h a_i(x_i) v_i, \ldots, x_n),$$

and h is a small constant such as 0.01 or 0.001. Thus, the expected quality loss is further approximated by

$$R(x, v) = k\{(1/h^2) \sum_{i=1}^{n} (Y_i - Y_0)^2 + (Y_0 - T)^2\}.$$

4.5. Example

To illustrate the optimization strategies proposed in this chapter, we consider the combined robust design of a simple electric circuit that translates alternating current into direct current. The transfer function is

$$y = \frac{V}{\sqrt{R^2 + (2\pi f L)^2}},$$

where R is the resistance, L the self-inductance, and V the input voltage with frequency f. The target value of the output characteristic is T=10 A. The components R and L are control factors whose nominal values are chosen by design engineers. V and f are noise factors with nominal values V=120 V and f=50 Hz that are influenced by environmental variables such as temperature and humidity. There is also unit-to-unit variation in both R and L. Therefore, R and L are both control factors and noise factors. The unit-to-unit variation in both R and L can be reduced by using higher-grade components, while the variation in both V and f can not be reduced in this example. Note that the current example is different from the earlier assumption that each factor is both a control factor and a noise factor. Now V and f are just noise factors, not control factors. Despite this difference, the earlier results can be applied very easily with a slight modification that is evident in the approximation of the expected quality loss.

Let $(x_1, x_2)=(R, L)$. Suppose that the control-factor setting is defined over

$$1 \le x_1 \le 20, \qquad 0.003 \le x_2 \le 0.045.$$

It is also assumed that

$$\mathrm{Var}(X_1) = (x_1 v_1)^2,$$

$$\mathrm{Var}(X_2) = (x_2 v_2)^2,$$

$$\mathrm{Var}(V) = (120 \times 0.005)^2,$$

$$\mathrm{Var}(f) = (50 \times 0.005)^2.$$

Let $\mathbf{v}=(v_1, v_2)$ and $\mathbf{x}=(x_1, x_2, 120, 50)=(R, L, V, f)$. Then the quality loss function is approximated by

$$R(\mathbf{x}, \mathbf{v}) = k\{[f_1(\mathbf{x})]^2(x_1 v_1)^2 + [f_2(\mathbf{x})]^2(x_2 v_2)^2 + [f_3(\mathbf{x})]^2[(120)(.005)]^2 + [f_4(\mathbf{x})]^2[(50)(.005)]^2 + [F(\mathbf{x})-10]^2\}$$

$$= k\{[f_1(\mathbf{x})x_1]^2(v_1)^2 + [f_2(\mathbf{x})x_2]^2(v_2)^2 + [f_3(\mathbf{x})(120)]^2(.005)^2 + [f_4(\mathbf{x})(50)]^2(.005)^2 + [F(\mathbf{x})-10]^2\}.$$

The constant k is assumed to be 50. The component cost structure is given in Table 4.1.

Table 4.1. Component Costs

Factor	Grade 1		Grade 2	
	v	Cost	v	Cost
Resistor (Ω)	0.020	\$0.30	0.005	\$0.60
Coil (H)	0.020	\$0.70	0.005	\$1.20

We use the IMSL subroutine ZXMWD to maximize $R(\mathbf{x}, \mathbf{v}_0)$ with the initial tolerance-coefficient setting $\mathbf{v}_0=(.020, .020)$. The solution is $\mathbf{x}_0=(7.9707, .0241)$. Thus, the first round solution of the combined robust design is

$$(\mathbf{x}_0, \mathbf{v}_0) = (7.9707, .0241, .020, .020).$$

The quality loss at (x_0, v_0) is

$$R(x_0, v_0) = 50\{(27.6123)(.020)^2 + (22.4653)(.020)^2 + (100.6541)(.005)^2 + (21.8370)(.005)^2 + (.0059)^2\}$$

$$= 50x.0231 = \$1.16.$$

The total loss is computed by

$$M(x_0, v_0) = 1.16 + .30 + .70 = \$2.16.$$

We now perform tolerance design to further reduce the expected quality loss on the basis of cost-effectiveness. Since the expected quality loss is a separable function of v_1 and v_2, the upgrading of components can be done separately for both the resistor and the coil. If the resistor is upgraded from grade 2 to grade 1, the cost is increased by 0.60-0.30= \$.30, but the expected quality loss is reduced to

$$R = 50\{(27.6123)(.005)^2 + (22.4653)(.020)^2 + (100.6541)(.005)^2 + (21.8370)(.005)^2 + (.0059)^2\}$$

$$= 50x.0128 = \$.64.$$

The total loss decreases from \$2.16 to

$$M = .64 + .60 + .70 = \$1.94.$$

The cost reduction due to switching the resistor from grade 2 to grade 1 is

$$2.16 - 1.94 = \$.22.$$

Thus, upgrading the resistor is cost-effective. If the coil is also changed from grade 2 to grade 1, the expected quality loss is further reduced to

$$R = 50\{(27.6123)(.005)^2 + (22.4653)(.005)^2 + (100.6541)(.005)^2 + (21.8370)(.005)^2 + (.0059)^2\}$$

$$= 50x.0043 = \$.22.$$

The total loss, using a first-grade resistor and coil, is

$$M = .22 + .60 + 1.20 = \$2.02.$$

which is eight cents more than \$1.94. Thus, it is not worthwhile to upgrade the coil. Thus, tolerance design provides $\mathbf{v}_1$=(.005, .020). Taguchi's two-stage procedure provides the following solution

$$(\mathbf{x}_0, \mathbf{v}_1) = (7.9707, .0241, .005, .020)$$

with M=\$1.94.

Following our iterative procedure, however, we perform another round of parameter design resulting in $\mathbf{x}_1$=(10.8413, .0059). The second-round optimization solution is

$$(\mathbf{x}_1, \mathbf{v}_1) = (10.8413, .0059, .005, .020)$$

with R=\$.24 and M=\$1.54. The total loss is reduced by

$$1.94 - 1.54 = \$.40.$$

This is a (.40/1.94)x100% or 20.6% reduction compared to the solution given by the two-stage procedure. More importantly, without increasing the component cost, the expected quality loss is reduced in the second-round parameter design by a ratio of

$$R(\mathbf{x}_0, \mathbf{v}_1)/R(\mathbf{x}_1, \mathbf{v}_1) = .64/.24 = 2.66.$$

An additional round tolerance design provides no reduction in total loss, this is, $\mathbf{v}_2=\mathbf{v}_1$, and the iteration thus ends here.

In this example, since the number of the control factors is small, the exhaustive search procedure can also be used. For each combination of $\mathbf{v}$, we find that

$$\mathbf{x}^*(.020, .020) = (7.9707, .0241),$$

$$\mathbf{x}^*(.005, .020) = (10.8413, .0059),$$

$$\mathbf{x}^*(.020, .005) = (4.2007, .0324),$$

$$\mathbf{x}^*(.005, .005) = (8.8838, .0206).$$

The total loss for each setting of control and noise factors are

$$M(7.9707, .0241, .020, .020) = 2.16,$$

$$M(10.8413, .0059, .005, .020) = 1.54,$$

$$M(4.2007, .0324, .020, .005) = 1.85,$$

$$M(8.8838, .0206, .005, .005) = 2.01.$$

Clearly, the combined robust design solution is (10.8413, .0059, .005, .020) which is identical to the solution we found using the iterative procedure.

4.6. Comment

Robust design has been recognized as a very effective way of reducing performance variation. In the case in which the theoretical model is known, the objective is to minimize the total loss, the sum of the expected quality loss and the component cost. Under this general formulation, we have proposed a procedure for further reducing the total loss as well as the expected quality loss. This procedure requires iterating parameter design and tolerance design. The significant reduction in total loss as well as the expected quality loss can be achieved using the proposed procedure instead of Taguchi's two-stage procedure. The example demonstrated that a much better design can be found by simply performing an additional round parameter design using the updated tolerance setting, compared to Taguchi's two-stage procedure.

Appendix

Theorem. Let $(\mathbf{x}_i, \mathbf{v}_i)$ represents the i-round solution of the combined robust design. Then, for any $i \geq 1$,

$$M(\mathbf{x}_i, \mathbf{v}_i) \leq M(\mathbf{x}_{i-1}, \mathbf{v}_{i-1}).$$

PROOF. For simplicity, we proceed to prove the case i=1. By definition

$$R(\mathbf{x}_1, \mathbf{v}_1) = \min_{\mathbf{x}} R(\mathbf{x}, \mathbf{v}_1).$$

So

$$\begin{aligned} M(\mathbf{x}_1, \mathbf{v}_1) &= R(\mathbf{x}_1, \mathbf{v}_1) + C(\mathbf{v}_1) = \min_{\mathbf{x}} R(\mathbf{x}, \mathbf{v}_1) + C(\mathbf{v}_1) \\ &= \min_{\mathbf{x}} [R(\mathbf{x}, \mathbf{v}_1) + C(\mathbf{v}_1)] \\ &= \min_{\mathbf{x}} M(\mathbf{x}, \mathbf{v}_1) \leq M(\mathbf{x}_0, \mathbf{v}_1) \\ &= \min_{\mathbf{v}} M(\mathbf{x}_0, \mathbf{v}) \leq M(\mathbf{x}_0, \mathbf{v}_0). \end{aligned}$$

5. Weighted Attribute Control Charts for Variable Sample Size

5.1. Introduction

Attribute charts have been widely used in statistical process control. They are important because (a) attribute situations exist in many industrial and administrative processes, (b) attribute data are frequently available, and (c) in general attribute data are quick and inexpensive to obtain (see Ford Manual, 1987). There are mainly four types of attribute control charts: the p chart, the np chart, the c chart, and the u chart. Of the four types, the p chart and the u chart do not require constant subgroup sample size.

When the subgroup samples are of constant size, the construction and the interpretation of attribute control charts are usually simple and straightforward. However, there are many situations in which the sample size varies. For example, if samples are from daily or weekly production, the sample size often varies.

There are three traditional procedures that could be used for constructing attribute control charts such as the p chart for variable sample size (see Duncan, 1974; Messina, 1987; Montgomery, 1991; Nelson, 1989). The first procedure is to calculate separate control limits for each sample. This procedure is straightforward, but computation is tedious. The second procedure is to base the control limits on the average sample size; these limits are approximate ones, the interpretation is inexact, and separate limits have to be computed for especially large or small samples. For both procedures it is difficult sometimes to appropriately identify patterns or trends. To overcome this problem, the third procedure is to plot the standardized values

$$z_i = \sqrt{n_i}\,(p_i - \bar{p})/\sqrt{\bar{p}(1 - \bar{p})} \quad .$$

On this chart the limits are set at UCL=3, LCL=-3, and CL=0. However, the major disadvantage of this procedure is that the values of fraction nonconforming are not shown.

In this chapter we propose a new procedure for constructing the p chart and the u chart through a transformation of the sample data. The control limits on these new charts will remain constant over all sample sizes.

On a new chart, say the p chart, the transformed fraction nonconforming p_i*'s are measured in terms of the same degree of accuracy. Hence, non-random patterns and trends can be more meaningfully interpreted and identified. The transformation could be made so that p_i* and the original fraction nonconforming (before the transformation) p_i are almost identical for most samples on the chart. For samples whose sizes are particularly large or small in comparison with those of the majority of the samples, one could also plot their p_i along with the p_i* .

5.2. The Weighted p Chart

The p chart is used to study graphically variations in fraction nonconforming, which is defined to be the ratio of the number of nonconforming items x_i to the total number n_i in sample i: $p_i=x_i/n_i$. We assume in this section that the process fraction nonconforming p' under consideration is either known or could be estimated from previous samples. Then the conventional 3-standard-deviation control limits for the fraction nonconforming from sample i are as follows

$$UCL = p' + 3\sqrt{\frac{p'(1-p')}{n_i}},$$

$$CL = p',$$

$$LCL = p' - 3\sqrt{\frac{p'(1-p')}{n_i}}.$$

When the sample size varies, these limits vary from sample to sample. Note that the width of the limits is inversely proportional to the square root of the sample size. One way out of this difficulty is to make a transformation to stabilize the variable. The use of a standardized p chart is such an approach, which was modified by Soffer (1981). In this

chapter we suggest a different transformation. Suppose n is the sample size on which the control limits for the transformed data will be based. Here n can be any convenient positive integer. We will discuss how to choose n later on. The fraction nonconforming p_i for sample i is transformed to

$$p_i^* = p' + \sqrt{\frac{n_i}{n}}(p_i - p').$$

It is equivalent to

$$p_i^* = p_i + (\sqrt{\frac{n_i}{n}} - 1)(p_i - p').$$

Note that p_i^* is p_i plus a weighted deviation of p_i from p'. So the graph of the p_i^* could be called a weighted p chart.

When the sample size n_i is equal to n, $p_i^*=p_i$. Under the assumption that the process is in control, it is easy to show that the mean and standard deviation of p_i^* are given by

$$E(p_i^*) = p'$$

and

$$\sigma(p_i^*) = \sqrt{\frac{p'(1-p')}{n}}$$

which are independent of n_i. Therefore, the control limits for the transformed fraction nonconforming p_i^* are constant and are set at

$$UCL = p' + 3\sqrt{\frac{p'(1-p')}{n}},$$

$$CL = p',$$

$$LCL = p' - 3\sqrt{\frac{p'(1-p')}{n}}.$$

On this chart the center line immediately shows the process fraction nonconforming p'.

The fact that p_i^*'s having the same standard deviation implies that the subgroup fraction nonconformities are measured by estimates of the same accuracy. Consequently, it makes more sense to identify non-random patterns and trends through the p_i^*'s than

through the p_i's.

Note that p_i^* is actually a linear function of the standardized value

$$z_i = \sqrt{n_i}(p_i - p')/\sqrt{p'(1-p')}.$$

In fact,

$$p_i^* = z_i\sqrt{p'(1-p')/n} + p'.$$

When n_i deviates from n within 25%, p_i^* is only slightly different from p_i as can be seen from the following inequality

$$|p_i^* - p_i| = |\sqrt{\frac{n_i}{n}} - 1||p_i - p'| < 0.134\,|p_i - p'|.$$

If p_i is outside of its individual control limits, then p_i^* will be outside of the control limits of the new p chart; the converse is also true. For example, when p_i is above its individual UCL, i.e.,

$$p_i > p' + 3\sqrt{\frac{p'(1-p')}{n_i}},$$

then

$$p_i^* = p' + \sqrt{\frac{n_i}{n}}(p_i - p') > p' + 3\sqrt{\frac{p'(1-p')}{n}}$$

which means that p_i^* is plotted above the transformed UCL. Similarly, the lower limit case can be shown.

One problem of the new transformed chart is that the p_i^* value could be less than 0 or greater than 1. More specifically,

$$p_i^* < 0 \text{ when } p_i < (1 - \sqrt{n/n_i})p',$$

and

$$p_i^* > 1 \text{ when } p_i > p' + \sqrt{n/n_i}(1 - p').$$

Here we note that p_i^* is not likely to be less than 0 unless p_i is well below p' and n_i is substantially larger than n. Similarly, p_i^* is not likely to be greater than 1 unless p_i is well above p' and n_i is substantially greater than n. One way to avoid having $p_i^*<0$ or $p_i^*>1$ is to choose the n to be greater than all the n_i's .

5.3. The Case When p' Is Unknown

Generally, p', the process fraction nonconforming, is unknown. It must be estimated from the current sample data. Let $\bar{p}$ be calculated from m samples as follows

$$\bar{p} = \frac{\sum_{i=1}^{m} n_i p_i}{\sum_{i=1}^{m} n_i}.$$

The statistic $\bar{p}$ is used to estimate p'. Now the transformation becomes

$$p_i^* = \bar{p} + \sqrt{\frac{n_i}{n}}(p_i - \bar{p}).$$

It is equivalent to

$$p_i^* = p_i + (\sqrt{\frac{n_i}{n}} - 1)(p_i - \bar{p}).$$

Consequently, the control limits for the transformed fraction nonconforming are set at

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}},$$

$$CL = \bar{p},$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}.$$

The center line shows the estimated process fraction nonconforming. Again $p_i^*=p_i$ when $n_i=n$. Similar to the case in which p' is known, it is easy to see that both of the transformed chart and the chart with individual control limits will simultaneously indicate a sample being between the control limits or outside of the control limits.

Example 5.1

The samples in this example were taken from daily production. The sample size varied from sample to sample. On the transformed p chart the control limits are based on the average sample size, i.e., n=950. The range of minus to plus 25% of the average

sample size is from 713 to 1187. There are 11 samples that fall in this range. Using the average sample size as the value of n, the original and transformed data are given in the last two columns of Table 5.1.

Table 5.1. Data for Example 5.1

i	n_i	x_i	p_i	p_i*
1	950	9	0.009	0.009
2	850	15	0.018	0.018
3	600†	15	0.025	0.023
4	950	15	0.016	0.016
5	1000	5	0.005	0.005
6	950	8	0.008	0.008
7	1350†	17	0.013	0.012
8	950	10	0.011	0.011
9	850	8	0.009	0.010
10	850	15	0.018	0.018
11	950	12	0.013	0.013
12	950	16	0.017	0.017
13	600†	14	0.023	0.022
14	1500†	31	0.021	0.022
15	950	24	0.025	0.025

†: Sample size deviates more than 25% from the average size 950.

We notice that the p_i and p_i* values are identical for most samples except for 3, 7, 13 and 14 where the n_i values are, respectively, 37%, 42%, 37% and 58% greater or

less than n=950. The estimated process nonconforming fraction $\bar{p}$=0.015 is the central line CL. The control limits for the average sample size are UCL=0.027 and LCL=0.003. Figure 5.1 displays the p chart with the control limits for the average sample size. The individual control limits for the 3 specially large or small samples are also plotted. Figure 5.2 shows the new p chart with n=950. The p_i values are also given for comparison. Note that the p_i^* and p_i values are identical except for samples 3, 13 and 14. Also note that the upward trend of the p_i^*'s from sample 11 to 15. The trend is not shown by the corresponding p_i values.

The traditional p chart based on average sample size is familiar to and used frequently by most practitioners. In practice, they could continue to construct this chart and then obtain the new p chart (with n=average sample size) by simply adding the p_i^* values to the samples with sizes deviating more than 25% from the average sample size (e.g., samples 3, 7, 13, and 14 of Example 5.1), to the samples whose p_i values are close to the control limits, and to the samples that may show non-random pattern or trend if p_i^* values are plotted (e.g., samples 13 and 14 of Example 5.1).

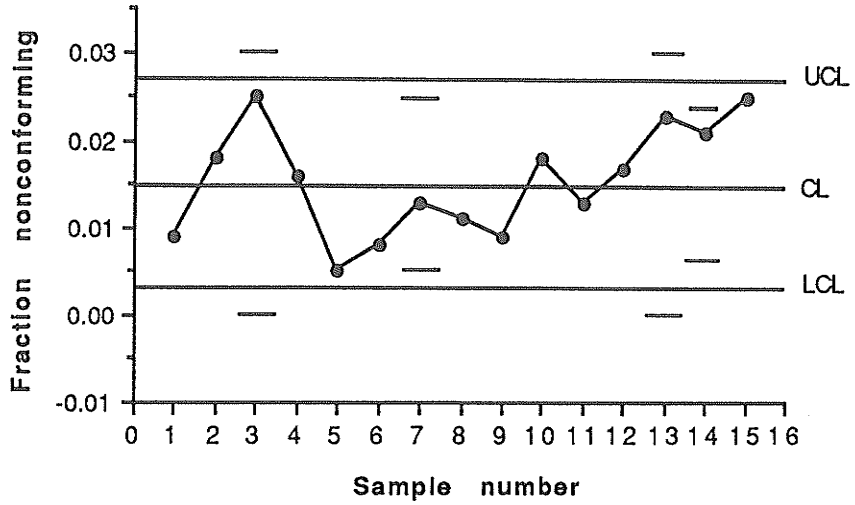

Figure 5.1. Conventional p chart with control limits based on the average sample size n=950; individual control limits for 4 especially large and small samples (3, 7, 13 and 14) are also given.

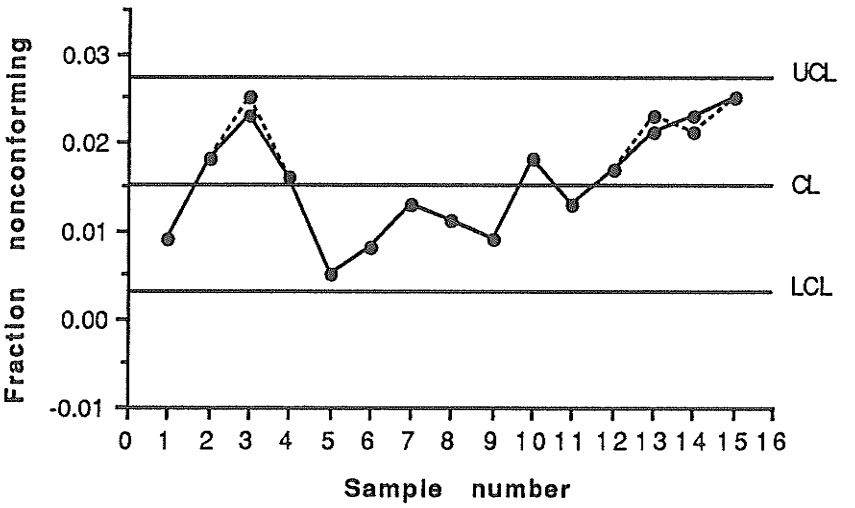

Figure 5.2. Weighted p chart with average size n=950; the dashed lines are the p_i values that differ from the corresponding p_i*.

5.4. The Weighted u Chart

The u chart is used to study the variations in the number of nonconformities per unit. This chart is generally used when the number of units that are combined to form a sample is not constant. The discussion here is similar to that for the transformed p chart. Thus we only need to consider the case in which u', the average number of process nonconformities per unit, is unknown. Let u_i represent the number of nonconformities per unit in the sample i of size n_i units. Let n be the sample size on which the control chart for the transformed number of nonconformities per unit will be based. Then the transformed number of nonconformities per unit is defined as follows

$$u_i^* = \bar{u} + \sqrt{\frac{n_i}{n}}(u_i - \bar{u}).$$

It is equivalent to

$$u_i^* = u_i + (\sqrt{\frac{n_i}{n}} - 1)(u_i - \bar{u}),$$

where $\bar{u}$ is the estimated average nonconformities per unit and is calculated from m samples by

$$\bar{u} = \frac{\sum_{i=1}^{m} n_i u_i}{\sum_{i=1}^{m} n_i}.$$

Note that u_i^* is u_i plus a weighted deviation of u_i from $\bar{u}$. The resulting plot could be called a weighted u chart.

The control limits for the transformed data are constant and set at

$$UCL = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}},$$

$$CL = \bar{u},$$

$$LCL = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}}.$$

One problem of the transformed u chart is that u_i^* could be less than 0.

Specifically,

$$u_i^* < 0 \text{ when } u_i < (1-\sqrt{\frac{n}{n_i}})\bar{u}.$$

However, we note that u_i^* is unlikely to be less than 0 unless u_i is well below $\bar{u}$ and n_i is substantially larger than n.

If u' is known, $\bar{u}$ in the above equations is to be replaced by u'.

Example 5.2

The data set in this example is from the Ford Manual (1987, p.48a), and the sample size varies from sample to sample. The average sample size is n=8 units and the estimated process nonconformities per unit is $\bar{u}$=1.89. The range of minus to plus 25% of the average sample size is from 6 to 10, and 19 of 25 samples fall in this range. Based on the average sample size, the transformed data are given in the last column of Table 5.2. We note that the u_i and u_i^* values are almost identical for most samples except for 17, 21 and 23 where the n_i values deviate more than 50% from n=8. The control limits for the average sample size are UCL=3.35 and LCL=0.43. Figure 5.3 gives the conventional u chart with the control limits for the average sample size, and the individual control limits for specially large or small samples are also plotted. Figure 5.4 gives the new transformed control chart.

If a practitioner prefers to keep the traditional u chart based on the average sample size, the special u_i^* values could be simply added to the traditional u chart in a similar to that described at the end of the last section for the p chart.

Table 5.2. Data for Example 5.2

i	n_i	$n_i u_i$	u_i	u_i*
1	8	8	1.0	1.0
2	8	17	2.1	2.1
3	9	18	2.0	2.0
4	8	15	1.9	1.9
5	8	23	2.9	2.9
6	7	9	1.3	1.3
7	7	19	2.7	2.7
8	8	6	0.8	0.8
9	8	14	1.8	1.8
10	8	17	2.1	2.1
11	7	13	1.9	1.9
12	8	15	1.9	1.9
13	9	16	1.8	1.8
14	9	22	2.4	2.5
15	8	13	1.6	1.6
16	8	10	1.3	1.3
17	4†	14	3.5	3.0
18	8	9	1.1	1.1
19	12†	23	1.9	1.9
20	12†	21	1.8	1.7
21	16†	51	3.2	3.7
22	8	31	3.9	3.9
23	4†	3	0.8	1.1
24	4†	8	2.0	2.0
25	8	12	1.5	1.5

†: Sample size deviates more than 50% from the average size of 8 units.

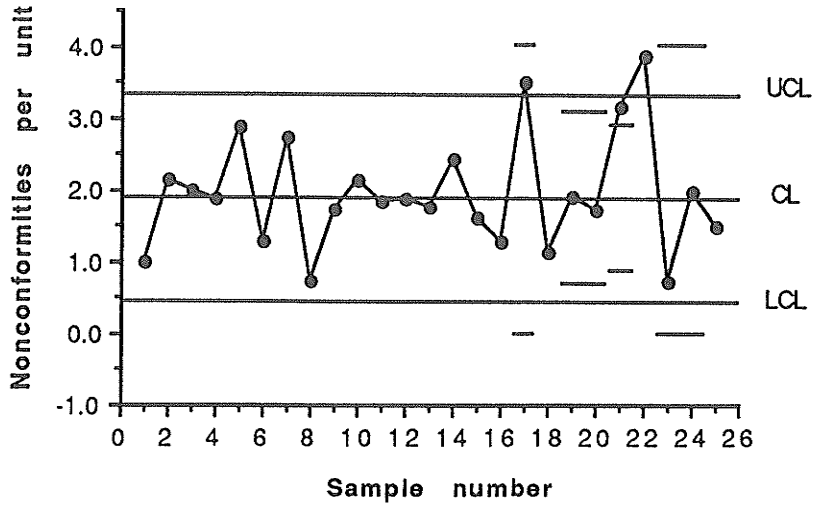

Figure 5.3. Conventional u chart with control limits based on the average sample size n=8; individual control limits for 6 especially large and small samples (17, 19, 20, 21, 23 and 24) are also given.

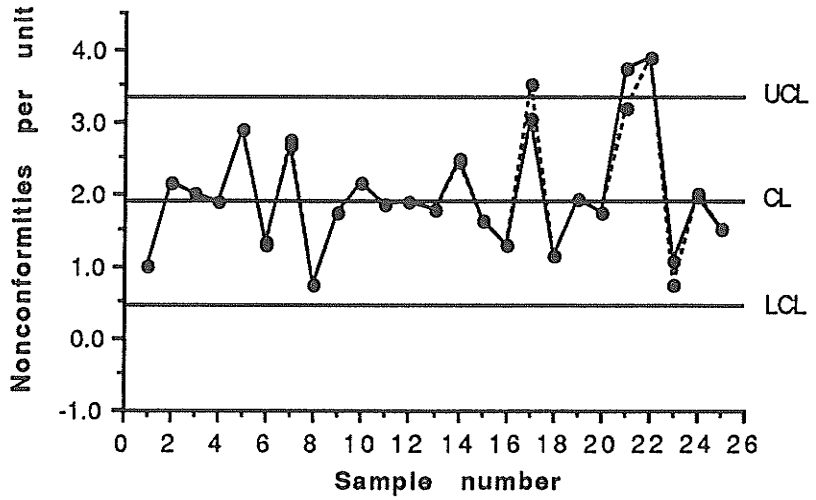

Figure 5.4. Weighted u chart with average size n=8; the dashed lines are the u_i values that differ from the corresponding u_i*.

5.5. How to Choose n

Although the selection of the common sample size n on which the transformed control limits will be based could be arbitrary, an appropriate n will simplify the construction of the new chart. Here three rules to select n for the new p chart are recommended. These rules can also be applied to the new u chart.

Average Sample Size

A straightforward choice for n is the average of all sample sizes. The use of the average sample size is familiar to practitioners. The advantage of this selection is that in many applications the range of minus to plus 25% of the average sample size usually includes most sample sizes. As shown earlier in Example 5.1, the differences between most of the p_i^* and the p_i are so small that they can be neglected.

Most Common Sample Size

Another choice for n is the most common sample size. Since if $n_i=n$ then $p_i^*=p_i$, this selection reduces computational work to obtain the exact p_i^* values. For example, for 15 samples of sizes 100, 80, 120, 100, 110, 120, 100, 100, 90, 90, 100, 120, 120, 100, 100, the average sample size is 103, but the most common sample size is 100. If 100 is selected as n, there are only 8 out of 15 samples for which the p_i^* values need to be calculated.

Maximum Sample Size

We mentioned earlier that the p_i^* could be less than 0 or greater than 1 when n_i is unusually large compared to n. In this case, the maximum sample size n will eliminate the problem. Furthermore, the new LCL can be set at 0 if LCL<0 and the new UCL can be set at 1 if UCL>1. For the proceeding 15 samples the maximum sample size is 120.

5.6. Comment

New p and u charts based on transformations p_i^* of the fraction nonconforming p_i and u_i^* of the number of nonconformities per unit u_i, called the weighted attribute charts,

are proposed. These transformed values in general are very close to the original p_i and u_i values. For all the samples in the control charts, the standard deviations of the p_i*'s and u_i*'s remain constant. Consequently, the control limits remain constant and the charts can be easily visualized. Furthermore, since the p_i*'s and u_i* 's are measured in terms of the same degree of accuracy for all samples, non-random patterns and trends can be more meaningfully interpreted and identified. Practitioners who still feel more comfortable using traditional control charts based on the average sample size, can simply plot the p_i* or u_i* values for samples with specially large or small size, or those samples which potentially may be beyond the control limits or which show a non-random pattern or trend.

6. Attribute Control Charts for Short-Run Processes

6.1. Introduction

A conventional attribute control chart is usually constructed on the basis of 20 to 25 samples (also called subgroups). The control limits are set at the mean of an appropriate statistic plus and minus 3 standard deviations. In many applications, unfortunately, there are not sufficient subgroups available to accurately estimate the 3-standard-deviation control limits. This may occur when short runs are involved due to high-speed operations or short production runs (Proschan and Savage, 1960; Koons and Luner, 1988; Bothe, 1989). Here we define a short run as a small number of samples obtained from a process. In general, the conventional control charts are not effective tools for dealing with the short-run situations. First, their control limits are not very accurate estimates when only a small number of samples are available at the start-up stage of a process. Second, it is difficult to use them to plot multiple short-run data (i.e., a sequence of short runs obtained under a sequence of different sets of conditions), because a separate control chart is needed for each short run. A short-run process needs new charting techniques that provide real-time control and monitoring of the process so that appropriate actions can be taken at the point where the process is out of control.

Quesenberry (1991a, b) proposed Q control charts for attribute short-run processes. In his approach, the original statistics of interest are transformed, using non-linear transformations, into normalized values, called Q statistics, that approximately have the standard normal distribution with mean 0 and standard deviation 1. The Q statistics are then plotted on the Q chart with UCL=3, LCL=-3, and CL=0. The significance of this approach is that the Q statistic is computed for each sample as it is obtained using the accumulated sample data. Thus, the process control and monitoring can in most cases begin at the second sample, far earlier than a traditional control chart approach that usually requires 20

to 25 samples. In addition, the Q charts can be applied to multiple short-run situations by simply plotting run after run.

In this chapter we propose simple standardizations of attribute statistics. The purpose of standardization is to achieve constant control limits for the standardized statistics. For practitioners who are familiar with traditional attribute control charts, they will find that the standardized charts introduced here are natural extensions of the traditional attribute charts to the short-run case. Although the standardizations are slightly less accurate than the nonlinear transformations to the standard normal distribution discussed by Quesenberry (1991a, b), their conceptual simplicity and ease of implementation are the key advantages over the nonlinear transformations. Furthermore, the implementation of these charts does not require a specifically designed computer program.

In section 6.2 we introduce a standardized chart for the control of fraction nonconforming. Sections 6.3 and 6.4 present standardized charts for controlling the number of nonconformities. In section 6.5 we compare the proposed standardized charts with the Q charts and the conventional standardized charts. Two examples are given in section 6.6. A *standardized chart* has

$$\begin{aligned} \text{UCL} &= 3, \\ \text{CL} &= 0, \\ \text{LCL} &= -3. \end{aligned} \tag{6.1}$$

6.2. The Q Charts for Attribute Processes

Quesenberry (1991a, b) devised Q control charts for the binomial and Poisson processes. Since the binomial and Poisson variables are not normally distributed, he uses various transformations to normalize the sample statistics so that they approximately have the normal distribution with mean 0 and standard deviation 1. These transformed statistics, called Q statistics, can then be plotted on the Q chart. To briefly describe Quesenberry's approach, the following notation is required.

Notation:

$\Phi(.)$ -- The cumulative standard normal distribution function.

$\Phi^{-1}(.)$ -- The inverse of the cumulative standard normal distribution function.

$B(x; n, p)$- - The cumulative binomial distribution function with parameters n and p.

$H(x; n, N_1, N_2)$ -- The cumulative hypergeometric distribution function with parameters n, N_1, and N_2.

$P(x; \lambda)$ -- The cumulative Poisson distribution function with parameter λ.

The Q statistics for fraction nonconforming

Let x_k be the number of nonconforming items in the kth sample of size n_k. When the process is in control, x_k will have a binomial distribution with parameters n_k and p, where p is the expected fraction of nonconforming items.

Case I: p is known.

The Q statistics are obtained by computing

$$u_k = B(x_k; n_k, p),$$

$$Q_k = \Phi^{-1}(u_k), \qquad k=1, 2, ...$$

These Q_k are plotted on the standardized chart with UCL=3, CL=0, and LCL=-3.

Case II: p is unknown.

Let

$$N_k = n_1 + ... + n_k,$$

and

$$t_k = x_1 + ... + x_k.$$

At time k, compute

$$u_k = H(x_k; t_k, n_k, N_{k-1}),$$

and

$$Q_k = \Phi^{-1}(u_k), \qquad k=2, 3, ...$$

The hypergeometric distribution is used because it is the UMVUE (uniformly minimum variance unbiased estimator) of the binomial distribution function in Case I (see Quesenberry, 1991a).

The Q charts for the number of nonconformities

Let x_k be the number of nonconformities in the kth sample of size n_k with u being the expected number of nonconformities per unit when the process is operating stably.

Case I: u is known.

Compute

$$u_k = P(x_k; n_k u),$$

and

$$Q_k = \Phi^{-1}(u_k), \qquad k=1, 2, ...$$

Case II: u is unknown.

Let

$$t_k = x_1 + ... + x_k.$$

Then, compute

$$u_k = B(x_k; t_k, n_k/N_k),$$

and

$$Q_k = \Phi^{-1}(u_k), \qquad k=2, 3, ...$$

The binomial distribution function is used because $B(x_k; t_k, n_k/N_k)$ is the UMVUE of $P(x_k; n_k u)$ (see Quesenberry, 1991b).

6.3. Standardized Chart for Fraction Nonconforming

Let (n_k, x_k) be the sample size and number of nonconforming items in the kth sample that is taken at time k, k=1, 2, ... The sample fraction nonconforming is $p_k=x_k/n_k$.

Case I: p is known.

In this case, the true fraction nonconforming is known. The standardized fraction nonconforming is defined as follows

$$z_k = \frac{\sqrt{n_k}\,(p_k - p)}{\sqrt{pq}}, \quad \text{where } q=1-p, \quad k=1, 2, .. \quad .. \qquad (6.2)$$

The control chart for this case is in fact the conventional standardized p chart. If the sample size is constant and equal to n, then the statistic in (6.2) becomes

$$z_k = \frac{\sqrt{n}(p_k - p)}{\sqrt{pq}}, \qquad k= 1, 2, ...$$

It is well known in statistical theory that the standardized statistic z_k has approximately a normal distribution with mean 0 and standard deviation 1 when n_k is large. Therefore, the standardized fraction nonconforming can be plotted on the standardized chart (6.1).

When the process switches to a new production run with a different process fraction nonconforming, say p', the above procedure will compute the standardized fraction nonconforming with p' replacing p in (6.2). It has the same approximate standard normal distribution that any other short run. This important fact means that we can plot the standardized statistics from different runs in the time sequence in which each sample is taken on the same standardized chart. In this way, the overall production process of various runs can be plotted and examined on a single chart with constant control limits.

Case II: p is unknown.

In practice, the true fraction nonconforming is frequently unknown and has to be estimated from sample data. Define the following accumulated quantities

$$N_k = n_1 + n_2 + ... + n_k$$

and

$$\bar{p}_k = (n_1p_1 + ... + n_kp_k)/N_k = (x_1 + ... + x_k)/N_k.$$

In fact, $\bar{p}_k$ can also be written into an updating formula as the weighted average of $\bar{p}_{k-1}$ and p_k as follows

$$\bar{p}_k = (N_{k-1}/N_k)\bar{p}_{k-1} + (n_k/N_k)p_k.$$

When the process is in control, the true fraction nonconforming can be estimated by $\bar{p}_k$ at time k. The standardized fraction nonconforming is defined as follows

$$z_k = \sqrt{\frac{N_{k-1}}{N_k}}\frac{\sqrt{n_k}(p_k-\bar{p}_{k-1})}{\sqrt{\bar{p}_{k-1}\bar{q}_{k-1}}}, \quad \text{where } \bar{q}_{k-1}=1-\bar{p}_{k-1}, \quad k=2, 3, ... \tag{6.3}$$

The test of control starts at time k=2.

If the subgroup size is constant and equal to n, then

$$\bar{p}_k = [(k-1)/k]\bar{p}_{k-1} + (1/k)p_k$$

and

$$z_k = \sqrt{\frac{k-1}{k}}\frac{\sqrt{n}(p_k-\bar{p}_{k-1})}{\sqrt{\bar{p}_{k-1}\bar{q}_{k-1}}}, \quad k=2, 3, ... \tag{6.4}$$

Again the standardized statistic z_k in (6.3) or (6.4) has approximately a standard normal distribution and can be plotted on the standardized chart. The reason that the z_k in (6.4) now has a multiply factor $\sqrt{k/(k-1)}$ can be seen by computing the following variance

$$\begin{aligned} \text{Var}(p_k-\bar{p}_k) &= [(k-1)/k]^2\text{Var}(p_k-\bar{p}_{k-1}) \\ &= [(k-1)/k]^2\{pq/n + pq/[(k-1)n]\} \\ &= (k-1)pq/(kn). \end{aligned}$$

The case for unequal sample size in (6.3) is similar.

The calculation of the standardized statistics begins with sample 2. As a new sample is obtained, the estimate of the true fraction nonconforming is updated. In this way, the process is being monitored in real-time and immediate action can be taken if the control chart indicates an out-of-control signal. When a different run is introduced, the standardized values can be plotted on the same standardized chart. The procedures described in the next two sections are similar to that discussed in this section.

6.4. Standardized Chart for Number of Nonconformities

Let c_k be the number of nonconformities in the kth sampled unit with c being the expected number of nonconformities when the process is operating stably.

Case I: c is known.

Define the z_k statistic as follows

$$z_k = \frac{c_k - c}{\sqrt{c}}, \qquad k=1, 2, ... \tag{6.5}$$

Note that the above standardized statistics are not even approximately normal. Since the conventional c chart is given by UCL=c+3$\sqrt{c}$, CL=c, and LCL =c-3$\sqrt{c}$, the standardized statistics will be plotted on the standardized chart. Except for the scale of the chart, the nature of the standardized chart in this case is equivalent to that of the conventional c chart.

Case II: c is unknown.

This is a more practical case in which c is estimated from sample data. Denote

$$\bar{c}_k = (c_1 + ... + c_k)/k,$$

i.e., $\bar{c}_k$ is the accumulated average number of nonconformities up to sample k. It can also be expressed as the weighted average of $\bar{c}_{k-1}$ and c_k

$$\bar{c}_k = [(k-1)/k]\bar{c}_{k-1} + (1/k)c_k.$$

The z_k statistic is defined by

$$z_k = \sqrt{\frac{k-1}{k}}\ \frac{(c_k - \bar{c}_{k-1})}{\sqrt{\bar{c}_{k-1}}}, \qquad k=2, 3, ... \tag{6.6}$$

This standardized statistic is plotted on the standardized chart. The monitoring of the process can be initiated at the second sample when c is unknown.

6.5. Standardized Chart for Number of Nonconformities Per Unit

In last section the sample size considered is exactly equal to one unit. In practice, several units are often combined into a single sample to increase the area of exposure to the occurrence of nonconformities. Let (n_k, c_k) be the number of units and the total number of nonconformities in the kth independent sample, k=1, 2, ... The number of nonconformities per unit in the kth sample is $u_k=c_k/n_k$. Let u denote the true number of nonconformities per unit when the process is in control.

Case I: u is known.

For the number of nonconformities per unit, the conventional u chart is given by $UCL=u+3\sqrt{u/n_k}$, CL=u, and $LCL=u-3\sqrt{u/n_k}$. The standardized statistic is now defined by

$$z_k = \frac{\sqrt{n_k}\,(u_k-u)}{\sqrt{u}}, \qquad k=1, 2, ... \tag{6.7}$$

The standardized statistic is plotted on the standardized chart. Again the standardized chart for (6.7) is equivalent to the conventional u chart except for the scale of the chart. If the sample size is constant and equal to n, then the z_k statistic becomes

$$z_k = \frac{\sqrt{n}(u_k-u)}{\sqrt{u}}, \qquad k=1, 2, ... \tag{6.8}$$

Case II: u is unknown.

When the true number of nonconformities per unit is unknown, it is estimated from sample data. Let

$$\begin{aligned}\bar{u}_k &= (n_1u_1 + ... + n_ku_k)/N_k \\ &= (c_1 + ... + c_k)/N_k,\end{aligned}$$

where $N_k = n_1 + ... + n_k$. Here $\bar{u}_k$ is the accumulated quantity up to sample k, and it can be expressed as the weighted average of $\bar{u}_{k-1}$ and u_k

$$\bar{u}_k = [N_{k-1}/N_k]\bar{u}_{k-1} + (n_k/N_k)u_k.$$

The z_k statistic now is

$$z_k = \sqrt{\frac{N_{k-1}}{N_k}}\,\frac{\sqrt{n_k}\,(u_k-\bar{u}_{k-1})}{\sqrt{\bar{u}_{k-1}}}, \qquad k=2, 3, ... \tag{6.9}$$

If the sample size is constant and equal to n, then

$$\bar{u}_k = [(k-1)/k]\bar{u}_{k-1} + (1/k)u_k$$

and the statistic z_k becomes

$$z_k = \sqrt{\frac{k-1}{k}}\,\frac{\sqrt{n}(u_k-\bar{u}_{k-1})}{\sqrt{\bar{u}_{k-1}}}, \qquad k=2, 3, ... \tag{6.10}$$

6.6. Comparisons of the Standardized, Q and Conventional Standardized Charts

As we pointed out earlier, the advantage of the standardized charts is their simplicity. The transformation employed is essentially linear, and can be easily calculated and implemented. The control limits are set at 3 and -3.

We consider the standardized chart for fraction nonconforming. When p is unknown, the z_k statistic with constant sample size is

$$z_k = \sqrt{\frac{k-1}{k}} \frac{\sqrt{n}(p_k - \bar{p}_{k-1})}{\sqrt{\bar{p}_{k-1}\bar{q}_{k-1}}}. \tag{6.4}$$

The distribution of this statistic is not normal, however it is approximately normal when n is large. Tables 6.1, 6.2 and 6.3 show the appropriate probabilities of falsely indicating out-of-control signals when the process is in fact in control. The first, third and fifth columns provide the probabilities below the lower control limit, -3; while the second, fourth and sixth columns give the probabilities above the upper control limit, 3. The case of k equal to infinity is equivalent to the case (6.2), i.e., $\bar{p}_{k-1}$ is replaced by the known p. We call this the conventional standardized chart. For the other cases of k values, $\bar{p}_{k-1}$ is the average of the first k-1 sample p values. If n is large enough, these probabilities will be close to .0027/2=.00135.

These probabilities are computed by simulation. For each combination of n and p, 80,000 observations are generated from the binomial distribution with n and p, the sample fractions are obtained by dividing the individual observations by the sample size n. For every k samples, the z_k statistic is computed and counted whether it falls below LCL, above UCL, or between LCL and UCL. Each entry of the tables is obtained by dividing the count by the total number of z_k statistics.

Table 6.1. Probabilities < LCL and > UCL for p=.01 and n=200, 600 and 1000

n	200		600		1000	
k	<LCL	>UCL	<LCL	>UCL	<LCL	>UCL
3	.00000	.03326	.00000	.01684	.00000	.01388
4	.00000	.02525	.00000	.01305	.00000	.00880
5	.00000	.02756	.00000	.00975	.00000	.00738
6	.00000	.01725	.00000	.00773	.00000	.00683
7	.00000	.01312	.00000	.00814	.00000	.00735
8	.00000	.01380	.00000	.00650	.00000	.00500
9	.00000	.01316	.00000	.00686	.00000	.00484
inf. (con.)	.00000	.00430	.00000	.00345	.00004	.00329

Table 6.2. Probabilities < LCL and > UCL for p=.05 and n=100, 300 and 500

n	100		300		500	
k	<LCL	>UCL	<LCL	>UCL	<LCL	>UCL
3	.00000	.01796	.00011	.00833	.00056	.00720
4	.00000	.01305	.00025	.00675	.00040	.00450
5	.00000	.00969	.00019	.00456	.00031	.00381
6	.00000	.00795	.00015	.00533	.00045	.00488
7	.00000	.00761	.00018	.00429	.00061	.00280
8	.00000	.00720	.00030	.00430	.00010	.00300
9	.00000	.00675	.00023	.00349	.00034	.00371
inf. (con.)	.00000	.00427	.00016	.00257	.00046	.00270

Table 6.3. Probabilities < LCL and > UCL for p=.10
and n=100, 300 and 500

n	100		300		500	
k	<LCL	>UCL	<LCL	>UCL	<LCL	>UCL
3	.00000	.01020	.00038	.00589	.00064	.00427
4	.00000	.00680	.00040	.00470	.00095	.00390
5	.00000	.00519	.00044	.00363	.00094	.00325
6	.00000	.00503	.00030	.00390	.00098	.00270
7	.00000	.00438	.00000	.00420	.00070	.00254
8	.00000	.00420	.00050	.00270	.00050	.00320
9	.00000	.00428	.00034	.00304	.00056	.00180
inf. (con.)	.00003	.00198	.00057	.00242	.00058	.00169

From these three tables we see that in general the probabilities below LCL are much less than .0027/2 or .0013, and that the probabilities above UCL are larger than .0013, even for the last rows which represent the conventional case of known p. This fact is due to the skewness of the binomial distribution for small p. The tables also show that the standardized chart for unknown p is almost as effective as the conventional standardized chart for known p for moderate or large values of n and p. Quesenberry (1991a) provided the probabilities below LCL and above UCL of the Q chart as well as the conventional p chart for known p. His result shows that the Q statistic is closer to normal than the sample fraction.

To compare the average run length (ARL) of the standardized chart with that of the Q chart and conventional standardized p chart, Table 6.4 is provided. The simulation is similar to that of Tables 6.1, 6.2 and 6.3. The difference is that we now count the number of subgroups until the standardized statistic (or Q statistic) falls below or above the control

limits, -3 and 3, respectively. Each entry in this table is the ARL or average number of subgroups required to find a point outside of the control limits when the process is in control. The rows associated with letter z display the ARLs for the standardized chart (6.4), and those associated with letter Q give the ARLs for the corresponding Q chart. The ARLs for the conventional standardized p chart are also obtained and identified as "con.".

Table 6.4 shows that the ARLs are quite close for the three types of control charts. Under a normal distribution, the ARL is 1/.0027 or 370.4. In general, the ARL of the Q chart is closer to 370.4 than that of the other two charts, and it is especially true for small n and p. However, for moderate and large values of n and p, the difference in ARLs is not very significant for practical purposes.

Table 6.4. ARL for Various n and p

		p		
	n	.01	.05	.10
z	100	110.4	235.2	306.0
Q		205.5	359.6	415.7
con.		50.4	234.3	497.5
z	200	153.9	289.6	318.6
Q		237.6	336.8	465.4
con.		232.6	269.3	294.1
z	300	180.8	295.9	348.3
Q		311.4	324.2	373.1
con.		207.5	286.1	334.4
z	500	276.6	334.7	373.1
Q		280.2	361.5	375.3
con.		289.9	316.5	440.5
z	inf.	370.4	370.4	370.4
Q		370.4	370.4	370.4
con.		370.4	370.4	370.4

6.7. Examples

Example 6.1

In this example, the multiple-run situation has two short runs. For the first run, samples of size 500 were first taken from the process in a time sequence. In the second run samples of size 350 were obtained from the process under a different operating condition with a higher p. We here compute separately the standardized values for each short run but plot them together on the same standardized chart. The data and calculations are given in Table 6.5. The number and fraction of nonconforming items in a sample are listed in columns 3 and 4. The up-to-date average fraction nonconforming is given in column 5. Column 6 shows the standardized statistic z_k.

Table 6.5. Data for Example 6.1

Sample k	n_k	x_k	p_k	$\bar{p}_k$	z_k
1	500	6	0.012	0.012	
2	500	2	0.004	0.008	-1.16
3	500	4	0.008	0.008	0.00
4	500	7	0.014	0.010	1.30
5	500	1	0.002	0.008	-1.55
6	500	5	0.010	0.008	0.46
7	500	8	0.016	0.009	1.75
8	500	7	0.014	0.010	0.99
9	500	3	0.006	0.010	-0.85
1	350	4	0.011	0.011	
2	350	1	0.003	0.007	-1.07
3	350	7	0.020	0.011	2.33
4	350	4	0.011	0.011	0.00
5	350	6	0.017	0.013	0.90
6	350	5	0.014	0.013	0.26

The standardized chart, with UCL=3, CL=0, and LCL=-3, is shown in Figure 6.1. Run 1 is first plotted point by point when samples 1 to 9 are taken. Run 2 of samples 10 to 15 is then plotted in the same manner immediately after Run 1. No apparent out-of-control signal can be found from either run.

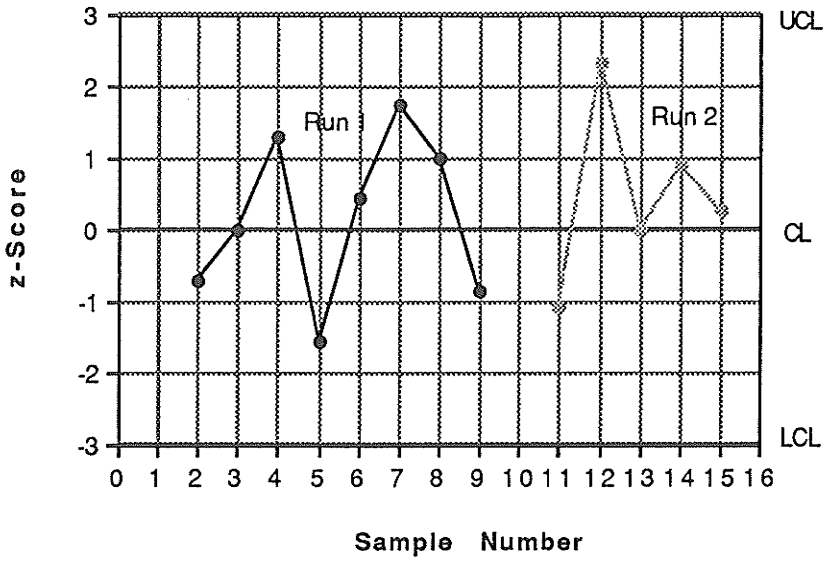

Figure 6.1. Standardized Chart for Example 6.1.

Example 6.2

The data are from the Ford Manual (1987, p.47a). We here construct the standardized chart for part of the original data set. The data are given in Table 6.6. Columns 2 and 3 give the sample size and the number of nonconformities. Columns 4 and 5 give the number and the up-to-date average number of nonconformities per unit. The z_k statistic is showed in column 6.

The standardized chart for this example is given in Figure 6.2. The standardized values are plotted one by one in the time sequence in which each sample is obtained. Clearly, sample 21 indicates an out-of-control situation, and further investigation is required.

Table 6.6. Data for Example 6.2

Sample k	n_k	c_k	u_k	$\bar{u}_k$	z_k
1	8	8	1.00	1.00	0.00
2	8	17	2.13	1.56	2.25
3	9	18	2.00	1.72	0.84
4	8	15	1.88	1.76	0.29
5	8	23	2.88	1.98	2.14
6	7	9	1.29	1.88	-1.20
7	7	19	2.71	1.98	1.51
8	8	6	0.75	1.83	-2.31
9	8	14	1.75	1.82	-0.15
10	8	17	2.13	1.85	0.61
11	7	13	1.86	1.85	0.02
12	8	15	1.88	1.85	0.05
13	9	16	1.78	1.84	-0.15
14	9	22	2.44	1.89	1.27
15	8	13	1.63	1.88	-0.53
16	8	10	1.25	1.84	-1.25
17	4	14	3.50	1.89	2.42
18	8	9	1.13	1.84	-1.52
19	12	23	1.92	1.85	0.18
20	12	21	1.75	1.84	-0.24
21	16	51	3.19	1.96	3.79

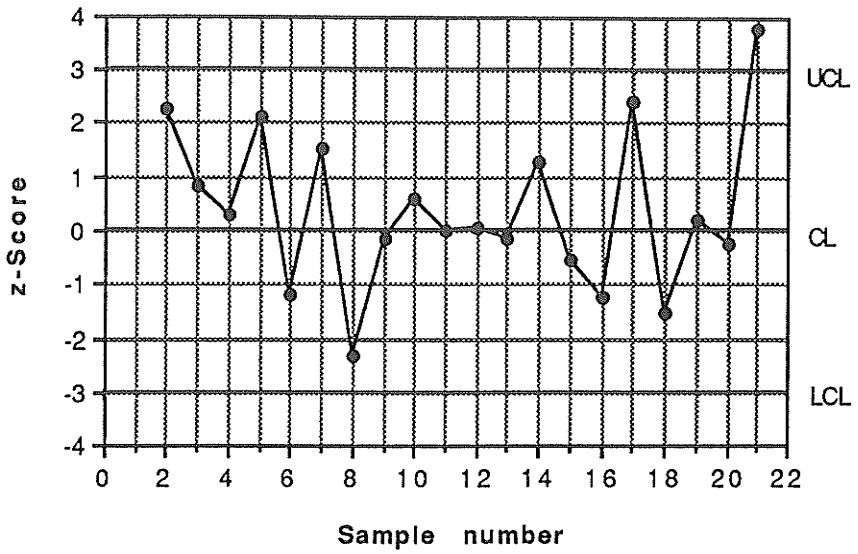

Figure 6.2. Standardized Chart for Example 6.2.

6.8. Concluding Remark

Attribute control charts are effective tools for studying the statistical control of attribute processes. However, the conventional control charts require at least 20 to 25 samples for accurately computing the control limits. The standardized attribute charts for short runs proposed in this chapter are natural extensions of the traditional control charts that are widely used by practitioners. One of the two major advantages of these charts is that they permit real-time process control in which we can start charting from as early as the second sample. The other is that we can present multiple short runs under different operating conditions on a single standardized chart. Furthermore, the standardized charts are easy to understand and simple to implement; these are features important to practitioners. The simulation studies have shown that the standardized charts are practically as effective as the Q charts for moderate and large sample size.

7. Variable Control Charts for Short-Run Processes

7.1. Introduction

Variable control charts are used to study the statistical control of processes for continuous quality characteristics. For example, the $\bar{x}$ and R charts have been used together to control the process average and variability. Samples (also called subgroups) are taken periodically, and the sample mean $\bar{x}$ and range R are calculated for each subgroup. When the expected subgroup mean μ and range R' are known, the control limits are set at UCL=μ + A_2R', LCL=μ - A_2R' for the $\bar{x}$ chart, and UCL=D_4R', LCL=D_3R' for the R chart, where A_2, D_3, and D_4 are appropriate control limit factors. If a plotted point is outside of its control limits, an out-of-control signal is indicated. In practice, μ and R' are often unknown and to be estimated from the subgroups. This is done by taking a number of subgroups, usually as many as 20 to 25, calculating the grand average $\bar{\bar{x}}$ and the average range $\bar{R}$, and setting the control limits at UCL=$\bar{\bar{x}}$ + $A_2\bar{R}$, LC=$\bar{\bar{x}}$ - $A_2\bar{R}$ for the $\bar{x}$ chart, and UCL=$D_4\bar{R}$, LCL=$D_3\bar{R}$ for the R chart. The center lines will be $\bar{\bar{x}}$ for the $\bar{x}$ chart and $\bar{R}$ for the R chart.

As we pointed out in chapter 6, in many manufacturing and other situations there exist short-run processes. For short runs with continuous quality characteristics, the conventional variable control charts are not effective tools dealing with these short-run cases, because they require at least 20 to 25 subgroups for calculating reliable control limits. There is a need for new charting techniques that are effective as well as easy to implement.

In spite of their significance in practical applications in modern industry, relatively little research has been done for short runs. Also short-run control charts currently used in industry are not well documented in the literature. Sections 7.2, 7.3 and 7.4 summarize several useful existing charts. New charts developed in this chapter are given in sections

7.5 and 7.6; they are compared with the existing charts in section 7.7. An example is given in section 7.8. It is assumed in this chapter that the process distribution is normal.

7.2. Control Charts with Modified Control Limit Constants

In low-volume manufacturing operations where sample data are obtained slowly, it is natural to calculate preliminary control limits from a small number of initial subgroups for assessing the statistical control of a process in the early stage, with subsequent modifications of those control limits as more samples are obtained. This approach provides early information about the process so that quicker action can be taken to improve the process.

Hillier (1969) proposed a two-stage approach for computing control limits for the $\bar{x}$ and R charts such that these control limits will be statistically reliable regardless of the number of subgroups obtained. In the first stage, control limits are calculated from the short run to assess whether the process was in control while the initial samples were being taken. After using these limits to establish control, the second stage then recalculates the control limits to make them appropriate to test whether the process remains in control when new samples are taken in the future.

Suppose a short run of length m was taken, and $\bar{x}$ and R are calculated for each of the m subgroups. The average range $\bar{R}$ is then calculated, and the R chart is constructed first. The first stage control limits for the R chart are set at

$$UCL = D_{4F}\bar{R},$$

$$CL = \bar{R},$$

$$LCL = D_{3F}\bar{R},$$

where D_{3F} and D_{4F} are modified control limit constants which depend on the number of subgroups used as well as the sample size. If there are points lying outside of the the limits and are identified as out-of-control subgroups, the average range, along with the control limits, will be recalculated by dropping the out-of-control subgroups. This process should

be repeated until no out-of-control subgroups can be found. Then the grand average $\bar{\bar{x}}$ is computed with the in-control subgroups, and the control limits for the $\bar{x}$ chart are set at

$$UCL = \bar{\bar{x}} + A_{2F}\bar{R},$$

$$CL = \bar{\bar{x}},$$

$$LCL = \bar{\bar{x}} - A_{2F}\bar{R},$$

where A_{2F} is the modified control limit constant which also depends on the number of subgroups used as well as the sample size. Again, if there are sample points lying outside of the control limits and are identified as an out-of-control subgroup, the control limits will be recalculated by dropping the out-of-control subgroups.

Now suppose that the initial testing has been done and a number of in-control subgroups have been identified. To test whether the process remains in control in the future, one should notice that a future $\bar{x}$ and R will not be included in $\bar{\bar{x}}$ and $\bar{R}$. Therefore, the constants used for calculating the first-stage control limits will no longer be applied, and the appropriate control limit constants, denoted as A_{2S}, D_{3S}, and D_{4S}, should be determined. Then the second-stage control limits would be set at

$$UCL = D_{4S}\bar{R},$$

$$CL = \bar{R},$$

$$LCL = D_{3S}\bar{R},$$

for the R chart and at

$$UCL = \bar{\bar{x}} + A_{2S}\bar{R},$$

$$CL = \bar{\bar{x}},$$

$$LCL = \bar{\bar{x}} - A_{2S}\bar{R},$$

for the $\bar{x}$ chart, respectively.

Hillier (1969) provided appropriate formulas for computing the above control limit constants, and tables were also given for various numbers and sizes of subgroups. Pyzdek (1990) provided a table of the same modified control limit factors. Pyzdek (1992) republicized the two-stage approach and suggested some modifications.

Yang and Hillier (1970) examined the two-step charting technique applied to the mean and variance control charts. Tables and formulas of modified control limit constants were also provided in their paper.

7.3. Transformed Control Charts

Bothe (1989) introduced the transformed $\bar{x}$ and R charts for short runs. Assuming that the expected subgroup average μ and range R' are known, he suggested the transformed statistics

$$\bar{x} \text{ PLOT POINT} = \frac{\bar{x}-\mu}{R'}$$

and

$$\text{R PLOT POINT} = \frac{R}{R'}$$

be plotted on the transformed $\bar{x}$ and R charts with the control limits set at

$$UCL = A_2,$$

$$CL = 0,$$

$$LCL = - A_2.$$

for the transformed $\bar{x}$ chart and at

$$UCL = D_4,$$

$$CL = 1,$$

$$LCL = D_3.$$

for the transformed R chart, respectively. These control limits are constant and apply to multiple short runs.

The major disadvantage of this approach is that the μ and R' have to be determined from prior information on the same or similar operations.

7.4. The Q Charts

To obtain control charts right from the start of a process and monitor the process in real time, Quesenberry (1991c) proposed Q control charts for mean μ and variance σ^2. His

approach is to transform the values of conventional statistics to normalized values so that all points can be plotted on the same chart with the control limits set at UCL=3 and LCL=-3, and CL=0. These Q charts can naturally be applied to multiple short runs by simply plotting a newer run right after the last run. Since all Q statistics are normalized, the statistic for variance can even be plotted together with that for mean. However, it is usually recommended that the Q statistics for mean and variance be plotted on separate charts.

The notation given below is used in this section.

$\Phi(.)$ -- The cumulative standard normal distribution function.

$\Phi^{-1}(.)$ -- The inverse of the cumulative standard normal distribution function.

$G_r(.)$ -- The cumulative student t distribution function with r degrees of freedom.

$H_r(.)$ -- The cumulative chi-square distribution function with r degrees of freedom.

$F_{r,v}(.)$ -- The cumulative F distribution function with r and v degrees of freedom.

For the case of control charts based on the mean $\bar{x}_k$ and variance s_k^2 from a subgroup of size n_k, define the following accumulated quantities

$$N_k = n_1 + n_2 + \ldots + n_k,$$

$$\bar{\bar{x}}_k = (n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k)/N_k,$$

$$\bar{s}_k^2 = \{(n_1-1)s_1^2 + (n_2-1)s_2^2 + \ldots + (n_k-1)s_k^2\}/(N_k-k),$$

where $\bar{x}_k$ the s_k^2 are the commonly used sample mean and variance for the kth subgroup, respectively.

<u>The Q statistics for μ:</u>

Case I: $\mu=\mu_0$, $\sigma=\sigma_0$

$$Q_k(\bar{x}_k) = \sqrt{n_k}(\bar{x}_k - \mu_0)/\sigma_0, \quad k=1, 2, \ldots.$$

Case II: μ unknown, $\sigma=\sigma_0$

$$Q_k(\bar{x}_k) = [n_k N_{k-1}/N_k]^{1/2}(\bar{x}_k - \bar{\bar{x}}_{k-1})/\sigma_0, \quad k=2, 3, \ldots.$$

Case III: $\mu=\mu_0$, σ unknown

Let

$$\bar{s}^2_{0,k} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\mu_0)^2/N_k.$$

Then

$$Q_k(\bar{x}_k) = \Phi^{-1}\{G_{Nk}[\sqrt{n_k}(\bar{x}_k - \mu_0)/\bar{s}_{0,k}]\}, \qquad k=2, 3, ...$$

Case IV: μ, σ unknown

Let

$$w_k = [n_k N_{k-1}/N_k]^{1/2}(\bar{x}_k - \bar{\bar{x}}_{k-1})/\bar{s}_k.$$

Then

$$Q_k(\bar{x}_k) = \Phi^{-1}\{G_{Nk-k}(w_k)\}, \qquad k=2, 3, ...$$

The Q statistics for σ^2:

Case V: $\sigma=\sigma_0$

$$Q_k(s_k^2) = \Phi^{-1}\{H_{nk-1}[(n_k-1)s_k^2/\sigma_0^2]\}, \qquad k=1, 2, ...$$

Case V: σ unknown

Let

$$w_k = s_k^2/\bar{s}_{k-1}^2.$$

Then

$$Q_k(s_k^2) = \Phi^{-1}\{F_{n_{k-1},N_{k-1}-k+1}(w_k)\}, \qquad k=2, 3, ...$$

Quesenberry (1991) also provided Q statistics for the case of control charts based on individual observations. Define the following accumulated sample mean and variance

$$\bar{x}'_k = (x_1 + x_2 + ... + x_k)/k$$

and

$$s'^2_k = \{(x_1-\bar{x}'_k)^2 + ... + (x_k-\bar{x}'_k)^2\}/(k-1).$$

The Q statistics for μ:

Case I: $\mu=\mu_0$, $\sigma=\sigma_0$

$$Q_k(x_k) = (x_k - \mu_0)/\sigma_0, \qquad k=1, 2, ...$$

Case II: μ unknown, $\sigma=\sigma_0$

$$Q_k(x_k) = [(k-1)/k]^{1/2}(x_k - \bar{x}'_{k-1})/\sigma_0, \qquad k=2, 3, ...$$

Case III: $\mu=\mu_0$, σ unknown

Let

$$s^2_{0;k} = \{(x_1-\mu_0)^2 + ... + (x_k-\mu_0)^2\}/k.$$

Then

$$Q_k(x_k) = \Phi^{-1}\{G_{k-1}[(x_k - \mu_0)/s_{0;k-1}]\}, \qquad k=2, 3, ...$$

Case IV: μ, σ unknown

$$Q_k(x_k) = \Phi^{-1}\{G_{k-2}[(k-1)/k]^{1/2}(x_k - \bar{x}'_{k-1})/s'_{k-1}]\}, \qquad k=3, 4, ...$$

Q statistics for σ^2:

Let

$$R_k = x_k - x_{k-1}.$$

Case V: $\sigma=\sigma_0$

$$Q_k(R_k) = \Phi^{-1}\{H_1[R_k^2/(2\sigma_0^2)]\}, \qquad k = 2, 4, 6, ...$$

Case VI: σ unknown

$$Q_k(R_k) = \Phi^{-1}\{F_{1,v}[vR_k^2/(R_2^2+R_4^2+...+R_{k-2}^2)]\},$$

$$k=4, 6, ..., \text{ and } v=(k/2)-1.$$

The drawback of the above Q charts is their requirement of evaluatiing certain distribution functions to obtain normalized statistics. Their implementation can only be done with packaged computer program.

7.5. Standardized Charts Based on the Subgroup Means and Variances

The 3-standard-deviation control limits have been recognized as the most appropriate control limits for control charting since Shewhart invented control charts in the 1930's. The reason is that they provide the right balance, for any type of distribution, between the ability to detect promptly out-of-control points and the chance of falsely giving out-of-control signals when the process is in fact in control.

In this section, we propose to plot the standardized statistics with mean 0 and variance 1, on the standardized chart, regardless of their distributions. Specifically, let w_k be the kth sample statistic with known mean $E(w_k)$ and standard deviation $\sigma(w_k)$. Then the control limits are computed by

$$UCL = E(w_k) + 3\sigma(w_k),$$

$$CL = E(w_k),$$

$$LCL = E(w_k) - 3\sigma(w_k).$$

When the mean and standard deviation of w_k vary from sample to sample, the standardized statistic defined below

$$z_k = \{w_k - E(w_k)\}/\sigma(w_k),$$

will be a better statistic to be plotted because the control limits for z_k will always be set at

$$UCL = 3,$$

$$CL = 0,$$

$$LCL = -3.$$

The standardized chart can be applied to processes of multiple sources.

For the case of control charts based on the mean $\bar{x}_k$ and variance s_k^2 of subgroups of equal size n, define

$$\bar{\bar{x}}_k = (\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_k)/k$$

$$\bar{s}_k^2 = (s_1^2 + s_2^2 + \ldots + s_k^2)/k.$$

It is assumed the distribution of the process population is normal and sample size n>1. The z_k statistics for the various cases are given below.

<u>The standardized statistics for μ:</u>

Case I: $\mu=\mu_0$, $\sigma=\sigma_0$

Let $w_k = \bar{x}_k$. Then

$$E(w_k) = \mu_0,$$

$$V(w_k) = \sigma_0^2/n$$

and

$$z_k = \sqrt{n}\,(\bar{x}_k - \mu_0)/\sigma_0, \qquad k=1,\ 2,\ ....$$

Case II: μ unknown, $\sigma=\sigma_0$

Let $w_k = (\bar{x}_k - \bar{\bar{x}}_{k-1})/\sigma_0$. Then

$$E(w_k) = 0,$$

$$V(w_k) = k/[n(k-1)]$$

and

$$z_k = [n(k-1)/k]^{1/2}(\bar{x}_k - \bar{\bar{x}}_{k-1})/\sigma_0, \qquad k=2,\ 3,\ ....$$

The above two z_k statistics are identical to the Q statistics for the same cases.

Case III: $\mu=\mu_0$, σ unknown

Let $w_k = (\bar{x}_k - \mu_0)/\bar{s}_k$. Then

$$E(w_k) = 0,$$

$$V(w_k) = \frac{(nk-k)}{n(nk-k-2)}$$

and

$$z_k = \left(\frac{n(nk-k-2)}{nk-k}\right)^{1/2} \frac{\bar{x}-\mu_0}{\bar{s}_k}, \qquad k=1,\ 2,\ ....$$

Case IV: μ and σ unknown

Let $w_k = (\bar{x}_k - \bar{\bar{x}}_{k-1})/\bar{s}_k$. Then

$$E(w_k) = 0,$$

$$V(w_k) = \frac{k(nk-k)}{n(k-1)(nk-k-2)}$$

and

$$z_k = \left(\frac{n(k-1)(nk-k-2)}{k(nk-k)}\right)^{1/2} \frac{\bar{x}_k-\bar{\bar{x}}_{k-1}}{\bar{s}_k}, \qquad k=2,\ 3,\ ....$$

<u>The standardized statistics for σ^2</u>:

Case V: $\sigma=\sigma_0$

Let

$$w_k = s_k^2/\sigma_0^2.$$

Then $(n-1)w_k$ has a chi-square distribution with n-1 degrees of freedom. Thus,

$$E(w_k) = 1,$$

$$V(w_k) = 2/(n-1),$$

and

$$z_k = [(n-1)/2]^{1/2}(w_k - 1), \quad k=1, 2, ...$$

Case VI: σ unknown

Let

$$w_k = s_k^2/\bar{s}_{k-1}^2.$$

Then w_k has an F distribution with n_k-1 and n(k-1)-k+1 degrees of freedom, and

$$E(w_k) = \frac{n(k-1)-k+1}{n(k-1)-k-1},$$

$$V(w_k) = \frac{2[n(k-1)-k+1]^2(nk-k)-2}{(n-1)[n(k-1)-k-1]^2[n(k-1)-k-3]}.$$

The z_k statistic is

$$z_k = \{w_k - E(w_k)\}/[V(w_k)]^{1/2}, \qquad k=2, 3, ...$$

The standardized charts can be extended to the case of unequal subgroup size. We consider here the most common cases, IV and VI, where μ and σ are unknown. The other cases are very similar. Suppose that the size of the kth subgroup is n_k. As in section 7.4, define the following accumulated quantities

$$N_k = n_1 + n_2 + ... + n_k$$

$$\bar{\bar{x}}_k = (n_1\bar{x}_1 + n_2\bar{x}_2 + ... + n_k\bar{x}_k)/N_k$$

$$\bar{s}_k^2 = \{(n_1-1)s_1^2 + (n_2-1)s_2^2 + ... + (n_k-1)s_k^2\}/(N_k-k).$$

Case IV': μ and σ unknown, unequal sample size

Let $w_k = (\bar{x}_k - \bar{\bar{x}}_{k-1})/\bar{s}_k$. Then

$$E(w_k) = 0,$$

$$V(w_k) = \frac{N_k(N_k-k)}{(N_{k-1}n_k)(N_k-k-2)}$$

and

$$z_k = \left(\frac{n_k N_{k-1}(N_k-k-2)}{N_k(N_k-k)}\right)^{1/2} \frac{\bar{x}_k - \bar{\bar{x}}_{k-1}}{\bar{s}_k}, \qquad k=2, 3, ...$$

Case VI': σ unknown, unequal sample size

Let

$$w_k = s_k^2/\bar{s}_{k-1}^2.$$

Then w_k has an F distribution with n_k-1 and (N_{k-1}-k+1) degrees of freedom.

$$E(w_k) = \frac{N_{k-1}-k+1}{N_{k-1}-k-1},$$

$$V(w_k) = \frac{2(N_{k-1}-k+1)^2(N_k-k)-2}{(n_k-1)(N_{k-1}-k-1)^2(N_{k-1}-k-3)}.$$

The z statistic is

$$z_k = \{w_k - E(w_k)\}/[V(w_k)]^{1/2}, \qquad k=2, 3, ...$$

7.6. Standardized Charts Based on Individual Measurements

The standardized statistics can be computed for control charts based on individual observations. They are given for the following cases.

<u>The standardized statistics for μ:</u>

Case I: $\mu=\mu_0$, $\sigma=\sigma_0$

Let $w_k = x_k$. Then

$$E(w_k) = \mu_0,$$

$$V(w_k) = \sigma_0^2,$$

and

$$z_k = (x_k - \mu_0)/\sigma_0, \qquad k=1, 2, ...$$

Case II: μ unknown, $\sigma=\sigma_0$.

Let $w_k = x_k - \bar{x}'_{k-1}$. Then

$$E(w_k) = 0,$$

$$V(w_k) = k/(k-1)\sigma_0^2,$$

and

$$z_k = [(k-1)/k]^{1/2}(x_k - \bar{x}'_{k-1})/\sigma_0, \qquad k=2, 3, ...$$

The above two z_k statistics are identical to the Q statistics for the same cases.

Case III: $\mu=\mu_0$, σ unknown

Let

$$s^2_{0;k} = \{(x_1-\mu_0)^2+....+(x_k-\mu_0)^2\}/k,$$

and

$$w_k = (x_k - \mu_0)/s_{0;k-1}.$$

Then

$$E(w_k) = 0,$$

$$V(w_k) = (k-1)/(k-3),$$

and

$$z_k = \left(\frac{k-3}{k-1}\right)^{1/2} \frac{x_k-\mu_0}{s_{0;k-1}}, \qquad k=4, 5, ...$$

Case IV: μ, σ unknown

Let $w_k = (x_k - \bar{x}'_{k-1})/s'_{k-1}$. Then

$$E(w_k) = 0,$$

$$V(w_k) = \frac{k(k-2)}{(k-1)(k-4)},$$

and

$$z_k = \left(\frac{(k-1)(k-4)}{k(k-2)}\right)^{1/2} \frac{x_k-\bar{x}'_{k-1}}{s'_{k-1}}, \qquad k=5, 6, ...$$

<u>The standardized statistics for σ^2</u>:

Case V: $\sigma=\sigma_0$

Let

$$w_k = v_k^2/\sigma_0^2,$$

where $v_k=(1/2)^{1/2}\,|x_k - x_{k-1}|$. w_k has a chi-square distribution with 1 degree of freedom. Then

$$E(w_k) = 1,$$

$$V(w_k) = 2,$$

and

$$z_k = (1/2)^{1/2}(w_k - 1), \qquad k=2, 3, ...$$

Case VI: σ unknown

Let

$$w_k = v_k^2/s'^2_{k-2}.$$

w_k has an F distribution with 1 and k-3 degrees of freedom. Then

$$E(w_k) = (k-3)/(k-5),$$

$$V(w_k) = \frac{2(k-3)^2(k-2)-2}{(k-5)^2(k-7)},$$

and

$$z_k = \{w_k - E(w_k)\}/\{V(w_k)\}^{1/2}, \qquad k=8, 9, ...$$

The implementation of the standardized charts will be relatively less time-consuming than that of Q charts because the calculation of the mean and variance is easier than the approximation of a distribution function.

7.7. Comparison

In this section we compare the standardized charts with the Q charts and the conventional control charts based on subgroup means and variances. We only consider the case in which the process mean and variance are unknown. As pointed out earlier in this chapter, a conventional $\bar{x}$ chart gives false out-of-control signals with a probability equal to .0027, assuming the underlying population has a normal distribution (e.g., see Montgomery, 1991). Similarly, the probability of falsely indicating out-of-control points is

also .0027 for the Q charts because all the Q statistics are transformed into standard normal variables (see Quesenberry, 1991c).

To compare the standardized chart with the Q chart, we consider in this section case IV of section 7.5 where the process mean and variance are unknown. The standardized chart, as well as the corresponding Q chart, is constructed for controlling the process mean, based on the sample averages. Denote $G_{v,\lambda}(.)$ the cumulative t distribution with v degrees of freedom and non-centrality parameter λ. Let

$$w_k = [n(k-1)/k]^{1/2}(\bar{x}_k - \bar{\bar{x}}_{k-1})/\bar{s}_k.$$

Then the z_k statistic is

$$z_k = \left(\frac{(nk-k-2)}{(nk-k)}\right)^{1/2} w_k,$$

and the corresponding Q statistic is

$$Q_k(\bar{x}_k) = \Phi^{-1}\{G_{nk-k}(w_k)\}.$$

Given that the first k-1 points are in control with mean μ and the kth point has the mean $\mu^*=\mu + \delta\sigma$, then w_k has a t distribution with nk-k degrees of freedom and non-centrality parameter $\lambda=[n(k-1)/k]^{1/2}\delta$. For the standardized chart, the probability of the kth point plotting outside of the control limits is

$$\begin{aligned}\pi &= P\{z_k<-3 \text{ or } z_k>3\} = P\{z_k<-3\} + P\{z_k>3\} \\ &= P\{w_k<-3/C(n,k)\} + P\{w_k>3/C(n,k)\} \\ &= 1 - G_{nk-k,\lambda}\{3/C(n,k)\} + G_{nk-k,\lambda}\{-3/C(n,k)\},\end{aligned}$$

where

$$C(n,k) = \left(\frac{(nk-k-2)}{(nk-k)}\right)^{1/2}.$$

For the Q chart, the probability of the same event is

$$\begin{aligned}\pi^* &= P\{Q_k<-3 \text{ or } Q_k>3\} \\ &= P\{w_k<G_{nk-k}^{-1}[\Phi(-3)]\} + P\{w_k>G_{nk-k}^{-1}[\Phi(3)]\} \\ &= 1 - G_{nk-k,\lambda}\{G_{nk-k}^{-1}[\Phi(3)]\} + G_{nk-k,\lambda}\{G_{nk-k}^{-1}[\Phi(-3)]\}.\end{aligned}$$

Table 7.1 shows the above probabilities for the standardized and Q charts based on samples of size n=5. The first column indicates the number of samples used. Column 2 gives the probability of Type I error for the kth standardized statistic plotting outside of the control limit when the process is actually in control, i.e., $\delta=0$. As pointed out earlier, this probability for the Q statistic is always .0027 for any k. Columns 3 and 4 show the probabilities of Type II error for the kth standardized and Q statistics, respectively, plotting outside of the control limits when the first k-1 samples are in control and the kth sample is changed to the new mean $\mu^*=\mu + \delta\sigma$. We only consider $\delta=3/\sqrt{5}$, which means that the shift in the process mean is 3 standard deviations of the sample mean.

Table 7.1. Probabilities of the Standardized and Q Charts

k	Type I (S)	Type II (S)	Type II (Q)
2	.0085	.1737	.0470
3	.0065	.2620	.1071
4	.0055	.3139	.1575
5	.0049	.3476	.1963
6	.0045	.3711	.2260
7	.0042	.3884	.2491
8	.0040	.4016	.2675
9	.0039	.4121	.2824
10	.0038	.4205	.2947
15	.0034	.4464	.3336
20	.0032	.4595	.3540
inf.	.0027	.5000	.5000

From Table 7.1, we see that the probability in column 2 of the standardized chart falsely indicating an out-of-control signal is not much higher than .0027, that of the corresponding Q chart. On the other hand, the probability in column 3 of the standardized chart correctly indicating an out-of-control signal is higher than that in column 4 of the Q chart. When k is large the two charts are equivalent.

7.8. Example

The data of this example are taken from Messina (1987, p.126). A manufacturing manager of a computer assembly line wants to evaluate the line to determine if it exhibits statistical control. Samples of size 5 were taken periodically, and their averages and variances are given in the second and third columns of Table 7.2.

Following the formulas of cases IV and VI given in section 7.5, we obtain the standardized statistics shown in the last two columns of Table 7.2. These plotted statistics are calculated sequentially as the samples are being taken. For the standardized chart for average, we can begin plotting the control chart at time as early as k=2. And for the standardized chart for variance, the plotting can be initiated at time k=3. The two control charts are shown in Figures 7.1 and 7.2.

For both charts, control limits are set at UCL=3 and LCL=-3, and with the center line at CL=0. No apparent out-of-control points can be seen on either chart.

Table 7.2. Data for the Computer Assembly Line Example

k	Average	Variance	z-avg	z-var
1	20.12	0.377		
2	19.36	0.453	-1.615	
3	20.98	2.017	2.121	2.161
4	19.40	0.490	-1.494	-0.570
5	20.84	1.143	1.754	0.218
6	19.44	0.053	-1.574	-1.093
7	18.88	0.067	-2.814	-1.098
8	19.72	0.437	-0.357	-0.468
9	20.46	1.573	1.476	1.674
10	19.40	0.330	-1.269	-0.728
11	20.40	1.435	1.289	1.234
12	19.38	0.467	-1.292	-0.535
13	19.74	1.493	-0.295	1.225
14	20.20	1.135	0.806	0.488
15	19.46	0.073	-1.017	-1.203
16	20.30	1.410	1.061	1.020

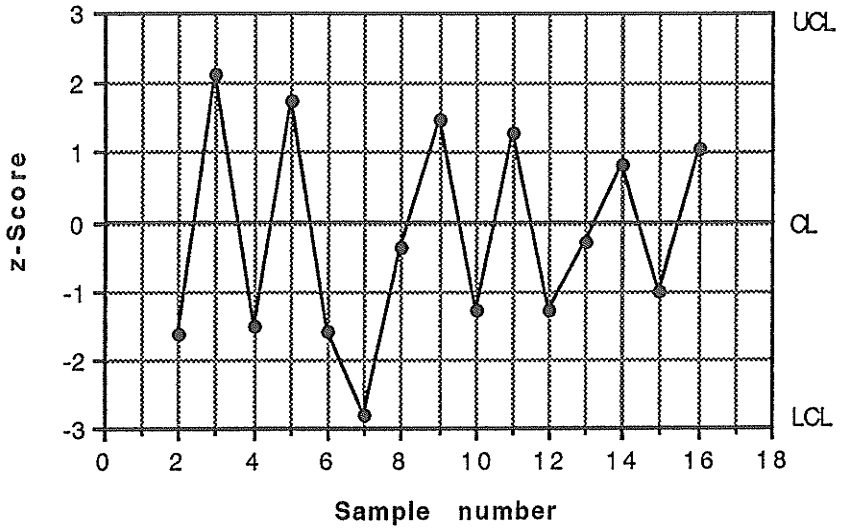

Figure 7.1. The standardized chart for average.

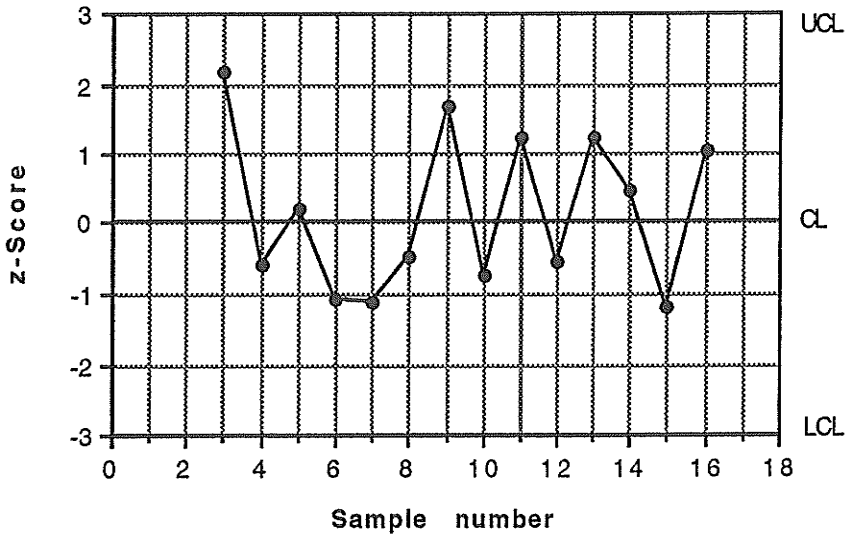

Figure 7.2. The standardized chart for variance.

7.9. Comments

The control chart techniques are powerful tools for statistical process control. In this chapter we have reviewed some of the new control charts that can be applied to new manufacturing environments where short runs are present or real-time monitoring is desired. The two-step approach with modified control limits was an early attempt to deal with short runs. Transformed control charts are conceptually simple and can be easily applied to multiple short runs. The Q charts and standardized charts are the newest efforts in devising more effective control chart techniques for controlling and monitoring the process in real time. The real-time process control appeals to practitioners because the process can be monitored as soon as the process is started.

The standardized charts proposed in this chapter are easier to implement than the Q charts because no computation of the transformation function is required. Although the Q charts give the precise probability control limits, the comparison study in section 7.7 showed that the difference between the standardized charts and Q charts are small. Overall, the standardized charts are good alternatives to the Q charts.

8. A Cumulative Score Control Scheme

8.1. Introduction

The $\bar{x}$ chart, along with the R chart, is one of the most widely used techniques in statistical process control. When a point plots outside of the control limits, an out-of-control signal is indicated. The $\bar{x}$ chart can also indicate an out-of-control condition even though no single point is outside of the control limits, if the pattern of the plotted points exhibits non-random or systematic behavior such as shift, trend, and stratification (Western Electric, 1956).

The CUSUM scheme was introduced by Page (1954) to detect a shift in the process mean. The CUSUM scheme is more effective in detecting relatively small shifts than the $\bar{x}$ chart. However, diagnosis of patterns other than shift on the CUSUM scheme is very difficult, although not impossible, because the sequence of plotted points is not uncorrelated.

Munford (1980) originally proposed cumulative score schemes in which a score of -1, 1 or 0 is assigned to each sample according to whether the sample mean lies below, above or within the range of -k to k, assuming that the process mean is 0 and the standard deviation 1. Scores are then accumulated and an out-of-control signal will be indicated if the cumulative score reaches a critical value. Lewis (1981) further discussed the determination of the parameters of Munford's schemes.

In this chapter we propose a new cumulative score control scheme to detect a shift in the process mean. The cumulative score scheme divides the $\bar{x}$ chart into zones of width $\sigma_{\bar{x}}$. For a value of the sample mean, a score will be assigned. Specifically, if y_i denotes the score of the ith sample mean $\bar{x}_i$, then y_i is assigned by

$$y_i = d + 0.5, \qquad \text{if } \bar{\bar{x}} + d\sigma_{\bar{x}} \le \bar{x}_i < \bar{\bar{x}} + (d+1)\sigma_{\bar{x}},$$

$$= -(d + 0.5), \qquad \text{if } \bar{\bar{x}} - (d+1)\sigma_{\bar{x}} \le \bar{x}_i < \bar{\bar{x}} - d\sigma_{\bar{x}},$$

where d is a non-negative integer. This is illustrated in Figure 8.1.

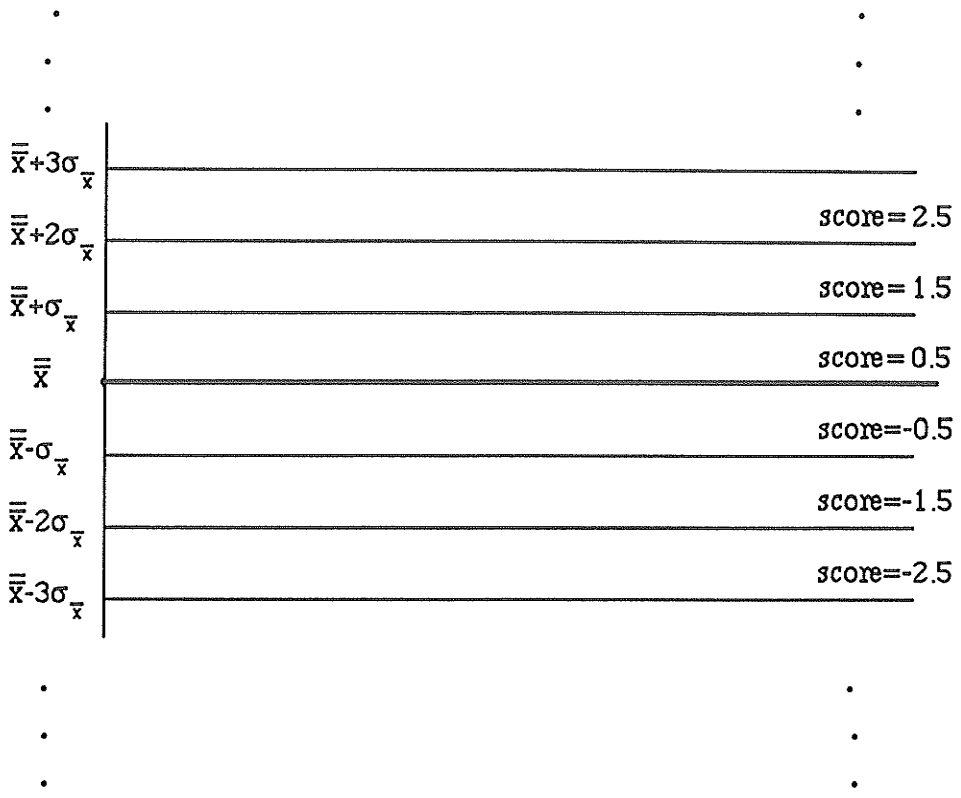

Figure 8.1. Assigning a score.

For detecting an increase in the mean, the cumulative score scheme with some critical value h and reference value k is performed. When the accumulated score reaches h, an out-of-control signal is indicated. In the two-sided case for detecting either an increase or a decrease in the mean, a second cumulative score is added to detect a decrease.

This cumulative score scheme can be supplemented to the $\bar{x}$ chart. On the composite scheme, an out-of-control is indicated if either the current sample is outside of the control limits of the $\bar{x}$ chart or the accumulated score reaches the critical value.

The performance of the cumulative score scheme is assessed by evaluating its average run length (ARL) values. The results are shown in Tables 8.1, 8.2, 8.3, and 8.4. It is assumed that the variable which measures the quality characteristic is normally distributed.

8.2. Cumulative Score Scheme

Let $\bar{x}_1$, $\bar{x}_2$, ... represent the sample means which are successively observed from the process and assumed to be mutually independent and normally distributed with mean μ and known standard deviation $\sigma_{\bar{x}}$. Because of the central limit theorem moderate departures of the process distribution from normality will have little effect on the results in this chapter, for a reasonable sample size. We assume that the target value is $\bar{\bar{x}}=\mu_0$.

8.2.1. The One-Sided Case

Let y_i be the score of the sample mean $\bar{x}_i$. We first consider the one-sided case where we want to detect an increase in the process mean. The criterion for accumulating score is

$$S_i = \max(0, S_{i-1} + y_i - k), \qquad i=1, 2, ...,$$

with $S_0=0$. If $S_i \geq h$, then an out-of-control signal is indicated. The values h and k here are referred to as critical value and reference value, respectively. In order to detect small and moderate departures from the target value quickly we recommend k=0.5, although k can be any positive number. This value of k also facilitates the implementation of the cumulative score scheme. In the rest of this chapter we only consider cases where k=0.5.

A control scheme is usually evaluated by calculating ARLs. The ARL is the average number of samples taken before a corrective signal is obtained. The ARL should be long when the mean is at its target value μ_0 and short when the process shifts to an undesirable level, say $\mu_0+\delta\sigma_{\bar{x}}$. The performance of the cumulative score scheme is determined by its critical value h. We use the Markov chain approach introduced by Brook and Evans (1972) to calculate the ARLs of the cumulative score scheme. A description of the approach is contained in the appendix. Table 8.1 shows the result of the one-sided case.

Table 8.1. ARL for the One-Sided Cumulative Score Scheme

					δ				
h	0.0	0.25	0.5	0.75	1.0	1.5	2.0	3.0	4.0
4	178.3	53.47	21.78	11.64	7.53	4.31	3.04	2.00	1.51
5	465.3	97.48	31.72	15.31	9.51	5.30	3.70	2.39	1.88
6	1190.9	169.7	43.50	19.10	11.51	6.30	4.37	2.78	2.13
7	3018.8	286.5	57.13	22.97	13.51	7.30	5.03	3.19	2.38

8.2.2. The Two-Sided Case

We now consider the two-sided cumulative score scheme in which a deviation in the process mean from the target value in either direction indicates an out-of control signal. In this case, the criterion for accumulating scores is

$$S_i = \max(0, S_{i-1} + y_i - 0.5),$$

and

$$T_i = \max(0, T_{i-1} - y_i - 0.5), \qquad i=1, 2, ...,$$

with $S_0=0$ and $T_0=0$. If $S_i \geq h$ or $T_i \geq h$, an out-of-control signal is indicated. The first formula is used to detect an increase in the mean, and the second is used to detect a decrease in the mean.

The properties of the two-sided cumulative score scheme can be obtained by combining the results of two one-sided schemes. Let ARL_U be the average run length of the one-sided scheme which is used to detect an increase in the mean and ARL_L the average run length of the one-sided scheme which is used to detect a decrease in the mean. Then the ARL of the two-sided scheme is given by

$$\frac{1}{ARL} = \frac{1}{ARL_U} + \frac{1}{ARL_L} .$$

Results of the two-sided cumulative score scheme for the ARL values are given in Table 8.2.

Table 8.2. ARL for the Two-Sided Cumulative Score Scheme

					δ				
h	0.0	0.25	0.5	0.75	1.0	1.5	2.0	3.0	4.0
4	89.14	50.00	21.66	11.64	7.53	4.31	3.04	2.00	1.51
5	232.7	94.51	31.68	15.31	9.51	5.30	3.70	2.39	1.88
6	595.5	167.4	43.49	19.10	11.51	6.30	4.37	2.78	2.13
7	1509.4	284.8	57.13	22.93	13.51	7.30	5.03	3.19	2.38

8.3. Cumulative Score Scheme with Control Limits

It is natural to combine the use of the cumulative score scheme and the $\bar{x}$ chart. The $\bar{x}$ chart has the control limits placed at $\bar{\bar{x}} \pm a\sigma_{\bar{x}}$, where the control limit constant a is usually 3 or 4. On the composite scheme, an out-of-control signal is indicated if the most recent sample is outside of the $\bar{x}$ chart control limits or if a cumulative score signal is given. The composite scheme can be actually implemented as the $\bar{x}$ chart with an extra feature of using the information contained in the past samples.

The properties of the cumulative score scheme with control limits are determined by two parameters, critical value h and control limit constant a. The control limit constants of 3 and 4 are examined. The ARLs of the one-sided case are shown in Table 8.3. Those of the two-sided are listed in Table 8.4.

Table 8.3. ARL for the One-Sided Cumulative Score Scheme with Control Limits

		δ								
h	a	0.0	0.25	0.5	0.75	1.0	1.5	2.0	3.0	4.0
4	3	159.2	50.87	21.21	11.42	7.38	4.16	2.84	1.65	1.17
5	3	317.1	84.68	29.82	14.70	9.14	4.99	3.31	1.79	1.18
5	4	462.1	97.25	31.69	15.30	9.50	5.29	3.67	2.25	1.54
6	3	494.3	127.1	39.11	17.91	10.82	5.73	3.69	1.85	1.19
6	4	1159.2	168.5	43.40	19.07	11.48	6.27	4.31	2.57	1.64
7	3	623.9	173.5	48.75	21.02	12.43	6.43	4.04	1.91	1.19
7	4	2788.8	282.1	56.88	22.91	13.46	7.25	4.93	2.87	1.76

The difference between the cumulative score scheme without control limits and the cumulative score scheme with control limits can be found by comparing Table 8.1 to Table 8.3 and Table 8.2 to Table 8.4. The scheme with control limits is more effective in detecting large shifts in the process mean than the scheme without control limits.

The cumulative score scheme may be also added to attribute charts such as the p chart and the c chart. If the sample is reasonably large, the plotted statistic, for example, the sample fraction nonconforming for the p chart, would approximately have a normal distribution. Therefore, the ARLs of the cumulative score scheme for attribute data may be well approximated by those obtained in the previous sections.

Table 8.4. ARL for the Two-Sided Cumulative Score Scheme with Control Limits

		δ								
h	a	0.0	0.25	0.5	0.75	1.0	1.5	2.0	3.0	4.0
4	3	79.61	46.87	21.02	11.40	7.38	4.16	2.84	1.65	1.17
5	3	158.6	79.08	29.59	14.68	9.14	4.99	3.31	1.79	1.18
5	4	231.0	94.23	31.64	15.30	9.50	5.29	3.67	2.25	1.54
6	3	247.2	117.5	38.75	17.88	10.82	5.73	3.69	1.85	1.19
6	4	579.6	166.0	43.38	19.07	11.48	6.27	4.31	2.57	1.64
7	3	311.9	157.3	48.20	20.98	12.42	6.43	4.04	1.91	1.19
7	4	1394.4	279.7	56.87	22.90	13.46	7.25	4.93	2.87	1.76

8.4. Comparison with CUSUM Schemes

We consider the one-sided cumulative score scheme. Table 8.5 compares the cumulative score scheme with critical value h=5 to two CUSUM schemes. When the process is in control, all of three control schemes give a false signal at average in 465.3 samples.

We observe from Table 8.5 that the cumulative score control scheme is as effective as the CUSUM schemes. When the process shifts to an undesirable level, all of them give an almost equally quick detection of an out-of-control situation. However, the cumulative score scheme is operationally simpler and maintains all the features of the $\bar{x}$ chart.

Table 8.5. Comparison of Schemes

	Cumulative Score	CUSUMs	
δ	h=5	h=4.37, k=0.5	h=2.33, k=1.0
0.0	465.3	465.3	465.3
0.5	31.7	31.5	56.5
1.0	9.5	9.7	12.5
1.5	5.3	5.2	5.6
2.0	3.7	3.6	3.2
2.5	2.9	2.8	2.7

The run-sum control chart of Roberts (1958) is similar to the cumulative score control scheme proposed here. The major difference is that the cumulative score is reset to 0 again when a score changes the sign. The ARL for the run-sum chart should be larger since the preceding information is not utilized when the cumulative score is reset.

8.5. Example

We use the data of Hockman and Lucas (1987) to illustrate the implementation of a cumulative score scheme. The data was taken from a production process where the target value is 100 and $\sigma_{\bar{x}}=10$. We use the parameter values h=6 and a=3. The implementation is very simple and can be achieved without the aid of a computer. It is illustrated in Figure 8.2 and Table 8.6.

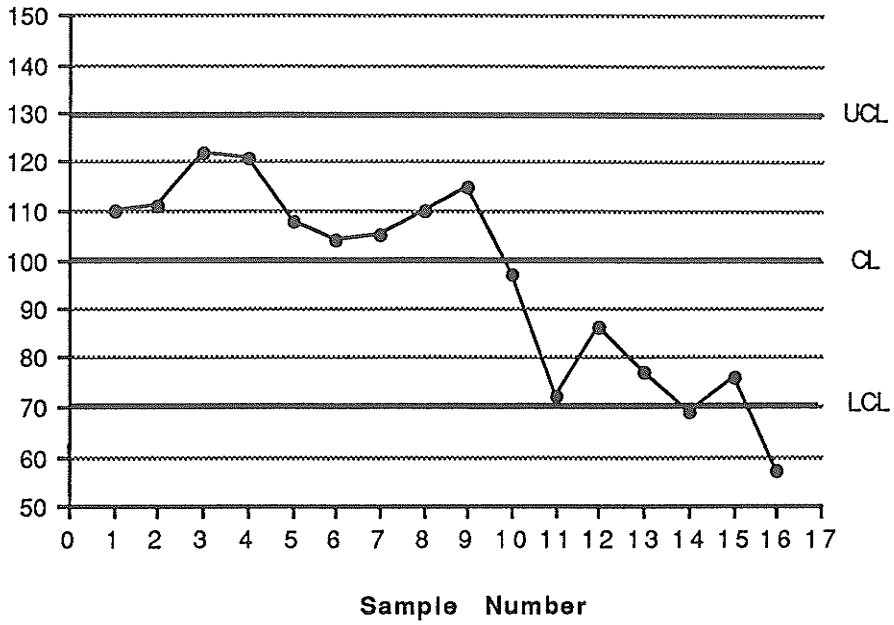

Figure 8.2. Example of implementation.

In Table 8.6, the $\bar{x}_i$ column presents the data. The scores, y_i, are shown in column 3, the positive contributions to the cumulative score in column 4 and the upper cumulative score in column 5. Similarly, the negative contributions and lower score are given in columns 6 and 7. An out-of-control signal is indicated by the upper score, S_i at sample i=4. At i=14 both $\bar{x}$ and T_i give lower side signals.

The cumulative score scheme gives a quicker detection of an out-of-control situation than the $\bar{x}$ chart. In fact, the first sample which plots outside the $\pm 3\sigma_{\bar{x}}$ control limits is the 14th sample. However, the $\bar{x}$ chart in Figure 8.2 gives a clear picture of what was happening in the production process. There was first an increase in the process mean; then a serious decrease occurred.

Table 8.6. Example of Implementation, h=6, a=3

i	$\bar{x}_i$	y_i	y_i-0.5	S_i	-y_i-0.5	T_i
1	110	1.5	1	1	-2	0
2	111	1.5	1	2	-2	0
3	122	2.5	2	4	-3	0
4	121	2.5	2	6†	-3	0
5	108	0.5	0	6†	-1	0
6	104	0.5	0	6†	-1	0
7	105	0.5	0	6†	-1	0
8	110	1.5	1	7†	-2	0
9	115	1.5	1	8†	-2	0
10	97	-0.5	-1	7†	0	0
11	72	-2.5	-3	4	2	2
12	86	-1.5	-2	2	1	3
13	77	-2.5	-3	0	2	5
14	69	-3.5	-4	0	3	8††
15	76	-2.5	-3	0	2	10††
16	57	-4.5	-5	0	4	14††

†: an out-of-control signal for the upper score;
††: an out-of-control signal for the lower score.

8.6. Concluding Remark

Since the cumulative score scheme only involves discrete scores, its implementation is easier than that of the CUSUM scheme. However, a comparison shows that the cumulative score scheme is practically as effective as the CUSUM scheme in detecting shifts in the process mean, although some of the diagnostic features of a CUSUM are

sacrificed. More importantly, the cumulative score scheme can be added to the $\bar{x}$ chart and implemented as an extra feature to increase the ability to detect small shifts. All the work can be done without a computer.

Appendix

The Markov chain approach introduced by Brooke and Evans (1972) can be used to calculate the ARLs for the CUSUM scheme. The method is based on the probability transition matrix $P=(p_{ij})$, which represents the transition probability of moving from any state i to any another state j where the different states represent different CUSUM accumulations. For a Markov chain, the transition probability matrix P has the form

$$P = \begin{bmatrix} R & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

where the R matrix represents various in-control states and the **p**-vector is the probability of receiving an out-of-control signal and **0** is the zero vector. Brooke and Evans (1972) showed that the ARL of the one-sided CUSUM for the initial state (i.e., $S_0=0$) can be obtained by adding all the elements in the first row of $(I-R)^{-1}$.

The cumulative score scheme is essentially a discrete CUSUM. Hence we can use the above method to evaluate the ARLs of the one-sided cumulative score procedure. We say that the cumulative score scheme is in the state i ($i<h$), when $S_k=i$; we say that it is in the state h which is the out-of-control state or absorbing state, when $S_k \geq h$. The transition probabilities are given by

$$p_{i0} = \Phi(-i+1-\delta),$$
$$p_{ij} = \Phi(j-i+1-\delta) - \Phi(j-i-\delta),$$

for i=0, 1, ..., h-1, j=1, 2, ..., h-1, where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution.

For the cumulative score scheme with control limits, we can reassign the score h+0.5 to a sample when its mean is outside of the upper control limit. Thus the transition

probabilities are given by

$$p_{i0} = \Phi(-i+1-\delta),$$

$$p_{ij} = \Phi(j-i+1-\delta) - \Phi(j-i-\delta), \qquad j=1, \ldots, i+a-1,$$

$$p_{ij} = 0, \qquad j=i+a, \ldots, h-1,$$

for i=0, 1, ..., h-1.

9. Summary and Future Work

9.1. Summary of the Thesis

In chapter 1 the topics of both parameter design and control charts are reviewed.

In chapter 2 two data analysis strategies for parameter design under a general location-variance model are proposed. These strategies are applied to experiments based on a combined array design. Under this type of design the response model is fitted, and then the expected quality loss function is estimated and minimized. The first strategy deals with the case in which the experiment is replicated. It is implemented in the following five steps:

Step 1. Identify significant dispersion effects.

Step 2. Identify significant location effects.

Step 3. Estimate the location and variance functions.

Step 4. Estimate the expected quality loss.

Step 5. Find the setting of the control factors that minimizes this loss.

The second strategy deals with the case in which the experiment is not replicated. Since there is only a single replicate for each run, the strategy is provided as follows

Step 1. Identify significant location effects.

Step 2. Identify significant dispersion effects using the residuals.

Step 3. Estimate the location and variance functions.

Step 4. Estimate the expected quality loss.

Step 5. Find the optimal setting of the control factors that minimizes this loss.

The appropriate statistical techniques for identifying significant location and dispersion effects and fitting the response model are also proposed and two examples are provided as illustrations.

In chapter 3 a procedure for selecting a transformation is proposed for experiments based on a combined array design. For a value of λ, indexed for a family of

transformations, the transformed response model is fitted. Then the coefficients of the mean and variance functions of the transformed response variable are plotted against λ to identify an appropriate transformation. The chosen transformation would separate the control factors into two groups: adjustment factors and the rest. The isolation of the adjustment factors is very useful because the optimization can be achieved in one of the two-step procedures reviewed in chapter 1.

In chapter 4 a unified formulation, called the *combined robust design*, which integrates parameter design and tolerance design into one single optimization strategy, is proposed. When the objective is to minimize the total cost (the sum of the expected quality loss and the component costs), Taguchi's approach of parameter design followed by tolerance design becomes a sub–optimal process of the combined robust design. To implement the combined robust design, an iterative procedure is proposed. This procedure can be carried out as follows

Iterative Procedure:

Step 1. Parameter design: given an initial setting of the variation-coefficients $\mathbf{v}_0$, find the setting $\mathbf{x}_0$ of the control factors $\mathbf{x}$ such that

$$R(\mathbf{x}_0, \mathbf{v}_0) = \min_{\mathbf{x}} R(\mathbf{x}, \mathbf{v}_0).$$

Step 2. Tolerance design: find $\mathbf{v}_1$ such that

$$M(\mathbf{x}_0, \mathbf{v}_1) = \min_{\mathbf{v}} M(\mathbf{x}_0, \mathbf{v}) = \min_{\mathbf{v}} [R(\mathbf{x}_0, \mathbf{v}) + C(\mathbf{v})].$$

Step 3. If $\mathbf{v}_1 = \mathbf{v}_0$, then $(\mathbf{x}_0, \mathbf{v}_0)$ is the optimal solution. If not, go to Step 1 with $\mathbf{v}_0$ replaced by $\mathbf{v}_1$.

It is shown by an example that the iterative procedure is much effective than the traditional approach by Taguchi.

In chapter 5 weighted p and u control charts for variable sample size are proposed. The control limits of these charts are always constant in spite of variable sample size. This is done by transforming the sample data. For the weighted charts, the transformed statistics are measured in terms of the same degree of accuracy. Hence, non-

random patterns and trends can be more meaningfully interpreted and identified. The transformation could be made so that the transformed statistic and original statistic are almost identical for most samples on the chart. For samples whose sizes are particularly large or small in comparison with those of the majority of the samples, one could also plot the original statistic along with the transformed statistic.

In chapter 6 three standardized attribute control charts for short runs are proposed. They are the standardized p chart, the standardized c chart, and the standardized u chart. These charts are constructed on the basis of the samples collected continually. The standardized statistics are computed and plotted on the standardized chart with constant control limits UCL=3, LCL=-3, and CL=0.

In chapter 7 some existing variable control charts for short runs, including the two-stage control charts, the transformed control charts, and the Q charts, are reviewed. Standardized control charts for the mean and variance are proposed as alternatives to the reviewed control charts. The comparison study is conducted to show the similarities and the differences of the Q charts and the standardized charts.

In chapter 8 a cumulative score control scheme is devised to increase the ability of detecting a change in the process mean. This control scheme divides the conventional $\bar{x}$ chart into zones with width equal to one standard deviation. An appropriate score is assigned for each sample point according to which zone it falls. The out-of-control signal will be obtained when the accumulated score reaches a critical level. The control scheme can be supplemented to the conventional $\bar{x}$ chart to increase its sensitivity in detecting shifts in the process.

9.2. Future Work

For parameter design, modeling and optimization are two important topics. The loss-model approach based on a product-array design is straightforward, but it requires a large number of experimental runs. The recently developed response-model approach

based on a combined array design can result in significant reduction of the size of experiments and provide more flexible analysis of the data.

The location-variance model based on a combined array design is a new and more general approach that attains the advantages of the response-model approach because of using the same combined array design, while it requires less assumptions. Although two data analysis strategies are proposed in this dissertation, further studies in this area can undoubtedly improve these strategies. Especially, research can be done to develop more techniques for estimating the variance function. In addition, as more actual experiments are done and published, it is possible to compare the loss-model approach, the response-model approach, and the location-variance model approach more thoroughly.

Combined robust design proposed in chapter 4 is a new formulation that unifies parameter design and tolerance design. This formulation can be extended to dynamic systems and multi-variate systems. For a known transfer function, computational consideration is important in combined robust design. there is a need to research more effective numerical approximations of the expected quality loss function.

There has recently been a lot of interest in control charts for short runs. The standardized control charts and many existing short-run control charts are very useful techniques. However, more research is needed to develop new techniques as well as to refine the existing ones. There is also a need to clarify the assumptions that are essential for the correct applications of various control charts. Finally, comparison studies are needed to assess the sensitivity of the various control charts to out-of-control situations.

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