MINIMAL PRESENTATIONS OF FREE METABELIAN NILPOTENT GROUPS

BY

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Abstract

Minimal presentations of free metabelian nilpotent groups, in terms of basic commutators, are investigated. For $m, n \geq 2$, let $M(m, n)$ be a free metabelian nilpotent group of rank m and of nilpotency class $n - 1$. In Chapter 2 we have shown that for $n = 2, 3, 4$, $M(m, n)$ admits a minimal presentation whose set of defining relators is the set of all basic commutators of weight n; this is in fact a yes answer for these values of n to the question raised by Charles C. Sims in this regard. In Chapter 3 the same result is obtained for $M(2, 5)$.

For $m = 2$ and $n \geq 6$ in Chapter 3 we have found a minimal presentation of $M(2, n)$ with the set of relators consisting of certain types of basic commutators of weight at most n.

Finally for $m \geq 3$ and $n \geq 5$, first in Section 2 of Chapter 2 we present a finite presentation of $M(m, n)$, and then in Chapter 4 we refine this presentation to a sharper one. In Chapter 5 we offer a last refinement and introduce a very sharp presentation of $M(m, n)$.

All of the results are obtained using only pure group theoretical techniques without involving any computer methods.

Contents

CHAPTER 1

Preliminaries

1.1. Notation and Definitions

Although the notations we use throughout this thesis are mostly standard and consistent with Derek J.S.Robinson[1996][**16**], Marshall Hall, Jr.[1959][**6**] and Hanna Neumann[1967],[**12**] however for the sake of completeness we list those which are frequently used.

In a group G, the commutator $[x,y]$ is defined as $[x,y] = x^{-1}y^{-1}xy$. For $n \geq 3$, having defined the left-normed commutator $[x_1, x_2, \ldots, x_{n-1}]$ we define $[x_1, x_2, \ldots, x_{n-1}, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$. Throughout this thesis we always consider commutators to be left-normed. Also we denote $[[x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_m]]$ by $[x_1, x_2, \ldots, x_n : y_1, y_2, \ldots, y_m]$.

If $S \subseteq G$ then $\langle S \rangle$ denotes the subgroup generated by S. For the subgroups H_1 and H_2 of G, the subgroup $[H_1, H_2]$ is defined as

$$[H_1, H_2] = \langle \{ [h_1, h_2] \mid h_1 \in H_1 \text{ and } h_2 \in H_2 \} \rangle$$

In particular if $H_1 = H_2 = G$ then $G' = [G, G]$ is called the **derived subgroup**. In general for $n \geq 2$ the n^{th} derived subgroup $G^{(n)}$ of G is defined as $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$, and G is called **solvable** of length n if n is the least integer such that $G^{(n)} = 1$. When $n = 2$, i.e $G'' = [G', G'] = 1$, G is called **metabelian**.

Also let $\gamma_1(G) = G$ and for $n \geq 2$ define $\gamma_n(G) = [\gamma_{n-1}(G), G]$. A group G is said to be **nilpotent** of class $n-1$ if n is the least integer such that $\gamma_n(G) = 1$.

Let $F = \langle X \, ; \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$ where $m \geq 2$, and define an order on X by $x_1 < x_2 < \cdots < x_m$. Considering every element of X as a commutator of weight 1, for $n \geq 2$ commutator C in F is of weight n, if $C = [C_1, C_2]$ such that C_1 and C_2 are of weight n_1 and n_2 respectively and $n_1 + n_2 = n$.

The (standard) **basic commutators** of weight 1 in X are $x_1, x_2, \ldots, x_{m-1}, x_m$ in some fixed but arbitrary order. Having defined and ordered basic commutators of weight less than n we define those of weight n as follows: let C_1 and C_2 be basic commutators of weight n_1 and n_2 where $n_1 + n_2 = n$; then $C = [C_1, C_2]$ is said to be basic if $C_1 > C_2$ and if $C_1 = [C_{11}, C_{12}]$ then $C_2 \geqslant C_{12}$. Basic commutators of of the same weight are put in a fixed but arbitrary order. Moreover, a basic commutator of weight n is greater than any of weight less than n.

In this thesis we only deal with those type of basic commutators which are either simple (i.e. of form $[y_2, y_1, y_3, y_4, \ldots, y_s]$) or double basic of form $[y_2, y_1, y_3, y_4, \ldots, y_s \, ; z_2, z_1]$ where $s \geq 2$. $\{y_1, y_2, \ldots, y_s, z_1, z_2\} \subseteq X$ and $y_2 > y_1 \leq y_3 \leq y_4 \leq \cdots \leq y_s$ and $z_1 < z_2$.

For any group G and epimorphism $\pi : F \to G$ let $S \subseteq F$ be such that ker π is the normal closure of S and let $\pi(X) = Y$; then π together with Y and S determines a set of generators and defining relators for G. In symbols, $G = \langle X \, | \, S \rangle$ denotes a **presentation** of G with set of generators X and set of relators S. When X and S are both finite, G is called a **finitely presented** group and such a presentation is called **minimal** if S is the smallest subset with this property.[1]

[1]For more information about presentations of groups see [9] or [10]

1.2. Group Identities

The following lemma contains (standard) identities which will be used frequently and most of the time without any specific reference.

Lemma 1.1. *(Identities) For* a, b, c, d *in a group* G.

1. $a^b = b^{-1}a\,b = a\,[a,b]$
2. $a\,b = b\,a^b = b\,a\,[a,b] = [a^{-1}, b^{-1}]\,b\,a$
3. $[a,b] = a^{-1}a^b = {b^{-1}}^{a}b$
4. $[a,b] = [b,a]^{-1} = [b^{-1}, a]^b = [b, a^{-1}]^a$
5. $[a,b]^c = [a,b][a,b,c]$
6. $[a,b]^{cd} = [a,b]^d\,[a,b,c]^d$
7. $[a, b\,c] = [a,c]\,[a,b]^c$
8. $[a\,b, c] = [a,c]^b\,[b,c]$
9. $[a, b\,c][c, a\,b][b, c\,a] = 1$
10. $[a, b, c^a][c, a, b^c][b, c, a^b] = 1$ *(Magnus)*
11. $[a, b^{-1}, c]^b\,[b, c^{-1}, a]^c\,[c, a^{-1}, b]^a = 1$ *(Hall – Witt)*

PROOF. The proof is straightforward and can be verified by expanding each commutator explicitly as a product of group elements and then simplifying. ☐

Theorem 1.2. [2] *Let* G *be a metabelian group and let* d *in* G'. *For* $n \geq 2$ *and* $g_1, g_2, \ldots, g_n$ *in* G,

$$[d, g_1, g_2, \ldots, g_n] = [d, g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)}]$$

where σ *is any permutation of* $\{1, 2, \ldots, n\}$.

[2]The converse of this is also true and the proof is based on a lemma by Frank Levin **[13]**; see **[2]**.

PROOF. Since $G'' = 1$ so G' is abelian. It is enough to prove it only for $n = 2$ and then an easy induction would prove it for any n. So must show that,

$$[d, g_1, g_2] = [d, g_2, g_1].$$

But using 1.1 we have,

$$
\begin{aligned}
&[d, g_1, g_2][d, g_2, g_1]^{-1} \\
&\quad = [d, g_1]^{-1}[d, g_1]^{g_2} \left([d, g_2]^{-1}[d, g_2]^{g_1} \right)^{-1} \quad (\text{since } G' \text{ is abelian}) \\
&\quad = [d, g_2][d, g_1]^{g_2} \left([d, g_1][d, g_2]^{g_1} \right)^{-1} \\
&\quad = [d, g_1 g_2][d, g_2 g_1]^{-1} \\
&\quad = \left[d, g_2 g_1 [g_1, g_2]\right] [d, g_2 g_1]^{-1} \\
&\quad = [d; g_1, g_2][d, g_2 g_1]^{[g_1, g_2]} [d, g_2 g_1]^{-1} \qquad (\text{ since } G'' = 1 \ ) \\
&\quad = [d, g_2 g_1][d, g_2 g_1 ; g_1, g_2][d, g_2 g_1]^{-1} \\
&\quad = 1.
\end{aligned}
$$

Hence, $[d, g_1, g_2] = [d, g_2, g_1]$ and the proof is complete.

□

Theorem 1.3. *(Witt formula)(*[6, Theorem 11.2.2]*) Let $N_m(n)$ be the number of basic commutators of weight n in m generators then*

$$N_m(n) = \frac{1}{n} \sum_{d|n} \mu(d)\, m^{\frac{n}{d}},$$

where μ is the Möbius function which is defined for positive integers by the rules $\mu(1) = +1$, and for $n = p_1^{\varepsilon_1} p_2^{\varepsilon_2} \dots p_s^{\varepsilon_s}$; $p_1, p_2, \dots, p_s$ being distinct primes, $\mu(n) = 0$ if any $\varepsilon_i > 1$, and $\mu(p_1 p_2 \dots p_s) = (-1)^s$.

Lemma 1.4. *(Collection Lemma) Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{ x_1, x_2, \dots, x_m \}$, $(m \geq 2)$, such that $x_1 < x_2 < \dots < x_m$. Also* *for* $r \geq 2$ *let* $\{a_1, a_2, \dots, a_r\} \subseteq X$ *such that* $a_1 \leq a_2 \leq \cdots \leq a_r$. *Then,*

$$(1) \qquad (a_2a_3a_4\dots a_r)a_1 = a_1a_2a_3\dots a_r \cdot \prod_{t=2}^{r} \prod_{\substack{\lambda_k\in\{0,1\}\\ t+1\leq k\leq r}} [a_t, a_1, {}_{\lambda_{t+1}}a_{t+1}, {}_{\lambda_{t+2}}a_{t+2}, \dots, {}_{\lambda_r}a_r];$$

$$(2) \qquad a_r(a_1a_2a_3\dots a_{r-1}) = a_1a_2a_3\dots a_r \cdot \prod_{t=1}^{r-1} \prod_{\substack{\lambda_k\in\{0,1\}\\ t+1\leq k\leq r-1}} [a_r, a_t, {}_{\lambda_{t+1}}a_{t+1}, {}_{\lambda_{t+2}}a_{t+2}, \dots, {}_{\lambda_{r-1}}a_{r-1}].$$

For $2 \leq i \leq r-1$.

$$(3) \qquad (a_1a_2a_3\dots a_{i-1}a_{i+1}\dots a_r)a_i = a_1a_2a_3\dots a_r \cdot \prod_{t=i+1}^{r} \prod_{\substack{\lambda_k\in\{0,1\}\\ t+1\leq k\leq r}} [a_t, a_i, {}_{\lambda_{t+1}}a_{t+1}, {}_{\lambda_{t+2}}a_{t+2}, \dots, {}_{\lambda_r}a_r];$$

$$(4) \qquad a_i(a_1a_2a_3\dots a_{i-1}a_{i+1}\dots a_r) = a_1a_2a_3a_4\dots a_r \cdot \prod_{t=1}^{i-1} \prod_{\substack{\lambda_k\in\{0,1\}\\ t+1\leq k\leq r}} [a_i, a_t, {}_{\lambda_{t+1}}a_{t+1}, {}_{\lambda_{t+2}}a_{t+2}, \dots, {}_{\lambda_{i-1}}a_{i-1}, {}_{\lambda_{i+1}}a_{i+1}, \dots, {}_{\lambda_r}a_r];$$

$$(5) \qquad a_ra_{r-1}a_{r-2}\dots a_3a_2a_1 = a_1a_2a_3a_4\dots a_r \cdot \prod_{t=s+1}^{r} \prod_{s=1}^{r-1} \prod_{\substack{\lambda_k\in\{0,1\}\\ s+1\leq k\leq t-1}} [a_t, a_s, {}_{\lambda_{s+1}}a_{s+1}, {}_{\lambda_{s+2}}a_{s+2}, \dots, {}_{\lambda_{t-1}}a_{t-1}].$$

the rest at most

PROOF. The proof is based on the collection process method as it has been described on Chapter 11 of Marshall Hall Jr. [1959][**6**].

To prove the first one using the identity $ab = ba[a, b]$, first we collect a_1; So a_1 passes $a_r, a_{r-1}, \dots, a_3$ and a_2 respectively and would create $[a_t, a_1]$

where $2 \leq t \leq r$. Now each of these commutators must move to the right side, so for each t the commutator $[a_t, a_1]$ has to pass $a_{t+1}, a_{t+2}, \ldots, a_{r-1}$ and a_r, respectively, and therefore would create simple basic commutators of the form $[a_t, a_1, {}_{\lambda_{t+1}}a_{t+1}, {}_{\lambda_{t+2}}a_{t+2}, \ldots, {}_{\lambda_r}a_r]$ where $\{\lambda_{t+1}, \lambda_{t+2}, \ldots, \lambda_r\} \subseteq \{0, 1\}$.

The proofs of (2), (3), (4) are similar. To prove (5), similar to the above, we first collect a_1. So a_1 passes through $a_2, a_3, \ldots, a_{r-1}$ and a_r, respectively, and would create $[a_t, a_1]$ where $2 \leq t \leq r$. Then we begin collecting a_2; this time a_2 will pass through $a_3, a_4, \ldots, a_{r-1}$ and a_r as well as through $[a_t, a_1]$ where $3 \leq t \leq r$. Therefore it will create commutators of the form $[a_t, a_2]$ as well as of the form $[a_t, a_1, a_2]$ where $3 \leq t \leq r$. Now continuing this process for all $a_3, a_4, \ldots, a_{r-2}, a_{r-1}$ completes the collection and the proof. □

NOTE: In the above lemma if for each i ($1 \leq i \leq r$) we replace a_i by $a_i^{\varepsilon_i}$ where $\varepsilon_i \in \{1, -1\}$, the lemma is still valid and the proof is similar.

Theorem 1.5. *(Basis Theorem)(* [6] *and* [7] *) If F is the free group with generators $y_1, y_2, \ldots, y_r$ and if in a sequence of basic commutators $c_1, c_2, \ldots, c_t$ are those of weights $1, 2, \ldots, n$, then an arbitrary element f of F has a unique representation.*

$$f = c_1^{e_1} c_2^{e_2} \ldots c_t^{e_t} \mod \gamma_{n+1}(F).$$

The basic commutators of weight n form a basis for the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$.

CHAPTER 2

Minimal Presentations

Finitely generated nilpotent groups are finitely related and it is natural to seek a presentation of such groups requiring only a minimal set of defining relations. This is in general an extremely difficult problem even when restricted to finitely generated free nilpotent groups and even for small nilpotency class, say 4 or 5 (see [**17**] and [**1**]). In this thesis we study minimal presentations of free metabelian nilpotent groups.

Let $F = \langle X : \varnothing \rangle$ be a free group of rank m ($m \geq 2$), with basis $X = \{x_1, x_2, \ldots, x_m\}$. Our goal in this thesis is to find a finite presentation for **free metabelian nilpotent** groups and make it as sharp as possible.

Consider the group $M(m, n)$ $(n \geq 2)$, defined by the presentation

$$M(m, n) = \langle\, X ; \gamma_n(F) \,.\, F'' \,\rangle,$$

where $\gamma_n(F)$ is the n-th term of the lower central series of F and $F'' = [\,F', F'\,] = \gamma_2(\gamma_2(F))$ is the second commutator subgroup of F. Then,

$$M(m, n) \cong F/\gamma_n(F)F''.$$

is the relatively free group of rank m in the variety of metabelian nilpotent groups of class at most $n-1$. Define an order on X by $x_1 < x_2 < \cdots < x_m$.

In order to find a finite presentation of $M(m, n)$ for arbitrary but fixed integer n , since $\gamma_n(F)$ is generated by the set of all left-normed simple commutators of weight n with entries from X , we may assume that $m \leq n$. In the following section we introduce a minimal presentation for $M(m, 2)$, $M(m, 3)$ and $M(m, 4)$.

2.1. Minimal Presentations of $M(m,n)$ for $n = 2,3,4$

Definition 2.1. Let $F = \langle X : \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$ $(m \geq 2$); define $B(m,n,F)$ to be the set of all basic commutators of weight n on X and denote normal closure of $B(m,n,F)$ with $\langle B(m,n,F) \rangle^F$.

Theorem 2.2. *For $n = 2$ or 3, $M(m,n)$ admits the following presentation with exactly $\frac{1}{n}(m^n - m)$ relators.*

$$M(m,n) = \langle x_1, x_2, \ldots, x_m ; B(m,n,F) \rangle$$

PROOF. For $n = 2$ or 3 , $F'' \subseteq \gamma_n(F)$. When $n = 2$ the proof is trivial. For $n = 3$ let $U = [u_1, u_2, u_3] \in \gamma_3(F)$: since U can be written as a product of conjugates of commutators of the form$[a,b,c]$ where $\{a,b,c\} \subseteq X$, it is enough to show that $[a,b,c] \in \langle B(m,3,F) \rangle^F$. But since

$$[a,b,c] = ([b,a,c]^{-1})^{[a,b]}$$

we can always assume that $b < a$.
Now if $b \leq c$ then $[a,b,c]$ is itself basic and so is in $B(m,n,F)$. If $c < b$, then $c < b < a$ and by the Magnus identity we have.

$$[a,b,c^a][c,a,b^c][b,c,a^b] = 1. \tag{2.1}$$

Working modulo $\langle B(m,3,F) \rangle^F$ we have.

$$\begin{aligned}
[a,b,c^a] &= [a,c\,;\,a,b]^{[c,a]}[a,b,c]^{[c,a]} \\
&= \prod_{i=1}^{4}[a,c,u_i]^{\alpha_i g_i}[a,b,c]^{[c,a]} && (\alpha_i \in \mathbb{Z}\,, g_i \in F\,, u_i \in \{a,b\}) \\
&\equiv [a,b,c]^{[c,a]} && (\text{since } c < b < a).
\end{aligned}$$

And

$$\begin{aligned}
[c, a, b^c] &= \left([a, c; b, c]^{-1}\right)^{[c,a]}\left([a, c, b]^{-1}\right)^{[c,a][b,c]} \\
&\equiv \prod_{i=1}^{4}[a, c, u_i]^{\alpha_i g_i} && (\alpha_i \in \mathbb{Z}, g_i \in F, u_i \in \{b, c\}) \\
&\equiv 1 && (\text{ since } c < b \text{ }).
\end{aligned}$$

Also,

$$\begin{aligned}
[b, c, a^b] &= [b, c; a, b][b, c, a]^{[a,b]} \\
&\equiv \prod_{i=1}^{4}[b, c, u_i]^{\alpha_i g_i} && (\alpha_i \in \mathbb{Z}, g_i \in F, u_i \in \{a, b\}) \\
&\equiv 1 && (\text{ since } c < b < a \text{ }).
\end{aligned}$$

Therefore by substitution in 2.1 we get $[a, b, c] \equiv 1 \pmod{\langle B(m, 3, F)\rangle^F}$. Minimality comes from the fact that by the Basis Theorem, $B(m, 3, F)$ is an independent set modulo $\gamma_4(F)$. The number of relators comes from the *Witt formula.* □

Lemma 2.3. *Let $\{x, y, z\} \subseteq \{x_1, x_2, \ldots, x_m\}$ such that $x > y \leq z$; then*

(i) $[x, y, z, y] \equiv 1 \pmod{\langle B(m, 4, F)\rangle^F}$;

(ii) $[x, y, z, x] \equiv 1 \pmod{\langle B(m, 4, F)\rangle^F}$.

PROOF. (*i*) By using the the Hall-Witt identity for $[x, y]$, z and y we have,

$$[x, y, z^{-1}, y]^z[z, y^{-1}; x, y]^y[y, [x, y]^{-1}, z]^{[x,y]} = 1. \tag{2.2}$$

Working modulo $\langle\, B(m,4,F)\,\rangle^F$ we have,

$$
\begin{aligned}
&[\,x,y,z^{-1},y\,]^z = \big[\,[\,x,y,z^{-1}\,]^z,\ y^z\,\big] \\
&= \big([\,x,y,z,\ z^{-1}yz\,]^{-1}\big)^{[x,y,z]^{-1}} \\
&= \big([\,x,y,z,\ z^{-1}\,]^{-1}\big)^{yz[x,y,z]^{-1}} . \big([\,x,y,z,y\,]^{-1}\big)^{z[x,y,z]^{-1}} . \big([\,x,y,z,z\,]^{-1}\big)^{[x,y,z]^{-1}} \\
&\equiv [\,x,y,z,y\,]^{-1}\big)^{z[x,y,z]^{-1}}.
\end{aligned}
$$

And

$$
\begin{aligned}
&[\,z,y^{-1}\,;\ x,y\,]^y = [\,y,z\,;\ [\,x,y\,][\,x,y,y\,]\,] \\
&\equiv [\,x,y,y\,;\ y,z\,]^{-1} \\
&\equiv \prod_{i=1}^{4}[\,x,y,y,u_i\,]^{\alpha_i g_i} \qquad\qquad (\alpha_i \in \mathbb{Z}\,.\,g_i \in F\,.\,u_i \in \{y,z\}) \\
&\equiv 1 \qquad\qquad (\text{ since } x > y \leq z\ ).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\big[\,y,[x,y]^{-1},\ z\,\big]^{[x,y]} = \big[\,x,y,y,\ z^{[x,y]}\,\big] \\
&\equiv \prod_{i=1}^{k}[\,x,y,y,u_i\,]^{\alpha_i g_i} \quad (\text{ for some } k \text{ and } \alpha_i \in \mathbb{Z}\,.\,g_i \in F\,.\,u_i \in \{x,y,z\}) \\
&\equiv 1 \quad (\text{ since } x > y \leq z).
\end{aligned}
$$

Substitution in 2.2 gives,

$$[\,x,y,z,y\,] \equiv 1 \pmod{\langle\, B(m,4,F)\,\rangle^F}.$$

which completes the proof of part (i).

In order to prove part (ii), if $z \leq x$ then $[\,x,y,z,x\,]$ is basic and so is in $B(m,4,F)$. For $x < z$ again using the the Hall-Witt identity this time for $[\,x,y\,]$, z and x we have,

$$
(2.3) \qquad [\,x,y,z^{-1},x\,]^z[\,z,x^{-1}:x,y\,]^x[\,x,\ [x,y]^{-1},\ z\,]^{[x,y]} = 1.
$$

Working modulo $\langle\, B(m,4,F)\,\rangle^F$ we have,

$$\begin{aligned}[x,y,z^{-1},x]^z &= \left([x,y,z,\; z^{-1}xz\,]^{-1}]\right)^{[x,y,z]^{-1}} \\ &\equiv \left([x,y,z,x]^{-1}\right)^{z[x,y,z]^{-1}}.\end{aligned}$$

And

$$\begin{aligned}&[z,x^{-1};x,y]^x = \left[x,z:[x,y][x,y,x]\right] \\ &\equiv \prod_{i=1}^{4}[x,y,x,u_i]^{\alpha_i g_i} \qquad\qquad (\alpha_i \in \mathbb{Z}\,, g_i \in F\,, u_i \in \{x,z\}) \\ &\equiv 1 \qquad\qquad (\text{ since } y < x < z\ ).\end{aligned}$$

Also,

$$\begin{aligned}&[x,\,[x,y]^{-1},\; z\,]^{[x,y]} \\ &= \left[x,y,x,\; z^{[x,y]}\right] \\ &\equiv \prod_{i=1}^{k}[x,y,x,u_i]^{\alpha_i g_i} \quad (\text{ for some } k \text{ and } \alpha_i \in \mathbb{Z}\,, g_i \in F\,, u_i \in \{x,y,z\}) \\ &\equiv 1 \qquad\qquad (\text{ by part } i \text{ and the fact that } x < z\ ).\end{aligned}$$

Substitution in 2.3 gives

$$[x,y,z,x] \equiv 1 \pmod{\langle\, B(m,4,F)\,\rangle^F}.$$

which completes the proof of part (ii) and the lemma. □

Lemma 2.4. *Let* $\{a_1,a_2,a_3\} \subseteq \{x_1,x_2,\ldots,x_m\}$ *such that* $a_1 < a_2 < a_3$ *; then,*

$$[a_3,a_2,a_1] \equiv [a_2,a_1,a_3]^{-1}[a_3,a_1,a_2] \pmod{\langle\, B(m,4,F)\,\rangle^F}.$$

PROOF. By the Magnus identity,

$$[a_3,a_2,a_1^{a_3}][a_1,a_3,a_2^{a_1}][a_2,a_1,a_3^{a_2}] = 1. \tag{2.4}$$

Working modulo $\langle B(m,4,F)\rangle^F$ we have,

$$[a_3, a_2, a_1^{a_3}] = [a_3, a_2 ; a_1, a_3][a_3, a_2, a_1]^{[a_1, a_3]}$$
$$\equiv [a_3, a_2, a_1][a_3, a_2, a_1 ; a_1, a_3].$$

On the other hand, since

$$\begin{aligned}
&[a_3, a_2, a_1 ; a_1, a_3] \\
&= [[a_3, a_2]^{-1}[a_3, a_2]^{a_1} ; a_1, a_3] \\
&\equiv [[a_3, a_2]^{a_1} ; a_1, a_3] \\
&= [a_3, a_1, [a_3, a_2]^{a_1}]^{[a_1, a_3]} \\
&\equiv [a_3, a_1, a_1^{-1} ; a_3, a_2]^{a_1[a_1, a_3]} \\
&\equiv \prod_{i=1}^{k}[a_3, a_1, a_1, d_i]^{\alpha_i g_i} \quad (\text{ for some } k \text{ and } \alpha_i \in \mathbb{Z}, g_i \in F, d_i \in \{a_1, a_2, a_3\}) \\
&\equiv 1 \quad (\text{ since } a_1 < a_2 < a_3).
\end{aligned}$$

therefore $[a_3, a_2, a_1^{a_3}] \equiv [a_3, a_2, a_1]$ (1).

Also,

$$\begin{aligned}
[a_1, a_3, a_2^{a_1}] &= ([a_3, a_1, a_2^{a_1}]^{-1})^{[a_1, a_3]} \\
&\equiv ([a_3, a_1, a_2]^{-1})^{[a_2, a_1][a_1, a_3]} \\
&\equiv \prod_{i=1}^{k}[a_3, a_1, a_2, d_i]^{\alpha_i g_i} . [a_3, a_1, a_2]^{-1} \\
&\quad (\text{ for some } k \text{ and } \alpha_i \in \mathbb{Z}, g_i \in F, d_i \in \{a_1, a_2, a_3\}) \\
&\equiv [a_3, a_1, a_2]^{-1} \quad (\text{by Lemma 2.3 });
\end{aligned}$$

so $[a_1, a_3, a_2^{a_1}] \equiv [a_3, a_1, a_2]^{-1}$ (2).

In addition, similar to the above, $[a_2, a_1, a_3^{a_2}] \equiv [a_2, a_1, a_3]$ (3).

Substitution of (1), (2) and (3) in 2.4 gives,

$$[a_3, a_2, a_1] \equiv [a_2, a_1, a_3]^{-1}[a_3, a_1, a_2] \pmod{\langle B(m,4,F)\rangle^F},$$

which completes the proof. ∎

Lemma 2.5. *Let* $\{x, y, z, u\} \subseteq \{x_1, x_2, \ldots, x_m\}$ *such that* $x > y \leq z$ *and* $y \leq u$ *; then*

$$[x, y, u, z] \equiv 1 \pmod{\langle B(m,4,F)\rangle^F}.$$

PROOF. Since y is the minimum of the set $\{x, y, z, u\}$, if $u \leq z$, then $[x, y, u, z]$ is basic and so is in $B(m,4,F)$. For $z < u$ using the the Hall-Witt identity for $[x, y]$, u and z we have,

$$(2.5) \qquad [x, y, u^{-1}, z]^u [u, z^{-1} ; x, y]^z [z, [x, y]^{-1} u,]^{[x,y]} = 1.$$

Working modulo $\langle B(m,4,F)\rangle^F$ we have,

$$\begin{aligned} &[x, y, u^{-1}, z]^u \\ &= [[x, y, u, u_{-1}]^{-1}[x, y, u]^{-1}, z]^u \\ &\equiv [[x, y, u]^{-1}, z]^u \\ &([x, y, u, z]^{-1})^{[x,y,u]^{-1}u}. \end{aligned}$$

And

$$\begin{aligned} &[u, z^{-1} ; x, y]^z \\ &\qquad \equiv [x, y, z ; z, u]^{-1} \\ &\qquad \equiv \prod_{i=1}^{4} [x, y, z, v_i]^{\alpha_i g_i} \qquad (\alpha_i \in \mathbb{Z}, g_i \in F, v_i \in \{z, u\}) \\ &\qquad \equiv 1 \qquad\qquad\qquad (\text{ since } z < u). \end{aligned}$$

Also,

$$\begin{aligned}
&\left[z, [x,y]^{-1} u\right]^{[x,y]} \\
&= \left[x, y, z, u^{[x,y]}\right] \\
&\equiv \prod_{i=1}^{k} [x, y, z, v_i]^{\alpha_i g_i} \qquad \text{(for some } k \text{ and } \alpha_i \in \mathbb{Z}, g_i \in F, v_i \in \{x, y, u\}) \\
&\equiv 1 \qquad \text{(by Lemma 2.3 and the fact that } z < u).
\end{aligned}$$

Hence substitution in 2.5 gives,

$$[x, y, u, z] \equiv 1 \pmod{\langle B(m,4,F)\rangle^F}.$$

□

Theorem 2.6. [1] *$M(m,4)$ admits the following minimal presentation with exactly $\frac{1}{4}(m^4 - m^2)$ relators.*

$$M(m,4) = \langle\, x_1, x_2, \ldots, x_m : B(m,4,F) \,\rangle.$$

PROOF. Let $U = [u_1, u_2, u_3, u_4]$ be an arbitrary element of $\gamma_4(F)$; since U can be written as a product of conjugates of commutators of the form $[a', b', c', d']$ where $\{a', b', c', d'\} \subseteq X$, it is enough to prove that $[a', b', c', d'] \equiv 1 \pmod{\langle B(m,4,F)\rangle^F}$.
But without loss of generality one can assume that $b' < a'$. Now if $c' < b'$, then since $c' < b' < a'$ using Lemma 2.4 we have,

$$\begin{aligned}
[a', b', c', d'] &\equiv \left[[b', c', a']^{-1}[a', c', b'], d'\right] \\
&\equiv \left([b', c', a', d']^{-1}\right)^g [a', c', b', d'] \qquad (g \in F).
\end{aligned}$$

[1] Gorge Havas in [8] has mentioned a proof of the special case of this theorem for $m = 2$ which is originally due to J.R.J. Groves; however our proof here is different than his.

Therefore $[a', b', c', d']$ with $b' < a'$ is congruent to a product of conjugates of commutators of the form $[a, b, c, d]$ where $a > b \leq c$ and $\{a, b, c, d\} \subseteq X$. Thus it remains to show that $[a, b, c, d] \equiv 1 \pmod{\langle B(m, 4, F) \rangle^F}$.

- If $b \leq d$ then since also $a > b \leq c$, by Lemma 2.5, $[a, b, c, d] \equiv 1 \pmod{\langle B(m, 4, F) \rangle^F}$.
- If $d < b$ then since $d < b < a$ by using the Hall-Witt identity for $[a, b], c$ and d we have,

$$[a, b, c^{-1}, d]^c [c, d^{-1} ; a, b]^d [d, [a, b]^{-1}, c]^{[a,b]} = 1. \tag{2.6}$$

Working modulo $\langle B(m, 4, F) \rangle^F$ we have,

$$\begin{aligned}[a, b, c^{-1}, d]^c &= [[a, b, c, c^{-1}]^{-1}[a, b, c]^{-1}, d]^c \\ &\equiv ([a, b, c, d]^{-1})^{[a,b,c]^{-1}c}.\end{aligned}$$

And

$$\begin{aligned}&[c, d^{-1} ; a, b]^d \\ &= ([[c, d]^{d^{-1}} : a, b]^{-1})^{[d,c]^{d^{-1}}d} \\ &\equiv ([[c, d, d^{-1} ; a, b]^{-1})^{[d,c]^{d^{-1}}d} \\ &\equiv \prod_{i=1}^{k}[c, d, d, e_i]^{\alpha_i g_i} && (\alpha_i \in \mathbb{Z},\ g_i \in F,\ e_i \in \{a, b, d\}) \\ &\equiv 1 && (\text{ since } d < b < a).\end{aligned}$$

Also,

$$\begin{aligned}&[d, [a, b]^{-1}, c]^{[a,b]} \\ &= [a, b, d, c^{[a,b]}] \\ &\equiv \prod_{i=1}^{k}[a, b, d, e_i]^{\alpha_i g_i} && (\alpha_i \in \mathbb{Z},\ g_i \in F,\ e_i \in \{a, b, c\}).\end{aligned}$$

Substitution in 2.6 gives,

$$[a,b,c,d] \equiv \prod_{i=1}^{k} [a,b,d,e_i]^{\alpha_i f_i}, \tag{2.7}$$

where $\alpha_i \in \mathbb{Z}$, $f_i \in F$ and $e_i \in \{a,b,c\}$.

Now since $d < b < a$, by Lemma 2.4,

$$[a,b,d] \equiv [b,d,a]^{-1}[a,d,b] \pmod{\langle B(m,4,F)\rangle^F},$$

so for each $i, (1 \leq i \leq k)$,

$$[a,b,d,e_i]$$

$$= \left([b,d,a,e_i]^{-1}\right)^{[b,d,a]^{-1}[a,d,b]}[a,d,b,e_i]$$

$$\equiv 1 \text{ (by Lemma 2.5 considering } d < b < a \text{ and } d < e_i).$$

Hence by substitution in 2.7 we get,

$$[a,b,c,d] \equiv 1 \pmod{\langle B(m,4,F)\rangle^F}.$$

Minimality comes from the fact that by the Basis Theorem, the set of all basic commutators of weight 4 is an independent set modulo $\gamma_5(F)$. The number of relators comes from the *Witt formula*.

□

2.2. A Finite Presentation of $M(m,n)$ for $(n \geq 5)$

In this section, for arbitrary but fixed integer n $(n \geq 5)$, we take the first step toward our goal and will find a finite presentation for $M(m,n)$ for $m \geq 2$. Although this presentation is not the best possible, it will help us to find the suitable ones in the next chapters.

Definition 2.7. Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 2$, and let $n \geq 5$. Define $\Omega_0(m,n,F)$ to be the set of all simple basic commutators of weight n on X. In other words,

$\Omega_0(m,n,F) =$

$\{[z_i, z_1, z_2, z_3, \ldots, \hat{z}_i, \ldots, z_n] \mid z_i > z_1 \leq z_2 \leq \cdots \leq z_n,\ z_t \in X\,(1 \leq t \leq n)\}$,

where $\hat{z}_i$ means z_i is missing.

Theorem 2.8. *Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 2$, and let $n \geq 5$. Then*

(a) *F' is generated by S where*
$S = \{ [x_i, x_j]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_r^{\varepsilon_r}} \mid r \in \mathbb{Z},\ \{x_i, x_j, y_1, y_2, \ldots, y_r\} \subseteq X,$
$x_i > x_j \leq y_1 \leq y_2 \leq \cdots \leq y_r,\ \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\} \subseteq \{1, -1\}\}$.

(b) *Modulo $\Omega_0(m,n,F)$ every element d in F' can be written as a product of simple basic commutators of weight $2, 3, 4, \ldots, n-1$; i.e.*

$$d \equiv C_1^{\varepsilon_1} C_2^{\varepsilon_2} \ldots C_k^{\varepsilon_k} \quad \textit{(for some integer } k),$$

where C_i is a simple basic commutator of weight at most $n-1$ and $\varepsilon_i \in \{1, -1\}$.

Proof. **(a):** Let $\langle S \rangle$ denote the subgroup of F generated by S. In order to proof part (a) we need to show that $\langle S \rangle = F'$: But clearly $\langle S \rangle \leqslant F'$. In order to prove $F' \leqslant \langle S \rangle$ first we need to proof the following : For $t \geq 2$ let $\{y_1, y_2, \ldots, y_t\} \subseteq X$ such that $y_1 \leq y_2 \cdots \leq y_t$; also let $\{i_1, i_2, \ldots, i_t\} = \{1, 2, \ldots, t\}$ and $\varepsilon_{i_j} \in \{1, -1\}$. Then

$$y_{i_1}^{\varepsilon_{i_1}} y_{i_2}^{\varepsilon_{i_2}} \ldots y_{i_t}^{\varepsilon_{i_t}} \equiv y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_t^{\varepsilon_t} \pmod{\langle S \rangle} \qquad \textbf{(1)}.$$

We prove (1) by induction on t. If $t = 2$ then for $i_1 = 1$ there is nothing to prove and for $i_2 = 1$ we have $y_2^{\varepsilon_2} y_1^{\varepsilon_1} = y_1^{\varepsilon_1} y_2^{\varepsilon_2} [y_2^{\varepsilon_2}, y_1^{\varepsilon_1}]$. Since

$$[y_2^{\varepsilon_2}, y_1^{\varepsilon_1}] = [y_2, y_1]^{\varepsilon_1 \varepsilon_2 y_1^{\frac{1}{2}(\varepsilon_1 - 1)} y_2^{\frac{1}{2}(\varepsilon_2 - 1)}} \equiv 1 \pmod{\langle S \rangle},$$

therefore $y_2^{\varepsilon_2} y_1^{\varepsilon_1} \equiv y_1^{\varepsilon_1} y_2^{\varepsilon_2} \pmod{\langle S \rangle}$.

Assuming it is also true for any number less than or equal to $t-1$, to prove it for t, if $i_1 = 1$, it is true by the induction hypothesis. If $i_1 > 1$, then there is $2 \leq t_1 \leq t$ such that $y_1 = y_{i_{t_1}}$. Collecting and transferring y_1 to the very left, would create commutators $[y_i^{\varepsilon_i}, y_1^{\varepsilon_1}]$, where $y_i = y_{i_\ell}$, $(1 \leq \ell \leq t_1 - 1)$ and $y_1 < y_i$. Continuing the collection process, assuming that y_{j-1} is already collected where $2 \leq j \leq t$ and for $1 \leq t_j \leq t$, $y_j = y_{i_{t_j}}$: then collecting y_j would create commutators $C_{(i,j)} = [y_i^{\varepsilon_i}, y_j^{\varepsilon_j}]^{y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}}$ where $y_i = y_{i_\ell}$, $(1 \leq \ell \leq t_j - 1)$ and $y_j < y_i$. So it remains to show that for each each i and j, $C(i,j) \equiv 1 \pmod{\langle S \rangle}$. If $\varepsilon_i = +1$ then

$$C_{(i,j)} = [y_i, y_j^{\varepsilon_j}]^{y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}} = [y_i, y_j]^{\varepsilon_j y_j^{\frac{1}{2}(\varepsilon_j - 1)} y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}} \in S.$$

If $\varepsilon_i = -1$ then

$$C_{(i,j)} = [y_i^{-1}, y_j^{\varepsilon_j}]^{y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}} = ([y_i, y_j^{\varepsilon_j}]^{-1})^{y_i^{-1} y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}} \tag{2.8}$$

Since $y_i^{-1} y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \ldots y_t^{\varepsilon_t}$ is of length at most $t-1$, then by the induction hypothesis.

$$y_i^{-1} y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \ldots y_t^{\varepsilon_t} \equiv z_1^{\varepsilon_1} z_2^{\varepsilon_2} \ldots z_r^{\varepsilon_r} \pmod{\langle S \rangle}$$

where $\{z_1, z_2, \ldots, z_r\} = \{y_i, y_{j+1}, y_{j+2}, \ldots, y_t\}$ and $y_j \leq z_1 \leq z_2 \leq \cdots \leq z_r$. Substitution in 2.8 gives,

$$\begin{aligned} C_{(i,j)} &= ([y_i, y_j^{\varepsilon_j}]^{-1})^{y_i^{-1} y_{j+1}^{\varepsilon_{j+1}} y_{j+2}^{\varepsilon_{j+2}} \cdots y_t^{\varepsilon_t}} \\ &\equiv ([y_i, y_j]^{-1})^{\varepsilon_j y_j^{\frac{1}{2}(\varepsilon_j - 1)} z_1^{\varepsilon_1} z_2^{\varepsilon_2} \cdots z_r^{\varepsilon_r}} \\ &\equiv 1 \pmod{\langle S \rangle}. \end{aligned}$$

Therefore the proof of (1) is complete.

Now we are in position to prove $F' \leqslant \langle S \rangle$. By definition F' is generated by commutators of the form $[f_1, f_2]$ $(f_1, f_2 \in F)$, and each one these commutators can be written as a product of commutators of form $[x, y]^f$

and their inverses where $x, y \in X$, $x > y$ and $f \in F$. So it remains to show that $[x,y]^f \in \langle S \rangle$. Using **(1)**,

$$f \equiv y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_r^{\varepsilon_r} \pmod{\langle S \rangle}.$$

where $y_i \in X$, $y_1 \leq y_2 \cdots \leq y_r$ and $r \geq 1$. Therefore

$$[x,y]^f \equiv [x,y]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_r^{\varepsilon_r}} \pmod{\langle S \rangle}.$$

Call $D(r) = [x,y]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_r^{\varepsilon_r}}$. If $y \leq y_1$, then by definition $D(r) \in S$. If $y > y_1$, then by induction on r we show that $D(r) \equiv 1 \pmod{\langle S \rangle}$. If $r = 1$, then $D(1) = [x,y]^{y_1^{\varepsilon_1}} = [x,y][x,y,y_1^{\varepsilon_1}]$ where $y_1 < y < x$. By the Magnus identity,

$$[x,y,(y_1^{\varepsilon_1})^x][y_1^{\varepsilon_1}, x, y^{y_1^{\varepsilon_1}}][y, y_1^{\varepsilon_1}, x^y] = 1.$$

Using the Magnus identity it is not difficult to see that $[x,y,y_1^{\varepsilon_1}]$ can be written as a product of the following commutators and their inverses :

$$[x,y],\ [x,y_1^{\varepsilon_1}],\ [x,y_1^{\varepsilon_1}]^y,\ [y,y_1^{\varepsilon_1}],\ [y,y_1^{\varepsilon_1}]^x.$$

Since each one of these commutators is in $\langle S \rangle$ so $[x,y,y_1^{\varepsilon_1}] \equiv 1 \pmod{\langle S \rangle}$ and thus $D(1) \equiv 1 \pmod{\langle S \rangle}$. Assume that the statement is true for integers less than or equal $r-1$, then for r we have,

$$\begin{aligned} D(r) &= [x,y]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_r^{\varepsilon_r}} \\ &= [x,y]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \dots y_r^{\varepsilon_r}} [x,y,y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \dots y_r^{\varepsilon_r}} \\ &\equiv [x,y,y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \dots y_r^{\varepsilon_r}} \quad \text{(by the induction hypothesis).} \end{aligned} \tag{2.9}$$

As a result of the Magnus identity, $[x, y, y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}$ is equal to a product of following commutators and their inverses:

$$\begin{aligned}
&(1)\quad [x, y]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}\\
&(2)\quad [x, y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}\\
&(3)\quad [y, y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}\\
&(4)\quad [x, y_1^{\varepsilon_1}]^{y y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}\\
&(5)\quad [y, y_1^{\varepsilon_1}]^{x y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}
\end{aligned}$$

By induction hypothesis (1) is in $\langle S \rangle$. Since $y_1 < y$ so (2) and (3) are in $\langle S \rangle$. To prove (4) is in $\langle S \rangle$, note that

$$[x, y_1^{\varepsilon_1}]^{y y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}} = [x, y_1]^{\varepsilon_1 y_1^{\frac{1}{2}(\varepsilon_1 - 1)} y y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}}.$$

Using (1) modulo $\langle S \rangle$, $y_1^{\frac{1}{2}(\varepsilon_1 - 1)} y y_2^{\varepsilon_2} y_3^{\varepsilon_3} \ldots y_r^{\varepsilon_r}$ can be ordered and therefore

$$[x, y_1^{\varepsilon_1}]^{y y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}} \equiv 1 \pmod{\langle S \rangle}.$$

By a similar argument (5) is also done. As a result,

$$[x, y, y_1^{\varepsilon_1}]^{y_2^{\varepsilon_2} y_3^{\varepsilon_3} \cdots y_r^{\varepsilon_r}} \equiv 1 \pmod{\langle S \rangle}.$$

Substitution in 2.9 gives, $D(r) \equiv 1 \pmod{\langle S \rangle}$ which completes the proof of part **(a)**.

Proof of (b): Using part (a) it is enough to prove that each commutator $[x_i, x_j]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_r^{\varepsilon_r}}$ modulo $\langle \Omega_0(m, n, F) \rangle^F$ can be written as a product of simple basic commutators of weight at most $n-1$. But $[x_i, x_j]^{y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_r^{\varepsilon_r}}$ can be written as a product of commutators of the form $[x_i, x_j, {}_{\lambda_1} y_1^{\varepsilon_1}, {}_{\lambda_2} y_2^{\varepsilon_2}, \ldots, {}_{\lambda_r} y_r^{\varepsilon_r}]$ where for each $1 \leq \ell \leq r$, $\lambda_\ell \in \{0, 1\}$ and $\varepsilon_\ell \in \{1, -1\}$. In addition since $x_i > x_j \leq y_1 \leq y_2 \leq \cdots \leq y_r$, by using the identities lemma, each one of these last commutators modulo $\langle \Omega_0(m, n, F) \rangle^F$ can be written as a product of simple basic commutators of weight at most $n-1$. Hence the proof of part (**b**) and the theorem is complete. □

Definition 2.9. Let $F = \langle X : \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 2$; for arbitrary but fixed integer $n \geq 5$ and each $4 \leq c \leq n+1$, let $\{y_1, y_2, \ldots, y_c, z_1, z_2\} \subseteq X$. Define,

$\Theta(m, c, F) =$

$$\{ [y_2, y_1, y_3, y_4, \ldots, y_{c-2}; z_2, z_1] \mid \{y_1, y_2, \ldots, y_{c-2}, z_1, z_2\} \subseteq X,\ z_1 < z_2,$$
$$y_2 > y_1 \leq y_3 \leq y_4 \leq \ldots \leq y_{c-2},\ z_1 \leq y_1 \}.$$

In addition define,

$$\Lambda(m, n, F) = \Big(\bigcup_{c=4}^{n+1} \Theta(m, c, F) \Big) \bigcup \Omega_0(m, n, F).$$

where $\Omega_0(m, n, F)$ is as defined in Definition 2.7. Denote the normal closure of $\Lambda(m, n, F)$ by $\langle \Lambda(m, n, F) \rangle^F$.

Theorem 2.10. *For $m \geq 2$ and $n \geq 5$, $M(m,n)$ admits the following finite presentation.*

$$M(m,n) = \langle x_1, x_2, \ldots, x_m : \Lambda(m, n, F) \rangle.$$

PROOF. By definition of $M(m,n)$,

$$M(m,n) = \langle x, y ; \gamma_n(F), F'' : \rangle$$

$\gamma_n(F)$, by definition, is generated by commutators of form $[f_1, f_2, \ldots, f_n]$ where $f_i \in F$. Also $[f_1, f_2, \ldots f_n]$ can be written as a product of conjugates of commutators of the form $[z_1, z_2, \ldots, z_n]$ where $z_i \in X$. On the other hand by Theorem 1.2,

$$[z_1, z_2, \ldots, z_n] \equiv [z_1, z_2, z_{\sigma(3)}, z_{\sigma(4)}, \ldots, z_{\sigma(n)}] \pmod{F''},$$

where σ is any arbitrary permutation on $\{3,4,\ldots,n\}$. Let $\{y_1, y_2, \ldots, y_{n-2}\} = \{z_3, z_4, \ldots, z_n\}$ such that $y_1 \leq y_2 \leq \ldots \leq y_{n-2}$; also without loss of generality, modulo F'', we may assume that $z_1 < z_2$. Therefore,

$$[z_1, z_2, \ldots, z_n] \equiv [z_2, z_1, y_1, y_2, \ldots, y_{n-2}]^{-1} \pmod{F''}.$$

If $z_1 \leq y_1$, then the above commutator is congruent to 1. If $y_1 < z_1$ then since

$$[z_2, z_1, y_1] \equiv [z_1, y_1, z_2]^{-1}[z_2, y_1, z_1] \pmod{F''};$$

so modulo $\langle \Omega_0(m,n,F) \rangle^F F''$ we have,

$$\begin{aligned} [z_1, z_2, \ldots, z_n] &\equiv [z_2, z_1, y_1, y_2, \ldots, y_{n-2}]^{-1} \\ &\equiv [z_1, y_1, z_2, y_2, \ldots, y_{n-2}][z_2, y_1, z_1, y_2, \ldots, y_{n-2}]^{-1} \\ &\equiv 1 \quad \text{(by Theorem 1.2).} \end{aligned}$$

Therefore,

$$\gamma_n(F) \leqslant \langle \Omega_0(m,n,F) \rangle^F F''.$$

So $M(m,n)$ admits the following presentation.

$$M(m,n) = \langle\, x, y : \Omega_0(m,n,F),\ F''\, \rangle.$$

In order to show that $F'' \leqslant \langle \Lambda(m,n,F) \rangle^F$, let $[d_1, d_2]$ be an arbitrary element of F''. By part(a) of Theorem 2.8 $[d_1, d_2]$ can be written as a product of conjugates of commutators of the form $[[u_2, u_1]^{v_1^{\varepsilon_1} v_2^{\varepsilon_2} \ldots v_\ell^{\varepsilon_\ell}}, [z_2, z_1]^{w_1^{\delta_1} w_2^{\delta_2} \ldots w_k^{\delta_k}}]$ where $\{u_1, u_2, v_1, \ldots, v_\ell, w_1, \ldots, w_k, z_1, z_2\} \subseteq X$, $u_2 > u_1 \leq v_1 \leq \cdots \leq v_\ell$ and $z_2 > z_1 \leq w_1 \leq \cdots \leq w_k$. Without loss of generality we can assume that $z_1 \leq u_1$. On the other hand,

$$\begin{aligned} &[[u_2, u_1]^{v_1^{\varepsilon_1} v_2^{\varepsilon_2} \ldots v_\ell^{\varepsilon_\ell}}, [z_2, z_1]^{w_1^{\delta_1} w_2^{\delta_2} \ldots w_k^{\delta_k}}] \\ &\qquad = [[u_2, u_1]^{v_1^{\varepsilon_1} v_2^{\varepsilon_2} \ldots v_\ell^{\varepsilon_\ell} w_k^{-\delta_k} \ldots w_2^{-\delta_2} w_1^{-\delta_1}}, [z_2, z_1]]^{w_1^{\delta_1} w_2^{\delta_2} \ldots w_k^{\delta_k}}, \end{aligned}$$

and by part(b) of Theorem 2.8, modulo $\langle\, \Omega_0(m,n,F)\,\rangle^F$. each commutator $\big[\,[\,u_2,u_1\,]^{v_1^{\varepsilon_1} v_2^{\varepsilon_2} \ldots v_\ell^{\varepsilon_\ell} w_k^{-\delta_k} \ldots w_2^{-\delta_2} w_1^{-\delta_1}}, [\,z_2,z_1\,]\,\big]$ can be written as a product of double basic commutators of weight $(4 \leq c \leq n+1)$ and of form $[\,y_2,y_1,y_3,y_4,\ldots,y_{c-2}\,;\,z_2,z_1\,]$ where $\{\,y_1,y_2,y_3,y_4,\ldots,y_{c-2},z_1,z_2\} \subseteq X, z_1 < z_2$, $y_2 > y_1 \leq y_3 \leq y_4 \leq \cdots \leq y_{c-2}$ and $z_1 \leq y_1$. But by the definition of $\Theta(m,c,F)$, each one of these last commutators belongs to $\Theta(m,c,F)$: therefore $F'' \leqslant \langle\, \Lambda(m,n,F)\,\rangle^F$. So in fact $\gamma_n(F)F'' = \langle\, \Lambda(m,n,F)\,\rangle^F$ and the proof is complete. $\square$

CHAPTER 3

A Minimal Presentation of $M(2,n)$

In this chapter we study minimal presentations of free metabelian nilpotent groups of rank 2. So let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x, y\}$ and $x < y$; in order to find a minimal presentation of $M(2,n)$, first we deal with $2 \leq n \leq 5$ and then we study cases with $n \geq 6$.

Definition 3.1. Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x, y\}$ and $x < y$; for $n \geq 5$ and $5 \leq c \leq n+1$ we define the following subsets of F:

$$\Omega_0(2,n,F) = \{ [y, x,_t x,_s y] \mid t+s+2 = n,\ t \geq 0,\ s \geq 0 \},$$

$$\Omega(2,c,F) = \{ [y, x ; y, x,_t x,_s y] \mid t+s+4 = c,\ t \geq 0,\ s \geq 0 \},$$

$$\Delta(2,n,F) = \Omega_0(2,n,F) \cup \Big(\bigcup_{c=5}^{n+1} \Omega(2,c,F) \Big),$$

$$\Delta^*(2,n,F) = \Omega_0(2,n,F) \cup \Big(\bigcup_{c=5}^{n} \Omega(2,c,F) \Big).$$

Also the normal closure of $\Delta^*(2,n,F)$ in F is denoted by $\langle \Delta^*(2,n,F) \rangle^F$. Note that the definition of $\Omega_0(2,n,F)$ here coincides with the one in Definition 2.7 when $m = 2$. Also, $\Omega(2,c,F)$ and $\Delta(2,n,F)$ are the same as $\Theta(m,n,F)$ and $\Lambda(m,n,F)$ in Definition 2.9, respectively when $m = 2$.

Lemma 3.2. *Let $F = \langle X ; \varnothing \rangle$ be free group with basis $X = \{x, y\}$, and for $n \geq 5$ let $d = [y,_k x]$ where $1 \leq k \leq n-3$. Then modulo $\langle \Delta^*(2,n,F) \rangle^F$.*

(i) *if $k = n-3$, then $[d, y, x] = [y,_{(n-3)} x, y, x] \equiv 1$;*

(ii) *if* $1 \leq k \leq n-4$, *then* $[d, y, x] \equiv \prod_{t=1}^{r}[d, x, z_t]^{\epsilon_t f_t}$.
for some integer r *and* $\epsilon_t \in \{1, -1\}$. $f_t \in F$, $z_t \in \{x, y\}$.

PROOF. (i) : Let $d = [y,_{n-3} x]$ which is a commutator of weight $n-2$; Using the Hall-Witt for d, y and x yields.

$$(3.1) \qquad [d, y^{-1}, x]^y [y, x^{-1}, d]^x [x, d^{-1}, y]^d = 1.$$

Working modulo $\langle \Delta^*(2, n, F)\rangle^F$ we have.

$$[d, y^{-1}, x]^y = \big[[d, y, y^{-1}]^{-1}[d, y]^{-1}, x\big]^y \equiv \big([d, y, x]^{-1}\big)^{[y,d]y} \quad (1).$$

And

$$\begin{aligned}
[y, x^{-1}, d]^x &= \big[[y, x]^{-1}, d^x\big] \\
&= \big[d[d, x] ; y, x\big]^{[x,y]} \\
&= [d : y, x]^{[d,x][x,y]} [d, x : y, x]^{[x,y]} \\
&\equiv [d, x : y, x]^{[x,y]} \\
&\equiv \prod_{t=1}^{4}[d, x, z_t]^{\epsilon_t f_t} \qquad (\epsilon_t \in \{1, -1\}.\ f_t \in F.\ z_t \in \{x, y\}) \\
&\equiv 1 \qquad (2).
\end{aligned}$$

Also,

$$\begin{aligned}
[x, d^{-1}, y]^d &= [d, x, y^d] \\
&\equiv \prod_{t=1}^{r}[d, x, z_t]^{\epsilon_t f_t} \qquad (3)
\end{aligned}$$

(for some r and $\epsilon_t \in \{1, -1\}$. $f_t \in F$. $z_t \in \{x, y\}$).

Substitution of (1) , (2) and (3) in 3.1 gives,

$$[d, y, x] = [y,_{(n-3)} x, y, x] \equiv 1,$$

which completes the proof of (i) .

Proof of (ii): If $1 \leq k \leq n-4$ then the weight of commutator d is at most $n-3$. Using the Magnus identity for d, y and x yields,

(3.2) $$[d, y, x^d][x, d, y^x][y, x, d^y] = 1.$$

Working modulo $\langle \Delta^*(2, n, F)\rangle^F$ we have,

$$\begin{aligned}[d, y, x^d] &= [d, x; d, y]^{[x,d]}[d, y, x]^{[x,d]} \\ &= \prod_{t=1}^{r_1}[d, x, z_t]^{\epsilon_t g_t}[d, y, x]^{[x,d]} \qquad (1)\end{aligned}$$

(for some r_1 and $\epsilon_t \in \{1, -1\}$, $g_t \in F$, $z_t \in \{x, y\}$).

And

$$\begin{aligned}[x, d, y^x] &= ([d, x, y^x]^{-1})^{[x,d]} \\ &= \prod_{t=1}^{r_2}[d, x, z_t]^{\epsilon_t h_t}[d, y, x]^{[x,d]} \qquad (2)\end{aligned}$$

(for some r_2 and $\epsilon_t \in \{1, -1\}$, $h_t \in F$, $z_t \in \{x, y\}$).

Also ,

(3.3) $$[y, x, d^y] = [y, x; d, y][y, x, d]^{[d,y]}.$$

But $[y, x, d] = [y, x; y, {}_k x] \equiv 1$ (because it is a basic commutator of weight $k+3 \leq n-1$). Also, $[y, x; d, y] = [y, x; y, {}_k x, y] \equiv 1$ (because it is a basic commutator of weight $k+4 \leq n$).

Therefore by 3.3, $[y, x, d^y] \equiv 1$ (3) .

Finally, substitution of (1) , (2) and (3) in 3.2 gives,

$$[d, y, x] \equiv \prod_{t=1}^{r}[d, x, z_t]^{\epsilon_t f_t}.$$

for some integer r and $\epsilon_t \in \{1, -1\}$, $f_t \in F$, $z_t \in \{x, y\}$.

□

Theorem 3.3. [1] *Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x, y\}$; Then for $2 \le n \le 5$, $M(2,n)$ admits the following minimal presentation,*

$$M(2,n) = \langle\, x, y : B(2,n,F)\, \rangle,$$

where $B(2,n,F)$ is the set of all basic commutators of weight n on X.

PROOF. For $2 \le n \le 4$ this is done by Theorem 2.2 and Theorem 2.6. So assume that $n = 5$. In order to show that

$$M(2,5) = \langle\, x, y : B(2,5,F)\, \rangle,$$

considering the fact that by the definition $M(2,5) = \langle x, y : \gamma_5(F), F'' \rangle$ and since $F'' \subseteq \gamma_5(F)$, it is enough to show that $\gamma_5(F) \leqslant \langle B(2,5,F) \rangle^F$. But since every arbitrary element of $\gamma_5(F)$ can be written as a product of conjugates of commutators of the form $[y, x, a_1, a_2, a_3]$ where $\{a_1, a_2, a_3\} \subseteq \{x, y\}$, it remains to prove that, modulo $\langle B(2,5,F) \rangle^F$,

(1) $[y, x, x, y, x] \equiv 1$,

(2) $[y, x, y, x, x] \equiv 1$,

(3) $[y, x, y, x, y] \equiv 1$,

(4) $[y, x, y, y, x] \equiv 1$.

- To prove (1) let $d = [y, {}_2x]$: then by definition $\Delta^*(2,5,F) = B(2,5,F)$, so applying Lemma 3.2 when $n = 5$ gives,

$$[y, x, x, y, x] = [d, y, x] \equiv 1 \pmod{\langle B(2,5,F) \rangle^F}.$$

[1] Charles C. Sims in [17, page 246] has mentioned that this result can be verified by using computer; however our proof here is independent and is based on different approach.

- To prove (**2**) and (**3**) let $d = [y, x]$; then again using Lemma 3.2 for $n = 5$,

$$[d, y, x]$$
$$= [y, x, y, x] \equiv \prod_{t=1}^{r} [y, x, x, u_t]^{\epsilon_t f_t} \quad (r \in \mathbb{Z}\, \epsilon_t \in \{1, -1\},\ f_t \in F,\ u_t \in \{x, y\}).$$

Therefore for $a \in \{x, y\}$,

(3.4) $$[y, x, y, x, a]$$
$$\equiv \prod_{t=1}^{r_1} \prod_{s=1}^{r_2} [y, x, x, u_t, v_s]^{\epsilon_{(t,s)} f_{(t,s)}}$$
$$(r_1, r_2 \in \mathbb{Z},\ \epsilon_{(t,s)} \in \{1, -1\},\ f_{(t,s)} \in F,\ \{u_t, v_s\} \subseteq \{x, y\}).$$

But for each t and s **either** $u_t = x$ which implies $[y, x, x, x, v_s] \equiv 1$ **or** $u_t = y$ which by (**1**) yields $[y, x, x, y, v_s] \equiv 1$. So in any case, $[y, x, x, u_t, v_s] \equiv 1$. By substitution in 3.4 we get,

$$[y, x, y, x, a] \equiv 1 \pmod{\langle B(2, 5, F)\rangle^F} \quad (a \in \{x, y\}),$$

which completes the proof of (**2**) and (**3**).

- To prove (4), using the Hall-Witt identity for $[y, x, y]$, y and x yields

(3.5) $$[y, x, y, y^{-1}, x]^y\, [y, x^{-1} ; y, x, y]^x\, [x, [y, x, y]^{-1}, y]^{[y,x,y]} = 1.$$

Working modulo $\langle B(2, 5, F)\rangle^F$ we have,

(3.6) $$[y, x, y, y^{-1}, x]^y \equiv \left([y, x, y, y, x]^{-1}\right)^{[y,x,y]^{-1} y}.$$

And

$$[\,y, x^{-1}\,;\, y, x, y\,]^{x}$$

$$= [[\,y, x\,]^{-1}, [\,y, x, y\,][\,y, x, y, x\,]]$$

$$\equiv [\,y, x, y, x\,;\, y, x\,]^{[x, y]}$$

$$\equiv \prod_{t=1}^{4}[\,y, x, y, x, z_t\,]^{\epsilon_t f_t} \qquad (\epsilon_t \in \{1, -1\}\,,\ f_t \in F\,,\ z_t \in \{x, y\})$$

$$\equiv 1 \quad (\text{by } (\mathbf{3})\,). \tag{3.7}$$

Also,

$$[\,x, [\,y, x, y\,]^{-1}, y\,]^{[y, x, y]}$$

$$= [\,y, x, y, x, y^{[y, x, y]}\,]$$

$$\equiv \prod_{t=1}^{r}[\,y, x, y, x, z_t\,]^{\epsilon_t f_t} \tag{3.8}$$

(for some integer r, $\epsilon_t \in \{1, -1\}$, $f_t \in F$ and $z_t \in \{x, y\}$).

Now by substitution of 3.6, 3.7 and 3.8 in 3.5 we get,

$$[\,y, x, y, y, x\,] \equiv 1 \pmod{\langle\, B(2, 5, F)\,\rangle^{F}},$$

which completes the proof of (4). Minimality of the presentation comes from the fact that by the Basis Theorem, $B(2, 5, F)$ is an independent set modulo $\gamma_6(F)$.

□

Lemma 3.4. *Let $F = \langle\, X : \varnothing\,\rangle$ be a free group with basis $X = \{x, y\}$; for $n \geq 6$, $k \geq 1$ and $\ell \geq 1$ let $d = [\,y,_k x,_\ell\, y\,]$ be a simple basic commutator of weight at most $n-2$. Then*

$$[\,d, y, x\,] \equiv \prod_{t=1}^{r}[\,d, x, z_t\,]^{\epsilon_t f_t} \pmod{\langle\, \Delta^{*}(\,2, n, F\,)\,\rangle^{F}},$$

for some integer r, $\epsilon_t \in \{1,-1\}$, $f_t \in F$ *and* $z_t \in \{x,y\}$.

PROOF. We consider two cases, either d is of weight exactly $n-2$, in which case, similar to the proof of part (i) of the Lemma 3.2, using the Hall-Witt identity will give the result. Or d is of weight less than $n-2$, and then similar to the proof of part (ii) of the Lemma 3.2, the Magnus identity will again give the result. □

Proposition 3.5. *Let* $F = \langle X : \varnothing \rangle$ *be a free group with basis* $X = \{x,y\}$*; then for* $n \geq 6$,

$$[y,_{(n-i)} x,_j y, x, z_1, z_2, \ldots, z_{i-j-2}] \equiv 1 \pmod{\langle \Delta^*(2,n,F)\rangle^F}.$$

where $\{z_1, z_2, \ldots, z_{i-j-2}\} \subseteq \{x,y\}$, $3 \leq i \leq n-1$ *and* $1 \leq j \leq i-2$.

PROOF. The proof is based on double induction on i and j. First we prove it for $i=3$ and $j=1$. Then assuming it is also true for k where $3 \leq k \leq i-1$ and all j's such that $1 \leq j \leq k-2$, we will prove it for i and all j's such that $1 \leq j \leq i-2$.

If $i=3$ and $j=1$, then by part (i) of Lemma 3.2,

$$[y,_{(n-3)} x, y, x] \equiv 1 \pmod{\langle \Delta^*(2,n,F)\rangle^F}.$$

Assume it is also true for k where $3 \leq k \leq i-1$ and all j's such that $1 \leq j \leq k-2$; then for $i \geq 4$ and j where $1 \leq j \leq i-2$ we must prove that

$$[y,_{(n-i)} x,_j y, x, z_1, z_2, \ldots, z_{i-j-2}] \equiv 1 \pmod{\langle \Delta^*(2,n,F)\rangle^F}.$$

We do this by another induction on j:

If $j=1$, then let $d = [y,_{(n-i)} x]$; by part (ii) of Lemma 3.2 we have,

$$[d, y, x] = [y,_{(n-i)} x, y, x] \equiv \prod_{t=1}^{r} [y,_{(n-i+1)} x, v_t]^{\epsilon_t f_t},$$

for some integer r and $\epsilon_t \in \{1,-1\}$, $f_t \in F$ and $v_t \in \{x,y\}$.

Thus by adding $z_1, z_2, \ldots, z_{i-3}$ to the above, $[y,_{(n-i)} x, y, x, z_1, z_2, \ldots, z_{i-3}]$ modulo $\langle \Delta^*(2,n,F)\rangle^F$ is congruent to a product of conjugates of commutators of form $[y,_{(n-i+1)} x, u_1, u_2, \ldots, u_{i-2}]$ where $\{u_1, u_2, \ldots, u_{i-2}\} \subseteq \{x,y\}$. So it is enough to show that $[y,_{(n-i+1)} x, u_1, u_2, \ldots, u_{i-2}] \equiv 1$. If $u_1, u_2, \ldots, u_{i-3}$ and u_{i-2} are all x or all y then the commutator $[y,_{(n-i+1)} x, u_1, u_2, \ldots, u_{i-2}]$ is a simple basic commutator of weight n and so is in $\Omega_0(2,n,F)$. Otherwise, assume that s ($1 \le s \le i-3$) is the smallest number such that $u_s = y$ (if $s = i-2$ then again the commutator is in $\Omega_0(2,n,F)$). Also assume that t ($1 \le t \le i-s-2$) is the smallest number such that $u_{s+t} = x$, then by hypothesis of the first induction for $i-s$ and t we have,

$$[y,_{(n-i+1)} x, u_1, u_2, \ldots, u_{i-2}] = [y,_{n-(i-s)} x,_t y, x, u_{s+t+1}, u_{s+t+2}, \ldots, u_{i-2}] \equiv 1.$$

which completes the proof of the case $j=1$.

Assume it is also true for $j-1$, that is for i and $1 \le j-1 < i-2$,

$$[y,_{(n-i)} x,_{(j-1)} y, x, z_1, z_2, \ldots, z_{i-j-1}] \equiv 1.$$

In order to prove that $[y,_{(n-i)} x,_j y, x, z_1, z_2, \ldots z_{i-j-2}] \equiv 1$, let $d = [y,_{n-i} x,_{j-1} y]$; then by Lemma 3.4,

$$[d, y, x] = [y,_{(n-i)} x,_j y, x] \equiv \prod_{t=1}^{r} [y,_{(n-i)} x,_{j-1} y, x, v_t]^{\epsilon_t f_t},$$

for some integer r and $\epsilon_t \in \{1,-1\}$, $v_t \in F$ and $f_t \in F$.

Therefore by adding $z_1, z_2, \ldots, z_{i-j-2}$ to the above, modulo $\langle \Delta^*(2,n,F)\rangle^F$, the commutator $[y,_{(n-i)} x,_j y, x, z_1, z_2, \ldots, z_{i-j-2}]$ is congruent to a product of conjugates of commutators of form $[y,_{(n-i)} x,_{j-1} y, x, u_1, u_2, \ldots, u_{i-j-1}]$

where $\{u_1, u_2, \ldots, u_{i-j-1}\} \subseteq \{x, y\}$. But by hypothesis of the second induction on j, each one of these last commutators is congruent to 1. Thus

$$[y,_{(n-i)} x,_j y, x, z_1, z_2, \ldots, z_{i-j-2}] \equiv 1 \pmod{\langle \Delta^*(2, n, F)\rangle^F}.$$

□

Corollary 3.6. *For $n \geq 6$ and $2 \leq i \leq n-1$,*

$$[y,_{(n-i)} x,_{(i-2)} y, x] \equiv 1 \pmod{\langle \Delta^*(2, n, F)\rangle^F}.$$

PROOF. If $i = 2$, then by definition $[y,_{(n-i)} x, x]$ is in $\Omega_0(2, n, F)$. If $3 \leq i \leq n-1$, it is a direct consequence of Proposition 3.5 for $j = i-2$. □

Theorem 3.7. *Let $F = \langle X : \varnothing \rangle$ be a free group with basis $X = \{x, y\}$; Then for arbitrary but fixed integer $n \geq 6$, $M(2,n)$ admits the following minimal presentation with exactly $\binom{n-1}{2}$ relators.*

$$M(2,n) = \langle x, y : \Delta^*(2, n, F) \rangle$$

PROOF. By Theorem 2.10 when $m = 2$, considering the fact that $\Delta(2, n, F) = \Delta(2, m, F)$ we have,

$$M(2,n) = \langle x, y : \Delta(2, n, F) \rangle.$$

Comparing the definition of $\Delta^*(2, n, F)$ and $\Delta(2, n, F)$ we need to prove that $\Omega(2, n+1, F) \equiv 1 \pmod{\Delta^*(2, n, F)}$. Therefore using the definition of $\Omega(2, n, F)$, we must show that for $t + s + 4 = n + 1$, $t \geq 0$, $s \geq 0$,

$$[y, x,_t x,_s y; y, x] \equiv 1 \pmod{\langle \Delta^*(2, n, F)\rangle^F}.$$

So for convenience let $i = s+2$, then for $2 \leq i \leq n-1$.

$$\begin{aligned}
&[\, y, x,_t\, x,_s\, y \,;\, y, x \,] \\
&\qquad = [\, y,_{(n-s-2)}\, x,_s\, y \,;\, y, x \,] \\
&\qquad = [\, y,_{n-i}\, x,_{i-2}\, y \,;\, y, x \,] \\
&\qquad = \prod_{r=1}^{4} [\, y,_{n-i}\, x,_{i-2}\, y, z_r \,]^{\epsilon_r g_r} \quad (\epsilon_r \in \{1, -1\}),\ g_r \in F,\ z_r \in \{x, y\}).
\end{aligned}$$

For each r:

- If $z_r = x$ then by Corollary 3.6
$$[\, y,_{n-i}\, x,_{i-2}\, y, x \,] \equiv 1 \pmod{\langle\, \Delta^*(\, 2, n, F\,)\,\rangle^F}.$$
- If $z_r = y$ then $[\, y,_{n-i}\, x,_{i-2}\, y, y \,]$ is a simple basic commutator of weight n and so is in $\Delta^*(2, n, F)$.

Hence $[\, d_1, d_2 \,] \equiv 1 \pmod{\langle\, \Delta^*(\, 2, n, F\,)\,\rangle^F}$: i.e $F'' \leqslant \langle\, \Delta^*(\, 2, n, F\,)\,\rangle^F$ and

$$M(2, n) = \langle\, x, y : \Delta^*(2, n, F)\,\rangle.$$

To find the exact number of elements of $\Delta^*(2, n, F)$, there are exactly $n-1$ simple basic commutators of weight n. On the other hand for each $5 \leq c \leq n$ there are exactly $c-3$ commutators of the form $[\, y, x,_t\, x,_s\, y \,;\, y, x \,]$ $(\, t+s+4 = c\,, t \geq 0, s \geq 0\,)$. Therefore in total the number of relators is

$$(n-1) + \sum_{c=5}^{n}(c-3) = \binom{n-1}{2}.$$

Proof of minimality: In order to prove this presentation is minimal first we indicate the following general fact:

Let

$$G = \langle x_1, x_2, \ldots, x_m \,;\, r_1, r_2, \ldots, r_k \rangle,$$

be an m-generator, k-relator group, and let $R = \{r_1, r_2, \ldots, r_k\}$. Then a sufficient condition for G to be minimally related is,

$$P: \qquad r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_k^{\alpha_k} \in [R, F] \implies \alpha_i = 0 \ \ (1 \leq i \leq k).$$

In our case here, $k = \binom{n-1}{2}$ and $R = \Delta^{\bullet}(2, n, F)$. Therefore,

$[R, F]$

$$= [\Delta^{\bullet}(2, n, F), F] = [\langle \Delta^{\bullet}(2, n, F) \rangle^F, F] = [\gamma_n(F)F'', F] = \gamma_{n+1}(F)[F'', F].$$

So condition P becomes,

$$(3.9) \qquad P: \qquad r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_k^{\alpha_k} \in \gamma_{n+1}(F)[F'', F] \implies \alpha_i = 0 \ \ (1 \leq i \leq k).$$

Now we use some group ring methods to justify 3.9. Let $\mathbf{Z}F$ be the free integral group ring and let $\underline{\underline{\mathrm{f}}} = \mathbf{Z}F(F - 1)$ be the augmentation ideal and also let $\underline{\underline{\mathrm{a}}} = \mathbf{Z}F(F' - 1)$.[2] By Theorem B of Gupta and Levin [5], for $n \geq 5$,

$$\gamma_{n+1}(F)[F'', F] = F \cap (1 + \underline{\underline{\mathrm{r}}} + \underline{\underline{\mathrm{f}}}^{n+1}).$$

where $\underline{\underline{\mathrm{f}}}^{n+1} = \mathbf{Z}F(F - 1)^{n+1}$ and $\underline{\underline{\mathrm{r}}} = \mathbf{Z}F([F'', F] - 1)$ are ideals of $\mathbf{Z}F$. So assume that

$$r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_k^{\alpha_k} \equiv 1 \quad (\mathrm{mod}\ F \cap (1 + \underline{\underline{\mathrm{r}}} + \underline{\underline{\mathrm{f}}}^{n+1})).$$

Since $\underline{\underline{\mathrm{r}}} \subseteq \underline{\underline{\mathrm{faf}}}$, so $F \cap (1 + \underline{\underline{\mathrm{r}}} + \underline{\underline{\mathrm{f}}}^{n+1}) \subseteq F \cap (1 + \underline{\underline{\mathrm{faf}}} + \underline{\underline{\mathrm{f}}}^{n+1})$. Indeed working modulo $\underline{\underline{\mathrm{faf}}} + \underline{\underline{\mathrm{f}}}^{n+1}$,

$$r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_k^{\alpha_k} - 1 \equiv \sum_{i=1}^{k} \alpha_i (r_i - 1).$$

[2] For notion and definitions and more details see [2], [3], [14] and [15].

For each $r_i = r_{(t,s)} = [\,y, x\,;\, y, x,_t\, x,_s\, y\,] \in \Delta^*(2,n,F)$ where $5 \le t+s+4 < n$, working modulo $\underline{\underline{\mathrm{faf}}} + \underline{\underline{\mathrm{f}}}^{n+1}$,

$$\begin{aligned}
r_{(t,s)} - 1 &= [\,y, x\,;\, y, x,_t\, x,_s\, y\,] - 1 \\
&\equiv ([\,y,x\,]-1)([\,y,x,_t\,x,_s\,y\,]-1) - ([\,y,x,_t\,x,_s\,y\,]-1)([\,y,x\,]-1) \\
&\equiv \Big(([\,y,x\,]-1)((\overline{x}-1)^t(\overline{y}-1)^s)([\,y,x\,]-1)\Big) \\
&\quad - \Big(([\,y,x\,]-1)((x-1)^t(y-1)^s)([\,y,x\,]-1)\Big) \\
&\equiv ([\,y,x\,]-1)\Big((\overline{x}-1)^t(\overline{y}-1)^s - (x-1)^t(y-1)^s\Big)([\,y,x\,]-1),
\end{aligned}$$

where $\overline{x} = x^{-1}$ and $\overline{y} = y^{-1}$. As a result,

$$\sum_{5\le t+s+4<n} \alpha_{(t,s)}(r_{(t,s)} - 1)$$
$$\equiv ([\,y,x\,]-1) \sum_{5\le t+s+4<n} \alpha_{(t,s)}\Big((\overline{x}-1)^t(\overline{y}-1)^s - (x-1)^t(y-1)^s\Big)([\,y,x\,]-1).$$

Therefore if $\sum_{5\le t+s+4<n} \alpha_{(t,s)}(r_{(t,s)} - 1) = 0$, then

$$\sum_{5\le t+s+4<n} \alpha_{(t,s)}\Big((\overline{x}-1)^t(\overline{y}-1)^s - (x-1)^t(y-1)^s\Big) = 0.$$

But since $(\overline{x}-1) = -\overline{x}(x-1)$, $(\overline{y}-1) = -\overline{y}(y-1)$ and $\underline{\underline{\mathrm{f}}}$ is a free $\mathbf{Z}F$-module with the basis $\{x-1, y-1\}$, this implies that $\alpha_{(t,s)} = 0$.

Also we note that by the Basis Theorem (i.e. Theorem 1.5), modulo $\gamma_{n+1}(F)$, the set of all basic commutators of weight n is an independent set. So in any case for all $1 \le i \le k$, $\alpha_i = 0$ and the proof is complete. □

CHAPTER 4

A Finite Presentation of $M(m,n)$ for $m \geq 3$ and $n \geq 5$

In Chapter 2 we studied minimal presentations of free metabelian nilpotent groups of rank $m \geq 2$ and class $n-1$ for $2 \leq n \leq 4$. Also in Chapter 3 we found minimal presentations of free metabelian nilpotent groups of rank $m = 2$ and class $n-1$ for $n \geq 2$.

Our next goal is to find the best possible finite presentation, in terms of basic commutators, for free metabelian nilpotent groups of rank $m \geq 3$ and class $n-1$, $(n \geq 5)$. This chapter is in fact preparation for this goal and by the end of this chapter we will find a very sharp finite presentation of $M(m,n)$.

Through this chapter and from now on we always assume that $n \geq 5$ is an arbitrary but fixed integer and $F = \langle X ; \varnothing \rangle$ is a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $(m \geq 3)$. Also whenever we put $\hat{}$ at the top of an element it means that element is missing.

Definition 4.1. Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 3$. For arbitrary but fixed integer $n \geq 5$ and each $4 \leq c \leq n+1$ define,

$$\Omega_1(m,c,F) = \{ [z_i, z_1 ; z_j, z_2, z_3, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c] \mid z_1 \leq z_2 \leq \cdots \leq z_c,$$
$$z_2 < z_j \leq z_i \leq z_c, z_t \in X\,(1 \leq t \leq n) \};$$

$$\Omega_2(m,c,F) = \{ [z_c, z_2 ; z_i, z_1, z_3, \ldots, \hat{z}_i, \ldots, z_{c-1}] \mid z_1 < z_2 \leq \cdots \leq z_c,$$
$$z_i < z_c, z_t \in X\,(1 \leq t \leq n) \};$$

$$\Omega_3(m,c,F) = \{\, [\, z_2, z_1\, ;\, z_c, z_3, z_4, z_5, \ldots, z_{c-1}\,] \mid z_1 < z_2 < z_3 < \cdots < z_c,$$
$$z_t \in X\,(\,1 \le t \le n\,)\,\}.$$

Let $\Omega(m,c,F) = \Omega_1(m,c,F) \cup \Omega_2(m,c,F) \cup \Omega_3(m,c,F)$.

Define,

$$\Delta(m,n,F) = \Big(\bigcup_{c=4}^{n+1} \Omega(m,c,F)\Big) \bigcup \Omega_0(m,n,F)$$

Recall $\Omega_0(m,n,F)$, as is defined in Definition 2.7, is the set of all simple basic commutators of weight n on X. Denote the normal closure of $\Delta(m,n,F)$ by $\langle\, \Delta(m,n,F)\,\rangle^F$.

4.1. Basic Tools

In this section after introducing the Jacobi Lemma we show that every commutator of weight 4 and form $[\, y_{i_1}^{\varepsilon_{i_1}}, y_{i_2}^{\varepsilon_{i_2}} ; y_{i_3}^{\varepsilon_{i_3}}, y_{i_3}^{\varepsilon_{i_3}}\,]$ is congruent to 1 modulo $\langle\, \Delta(m,n,F)\,\rangle^F$.

Lemma 4.2. *Let* $\{\, y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_c\,\} \subseteq X$ *such that* $y_1 \le y_2 \le \cdots \le y_r,\,(\,r \ge 2\,)$ *and* $z_1 \le z_2 \le \cdots \le z_c\,(\,4 \le c \le n+1\,)$. *Then modulo* $\langle\, \Delta(m,n,F)\,\rangle^F$,

(a) *For* $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\} \subseteq \{1, -1\}$,

$$[\, y_i, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, \hat{y}_i, \ldots, y_r^{\varepsilon_r}\,] \equiv C_1^{\delta_1} C_2^{\delta_2} \ldots C_k^{\delta_k} \quad \textit{(for some integer } k\textit{)},$$

where for each $1 \le \ell \le k$, $\delta_\ell \in \{1, -1\}$ *and* C_ℓ *is a simple basic commutator of weight at most* $n-1$ *and of form* $C_\ell = [\, y_i, y_1, y_{t_1}, y_{t_2}, \ldots, y_{t_s}]$ *such that* $y_1 \le y_{t_1} \le y_{t_2} \le \cdots \le y_{t_s}$, $\{t_1, t_2, \ldots, t_s\} \subseteq \{1, 2, 3, \ldots, r\}$ *and* $0 \le s \le n-1$.

(b) *For* $z_2 \le z_j \le z_i \le z_c$ *and* $\{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_c\} \subseteq \{1, -1\}$,

(i) $[\, z_i, z_1\, ;\, z_j, z_2^{\varepsilon_2}, z_3^{\varepsilon_3}, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c^{\varepsilon_c}\,] \equiv 1$;

(ii) $[\, z_i, z_1\, ;\, z_3^{\varepsilon_3}, z_2^{\varepsilon_2}, z_4^{\varepsilon_4}, \ldots, \hat{z}_i, \ldots, z_c^{\varepsilon_c}\,] \equiv 1$.

(c) *For* $z_1 < z_2 \leq z_i < z_c$ *and* $\{\varepsilon_3, \varepsilon_4, \ldots, \varepsilon_c\} \subseteq \{1, -1\}$,

$[\, z_c^{\varepsilon_c}, z_2 \,;\, z_i, z_1, z_3^{\varepsilon_3}, z_4^{\varepsilon_4}, \ldots, \hat{z}_i, \ldots, z_{c-1}^{\varepsilon_{c-1}}, \,] \equiv 1.$

PROOF. The proof of part(a) is by induction on r. Let $r = 2$. If $\varepsilon_2 = +1$ there is noting to prove. If $\varepsilon_2 = -1$ Then first we notice that for each d and x in F,

$$[\, d, x^{-1} \,] = [\, d, x, x^{-1} \,]^{-1} [\, d, x \,]^{-1}. \tag{4.1}$$

Now using 4.1 twice we have,

$$\begin{aligned}[\, y_1, y_2^{-1} \,] &= [\, y_1, y_2, y_2^{-1} \,]^{-1} [\, y_1, y_2 \,]^{-1} \\ &= [\, y_1, y_2, y_2 \,][\, y_1, y_2, y_2, y_2^{-1} \,][\, y_1, y_2 \,]^{-1}.\end{aligned}$$

where $[\, y_1, y_2 \,]$ and $[\, y_1, y_2, y_2 \,]$ are simple basic commutators of weight 2 and 3. Applying 4.1 on $[\, y_1, y_2, y_2, y_2^{-1} \,]$ repeatedly would create simple basic commutators. Continuing this process, we reach the point that the commutator $[\, y_1, \underbrace{y_2, y_2, \ldots, y_2}_{n-2}, y_2^{-1} \,]$ would appear. But then, modulo $\langle\, \Omega_0(m, n, F) \,\rangle^F$,

$$[\, y_1, \underbrace{y_2, y_2, \ldots, y_2}_{n-2}, y_2^{-1} \,] = \big(\, [\, y_1, \underbrace{y_2, y_2, \ldots, y_2}_{n-1} \,]^{-1} \big)^{y_2^{-1}} \equiv 1,$$

so the proof is complete for $r = 2$. Assume it is also true for $r - 1$. Then modulo $\langle\, \Omega_0(m, n, F) \,\rangle^F$, using the induction hypothesis,

$$\begin{aligned}[\, y_i, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}, \ldots, \hat{y}_i, \ldots, y_r^{\varepsilon_r} \,] &\equiv [\, B_1^{\theta_1} B_2^{\theta_2} \ldots B_\ell^{\theta_\ell}\, y_r^{\varepsilon_r} \,] \\ &\equiv C_1^{\delta_1} C_2^{\delta_2} \ldots C_k^{\delta_k} \quad \text{(for some integer } k\text{)},\end{aligned}$$

where B_j $(1 \leq j \leq \ell)$ is a simple basic commutator of weight at most $n-1$. Now for each $1 \leq i \leq k$, either $C_i = B_j$ which is basic or $C_i = [\, B_j, y_r^{\varepsilon_r} \,]$. In C_i if $\varepsilon_r = +1$, it is basic, and if $\varepsilon_r = -1$, it is done by exactly same argument as in case $r = 2$.

Proof of (b-i): By part(a) modulo $\langle \Delta(m,n,F) \rangle^F$,

$$[z_j, z_2^{\varepsilon_2}, z_3^{\varepsilon_3}, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c^{\varepsilon_c}] \equiv C_1'^{\delta_1} C_2'^{\delta_2} \ldots C_k'^{\delta_k} \quad \text{(for some integer } k\text{)},$$

where for each $1 \leq \ell \leq k$, $\delta_\ell \in \{1, -1\}$ and C_ℓ is a simple basic commutator of weight at most $n-1$ and of form $C_\ell = [z_j, z_2, z_{t_1}, z_{t_2}, \ldots, z_{t_s}]$ such that $z_2 \leq z_{t_1} \leq z_{t_2} \leq \cdots \leq z_{t_s}$, $\{t_1, t_2, \ldots, t_s\} \subseteq \{1, 2, 3, \ldots, r\}$ and $0 \leq s \leq n-1$. Therefore modulo $\langle \Delta(m,n,F) \rangle^F$,

$$\begin{aligned}[z_j, z_2^{\varepsilon_2}, z_3^{\varepsilon_3}, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c^{\varepsilon_c}] &\equiv \prod_{\ell=1}^{k} [z_i, z_1 : C_\ell]^{\varepsilon_\ell f_\ell} \\ &\equiv 1 \quad (\text{by definition of } \Omega(m, c, F)).\end{aligned}$$

Proof of (b-ii): If $\varepsilon_3 = +1$, this follows from part(i). If $\varepsilon_3 = -1$, then since

$$[z_3^{-1}, z_2^{\varepsilon_2}] = [z_3, z_2^{\varepsilon_2}, z_3^{-1}]^{-1} [z_3, z_2^{\varepsilon_2}]^{-1};$$

therefore $[z_i, z_1 : z_3^{-1}, z_2^{\varepsilon_2}, z_4^{\varepsilon_4}, \ldots, \hat{z}_i, \ldots, z_c^{\varepsilon_c},]$, modulo $\langle \Delta(m,n,F) \rangle^F$, can be written as a product of conjugates of commutators of the form $[z_i, z_1 ; z_3, z_2^{\varepsilon_2}, z_{t_1}^{\varepsilon_{t_1}}, z_{t_2}^{\varepsilon_{t_2}}, \ldots, z_{t_s}^{\varepsilon_{t_s}}]$ where $z_2 \leq z_{t_1} \leq z_{t_2} \leq \cdots \leq z_{t_s}$. But by part(i) each one of these commutators is congruent to 1.

Proof of (c): Since $[z_c^{\varepsilon_c}, z_2] = [z_c, z_2]^{\varepsilon_c z_c^{(1-\varepsilon_c)/2}}$ so

$$\begin{aligned}C &= [z_c^{\varepsilon_c}, z_2 ; z_i, z_1, z_3^{\varepsilon_3}, z_4^{\varepsilon_4}, \ldots, \hat{z}_i, \ldots, z_{c-1}^{\varepsilon_{c-1}}] \\ &= \left[z_c, z_2 ; [z_i, z_1, z_3^{\varepsilon_3}, z_4^{\varepsilon_4}, \ldots, \hat{z}_i, \ldots, z_{c-1}^{\varepsilon_{c-1}}]^{z_c^{(\varepsilon_c-1)/2}}\right]^{\delta_1 g_1}.\end{aligned}$$

But by part (a), C is congruent to a product of conjugates of commutators of the form $C_r = [z_c, z_2 ; y_j, y_1, y_2, \ldots, \hat{y}_j, \ldots, y_r]$ where $2 \leq r \leq n-1$ and $\{y_1, y_2, \ldots, y_r\} \subseteq \{z_1, z_3, z_4, \ldots, z_c\}$ such that $y_1 \leq y_2 \cdots \leq y_r$. Now for each r, if $y_1 = z_1$, then since z_1 is repeated only once we have $z_1 = y_1 < z_2 \leq y_2$; therefore by definition $C_r \equiv 1$. If $y_1 > z_1$, then necessarily $z_2 \leq y_2$ and so by part (b), $C_r \equiv 1$. Hence, $C \equiv 1$. □

Lemma 4.3. *Let a,b and c be in group G; then*

$$[a,b,c] \equiv [b,c,a,]^{-1}[a,c,b] \pmod{\langle T\rangle^G},$$

where $\langle T\rangle^G$ is the normal closure of T in G and T is the subset defined as follows,

$$T = \{[a,b;\, a,c],\, [b,c;\, a,b],\, [b,c,a,:\, a,b],\, [b,c,a,:\, a,c],$$
$$[a,c;\, b,c],\, [a,c,b;\, b,c],\, [a,c,b;\, a,c]\}$$

PROOF. By the Hall identity, $[a,b,c^a][c,a,b^c][b,c,a^b] = 1 \quad (*)$. Working modulo $\langle T\rangle^G$ we have,

$$[a,b,c^a] = [a,b;\, a,c][a,b,c]^{[c,a]} \equiv [a,b,c]^{[c,a]}.$$

On the other hand,

$$\begin{aligned}[c,a,b^c] &= [c,a;\, b,c][c,a,b]^{[b,c]}\\ &\equiv \left([a,c,b]^{-1}\right)^{[c,a][b,c]}\\ &\equiv \left([a,c,b]^{-1}\right)^{[b,c][c,a]}.\end{aligned}$$

In addition, $[b,c,a^b] = [b,c;\, a,b][b,c,a]^{[a,b]} \equiv [b,c,a]^{[a,b]}$. Therefore by substitution in $(*)$ we get, modulo $\langle T\rangle^G$,

$$[a,b,c] \equiv \left([b,c,a]^{[a,b][a,c]}\right)^{-1}[a,c,b]^{[b,c]} \equiv [b,c,a,]^{-1}[a,c,b].$$

□

Lemma 4.4. *(Jacobi Lemma) Let $\{y_1, y_2, y_3\} \subseteq X$ such that $y_1 < y_2 < y_3$; then modulo $\langle\Delta(m,n,F)\rangle^F$,*

(a) $[y_3, y_2, y_1] \equiv [y_2, y_1, y_3]^{-1}[y_3, y_1, y_2]$;

(b) $[y_3^{\varepsilon_3}, y_2^{\varepsilon_2}, y_1^{\varepsilon_1}] \equiv [y_2^{\varepsilon_2}, y_1^{\varepsilon_1}, y_3^{\varepsilon_3}]^{-1}[y_3^{\varepsilon_3}, y_1^{\varepsilon_1}, y_2^{\varepsilon_2}]$,

where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{1, -1\}$.

PROOF. To prove part(a)and part(b), by using Lemma 4.3, it is enough to prove that

$$T \equiv 1 \pmod{\langle \Delta(m,n,F)\rangle^F},$$

where in part(a) the set T is

$$T = \{[y_3, y_2 ; y_3, y_1], [y_2, y_1 ; y_3, y_2], [y_2, y_1, y_3 ; y_3, y_2], [y_2, y_1, y_3 ; y_3, y_1], [y_3, y_1 ; y_2, y_1], [y_3, y_1, y_2 ; y_2, y_1], [y_3, y_1, y_2 ; y_3, y_1]\}.$$

But trivially all of the elements of T are congruent to 1 except $[y_3, y_1, y_2; y_2, y_1]$. For this one, using Lemma 4.2, we have,

$$[y_3, y_1, y_2 ; y_2, y_1] \equiv [y_3, y_1 ; y_2, y_1, y_2^{-1}]^{y_2} \equiv 1.$$

For part(b) the set T is

$$T = \{[y_3^{\varepsilon_3}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1^{\varepsilon_1}], [y_2^{\varepsilon_2}, y_1^{\varepsilon_1} ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2}], [y_2^{\varepsilon_2}, y_1^{\varepsilon_1}, y_3^{\varepsilon_3} ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2}], [y_2^{\varepsilon_2}, y_1^{\varepsilon_1}, y_3^{\varepsilon_3} ; y_3^{\varepsilon_3}, y_1^{\varepsilon_1}], [y_3^{\varepsilon_3}, y_1^{\varepsilon_1} ; y_2^{\varepsilon_2}, y_1^{\varepsilon_1}], [y_3^{\varepsilon_3}, y_1^{\varepsilon_1}, y_2^{\varepsilon_2} ; y_2^{\varepsilon_2}, y_1^{\varepsilon_1}], [y_3^{\varepsilon_3}, y_1^{\varepsilon_1}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1^{\varepsilon_1}]\}.$$

In order to prove each one of the elements of T is congruent to 1, we consider two different cases :

Case 1 : Two of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are $+1$ and the other one is -1.

Case 2 : Two of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are -1 and the other one is $+1$.

Proof of Case 1 : Case 1 itself is divided into two sub-cases:

$(1-i)$ **Either** $\varepsilon_1 = +1$ and then $\varepsilon_2 = +1$, $\varepsilon_3 = -1$ and or $\varepsilon_2 = -1$, $\varepsilon_3 = +1$.

$(1-ii)$ **Or** $\varepsilon_1 = -1$ then $\varepsilon_2 = \varepsilon_3 = +1$.

Proof of (1-i): Working modulo $\langle \Delta(m,n,F)\rangle^F$ and using Lemma 1.1, since either $\varepsilon_1 = \varepsilon_2 = +1$, $\varepsilon_3 = -1$ or $\varepsilon_1 = \varepsilon_3 = +1$, $\varepsilon_2 = -1$, we have

•

$$[\, y_3^{-1}, y_2 \,;\, y_3^{-1}, y_1 \,] = \left[\, ([\, y_3, y_2 \,]^{-1})^{y_3^{-1}} \,.\, ([\, y_3, y_1 \,]^{-1})^{y_3^{-1}} \,\right]$$

$$= \left[\, [\, y_3, y_2 \,]^{-1} \,,\, [\, y_3, y_1 \,]^{-1} \,\right]^{y_3^{-1}}$$

$$\equiv 1.$$

Also by Lemma 4.2, $[\, y_3, y_2^{-1} : y_3, y_1 \,] \equiv 1$. Therefore, $[\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} : y_3^{\varepsilon_3}, y_1 \,] \equiv 1$.

- $[\, y_2^{-1}, y_1 \,;\, y_3, y_2^{-1} \,] = \left[\, [\, y_2, y_1 \,]^{-1}, [\, y_3, y_2 \,]^{-1} \,\right]^{y_2^{-1}} \equiv 1$. Also by Lemma 4.2 , $[\, y_2, y_1 \,;\, y_3^{-1}, y_2 \,] \equiv 1$. Therefore, $[\, y_2^{\varepsilon_2}, y_1 : y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1$.
- By part (c) of Lemma 4.2 , $[\, y_2, y_1, y_3^{-1} : y_3^{-1}, y_2 \,] \equiv 1$. Also,

$$[\, y_2^{-1}, y_1, y_3 \,;\, y_3, y_2^{-1} \,] = \left[\, [\, y_2, y_1 \,]^{y_2^{-1}} ([\, y_2, y_1 \,]^{-1})^{y_3} . ([\, y_3, y_2 \,]^{-1})^{y_2^{-1}} \,\right]$$

$$= [\, y_3, y_2 \,;\, y_2, y_1 \,]^{g_1} \left[\, [\, y_3, y_2 \,], [\, y_2, y_1 \,]^{y_2^{-1} y_3 y_2} \,\right]^{g_2}$$

$$\equiv \left[\, [\, y_3, y_2 \,], [\, y_2, y_1 \,]^{y_3 [y_3, y_2]} \,\right]^{g_3}$$

$$\equiv \left[\, [\, y_3, y_2 \,], [\, y_2, y_1 \,]^{y_3} \,\right]^{g_4}$$

$$\equiv [\, y_3, y_2 : y_2, y_1 \,]^{g_5} [\, y_3, y_2 : y_2, y_1, y_2, y_3 \,]^{g_6}$$

$$\equiv 1,$$

for some $g_1, g_2, \ldots, g_5$ and g_6 in F.

Therefore, $[\, y_2^{\varepsilon_2}, y_1, y_3^{\varepsilon_3} : y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1$.

- With a similar argument as the previous one, $[\, y_2^{\varepsilon_2}, y_1, y_3^{\varepsilon_3} \,;\, y_3^{\varepsilon_3}, y_1 \,] \equiv 1$.
-

$$[\, y_3^{-1}, y_1 \,;\, y_2, y_1 \,] = ([\, y_3, y_1 \,;\, y_2, y_1, y_3 \,]^{-1})^{[y_1, y_3] y_3^{-1}}$$

$$. ([\, y_3, y_1 : y_2, y_1 \,]^{-1})^{[y_1, y_3][y_2, y_1, y_3] y_3^{-1}}$$

$$\equiv 1.$$

Also by Lemma 4.2, $[y_3, y_1 ; y_2^{-1}. y_1] \equiv 1$. Therefore $[y_3^{\varepsilon_3}. y_1 ; y_2^{\varepsilon_2}, y_1] \equiv 1$.

- Similar to the above, $[y_3, y_1, y_2^{-1} ; y_2^{-1} y_1] \equiv 1$. Also,

$$[y_3^{-1}, y_1, y_2 ; y_2 y_1] \equiv [[y_3, y_1,], [y_2, y_1]^{y_2^{-1} y_3}]^{g} \qquad (g \in F)$$
$$\equiv 1 \qquad \text{(by Lemma 4.2)}.$$

therefore, $[y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2} ; y_2^{\varepsilon_2}, y_1] \equiv 1$.

- By Lemma 4.2, $[y_3, y_1, y_2^{-1} ; y_3 y_1] \equiv 1$. Also,

$$[y_3^{-1}, y_1, y_2 ; y_3^{-1} y_1] \equiv ([[y_3, y_1,]^{y_3^{-1} y_2}, [y_3^{-1}, y_1]]^{-1})^{g_1}$$
$$\equiv [[y_3, y_1,]^{y_2}, [y_3, y_1]]^{g_2}$$
$$\equiv 1.$$

for some g_1 and g_2 in F. Therefore $[y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1] \equiv 1$. So the proof of case (1-i) is complete.

Proof of (1-ii) : In this case $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = +1$. Working modulo $\langle \Delta(m, n, F) \rangle^F$ and by using Lemma 1.1, part (a) and the proof of case (1-i) we have,

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$$[y_3, y_2 ; y_3, y_1^{-1}] = [y_3, y_1 ; [y_3, y_2]^{y_1}]^{g_1}$$
$$\equiv [y_3, y_1 ; y_3, y_2, y_1]^{g_1}$$
$$\equiv [y_3, y_1 ; y_3, y_1, y_2]^{g_2}([y_3, y_1 ; y_2, y_1, y_3,]^{-1})^{g_3}$$
$$\equiv 1 \quad \text{(by part (a))},$$

for some g_1, g_2 and g_3 in F.

•

$$\begin{aligned}[y_2, y_1^{-1}; y_3, y_2] &\equiv [y_3, y_2, y_1; y_2, y_1]^{g_1} \\ &\equiv [y_2, y_1; y_2, y_1, y_3]^{g_2}([y_2, y_1; y_2, y_1, y_3]^{-1})^{g_1} \quad (\text{by part (a)}) \\ &\equiv 1 \quad (g_1, g_2 \in F).\end{aligned}$$

•

$$\begin{aligned}&[y_2, y_1^{-1} \cdot y_3; y_3, y_2] \\ &\equiv [y_2, y_1^{-1}; y_3, y_2]^{\delta_1 g_1} [y_2, y_1^{-1}; [y_3, y_2]^{y_3^{-1}}]^{\delta_2 g_2} \\ &\equiv [y_2, y_1^{-1}; y_3^{-1} \cdot y_2]^{\delta_3 g_3} \quad (\text{by previous one}) \\ &\equiv [y_2, y_1; y_3^{-1}, y_2]^{\delta_4 g_4} [y_2, y_1; y_3^{-1} \cdot y_2, y_1]^{\delta_5 g_5} \\ &\equiv [y_2, y_1; y_3^{-1} \cdot y_1 \cdot y_2]^{\delta_6 g_6} [y_2, y_1; y_2, y_1, y_3^{-1}]^{\delta_7 g_7} \quad (\text{by case (1-i)}) \\ &\equiv 1 \quad (\text{by Lemma 4.2 and the proof of case (1-i)}).\end{aligned}$$

for some $g_1, g_2, \ldots, g_7$ in F and $\delta_1, \delta_2, \ldots, \delta_7$ in $\{1, -1\}$.

•

$$\begin{aligned}&[y_2, y_1^{-1}, y_3; y_3, y_1^{-1}] \\ &\qquad \equiv [y_2, y_1^{-1}; y_3, y_1^{-1}]^{\delta_1 g_1} [y_2, y_1^{-1}; y_3^{-1} \cdot y_1^{-1}]^{\delta_2 g_2} \\ &\qquad \equiv [y_2, y_1; y_3, y_1]^{\delta_3 g_3} [y_2, y_1; y_3^{-1} \cdot y_1]^{\delta_4 g_4} \\ &\qquad \equiv 1 \quad (\text{by the proof of case (1-i)}).\end{aligned}$$

for some g_1, g_2, g_3 in F and $\delta_1, \delta_2, \delta_3$ in $\{1, -1\}$.

- $[y_3, y_1^{-1}; y_2, y_1^{-1}] = [y_3, y_1; y_2, y_1]^g \equiv 1 \quad (g \in F)$.

- $$\begin{aligned}
&[\, y_3, y_1^{-1}, y_2 \,;\, y_2, y_1^{-1} \,] \\
&\equiv [\, y_3, y_1^{-1} : y_2, y_1^{-1} \,]^{\delta_1 g_1} [\, y_3, y_1^{-1} : y_2^{-1}, y_1^{-1} \,]^{\delta_2 g_2} \\
&\equiv ([\, y_3, y_1 : y_2, y_1 \,])^{\delta_3 g_3} ([\, y_3, y_1 : y_2^{-1}, y_1 \,])^{\delta_4 g_4} \\
&\equiv 1 \quad \text{(by Lemma 4.2)},
\end{aligned}$$

for some $g_1, g_2, g_3, g_4 \in F$, $\delta_1, \delta_2, \delta_3, \delta_4 \in \{1, -1\}$.

- $$\begin{aligned}
&[\, y_3, y_1^{-1}, y_2 \,;\, y_3, y_1^{-1} \,] \\
&\equiv [\, y_3, y_1 : y_3, y_1^{-1}, y_2 \,]^{\delta_1 g_1} \left[\, y_3, y_1 : [\, y_3, y_1^{-1}, y_2 \,]^{y_1} \,\right]^{\delta_2 g_2} \\
&\equiv \left[\, y_3, y_1 ; [\, y_3, y_1^{-1} \,]^{y_2 y_1} \,\right]^{\delta_3 g_3} \left[\, y_3, y_1 : [\, y_3, y_1^{-1}, \,]^{y_1} \,\right]^{\delta_4 g_4} \text{(by Lemma 4.2)} \\
&\equiv \left[\, y_3, y_1 ; [\, y_3, y_1^{-1} \,]^{y_1 y_2} \,\right]^{\delta_5 g_5} \\
&\equiv [\, y_3, y_1 : y_3, y_1, y_2 \,]^{\delta_6 g_6} \\
&\equiv 1.
\end{aligned}$$

for some $g_1, g_2, \ldots, g_6$ in F and $\delta_1, \delta_2, \ldots, \delta_6$ in $\{1, -1\}$. So the proof of case(1-ii) and therefore case 1 is complete.

Proof of case 2 **:** Case 2 also is divided into two sub-cases :

$(2-i)$ **Either** $\varepsilon_1 = +1$ and so $\varepsilon_2 = \varepsilon_3 = -1$.

$(2-ii)$ **Or** $\varepsilon_1 = -1$ and so $\varepsilon_2 = +1$, $\varepsilon_3 = -1$ or $\varepsilon_2 = -1$, $\varepsilon_3 = +1$.

Proof of (2-i) : We first note that by the proof of case 1,

$$[\, y_2^{\varepsilon_2}, y_1 \,;\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1 \quad \& \quad [\, y_3^{\varepsilon_3}, y_1 \,;\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1 \qquad (1),$$

where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{1, -1\}$.

Working modulo $\langle \Delta(m,n,F) \rangle^F$ and using Lemma 4.2 and (1) we have,

$$[y_3^{-1}, y_2^{-1} ; y_3^{-1}, y_1] \equiv [y_3, y_2^{-1} : y_3, y_1]^{g_1} \equiv 1 \qquad (g_1 \in F).$$

$$[y_2^{-1}, y_1 ; y_3^{-1}, y_2^{-1}] \equiv [y_2, y_1 : y_3^{-1}, y_2]^{g_2} \equiv 1 \qquad (g_2 \in F).$$

$$[y_2^{-1}, y_1, y_3^{-1} ; y_3^{-1}, y_2^{-1}] \equiv ([y_2^{-1}, y_1 ; y_3^{-1}, y_2^{-1}]^{-1})^{f_1}([y_2^{1}, y_1 : y_3, y_2^{-1}]^{-1})^{f_2}$$

$$\equiv 1 \qquad (f_1, f_2 \in F).$$

Similarly, $[y_2^{-1}, y_1, y_3^{-1} ; y_3^{-1}, y_1] \equiv 1$. Also. $[y_3^{-1}, y_1 : y_2^{-1}, y_1] \equiv 1$. In addition $[y_3^{-1}, y_1, y_2^{-1} : y_3^{-1}, y_1] \equiv 1$.

Proof of (2-ii): In this case either $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = +1$ or $\varepsilon_1 = \varepsilon_3 = -1, \varepsilon_2 = +1$. Working modulo $\langle \Delta(m,n,F) \rangle^F$ and using Lemma 4.2 and (1) and case 1 we have.

•

$$[y_3^{\varepsilon_3}, y_2^{\varepsilon_2} : y_3^{\varepsilon_3}, y_1^{-1}]$$

$$= [y_3^{\varepsilon_3}, y_1 ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2}]^{\delta_1 g_1}[y_3^{\varepsilon_3}, y_1 ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2}, y_1]^{\delta_2 g_2}$$

$$\equiv [y_3^{\varepsilon_3}, y_1 ; y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2}]^{\delta_3 g_3}[y_3^{\varepsilon_3}, y_1 : y_2^{\varepsilon_2}, y_1, y_3^{\varepsilon_3}]^{\delta_4 g_4} \quad \text{(by (1) and case 1)} \quad (2).$$

for some g_1, g_2, g_3, g_4 in F and $\delta_1, \delta_2, \delta_3, \delta_4$ in $\{1, -1\}$. Set $C_1 = [y_3^{\varepsilon_3}, y_1 ; y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2}]$ and $C_2 = [y_3^{\varepsilon_3}, y_1 ; y_2^{\varepsilon_2}, y_1, y_3^{\varepsilon_3}]$.
If $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = +1$, then by Lemma 4.2, $C_1 = [y_3, y_1 : y_3, y_1, y_2^{-1}] \equiv 1$ and $C_2 = [y_3, y_1 : y_2^{-1}, y_1, y_3] \equiv 1$. If $\varepsilon_1 = \varepsilon_3 = -1, \varepsilon_2 = +1$, then

$$C_1 = [y_3^{-1}, y_1 : y_3^{-1}, y_1, y_2] \equiv [y_3, y_1 ; [y_3, y_1]^{y_2}]^g$$

$$\equiv 1 \qquad \text{(by (1), where } g \in F).$$

Also, $C_2 = [y_3^{-1}, y_1 ; y_2, y_1, y_3^{-1}] \equiv [y_3, y_1 ; y_2, y_1, y_3]^g \equiv 1 \ (g \in F)$.
Therefore in any case $C_1 \equiv C_2 \equiv 1$ and by sustitution in (2) we get,

$$[y_3^{\varepsilon_3}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1^{-1}] \equiv 1.$$

- With a similar argument as the above. $[\, y_2^{\varepsilon_2}, y_1^{-1} ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1.$
- If $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = +1$, then

$$
\begin{aligned}
&[\, y_2^{-1}, y_1^{-1}, y_3 ; y_3, y_2^{-1} \,] \\
&\equiv [\, y_2^{-1}, y_1^{-1} : y_3, y_2^{-1} \,]^{\delta_1 g_1} [\, y_2^{-1}, y_1^{-1} : y_3^{-1}, y_2^{-1} \,]^{\delta_2 g_2} \\
&\equiv [\, y_2, y_1 ; y_3, y_2, y_1 \,]^{\delta_3 g_3} [\, y_2, y_1 : y_3^{-1}, y_2, y_1 \,]^{\delta_4 g_4} \\
&\equiv [\, y_2, y_1 ; y_3, y_1, y_2 \,]^{\delta_5 g_5} [\, y_2, y_1 : y_2, y_1, y_3 \,]^{\delta_6 g_6} \quad \text{(by part (a) and case 1)} \\
&\quad \cdot [\, y_2, y_1 ; y_3^{-1}, y_1, y_2 \,]^{\delta_7 g_7} [\, y_2, y_1 : y_2, y_1, y_3^{-1} \,]^{\delta_8 g_8} \\
&\equiv 1 \quad \text{(by Lemma 4.2).}
\end{aligned}
$$

for some $g_1, g_2, \ldots, g_8$ in F and $\delta_1, \delta_2, \ldots, \delta_8$ in $\{1, -1\}$.

If $\varepsilon_1 = \varepsilon_3 = -1, \varepsilon_2 = +1$, then

$$
\begin{aligned}
&[\, y_2, y_1^{-1}, y_3^{-1} ; y_3^{-1}, y_2 \,] \\
&\qquad \equiv [\, y_2, y_1^{-1}, y_3 : y_3, y_2 \,]^{\delta_1 g_1} \\
&\qquad \equiv [\, y_2, y_1 : y_3, y_2, y_1 \,]^{\delta_2 g_2} [\, y_2, y_1 : y_3^{-1}, y_2, y_1 \,]^{\delta_3 g_3} \\
&\qquad \equiv 1 \quad (\, g_1, g_2, g_3 \in F \text{ and } \delta_1, \delta_2, \delta_3, \in \{1, -1\}\,).
\end{aligned}
$$

Therefore.

$$[\, y_2^{\varepsilon_2}, y_1^{-1}, y_3^{\varepsilon_3} : y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1.$$

- With a similar argument as the previous one. $[\, y_2^{\varepsilon_2}, y_1^{-1}, y_3^{\varepsilon_3} : y_3^{\varepsilon_3}, y_1^{-1} \,] \equiv 1$.
- $[\, y_3^{\varepsilon_3}, y_1^{-1} : y_2^{\varepsilon_2}, y_1^{-1} \,] = [\, y_3^{\varepsilon_3}, y_1 ; y_2^{\varepsilon_2}, y_1 \,]^g \equiv 1 \qquad (\, g \in F\,).$
- If $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = +1$, then

$$[\, y_3^{-1}, y_1^{-1}, y_2 : y_2, y_1^{-1} \,] = [\, y_3^{-1}, y_1^{-1} : y_2, y_1^{-1} \,]^{g_1} [\, y_3^{-1}, y_1^{-1} : y_2^{-1}, y_1^{-1} \,]^{g_2} \equiv 1.$$

for some $g_1, g_2 \in F$.

If $\varepsilon_1 = \varepsilon_3 = -1, \varepsilon_2 = +1$, then

$$[\, y_3, y_1^{-1}, y_2^{-1} ; y_2^{-1}, y_1^{-1} \,] \equiv [\, y_3, y_1^{-1} ; y_2, y_1^{-1} \,]^{g_1} [\, y_3, y_1^{-1} ; y_2^{-1}, y_1^{-1} \,]^{g_2} \equiv 1,$$

for some $g_1, g_2 \in F$. Therefore,

$$[\, y_3^{\varepsilon_3}, y_1^{-1}, y_2^{\varepsilon_2} \,;\, y_2^{\varepsilon_2}, y_1^{-1} \,] \equiv 1.$$

- If $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = +1$, then

$$\begin{aligned}[\, y_3, y_1^{-1}, y_2^{-1} : y_3, y_1^{-1} \,] &\equiv \left[\, y_3, y_1^{-1} : [y_3, y_1]^{y_1^{-1} y_2^{-1}} \,\right]^{g_1} \\ &\equiv [\, y_3, y_1 : y_3, y_1, y_2 \,]^{g_2} \\ &\equiv 1 \qquad (\text{ for some } g_1, g_2 \in F\,).\end{aligned}$$

If $\varepsilon_1 = \varepsilon_3 = -1, \varepsilon_2 = +1$, then with a similar argument as the previous one, $[\, y_3^{-1}, y_1^{-1}, y_2 : y_3^{-1}, y_1^{-1} \,] \equiv 1$. Therefore,

$$[\, y_3^{\varepsilon_3}, y_1^{-1}, y_2^{\varepsilon_2} : y_3^{\varepsilon_3}, y_1^{-1} \,] \equiv 1.$$

Hence the proof of case 2, and therefore the proof of the lemma, is complete.

$\square$

Theorem 4.5. *Let* $\{a, b, c, d\} \subseteq X$ *and* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \subseteq \{1, -1\}$. *Then*

$$[\, a^{\varepsilon_1}, b^{\varepsilon_2} : c^{\varepsilon_3}, d^{\varepsilon_4} \,] \equiv 1 \quad (\text{mod } \langle\, \Delta(m, n, F)\,\rangle^F).$$

PROOF. Without loss of generality we may assume that $b < a$, $d < c$ and $c \leq a$. For the sake of argument we call $a = y_4$ and let y_3 be the greatest of b, c, d; y_2 the second greatest, and y_1 the smallest one. Considering all the possibilities we need to prove the following :

(1) $C_1 = [\, y_4^{\varepsilon_4}, y_1^{\varepsilon_1} : y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1$, where $y_1 \leq y_2 < y_3 \leq y_4$;

(2) $C_2 = [\, y_4^{\varepsilon_4}, y_2^{\varepsilon_2} : y_3^{\varepsilon_3}, y_1^{\varepsilon_1} \,] \equiv 1$, where $y_1 < y_2 \leq y_3 \leq y_4$;

(3) $C_3 = [\, y_4^{\varepsilon_4}, y_3^{\varepsilon_3} : y_2^{\varepsilon_2}, y_1^{\varepsilon_1} \,] \equiv 1$, where $y_1 < y_2 < y_3 < y_4$.

Proof of (1) : We consider four cases :

$$(1-i) \qquad \varepsilon_1 = \varepsilon_4 = +1:$$

$$(1-ii) \qquad \varepsilon_1 = \varepsilon_4 = -1:$$

$$(1-iii) \qquad \varepsilon_1 = +1\,,\ \varepsilon_4 = -1:$$

$$(1-iv) \qquad \varepsilon_1 = -1\,,\ \varepsilon_4 = +1.$$

To prove $(1-i)$, since in this case $\varepsilon_1 = \varepsilon_4 = +1$, so by Lemma 4.2 $C_1 = [\, y_4, y_1, \,;\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \equiv 1$.

To prove $(1-ii)$, since in this case $\varepsilon_1 = \varepsilon_4 = -1$, using the Identities Lemma, Lemma 4.2, and the Jacobi Lemma, and working modulo $\langle \Delta(m, n, F) \rangle^F$ we have

$$\begin{aligned}
C_1 &= \left[\, y_4^{-1}, y_1^{-1} \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \,\right] \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,]^{y_1 y_4} \,\right]^{\delta_1 f_1} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,]^{y_4} \,\right]^{\delta_2 f_2} \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2}, y_1 \,]^{y_4} \,\right]^{\delta_3 f_3} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2} \,] \,\right]^{\delta_4 f_4} \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_2^{\varepsilon_2}, y_4 \,] \,\right]^{\delta_5 f_5} \\
&\quad \cdot \left[\, y_4, y_1 \,;\, [\, y_2^{\varepsilon_2}, y_1, y_3^{\varepsilon_3} \,]^{y_4} \,\right]^{\delta_6 f_6} \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2} \,]^{y_4} \,\right]^{\delta_7 f_7} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{\varepsilon_3}, y_1, y_2^{\varepsilon_2} \,]^{y_4} \,\right]^{\delta_8 f_8} \quad \text{(by Lemma 4.2 and the Jacobi Lemma).}
\end{aligned}$$

Now if $\varepsilon_3 = +1$, then by Lemma 4.2, $\left[\, y_4, y_1 \,;\, [\, y_3, y_1, y_2^{\varepsilon_2} \,]^{y_4} \,\right] \equiv 1$, and $C_1 \equiv 1$. If $\varepsilon_3 = -1$, then

$$\begin{aligned}
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{-1}, y_1 \,]^{y_4} \,\right]^{\delta_9 f_9} \left[\, y_4, y_1 \,;\, [\, y_3^{-1}, y_1 \,]^{y_2^{\varepsilon_2} y_4} \,\right]^{\delta_{10} f_{10}} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3^{-1}, y_1 \,]^{y_2^{\varepsilon_2} y_4} \,\right]^{\delta_{10} f_{10}} \quad \text{(by Lemma 4.2)} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3, y_1 \,]^{y_3^{-1} y_2^{\varepsilon_2} y_4} \,\right]^{\delta_{11} f_{11}} \\
&\equiv \left[\, y_4, y_1 \,;\, [\, y_3, y_1 \,]^{y_2^{\varepsilon_2} y_3^{-1} y_4 [y_3^{-1}, y_2^{\varepsilon_2}][y_3^{-1}, y_2^{\varepsilon_2}, y_4]} \,\right]^{\delta_{12} f_{12}} \quad \text{(by Lemma 1.4),}
\end{aligned}$$

but by Lemma 4.2, both $[y_4, y_1 ; y_3^{-1}, y_2^{\varepsilon_2}]$ and $[y_4, y_1 ; y_3^{-1}, y_2^{\varepsilon_2}, y_4]$ are congruent to 1, so

$$\equiv \left[y_4, y_1 ; [y_3, y_1]^{y_2^{\varepsilon_2} y_3^{-1} y_4} \right]^{\delta_{13} f_{13}}$$
$$\equiv 1 \quad \text{(by Lemma 4.2)},$$

for some $f_i \in F$ and $\delta_i \in \{1, -1\}$ where $1 \leq i \leq 13$.
The proofs of $(1 - iii)$ and $(1 - iv)$ are similar to the proof of $(1 - ii)$. Therefore in any case,

$$C_1 = [y_4^{\varepsilon_4}, y_1^{\varepsilon_1} ; y_3^{\varepsilon_3}, y_2^{\varepsilon_2}] \equiv 1. \qquad \text{where} \quad y_1 \leq y_2 < y_3 \leq y_4.$$

Proof of (2) : We consider two cases : Either $\varepsilon_1 = +1$ or $\varepsilon_1 = -1$, and we refer to them as (2-i) and (2-ii), respectively.
Proof of case (2-i) : There are two sub-cases, **either** $\varepsilon_1 = \varepsilon_2 = +1$, **or** $\varepsilon_1 = +1$, $\varepsilon_2 = -1$.
For $\varepsilon_1 = \varepsilon_2 = +1$, since in this case $C_2 = [y_4^{\varepsilon_4}, y_2 ; y_3^{\varepsilon_3}, y_1]$, if $\varepsilon_4 = +1$, then by Lemma 4.2, $C_2 \equiv 1$ and if $\varepsilon_4 = -1$, then

$$C_2 = [y_4^{-1}, y_2 ; y_3^{\varepsilon_3}, y_1] = \left(\left[y_4, y_2 ; y_3^{\varepsilon_3}, y_1 \right]^{-1} \right)^f \equiv 1 \quad (f \in F).$$

For $\varepsilon_1 = +1$, $\varepsilon_2 = -1$, if $\varepsilon_3 = +1$, then

$$C_2 = [y_4^{\varepsilon_4}, y_2^{-1} ; y_3, y_1] = \left(\left[y_4^{\varepsilon_4}, y_2 ; [y_3, y_1]^{y_2} \right]^{-1} \right)^f \equiv 1 \quad (f \in F).$$

If $\varepsilon_3 = -1$, then $\varepsilon_4 = +1$ or $\varepsilon_4 = -1$. We prove this for $\varepsilon_4 = +1$; for the other case the proof would be similar. So,

$$\begin{aligned} C_2 &= [y_4, y_2^{-1} ; y_3^{-1}, y_1] \\ &= \left[y_4, y_2 ; [y_3^{-1}, y_1]^{y_2} \right]^{\delta_1 f_1} \\ &= \left[y_4, y_2 ; [y_3, y_1]^{y_2 y_3^{-1} [y_3^{-1}, y_2]} \right]^{\delta_2 f_2}; \end{aligned}$$

considering the fact that $[y_4, y_2 ; y_3^{-1}, y_2] \equiv 1$ we have,

$$\begin{aligned} C_2 &\equiv \left[y_4, y_2 ; [y_3, y_1]^{y_2 y_3^{-1}} \right]^{\delta_3 f_3} \\ &\equiv 1 \quad (\text{by Lemma 4.2 }) \quad (\delta_1, \delta_2, \delta_3 \in \{1, -1\}\, , \ f_1, f_2, f_3 \in F). \end{aligned}$$

Proof of case (2-ii) : Using (1) and case (2-i) we have.

$$\begin{aligned} C_2 &= [y_4^{\varepsilon_4}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1^{-1}] \\ &\equiv [y_4^{\varepsilon_4}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1]^{\delta_1 f_1} [y_4^{\varepsilon_4}, y_2^{\varepsilon_2}, y_1 : y_3^{\varepsilon_3}, y_1]^{\delta_2 f_2} \\ &\equiv [y_4^{\varepsilon_4}, y_2^{\varepsilon_2}, y_1 ; y_3^{\varepsilon_3}, y_1]^{\delta_2 f_2} \quad (\text{ by case (2-i) }) \\ &\equiv [y_4^{\varepsilon_4}, y_1, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1]^{\delta_3 f_3} [y_2^{\varepsilon_2}, y_1, y_4^{\varepsilon_4} : y_3^{\varepsilon_3}, y_1]^{\delta_4 f_4} \\ &\equiv 1 \qquad (\text{by (1) and case (2-i)}). \end{aligned}$$

Proof of (3) : If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_= +1$, then by definition $C_3 \equiv 1$. Otherwise we consider the following cases :

$$\begin{aligned} &(3-i) && \varepsilon_1 = \varepsilon_2 = +1; \\ &(3-ii) && \varepsilon_1 = -1\, ,\ \varepsilon_2 = +1; \\ &(3-iii) && \varepsilon_1 = +1\, ,\ \varepsilon_2 = -1; \\ &(3-iv) && \varepsilon_1 = \varepsilon_2 = -1. \end{aligned}$$

Proof of case (3-i): Either $\varepsilon_3 = +1, \varepsilon_4 = -1$ or $\varepsilon_3 = -1, \varepsilon_4 = +1$ or $\varepsilon_3 = \varepsilon_4 = -1$. We discuss the first two possibilities at once. So let $z \in \{y_3, y_4\}$; then since $y_1 < y_2 < z$ we have.

$$\begin{aligned} C_3 &= [y_4^{\varepsilon_4}, y_3^{\varepsilon_3} ; y_2, y_1] \\ &= \left[[y_2, y_1]^z ; y_4, y_3 \right]^{\delta_1 f_1} \\ &\equiv [y_2, y_1, z ; y_4, y_3]^{\delta_2 f_2} \\ &\equiv [z, y_2, y_1 ; y_4, y_3]^{\delta_3 f_3} [z, y_1, y_2 ; y_4, y_3]^{\delta_4 f_4} \quad (\text{by the Jacobi Lemma}) \qquad (*). \end{aligned}$$

Set $C_{(3,1)} = [z, y_2, y_1 ; y_4, y_3]$ and $C_{(3,2)} = [z, y_1, y_2 ; y_4, y_3]$. By $(*)$ it is enough to show that $C_{(3,1)} \equiv C_{(3,2)} \equiv 1$.

Since $C_{(3,1)} \equiv [z, y_2 ; y_4, y_1^{-1}, y_3]^{\delta_1 f_1}[z, y_2 ; y_3, y_1^{-1}, y_4]^{\delta_2 f_2} \qquad (**)$,

if $z = y_3$, then

$$[z, y_2 ; y_4, y_1^{-1}, y_3] \equiv [y_4, y_1^{-1} ; y_3, y_2]^{f_1}[y_4, y_1^{-1} : y_3^{-1}, y_2]^{f_2} \equiv 1 \quad \text{(by (2))}.$$

If $z = y_4$ then,

$$\begin{aligned}
&[z, y_2 ; y_4, y_1^{-1}, y_3] \\
&\quad = [y_4, y_1^{-1} ; y_4, y_2]^{\delta_1 f_1}[y_4, y_1^{-1} : [y_4, y_2]^{y_3^{-1}}]^{\delta_2 f_2} \\
&\quad \equiv [y_4, y_1 ; [y_4, y_2]^{y_1 y_3^{-1}}]^{\delta_3 f_3} \\
&\quad \equiv [y_4, y_1 ; [y_4, y_2]^{y_3^{-1}}]^{\delta_4 f_4}[y_4, y_1 ; [y_4, y_2, y_1]^{y_3^{-1}}]^{\delta_5 f_5} \\
&\quad \equiv [y_4, y_1 : [y_2, y_1, y_4]^{y_3^{-1}}]^{\delta_6 f_6} \text{ (by the Jacobi Lemma and Lemma 4.2)} \\
&\quad \equiv 1 \quad \text{(by (1) and Lemma 4.2)}.
\end{aligned}$$

So $[z, y_2 ; y_4, y_1^{-1}, y_3] \equiv 1$ (†). Similarly, $[z, y_2 ; y_3, y_1^{-1}, y_4] \equiv 1$ (‡). Therefore substitution of † and ‡ in $(**)$ gives $C_{(3,1)} = [z, y_2, y_1 ; y_4, y_3] \equiv 1$.

On the other hand by Lemma 4.2 and the Jacobi Lemma, similar to the above, $C_{(3,2)} = [z, y_1, y_2 ; y_4, y_3] \equiv 1$. Thus by substitution in $(*)$, in this case, $C_3 \equiv 1$.

If $\varepsilon_3 = \varepsilon_4 = -1$ then using Jacobi Lemma, the above case, (1) and (2) we have,

$$\begin{aligned}
C_3 &= [y_4^{-1}, y_3^{-1} ; y_2, y_1] \\
&= [[y_2, y_1]^{y_4} ; y_4, y_3^{-1}]^{\delta_1 f_1} \\
&\equiv [y_4, y_1, y_2 ; y_4, y_3^{-1}]^{\delta_2 f_2}[y_4, y_2, y_1 ; y_4, y_3^{-1}]^{\delta_3 f_3} \\
&\equiv [y_4, y_1 ; y_4, y_2^{-1}, y_3^{-1}]^{\delta_4 f_4}[y_4, y_1 ; y_3^{-1}, y_2^{-1}, y_4]^{\delta_5 f_5} \\
&\quad \cdot [y_4, y_2 ; y_4, y_1^{-1}, y_3^{-1}]^{\delta_6 f_6}[y_4, y_2 ; y_3^{-1}, y_1^{-1}, y_4]^{\delta_7 f_7}
\end{aligned}$$

$$\equiv [y_4, y_1^{-1}; [y_4, y_2]^{y_3}]^{\delta_8 f_8}[y_4, y_2; y_3^{-1}, y_1^{-1}]^{\delta_9 f_9}[y_4^{-1}, y_2; y_3^{-1}, y_1^{-1}]^{\delta_{10} f_{10}}$$

$$\equiv [y_4, y_1; [y_4, y_2]^{y_1 y_3 [y_3, y_1]}]^{\delta_{11} f_{11}}$$

$$\equiv [y_4, y_1; [y_4, y_1, y_2]^{y_3}]^{\delta_{12} f_{12}}[y_4, y_1; [y_2, y_1, y_4]^{y_3}]^{\delta_{13} f_{13}}$$

$$\equiv 1.$$

Proof of case (3-ii): Since $\varepsilon_1 = -1$, $\varepsilon_2 = +1$, using all of the information obtained so far leads to,

$$C_3 = [y_4^{\varepsilon_4}, y_3^{\varepsilon_3}; y_2, y_1^{-1}]$$

$$\equiv [y_2, y_1; [y_4^{\varepsilon_4}, y_3^{\varepsilon_3}]^{y_1}]^{\delta_1 f_1}$$

$$\equiv [y_2, y_1; y_4^{\varepsilon_4}, y_1, y_3^{\varepsilon_3}]^{\delta_2 f_2}[y_2, y_1; y_3^{\varepsilon_3}, y_1, y_4^{\varepsilon_4}]^{\delta_3 f_3}$$

$$\equiv [y_4^{\varepsilon_4}, , y_1; y_2, y_1, y_3^{-\varepsilon_3}]^{\delta_4 f_4}[y_2, y_1; [y_3^{\varepsilon_3}, y_1]]^{\delta_5 f_5}[y_2, y_1; [y_3^{\varepsilon_3}, y_1]^{y_4^{\varepsilon_4}}]^{\delta_6 f_6}$$

$$\equiv 1.$$

Proof of case (3-iii): Since $\varepsilon_1 = +1$, $\varepsilon_2 = -1$ using all the above we have,

$$C_3 = [y_4^{\varepsilon_4}, y_3^{\varepsilon_3}; y_2^{-1}, y_1]$$

$$\equiv [y_2, y_1; y_4^{\varepsilon_4}, y_2, y_3^{\varepsilon_3}]^{\delta_1 f_1}[y_2, y_1; [y_3^{\varepsilon_3}, y_2]]^{\delta_2 f_2}[y_2, y_1; [y_3^{\varepsilon_3}, y_2]^{y_4^{\varepsilon_4}}]^{\delta_3 f_3}$$

$$\equiv [y_4^{-\varepsilon_4}, y_1, y_2; y_3^{\varepsilon_3}, y_2]^{\delta_4 f_4}[y_4^{-\varepsilon_4}, y_2, y_1; y_3^{\varepsilon_3}, y_2]^{\delta_5 f_5}$$

$$\equiv [y_4^{-\varepsilon_4}, y_1; y_3^{\varepsilon_3}, y_2^{-1}]^{\delta_6 f_6}[y_4^{-\varepsilon_4}, y_2^{-1}; y_3^{\varepsilon_3}, y_1^{-1}]^{\delta_7 f_7}[y_4^{-\varepsilon_4}, y_2; y_2, y_1^{-1}, y_3^{\varepsilon_3}]^{\delta_8 f_8}$$

$$\equiv [y_4^{-\varepsilon_4}, y_1, y_2; y_2, y_1]^{\delta_9 f_9}[y_2, y_1, y_4^{-\varepsilon_4}; y_2, y_1]^{\delta_{10} f_{10}}$$

$$\cdot [y_4^{-\varepsilon_4}, y_2; [y_2, y_1]^{y_3^{\varepsilon_3} y_1^{-1}}]^{\delta_{11} f_{11}}$$

$$\equiv [y_4^{-\varepsilon_4}, y_2; [y_2, y_1]^{y_3^{\varepsilon_3}}]^{\delta_{12} f_{12}}[y_4^{-\varepsilon_4}, y_1, y_2; [y_2, y_1]^{y_3^{\varepsilon_3}}]^{\delta_{13} f_{13}}$$

$$\cdot [y_2, y_1, y_4^{\varepsilon_4}; [y_2, y_1]^{y_3^{\varepsilon_3}}]^{\delta_{14} f_{14}}$$

$$\equiv [y_4^{-\varepsilon_4}, y_1; [y_2, y_1]^{y_2^{-1} y_3^{\varepsilon_3}}]^{\delta_{15} f_{15}}$$

$$\equiv 1.$$

Proof of case (3-iv): Since $\varepsilon_1 = \varepsilon_2 = -1$, we have

$$\begin{aligned} C_3 = [\, y_4^{\varepsilon_4}, y_3^{\varepsilon_3} \,;\, y_2^{-1}, y_1^{-1} \,] &\equiv [\, y_2^{-1}, y_1 \,;\, y_4^{-\varepsilon_4}, y_1, y_3^{\varepsilon_3} \,]^{\delta_1 f_1} \\ &\equiv [\, y_2^{-1}, y_1 \,;\, y_3^{-\varepsilon_3}, y_1, y_4^{\varepsilon_4} \,]^{\delta_2 f_2} \\ &\equiv [\, y_2^{-1}, y_1 \,;\, y_3^{-\varepsilon_3}, y_1, y_4^{\varepsilon_4} \,]^{\delta_3 f_3}. \end{aligned}$$

If $\varepsilon_3 = +1$, then $[\, y_3, y_1 \,;\, y_2^{-1}, y_1, y_4^{-\varepsilon_4} \,] \equiv 1$ and so $C_3 \equiv 1$. If $\varepsilon_3 = -1$, then

$$\begin{aligned} [\, y_2^{-1}, y_1 \,;\, y_3^{-1}, y_1, y_4^{\varepsilon_4} \,] &\equiv [[\, y_2^{-1}, y_1 \,;\, [\, y_3, y_1 \,]^{y_4^{\varepsilon_4} y_3^{-1}} \,]^{y_1} \\ &\equiv \left([\, y_3, y_1 \,;\, [\, y_2^{-1}, y_1 \,]^{y_3 y_4^{-\varepsilon_4}} \,]^{-1} \right)^{g_2} \\ &\equiv 1. \end{aligned}$$

Hence in any case, $C_3 \equiv 1$ and the proof of the theorem is complete. ☐

Lemma 4.6. *(Mini-Max Lemma) For $4 \le c \le n+1$ let $\{z_1, z_2, \dots, z_c\} \subseteq X$ such that $z_1 \le z_2 \le \dots \le z_c$; then modulo $\langle \Delta(m, n, F) \rangle^F$.*

(a) $\quad [\, z_c^{\varepsilon_c}, z_1^{\varepsilon_1} \,;\, z_{i_1}^{\varepsilon_{i_1}}, z_{i_2}^{\varepsilon_{i_2}}, \dots, z_{i_{c-2}}^{\varepsilon_{i_{c-2}}} \,] \equiv 1$.

where $\{i_1, i_2, \dots, i_{c-2}\} = \{2, 3, 4, \dots, c-1\}$ and $\{\varepsilon_1, \varepsilon_c, \varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_{c-2}}\} \subseteq \{1, -1\}$. In particular $[\, z_c^{\varepsilon_c}, z_1^{\varepsilon_1} \,;\, z_i^{\varepsilon_i}, z_2^{\varepsilon_2}, z_3^{\varepsilon_3}, \dots, \hat{z}_i, \dots, z_{c-1}^{\varepsilon_{c-1}} \,] \equiv 1$.

(b) If $z_1 < z_2$, then

$$[\, z_c^{\varepsilon_c}, z_2^{\varepsilon_2} \,;\, z_{j_1}^{\varepsilon_{j_1}}, z_{j_2}^{\varepsilon_{j_2}}, \dots, z_{j_{c-2}}^{\varepsilon_{j_{c-2}}} \,] \equiv 1.$$

where $\{j_1, j_2, \dots, j_{c-2}\} = \{1, 3, 4, \dots, c-1\}$ and $\{\varepsilon_2, \varepsilon_c, \varepsilon_{j_1}, \varepsilon_{j_2}, \dots, \varepsilon_{j_{c-2}}\} \subseteq \{1, -1\}$ such that $j_{s_o} = 1$ implies $\varepsilon_{j_{s_0}} = +1$. In particular.
$[\, z_c^{\varepsilon_c}, z_2^{\varepsilon_2} \,;\, z_j^{\varepsilon_j}, z_1, z_3^{\varepsilon_3}, z_4^{\varepsilon_4}, \dots, \hat{z}_j, \dots, z_{c-1}^{\varepsilon_{c-1}} \,] \equiv 1$.

PROOF. (a): First assume $\varepsilon_1 = \varepsilon_c = +1$. Since $[\, z_{i_1}^{\varepsilon_{i_1}}, z_{i_2}^{\varepsilon_{i_2}}, \dots, z_{i_{c-2}}^{\varepsilon_{i_{c-2}}} \,] \in F'$ by Theorem 2.8, modulo $\langle \Omega_0(m, n, F) \rangle^F$, it is congruent to a product of simple basic commutators of the form $[\, y_2, y_1, y_3, y_4, \dots, y_r \,]$, where $2 \le r \le$

$n-1$ and $\{y_1, y_2, \ldots, y_r\} \subseteq \{z_2, z_3, \ldots, z_{c-1}\}$. Therefore,

$$[\, z_c, z_1 \,;\, z_{i_1}^{\varepsilon_{i_1}} z_{i_2}^{\varepsilon_{i_2}}, \ldots, z_{i_{c-2}}^{\varepsilon_{i_{c-2}}} \,].$$

is congruent to a product of conjugates of commutators of the form

$$[z_c, z_1 \,;\, y_2, y_1, y_3, y_4, \ldots, y_r\,].$$

But since $z_1 \leq y_1$ and $y_2 \leq z_c$, each of these last commutators is congruent to 1.

If either ε_1 or ε_2 or both are -1, then we first bring the inverse to the other side and then the rest is the same argument as the above.

Proof of (b): Similar to part (a), it is enough to prove it only for $\varepsilon_2 = \varepsilon_c = +1$. Since $[\, z_{j_1}^{\varepsilon_{j_1}}, z_{j_2}^{\varepsilon_{j_2}}, \ldots, z_{j_{c-2}}^{\varepsilon_{j_{c-2}}} \,] \in F'$ by part (b) of Theorem 2.8, modulo $\langle \Omega_0(m, n, F)\rangle^F$, it is congruent to a product of simple basic commutators of the form $[\, y_2, y_1, y_3, y_4, \ldots, y_r\,]$ where $2 \leq r \leq n-1$ and $\{y_1, y_2, \ldots, y_r\} \subseteq \{z_1, z_3, z_4, \ldots, z_{c-1}\}$. Therefore,

$$[\, z_c, z_2 \,;\, z_{j_1}^{\varepsilon_{j_1}}, z_{j_2}^{\varepsilon_{j_2}}, \ldots, z_{j_{c-2}}^{\varepsilon_{j_{c-2}}} \,]$$

is congruent to a product of conjugates of commutators of the form

$$[z_c, z_2 \,;\, y_2, y_1, y_3, y_4, \ldots, y_r\,].$$

In order to show each one these last commutators is congruent to 1 we consider two different cases :

If $z_1 < y_1$, then $y_1 \in \{z_3, z_4, \ldots, z_{c-1}\}$, so $z_2 \leq y_1$ and therefore by (a), $[z_c, z_2 \,;\, y_2, y_1, y_3, y_4, \ldots, y_r\,] \equiv 1$.

If $y_1 = z_1$, then since $z_1 < z_2$ and z_1 has appeared only once, so $y_1 < y_3$. Now if $y_2 = z_c$ then $[z_c, z_2 \,;\, z_c, z_1, y_3, y_4, \ldots, y_r\,]$ is congruent to a product of conjugates of commutators of the form $[\, z_c, z_1 \,;\, z_{i_1}^{\varepsilon_{i_1}}, z_{i_2}^{\varepsilon_{i_2}}, \ldots, z_{i_k}^{\varepsilon_{i_k}} \,]$, where $(k \geq 2)$ and each one of these last commutators is congruent to 1 by part (a).

If $y_2 < z_c$, then since $y_1 = z_1 < z_2$ and $y_1 < y_3$, by definition $[z_c, z_2\,;\, y_2, y_1, y_3, y_4, \ldots, y_r] \equiv 1$. □

Corollary 4.7. *(Mini-Max Corollary) For* $4 \le c \le n+1$, *let* $\{z_1, z_2, \ldots, z_c\} \subseteq X$ *such that* $z_1 \le z_2 \le \ldots \le z_c$; *then modulo* $\langle \Delta(m, n, F)\rangle^F$,

$$(a)\qquad C_1 = [\, z_i^{\varepsilon_i}, z_1\,;\, z_c^{\varepsilon_c}, z_2^{\varepsilon_2}, z_3^{\varepsilon_3}, \ldots, z_i^{\varepsilon_i}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,] \equiv 1;$$

$$(b)\qquad C_2 = [\, z_2^{\varepsilon_2}, z_1^{-1}\,;\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_c^{\varepsilon_c} \,] \equiv 1.$$

$$\textit{where } \varepsilon_t \in \{1, -1\},\ (2 \le t \le c).$$

PROOF. (a): If $z_i = z_c$, then by part (a) of the Mini-Max Lemma, $C_1 \equiv 1$. If $z_i < z_c$, then C_1 is congruent to a product of conjugates of commutators of the form $[\, z_c^{\varepsilon_c}, z_2^{\varepsilon_2} \,;\, z_i^{\varepsilon_i}, z_1, z_{j_1}^{\varepsilon_{j_1}}, z_{j_2}^{\varepsilon_{j_2}}, \ldots, z_{j_k}^{\varepsilon_{j_k}} \,]$, where $(k \ge 0)$ and $\{j_1, j_2, \ldots, j_k\} \subseteq \{3, 4, \ldots, c-1\}$. By the Mini-Max Lemma, each one of these last commutators is congruent to 1.

Proof of (b): If $z_1 = z_2$, then $C_2 = 1$. If $z_1 < z_2$, we prove (b) by induction on c. If $c = 4$, then by Theorem 4.5, $C_2 \equiv 1$. Assume it is also true for $c-1$; then for c we have

$$\begin{aligned}
C_2 &= [\, z_2^{\varepsilon_2}, z_1^{-1}\,;\, [\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,]^{z_c^{\varepsilon_c}} \,]^{\delta_1 f_1} \\
&\quad \cdot [\, z_2^{\varepsilon_2}, z_1^{-1}\,;\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,]^{\delta_2 f_2} \\
&\equiv [\, z_2^{\varepsilon_2}, z_1^{-1}\,;\, [\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,]^{z_c^{\varepsilon_c}} \,]^{\delta_3 f_3} \quad \text{(by the induction hypothesis)} \\
&\equiv [\, z_c^{-\varepsilon_c}, z_1^{-1}, z_2^{\varepsilon_2}\,;\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,]^{\delta_4 f_4} \\
&\quad \cdot [\, z_c^{-\varepsilon_c}, z_2^{\varepsilon_2}, z_1^{-1}\,;\, z_4^{\varepsilon_4}, z_3^{\varepsilon_3}, z_5^{\varepsilon_5}, \ldots, z_{c-1}^{\varepsilon_{c-1}} \,]^{\delta_5 f_5} \quad \text{(by the Jacobi Lemma)} \\
&\equiv 1 \quad \text{(by the Mini-Max Lemma).}
\end{aligned}$$

where $f_t \in F$ and $\delta_t \in \{1, -1\}$, $(1 \le t \le 5)$.

□

From now on for brevity we refer to both Mini-Max Lemma and Mini-Max Corollary as Mini-Max Lemma.

4.2. Type $[z_2, z_1 : z_c, z_3, z_4, \ldots, z_{c-1}]$

Our goal in this section is to show that for each $4 \le c \le n+1$ and $z_1 < z_2 < z_3 \le z_4 \le \cdots \le z_c$ the commutator $[z_2, z_1 : z_c, z_3, z_4, \ldots, z_{c-1}]$ is congruent to 1 modulo $\langle \Delta(m, n, F) \rangle^F$.

Lemma 4.8. *For* $4 \le c \le n+1$ *let* $z_1 \le z_2 \le \ldots \le z_c$*; then modulo* $\langle \Delta(m, n, F) \rangle^F$, $C = [z_c^{-1}, z_2 : z_j, z_1^{-1}, z_3, \ldots, \hat{z}_j, \ldots, z_{c-1}]$ *is congruent to either* 1 *or a product of conjugates of commutators of the form* $[y_3, y_1 ; y_i, y_2, y_4, \ldots, \hat{y}_i, \ldots, y_p]$ *where* $p \le c+1$ *and* $y_1 = y_2 < y_3 \le y_4 \le \cdots \le y_p$, $y_3 < y_i$ *such that* $y_1 = y_2 = z_1$, $y_3 = z_2$, $\{y_4, y_5, \ldots, y_p\} \subseteq \{z_3, z_4, \ldots, z_c\}$.

PROOF. If $z_1 = z_2$ then by the Mini-Max Lemma, $C \equiv 1$. If $z_1 < z_2$, we prove it by induction on c. Let N denote the normal closure of the set of all commutators of the form $[y_3, y_1 : y_i, y_2, y_4, \ldots, \hat{y}_i, \ldots, y_p]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:
If $c = 5$, in order to prove $[z_5^{-1}, z_2 : z_3, z_1^{-1}, z_4]$ and $[z_5^{-1}, z_2 : z_4, z_1^{-1}, z_3]$ are congruent to 1, let $\{j, k\} = \{3, 4\}$, then

$$
\begin{aligned}
&[z_5^{-1}, z_2 ; z_j, z_1^{-1}, z_k] \\
&\equiv [z_5^{-1}, z_2 ; z_j, z_1^{-1}]^{\varepsilon_1 f_1} [z_5^{-1}, z_2 ; [z_j, z_1^{-1}]^{z_k}]^{\varepsilon_2 f_2} \\
&\equiv [z_5^{-1}, z_2 ; [z_j, z_1]^{z_k z_1^{-1}}]^{\varepsilon_3 f_3} \quad \text{(by Theorem 4.5)} \\
&\equiv [z_5^{-1}, z_1, z_2 ; [z_j, z_1]^{z_k}]^{\varepsilon_4 f_4} [z_2, z_1, z_5^{-1} ; [z_j, z_1]^{z_k}]^{\varepsilon_5 f_5} \\
&\equiv [z_2, z_1, z_5^{-1} ; [z_j, z_1]^{z_k}]^{\varepsilon_6 f_6} \quad \text{(by the Mini-Max Lemma)} \\
&\equiv [z_2, z_1 ; [z_j, z_1]^{z_k}]^{\varepsilon_7 f_7} [z_2, z_1, ; [z_j, z_1]^{z_k z_5}]^{\varepsilon_8 f_8} \quad (f_t \in F, \varepsilon_t \in \{1, -1\})
\end{aligned}
$$

$\equiv 1.$

Assume the statement is true for integers less than c. Then for c we have

$$C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \le k \le c-1,\, k \ne j}} \left[z_c^{-1}, z_2 ; [\, z_j, z_1^{-1} \,]^{z_3^{\lambda_3} z_4^{\lambda_4} \ldots \hat{z}_j \ldots z_{c-1}^{\lambda_{c-1}}} \right]^{\varepsilon(\lambda_3,\lambda_4,\ldots,\lambda_{c-1}) f(\lambda_3,\lambda_4,\ldots,\lambda_{c-1})}$$

$$\equiv \left[z_c^{-1}, z_2 ; [\, z_j, z_1^{-1} \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)}$$

$$\equiv \left[z_c^{-1}, z_2 ; [\, z_j, z_1 \,]^{z_1^{-1} z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_2 f_2}.$$

Now using the Collection Lemma we have

$$z_1^{-1} z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1} = z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1} z_1^{-1} \cdot \prod_{\substack{t=3 \\ t \ne j}}^{c-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \le k \le c-1}} [\, z_t, z_1^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,].$$

For each choice of t and λ's,

$$T_{(t,\lambda)} = [\, z_c^{-1}, z_2 : z_t, z_1^{1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,]$$

is of the form $[\, y_p^{-1}, y_2 ; y_3, y_1^{-1}, y_4, \ldots, y_p \,]$ where $p \le c-1$ and $y_1 < y_2 \le y_3 \le \cdots \le y_p$, and so by the induction hypothesis, $T_{(t,\lambda)} \equiv 1$. Therefore,

$$C \equiv \left[z_c^{-1}, z_2 ; [\, z_j, z_1 \,]^{z_1^{-1} z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_2 f_2}$$

$$\equiv \left[z_c^{-1}, z_2 ; [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1} z_1^{-1}} \right]^{\varepsilon_3 f_3}$$

$$\equiv \left[z_c^{-1}, z_2 ; [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_4 f_4} \left[z_c^{-1}, z_1, z_2 : [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_5 f_5}$$

$$\cdot \left[z_2, z_1, z_c^{-1} ; [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_6 f_6} \quad \text{(by the Jacobi Lemma)}$$

$$\equiv \left[z_2, z_1 ; [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1}} \right]^{\varepsilon_7 f_7} \left[z_2, z_1 ; [\, z_j, z_1 \,]^{z_3 z_4 \ldots \hat{z}_j \ldots z_{c-1} z_c} \right]^{\varepsilon_8 f_8}$$

$$\equiv 1 \quad \text{(by Lemma 4.2 and the Mini-Max Lemma)} \quad (f_t \in F,\ \delta_t \in \{1, -1\}).$$

So the proof is complete. □

Lemma 4.9. *For $5 \leq p \leq n+1$, let $y_1 = y_2 \leq y_3 \leq y_4 \leq \cdots \leq y_p$ and $y_3 \leq y_j < y_i \leq y_p$; then modulo $\langle \Delta(m, n, F) \rangle^F$,*
$C = [y_j, y_1 \,;\, y_i, y_2, y_3, \ldots, \hat{y}_j, \ldots, \hat{y}_i, \ldots, y_p]$ is congruent to either 1 or a product of conjugates of commutators of the form $[u_2, u_1 \,:\, u_q, u_3, u_4, \ldots, u_{q-1}]$ where $q \leq p-2$ and $u_1 < u_2 < u_3 \leq u_4 \leq \cdots \leq u_q$ such that $u_1 = y_1 = y_2$, $\{u_2, u_3, \ldots, u_q\} \subseteq \{y_3, y_4, \ldots, \hat{y}_j, \ldots, \hat{y}_i, \ldots, y_p\}$.

Proof. If $y_i = y_p$, then by the Mini-Max Lemma, $C \equiv 1$. If $y_i < y_p$, then we prove it by induction on p. Let N denote the normal closure of the set of all commutators of the form $[u_2, u_1 \,;\, u_q, u_3, u_4, \ldots, u_{q-1}]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:
If $p = 5$, then since $y_1 = y_2$ and $y_3 \leq y_4$,

$$[y_3, y_1 \,;\, y_4, y_2, y_5] \equiv ([y_4, y_1 \,:\, y_3, y_2, y_5^{-1}]^{-1})^{y_5} \equiv 1.$$

Assume the statement is true for integers less than p: then for p we have

$$C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \leq k \leq p,\, k \neq i,j}} [y_j, y_1 \,:\, [y_i, y_2]^{y_3^{\lambda_3} y_4^{\lambda_4} \ldots \hat{y}_j \ldots \hat{y}_i \ldots y_p^{\lambda_p}}]^{\varepsilon(\lambda_3, \lambda_4, \ldots, \lambda_p) f(\lambda_3, \lambda_4, \ldots, \lambda_p)}$$
$$\equiv [y_j, y_1 \,;\, [y_i, y_2]^{y_3 y_4 \ldots \hat{y}_j \ldots \hat{y}_i \ldots y_p}]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis).}$$

Now using the Collection Lemma we have

$$y_3 y_4 y_5 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_p = y_p y_{p-1} y_{p-2} \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_3 y_2 y_1$$
$$\cdot \Big(\prod_{\substack{t=s+1 \\ t \neq i,j}}^{p} \prod_{\substack{s=1 \\ s \neq i,j}}^{p-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ s+1 \leq k \leq t-1}} [y_t, y_s, {}_{\lambda_{s+1}} y_{s+1}, {}_{\lambda_{s+2}} y_{s+2}, \ldots, {}_{\lambda_{t-1}} y_{t-1}] \Big)^{-1}.$$

For each choice of t, s and λ's, set

$$T_{(t,s,\lambda)} = [y_j, y_1 \,;\, y_t, y_s, {}_{\lambda_{s+1}} y_{s+1}, {}_{\lambda_{s+2}} y_{s+2}, \ldots, {}_{\lambda_{t-1}} y_{t-1}].$$

Either $y_s \leq y_j$ so that by the Mini-Max Lemma, $T_{(t,s,\lambda)} \equiv 1$, or $y_s > y_j$ which means $T_{(t,s,\lambda)}$ is of the form $[u_2, u_1; u_q, u_3, u_4, \ldots, u_{q-1}]$ where $q \leq p-2$, and $u_1 < u_2 < u_3 \leq u_4 \leq \cdots \leq u_q$ and therefore congruent to 1.

On the other hand,

$$\begin{aligned}
&[y_j, y_1; [y_i, y_2]^{y_p y_{p-1} y_{p-2} \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_3 y_2 y_1}] \\
&\qquad \equiv [y_i, y_2; [y_j, y_1]^{y_3^{-1} y_4^{-1} y_5^{-1} \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}^{-1} y_p^{-1}}]^{\varepsilon f} \\
&\qquad \equiv 1 \quad \text{(by Lemma 4.2)} \qquad (\varepsilon \in \{1, -1\}, f \in F).
\end{aligned}$$

Hence, C can be written as a product of conjugates of commutators of required form and the proof is complete. □

Corollary 4.10. *For* $4 \leq c \leq n+1$, *let* $z_1 \leq z_2 \leq \ldots \leq z_c$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$, $C = [z_c^{-1}, z_2; z_j, z_1^{-1}, z_3, \ldots, \hat{z}_j, \ldots, z_{c-1}]$ *is congruent to either* 1 *or a product of conjugates of commutators of the form* $[u_2, u_1; u_q, u_3, u_4, \ldots, u_{q-1}]$ *where* $q \leq c-1$ *and* $u_1 < u_2 < u_3 \leq u_4 \leq \cdots \leq u_q$, *such that* $u_1 = z_1, u_2 = z_2$, $\{u_3, u_4, \ldots, u_q\} \subseteq \{z_3, z_4, \ldots, \hat{z}_j, \ldots, z_c\}$.

Proof. This is direct consequence of Lemma 4.8 and Lemma 4.9. We just note that $q \leq (c+1) - 2 = c - 1$. □

Theorem 4.11. *For* $5 \leq c \leq n+1$, *let* $z_1 < z_2 < z_3 \leq z_4 \leq \cdots \leq z_c$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$ *for* $4 \leq i \leq c-1$.
$C = [z_2, z_1; z_i, z_3, z_4, \ldots, \hat{z}_i, \ldots, z_c]$ *can be written as a product of conjugates of commutators of the form* $[u_2, u_1; u_q, u_3, u_4, \ldots, u_{q-1}]$ *where* $q \leq c-1$ *and* $u_1 < u_2 < u_3 \leq u_4 \leq \cdots \leq u_q$, *such that* $u_1 = z_1, u_2 = z_2$, $\{u_3, u_4, \ldots, u_q\} \subseteq \{z_3, z_4, \ldots, \hat{z}_i, \ldots, z_c\}$.

Proof. The proof is based on induction on c by frequent use of Theorem 4.5, the Jacobi Lemma, the Mini-Max Lemma and Lemma 4.2. Let N denote the normal closure of the set of all commutators of the form

$[u_2, u_1 ; u_q, u_3, u_4, \ldots, u_{q-1}]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:
If $c = 5$, then

$$\begin{aligned}
[z_2, z_1 ; z_4, z_3, z_5] &\equiv [z_5^{-1}, z_1, z_2 : z_4, z_3]^{\varepsilon_1 f_1} [z_5^{-1}, z_2, z_1 : z_4, z_3]^{\varepsilon_2 f_2} \\
&\equiv [z_5^{-1}, z_2 ; z_4, z_1^{-1}, z_3]^{\varepsilon_3 f_3} [z_5^{-1}, z_2 : z_3, z_1^{-1}, z_4]^{\varepsilon_4 f_4} \\
&\equiv [z_5^{-1}, z_2 ; [z_4, z_1]^{z_3 z_1^{-1}}]^{\varepsilon_5 f_5} [z_5^{-1}, z_2 ; [z_3, z_1]^{z_4 z_1^{-1}}]^{\varepsilon_6 f_6} \\
&\equiv [z_5^{-1}, z_2, z_1 : [z_4, z_1]^{z_3}]^{\varepsilon_7 f_7} [z_5^{-1}, z_2, z_1 : [z_3, z_1]^{z_4}]^{\varepsilon_8 f_8} \\
&\equiv [z_5^{-1}, z_1, z_2 : [z_4, z_1]^{z_3}]^{\varepsilon_9 f_9} [z_2, z_1, z_5^{-1} : [z_4, z_1]^{z_3}]^{\varepsilon_{10} f_{10}} \\
&\quad \cdot [z_5^{-1}, z_1, z_2 : [z_3, z_1]^{z_4}]^{\varepsilon_{11} f_{11}} [z_2, z_1, z_5^{-1} : [z_3, z_1]^{z_4}]^{\varepsilon_{12} f_{12}} \\
&\equiv 1 \qquad (\varepsilon_t \in \{1, -1\}, f_t \in F, 1 \leq t \leq 12).
\end{aligned}$$

Assume the statement is true for integers less than c; then for c we have

$$\begin{aligned}
C = &\prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \leq k \leq c, k \neq i}} [z_2, z_1 ; [z_i, z_3]^{z_4^{\lambda_4} z_5^{\lambda_5} \ldots \hat{z}_i \ldots z_c^{\lambda_c}}]^{\varepsilon(\lambda_4, \lambda_5, \ldots, \lambda_c) f(\lambda_4, \lambda_5, \ldots, \lambda_c)} \\
&\equiv [z_2, z_1 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_c}]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)} \\
&= [[z_2, z_1]^{z_c^{-1}} ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_2 f_2} \\
&\equiv [z_c^{-1}, z_1, z_2 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_3 f_3} [z_c^{-1}, z_2 : [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_4 f_4} \\
&\quad \cdot [z_c^{-1}, z_2 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} z_1^{-1}}]^{\varepsilon_5 f_5} \\
&\equiv [z_c^{-1}, z_2 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} z_1^{-1}}]^{\varepsilon_5 f_5} \quad \text{(by the Mini-Max Lemma).}
\end{aligned}$$

But by the Collection Lemma.

$$\begin{aligned}
z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} z_1^{-1} = &\; z_1^{-1} z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} \\
&\cdot \prod_{\substack{t=4 \\ t \neq i}}^{c-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \leq k \leq c-1}} [z_t, z_1^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1}].
\end{aligned}$$

For each choice of t and λ' s, by Corollary 4.10,

$$T(t,\lambda) = [z_c^{-1}, z_2 ; z_t, z_1^{-1}, {}_{\lambda_{t+1}}z_{t+1}, {}_{\lambda_{t+2}}z_{t+2}, \ldots, {}_{\lambda_{c-1}}z_{c-1}] \equiv 1.$$

Therefore,

$$\begin{aligned}
C &\equiv [z_c^{-1}, z_2 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} z_1^{-1}}]^{\varepsilon_5 f_5} \\
&\equiv [z_c^{-1}, z_2 ; [z_i, z_3]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_6 f_6} \\
&\equiv [z_c^{-1}, z_2 ; [z_i, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_7 f_7} [z_c^{-1}, z_2 ; [z_i, z_1^{-1}, z_3]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_8 f_8} \\
&\quad \cdot [z_c^{-1}, z_2 ; [z_3, z_1^{-1}, z_i]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_9 f_9} \\
&\equiv [z_c^{-1}, z_2 ; [z_3, z_1^{-1}, z_i]^{z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_{10} f_{10}} \text{ (by Lemma 4.2 and Corollary 4.10)} \\
&\equiv [z_c^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_i z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_{11} f_{11}} \quad \text{(by Corollary 4.10)}.
\end{aligned}$$

One last time, by the Collection Lemma, if $z_4 < z_i$, then

$$\begin{aligned}
z_i z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1} &= z_4 z_5 \ldots z_i \ldots z_{c-1} \\
&\quad \cdot \prod_{t=4}^{i-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \le k \le c-1,\, k \ne i}} [z_i, z_t, {}_{\lambda_{t+1}}z_{t+1}, {}_{\lambda_{t+2}}z_{t+2}, \ldots, {}_{\lambda_{c-1}}z_{c-1}].
\end{aligned}$$

For each choice of t and λ' s, by the Mini-Max Lemma,

$$[z_c^{-1}, z_2 ; z_i, z_t, {}_{\lambda_{t+1}}z_{t+1}, {}_{\lambda_{t+2}}z_{t+2}, \ldots, {}_{\lambda_{c-1}}z_{c-1}] \equiv 1.$$

Therefore,

$$\begin{aligned}
C &\equiv [z_c^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_i z_4 z_5 \ldots \hat{z}_i \ldots z_{c-1}}]^{\varepsilon_{11} f_{11}} \\
&\equiv [z_c^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_4 z_5 \ldots z_i \ldots z_{c-1}}]^{\varepsilon_{12} f_{12}} \\
&\equiv 1 \quad \text{(by Corollary 4.10)},
\end{aligned}$$

which completes the proof . □

Lemma 4.12. *For* $5 \le c \le n+1$, *let* $y_1 < y_2 = y_3 < y_4 \le y_5 \le \cdots \le y_c$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$. $C = [y_3, y_1 ; y_4, y_2, y_5, \dots, y_c]$ *is congruent to either* 1 *or a product of conjugates of commutators of the form* $[u_2, u_1 ; u_q, u_3, u_4, \dots, u_{q-1}]$ *where* $q \le c-1$ *and* $u_1 < u_2 < u_3 \le u_4 \le \cdots \le u_q$ *such that* $u_1 = y_1$, $u_2 = y_2 = y_3$, $\{u_3, u_4, \dots, u_q\} \subseteq \{y_4, y_5, \dots, y_c\}$.

PROOF. Proof is based on induction on c. Let N denote normal closure of the set of all commutators of form $[u_2, u_1 : u_q, u_3, u_4, \dots, u_{q-1}]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:
If $c = 5$ then since $y_3 = y_2$,

$$[y_3, y_1 ; y_4, y_2, y_5] \equiv [y_5^{-1} \cdot y_1 \cdot y_3 ; y_4, y_2]^{\varepsilon_1 f_1} [y_5^{-1} \cdot y_3 \cdot y_1 ; y_4, y_2]^{\varepsilon_2 f_2}$$
$$\equiv [y_5^{-1} \cdot y_2 ; y_4, y_1^{-1}, y_3]^{\varepsilon_3 f_3} [y_5^{-1}, y_2 ; y_3, y_1^{-1} \cdot y_4]^{\varepsilon_4 f_4}.$$

But

$$[y_5^{-1} \cdot y_2 ; y_4, y_1^{-1}, y_3] \equiv [y_5^{-1}, y_2 ; [y_4, y_1]^{y_3 y_1^{-1}}]^{\delta_1 g_1}$$
$$\equiv [y_5^{-1} \cdot y_1 \cdot y_2 ; [y_4 \cdot y_1]^{y_3}]^{\delta_2 g_2} [y_2 \cdot y_1 \cdot y_5^{-1} ; [y_4 \cdot y_1]^{y_3}]^{\delta_3 g_3}$$
$$\equiv 1.$$

Similarly $[y_5^{-1}, y_2 ; y_3, y_1^{-1}, y_4] \equiv 1$: therefore, $[y_3, y_1 ; y_4, y_2, y_5] \equiv 1$.
Assume the statement is true for integers less than c: then for c we have,

$$C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 5 \le k \le c}} [y_3, y_1 ; [y_4, y_2]^{y_5^{\lambda_5} y_6^{\lambda_6} \cdots y_c^{\lambda_c}}]^{\varepsilon(\lambda_5, \lambda_6, \dots, \lambda_c) f(\lambda_5, \lambda_6, \dots, \lambda_c)}$$
$$\equiv [y_3, y_1 ; [y_4, y_2]^{y_5 y_6 \cdots y_c}]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)}$$
$$\equiv [y_3, y_1 ; [y_4, y_2]^{y_5 y_6 \cdots y_{c-1}}]^{\varepsilon_2 f_2} [y_c^{-1} \cdot y_1, y_3 ; [y_4, y_2]^{y_5 y_6 \cdots y_{c-1}}]^{\varepsilon_3 f_3}$$
$$\cdot [y_c^{-1}, y_3, y_1 ; [y_4, y_2]^{y_5 y_6 \cdots y_{c-1}}]^{\varepsilon_4 f_4}$$
$$\equiv [y_c^{-1}, y_3, y_1 ; [y_4, y_2]^{y_5 y_6 \cdots y_{c-1}}]^{\varepsilon_4 f_4},$$

by the induction hypothesis and the Mini-Max Lemma. Now since $y_3 = y_2$,

$$\equiv \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_5 y_6 \cdots y_{c-1}}\right]^{\varepsilon_5 f_5} \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_5 y_6 \cdots y_{c-1} y_1^{-1}}\right]^{\varepsilon_6 f_6}$$
$$\equiv \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_5 y_6 \cdots y_{c-1} y_1^{-1}}\right]^{\varepsilon_6 f_6} \quad \text{(by the Mini-Max Lemma).}$$

But by the Collection Lemma,

$$y_5 y_6 \cdots y_{c-1} y_1^{-1} = y_1^{-1} y_5 y_6 \cdots y_{c-1} \cdot \prod_{t=5}^{c-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \le k \le c-1}} [y_t, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{c-1}} y_{c-1}].$$

For each choice of t and λ' s,

$$T(t, \lambda) = [y_c^{-1}, y_2 ; y_t, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{c-1}} y_{c-1}],$$

is of weight at most $c-2$, and by Corollary 4.10. $T(t, \lambda) \equiv 1$. Therefore,

$$C \equiv \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_5 y_6 \cdots y_{c-1} y_1^{-1}}\right]^{\varepsilon_6 f_6}$$
$$\equiv \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_1^{-1} y_5 y_6 \cdots y_{c-1}}\right]^{\varepsilon_7 f_7}$$
$$\equiv \left[y_c^{-1}, y_2 ; \left[y_4, y_3\right]^{y_5 y_6 \cdots y_{c-1}}\right]^{\varepsilon_8 f_8} \left[y_c^{-1}, y_2 ; \left[y_4, y_1^{-1}, y_3\right]^{y_5 y_6 \cdots y_{c-1}}\right]^{\varepsilon_9 f_9}$$
$$\cdot \left[y_c^{-1}, y_2 ; \left[y_3, y_1^{-1}, y_4\right]^{y_5 y_6 \cdots y_{c-1}}\right]^{\varepsilon_{10} f_{10}}$$
$$\equiv 1 \quad \text{(by the Mini-Max Lemma and Corollary 4.10).}$$

which completes the proof. □

Lemma 4.13. *For $5 \le p \le n+1$, let $y_1 < y_2 \le y_3 < y_4 \le \cdots \le y_p$ and $y_3 \le y_j \le y_i \le y_p$; then modulo $\langle \Delta(m, n, F) \rangle^F$ for $\varepsilon \in \{1, -1\}$, $C = [y_i^{-1}, y_1 ; y_j, y_2^{\varepsilon}, y_3, \dots, \hat{y}_j, \dots, \hat{y}_i, \dots, y_p]$ is congruent to either 1 or a product of conjugates of commutators of the form $[u_2, u_1 ; u_q, u_3, u_4, \dots, u_{q-1}]$ where $q \le p-1$ and $u_1 < u_2 < u_3 \le u_4 \le \cdots \le u_q$ such that $\{u_1, u_2, \dots, u_q\} \subseteq \{y_1, y_2, \dots, \hat{y}_j, \dots, y_p\}$.*

PROOF. If $y_i = y_p$, then by the Mini-Max Lemma, $C \equiv 1$. If $y_i < y_p$ then we prove it by induction on p. Let N denote the normal closure of the set of all commutators of the form $[u_2, u_1; u_q, u_3, u_4, \ldots, u_{q-1}]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:

If $p = 5$, then

$$\begin{aligned}
[y_4^{-1}, y_1 : y_3, y_2^{\varepsilon}, y_5] &\equiv [y_4^{-1}, y_1 : [y_3, y_2^{\varepsilon}]^{y_5}]^{\delta_1 g_1} \\
&\equiv [y_4, y_1 ; [y_3, y_2^{\varepsilon}]^{y_4 y_5}]^{\delta_2 g_2} \\
&\equiv 1 \quad \text{(by Lemma 4.2).}
\end{aligned}$$

Assume the statement is true for integers less than p: then for p we have

$$\begin{aligned}
C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \le k \le p,\, k \ne i,j}} & [y_i^{-1}, y_1 : [y_j, y_2^{\varepsilon}]^{y_3^{\lambda_3} y_4^{\lambda_4} \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_p^{\lambda_p}}]^{\varepsilon(\lambda_3, \lambda_4, \ldots, \lambda_p) f(\lambda_3, \lambda_4, \ldots, \lambda_p)} \\
&\equiv [y_i^{-1}, y_1 ; [y_j, y_2^{\varepsilon}]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_p}]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)} \\
&\equiv [y_i, y_1 ; [y_j, y_2^{\varepsilon}]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_p y_i}]^{\varepsilon_2 f_2}.
\end{aligned}$$

But by the Collection Lemma.

$$\begin{aligned}
y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_p y_i = {} & y_3 y_4 \cdots \hat{y}_j \cdots y_i \cdots y_p \\
& \cdot \prod_{t=i+1}^{p} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \le k \le p}} [y_t, y_i, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_p} y_p].
\end{aligned}$$

For each choice of t and λ' s.

$$T(t, \lambda) = [y_i, y_1 ; y_t, y_i, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_p} y_p].$$

is of weight at most $p - 1$, and by Lemma 4.12, $T(t, \lambda) \equiv 1$. Therefore.

$$\begin{aligned}
C &\equiv [y_i, y_1 ; [y_j, y_2^{\varepsilon}]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_p y_i}]^{\varepsilon_2 f_2} \\
&\equiv [y_i, y_1 ; [y_j, y_2^{\varepsilon}]^{y_3 y_4 \cdots \hat{y}_j \cdots y_i \cdots y_p}]^{\varepsilon_3 f_3} \\
&\equiv 1 \quad \text{(by Lemma 4.2).}
\end{aligned}$$

which completes the proof. $\square$

Lemma 4.14. *For $5 \leq p \leq n+1$, let $z_1 < z_2 < z_3 \leq z_4 \leq \cdots \leq z_p$: then modulo $\langle \Delta(m, n, F) \rangle^F$, $C = [z_2^{-1}, z_1 ; z_p, z_3, z_4, \dots, z_{p-1}]$ can be written as a product of conjugates of commutators of the form $[u_2, u_1 : u_q, u_3, u_4, \dots, u_{q-1}]$, where $q \leq p$ and $u_1 < u_2 < u_3 \leq u_4 \leq \cdots \leq u_q$, such that $\{u_1, u_2, \dots, u_q\} \subseteq \{z_1, z_2, \dots, z_p\}$.*

PROOF. The proof is based on induction on p. Let N denote the normal closure of the set of all commutators of the form $[u_2, u_1 : u_q, u_3, u_4, \dots, u_{q-1}]$ with the above conditions. We work modulo $\langle \Delta(m, n, F) \rangle^F N$:
If $p = 5$, then

$$[z_2^{-1}, z_1 ; z_5, z_3, z_4] \equiv [z_2, z_1 : [z_5, z_3]^{z_2}]^{\varepsilon_1 f_1} [z_2, z_1 : [z_5, z_3]^{z_2 z_4}]^{\varepsilon_2 f_2} \tag{4.2}$$

But

$$\begin{aligned}
[z_2, z_1 ; [z_5, z_3]^{z_2 z_4}] &\equiv [z_2, z_1 ; z_5, z_3, z_4]^{\delta_1 g_1} [z_2, z_1 : [z_5, z_2, z_3]^{z_4}]^{\delta_2 g_2} \\
&\quad \cdot [z_2, z_1 ; [z_3, z_2, z_5]^{z_4}]^{\delta_3 g_3} \\
&\equiv [z_2, z_1 : [z_3, z_2, z_5]^{z_4}]^{\delta_3 g_3} \quad \text{(by Lemma 4.2)} \\
&\equiv [z_2, z_1 : [z_3, z_2]^{z_4}]^{\delta_4 g_4} [z_2, z_1 : [z_3, z_2]^{z_4 z_5}]^{\delta_5 g_5}. \qquad (4.3)
\end{aligned}$$

On one hand,

$$\begin{aligned}
[z_2, z_1 ; [z_3, z_2]^{z_4}] &\equiv [z_4^{-1}, z_1, z_2 ; z_3, z_2]^{\sigma_1 h_1} [z_4^{-1}, z_2, z_1 ; z_3, z_2]^{\sigma_2 h_2} \\
&\equiv [z_4^{-1}, z_2 ; z_3, z_1^{-1}, z_2]^{\sigma_3 h_3} [z_4^{-1}, z_2 ; z_2, z_1^{-1}, z_3]^{\sigma_4 h_4} \\
&\equiv 1 \quad \text{(by Corollary 4.10)}. \qquad (4.4)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\left[z_2, z_1 ; [z_3, z_2]^{z_4 z_5} \right] &\equiv \left[z_2, z_1 : [z_3, z_2]^{z_4} \right]^{\theta_1 a_1} \left[z_5^{-1} \cdot z_1 \cdot z_2 ; [z_3, z_2]^{z_4} \right]^{\theta_2 a_2} \\
&\quad \cdot \left[z_5^{-1}, z_2, z_1 : [z_3, z_2]^{z_4} \right]^{\theta_3 a_3} \\
&\equiv \left[z_5^{-1} \cdot z_2, z_1 : [z_3, z_2]^{z_4} \right]^{\theta_3 a_3} \quad \text{(by 4.4)} \\
&\equiv \left[z_5^{-1}, z_2 : [z_3, z_2]^{z_4} \right]^{\theta_4 a_4} \left[z_5^{-1} \cdot z_2 : [z_3, z_1^{-1} \cdot z_2]^{z_4} \right]^{\theta_5 a_5} \\
&\quad \cdot \left[z_5^{-1} \cdot z_2 : [z_2 \cdot z_1^{-1} \cdot z_3]^{z_4} \right]^{\theta_6 a_6} \\
&\equiv 1 \quad \text{(by Corollary 4.10).} \qquad (4.5)
\end{aligned}
$$

Substitution of 4.4 and 4.5 in 4.3 gives

$$\left[z_2, z_1 : [z_5, z_3]^{z_2 z_4} \right] \equiv 1.$$

Similarly, $\left[z_2, z_1 ; [z_5, z_3]^{z_2} \right] \equiv 1$. So by substitution in 4.3 we have,

$$\left[z_2^{-1} \cdot z_1 : z_5, z_3, z_4 \right] \equiv 1.$$

Assume the statement is true for integers less than p: then for p we have

$$
\begin{aligned}
C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \le k \le p-1}} & \left[z_2^{-1} \cdot z_1 : [z_p, z_3]^{z_4^{\lambda_4} z_5^{\lambda_5} \cdots z_{p-1}^{\lambda_{p-1}}} \right]^{\varepsilon(\lambda_4, \lambda_5, \ldots, \lambda_{p-1}) f(\lambda_4, \lambda_5, \ldots, \lambda_{p-1})} \\
&\equiv \left[z_2^{-1}, z_1 ; [z_p, z_3]^{z_4 z_5 \cdots z_{p-1}} \right]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)} \\
&\equiv \left[z_2, z_1 ; [z_p, z_3]^{z_4 z_5 \cdots z_{p-1} z_2} \right]^{\varepsilon_2 f_2}.
\end{aligned}
$$

But by the Collection Lemma,

$$
\begin{aligned}
z_4 z_5 \ldots z_{p-1} z_2 = z_2 z_4 z_5 \ldots z_{p-1} & \\
\cdot \prod_{t=4}^{p-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \le k \le p-1}} & [z_t, z_2, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{p-1}} z_{p-1}].
\end{aligned}
$$

For each choice of t and λ' s, since

$$T(t, \lambda) = [z_2, z_1 ; z_t, z_2, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{p-1}} z_{p-1}],$$

is of weight at most $p-1$, by Lemma 4.12. $T(t, \lambda) \equiv 1$. Therefore,

$$
\begin{aligned}
C &\equiv \left[z_2, z_1 ; \left[z_p, z_3\right]^{z_4 z_5 \dots z_{p-1} z_2}\right]^{\varepsilon_2 f_2} \\
&\equiv \left[z_2, z_1 ; \left[z_p, z_3\right]^{z_2 z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_3 f_3} \\
&\equiv \left[z_2, z_1 ; \left[z_p, z_3\right]^{z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_4 f_4} \left[z_2, z_1 ; \left[z_p, z_2, z_3\right]^{z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_5 f_5} \\
&\quad \cdot \left[z_2, z_1 ; \left[z_3, z_2\right]^{z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_6 f_6} \left[z_2, z_1 ; \left[z_3, z_2\right]^{z_p z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_7 f_7} \\
&\equiv \left[z_2, z_1 ; \left[z_3, z_2\right]^{z_p z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_7 f_7} \text{ (by Lemma 4.12 and the Mini-Max Lemma).}
\end{aligned}
$$

But by the Collection Lemma,

$$
z_p z_4 z_5 \dots z_{p-1} = z_4 z_5 \dots z_{p-1} z_p \cdot \prod_{t=4}^{p-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ t+1 \leq k \leq p-1}} [z_p, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \dots, {}_{\lambda_{p-1}} z_{p-1}].
$$

For each choice of t and λ' s, since

$$
T(t, \lambda) = [z_2, z_1 ; z_p, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \dots, {}_{\lambda_{p-1}} z_{p-1}],
$$

is of required form of weight at most $p-1$, so $T(t, \lambda) \equiv 1$. Therefore,

$$
\begin{aligned}
C &\equiv \left[z_2, z_1 ; \left[z_3, z_2\right]^{z_p z_4 z_5 \dots z_{p-1}}\right]^{\varepsilon_7 f_7} \\
&\equiv \left[z_2, z_1 ; \left[z_3, z_2\right]^{z_4 z_5 \dots z_{p-1} z_p}\right]^{\varepsilon_8 f_8} \\
&\equiv 1 \quad \text{(by Lemma 4.12);}
\end{aligned}
$$

hence the proof is complete. □

Lemma 4.15. *For* $5 \leq p \leq n+1$, *let* $y_1 < y_2 < y_3 \leq y_4 \leq \dots \leq y_p$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$, *each one of the following commutators can be written as a product of conjugates of commutators of the form,* $[u_2, u_1 ; u_q, u_3, u_4, \dots, u_{q-1}]$, *where* $u_1 < u_2 < u_3 \leq u_4 \leq \dots \leq u_q$, *and*

$\{u_1, u_2, \dots, u_q\} \subseteq \{y_1, y_2, \dots, y_p\}$.

$$(a) \qquad C_1 = [\, y_3^{-1} \cdot y_2 \,;\, y_4, y_1^{-1} \cdot y_5, \dots, y_p \,].$$

$$(b) \qquad C_2 = [\, y_3^{-1} \cdot y_1 \,;\, y_4, y_2^{-1} \cdot y_5, \dots, y_p \,],$$

$$(c) \qquad C_3 = [\, y_i^{-1} \cdot y_2 \,;\, y_3, y_1^{-1} \cdot y_4, \dots, \hat{y}_i, \dots, y_p \,],$$

such that $q \le p-1$ *in* (a) *and* $q \le p$ *in* (b) *and* (c).

PROOF. (a) The proof is based on induction on p. Let N_1 denote the normal closure of the set of all commutators of the form $[\, u_2, u_1 \,;\, u_q, u_3, u_4, \dots, u_{q-1} \,]$ such that $q \le p-1$. We use induction on p and work modulo $\langle \Delta(m, n, F) \rangle^F N_1$. If $p = 5$, then

$$
\begin{aligned}
[\, y_3^{-1} \cdot y_2 \,;\, y_4, y_1^{-1}, y_5 \,] &\equiv [\, y_3^{-1} \cdot y_2 \,;\, [\, y_4, y_1 \,]^{y_5 y_1^{-1}} \,]^{\varepsilon_1 f_1} \\
&\equiv [\, y_4, y_1 \,;\, [\, y_3^{-1} \cdot y_2 \,]^{y_5^{-1}} \,]^{\varepsilon_2 f_2} [\, y_4, y_1 \,;\, [\, y_3^{-1} \cdot y_1, y_2 \,]^{y_5^{-1}} \,]^{\varepsilon_3 f_3} \\
&\quad \cdot [\, y_4, y_1 \,;\, [\, y_2, y_1 \cdot y_3^{-1} \,]^{y_5^{1}} \,]^{\varepsilon_4 f_4} \\
&\equiv 1 \quad \text{(by Lemma 4.2).}
\end{aligned}
$$

Assume the statement is true for integers less than p: then for p we have

$$
\begin{aligned}
C_1 &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 5 \le k \le p}} [\, y_3^{-1} \cdot y_2 \,;\, [\, y_4, y_1^{-1} \,]^{y_5^{\lambda_5} y_6^{\lambda_6} \cdots y_p^{\lambda_p}} \,]^{\varepsilon(\lambda_5, \lambda_6, \dots, \lambda_p) f(\lambda_5, \lambda_6, \dots, \lambda_p)} \\
&\equiv [\, y_3^{-1}, y_2 \,;\, [\, y_4, y_1 \,]^{y_1^{-1} y_5 y_6 \cdots y_p} \,]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis).}
\end{aligned}
$$

Now using the Collection Lemma we have

$$
\begin{aligned}
y_1^{-1} y_5 y_6 \cdots y_{p-1} y_p &= y_p y_{p-1} \cdots y_6 y_5 y_1^{-1} \\
&\quad \cdot \prod_{t=s+1}^{p} \prod_{s=5}^{p-1} \prod_{\substack{\lambda_k \in \{0,1\} \\ s+1 \le k \le t-1}} [\, y_t, y_s, {}_{\lambda_{s+1}} y_{s+1}, {}_{\lambda_{s+2}} y_{s+2}, \dots, {}_{\lambda_{t-1}} y_{t-1} \,] \\
&\quad \cdot \prod_{t=5}^{p} [\, y_t, y_1^{-1}, {}_{\lambda_5} y_5, {}_{\lambda_6} y_6, \dots, {}_{\lambda_{t-1}} y_{t-1} \,].
\end{aligned}
$$

For each choice of t, s and λ' s,

$$T(t, \lambda) = [y_3^{-1}, y_2 : y_t, y_1^{-1}, {}_{\lambda_5}y_5, {}_{\lambda_6}y_6, \dots, {}_{\lambda_{t-1}}y_{t-1}]$$

is of weight at most $p-1$, and by the Mini-Max Lemma, $T(t, \lambda) \equiv 1$. Also,

$$T(t, s, \lambda) = [y_3^{-1}, y_2 : y_t, y_s, {}_{\lambda_{s+1}}y_{s+1}, {}_{\lambda_{s+2}}y_{s+2}, \dots, {}_{\lambda_{t-1}}y_{t-1}]$$

is of weight at most $p-1$. If $y_3 = y_s$, by the Mini-Max Lemma, and if $y_3 < y_s$, by Lemma 4.14, $T(t, s, \lambda) \equiv 1$. Therefore,

$$\begin{aligned} C_1 &\equiv [y_3^{-1}, y_2 : [y_4, y_1]^{y_1^{-1}y_5y_6\cdots y_p}]^{\varepsilon_1 f_1} \\ &\equiv [y_3^{-1}, y_2 : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5y_1^{-1}}]^{\varepsilon_2 f_2} \\ &\equiv [y_3^{-1}, y_2 : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5}]^{\varepsilon_3 f_3} [y_3^{-1}, y_1, y_2 : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5}]^{\varepsilon_4 f_4} \\ &\quad \cdot [y_2, y_1, y_3^{-1} : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5}]^{\varepsilon_5 f_5}. \end{aligned} \tag{4.6}$$

But

$$\begin{aligned} [y_3^{-1}, y_2 : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5}] & \\ &\equiv [y_4, y_1 : [y_3^{-1}, y_2]^{y_5^{-1}y_6^{-1}\cdots y_{p-1}^{-1}y_p^{-1}}]^{\varepsilon f} \\ &\equiv 1 \quad \text{(by Lemma 4.2)}. \end{aligned} \tag{4.7}$$

Also,

$$\begin{aligned} [y_2, y_1, y_3^{-1} : [y_4, y_1]^{y_py_{p-1}\cdots y_6y_5}] & \\ &\equiv [y_4, y_1 : [y_2, y_1, y_3^{-1}]^{y_5^{-1}y_6^{-1}\cdots y_{p-1}^{-1}y_p^{-1}}]^{\varepsilon f} \\ &\equiv 1 \quad \text{(by Lemma 4.2)}. \end{aligned} \tag{4.8}$$

In addition,

$$[\, y_3^{-1}, y_1, y_2 ; [\, y_4, y_1\,]^{y_p y_{p-1} \cdots y_6 y_5}\,]$$

$$\equiv [\, y_3^{-1}, y_1 ; [\, y_4, y_1\,]^{y_p y_{p-1} \cdots y_6 y_5}\,]^{\varepsilon_1 f_1} [\, [\, y_3, y_1\,]^{y_2 y_3^{-1} [y_3^{-1}, y_2]} ; [\, y_4, y_1\,]^{y_p y_{p-1} \cdots y_6 y_5}\,]^{\varepsilon_2 f_2}$$

$$\equiv [\, y_4, y_1 ; [\, y_3, y_1\,]^{y_2 y_3^{-1} y_5^{-1} y_6^{-1} \cdots y_{p-1}^{-1} y_p^{-1}}\,]^{\varepsilon_3 f_3} \quad (\text{by } 4.7)$$

(4.9)

$$\equiv 1 \quad (\text{by Lemma 4.2}).$$

Substitution of 4.7, 4.8 and 4.9 in 4.6 gives $C_1 \equiv 1$.

Proof of (b) : Let N_2 denote the normal closure of the set of all commutators of the form $[\, u_2, u_1 ; u_q, u_3, u_4, \ldots, u_{q-1}\,]$ such that $q \leq p$. We work modulo $\langle \Delta(m, n, F) \rangle^F N_2$ and we prove the statement by induction on p : If $p = 5$, then

$$\begin{aligned}
C_2 &= [\, y_3^{-1}, y_1 ; y_4, y_2^{-1}, y_5\,] \\
&\equiv [\, y_3^{-1}, y_1 ; y_5, y_2^{-1}, y_4\,]^{\varepsilon_1 f_1} [\, y_3^{-1}, y_1 ; y_5, y_4, y_2^{-1}\,]^{\varepsilon_2 f_2} \\
&\equiv [\, y_3^{-1}, y_2, y_1 ; y_5, y_4\,]^{\varepsilon_3 f_3} [\, y_2, y_1, y_3^{-1} ; y_5, y_4\,]^{\varepsilon_4 f_4} \\
&\equiv [\, y_3^{-1}, y_2 ; y_5, y_1^{-1}, y_4\,]^{\varepsilon_5 f_5} [\, y_3^{-1}, y_2 ; y_4, y_1^{-1}, y_5\,]^{\varepsilon_6 f_6} \\
&\quad \cdot [\, y_2, y_1 ; y_5, y_3, y_4\,]^{\varepsilon_7 f_7} [\, y_2, y_1 ; y_4, y_3, y_5\,]^{\varepsilon_8 f_8} \\
&\equiv 1 \quad (\text{by Theorem 4.11, the Mini-Max Lemma and part (a)}).
\end{aligned}$$

Assume the statement is true for integers less than p: then for p we have

$$\begin{aligned}
C_2 &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 5 \leq k \leq p}} [\, y_3^{-1}, y_1 ; [\, y_4, y_2^{-1}\,]^{y_5^{\lambda_5} y_6^{\lambda_6} \cdots y_p^{\lambda_p}}\,]^{\varepsilon(\lambda_5, \lambda_6, \ldots, \lambda_p) f(\lambda_5, \lambda_6, \ldots, \lambda_p)} \\
&\equiv [\, y_3^{-1}, y_1 ; [\, y_4, y_2^{-1}\,]^{y_5 y_6 \cdots y_p}\,]^{\varepsilon_1 f_1} \quad (\text{by the induction hypothesis}).
\end{aligned}$$

Each transferring commutator,

$$T(t, \lambda) = [\, y_3^{-1}, y_1 ; y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1}\,].$$

is of weight at most $p-2$. If $y_3 = y_t$ by the Mini-Max Lemma, and if $y_3 < y_t$, by Lemma 4.14, we have $T(t, \lambda) \equiv 1$. So,

$$
\begin{aligned}
C_2 &\equiv \left[y_3^{-1}, y_1 ; [y_4, y_2^{-1}]^{y_p y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_2 f_2} \\
&\equiv \left[y_3^{-1}, y_1 ; [y_4, y_2^{-1}]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_3 f_3} \left[y_3^{-1}, y_1 ; [y_p, y_2^{-1}, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_4 f_4} \\
&\quad \cdot \left[y_3^{-1}, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_5 f_5} \left[y_3^{-1}, y_1 ; [y_p, y_4]^{y_2^{-1} y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_6 f_6}.
\end{aligned} \tag{4.10}
$$

In 4.10, the first commutator is congruent to 1 by the induction hypothesis. The second commutator is congruent to 1 by Mini-Max Lemma. The third commutator is congruent to one by Lemma 4.14. About the fourth one, since each transferring commutator,

$$T(t, \lambda) = [\, y_3^{-1}, y_1 ; y_t, y_2^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-2$, by the induction hypothesis, $T(t, \lambda) \equiv 1$. Therefore,

$$
\begin{aligned}
C_2 &\equiv \left[y_3^{-1}, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1} y_2^{-1}} \right]^{\varepsilon_7 f_7} \\
&\equiv \left[y_3^{-1}, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_8 f_8} \left[y_2, y_1, y_3^{-1} ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_9 f_9} \\
&\quad \cdot \left[y_3^{-1}, y_2, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_{10} f_{10}}.
\end{aligned} \tag{4.11}
$$

In 4.11, we show that each one of the three commutators is congruent to 1 separately: $\left[y_3^{-1}, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right] \equiv 1$ by the Mini-Max Lemma when $y_3 = y_4$, and by Lemma 4.14 when $y_3 < y_4$.
Set $\alpha = \left[y_2, y_1, y_3^{-1} ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]$; then

$$
\begin{aligned}
\alpha &\equiv \left[y_2, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1}} \right]^{\delta_1 g_1} \left[y_2, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1} y_3} \right]^{\delta_2 g_2} \\
&\equiv \left[y_2, y_1 ; [y_p, y_4]^{y_5 y_6 \cdots y_{p-1} y_3} \right]^{\delta_2 g_2} \quad \text{(by definition of } N_2\text{)}.
\end{aligned}
$$

Each transferring commutator,

$$T(t,\lambda) = [\, y_2, y_1 ; y_t, y_3, {}_{\lambda_{t+1}}y_{t+1}, {}_{\lambda_{t+2}}y_{t+2}, \ldots, {}_{\lambda_{p-1}}y_{p-1} \,],$$

is of weight at most $p-2$. By Theorem 4.11. we have $T(t,\lambda) \equiv 1$. So.

$$\begin{aligned}
\alpha &\equiv \left[\, y_2, y_1 ; [\, y_p, y_4 \,]^{y_3y_5y_6\cdots y_{p-1}} \,\right]^{\delta_3 g_3} \\
&\equiv \left[\, y_2, y_1 ; [\, y_p, y_3, y_4 \,]^{y_5y_6\cdots y_{p-1}} \,\right]^{\delta_4 g_4} \left[\, y_2, y_1 ; [\, y_4, y_3 \,]^{y_5y_6\cdots y_{p-1}} \,\right]^{\delta_5 g_5} \\
&\quad \cdot \left[\, y_2, y_1 ; [\, y_4, y_3 \,]^{y_py_5y_6\cdots y_{p-1}} \,\right]^{\delta_6 g_6} \\
&\equiv \left[\, y_2, y_1 ; [\, y_4, y_3 \,]^{y_py_5y_6\cdots y_{p-1}} \,\right]^{\delta_6 g_6} \text{ (by definition of } X_2 \text{ and Theorem 4.11).}
\end{aligned}$$

Each transferring commutator.

$$T(t,\lambda) = [\, y_2, y_1 ; y_p, y_t, {}_{\lambda_{t+1}}y_{t+1}, {}_{\lambda_{t+2}}y_{t+2}, \ldots, {}_{\lambda_{p-1}}y_{p-1} \,],$$

is of weight at most $p-2$; and by definition of X_2, $T(t,\lambda) \equiv 1$. So.

$$\alpha \equiv \left[\, y_2, y_1 ; [\, y_4, y_3 \,]^{y_5y_6\cdots y_{p-1}y_p} \,\right]^{\delta_7 g_7} \equiv 1 \quad \text{(by Theorem 4.11).}$$

Also, set $\beta = \left[\, y_3^{-1}, y_2, y_1 ; [\, y_p, y_4 \,]^{y_5y_6\cdots y_{p-1}} \,\right]$; then

$$\begin{aligned}
\beta &\equiv \left[\, y_3^{-1}, y_2 ; [\, y_p, y_4 \,]^{y_5y_6\cdots y_{p-1}} \,\right]^{\sigma_1 h_1} \left[\, y_3^{-1}, y_2 ; [\, y_p, y_4 \,]^{y_5y_6\cdots y_{p-1}y_1^{-1}} \,\right]^{\sigma_2 h_2} \\
&\equiv \left[\, y_3^{-1}, y_2 ; [\, y_p, y_4 \,]^{y_5y_6\cdots y_{p-1}y_1^{-1}} \,\right]^{\sigma_2 h_2}
\end{aligned}$$

(by Lemma 4.14 or the Mini-Max Lemma).

Each transferring commutator,

$$T(t,\lambda) = [\, y_3^{-1}, y_2 ; y_t, y_1^{-1}, {}_{\lambda_{t+1}}y_{t+1}, {}_{\lambda_{t+2}}y_{t+2}, \ldots, {}_{\lambda_{p-1}}y_{p-1} \,],$$

is of weight at most $p-2$, and by part (a), $T(t,\lambda) \equiv 1$. So,

$$\begin{aligned}
\beta &\equiv [\, y_3^{-1}, y_2 \,;\, [\, y_p, y_4 \,]^{y_1^{-1} y_5 y_6 \cdots y_{p-1}} \,]^{\sigma_3 h_3} \\
&\equiv [\, y_3^{-1}, y_2 \,;\, [\, y_p, y_1^{-1}, y_4 \,]^{y_5 y_6 \cdots y_{p-1}} \,]^{\sigma_4 h_4} [\, y_3^{-1}, y_2 \,;\, [\, y_4, y_1^{-1} \,]^{y_5 y_6 \cdots y_{p-1}} \,]^{\sigma_5 h_5} \\
&\quad \cdot [\, y_3^{-1}, y_2 \,;\, [\, y_4, y_1^{-1} \,]^{y_p y_5 y_6 \cdots y_{p-1}} \,]^{\sigma_6 h_6} \\
&\equiv [\, y_3^{-1}, y_2 \,;\, [\, y_4, y_1^{-1} \,]^{y_p y_5 y_6 \cdots y_{p-1}} \,]^{\sigma_6 h_6} \text{ (by part } (a) \text{ and the Mini-Max Lemma).}
\end{aligned}$$

Each transferring commutator,

$$T(t,\lambda) = [\, y_3^{-1}, y_2 \,;\, y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-2$. If $y_3 = y_t$, by the Mini-Max Lemma, and if $y_3 < y_t$, by Lemma 4.14, we have $T(t,\lambda) \equiv 1$. Therefore,

$$\begin{aligned}
\beta &\equiv [\, y_3^{-1}, y_2 \,;\, [\, y_4, y_1^{-1} \,]^{y_5 y_6 \cdots y_{p-1} y_p} \,]^{\sigma_7 h_7} \\
&\equiv 1 \quad \text{(by part } (a) \text{).}
\end{aligned}$$

As a result since $\alpha \equiv \beta \equiv 1$, by substitution in 4.11 we get $C_2 \equiv 1$.

Proof of (c) : Similar to the proof of part (b) here we also work modulo $\langle \Delta(m,n,F) \rangle^F N_2$. If $y_i = y_p$, then by Corollary 4.10, $C_3 \equiv 1$. If $y_i < y_p$, then we prove the statement by induction on p :

If $p = 5$ then,

$$\begin{aligned}
[\, y_4^{-1}, y_2 \,;\, y_3, y_1^{-1}, y_5 \,] &\equiv [\, y_4^{-1}, y_2 \,;\, y_5, y_1^{-1}, y_3 \,]^{\varepsilon_1 f_1} [\, y_4^{-1}, y_2 \,;\, y_5, y_3, y_1^{-1} \,]^{\varepsilon_2 f_2} \\
&\equiv [\, y_4^{-1}, y_1, y_2 \,:\, y_5, y_3 \,]^{\varepsilon_3 f_3} [\, y_2, y_1, y_4^{-1} \,:\, y_5, y_3 \,]^{\varepsilon_4 f_4} \\
&\equiv [\, y_4^{-1}, y_1 \,;\, y_3, y_2^{-1}, y_5 \,]^{\varepsilon_5 f_5} [\, y_4^{-1}, y_1 \,:\, y_5, y_2^{-1}, y_3 \,]^{\varepsilon_6 f_6} \\
&\quad \cdot [\, y_2, y_1 \,;\, y_5, y_3, y_4 \,]^{\varepsilon_7 f_7} \\
&\equiv 1 \quad \text{(by Lemma 4.13 and the Mini-Max Lemma).}
\end{aligned}$$

Assume the statement is true for integers less than p; then for p we have

$$C_3 = \prod_{\substack{\lambda_k \in \{0,1\} \\ 5 \le k \le p,\, k \ne i}} \left[y_i^{-1}, y_2 ; [y_3, y_1^{-1}]^{y_4^{\lambda_4} y_5^{\lambda_5} \cdots \hat{y}_i \cdots y_p^{\lambda_p}} \right]^{\varepsilon(\lambda_4, \lambda_5, \dots, \lambda_p) f(\lambda_4, \lambda_5, \dots, \lambda_p)}$$

$$\equiv \left[y_i^{-1}, y_2 ; [y_3, y_1^{-1}]^{y_4 y_5 \cdots \hat{y}_i \cdots y_p} \right]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis).}$$

Each transferring commutator.

$$T(t, \lambda) = [\, y_i^{-1}, y_2 ; y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p - 2$. If $y_t \le y_i$, by the Mini-Max Lemma. and if $y_i < y_t$, by Lemma 4.14. we have $T(t, \lambda) \equiv 1$. Therefore.

$$\begin{aligned}
C_3 &\equiv \left[y_i^{-1}, y_2 ; [y_3, y_1^{-1}]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_2 f_2} \\
&\equiv \left[y_i^{-1}, y_2 ; [y_3, y_1^{-1}]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_3 f_3} \left[y_i^{-1}, y_2 ; [y_p, y_1^{-1}, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_4 f_4} \\
&\quad \cdot \left[y_i^{-1}, y_2 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_5 f_5} \left[y_i^{-1}, y_2 ; [y_p, y_3]^{y_1^{-1} y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_6 f_6} \\
&\equiv \left[y_i^{-1}, y_2 ; [y_p, y_3]^{y_1^{-1} y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_6 f_6}
\end{aligned}$$

(by the induction hypothesis and Mini-Max Lemma).

But each transferring commutator.

$$T(t, \lambda) = [\, y_i^{-1}, y_2 ; y_t, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \dots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p - 2$. If $y_i \le y_t$, by part (a). and if $y_t < y_i$, by the induction hypothesis we have $T(t, \lambda) \equiv 1$. Therefore.

(4.12)

$$\begin{aligned}
C_3 &\equiv \left[y_i^{-1}, y_2 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_1^{-1}} \right]^{\varepsilon_7 f_7} \\
&\equiv \left[y_i^{-1}, y_1, y_2 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_8 f_8} \left[y_2, y_1, y_i^{-1} ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_9 f_9}.
\end{aligned}$$

In 4.12 we show that each of the two commutators is congruent to 1 separately :

Set $\alpha = \left[y_i^{-1}, y_1, y_2 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]$; then by the Mini-Max Lemma,

$$\alpha \equiv \left[y_i^{-1}, y_1 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_2^{-1}} \right]^{\delta_1 g_1}.$$

But each transferring commutator,

$$T(t, \lambda) = [\, y_i^{-1}, y_2 ; y_t, y_2^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-2$. If $y_i \leq y_t$, by part (b), and if $y_t < y_i$, by Lemma 4.13, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\begin{aligned}
\alpha &\equiv \left[y_i^{-1}, y_1 ; [y_p, y_3]^{y_2^{-1} y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_2 g_2} \\
&\equiv \left[y_i^{-1}, y_1 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_3 g_3} \left[y_i^{-1}, y_1 ; [y_p, y_2^{-1}, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_4 g_4} \\
&\quad \cdot \left[y_i^{-1}, y_1 ; [y_3, y_2^{-1}]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_5 g_5} \left[y_i^{-1}, y_1 ; [y_3, y_2^{-1}]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_6 g_6} \\
&\equiv \left[y_i^{-1}, y_1 ; [y_3, y_2^{-1}]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_6 g_6}
\end{aligned}$$

(by Lemma 4.13 and the Mini-Max Lemma).

But each transferring commutator,

$$T(t, \lambda) = [\, y_i^{-1}, y_2 ; y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-2$. If $y_t \leq y_i$, by the Mini-Max Lemma, and if $y_i < y_t$, by Lemma 4.14, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\alpha \equiv \left[y_i^{-1}, y_1 ; [y_3, y_2^{-1}]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_p} \right]^{\delta_7 g_7} \equiv 1 \quad \text{(by Lemma 4.13)}.$$

Also set $\beta = \left[y_2, y_1, y_i^{-1} ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]$; then

$$\begin{aligned}
\beta &\equiv \left[y_2, y_1 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\sigma_1 h_1} \left[y_2, y_1 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_i} \right]^{\sigma_2 h_2} \\
&\equiv \left[y_2, y_1 ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_i} \right]^{\sigma_2 h_2} \quad \text{(by definition of } N_2).
\end{aligned}$$

But each transferring commutator,

$$T(t, \lambda) = [\, y_2, y_1 ; y_t, y_i, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-2$, and by Theorem 4.11, we have $T(t, \lambda) \equiv 1$. So,

$$\beta \equiv \left[y_2, y_1 ; [y_p, y_3]^{y_4 y_5 \dots y_i \dots y_{p-1}} \right]^{\sigma_3 h_3} \equiv 1 \quad (\text{by definition of } N_2).$$

Hence, $\alpha \equiv \beta \equiv 1$, and by substitution in 4.12, we get $C_3 \equiv 1$ which completes the proof of part (c) and thus the proof of the lemma. □

Theorem 4.16. *For* $5 \le c \le n+1$, *let* $z_1 < z_2 < z_3 \le z_4 \le \dots \le z_c$; *then*

$$C = [z_2, z_1 ; z_c, z_3, z_4, \dots, z_{c-1}] \equiv 1 \quad (\text{mod } \langle \Delta(m, n, F) \rangle^F).$$

PROOF. If there is no repeat, i.e. $z_1 < z_2 < z_3 < z_4 < \dots < z_c$, then by definition, $C \equiv 1$. If there is a repeat, let $3 \le k \le c-1$ be the smallest for which $z_k = z_{k+1}$. We consider two different cases :

<u>Case 1</u>: If $k = c-1$ i.e. $z_1 < z_2 < z_3 < z_4 < \dots < z_{c-1} = z_c$ then use induction on c. If $c = 5$, then since $z_4 = z_5$,

$$\begin{aligned}
[z_2, z_1 ; z_5, z_3, z_4] &\equiv [z_4^{-1}, z_1, z_2 ; z_5, z_3]^{\varepsilon_1 f_1} [z_4^{-1}, z_2, z_1 ; z_5, z_3]^{\varepsilon_2 f_2} \\
&\equiv [z_4^{-1}, z_2 ; z_5, z_1^{-1}, z_3]^{\varepsilon_3 f_3} [z_4^{-1}, z_2 ; z_3, z_1^{-1}, z_5]^{\varepsilon_4 f_4} \\
&\equiv 1 \quad (\text{by Corollary4.10}).
\end{aligned}$$

Assume the statement is true for integers less than c. Then for c, since $z_{c-1} = z_c$ and renaming z_c to z_{c-1}, by Theorem 4.11, C is congruent to a product of conjugates of commutators of the form $[u_2, u_1 : u_q, u_3, u_4, \dots, u_{q-1}]$ where $q \le c-1$ and $u_1 < u_2 < u_3 \le u_4 \le \dots \le u_q$, such that $\{u_1, u_2, \dots, u_q\} \subseteq \{z_1, z_2, \dots, \hat{z}_i, \dots, z_c\}$. In $U = [u_2, u_1 : u_q, u_3, u_4, \dots, u_{q-1}]$ if $u_{q-1} < z_{c-1}$, then there is no repeat, and thus $U \equiv 1$: if $u_{q-1} = z_{c-1}$, then $u_{q-1} = z_{c-1} = z_c = u_q$, i.e. $u_1 < u_2 < u_3 < u_4 < \dots < u_{q-2} < u_{q-1} = u_q$, and therefore by the induction hypothesis, $U \equiv 1$. Hence, $C \equiv 1$.

<u>Case 2</u>: If $3 \le k \le c-2$, we use induction on c. If $c = 5$, then since

$z_3 = z_4$,

$$
\begin{aligned}
[z_2, z_1\,;\, z_5, z_3, z_4] &= [z_2, z_1\,;\, z_5, z_3, z_3] \\
&\equiv [z_3^{-1}, z_1, z_2\,;\, z_5, z_3]^{\varepsilon_1 f_1}[z_3^{-1}, z_2, z_1\,:\, z_5, z_3]^{\varepsilon_2 f_2} \\
&\equiv [z_3^{-1}, z_1\,;\, z_5, z_2^{-1}, z_3]^{\varepsilon_3 f_3}[z_3^{-1}, z_1\,:\, z_3, z_2^{-1}, z_5]^{\varepsilon_4 f_4} \\
&\quad \cdot [z_3^{-1}, z_2\,;\, z_5, z_1^{-1}, z_3]^{\varepsilon_5 f_5}[z_3^{-1}, z_2\,;\, z_3, z_1^{-1}, z_5]^{\varepsilon_6 f_6} \\
&\equiv 1 \quad \text{(by the Mini-Max Lemma and Corollary 4.10).}
\end{aligned}
$$

Assume the statement is true for integers less than c. Then for c we have

$$
\begin{aligned}
C &= \prod_{\substack{\lambda_s \in \{0,1\} \\ 3 \le s \le c-1}} [z_2, z_1\,:\, [z_c, z_3]^{z_4^{\lambda_4} z_5^{\lambda_5} \cdots z_k^{\lambda_k} z_{k+1}^{\lambda_{k+1}} \cdots z_{c-1}^{\lambda_{c-1}}}]^{\varepsilon(\lambda_4, \lambda_5, \ldots, \lambda_{c-1}) f(\lambda_4, \lambda_5, \ldots, \lambda_{c-1})} \\
&\equiv [z_2, z_1\,;\, [z_c, z_3]^{z_4 z_5 \cdots z_k z_{k+1} \cdots z_{c-1}}]^{\varepsilon_1 f_1}
\end{aligned}
$$

(by the induction hypothesis and case 1).

Since each transferring commutator.

$$
T(t, \lambda) = [z_2, z_1\,;\, z_t, z_{k+1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1}],
$$

is of weight at most $c-2$; and by Theorem 4.11 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. So,

$$
\begin{aligned}
C &\equiv [z_2, z_1\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1} z_{k+1}}]^{\varepsilon_2 f_2} \\
&\equiv [z_2, z_1\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}]^{\varepsilon_3 f_3}[z_{k+1}^{-1}, z_1, z_2\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}]^{\varepsilon_4 f_4} \\
&\quad \cdot [z_{k+1}^{-1}, z_2, z_1\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}]^{\varepsilon_5 f_5}.
\end{aligned}
$$

But by the induction hypothesis and case 1, $[z_2, z_1; [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}] \equiv 1$. In addition set :

$$
\begin{aligned}
\alpha &= [z_{k+1}^{-1}, z_1, z_2\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}], \\
\beta &= [z_{k+1}^{-1}, z_2, z_1\,;\, [z_c, z_3]^{z_4 z_5 \cdots \hat{z}_{k+1} \cdots z_{c-1}}].
\end{aligned}
$$

The proof would be complete if we show that $\alpha \equiv \beta \equiv 1$:

By the Mini-Max Lemma,

$$\alpha \equiv \left[z_{k+1}^{-1}, z_1 ; [z_c, z_3]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1} z_2^{-1}} \right]^{\delta_1 g_1}.$$

Since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_1 ;\; z_t, z_2^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,],$$

is of weight at most $c-2$, if $z_t \leq z_{k+1}$, then by Lemma 4.13 combined with the induction hypothesis and case 1, we have, $T(t, \lambda) \equiv 1$. If $z_{k+1} < z_t$, then by part (b) of Lemma 4.15 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. So,

$$\alpha \equiv \left[z_{k+1}^{-1}, z_1 ; [z_c, z_3]^{z_2^{-1} z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\delta_2 g_2}$$

(4.13)

$$\equiv \left[z_{k+1}^{-1}, z_1 ; [z_c, z_2^{-1}, z_3]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\delta_3 g_3}$$

$$\cdot \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\delta_4 g_4} \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_c z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\delta_5 g_5}.$$

Now in 4.13 the first commutator is congruent to 1 by the Mini-Max Lemma. The second commutator is congruent to 1 by Lemma 4.13 combined with the induction hypothesis and case 1. About the third commutator, since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_1 ;\; z_c, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,],$$

is of weight at most $c-2$; if $z_t \leq z_{k+1}$, then by the Mini-Max Lemma, and if $z_{k+1} < z_t$, by Lemma 4.14 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\alpha \equiv \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\delta_6 g_6}$$

$$\equiv 1 \quad \text{(by Lemma 4.13)}.$$

In order to prove that $\beta \equiv 1$, by the Mini-Max Lemma we have,

$$\beta \equiv \left[z_{k+1}^{-1}, z_2 ; \left[z_c, z_3 \right]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\sigma_1 h_1}.$$

Since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_2 ;\ z_t, z_1^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,],$$

is of weight at most $c-2$. if $z_t \leq z_{k+1}$, then by part (c) of Lemma 4.15. and if $z_{k+1} < z_t$, then by part (a) of Lemma 4.15 combined with the induction hypothesis and case 1. we have $T(t, \lambda) \equiv 1$. So by the Mini-Max Lemma.

$$
\begin{aligned}
\beta &\equiv \left[z_{k+1}^{-1}, z_2 ; \left[z_c, z_3 \right]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1} z_1^{-1}} \right]^{\sigma_2 h_2} \\
&\equiv \left[z_{k+1}^{-1}, z_2 ; \left[z_3, z_1^{-1} \right]^{z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\sigma_3 h_3} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_2 ; \left[z_3, z_1^{-1} \right]^{z_c z_4 z_5 \ldots \hat{z}_{k+1} \ldots z_{c-1}} \right]^{\sigma_4 h_4}.
\end{aligned} \tag{4.14}
$$

In 4.14 using part (c) of Lemma 4.15. the first commutator is congruent to a product of conjugates of commutators of the form $[\, u_2, u_1 : u_q, u_3, u_4, \ldots, u_{q-1} \,]$ where $q \leq c-1$ so by the induction hypothesis and case 1 the commutator is congruent to 1. About the second commutator. since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_2 :\ z_c, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,],$$

is of weight at most $c-2$, if $z_t \leq z_{k+1}$, then by the Mini-Max Lemma, and if $z_{k+1} < z_t$, by Lemma 4.14 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\beta \equiv \left[z_{k+1}^{-1}, z_2 ; \left[z_3, z_1^{-1} \right]^{z_4 z_5 \ldots z_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_5 h_5}.$$

Now there are two possibilities either $k = 3$ or $4 \leq k \leq c-2$.

<u>**If k = 3**</u>: then since $z_3 = z_4$, we have

$$\beta = \left[z_4^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_5 z_6 \ldots z_{c-1} z_c} \right]^{\sigma_5 h_5}$$

$$\equiv \left[z_4, z_2 ; [z_3, z_1^{-1}]^{z_5 z_6 \ldots z_{c-1} z_c z_4} \right]^{\sigma_6 h_6}.$$

Since each transferring commutator.

$$T(t, \lambda) = [\, z_4, z_2 ; z_t, z_4, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,]$$

is of weight at most $c-1$, so by Lemma 4.12 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\beta \equiv \left[z_4, z_2 ; [z_3, z_1]^{z_1^{-1} z_4 z_5 z_6 \ldots z_{c-1} z_c} \right]^{\sigma_7 h_7}.$$

Now one more time for each choice of t, s and λ' s, not only is each transferring commutator,

$$T(t, \lambda) = [\, z_4, z_2 ; z_t, z_1^{-1}, {}_{\lambda_4} z_4, {}_{\lambda_5} z_5, \ldots, {}_{\lambda_{t-1}} z_{t-1} \,],$$

of weight at most c and so by the Mini-Max Lemma congruent to 1, but also,

$$T(t, s, \lambda) = [\, z_4, z_2 ; z_t, z_s, {}_{\lambda_{s+1}} z_{s+1}, {}_{\lambda_{s+2}} z_{s+2}, \ldots, {}_{\lambda_{t-1}} z_{t-1} \,]$$

is of weight at most $c-1$, and so by Lemma 4.12 combined with the induction hypothesis and case 1, we have $T(t, s, \lambda) \equiv 1$. Therefore,

$$\beta \equiv \left[z_4, z_2 ; [z_3, z_1]^{z_c z_{c-1} \ldots z_6 z_5 z_4 z_1^{-1}} \right]^{\sigma_8 h_8}$$

$$\equiv \left[z_3, z_1 ; [z_4, z_2]^{z_1 z_4^{-1} z_5^{-1} z_6^{-1} \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\sigma_9 h_9}$$

$$\equiv 1 \quad \text{(by Lemma 4.2)}.$$

<u>**If $4 \leq k \leq c-2$**</u>: then since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_2 ; z_k, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most $c-2$, so by Lemma 4.13 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$
\begin{aligned}
\beta &\equiv \left[z_{k+1}^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_k z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_6 h_6} \\
&\equiv \left[z_{k+1}^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_7 h_7} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_2 ; [z_k, z_1^{-1}, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_8 h_8} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_2 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_9 h_9} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_2 ; [z_k, z_3]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_{10} h_{10}}.
\end{aligned} \tag{4.15}
$$

In 4.15 we call the four commutators β_1, β_2, β_3, β_4, respectively, and we show that each one these commutators is congruent to 1, separately.

$\beta_1 = \left[z_{k+1}^{-1}, z_2 ; [z_3, z_1^{-1}]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$ is of weight $c-1$, and so by Lemma 4.15 combined with the induction hypothesis and case 1, we have $\beta_1 \equiv 1$.

Similarly, $\beta_3 = \left[z_{k+1}^{-1}, z_2 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$ is of weight $c-1$, and so by Lemma 4.13 combined with the induction hypothesis and case 1, we have $\beta_3 \equiv 1$. On the other hand,

$$
\begin{aligned}
\beta_2 &= \left[z_{k+1}^{-1}, z_2 ; [z_k, z_1^{-1}, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right] \\
&\equiv \left[z_{k+1}^{-1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\varepsilon_1 f_1} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\varepsilon_2 f_2}.
\end{aligned} \tag{4.16}
$$

In 4.16 for the second commutator we have

$$
\begin{aligned}
&\left[z_{k+1}^{-1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right] \\
&\qquad \equiv \left[z_{k+1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c z_{k+1}} \right]^{\delta_1 g_1}.
\end{aligned}
$$

Since each transferring commutator,

$$
T(t, \lambda) = [\, z_{k+1}, z_2 ; z_t, z_{k+1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,],
$$

is of weight at most $c-2$, so by Lemma 4.13 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\left[z_{k+1}^{-1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$$
$$\equiv \left[z_{k+1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k z_{k+1} \ldots z_{c-1} z_c} \right]^{\delta_2 g_2}.$$

Now one more time for each choice of t, s and λ's, each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}, z_2 ; z_t, z_1^{-1}, {}_{\lambda_3} z_3, {}_{\lambda_4} z_4, \ldots, {}_{\lambda_{t-1}} z_{t-1} \,],$$

is of weight at most c. So if $z_{k+1} \leq z_t$ by Mini-Max Lemma, and if $z_t < z_{k+1}$ by Corollary 4.10 combined with the induction hypothesis and case 1 we have, $T(t, \lambda) \equiv 1$. In addition,

$$T(t, s, \lambda) = [\, z_{k+1}, z_2 ; z_t, z_s, {}_{\lambda_{s+1}} z_{s+1}, {}_{\lambda_{s+2}} z_{s+2}, \ldots, {}_{\lambda_{t-1}} z_{t-1} \,],$$

is of weight at most $c-1$. So if $z_t \leq z_{k+1}$, then by the Mini-Max Lemma, $T(t, s, \lambda) \equiv 1$. If $z_{k+1} < z_t$, then either $z_s \leq z_{k+1}$ in which case by the Mini-Max Lemma, or if $z_{k+1} < z_s$, in which case by the induction hypothesis and case 1, we have $T(t, s, \lambda) \equiv 1$. Therefore,

$$\left[z_{k+1}^{-1}, z_2 ; [z_k, z_1]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$$
$$\equiv \left[z_{k+1}, z_2 ; [z_k, z_1]^{z_c z_{c-1} \ldots z_{k+1} \hat{z}_k \ldots z_5 z_4 z_3 z_1^{-1}} \right]^{\delta_3 g_3}$$
$$\equiv \left[z_k, z_1 ; [z_{k+1}, z_2]^{z_1 z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_4 g_4}$$
$$\equiv \left[z_k, z_1 ; [z_{k+1}, z_2]^{z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_5 g_5}$$
$$\cdot \left[z_k, z_1 ; [z_{k+1}, z_1, z_2]^{z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_6 g_6}$$
$$\cdot \left[z_k, z_1 ; [z_2, z_1]^{z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_7 g_7}$$
$$\cdot \left[z_k, z_1 ; [z_2, z_1]^{z_{k+1} z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_8 g_8}$$
$$\equiv \left[z_k, z_1 ; [z_2, z_1]^{z_{k+1} z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k \ldots z_{c-1}^{-1} z_c^{-1}} \right]^{\delta_8 g_8} \quad \text{(by Lemma 4.2).}$$

For each $3 \leq t \leq k-1$ each transferring commutator,

$$T(t, \lambda) = [\, z_k, z_1 \,;\, z_{k+1}, z_t^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{k-1}} z_{k-1}, {}_{\lambda_{k+1}} z_{k+1}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most c. Since $z_1 < z_t < z_k = z_{k+1}$, by Lemma 4.2 we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned}
&[\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_1 \,]^{z_1^{-1} z_3 z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,] \\
&\qquad \equiv [\, z_k, z_1 \,;\, [\, z_2, z_1 \,]^{z_3^{-1} z_4^{-1} z_5^{-1} \ldots \hat{z}_k z_{k+1} z_{k+1}^{-1} z_{k+2}^{-1} \ldots z_{c-1}^{-1} z_c^{-1}} \,]^{\delta_9 g_9} \\
&\qquad \equiv 1 \quad \text{(by Lemma 4.2)}.
\end{aligned}$$

Similarly, $[\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_1 \,]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,] \equiv 1$. Therefore, by substitution in 4.16, we have $\beta_2 \equiv 1$.

In order to prove that $\beta_4 = [\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_3 \,]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,] \equiv 1$, since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_2 \,;\, z_t, z_1^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most $c-2$, if $z_t \leq z_{k+1}$, by part(c) of Lemma 4.15, and if $z_{k+1} < z_t$, by part(a) of Lemma 4.15 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned}
\beta_4 &= [\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_3 \,]^{z_1^{-1} z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,] \\
&\equiv [\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_3 \,]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c z_1^{-1}} \,]^{\varepsilon_1 f_1} \\
&\equiv [\, z_{k+1}^{-1}, z_2 \,;\, [\, z_k, z_3 \,]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,]^{\varepsilon_2 f_2} \\
&\quad \cdot [\, z_2, z_1, z_{k+1}^{-1} \,;\, [\, z_k, z_3 \,]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,]^{\varepsilon_3 f_3} \\
&\quad \cdot [\, z_{k+1}^{-1}, z_1, z_2 \,;\, [\, z_k, z_3 \,]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \,]^{\varepsilon_4 f_4}.
\end{aligned} \tag{4.17}$$

In 4.17, calling the three commutators β_{41}, β_{42}, β_{43}, respectively, we show that each one these commutators is congruent to 1, separately.

$\beta_{41} = \left[z_{k+1}^{-1}, z_2 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$ is of weight $c-1$, and by Lemma 4.13 combined with the induction hypothesis and case 1, we have $\beta_{41} \equiv 1$.

$$\begin{aligned} \beta_{42} &= \left[z_2, z_1, z_{k+1}^{-1} ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right] \\ &\equiv \left[z_2, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\delta_1 g_1} \\ &\quad \cdot \left[z_2, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c z_{k+1}} \right]^{\delta_2 g_2}. \end{aligned}$$

Now $\left[z_2, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$, is of weight $c-1$ and so by Theorem 4.11 combined with the induction hypothesis and case 1, it is congruent to 1. Also, since each transferring commutator,

$$T(t, \lambda) = [\, z_2, z_1 ; z_t, z_{k+1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most $c-2$, so by Theorem 4.11 combined with the induction hypothesis and case 1, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\beta_{42} \equiv \left[z_2, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\delta_3 g_3}.$$

and this commutator is of weight c; so by Theorem 4.11 combined with the induction hypothesis and case 1, we have $\beta_{42} \equiv 1$.

$$\begin{aligned} \beta_{43} &= \left[z_{k+1}^{-1}, z_1, z_2 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right] \\ &\equiv \left[z_{k+1}^{-1}, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]^{\sigma_1 h_1} \\ &\quad \cdot \left[z_{k+1}^{-1}, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c z_2^{-1}} \right]^{\sigma_2 h_2}. \end{aligned}$$

But $\left[z_{k+1}^{-1}, z_1 ; [z_k, z_3]^{z_4 z_5 \ldots \hat{z}_k \hat{z}_{k+1} \ldots z_{c-1} z_c} \right]$ is of weight $c-1$, so by Lemma 4.13 combined with the induction hypothesis and case 1, it is congruent to 1. Also, since each transferring commutator,

$$T(t, \lambda) = [\, z_{k+1}^{-1}, z_1 ; z_t, z_2^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most $c-2$, if $z_t \leq z_{k+1}$ by part(c) of Lemma 4.15, and if $z_{k+1} < z_t$, by part(a) of Lemma 4.15 combined with the induction hypothesis and case 1, we have $T(t,\lambda) \equiv 1$. Thus,

$$\begin{aligned}
\beta_{43} &\equiv \left[z_{k+1}^{-1}, z_1 ; [z_k, z_3]^{z_2^{-1} z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_3 h_3} \\
&\equiv \left[z_{k+1}^{-1}, z_1 ; [z_k, z_3]^{z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_4 h_4} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_1 ; [z_k, z_2^{-1}, z_3]^{z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_5 h_5} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_6 h_6} \\
&\quad \cdot \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_k z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_7 h_7} \\
&\equiv \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_k z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_7 h_7}
\end{aligned}$$

(by Lemma4.13 and the induction hypothesis).

For each $4 \leq t \leq k-1$, each transferring commutator,

$$T(t,\lambda) = [\, z_{k+1}^{-1}, z_1 ; z_k, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{k-1}} z_{k-1}, {}_{\lambda_{k+2}} z_{k+2}, \ldots, {}_{\lambda_c} z_c \,],$$

is of weight at most $c-2$; so by Lemma 4.13 combined with the induction hypothesis and case 1, we have $T(t,\lambda) \equiv 1$. Thus,

$$\beta_{43} \equiv \left[z_{k+1}^{-1}, z_1 ; [z_3, z_2^{-1}]^{z_4 z_5 \cdots \hat{z}_k \hat{z}_{k+1} \cdots z_{c-1} z_c} \right]^{\sigma_8 h_8}.$$

This last commutator is of weight c; so by Lemma 4.13 combined with the induction hypothesis and case 1, we have $\beta_{43} \equiv 1$. Therefore, $\beta_{41} \equiv \beta_{42} \equiv \beta_{43} \equiv 1$, and by substitution in 4.17 we get $\beta_4 \equiv 1$. Since we have also shown that $\beta_1 \equiv \beta_2 \equiv \beta_3 \equiv 1$, by substitution in 4.15 we get $\beta \equiv 1$. Hence, the proof is complete. □

Theorem 4.17. *For* $5 \leq c \leq n+1$, *let* $\{z_1, z_2, \ldots, z_c\} \subseteq X$; *then modulo* $\langle \Delta(m,n,F) \rangle^F$,

(a) *if* $z_1 < z_2 < z_3 \leq z_4 \leq \cdots \leq z_c$, *then*

(1) $[z_2, z_1 ; z_i, z_3, z_4, \ldots, \hat{z}_i, \ldots, z_c] \equiv 1 \quad (4 \leq i \leq c)$,

(2) $[z_2, z_1 ; z_{i_1}^{\varepsilon_1}, z_{i_2}^{\varepsilon_2}, z_{i_3}^{\varepsilon_3}, \ldots, z_{i_{c-2}}^{\varepsilon_{c-2}}] \equiv 1$,
where $\{i_1, i_2, \ldots, i_{c-2}\} = \{3, 4, \ldots, c\}$ *and* $\varepsilon_j \in \{1, -1\} (1 \leq j \leq c-2)$;

(b) *if* $z_1 = z_2 \leq z_3 \leq z_4 \leq \cdots \leq z_c$ *then for* $3 \leq i, j \leq c$,

$$[z_j, z_1 ; z_i, z_2, z_3, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c] \equiv 1.$$

PROOF. (a) : (1) is direct consequence of Theorem 4.11 and Theorem 4.16. To prove (2), since $[z_{i_1}^{\varepsilon_1}, z_{i_2}^{\varepsilon_2}, z_{i_3}^{\varepsilon_3}, \ldots, z_{i_{c-2}}^{\varepsilon_{c-2}}] \in F'$, by Theorem 2.8, modulo $\langle \Delta(m, n, F) \rangle^F$, it can be written as a product of simple basic commutators of form $[y_i, y_3, y_4, \ldots, \hat{y}_i, \ldots, y_r]$ where $2 \leq r \leq n-1$, $y_3 \leq y_4 \leq \cdots \leq y_r$ and $\{y_3, y_4, \ldots, y_r\} \subseteq \{z_3, z_4, \ldots, z_c\}$. Therefore, by setting $z_1 = y_1$, $z_2 = y_2$, $[z_2, z_1 ; z_{i_1}^{\varepsilon_1}, z_{i_2}^{\varepsilon_2}, z_{i_3}^{\varepsilon_3}, \ldots, z_{i_{c-2}}^{\varepsilon_{c-2}}]$ is congruent to a product of conjugates of commutators of the form $[y_2, y_1 ; y_i, y_3, y_4, \ldots, \hat{y}_i, \ldots, y_r]$, which are congruent to 1 by (1).
To prove (b), if $z_j \geq z_i$, then by definition the commutator is congruent to 1; If $z_j < z_i$, then it is direct consequence of Lemma 4.9 and Theorem 4.11. □

4.3. Type $[z_j, z_1 : z_i, z_2, z_3, \ldots, \hat{z}_j, \hat{z}_i, \ldots, z_c]$

Our goal in this section is to prove that, modulo $\langle \Delta(m, n, F) \rangle^F$, commutators of the form $[z_j, z_1 ; z_i, z_2, z_3, \ldots, \hat{z}_j, \hat{z}_i, \ldots, z_c]$ are congruent to 1.

Lemma 4.18. *For* $5 \leq c \leq n+1$, *let* $y_1 = y_2 \leq y_3 \leq \cdots \leq y_c$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$,

(a) $C_1 = [y_j^{\varepsilon_j}, y_1 : y_i, y_2^{\varepsilon_2}, y_3^{\varepsilon_3}, \ldots, \hat{y}_j, \ldots, \hat{y}_i, \ldots, y_c^{\varepsilon_c}] \equiv 1$;

(b) $C_2 = [y_j^{\varepsilon_j}, y_1 : y_i^{\varepsilon_i}, y_2^{\varepsilon_2}, y_{i+1}^{\varepsilon_{i+1}}, y_{i+2}^{\varepsilon_{i+2}}, \ldots, y_c^{\varepsilon_c}] \equiv 1$,

where $3 \leq i, j \leq c$ *and* $\{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_c\} \subseteq \{1, -1\}$.

PROOF. (a) By Lemma 4.2, C_1 is congruent to a product of conjugates of commutators of the form $\alpha_r = [\, y_j^{\varepsilon_j} . y_1 ; y_i, y_2, y_{t_1}, y_{t_2}, \dots, y_{t_r} \,]$, where $1 \leq r \leq n-3$, $y_2 \leq y_{t_1} \leq y_{t_2} \leq \cdots \leq y_{t_r}$ and $\{t_1, t_2, \dots, t_r\} \subseteq \{2, 3, \dots, c\}$. So it is enough to prove that for each r, $\alpha_r \equiv 1$. **If** $\varepsilon_j = +1$: then by Theorem 4.17, we have $\alpha_r \equiv 1$.

If $\varepsilon_j = -1$: then we use induction on r. If $r = 1$, then by Theorem 4.17,

$$[\, y_j^1, y_1 ; y_i, y_2, y_{t_1} \,] = [y_j, y_1 : [\, y_i, y_2, y_{t_1} \,]^{y_j} \,]^{\delta y} \equiv 1.$$

Assume the statement is true for integers less than r. Then for r we have

$$\begin{aligned}
\alpha_r &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 1 \leq k \leq r}} [y_j^{-1} . y_1 ; [\, y_i, y_2 \,]^{y_{t_1}^{\lambda_1} y_{t_2}^{\lambda_2} \cdots y_{t_r}^{\lambda_r}} \,]^{\delta(\lambda_1, \lambda_2, \dots, \lambda_r) g(\lambda_1, \lambda_2, \dots, \lambda_r)} \\
&\equiv [\, y_j^{-1} . y_1 : [\, y_i, y_2 \,]^{y_{t_1} y_{t_2} \cdots y_{t_r}} \,]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)} \\
&\equiv [\, y_j, y_1 : [\, y_i, y_2 \,]^{y_{t_1} y_{t_2} \cdots y_{t_r} y_j} \,]^{\delta_2 g_2}.
\end{aligned}$$

Now if $y_{t_r} \leq y_j$, then again by Theorem 4.17, $\alpha_r \equiv 1$: otherwise let $1 \leq k \leq r$ be the smallest integer such that $y_j < y_{t_k}$. For each $k \leq s \leq r$, each transferring commutator,

$$T(t_s, \lambda) = [\, y_j, y_1 ; y_{t_s}, y_j, {}_{\lambda_{t_{s+1}}} y_{t_{s+1}}, {}_{\lambda_{t_{s+2}}} y_{t_{s+2}}, \dots, {}_{\lambda_{t_r}} y_{t_r} \,],$$

is congruent to 1 by Lemma 4.12 together with Theorem 4.16. Therefore,

$$\begin{aligned}
\alpha_r &\equiv [\, y_j, y_1 : [\, y_i, y_2 \,]^{y_{t_1} y_{t_2} \cdots y_j y_{t_k} \cdots y_{t_r}} \,]^{\delta_3 g_3} \\
&\equiv 1 \quad \text{(by Theorem 4.17).}
\end{aligned}$$

Proof of (b) : The proof is exactly the same as part (a). We just note that $\varepsilon_i = -1$ will cause no problem because $y_i \leq y_{i+1} \leq \cdots \leq y_c$. □

Lemma 4.19. *For* $5 \leq c \leq n+1$, *let* $y_1 < y_2 \leq y_3 \leq \cdots \leq y_c$*; then modulo* $\langle \Delta(m, n, F) \rangle^F$,

(a) $[\, y_c^{\varepsilon_c}, y_2^{\varepsilon_2} ; y_3^{\varepsilon_3}, y_1^{\varepsilon_1}, y_4^{\varepsilon_4}, y_5^{\varepsilon_5}, \dots, y_{c-1}^{\varepsilon_{c-1}} \,] \equiv 1;$

(b) $\quad [y_c^{\varepsilon_c}, y_2^{\varepsilon_2}; y_i, y_1^{\varepsilon_1}, y_3^{\varepsilon_3}, y_4^{\varepsilon_4}, \dots, \hat{y}_i, \dots, y_{c-1}^{\varepsilon_{c-1}}] \equiv 1,$

where $4 \le i \le c-1$ *and* $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_c\} \subseteq \{1, -1\}$.

PROOF. (a) We consider two different cases, either $\varepsilon_1 = +1$ or $\varepsilon_1 = -1$.
Case(1) : If $\varepsilon_1 = +1$, then since $[y_c^{\varepsilon_c}, y_2^{\varepsilon_2}] = [y_c, y_2]^{\varepsilon_2\varepsilon_c y_2^{(1-\varepsilon_2)/2} y_c^{(1-\varepsilon_c)/2}}$ so,

$$\begin{aligned} C =& [y_c^{\varepsilon_c}, y_2^{\varepsilon_2}; y_3, y_1, y_4^{\varepsilon_4}, y_5^{\varepsilon_5}, \dots, y_{c-1}^{\varepsilon_{c-1}},] \\ =& [y_c, y_2; [y_3, y_1, y_4^{\varepsilon_4}, y_5^{\varepsilon_5}, \dots, y_{c-1}^{\varepsilon_{c-1}},]^{y_c^{(\varepsilon_c-1)/2} y_2^{(\varepsilon_2-1)/2}}]^{\delta_1 g_1}. \end{aligned}$$

But by Theorem 2.8, $[y_3, y_1, y_4^{\varepsilon_4}, y_5^{\varepsilon_5}, \dots, y_{c-1}^{\varepsilon_{c-1}},]^{y_c^{(\varepsilon_c-1)/2} y_2^{(\varepsilon_2-1)/2}}$ is congruent to a product of simple basic commutators of the form $[y_j, y_1, y_2, \dots, \hat{y}_j, \dots, y_r]$, where $2 \le r \le n-1$ and $\{y_1, y_2, \dots, y_r\} \subseteq \{y_1, y_2, \dots, y_c\}$. Therefore, C is is congruent to a product of conjugates of commutators of the form $\alpha_r = [y_c, y_2; y_j, y_1, y_2, \dots, \hat{y}_j, \dots, y_r]$. Now for each r, if $y_1 = y_1$, then for $y_j = y_c$ by the Mini-Max Lemma, and for $y_j < y_c$ by definition, we have $\alpha_r \equiv 1$. If $y_1 > y_1$, then since $y_2 \le y_1$, again by the Mini-Max Lemma, $\alpha_r \equiv 1$. Hence, $C \equiv 1$.
Case(2) : If $\varepsilon_1 = -1$, we prove it by induction on c. If $c = 5$, then

$$\begin{aligned} & [y_5^{\varepsilon_5}, y_2^{\varepsilon_2}; y_3^{\varepsilon_3}, y_1^{-1}, y_4^{\varepsilon_4}] \\ & \qquad \equiv [y_2^{\varepsilon_2}, y_1; [y_3^{\varepsilon_3}, y_1]^{y_4^{\varepsilon_4}}]^{\delta_1 g_1} [y_2^{\varepsilon_2}, y_1; [y_3^{\varepsilon_3}, y_1]^{y_4^{\varepsilon_4} y_5^{-\varepsilon_5}}]^{\delta_2 g_2} \\ & \qquad \equiv 1 \quad \text{(by Lemma 4.18).} \end{aligned}$$

Assume the statement is true for integers less than c. Then for c we have

$$\begin{aligned} C &= [y_c^{\varepsilon_c}, y_2^{\varepsilon_2}; y_3, y_1^{-1}, y_4^{\varepsilon_4}, y_5^{\varepsilon_5}, \dots, y_{c-1}^{\varepsilon_{c-1}}] \\ &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \le k \le c-1}} [y_c^{\varepsilon_c}, y_2^{\varepsilon_2}: [y_3^{\varepsilon_3}, y_1^{-1}]^{y_4^{\varepsilon_4\lambda_4} y_5^{\varepsilon_5\lambda_5} \cdots y_{c-1}^{\varepsilon_{c-1}\lambda_{c-1}}}]^{\delta(\lambda_4, \lambda_5, \dots, \lambda_{c-1}) g(\lambda_4, \lambda_5, \dots, \lambda_{c-1})} \\ &\equiv [y_c^{\varepsilon_c}, y_2^{\varepsilon_2}; [y_3^{\varepsilon_3}, y_1]^{y_1^{-1} y_4^{\varepsilon_4} y_5^{\varepsilon_5} \cdots y_{c-1}^{\varepsilon_{c-1}}}]^{\delta_1 g_1} \quad \text{(by the induction hypothesis).} \end{aligned}$$

But each transferring commutator.

$$T(t,\lambda) = [\, y_c^{\varepsilon_c}, y_2^{\varepsilon_2} ; y_t^{\varepsilon_t}, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}^{\varepsilon_{t+1}}, {}_{\lambda_{t+2}} y_{t+2}^{\varepsilon_{t+2}}, \ldots, {}_{\lambda_{c-1}} y_{c-1}^{\varepsilon_{c-1}} \,],$$

is of weight at most $c-1$; therefore. by the induction hypothesis. $T(t,\lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[y_c^{\varepsilon_c}, y_2^{\varepsilon_2} ; [\, y_3^{\varepsilon_3}, y_1 \,]^{y_4^{\varepsilon_4} y_5^{\varepsilon_5} \cdots y_{c-1}^{\varepsilon_{c-1}}} \right]^{\delta_2 g_2} \\ &\quad \cdot \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3^{\varepsilon_3}, y_1 \,]^{y_4^{\varepsilon_4} y_5^{\varepsilon_5} \cdots y_{c-1}^{\varepsilon_{c-1}}} \right]^{\delta_3 g_3} \\ &\quad \cdot \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3^{\varepsilon_3}, y_1 \,]^{y_4^{\varepsilon_4} y_5^{\varepsilon_5} \cdots y_{c-1}^{\varepsilon_{c-1}} y_c^{-\varepsilon_c}} \right]^{\delta_4 g_4} \\ &\equiv 1 \quad \text{(by Lemma 4.18)}. \end{aligned}$$

Proof of (b) : The proof is similar to the proof of part (a); we just note that the transferring commutators are congruent to 1 by part (a). □

Lemma 4.20. *For* $5 \le c \le n+1$. *let* $y_1 < y_2 < y_3 \le y_4 \le \cdots \le y_c$; *then modulo* $\langle \Delta(m,n,F) \rangle^F$.

$$C = [\, y_2^{\varepsilon_2}, y_1^{\varepsilon_1} ; y_c^{\varepsilon_c}, y_3, y_4, \ldots, y_{c-1} \,] \equiv 1.$$

Proof. We consider two different cases. either $\varepsilon_1 = +1$ or $\varepsilon_1 = -1$.

Case(1) : If $\varepsilon_1 = +1$, then for $\varepsilon_2 = +1$, by Theorem 4.17. $C \equiv 1$. For $\varepsilon_2 = -1$, we prove it by induction on c. If $c = 5$. then

$$\begin{aligned} &[\, y_2^{-1}, y_1 ; y_5^{\varepsilon}, y_3, y_4 \,] \\ &\qquad \equiv \left[y_2, y_1 ; [\, y_5^{\varepsilon}, y_3 \,]^{y_4 y_2} \right]^{\delta_1 g_1} \\ &\qquad \equiv \left[y_2, y_1 ; [\, y_5^{\varepsilon}, y_2, y_3 \,]^{y_4} \right]^{\delta_2 g_2} \left[y_2, y_1 ; [\, y_3, y_2 \,]^{y_4} \right]^{\delta_3 g_3} \\ &\qquad\quad \cdot \left[y_2, y_1 ; [\, y_3, y_2 \,]^{y_4 y_5^{\varepsilon}} \right]^{\delta_4 g_4} \\ &\qquad \equiv 1 \text{ (by the Mini-Max Lemma, Lemma 4.12 and Theorem 4.16).} \end{aligned}$$

Assume the statement is true for integers less than c. Then for c, we have

$$C = \prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \le k \le c-1}} \left[y_2^{-1}, y_1; [y_c^{\varepsilon_c}, y_3]^{y_4^{\lambda_4} y_5^{\lambda_5} \cdots y_{c-1}^{\lambda_{c-1}}}\right]^{\delta(\lambda_4, \lambda_5, \ldots, \lambda_{c-1}) g(\lambda_4, \lambda_5, \ldots, \lambda_{c-1})}$$

$$\equiv \left[y_2^{-1}, y_1; [y_c^{\varepsilon_c}, y_3]^{y_4 y_5 \cdots y_{c-1}}\right]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)}$$

$$\equiv \left[y_2, y_1; [y_c^{\varepsilon_c}, y_3]^{y_4 y_5 \cdots y_{c-1} y_2}\right]^{\delta_2 g_2}.$$

But each transferring commutator.

$$T(t, \lambda) = [y_2, y_1; y_t, y_2, {}_{\lambda_{t+1}} y_{t+1}^{\varepsilon_{t+1}}, {}_{\lambda_{t+2}} y_{t+2}^{\varepsilon_{t+2}}, \ldots, {}_{\lambda_{c-1}} y_{c-1}^{\varepsilon_{c-1}}],$$

is of weight at most $c-1$; therefore, by Lemma 4.12 together with Theorem 4.16, $T(t, \lambda) \equiv 1$. Thus,

$$C \equiv \left[y_2, y_1; [y_c^{\varepsilon_c}, y_3]^{y_4 y_5 \cdots y_{c-1}}\right]^{\delta_3 g_3} \left[y_2, y_1; [y_c^{\varepsilon_c}, y_2, y_3]^{y_4 y_5 \cdots y_{c-1}}\right]^{\delta_4 g_4}$$

$$\cdot \left[y_2, y_1; [y_3, y_2]^{y_4 y_5 \cdots y_{c-1}}\right]^{\delta_5 g_5} \left[y_2, y_1; [y_3, y_2]^{y_c^{\varepsilon_c} y_4 y_5 \cdots y_{c-1}}\right]^{\delta_6 g_6}$$

$$\equiv \left[y_2, y_1; [y_3, y_2]^{y_c^{\varepsilon_c} y_4 y_5 \cdots y_{c-1}}\right]^{\delta_6 g_6} \text{ (by Lemma 4.12 and Theorem 4.17).}$$

But each transferring commutator.

$$T(t, \lambda) = [y_2, y_1; y_c^{\varepsilon_c}, y_t, {}_{\lambda_{t+1}} y_{t+1}^{\varepsilon_{t+1}}, {}_{\lambda_{t+2}} y_{t+2}^{\varepsilon_{t+2}}, \ldots, {}_{\lambda_{c-1}} y_{c-1}^{\varepsilon_{c-1}}],$$

is of weight at most $c-1$; therefore by Theorem4.17, $T(t, \lambda) \equiv 1$. Thus,

$$C \equiv \left[y_2, y_1; [y_3, y_2]^{y_4 y_5 \cdots y_{c-1} y_c^{\varepsilon_c}}\right]^{\delta_7 g_7}$$

$$\equiv 1 \quad \text{(by Lemma 4.12 and Theorem 4.17)}$$

Case(2) : If $\varepsilon_1 = -1$, then we use induction on c. If $c = 5$, then

$$[y_2^{\varepsilon_2}, y_1^{-1}; y_5^{\varepsilon_5}, y_3, y_4] \equiv \left[y_2^{\varepsilon_2}, y_1; [y_5^{\varepsilon_5}, y_3]^{y_4}\right]^{\delta_1 g_1} \left[y_2^{\varepsilon_2}, y_1; [y_5^{\varepsilon_5}, y_1, y_3]^{y_4}\right]^{\delta_2 g_2}$$

$$\cdot \left[y_2^{\varepsilon_2}, y_1; [y_3, y_1]^{y_4}\right]^{\delta_3 g_3} \left[y_2^{\varepsilon_2}, y_1; [y_3, y_1]^{y_4 y_5^{\varepsilon_5}}\right]^{\delta_4 g_4}$$

$$\equiv 1 \text{ (by case(1), Lemma 4.18 and the Mini-Max Lemma).}$$

Assume the statement is true for integers less than c. Then for c, we have

$$
\begin{aligned}
C &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \le k \le c-1}} \left[y_2^{\varepsilon_2}, y_1^{-1} ; [\, y_c^{\varepsilon_c}, y_3 \,]^{y_4^{\lambda_4} y_5^{\lambda_5} \cdots y_{c-1}^{\lambda_{c-1}}} \right]^{\delta(\lambda_4,\lambda_5,\ldots,\lambda_{c-1}) g(\lambda_4,\lambda_5,\ldots,\lambda_{c-1})} \\
&\equiv \left[y_2^{\varepsilon_2}, y_1^{-1} ; [\, y_c^{\varepsilon_c}, y_3 \,]^{y_4 y_5 \cdots y_{c-1}} \right]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)} \\
&\equiv \left[y_2^{\varepsilon_2}, y_1 ; [\, y_c^{\varepsilon_c}, y_3 \,]^{y_4 y_5 \cdots y_{c-1} y_1} \right]^{\delta_2 g_2}.
\end{aligned}
$$

But each transferring commutator.

$$T(t, \lambda) = [\, y_2^{\varepsilon_2}, y_1 ; y_t, y_1, {}_{\lambda_{t+1}} y_{t+1}^{\varepsilon_{t+1}}, {}_{\lambda_{t+2}} y_{t+2}^{\varepsilon_{t+2}}, \ldots, {}_{\lambda_{c-1}} y_{c-1}^{\varepsilon_{c-1}} \,].$$

is of weight at most $c - 1$; therefore by Lemma 4.18. $T(t, \lambda) \equiv 1$. Thus.

$$
\begin{aligned}
C &\equiv \left[y_2^{\varepsilon_2}, y_1 ; [\, y_c^{\varepsilon_c}, y_3 \,]^{y_1 y_4 y_5 \cdots y_{c-1}} \right] \\
&\equiv \left[y_2^{\varepsilon_2}, y_1 ; [\, y_c^{\varepsilon_c}, y_3 \,]^{y_4 y_5 \cdots y_{c-1}} \right] \\
&\quad \cdot \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3, y_1 \,]^{y_4 y_5 \cdots y_{c-1}} \right] \\
&\quad \cdot \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3, y_1 \,]^{y_c^{\varepsilon_c} y_4 y_5 \cdots y_{c-1}} \right] \\
&\equiv \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3, y_1 \,]^{y_c^{\varepsilon_c} y_4 y_5 \cdots y_{c-1}} \right]^{s} \quad \text{(by Case (1) and Lemma 4.18).}
\end{aligned}
$$

Again each transferring commutator.

$$T(t, \lambda) = [\, y_2^{\varepsilon_2}, y_1 ; y_c^{\varepsilon_c}, y_t, {}_{\lambda_{t+1}} y_{t+1}^{\varepsilon_{t+1}}, {}_{\lambda_{t+2}} y_{t+2}^{\varepsilon_{t+2}}, \ldots, {}_{\lambda_{c-1}} y_{c-1}^{\varepsilon_{c-1}} \,],$$

is of weight at most $c - 1$; therefore by Case (1). $T(t, \lambda) \equiv 1$. Thus,

$$C \equiv \left[y_2^{\varepsilon_2}, y_1 ; [\, y_3, y_1 \,]^{y_c^{\varepsilon_c} y_4 y_5 \cdots y_{c-1}} \right] \equiv 1 \quad \text{(by Lemma 4.18),}$$

which completes the proof. □

Lemma 4.21. *For* $5 \le p \le n + 1$, *let* $y_1 < y_2 \le y_3 \le \cdots \le y_p$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$,

(a) $[\, y_j^{\varepsilon_j}, y_1^{-1} ; y_p^{\varepsilon_p}, y_2, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1} \,] \equiv 1 \quad (3 \le j \le p - 1)$;

(b) $[\, y_j, y_1^{-1} ; y_3, y_2, y_4, y_5, \ldots, \hat{y}_j, \ldots, y_p \,] \equiv 1 \quad (4 \le j \le p)$;

(c) $[\, y_3, y_2^{-1} ; y_4, y_1, y_5, y_6, \ldots, y_p \,] \equiv 1$;

(d) $\quad [y_3, y_1^{-1}; y_4, y_2, y_5, y_6, \ldots y_p] \equiv 1$.

PROOF. (a) We prove it by induction on p. If $p = 5$, then either $j = 3$ or $j = 4$. So,

$$\begin{aligned}
[y_3^{\varepsilon_3}, y_1^{-1}; y_5^{\varepsilon_5}, y_2, y_4] &\equiv [y_3^{\varepsilon_3}, y_1; [y_5^{\varepsilon_5}, y_2]^{y_1}]^{\delta_1 g_1} [y_3^{\varepsilon_3}, y_1; [y_5^{\varepsilon_5}, y_2]^{y_1 y_4}]^{\delta_2 g_2} \\
&\equiv [y_3^{\varepsilon_3}, y_1; y_2, y_1, y_5^{\varepsilon_5}]^{\delta_3 g_3} [y_3^{\varepsilon_3}, y_1; [y_2, y_1]^{y_4}]^{\delta_4 g_4} \\
&\quad \cdot [y_3^{\varepsilon_3}, y_1; [y_2, y_1]^{y_4 y_5^{\varepsilon_5}}]^{\delta_5 g_5} \\
&\equiv 1 \quad \text{(by Lemma 4.18)}.
\end{aligned}$$

Similarly, $[y_4^{\varepsilon_4}, y_1^{-1}; y_5^{\varepsilon_5}, y_2, y_3] \equiv 1$. Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned}
&[y_j^{\varepsilon_j}, y_1^{-1}; y_p^{\varepsilon_p}, y_2, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1}] \\
&= \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \le k \le p-1,\, k \ne j}} [y_j^{\varepsilon_j}, y_1^{-1}; [y_p^{\varepsilon_p}, y_2]^{y_3^{\lambda_3} y_4^{\lambda_4} \cdots \hat{y}_j \cdots y_{p-1}^{\lambda_{p-1}}}]^{\delta(\lambda_3, \lambda_4, \ldots, \lambda_{p-1}) g(\lambda_3, \lambda_4, \ldots, \lambda_{p-1})} \\
&\equiv [y_j^{\varepsilon_j}, y_1; [y_p^{\varepsilon_p}, y_2]^{y_3 y_4 \cdots y_{p-1} y_1}]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)}.
\end{aligned}$$

Each transferring commutator is congruent to 1 by Lemma 4.18, so

$$\begin{aligned}
&\equiv [y_j^{\varepsilon_j}, y_1; [y_p^{\varepsilon_p}, y_2]^{y_1 y_3 y_4 \cdots \hat{y}_j \cdots y_{p-1}}]^{\delta_2 g_2} \\
&\equiv 1 \text{ (by the Mini-Max Lemma, Lemma 4.18 and Lemma 4.20)}.
\end{aligned}$$

Proof of (b): By induction, if $p = 5$, then

$$\begin{aligned}
[y_4^{\varepsilon_4}, y_1^{-1}; y_3, y_2, y_5] &\equiv [y_4^{\varepsilon_4}, y_1; [y_3, y_2]^{y_1}] [y_4^{\varepsilon_4}, y_1; [y_3, y_2]^{y_1 y_5}] \\
&\equiv 1 \quad \text{(by Lemma 4.18)}.
\end{aligned}$$

Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned}
&[\,y_j^{\varepsilon_j}, y_1^{-1};\ y_3, y_2, y_4, y_5, \ldots, \hat{y}_j, \ldots, y_p\,] \\
&= \prod_{\substack{\lambda_k\in\{0,1\}\\ 4\le k\le p,\,k\ne j}} [y_j^{\varepsilon_j}, y_1^{-1} : [\,y_3, y_2\,]^{y_4^{\lambda_4} y_5^{\lambda_5}\cdots\hat{y}_j\cdots y_p^{\lambda_p}}\,]^{\delta(\lambda_4,\lambda_5,\ldots,\lambda_p)g(\lambda_4,\lambda_5,\ldots,\lambda_p)} \\
&\equiv [\,y_j^{\varepsilon_j}, y_1 : [\,y_3, y_2\,]^{y_4 y_5\cdots y_p y_1}\,]^{\delta_1 g_1} \quad \text{(by the induction hypothesis).}
\end{aligned}$$

Each transferring commutator is congruent to 1 by Lemma 4.18; so,

$$\begin{aligned}
&\equiv [\,y_j^{\varepsilon_j}, y_1 ; [\,y_3, y_2\,]^{y_1 y_4 y_5\cdots\hat{y}_j\cdots y_p}\,]^{\delta_2 g_2} \\
&\equiv 1 \text{ (by Lemma 4.18).}
\end{aligned}$$

Proof of (c): If $y_2 = y_3$, it is trivial. If $y_2 < y_3$, then use induction. If $p = 5$, we have

$$[\,y_3, y_2^{-1} : y_4, y_1, y_5\,] \equiv [\,y_4, y_1 : y_3, y_2^{-1}, y_5^{-1}\,]^{\delta g} \equiv 1 \quad \text{(by Lemma 4.2).}$$

Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned}
&[\,y_3, y_2^{-1} : y_4, y_1, y_5, y_6, \ldots, y_p\,] \\
&= \prod_{\substack{\lambda_k\in\{0,1\}\\ 5\le k\le p}} [y_3, y_2^{-1} : [\,y_4, y_1\,]^{y_5^{\lambda_5} y_6^{\lambda_6}\cdots y_p^{\lambda_p}}\,]^{\delta(\lambda_5,\lambda_6,\ldots,\lambda_p)g(\lambda_5,\lambda_6,\ldots,\lambda_p)} \\
&\equiv [\,y_3, y_2^{-1} ; [\,y_4, y_1\,]^{y_5 y_6\cdots y_p}\,]^{\delta_1 g_1} \quad \text{(by the induction hypothesis).}
\end{aligned}$$

For each choice of t, s and λ' s, each transferring commutator,

$$T(t, s, \lambda) = [\,y_3^{-1}, y_2 : y_t, y_s, {}_{\lambda_{s+1}}y_{s+1}, {}_{\lambda_{s+2}}y_{s+2}, \ldots, {}_{\lambda_{t-1}}y_{t-1}],$$

is of weight at most $p-2$. If $y_3 = y_s$, by part (a), and if $y_3 < y_s$, by Lemma 4.20, we have $T(t,s,\lambda) \equiv 1$. Therefore.

$$
\begin{aligned}
&[\, y_3, y_2^{-1} ; y_4, y_1, y_5, y_6, \ldots, y_p \,] \\
&\qquad \equiv \left[\, y_3, y_2^{-1} : [\, y_4, y_1 \,]^{y_p y_{p-1} \cdots y_7 y_6 y_5} \,\right]^{\delta_2 g_2} \\
&\qquad \equiv \left[\, y_4, y_1 ; [\, y_3, y_2^{-1} \,]^{y_5^{-1} y_6^{-1} \cdots y_p^{-1}} \,\right]^{\delta_3 g_3} \\
&\qquad \equiv 1 \quad \text{(by Lemma 4.2).}
\end{aligned}
$$

Proof of (d) : If $y_2 = y_3$, use the Mini-Max Lemma. If $y_2 < y_3$, then use induction. If $p = 5$, we have

$$
\begin{aligned}
&[\, y_3, y_1^{-1} : y_4, y_2, y_5 \,] \\
&\qquad \equiv [\, y_5^{-1}, y_3, y_1^{-1} ; y_4, y_2 \,]^{\delta_1 g_1} \\
&\qquad \equiv [\, y_5^{-1}, y_3 ; y_4, y_1, y_2 \,]^{\delta_2 g_2} [\, y_5^{-1}, y_3, ; y_2, y_1, y_4 \,]^{\delta_3 g_3} \\
&\qquad \equiv [\, y_4, y_1 : y_5^{-1}, y_2^{-1}, y_3 \,]^{\delta_4 g_4} [\, y_4, y_1 : y_3, y_2^{-1}, y_5^{-1} \,]^{\delta_5 g_5} \\
&\qquad\quad \cdot [\, y_2, y_1 ; y_5^{-1}, y_3, y_4^{-1} \,]^{\delta_6 g_6} \\
&\qquad \equiv 1 \text{ (by the Mini-Max Lemma, Lemma 4.2 and Lemma 4.20).}
\end{aligned}
$$

Assume the statement is true for integers less than p. Then for p, we have

$$
\begin{aligned}
C &= [\, y_3, y_1^{-1} ; y_4, y_2, y_5, y_6, \ldots, y_p \,] \\
&= \prod_{\substack{\lambda_k \in \{0,1\} \\ 5 \le k \le p}} \left[y_3, y_1^{-1} : [\, y_4, y_2 \,]^{y_5^{\lambda_5} y_6^{\lambda_6} \cdots y_p^{\lambda_p}} \right]^{\delta(\lambda_5, \lambda_6, \ldots, \lambda_p) g(\lambda_5, \lambda_6, \ldots, \lambda_p)} \\
&\equiv \left[\, y_3, y_1^{-1} ; [\, y_4, y_2 \,]^{y_5 y_6 \cdots y_p} \,\right]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)} \\
&\equiv \left[\, y_p^{-1}, y_3, y_1^{-1} ; [\, y_4, y_2 \,]^{y_5 y_6 \cdots y_{p-1}} \,\right]^{\delta_2 g_2} \\
&\equiv \left[\, y_p^{-1}, y_3 ; [\, y_4, y_2 \,]^{y_5 y_6 \cdots y_{p-1}} \,\right]^{\delta_3 g_3} \left[\, y_p^{-1}, y_3 ; [\, y_4, y_2 \,]^{y_5 y_6 \cdots y_{p-1} y_1} \,\right]^{\delta_4 g_4};
\end{aligned}
$$

In the above by Lemma 4.2, the first commutator and also each transferring commutator is congruent to 1; so,

$$
\begin{aligned}
C &\equiv \left[y_p^{-1}, y_3 ; \left[y_4, y_2 \right]^{y_1 y_5 y_6 \cdots y_{p-1}} \right]^{\delta_5 g_5} \\
(4.18) \qquad &\equiv \left[y_p^{-1}, y_3 ; \left[y_1, y_2 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\delta_6 g_6} \left[y_p^{-1}, y_3 ; \left[y_2, y_1, y_4 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\delta_7 g_7} \\
&\quad \cdot \left[y_p^{-1}, y_3 ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\delta_8 g_8} \left[y_p^{-1}, y_3 ; \left[y_4, y_1 \right]^{y_2 y_5 y_6 \cdots y_{p-1}} \right]^{\delta_9 g_9}.
\end{aligned}
$$

In 4.18 call the four commutators $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. The proof is complete if we show that each one of these commutators is congruent to 1 separately. But α_1 and α_3 are both congruent to 1 by Lemma 4.2. In addition,

$$
\begin{aligned}
\alpha_2 &= \left[y_p^{-1}, y_3 ; \left[y_2, y_1, y_4 \right]^{y_5 y_6 \cdots y_{p-1}} \right] \\
&\equiv \left[y_2, y_1 ; \left[y_p^{-1}, y_3 \right]^{y_{p-1}^{-1} y_{p-2}^{-1} \cdots y_5^{-1}} \right]^{\sigma_1 h_1} \left[y_2, y_1 ; \left[y_p^{-1}, y_3 \right]^{y_{p-1}^{-1} y_{p-2}^{-1} \cdots y_5^{-1} y_4^{-1}} \right]^{\sigma_2 h_2} \\
&\equiv 1 \quad \text{(by Theorem 4.17)}.
\end{aligned}
$$

On the other hand, $\alpha_4 = \left[y_p^1, y_3 ; \left[y_4, y_1 \right]^{y_2 y_5 y_6 \cdots y_{p-1}} \right]$, where each transferring commutator is congruent to 1 by Lemma 4.2; so

$$
\begin{aligned}
\alpha_4 &\equiv \left[y_p^{-1}, y_3 ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1} y_2} \right]^{\varepsilon_1 f_1} \\
&\equiv \left[y_p^{-1}, y_3 ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_2 f_2} \left[y_p^{-1}, y_2^{-1}, y_3 ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_3 f_3} \\
&\quad \cdot \left[y_3, y_2^{-1}, y_p^{-1} ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_4 f_4} \\
&\equiv \left[y_3, y_2^{-1}, y_p^{-1} ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_4 f_4} \quad \text{(by the Mini-Max Lemma)} \\
&\equiv \left[y_3, y_2^{-1} ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1}} \right]^{\varepsilon_5 f_5} \left[y_3, y_2^{-1} ; \left[y_4, y_1 \right]^{y_5 y_6 \cdots y_{p-1} y_p} \right]^{\varepsilon_6 f_6} \\
&\equiv 1 \quad \text{(by part (c))}.
\end{aligned}
$$

Hence, $C \equiv 1$ and the proof is complete. $\square$

Lemma 4.22. *For $5 \leq p \leq n+1$, let $y_1 < y_2 \leq y_3 \leq \cdots \leq y_p$; then modulo $\langle \Delta(m, n, F) \rangle^F$,*

$$C = [\, y_i, y_2^{-1} ; y_3, y_1, y_4, y_5, \ldots, \hat{y}_i, \ldots, y_p \,] \equiv 1 \quad (4 \leq i \leq p).$$

Proof. If $y_i = y_p$, then by the Mini-Max Lemma, $C \equiv 1$. If $y_i < y_p$, we prove it by induction on p. If $p = 5$, then

$$\begin{aligned}
C &= [\, y_4, y_2^{-1} ; y_3, y_1, y_5 \,] \\
&\equiv [\, y_4, y_2^{-1} ; y_5, y_3, y_1 \,]^{\varepsilon_1 f_1} \\
&\equiv [\, y_4, y_1^{-1}, y_2^{-1} ; y_5, y_3 \,]^{\varepsilon_2 f_2} [\, y_2^{-1}, y_1^{-1}, y_4 ; y_5, y_3 \,]^{\varepsilon_3 f_3} \\
&\equiv [\, y_5, y_2 ; y_4, y_1^{-1}, y_3^{-1} \,]^{\varepsilon_4 f_4} [\, y_4, y_1^{-1} ; y_3, y_2, y_5 \,]^{\varepsilon_5 f_5} [\, y_2^{-1}, y_1^{-1} ; y_5, y_3, y_4^{-1} \,]^{\varepsilon_6 f_6} \\
&\equiv [\, y_2^{-1}, y_1^{-1} ; y_5, y_3, y_4^{-1} \,]^{\varepsilon_6 f_6}
\end{aligned}$$

(by part (b) of Lemma 4.19 and part (b) of Lemma 4.21).

Now if $y_2 = y_3$, then by part (a) of Lemma 4.19,

$$[\, y_2^{-1}, y_1^{-1} ; y_5, y_3, y_4^{-1} \,] \equiv [\, y_5, y_2 ; y_3^{-1}, y_1^{-1}, y_4 \,]^{\varepsilon_7 f_7} \equiv 1.$$

If $y_2 < y_3$, then by Lemma 4.20, $[\, y_2^{-1}, y_1^{-1} ; y_5, y_3, y_4^{-1} \,] \equiv 1$. Therefore in this case, $C \equiv 1$.

Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned}
C &= [\, y_i, y_2^{-1} ; y_3, y_1, y_4, y_5, \ldots, \hat{y}_i, \ldots, y_p \,] \\
&= \prod_{\substack{\lambda_k \in \{0,1\} \\ 4 \leq k \leq p,\, k \neq i}} [\, y_i, y_2^{-1} ; [\, y_3, y_1 \,]^{y_4^{\lambda_4} y_5^{\lambda_5} \cdots \hat{y}_i \cdots y_p^{\lambda_p}} \,]^{\delta_{(\lambda_4, \lambda_5, \ldots, \lambda_p)} g_{(\lambda_4, \lambda_5, \ldots, \lambda_p)}} \\
&\equiv [\, y_i, y_2^{-1} ; [\, y_3, y_1 \,]^{y_4 y_5 \cdots \hat{y}_i \cdots y_p} \,]^{\delta_1 g_1} \quad \text{(by the induction hypothesis).}
\end{aligned}$$

But in each transferring commutator,

$$T(t, \lambda) = [\, y_i, y_2^{-1} ; y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

if $y_2 = y_t$, by the Mini-Max Lemma, and if $y_2 < y_t$, by part (a) of Lemma 4.21 we have, $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[y_i, y_2^{-1} ; [y_3, y_1]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_2 g_2} \\ &\equiv \left[y_i, y_2^{-1} ; [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_3 g_3} \end{aligned}$$

(by the induction hypothesis, the Mini-Max Lemma, and Lemma 4.21). Once again for each transferring commutator,

$$T(t, \lambda) = [y_i, y_2^{-1} ; y_t, y_1, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1}],$$

if $y_i \leq y_t$, by part (c) of Lemma 4.21, and if $y_i > y_t$, by the induction hypothesis, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C \equiv {} & \left[y_i, y_2^{-1} : [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_4 y_4} \left[y_i, y_1^{-1}, y_2^{-1} : [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_5 g_5} \\ & \cdot \left[y_2^{-1}, y_1^{-1}, y_i ; [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\delta_6 g_6}. \end{aligned}$$

Now in the above, the first commutator is congruent to 1 by part (a) of Lemma 4.21; calling the second and the third commutators α and β, respectively, the proof is complete if we show that $\alpha \equiv \beta \equiv 1$. But

$$\begin{aligned} \alpha &= \left[y_i, y_1^{-1}, y_2^{-1} ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right] \\ &\equiv \left[y_i, y_1^{-1} ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_2} \right]^{\varepsilon_1 f_1} \quad \text{(by part (a) of Lemma 4.21)} \end{aligned}$$

For each transferring commutator,

$$T(t, \lambda) = [y_i, y_1^{-1} ; y_t, y_2, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1}],$$

if $y_i \geq y_t$, by part (b) of Lemma 4.21, and if $y_i < y_t$, by part (d) of Lemma 4.21 we have, $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned}
\alpha &\equiv \left[y_i, y_1^{-1} ; [y_p, y_3]^{y_2 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_2 f_2} \\
&\equiv \left[y_i, y_1^{-1} ; [y_p, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_3 f_3} \left[y_i, y_1^{-1} ; [y_p, y_2, y_3]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_4 f_4} \\
&\quad \cdot \left[y_i, y_1^{-1} ; [y_3, y_2]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_5 f_5} \left[y_i, y_1^{-1} ; [y_3, y_2]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_6 f_6} \\
&\equiv \left[y_i, y_1^{-1} ; [y_3, y_2]^{y_p y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_6 f_6} \text{(by parts (a) and (b) of Lemma 4.21).}
\end{aligned}$$

Once again for each transferring commutator,

$$T(t, \lambda) = [\, y_i, y_1^{-1} ; y_p, y_t, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

if $y_t \leq y_i$, by part (a) of Lemma 4.21, and if $y_i < y_t$ by Lemma 4.20, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned}
\alpha &\equiv \left[y_i, y_1^{-1} : [y_3, y_2]^{y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_p} \right]^{\varepsilon_7 f_7} \\
&\equiv 1 \quad \text{(by part (b) of Lemma 4.21)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\beta &= \left[y_2^{-1}, y_1^{-1}, y_i ; [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right] \\
&\equiv \left[y_2^{-1}, y_1^{-1} : [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\sigma_1 h_1} \left[y_2^{-1}, y_1^{-1} : [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1} y_i} \right]^{\sigma_2 h_2}.
\end{aligned}$$

For $i + 1 \leq t \leq p - 1$ for each transferring commutator,

$$T(t, \lambda) = [\, y_2^{-1}, y_1^{-1} ; y_t, y_i^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

since $y_1 < y_2 < y_i$, by the Mini-Max Corollary, we have $T(t, \lambda) \equiv 1$. Therefore,

$$\beta \equiv \left[y_2^{-1}, y_1^{-1} ; [y_p, y_3]^{y_1 y_4 y_5 \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\sigma_1 h_1} \left[y_2^{-1}, y_1^{-1} : [y_p, y_3]^{y_1 y_4 y_5 \cdots y_i^{-1} \cdots y_{p-1}} \right]^{\sigma_3 h_3}.$$

But for each of the above two commutators, if $y_2 = y_3$, by part (a) of Lemma 4.21, and if $y_2 < y_3$, by Lemma 4.20, is congruent to 1: therefore, $\beta \equiv 1$, which in turn implies, $C \equiv 1$. Hence the proof is complete. $\square$

Proposition 4.23. *For* $5 \le p \le n+1$, *let* $y_1 < y_2 \le y_3 \le \cdots \le y_p$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$,

$$C = [\,y_p, y_j\,;\, y_2, y_1, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1}\,] \equiv 1 \quad (3 \le j \le p-1).$$

Proof. We prove it by induction on p. If $p = 5$, then for $y_j = y_2$, by definition, $C \equiv 1$; for $y_2 < y_j$, either $y_2 < y_3$ in which case by Theorem 4.17, $C \equiv 1$, or $y_2 = y_3$, then since $y_1 < y_2 = y_3 < y_4 < y_5$, we have

$$\begin{aligned} C = [\,y_5, y_4\,;\, y_3, y_1, y_2\,] &\equiv [\,y_5, y_2^{-1}, y_4\,;\, y_3, y_1\,]^{\varepsilon_1 f_1} [\,y_4, y_2^{-1}, y_5\,;\, y_3, y_1\,]^{\varepsilon_2 f_2} \\ &\equiv [\,y_5, y_2^{-1}\,;\, y_3, y_1, y_4^{-1}\,]^{\varepsilon_3 f_3} [\,y_4, y_2^{-1}\,;\, y_3, y_1, y_5^{-1}\,]^{\varepsilon_4 f_4} \\ &\equiv 1 \quad \text{(by part (a) of Lemma 4.19 and Lemma 4.22).} \end{aligned}$$

Assume the statement is true for integers less than p. Then for p, if $y_2 < y_3$, then by Theorem 4.17, $C \equiv 1$. If $y_2 = y_3$, then

$$\begin{aligned} C &\equiv \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \le k \le p-1,\, k \ne j}} [\,y_p, y_j\,;\, [\,y_2, y_1\,]^{y_3^{\lambda_3} y_4^{\lambda_4} \cdots \hat{y}_j \cdots y_{p-1}^{\lambda_{p-1}}}\,]^{\varepsilon(\lambda_3, \lambda_4, \ldots, \lambda_{p-1}) f(\lambda_3, \lambda_4, \ldots, \lambda_{p-1})} \\ &\equiv [\,y_p, y_j\,;\, [\,y_2, y_1\,]^{y_3 y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1}}\,]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis).} \end{aligned}$$

But each transferring commutator,

$$T(t, \lambda) = [\,y_p, y_j\,;\, y_t, y_3, {}_{\lambda_{t+1}}y_{t+1}, {}_{\lambda_{t+2}}y_{t+2}, \ldots, {}_{\lambda_{p-1}}y_{p-1}\,],$$

is of weight at most $p-2$. If $y_j \le y_t$, by definition, and if $y_t < y_j$, by the induction hypothesis, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv [\,y_p, y_j\,;\, [\,y_2, y_1\,]^{y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1} y_3}\,]^{\varepsilon_2 f_2} \\ &\equiv [\,y_p, y_j\,;\, [\,y_2, y_1\,]^{y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1}}\,]^{\varepsilon_3 f_3} [\,y_p, y_3^{-1}, y_j\,;\, [\,y_2, y_1\,]^{y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1}}\,]^{\varepsilon_4 f_4} \\ &\quad \cdot [\,y_j, y_3^{-1}\,;\, [\,y_2, y_1\,]^{y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1}}\,]^{\varepsilon_5 f_5} [\,y_j, y_3^{-1}\,;\, [\,y_2, y_1\,]^{y_4 y_5 \cdots \hat{y}_j \cdots y_{p-1} y_p^{-1}}\,]^{\varepsilon_6 f_6} \\ &\equiv 1 \text{ (by the induction hypothesis, the Mini-Max Lemma and Lemma 4.22).} \end{aligned}$$

□

Corollary 4.24. *For* $5 \leq p \leq n+1$, *let* $y_1 < y_2 \leq y_3 \leq \cdots \leq y_p$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$,

$$C = [\, y_p^{\varepsilon_p}, y_j \,;\, y_2, y_1^{\varepsilon_1}, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1} \,] \equiv 1.$$

where $3 \leq j \leq p-1$ *and* $\{\varepsilon_1, \varepsilon_p\} \subseteq \{1, -1\}$.

PROOF. **If** $\varepsilon_1 = +1$: then by Proposition 4.23.

$$\begin{aligned} C &= [\, y_p, y_j \,;\, y_2, y_1, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1} \,]^{\delta_1 g_1} \\ &\quad \cdot [\, y_p, y_j \,;\, y_2, y_1, y_3, y_4, \ldots, \hat{y}_j, \ldots, y_{p-1}, y_p^{(1-\varepsilon_p)/2} \,]^{\delta_2 g_2} \\ &\equiv 1. \end{aligned}$$

If $\varepsilon_1 = -1$: then by induction on p. if $p = 5$ for $y_2 = y_j$, by part (a) of Lemma 4.19, $C \equiv 1$. For $y_2 < y_j$, let $\{j, k\} = \{3, 4\}$: then we have $[\, y_5^{\varepsilon_5}, y_j \,;\, y_2, y_1^{-1}, y_k \,] \equiv \left[\, y_5^{\varepsilon_5}, y_j, y_1 : [\, y_2, y_1 \,]^{y_k} \,\right]^{\delta_1 g_1} \equiv \left[y_j, y_1 : [\, y_2, y_1 \,]^{y_k y_5^{-\varepsilon_5}} \,\right]^{\delta_2 g_2}$

$$\equiv 1 \quad \text{(by Lemma 4.18).}$$

Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned} C &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \leq k \leq p-1,\, k \neq j}} \left[y_p^{\varepsilon_p}, y_j \,;\, [\, y_2, y_1^{-1} \,]^{y_3^{\lambda_3} y_4^{\lambda_4} \cdots \hat{y}_j \cdots y_{p-1}^{\lambda_{p-1}}} \,\right]^{\delta_{\lambda_3, \lambda_4, \ldots, \lambda_{p-1}} g(\lambda_3, \lambda_4, \ldots, \lambda_{p-1})} \\ &\equiv \left[\, y_p^{\varepsilon_p}, y_j : [\, y_2, y_1 \,]^{y_1^{-1} y_3 y_4 \cdots y_{p-1}} \,\right]^{\delta_1 g_1} \quad \text{(by the induction hypothesis).} \end{aligned}$$

But each transferring commutator.

$$T(t, \lambda) = [\, y_p^{\varepsilon_p}, y_j \,;\, y_t, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

is of weight at most $p-1$; if $y_j \leq y_t$, by part (b) of Lemma 4.19, and if $y_t < y_j$ by the induction hypothesis, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[\, y_p^{\varepsilon_p}, y_j \,;\, [\, y_2, y_1 \,]^{y_3 y_4 \cdots y_{p-1} y_1^{-1}} \,\right]^{\delta_2 g_2} \equiv \left[\, y_j, y_1 \,;\, [\, y_2, y_1 \,]^{y_3 y_4 \cdots y_{p-1} y_p^{-\varepsilon_p}} \,\right]^{\delta_3 g_3} \\ &\equiv 1 \quad \text{(by part (a) of Lemma 4.18).} \end{aligned}$$

□

Proposition 4.25. *For* $5 \le p \le n+1$, *let* $y_1 \le y_2 \le y_3 \le \cdots \le y_p$ *and* $y_2 \le y_j < y_i \le y_p$*; then modulo* $\langle \Delta(m, n, F) \rangle^F$,

$$C = [\, y_p, y_j : y_i, y_1, y_2, y_3, \dots, \hat{y}_j, \dots, \hat{y}_i, \dots, y_{p-1} \,] \equiv 1.$$

PROOF. If $y_j = y_2$ or $y_i = y_p$, then by the Mini-Max Lemma, $C \equiv 1$. Assume $y_j \ge y_3$ and $y_i < y_p$; by induction on p, if $p = 5$, then

$$\begin{aligned} C = [\, y_5, y_3 : y_4, y_1, y_2 \,] &\equiv [\, y_5, y_2^{-1}, y_3 : y_4, y_1 \,]^{\varepsilon_1 f_1} [\, y_3, y_2^{-1}, y_5 : y_4, y_1 \,]^{\varepsilon_2 f_2} \\ &\equiv [\, y_5, y_2^{-1} : y_4, y_1, y_3^{-1} \,]^{\varepsilon_3 f_3} [\, y_4, y_1 : y_3, y_2^{-1}, y_5 \,]^{\varepsilon_4 f_4} \\ &\equiv 1. \end{aligned}$$

Assume the statement is true for integers less than p. Then for p, we have

$$\begin{aligned} C &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 2 \le k \le p-1,\, k \ne i,j}} \left[y_p, y_j : [\, y_i, y_1 \,]^{y_2^{\lambda_2} y_3^{\lambda_3} \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}^{\lambda_{p-1}}} \right]^{\varepsilon(\lambda_2, \lambda_3, \dots, \lambda_{p-1}) f(\lambda_2, \lambda_3, \dots, \lambda_{p-1})} \\ &\equiv \left[y_p, y_j : [\, y_i, y_1 \,]^{y_2 y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis).} \end{aligned}$$

But for each transferring commutator,

$$T(t, \lambda) = [\, y_p, y_j : y_t, y_2, {}_{\lambda_{t+1}}y_{t+1}, {}_{\lambda_{t+2}}y_{t+2}, \dots, {}_{\lambda_{p-1}}y_{p-1} \,],$$

if $y_j \le y_t$, by the Mini-Max Lemma, and if $y_t < y_j$, by Proposition 4.23, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[y_p, y_j : [\, y_i, y_1 \,]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_2} \right]^{\varepsilon_2 f_2} \\ &\equiv \left[y_j, y_2^{-1} : [\, y_i, y_1 \,]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}} \right]^{\varepsilon_3 f_3} \\ &\quad \cdot \left[y_j, y_2^{-1} : [\, y_i, y_1 \,]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_p^{-1}} \right]^{\varepsilon_4 f_4} \\ &\qquad \text{(by the induction hypothesis and the Mini-Max Lemma).} \end{aligned} \tag{4.19}$$

Now in the second commutator, for each $3 \le s \le p-2$, each $s+1 \le t \le p-1$ and λ' s,

$$T(t, s, \lambda) = [\, y_j, y_2^{-1} : y_t, y_s, {}_{\lambda_{s+1}}y_{s+1}, {}_{\lambda_{s+2}}y_{s+2}, \dots, {}_{\lambda_{t-1}}y_{t-1} \,],$$

is congruent to 1 by the Mini-Max Lemma if $y_j \geq y_t$. If $y_j < y_t$, then either $y_s \leq y_j$, in which case by part (a) of Lemma 4.21, or $y_j < y_s$ in which case by Lemma 4.20, we have $T(t, s, \lambda) \equiv 1$. In addition for each, $3 \leq s \leq p-1$, in

$$T(s, \lambda) = [\, y_j, y_2^{-1} : y_p, y_s, {}_{\lambda_{s+1}} y_{s+1}, {}_{\lambda_{s+2}} y_{s+2}, \dots, {}_{\lambda_{p-1}} y_{p-1} \,],$$

if $y_s \leq y_j$, by part (a) of Lemma 4.21, and if $y_j < y_s$ by Lemma 4.20, we have $T(s, \lambda) \equiv 1$. Therefore,

$$\begin{aligned} &[\, y_j, y_2^{-1} : [\, y_i, y_1 \,]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_p^{-1}} \,] \\ &\qquad \equiv [\, y_j, y_2^{-1} : [\, y_i, y_1 \,]^{y_p^{-1} y_{p-1} y_{p-2} \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_4 y_3} \,]^{\delta_1 g_1} \\ &\qquad \equiv [\, y_i, y_1 : [\, y_j, y_2^{-1} \,]^{y_3^{-1} y_4^{-1} \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-2}^{-1} y_{p-1}^{-1} y_p} \,]^{\delta_2 g_2} \\ &\qquad \equiv 1 \quad \text{(by Lemma 4.2).} \end{aligned}$$

Similarly, $[\, y_j, y_2^{-1} ; [\, y_i, y_1 \,]^{y_3 y_4 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}} \,] \equiv 1$. Hence by substitution in 4.19, we get $C \equiv 1$ and the proof is complete. □

Corollary 4.26. *For* $5 \leq p \leq n+1$, *let* $y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_p$ *and* $y_2 \leq y_j < y_i \leq y_p$*; then modulo* $\langle \Delta(m, n, F) \rangle^F$,

$$C = [\, y_p^{\varepsilon_p}, y_j ; y_i, y_1^{\varepsilon_1}, y_2, y_3, \dots, \hat{y}_j, \dots, \hat{y}_i, \dots, y_{p-1} \,] \equiv 1,$$

where $\{\varepsilon_1, \varepsilon_p\} \subseteq \{1, -1\}$.

Proof. **If** $\varepsilon_1 = +1$: then by Proposition 4.25,

$$\begin{aligned} C = {} & [\, y_p, y_j ; y_i, y_1, y_2, y_3, \dots, \hat{y}_j, \dots, \hat{y}_i, \dots, y_{p-1} \,]^{\delta_1 g_1} \\ & \cdot [\, y_p, y_j ; y_i, y_1, y_2, y_3, \dots, \hat{y}_j, \dots, \hat{y}_i, \dots, y_{p-1}, y_p^{(1-\varepsilon_p)/2} \,]^{\delta_2 g_2} \\ \equiv {} & 1. \end{aligned}$$

If $\varepsilon_1 = -1$: for $y_2 = y_j$ and or $y_i = y_p$, by part (a) of Lemma 4.19 and or the Mini-max Lemma, $C \equiv 1$. So assume $y_j \geq y_3$ and $y_i < y_p$; then use induction on p. If $p = 5$,

$$\begin{aligned}
[\,y_5^{\varepsilon_5}, y_3\,;\, y_4, y_1^{-1}, y_2\,] &\equiv \left[\,y_3, y_1, y_5^{\varepsilon_5}\,;\, [\,y_4, y_1\,]^{y_2}\,\right]^{\delta_1 g_1} \\
&\equiv \left[\,y_3, y_1\,;\, [\,y_4, y_1\,]^{y_2}\,\right]^{\delta_2 g_2} \left[\,y_3, y_1\,;\, [\,y_4, y_1\,]^{y_2 y_5^{-\varepsilon_5}}\,\right]^{\delta_3 g_3} \\
&\equiv 1 \quad \text{(by Lemma 4.18)}.
\end{aligned}$$

Assume the statement is true for integers less than p. Then for p we have

$$\begin{aligned}
C = &\prod_{\substack{\lambda_k \in \{0,1\} \\ 2 \le k \le p-1,\, k \ne i,j}} \left[\,y_p^{\varepsilon_p}, y_j\,;\, [\,y_i, y_1^{-1}\,]^{y_2^{\lambda_2} y_3^{\lambda_3} \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}^{\lambda_{p-1}}}\,\right]^{\delta(\lambda_2, \lambda_3, \ldots, \lambda_{p-1}) g(\lambda_2, \lambda_3, \ldots, \lambda_{p-1})} \\
&\equiv \left[\,y_p^{\varepsilon_p}, y_j\,;\, [\,y_i, y_1\,]^{y_1^{-1} y_2 y_3 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}}\,\right]^{\delta_1 g_1} \quad \text{(by the induction hypothesis)}.
\end{aligned}$$

But for each transferring commutator.

$$T(t, \lambda) = [\,y_p^{\varepsilon_p}, y_j\,;\, y_t, y_1^{-1}, {}_{\lambda_{t+1}} y_{t+1}, {}_{\lambda_{t+2}} y_{t+2}, \ldots, {}_{\lambda_{p-1}} y_{p-1}\,],$$

if $y_j \le y_t$, by part (a) of Lemma 4.19. and if $y_t < y_j$, by Corollary 4.24. we have $T(t, \lambda) \equiv 1$. Thus.

$$\begin{aligned}
C &\equiv \left[\,y_p^{\varepsilon_p}, y_j\,;\, [\,y_i, y_1\,]^{y_2 y_3 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_1^{-1}}\,\right]^{\delta_2 g_2} \\
&\equiv \left[\,y_p^{\varepsilon_p}, y_j\,;\, [\,y_i, y_1\,]^{y_2 y_3 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}}\,\right]^{\delta_3 g_3} \left[\,y_j, y_1\,;\, [\,y_i, y_1\,]^{y_2 y_3 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1}}\,\right]^{\delta_4 g_4} \\
&\quad \cdot \left[\,y_j, y_1\,;\, [\,y_i, y_1\,]^{y_2 y_3 \cdots \hat{y}_j \cdots \hat{y}_i \cdots y_{p-1} y_p^{-\varepsilon_p}}\,\right]^{\delta_5 g_5} \\
&\equiv 1 \quad \text{(by the Case } \varepsilon_1 = +1 \text{ and part (a) of Lemma 4.18)}.
\end{aligned}$$

□

Theorem 4.27. *For* $5 \le c \le n+1$, *let* $z_1 < z_2 \le z_3 \le z_4 \le \cdots \le z_c$ *and* $z_2 \le z_j < z_i \le z_c$; *then modulo* $\langle \Delta(m, n, F) \rangle^F$,

$$C = [\,z_j, z_1\,;\, z_i, z_2, z_3, z_4, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_{c-1}, z_c\,] \equiv 1.$$

PROOF. If $y_i = y_c$, then by the Mini-Max Lemma, $C \equiv 1$; so assume that $y_i < y_c$, use induction on c. If $c = 5$, then

$$\begin{aligned} C = [\, z_3, z_1 \,;\, z_4, z_2, z_5 \,] &\equiv [\, z_5^{-1}, z_3, z_1 : z_4, z_2 \,]^{\varepsilon_1 f_1} \\ &\equiv [\, z_5^{-1}, z_3 : z_4, z_1^{-1}, z_2 \,]^{\varepsilon_2 f_2} [\, z_5^{-1}, z_3 : z_2, z_1^{-1}, z_4 \,]^{\varepsilon_3 f_3} \\ &\equiv 1 \quad \text{(by Corollary 4.24 and Corollary 4.26).} \end{aligned}$$

Assume the statement is true for integers less than c. Then for c, we have

$$\begin{aligned} C &= \prod_{\substack{\lambda_k \in \{0,1\} \\ 3 \le k \le c,\, k \ne i,j}} \left[z_j, z_1 : [\, z_i, z_2 \,]^{z_3^{\lambda_3} z_4^{\lambda_4} \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_c^{\lambda_c}} \right]^{\varepsilon(\lambda_3, \lambda_4, \ldots, \lambda_c) f(\lambda_3, \lambda_4, \ldots, \lambda_c)} \\ &\equiv \left[\, z_j, z_1 \,;\, [\, z_i, z_2 \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_c} \,\right]^{\varepsilon_1 f_1} \quad \text{(by the induction hypothesis)} \\ &\equiv \left[z_c^{-1}, z_j, z_1 : [\, z_i, z_2 \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_2 f_2} \\ &\equiv \left[z_c^{-1}, z_j \,;\, [\, z_i, z_2 \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1} z_1^{-1}} \right]^{\varepsilon_3 f_3} \quad \text{(by Corollary 4.26).} \end{aligned}$$

But for each transferring commutator,

$$T(t, \lambda) = [\, z_c^{-1}, z_j \,;\, z_t, z_1^{-1}, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,]$$

if $z_j \le z_t$, by part (a) of Lemma 4.19, and if $z_t < z_j$, by Corollary 4.24 we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[z_c^{-1}, z_j : [\, z_i, z_2 \,]^{z_1^{-1} z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_4 f_4} \\ &\equiv \left[z_c^{-1}, z_j : [\, z_i, z_2 \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_5 f_5} \\ &\quad \cdot \left[z_c^{-1}, z_j \,;\, [\, z_i, z_1^{-1}, z_2 \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_6 f_6} \\ &\quad \cdot \left[z_c^{-1}, z_j \,;\, [\, z_2, z_1^{-1} \,]^{z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_7 f_7} \\ &\quad \cdot \left[z_c^{-1}, z_j \,;\, [\, z_2, z_1^{-1} \,]^{z_i z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_8 f_8} \\ &\equiv \left[z_c^{-1}, z_j \,;\, [\, z_2, z_1^{-1} \,]^{z_i z_3 z_4 \cdots \hat{z}_j \cdots \hat{z}_i \cdots z_{c-1}} \right]^{\varepsilon_8 f_8} \quad \text{(by Corollaries 4.24 and 4.26).} \end{aligned}$$

Once again for each transferring commutator,

$$T(t, \lambda) = [\, z_c^{-1}, z_j \,;\, z_i, z_t, {}_{\lambda_{t+1}} z_{t+1}, {}_{\lambda_{t+2}} z_{t+2}, \ldots, {}_{\lambda_{c-1}} z_{c-1} \,],$$

if $z_j \leq z_t$, by the Mini-max Lemma, and if $z_t < z_j$, by Corollary 4.26, we have $T(t, \lambda) \equiv 1$. Thus,

$$\begin{aligned} C &\equiv \left[z_c^{-1}, z_j \,;\, [\, z_2, z_1^{-1} \,]^{z_3 z_4 \ldots \hat{z}_j \ldots \hat{z}_i \ldots z_{c-1}} \right]^{\varepsilon_9 f_9} \\ &\equiv 1 \quad \text{(by Corollary 4.24)}, \end{aligned}$$

which completes the proof. □

Now we are in position to introduce a very sharp presentation of $M(m, n)$.

Theorem 4.28. *Let $F = \langle X : \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$ where $m \geq 3$; then for arbitrary but fixed integer $n \geq 5$, $M(m, n)$ admits the following finite presentation:*

$$M(m, n) = \langle\, x_1, x_2, \ldots, x_m, \,;\, \Delta(m, n, F) \,\rangle.$$

PROOF. By the definition of $M(m, n)$, we must prove that $\gamma_n(F)F'' = \langle \Delta(m, n, F) \rangle^F$. Clearly, $\langle \Delta(m, n, F) \rangle^F \leqslant \gamma_n(F)F''$. In order to prove $\gamma_n(F)F'' \leqslant \langle \Delta(m, n, F) \rangle^F$, since by Theorem 2.10 we have shown that $\gamma_n(F)F'' = \langle \Lambda(m, n, F) \rangle^F$, so it is enough to show that

$$\langle \Lambda(m, n, F) \rangle^F \leqslant \langle \Delta(m, n, F) \rangle^F.$$

In other words, using the definition of $\Lambda(m, n, F)$ as defined in Definition 2.9, we need to show that for each $4 \leq c \leq n+1$, $\Theta(m, c, F) \subseteq \langle \Delta(m, n, F) \rangle^F$. Let $C = [u, v ; x, y, \ldots, z]$, where $u < v$, $x > y \leq \cdots \leq z$, $v \leq y$, be an arbitrary element of $\Theta(m, c, F)$; then there are two cases, either $v = y$ or $v < y$.

Case(1): If $v = y$, then by calling $v = z_1, y = z_2, u = z_j, x = z_i, \ldots, z = z_c$ we have $C = [z_j, z_1 ; z_i, z_2, z_3, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c]$ where $z_1 = z_2 \leq z_3 \leq$

$\cdots \leq z_c$ and by part (b) of Theorem 4.17, $C \equiv 1 \pmod{\langle \Delta(m, n, F) \rangle^F}$.

Case(2): If $v < y$, then call $v = z_1$. **Either** $u < y$, in which case setting $u = z_2$, $y = z_3$, $x = z_i$, $\ldots$, $z = z_c$ we have $C = [z_2, z_1 : z_i, z_3, z_4, \ldots, \hat{z}_i, \ldots, z_c]$, where $z_1 < z_2 < z_3 \leq z_4 \leq \cdots \leq z_c$, and so by part (a) of Theorem 4.17, $C \equiv 1 \pmod{\langle \Delta(m, n, F) \rangle^F}$. **Or** $u \geq y$, so by setting $y = z_2$, $u = z_i$, $x = z_j$, $\ldots$, $z = z_c$ we have $C = [z_j, z_1 : z_i, z_2, z_3, \ldots, \hat{z}_j, \ldots, \hat{z}_i, \ldots, z_c]$, where $z_1 < z_2 \leq z_3 \leq \ldots z_c$, and so by Theorem 4.27, $C \equiv 1 \pmod{\langle \Delta(m, n, F) \rangle^F}$.

Hence, the proof is complete.

□

CHAPTER 5

A Last Refinement

In this chapter, using the last set of relators $\Delta(m, n, F)$ of $M(m,n)$ obtained by Theorem 4.28 in Chapter 4, we eliminate certain types of basic commutators of weight $n+1$ and introduce a refinement of $\Delta(m,n,F)$. The chapter ends with our concluding notes.

5.1. Last Eliminations

Definition 5.1. Let $F = \langle X : \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 3$. For arbitrary but fixed integer $n \geq 5$, let $\Omega_1(m, n+1, F)$ and $\Omega_2(m, n+1, F)$ be as defined in Definition 4.1. Define $\bar{\Omega}_1(m, n+1, F)$ and $\bar{\bar{\Omega}}_1(m, n+1, F)$, subsets of $\Omega_1(m, n+1, F)$, as follows:

$$\bar{\Omega}_1(m, n+1, F) = \{C \in \Omega_1(m, n+1, F) \mid C = [y_4, y_1 ; y_2, {}_{n-i}\, y_1, {}_{i-3}\, y_2, y_3],$$
$$y_1 < y_2 \leq y_3 \leq y_4,\ (3 \leq i \leq n-1)\};$$

$$\bar{\bar{\Omega}}_1(m, n+1, F) = \{C \in \Omega_1(m, n+1, F) \mid C = [y_2, y_1 ; y_2, y_1, {}_{n-4}\, y_2, y_3],$$
$$y_1 < y_2 < y_3\}.$$

Also, define $\bar{\Omega}_2(m, n+1, F)$ and $\bar{\bar{\Omega}}_2(m, n+1, F)$, subsets of $\Omega_2(m, n+1, F)$, as follows:

$$\bar{\Omega}_2(m, n+1, F) = \{C \in \Omega_2(m, n+1, F) \mid C = [y_5, y_2 : y_2, y_1, {}_{(n-5)}\, y_2, y_3, y_4],$$
$$y_1 < y_2 \leq y_3 \leq y_4 \leq y_5,\ y_2 < y_5\};$$

$$\bar{\bar{\Omega}}_2(m, n+1, F) = \{C \in \Omega_2(m, n+1, F) \mid C = [y_4, y_2 ; y_3, y_1, {}_{(n-3)}\, y_2],$$
$$y_1 < y_2 < y_3 < y_4\}.$$

In addition, define

$$\Omega_1^*(m,n+1,F) = \Omega_1(m,n+1,F) \smallsetminus \big(\bar{\Omega}_1(m,n+1,F) \cup \bar{\bar{\Omega}}_1(m,n+1,F)\big);$$

$$\Omega_2^*(m,n+1,F) = \Omega_2(m,n+1,F) \smallsetminus \big(\bar{\Omega}_2(m,n+1,F) \cup \bar{\bar{\Omega}}_2(m,n+1,F)\big).$$

Finally, define

$$\Omega^*(m,n+1,F) = \Omega_1^*(m,n+1,F) \cup \Omega_2^*(m,n+1,F) \cup \Omega_3(m,n+1,F);$$

$$\Delta^*(m,n,F) = \Big(\bigcup_{c=4}^{n} \Omega(m,c,F)\Big) \bigcup \Omega^*(m,n+1,F) \bigcup \Omega_0(m,n,F),$$

where $\Omega(m,c,F)$, $\Omega_3(m,n+1,F)$ and $\Omega_0(m,n,F)$ are as defined in Definition 4.1 and Definition 2.7, respectively. Denote the normal closure of $\Delta^*(m,n,F)$ by $\langle \Delta^*(m,n,F) \rangle^F$. Obviously, $\Delta^*(m,n,F) \subseteq \Delta(m,n,F)$.

Lemma 5.2. *Let* $\{y_1, y_2\} \subset X$ *such that* $y_1 < y_2$ *; then modulo* $\langle \Delta^*(m,n,F) \rangle^F$,

(a) *for* $d = [y_{2,\,k}\, y_{1,\,\ell}\, y_2]$, $(1 \le k + \ell \le n-3)$ *the commutator* $[d, y_2, y_1]$ *can be written as a product of conjugates of commutators of the form* $[d, y_1, u]$, *where* $u \in \{y_1, y_2\}$.

(b) $[y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1, z_1, z_2, \ldots, z_{i-j-2}] \equiv 1$, *where* $\{z_1, z_2, \ldots, z_{i-j-2}\} \subseteq \{y_1, y_2\}$, $3 \le i \le n-1$ *and* $1 \le j \le i-2$.

PROOF. By definition, $\Delta^*(2,n,F) \subset \Delta^*(m,n,F)$. So in fact, part (a) is the same as Lemma 3.2 for $\ell = 0$, and Lemma 3.4 for $\ell \ge 1$.
Also, part(b) is the same as Proposition 3.5. □

Lemma 5.3. *Let* $\{y_1, y_2, y_3\} \subseteq X$ *and* $y_1 < y_2 < y_3$*; then modulo* $\langle \Delta^*(m,n,F) \rangle^F$,

$$[y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1, z_1, z_2, \ldots, z_{i-j-3}, y_3] \equiv 1,$$

where $\{z_1, z_2, \ldots, z_{i-j-3}\} \subseteq \{y_1, y_2\}$, $4 \le i \le n-1$ *and* $1 \le j \le i-3$.

PROOF. The proof is similar to the proof of Proposition 3.5 and is based on double induction on i and j.

If $i = 4$ and $j = 1$, we need to show that $[y,_{(n-4)} y_1, y_2, y_1, y_3] \equiv 1$. But by part (a) of Lemma 5.2,

$$[y,_{(n-4)} y_1, y_2, y_1] \equiv \prod_{t=1}^{r} [y,_{(n-4)} y_1, y_1, u_t]^{\varepsilon_t f_t}, \tag{5.1}$$

for some integer r and $u_t \in \{y_1, y_2\}$, $f_t \in F$, $\varepsilon_t \in \{1, -1\}$. Therefore, $[y,_{(n-4)} y_1, y_2, y_1, y_3]$ is congruent to a product of conjugates of commutators of the form $C_{(u_t,v_t)} = [y,_{(n-3)} y_1, u_t, v_t]$, where $u_t \in \{y_1, y_2\}$ and $v_t \in \{y_1, y_2, y_3\}$. In $C_{(u_t,v_t)}$, if $v_t = y_3$ then $C_{(u_t,v_t)}$ is simple basic of weight n, and if $v_t \neq y_3$ then by part (b) of Lemma 5.2, $C_{(u_t,v_t)} \equiv 1$. So in this case, $[y,_{(n-4)} y_1, y_2, y_1, y_3] \equiv 1$.

Assume the statement is also true for k where $4 \leq k \leq i-1$ and all j's such that $1 \leq j \leq k-3$; then in order to prove it for i and all j's where $1 \leq j \leq i-3$, we do another induction on j. If $j = 1$ we must show that,

$$[y_2,_{(n-i)} y_1, y_2, y_1, z_1, z_2, \ldots, z_{i-4}, y_3] \equiv 1.$$

But adding $z_1, z_2, \ldots, z_{i-4}$ and y_3 to the both sides of 5.1 implies that $[y_2,_{(n-i)} y_1, y_2, y_1, z_1, z_2, \ldots, z_{i-4}, y_3]$ is congruent to a product of conjugates of commutators of the form $[y_2,_{(n-i)} y_1, y_1, u_1, u_2, \ldots, u_{i-3}, v]$, where $\{u_1, u_2, \ldots, u_{i-3}\} \subseteq \{y_1, y_2\}$ and $v \in \{y_1, y_2, y_3\}$. So it is enough to show that $[y_2,_{(n-i+1)} y_1, u_1, u_2, \ldots, u_{i-3}, v] \equiv 1$. Now if $u_1, u_2, \ldots, u_{i-3}$ are all y_1 or all y_2, then the commutator $[y_2,_{(n-i+1)} y_1, u_1, u_2, \ldots, u_{i-3}, v]$ is either a simple basic commutator of weight n and so is in $\Omega_0(n, 2, F)$, or is congruent to one by Lemma 5.2(b). Otherwise, assume that s $(1 \leq s \leq i-4)$ is the smallest integer such that $u_s = y_2$ (if $s = i-3$, then again it is congruent to 1 by part (b) of Lemma 5.2). Also, assume that t $(1 \leq t \leq i-s-3)$ is the smallest integer such that $u_{s+t} = y_1$; then by part (b) of Lemma

5.2 (when $v \neq y_3$), and or by the hypothesis of the first induction for $i-s$ and t (when $v = y_3$), we have

$$[\, y_{2,\,(n-i+1)}\, y_1, u_1, u_2, \dots, u_{i-3}, v \,]$$

$$= [\, y_{2,\,n-(i-s)}\, y_{1,\,t}\, y_2, y_1, u_{s+t+1}, u_{s+t+2}, \dots, u_{i-3}, v \,] \equiv 1.$$

which completes the proof of case $j = 1$. Assuming the statement is also true for $j-1$, that is, for i and $1 \leq j-1 < i-3$,

$$[\, y_{2,\,(n-i)}\, y_{1,\,(j-1)}\, y_2, y_1, z_1, z_2, \dots, z_{i-j-2}, y_3 \,] \equiv 1.$$

In order to prove that $[\, y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1, z_1, z_2, \dots, z_{i-j-3}, y_3 \,] \equiv 1$, using Lemma 5.2, let $d = [\, y_{2,\,(n-i)}\, y_{1,\,j-1}\, y_2 \,]$; then

$$[\, d, y_2, y_1 \,] = [\, y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1 \,] \equiv \prod_{t=1}^{r} [\, y_{2,\,(n-i)}\, y_{1,\,j-1}\, y_2, y_1, u_t \,]^{\epsilon_t f_t}.$$

for some integer r and $\epsilon_t \in \{1, -1\}$, $u_t \in F$ and $f_t \in F$.
Therefore by adding $z_1, z_2, \dots, z_{i-j-3}$ and y_3 to the both sides of the above congruence, the commutator $[\, y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1, z_1, z_2, \dots, z_{i-j-3}, y_3 \,]$ is congruent to a product of conjugates of commutators of the form.

$$[\, y_{2,\,(n-i)}\, y_{1,\,j-1}\, y_2, y_1, u_1, u_2, \dots, u_{i-j-2}, v \,],$$

where $\{u_1, u_2, \dots, u_{i-j-2}\} \subseteq \{y_1, y_2\}$ and $v \in \{y_1, y_2, y_3\}$. But by the hypothesis of the second induction on j (when $v = y_3$), and or by part (b) of Lemma 5.2 (when $v \neq y_3$), each one of these last commutators is congruent to 1. Therefore,

$$[\, y_{2,\,(n-i)}\, y_{1,\,j}\, y_2, y_1, z_1, z_2, \dots, z_{i-j-3}, y_3 \,] \equiv 1.$$

□

Corollary 5.4. *Let* $\{y_1, y_2, y_3\} \subseteq X$ *such that* $y_1 < y_2 < y_3$; *then modulo* $\langle \Delta^*(m,n,F)\rangle^F$, *for* $3 \le i \le n-1$,

$$[y_2,_{(n-i)} y_1,_{(i-3)} y_2, y_1, y_3] \equiv 1.$$

PROOF. If $i = 3$, then $[y_2,_{(n-2)} y_1, y_1, y_3]$ is a simple basic commutator of weight n; if $4 \le i \le n-1$, by Lemma 5.3 for $j = i-3$, we have

$$[y_2,_{(n-i)} y_1,_{i-3} y_2, y_1, y_3] \equiv 1.$$

□

Lemma 5.5. *Let* $\{y_1, y_2, y_3, y_4\} \subseteq X$ *such that* $y_1 < y_2 < y_3 \le y_4$; *also let* $d = [y_2, y_1,_k y_2,_\ell y_3]$ *where* $0 \le k + \ell \le n-4$. *Then modulo* $\langle \Delta^*(m,n,F)\rangle^F$ *for* $u \in \{y_1, y_2\}$, $[d, y_4, u]$ *can be written as a product of conjugates of commutators of the form* $[d, u, v]$ *where* $v \in \{y_1, y_2, y_3, y_4\}$.

PROOF. We consider two cases, either $k+\ell = n-4$ or $0 \le k+\ell \le n-5$. If $k+\ell = n-4$, then d is a commutator of weight $n-2$. Using the Hall-Witt identity for d, y_4 and u yields

$$[d, y_4^{-1}, u]^{y_4}[y_4, u^{-1}, d]^u[u, d^{-1}, y_4]^d = 1. \tag{5.2}$$

Working modulo $\langle \Delta^*(m,n,F)\rangle^F$, we have

$$[d, y_4^{-1}, u]^{y_4} = [[d, y_4, y_4^{-1}]^{-1}[d, y_4]^{-1}, u]^{y_4} \equiv ([d, y_4, u]^{-1})^{[y_4,d]y_4} \quad (1).$$

And

$$\begin{aligned}
[y_4, u^{-1}, d]^u &= [d; y_4, u]^{[d,u][u,y_4]}[d, u; y_4, u]^{[u,y_4]} \\
&\equiv [d, u; y_4, u]^{[u,y_4]} \\
&\equiv \prod_{t=1}^{4}[d, u, v_t]^{\epsilon_t f_t} \qquad (\epsilon_t \in \{1, -1\}, f_t \in F, v_t \in \{u, y_4\}) \\
&\equiv 1 \qquad (2).
\end{aligned}$$

Also,

$$[u, d^{-1}, y_4]^d = [d, u, y_4^d] \equiv \prod_{t=1}^{r} [d, u, v_t]^{\epsilon_t f_t} \qquad (3),$$

for some integer r and $\epsilon_t \in \{1, -1\}$, $f_t \in F$, $v_t \in \{y_1, y_2, y_3, y_4\}$. Substitution of (1), (2) and (3) in 5.2 completes the proof of this case.

If $0 \le k + \ell \le n - 5$, then d is a commutator of weight at most $n-3$. Using the Magnus identity for d, y_4 and u yields

$$(5.3) \qquad [d, y_4, u^d][u, d, y_4^u][y_4, u, d^{y_4}] = 1.$$

Working modulo $\langle \Delta^*(m, n, F) \rangle^F$, we have

$$\begin{aligned}[d, y_4, u^d] &= [d, u : d, y_4]^{[u,d]} [d, y_4, u]^{[u,d]} \\ &= \prod_{t=1}^{r_1} [d, u, v_t]^{\epsilon_t f_t} [d, y_4, u]^{[u,d]} \quad (\dagger)\end{aligned}$$

(for some integer r_1 and $\epsilon_t \in \{1, -1\}$, $f_t \in F$, $v_t \in \{x, y\}$).

And

$$\begin{aligned}[u, d, y_4^u] &= \left([d, u, y_4^u]^{-1}\right)^{[u,d]} \\ &= \prod_{t=1}^{r_2} [d, u, v_t]^{\delta_t g_t} [d, y_4, u]^{[u,d]} \quad (\dagger\dagger)\end{aligned}$$

(for some integer r_2 and $\delta_t \in \{1, -1\}$, $g_t \in F$, $v_t \in \{u, y_4\}$).

Also,

$$[y_4, u, d^{y_4}] = [y_4, u : d, y_4][y_4, u, d]^{[d, y_4]} \equiv 1 \qquad (\dagger\dagger\dagger).$$

Finally substitution of $(\dagger)$, $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$ in 5.3 completes the proof. $\square$

Lemma 5.6. *Let $\{y_1, y_2, y_3, y_4\} \subseteq X$ such that $y_1 < y_2 < y_3 \le y_4$; then modulo $\langle \Delta^*(m, n, F) \rangle^F$,*

$$[y_2, y_{1;\,(n-5)}\, y_2, y_3, y_4, y_2] \equiv 1.$$

PROOF. Using Lemma 5.5 for $d = [\,y_2, y_{1,\,(n-5)}\, y_2, y_3\,]$ implies that modulo $\langle\,\Delta^*(m,n,F)\,\rangle^F$, $C = [\,y_2, y_{1,\,(n-5)}\, y_2, y_3, y_4, y_2\,]$ is congruent to a product of conjugates of commutators of the form $[\,y_2, y_{1,\,(n-5)}\, y_2, y_3, y_2, u\,]$ where $u \in \{y_1, y_2, y_3, y_4\}$. But for each u first we observe that using Lemma 5.5 again, this time letting $d = [\,y_2, y_{1,\,(n-5)}\, y_2\,]$, implies that $[\,y_2, y_{1,\,(n-5)}\, y_2, y_3, y_2\,]$ is congruent to a product of conjugates of commutators of the form $[\,y_2, y_{1,\,(n-5)}\, y_2, y_2, v\,]$ where $v \in \{y_1, y_2, y_3\}$. Therefore, $[\,y_2, y_{1,\,(n-5)}\, y_2, y_3, y_2, u\,]$ is congruent to a product of conjugates of commutators of the form $[\,y_2, y_{1,\,(n-4)}\, y_2, v, w\,]$ where $v \in \{y_1, y_2, y_3\}$ and $w \in \{y_1, y_2, y_3, y_4\}$. Now we consider different cases for v and w:

If $v = y_1$, then $[\,y_2, y_{1,\,(n-4)}\, y_2, y_1, w\,] \equiv 1$, by Lemma 5.2 when $w = y_1$ or y_2, and or by Lemma 5.3 when $w = y_3$ or y_4.

If $v = y_2$, then $[\,y_2, y_{1,\,(n-4)}\, y_2, y_2, w\,] \equiv 1$, by Lemma 5.2.

If $v = y_3$, then for $w = y_3$ or y_4, $[\,y_2, y_{1,\,(n-4)}\, y_2, y_3, w\,] \in \Omega_0(m,n,F)$. For $w = y_1$ or y_2, let $d = [\,y_2, y_{1,\,(n-4)}\, y_2\,]$; using Lemma 5.5 one last time implies that $[\,y_2, y_{1,\,(n-4)}\, y_2, y_3, w\,]$ is congruent to a product of conjugates of commutators of the form $[\,y_2, y_{1,\,(n-4)}\, y_2, w, w_1\,]$ where $w_1 \in \{y_1, y_2, y_3\}$. But if $w = y_1$ then by Lemma 5.2 (when $w_1 = y_1$ or y_2), and or by Lemma 5.3 (when $w_1 = y_3$) we have $[\,y_2, y_{1,\,(n-4)}\, y_2, y_1, w_1\,] \equiv 1$. If $w = y_2$, by Lemma 5.2 we have $[\,y_2, y_{1,\,(n-4)}\, y_2, y_2, w_1\,] \equiv 1$. Hence, $C \equiv 1$. □

Theorem 5.7. $\langle\,\Delta^*(m,n,F)\,\rangle^F = \langle\,\Delta(m,n,F)\,\rangle^F$.

PROOF. By definition $\langle\,\Delta^*(m,n,F)\,\rangle^F \leqslant \langle\,\Delta(m,n,F)\,\rangle^F$. In order to prove that $\langle\,\Delta(m,n,F)\,\rangle^F \leqslant \langle\,\Delta^*(m,n,F)\,\rangle^F$, we need to show that modulo $\langle\,\Delta^*(m,n,F)\,\rangle^F$,

$$(1)\quad \bar{\Omega}_1(m,n+1,F) \equiv 1;$$

$$(2)\quad \bar{\bar{\Omega}}_1(m,n+1,F) \equiv 1;$$

(3) $\bar{\Omega}_2(m, n+1, F) \equiv 1$;

(4) $\bar{\bar{\Omega}}_2(m, n+1, F) \equiv 1$.

Proof of (1): Let $C \in \bar{\Omega}_1(m, n+1, F)$; so $C = [\, y_4, y_1 \,;\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_3 \,]$ where $y_1 < y_2 \leq y_3 \leq y_4$ and $3 \leq i \leq n-1$. Since C is equal to a product of conjugates of commutators of the form either $[\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_3, y_4 \,]$ (which are simple basics commutators of weight n and so in $\Omega_0(m, n+1, F)$), or $[\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_3, y_1 \,]$. But if $y_2 = y_3$, then by Lemma 5.2,

$$[\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_3, y_1 \,] \equiv 1.$$

If $y_2 < y_3$, then similar to the proof of Lemma 5.5, $[\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_3, y_1 \,]$ can be written as a product of conjugates of commutators of the form $[\, y_2, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_1, u \,]$ where $u \in \{y_1, y_2, y_3\}$. But if $u = y_3$, then by Corollary 5.4, and if $u = y_1$ or y_2 by Lemma 5.2, we have

$$[\, y_2, y_1, {}_{(n-i)}\, y_1, {}_{(i-3)}\, y_2, y_1, u \,] \equiv 1.$$

Hence, $C \equiv 1$.

Proof of (2): Let $C \in \bar{\bar{\Omega}}_1(m, n+1, F)$; so $C = [\, y_2, y_1 \,;\, y_2, y_1, {}_{(n-4)}\, y_2, y_3 \,]$ where $y_1 < y_2 < y_3$. But C is equal to a product of conjugates of commutators of the form $[\, y_2, y_1, {}_{(n-4)}\, y_2, y_3, u \,]$ where $u \in \{y_1, y_2\}$. For each u, by Lemma 5.5, $[\, y_2, y_1, {}_{(n-4)}\, y_2, y_3, u \,]$ can be written as a product of conjugates of commutators of the form $[\, y_2, y_1, {}_{(n-4)}\, y_2, u, v \,]$ where $v \in \{y_1, y_2, y_3\}$. Now if $u = y_1$, then by Lemma 5.2 (when $v = y_1$ or y_2), and by Corollary 5.4 (when $v = y_3$), we have $[\, y_2, y_1, {}_{(n-4)}\, y_2, y_1, v \,] \equiv 1$. If $u = y_2$, then for $v = y_1$, by Lemma 5.2, $[\, y_2, y_1, {}_{(n-4)}\, y_2, y_2, y_1 \,] \equiv 1$, and for $v = y_2$ or y_3, $[\, y_2, y_1, {}_{(n-4)}\, y_2, y_2, v \,]$ is a simple basic commutator of weight n and so in $\Omega_0(m, n+1, F)$. Therefore in any case, $[\, y_2, y_1, {}_{(n-4)}\, y_2, u, v \,] \equiv 1$, and thus $C \equiv 1$.

Proof of (3): Let $C \in \bar{\Omega}_2(m, n+1, F)$; so $C = [\, y_5, y_2 \,;\, y_2, y_1, {}_{(n-5)}\, y_2, y_3, y_4 \,]$

where $y_1 < y_2 \leq y_3 \leq y_4 \leq y_5$ and $y_2 < y_5$. But C is equal to a product of conjugates of commutators of the form $[y_2, y_1,_{(n-5)} y_2, y_3, y_4, u]$ where $u \in \{y_2, y_5\}$. If $u = y_5$, then $[y_2, y_1,_{(n-5)} y_2, y_3, y_4, y_5] \in \Omega_0(m, n+1, F)$. If $u = y_2$, then either $y_2 = y_3$ or $y_2 < y_3$. For $y_2 = y_3$, by Lemma 5.5, $[y_2, y_1,_{(n-5)} y_2, y_3, y_4, y_2]$ is congruent to a product of conjugates of commutators of the form $[y_2, y_1,_{(n-4)} y_2, y_2, v]$ where $v \in \{y_1, y_2, y_4\}$ and by Lemma 5.2 each one these commutators is congruent to 1. For $y_2 < y_3$, by Lemma 5.6 $[y_2, y_1,_{(n-5)} y_2, y_3, y_4, y_2] \equiv 1$. Hence, $C \equiv 1$.

Proof of (4): Let $C \in \bar{\bar{\Omega}}_2(m, n+1, F)$; so $C = [y_4, y_2; y_3, y_1,_{(n-3)} y_2]$ where $y_1 < y_2 < y_3 < y_4$. But C is equal to a product of conjugates of commutators of the form $[y_3, y_1,_{(n-3)} y_2, u]$ where $u \in \{y_2, y_4\}$, and each one of these commutators is in $\Omega_0(m, n+1, F)$; therefore, $C \equiv 1$, and the proof of the theorem is complete. □

Now we are in a position that allows us to introduce the sharpest presentation of $M(m, n)$.

Theorem 5.8. *Let $F = \langle X ; \varnothing \rangle$ be a free group with basis $X = \{x_1, x_2, \ldots, x_m\}$ where $m \geq 3$; then for arbitrary but fixed integer $n \geq 5$, $M(m, n)$ admits the following finite presentation.*

$$M(m, n) = \langle\, x_1, x_2, \ldots, x_m ; \; \Delta^*(m, n, F) \,\rangle.$$

PROOF. By definition of $M(m, n)$ we must prove that

$$\gamma_n(F)F'' = \langle\, \Delta^*(m, n, F) \,\rangle^F.$$

But by Theorem 4.28 we have shown that $\gamma_n(F)F'' = \langle\, \Delta(m, n, F) \,\rangle^F$. So it is enough to prove that $\langle\, \Delta^*(m, n, F) \,\rangle^F = \langle\, \Delta(m, n, F) \,\rangle^F$. But this is done by Theorem 5.7. □

Concluding Notes: As we mentioned at the beginning of Chapter 2, finding a minimal presentation of finitely generated nilpotent groups is in general an extremely difficult problem. From Chapter 2 to Chapter 5 we studied the free metabelian nilpotent case. But the very next step would be to seek a minimal presentation of *the free centre-by-metabelian nilpotent group* of rank m and nilpotency class $n-1$ ($m, n \geq 2$). The methods developed here can be used to study this case too, and the encouraging point is that Gupta, Hurley and Levin [**4**] have completely analyzed the lower central factors of free centre-by-metabelian groups.

The following interesting and important open problem was originally proposed by Sims [**17**] and then later by Gaglione and Spellman [**1**].

Problem : Is the n-th term of the lower central series of a free group of finite rank the normal closure of basic commutators of weight n ?

In Chapter 2 we gave a yes answer to this question for $n = 2, 3, 4$ and $m \geq 2$. Also in Chapter 3 we proved it for $n = 5$ and $m = 2$. Although the problem remains open in general for other values of n and m , we have some reason to believe that the general answer would be a NO answer. It seems that normal closure of basic commutators of weight 6 is a proper subgroup of $\gamma_6(F)$ where F is of rank 2, but it remains to be proved and we hope this will be answered soon.

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