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Curved-Space Quantization, and Dirac versus Reduced Quantization of Poincaré Invariant Gauge Theories

by

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CURVED-SPACE QUANTIZATION, AND DIRAC VERSUS REDUCED QUANTIZATION OF POINCARÉ INVARIANT GAUGE THEORIES

BY

RICHARD J. EPP

A Thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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Abstract

It is well known that the difference between Dirac and reduced quantization of gauge theories can be understood in terms of factor ordering ambiguities. The presence of symmetries (such as Poincaré invariance), which must be represented in the quantum theory without van Hove anomalies, leads to restrictions on the possible factor orderings. We consider scalar electrodynamics: its reduced field space is curved, which amplifies the van Hove obstructions and obscures the correct factor ordering of the Poincaré generators. In Dirac quantization on the other hand, the initial field space on which the quantum operators are defined is flat, and we show that the natural factor ordering automatically realizes the Poincaré algebra on physical states. We generalize this result to a broader class of observables and gauge theories, and apply Kaluza-Klein-like geometrical methods to determine what, if anything, makes the Dirac factor ordering special. Unlike reduced quantization, we discover remarkable similarities between Dirac quantization (on the physical state space) and other curved-space quantization schemes. Nevertheless, the Dirac factor ordering retains some dependence on the gauge structure of the theory—it could not have been guessed working strictly within the classical reduced theory. This sheds new light on the Dirac versus reduced quantization debate, as well as curved-space quantization in general.

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Contents

I Introduction 1

II Classical Analysis 10

1 Gauge Theories 10

1.1 Scalar Electrodynamics 12

1.2 Helix Model 18

2 Symmetries on the Extended Phase Space 20

2.1 Energy-Momentum Tensor and Conserved Currents 21

2.2 Poincaré Generators for Scalar Electrodynamics 24

2.3 The Poincaré Algebra 28

3 Classical Reduction 34

3.1 Geometrical Analysis of Classical Reduction 34

3.2 Phase Space Coordinates Adapted to the Gauge Structure 41

3.3 Adapted Coordinates for Scalar Electrodynamics and the Helix Model 45

4 Geometrical Structures in Gauge Theories 49

4.1 "Kaluza-Klein" Form of the Line Element 50

4.2 Horizontal/Vertical Nonholonomic Basis 54

4.3 Yang-Mills Connection and Curvature on the Extended Field Space 58

4.4 Levi-Civita Connections on the Extended and Reduced Field Spaces 64

4.5 Relationships Between the Curvatures of the Various Connections 68

III Quantum Analysis 73

5 Dirac Quantization 73

5.1 Symmetric Operators and Minimal Quantization 73

5.2 Van Hove Anomalies 78

5.3 Selection of the Physical State Space 84

5.4 Application to Scalar Electrodynamics and the Helix Model . . . 88

5.5 Quantum Reduction 92

5.6 Explicit Verification of the Reduced Quantum Poincaré Algebra . 97

6 Dirac versus Reduced: a Geometrical Analysis 107

6.1 Factor Ordering Ambiguity and Other Known Results 107

6.2 Curved-Space Quantization Schemes 110

6.3 Consequences of K Covariantly Constant 114

6.4 Consequences of M Ricci Flat 120

6.5 Comparison Between Dirac Quantization and Curved-Space Quantization Schemes 123

IV Conclusion 131

7 Appendix 140

7.1 Poincaré Charges are Classical Observables 140

7.2 Schouten Concomitants in Poincaré Algebra 143

Part I
Introduction

There are two central topics in our discussions: The first is van Hove anomalies in curved-space quantization, and the second is the Dirac versus reduced quantization factor ordering ambiguity in gauge theories. Note that a curved configuration space can arise when dealing with curved spacetime, but also in a gauge theory, the latter being of interest here. We introduce these topics in turn, and then describe how they fit into the main purpose of the thesis.

Dirac's [1] idea of quantization of a classical system is to find a representation of the Poisson algebra of classical observables by operators acting on some Hilbert space, $\mathcal{H}$. Physical predictions are extracted by using the inner product on $\mathcal{H}$, with respect to which operators should be self-adjoint. To be more precise, consider a $2n$-dimensional phase space γ, with canonical coordinates q^a, p_a, $a = 1 \ldots n$. Besides the usual linearity conditions, the quantization map from classical observables $f, g, \ldots$ (real functions on γ) to quantum observables $\hat{f}, \hat{g}, \ldots$ should satisfy (see, e.g. [2]):

(i) $\frac{1}{i\hbar}[\hat{f}, \hat{g}] = \widehat{\{f, g\}}$,

(ii) $\hat{1} = \mathrm{id}$ (identity operator),

(iii) $\hat{q}^a, \hat{p}_a$ act irreducibly on $\mathcal{H}$,

where $\{\ ,\ \}$ is the classical Poisson bracket and $[\ ,\ ]$ is the commutator bracket of operators.

However, since the work of Groenewold [3] and van Hove [4] it is known that this quantization program *cannot* be achieved consistently for the full Poisson algebra of classical observables—in fact, not even for the Poisson algebra of polynomials on a Euclidean phase space (see Gotay [5] and references therein). A detailed proof can be found in [2]. The breakdown of (i) manifests itself as the appearance of so-called van Hove anomalies.

Roughly speaking, it appears that this problem is being attacked from three different angles, which we will only briefly comment on. The first approach relaxes the condition (i). One gives up the notion of quantizing the classical Poisson algebra, and instead quantizes a modified classical algebra, in particular, a deformation of the Poisson algebra, with deformation parameter $\hbar$ [6,7]. The new Lie product has the form [8]

$$D_\hbar(f,g) = \{f,g\} + \hbar D_1(f,g) + \hbar^2 D_2(f,g) + \cdots, \tag{1}$$

where, like $\{\ ,\ \}$, $D_i : C^\infty(\gamma) \wedge C^\infty(\gamma) \to C^\infty(\gamma)$. [$C^\infty(\gamma)$ denotes smooth functions on γ.] The D_i are not arbitrary, however, and must satisfy certain integrability conditions in order that $D_\hbar$ obey the Jacobi identity:

$$D_\hbar(D_\hbar(f,g),h) + D_\hbar(D_\hbar(h,f),g) + D_\hbar(D_\hbar(g,h),f) = 0. \tag{2}$$

Notice that now $\hbar$ appears, in a sense, in the classical theory, as well as the quantum.

The second approach is the geometric quantization programme (see, e.g. [9,10]). It consists of two steps. In the first step, called prequantization, conditions (i) [and (ii)] are achieved for all classical observables, but at the expense of relaxing the irreducibility condition, (iii). For example, take $\mathcal{H}$ to be the set of complex valued functions ψ on the Euclidean phase space $\gamma = T^*\boldsymbol{R}^1 = \boldsymbol{R}^2$ (canonical coordinates x, p), square integrable with respect to the natural Liouville measure $dx\,dp$. [$T^*\boldsymbol{R}^1$ denotes the cotangent bundle of $\boldsymbol{R}^1$.] Then the prequantization map tells us that

$$\hat{x} = x + i\hbar\frac{\partial}{\partial p},\ \hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{3}$$

These operators are self adjoint and realize (i), but the problem is that by having wavefunctions on *phase space* we lose the uncertainty principle, which is so significant in quantum mechanics [11].

But (following Onofri and Pauri [12]) this representation can be reduced by restricting to a "subspace" defined by

$$\psi(x,p) = \exp(-\frac{i}{\hbar}\lambda p)\chi(x). \tag{4}$$

On this subspace $\hat{x}$ and $\hat{p}$ reduce to

$$\hat{x}' = x + \lambda,\ \hat{p}' = -i\hbar\frac{\partial}{\partial x}, \tag{5}$$

i.e. the usual (irreducible) Schrödinger representation with wavefunctions $\chi(x)$. This reestablishing of the irreducibility criterion, (iii), corresponds to the second step of geometric quantization, called choosing a polarization. Simply speaking, it means restricting the wavefunction to depend on only half of the phase space variables.

Unfortunately, though, one runs into serious difficulties choosing a consistent polarization when quantizing observables of degree two or higher in the momenta [5]. Nevertheless, it *is* possible to quantize the kinetic energy in the vertical polarization (i.e. Schrödinger representation) when $\gamma = T^*m$, even when the configuration space m is a generic Riemannian manifold. The result is [9,10]:

$$\hat{T} = -\frac{\hbar^2}{2}\left(\Delta - \frac{1}{6}\mathcal{R}\right), \tag{6}$$

where Δ is the Laplace-Beltrami operator, and $\mathcal{R}$ is the Ricci scalar of m. This latter curvature piece is what is interesting here: it plays a large role in our later discussions.

The third general response to the van Hove obstructions is to retain (i)-(iii), but be satisfied with quantizing only a restricted class of observables [7,13,14,12, 15]. An example of this type of approach is Weyl's [15] quantization scheme: Consider the n-dimensional Euclidean configuration space $m = \boldsymbol{R}^n$, and corresponding phase space $\gamma = T^*m = \boldsymbol{R}^{2n}$ with canonical coordinates $x^a, p_a,\ a = 1 \dots n$. An arbitrary classical observable f can be decomposed into the 'normal' form

$$f(x,p) = \int d^n\alpha\, d^n\beta\, \phi(\alpha,\beta) \exp\, i(\alpha_a x^a + \beta^a p_a), \tag{7}$$

which is then mapped to the quantum operator

$$\hat{f} = \int d^n\alpha\, d^n\beta\, \phi(\alpha,\beta) \exp\, i(\alpha_a \hat{x}^a + \beta^a \hat{p}_a), \tag{8}$$

where the basic observables

$$\hat{x}^a = x^a,\ \hat{p}_a = -i\hbar \frac{\partial}{\partial x^a} \tag{9}$$

in the Schrödinger representation.

Although this scheme does not preserve (i) in the generic case, it does so for arbitrary f when $g = T = \frac{1}{2}(p_1^2 + \cdots + p_n^2)$, the flat-space kinetic energy. As pointed out by Underhill [13], this means that the classical and quantum dynamics of a free particle on a Euclidean configuration space are identical. Unfortunately, his extension of Weyl's idea to curved configuration spaces does not have this property [13]. However, if m is a space of constant curvature, a modification can be made such that classical and quantum constants of the motion coincide [14]. Notice in this latter case that both the curvature of m, as well as the observables, are restricted. Furthermore, Bloore, Assimakopoulos and Ghobrial [7] have shown that *no* Schrödinger-type[1] quantization scheme exists in which the commutator of a generic quadratic operator with the kinetic energy is free of

[1] This term to be made precise later.

van Hove anomalies, unless m is of constant curvature or Ricci flat: If we want to work with a (nonconstant) curved configuration space, it is necessary (but perhaps not sufficient) that the observables be restricted in some way. Bloore *et. al.* give no hint at a suitable restriction, nor what sort of quantization scheme would be valid within that class of observables (and curvature).

These are some of the difficulties encountered in curved-space quantization. At this point we would just like to emphasize that van Hove obstructions are amplified when the configuration space is curved, and that some kind of restriction on observables and curvature appears to be unavoidable. To our knowledge, there is no known quantization scheme [that retains (i)-(iii)] which avoids van Hove anomalies for spaces of nonconstant and non-Ricci-flat curvature, when commutators of two operators quadratic (or higher order) in the momenta are involved. (See also remarks made by Underhill [13, page 1934].)

The other central topic present in our work is the Dirac versus reduced quantization factor ordering ambiguity in gauge theories. Important examples of gauge theories in physics are QED and QCD. Briefly, a gauge theory has a so-called extended configuration space, M, for which not all configurations are physically distinct—there are redundant, or gauge degrees of freedom (see, e.g. [1,16,17]). These extra degrees of freedom constitute so-called gauge orbits, and can be divided out, resulting in the reduced, or physical configuration space m. (This procedure is called classical reduction). For quantization one then has two choices: One can either quantize on the phase space T^*m (called reduced quantization), or one can quantize on T^*M and then perform a quantum reduction analogous to the corresponding classical reduction (this is usually called Dirac quantization[2]). It is well known that these two quantization procedures generally lead to physically distinct quantum systems [18,19,20,21,22,23]. It is also known that the

[2]The terms reduced and Dirac quantization will be made precise in due course.

distinction can be understood as a difference in factor ordering on the reduced space [23]. Given that both are self-consistent, one needs to invoke a further guiding principle in order to determine which factor ordering (if any) is correct for a given physical system.

For example, Kuchař [22] considers a toy model (the "helix model"), which consists of a particle in $M = \boldsymbol{R}^3$, where points that are related by a screw motion about a fixed axis are considered physically equivalent, and constitute a gauge orbit. He finds that although Dirac and reduced quantization are physically distinct (they give rise to different energy spectra), they are both consistent, and so the helix model cannot resolve the ambiguity. Kuchař also points out that the helix model is a finite dimensional analogue of scalar electrodynamics (the standard Yang-Mills theory of the complex scalar field in flat spacetime). Notice that, unlike the helix model, scalar electrodynamics has the Poincaré symmetry. What originally motivated the thesis was the idea that this extra symmetry, which must be represented in the quantum theory without van Hove anomalies, would restrict the possible factor orderings and thus resolve the Dirac versus reduced quantization ambiguity (at least for this theory, or perhaps even some class of Poincaré invariant gauge theories).

The boost generators, as well as the Hamiltonian, are observables quadratic in the momenta, so the Poincaré algebra involves quadratic-quadratic commutators. Furthermore, it turns out that although M is flat for scalar electrodynamics, the reduced configuration space m has nontrivial curvature [24]. Thus, reduced quantization would essentially be curved-space quantization of quadratic operators which, as discussed above, is highly nontrivial in general. On the other hand, Dirac quantization is less troublesome, and we shall show that it leads to a consistent quantization of the Poincaré symmetry on physical states. We are not aware of any similar, explicit demonstration of this result (in the functional

Schrödinger representation) in the literature (but see, e.g., Itzykson and Zuber [25] for a 'standard' treatment of scalar electrodynamics).

This establishes that the Dirac factor ordering (on the reduced space) yields a consistent curved-space quantization scheme for quadratic operators. Recall that in our introduction to curved-space quantization above we argued that likely both the class of observables and the curvature must be restricted. Here the observables are restricted to a Lie subalgebra of the full Poisson algebra corresponding to the generators of a symmetry group of the classical system (e.g. the Poincaré symmetry). This is consistent with an idea originally stressed by Onofri and Pauri [12] in their paper on dynamical quantization: "The point is that the basic feature of both the classical and quantum systems is not only a Lie structure of observables but also a mapping of observables to one-parameter subgroups of canonical or unitary transformations, respectively". In this case the Heisenberg subalgebra (spanned by $q, p, 1$) no longer plays a central role in quantization.

Furthermore, by working with a gauge theory (e.g. scalar electrodynamics) the curvature on the reduced space is automatically restricted to the special one induced by the gauge orbit structure on the (Ricci) flat M. A novel way of viewing this whole procedure is simply as a systematic means of generating a consistent (albeit restricted) curved-space quantization scheme: by quantizing on M instead of m we ease the problem of van Hove anomalies, but in exchange for additional problems and ambiguities associated with the gauge aspects of the theory, and quantum reduction. These result in special conditions on the observables, which we discuss in detail later.

We then generalize the scalar electrodynamics result to a broader class of observables, in a generic gauge theory (with Ricci flat M). In this wider context, and using a Kaluza-Klein-like analysis of the geometry of gauge theories, we provide a novel interpretation of the Dirac factor ordering: we discover deep connections with other curved-space quantization schemes—connections which reduced quantization simply does not have.

To recap: The first purpose of the thesis is to explicitly verify in a concrete example the (successful) Dirac quantization of a physically relevant symmetry algebra (containing at least quadratic-quadratic commutators). Also, the reduced configuration space should be curved, and the quantum reduction consistent, in order to yield, at least in a restricted sense, a curved-space quantization scheme for quadratic observables. We demonstrate that the Poincaré symmetry of scalar electrodynamics is such an example. Secondly, this provides motivation for generalization, and an in depth geometrical analysis of what makes the Dirac factor ordering so special. We discuss the consequences that these new results have for the Dirac versus reduced question, as well as curved-space quantization in general.

Our discussion is divided into two parts: The first is a comprehensive classical analysis, including in particular a Kaluza-Klein-like study of the geometrical structure of gauge theories in general, which is necessary to understand and interpret the quantum analysis, which forms the second part.

In the first chapter we briefly describe gauge theories with first class constraints linear in the momenta, through the action of a gauge group on the extended configuration space M. These notions are made concrete with the introduction of our two examples: the field theoretic scalar electrodynamics, and its finite dimensional analogue, the helix model.

The second chapter is an analysis of the classical Poincaré symmetry of scalar electrodynamics on M: We define the energy-momentum tensor and the associated Poincaré charges, and verify that the latter are constants of the motion and generate the Poincaré algebra when acting on physical classical states.

A geometrical treatment of classical reduction is given in chapter three, where it is then applied to our two examples. In particular, this establishes the action of the Poincaré group on the reduced phase space. We also introduce coordinates adapted to the gauge structure on M, which are convenient for later work.

In chapter four we analyze the geometrical structure on M, which plays a large role in our interpretation of the Dirac quantization. In particular, we discuss four connections: the first is a Yang-Mills-type connection, and the other three are Levi-Civita-type connections associated with certain metrics (e.g. the metrics on M and m). The curvatures of these connections are related to each other using a Kaluza-Klein-like analysis.

Chapter five is a discussion of Dirac quantization. We explicitly write down the van Hove anomalies, as well as discuss in detail subtleties associated with the gauge structure, and the conditions under which quantum reduction is consistent. We verify that the quantum Poincaré algebra for scalar electrodynamics satisfies these conditions, and is free of any van Hove anomalies, resulting in a realization of this algebra on the quantum physical state space. This establishes a successful quantization scheme for (a restricted class of) observables on a configuration space m with nonconstant (but not arbitrary) curvature.

Finally, in chapter six we discuss generalizations, leading to the novel interpretation of Dirac quantization mentioned above. We then present conclusions, and directions for future research.

Part II
Classical Analysis

1 Gauge Theories

We consider a classical system whose set of configurations can be identified with a manifold, M, of dimension N. A point in this manifold corresponds to an instantaneous configuration of the classical system. Suppose further that not all configurations are physically distinct—there are redundant, or gauge degrees of freedom. In this case we refer to M as the *extended* configuration space. This situation is usually associated with the action of a Lie group (the gauge group) on M, which forms a subgroup of the diffeomorphism group $\mathrm{Diff}(M)$. The infinitesimal gauge transformations are then generated by a set of vector fields ϕ_α on M ($\alpha = 1, ..., C$, the dimension of the gauge group), which satisfy

$$[\phi_\alpha, \phi_\beta] = f^\gamma_{\alpha\beta}\phi_\gamma, \tag{10}$$

where the $f^\gamma_{\alpha\beta}$ are the structure constants of the group (see, e.g. [26]).

As pointed out by Kuchař [21] these vector fields may not all be linearly independent, and the selection of a suitable subset of vector fields that *are* will result in structure *functions*. In any case, the only significance of (10), at least as far as the classical reduction is concerned, is that we learn that the ϕ_α are surface-forming: at each point $Q \in M$ the ϕ_α span a C (or less)-dimensional subspace of T_QM, and these subspaces are integrable, yielding a foliation of M by so called gauge orbits. All points in a given gauge orbit represent physically indistinguishable configurations, and the space of orbits, m, is called the reduced (or physical) configuration space.

This leads us to define, along with Kuchař, a gauge vector space

$$\mathcal{G} := \{\mu = \mu^\alpha \phi_\alpha \mid \mu^\alpha \in C^\infty(M)\} . \tag{11}$$

Any nonzero $\mu \in \mathcal{G}$ is everywhere tangent to the gauge orbits in M. At this point, then, we shall leave the exact nature of the $f^\gamma_{\alpha\beta}$ unspecified, requiring for further analysis only the gauge orbit structure encoded in $\mathcal{G}$. However, when we come to quantization we shall find that the gauge group structure encoded in the $f^\gamma_{\alpha\beta}$ will play a larger role.

In the Hamiltonian analysis the geometrical setting is either the extended phase space, $\Gamma := T^*M$, or the reduced phase space, $\gamma := T^*m$. In the former, the notion of first class constraints linear in the momenta arises as follows (see, e.g. [27]): Let q label a point in m, i.e. an orbit in M, and $Q \in M$ any point on that orbit (representing it). Then $T^*_q m$ would be the space of linear functions of variations, δQ of Q which depend only on which orbit the point $Q + \delta Q$ lies in. In other words, the allowed momenta $P \in T^*_Q M$ must satisfy

$$\langle P, \delta Q + \epsilon\mu \rangle = \langle P, \delta Q \rangle \;\forall\, \delta Q, \tag{12}$$

where ϵ is an arbitrary infinitesimal, and μ is any vector in $\mathcal{G}$ at Q. This linear condition on the momenta is equivalent to

$$\langle P, \mu \rangle = 0, \tag{13}$$

which defines a constraint surface in Γ. (An explicit local representation of γ as a surface in Γ would depend on which points Q in M were used to represent their respective orbits, q.) These constraints then canonically generate the gauge transformations of states in the phase space. These gauge transformations should preserve the constraints, so their Poisson algebra should close, i.e. the constraints should be first class.

1.1 Scalar Electrodynamics

To make this discussion more concrete we now introduce two classical systems as examples, which we shall turn to for inspiration and illustration throughout our work. The first is field theoretic: scalar electrodynamics, and as such will involve infinite-dimensional manifolds. For simplicity we shall not deal with possible subtleties in this regard, nor will we regularize the usual distributions that appear. To at least in part justify this approach we have chosen for the second example a finite-dimensional toy model which mimics many of the features of scalar electrodynamics. This is the helix model, briefly mentioned by DeWitt [28], and developed in detail by Kuchař [21,22]. The two models share essentially the same geometrical framework, so that a mathematically rigorous treatment of one lends credence to the other. We shall discuss the connection between the two more fully below.

The Lagrangian density for scalar electrodynamics in flat spacetime is

$$\mathcal{L} = \frac{1}{2}(D_\mu \varphi)\overline{(D^\mu \varphi)} - U - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{14}$$

where φ is a complex scalar field and U is a potential, for example a mass or self interaction term, which depends only on $|\varphi|$. The indices $\mu, \nu = 0, 1, 2, 3$, $x := (t, \boldsymbol{x})$ refer to a fixed inertial frame, and we use signature $(+\,-\,-\,-)$. The covariant derivative is $D_\mu := \partial_\mu + ieA_\mu$, with corresponding electromagnetic field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. The configuration of the system at time instant t is described by the electromagnetic potential $A_\mu(t, \boldsymbol{x})$ and the complex scalar field $\varphi(t, \boldsymbol{x}) =: \xi(t, \boldsymbol{x}) + i\eta(t, \boldsymbol{x})$, where ξ and η are real fields.

The Lagrangian $L(t) := \int d^3x \mathcal{L}(t, \boldsymbol{x})$ obtained from the Lagrangian density (14) can be cast into the following form:

$$L(\lambda, Q, \dot{Q}) = \frac{1}{2}G_{AB}(Q)\left(\dot{Q}^A - \lambda^\alpha \phi^A_\alpha(Q)\right)\left(\dot{Q}^B - \lambda^\beta \phi^B_\beta(Q)\right) - V(Q), \tag{15}$$

where we are using DeWitt's condensed notation (and suppressing the time argument) as follows:

$$
\begin{aligned}
Q^A &:= (A_i(\boldsymbol{x}), \xi(\boldsymbol{x}), \eta(\boldsymbol{x}))\,, && (16)\\
\lambda^\alpha &:= -eA_0(\boldsymbol{x}), && (17)\\
G_{AB}(Q) &:= \begin{pmatrix} \delta^{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \delta(\boldsymbol{x}-\boldsymbol{y}), && (18)\\
V(Q) &:= \int d^3x \left\{ \frac{1}{4}(F_{ij})^2 + \frac{1}{2}(\partial_i\xi - eA_i\eta)^2 + \frac{1}{2}(\partial_i\eta + eA_i\xi)^2 + U \right\}, && (19)\\
\phi^A_\beta(Q) &:= \left(-\frac{1}{e}\partial_{x^i}\delta(\boldsymbol{x}-\boldsymbol{y}), -\eta(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}), \xi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}) \right). && (20)
\end{aligned}
$$

Here the indices $A, B, ... := (I, \alpha), (J, \beta), ...$, with discrete part $I := (i, \xi, \eta)$, i running from 1 to 3, and continuous part $\alpha := \boldsymbol{x}$ running over $\boldsymbol{R}^3$. Thus contractions of A, B type indices involve ordinary discrete contractions of I, J type indices, as well as integrations over α, β type indices. $\dot{Q}^A$ denotes the time derivative of the Q^A configuration variables.

The advantage of the condensed notation is two-fold: It clarifies the inherently geometrical nature of the classical and quantum analysis and, secondly, the results we derive have greater generality, applying to any theory that can be put into the form of (15).

We initially take the configuration space to be the infinite-dimensional manifold with (global) coordinates (Q^A, λ^α). The phase space will then be the corresponding cotangent bundle with naturally induced momentum coordinates (P_A, κ_α) conjugate to the (Q^A, λ^α). Applying the Dirac-Bergmann algorithm (see, e.g. [1,16,17]) we obtain a map from the velocities to the momenta:

$$
\begin{aligned}
P_A &:= \frac{\partial L}{\partial \dot{Q}^A} = G_{AB}\left(\dot{Q}^B - \lambda^\beta \phi^B_\beta\right), && (21)\\
\kappa_\alpha &:= \frac{\partial L}{\partial \dot{\lambda}^\alpha} = 0. && (22)
\end{aligned}
$$

The first set of these equations can be inverted to solve for $\dot{Q}^A$ and the second are primary constraints. For scalar electrodynamics these read

$$\begin{aligned}
\Pi_{A_i(\boldsymbol{x})} &= \frac{\delta L}{\delta \dot{A}_i(\boldsymbol{x})} = \dot{A}_i(\boldsymbol{x}) - \partial_{x^i} A_0(\boldsymbol{x}) = F_{0i}(\boldsymbol{x}), && (23)\\
\Pi_{\xi(\boldsymbol{x})} &= \frac{\delta L}{\delta \dot{\xi}(\boldsymbol{x})} = \dot{\xi}(\boldsymbol{x}) - eA_0(\boldsymbol{x})\eta(\boldsymbol{x}), && (24)\\
\Pi_{\eta(\boldsymbol{x})} &= \frac{\delta L}{\delta \dot{\eta}(\boldsymbol{x})} = \dot{\eta}(\boldsymbol{x}) + eA_0(\boldsymbol{x})\xi(\boldsymbol{x}); && (25)\\
\Pi_{A_0(\boldsymbol{x})} &= \frac{\delta L}{\delta \dot{A}_0(\boldsymbol{x})} = 0, && (26)
\end{aligned}$$

where $\delta / \delta \dot{A}_i(\boldsymbol{x})$, etc., denotes functional derivative.

The canonical Hamiltonian generating time evolution is

$$H(\lambda, Q, P) = H_0(Q,P) + \lambda^\alpha C_\alpha(Q,P), \tag{27}$$

where

$$\begin{aligned}
H_0(Q,P) &:= \frac{1}{2} G^{AB}(Q) P_A P_B + V(Q)\\
&= \int d^3x \left\{ \frac{1}{2}\left(\Pi^2_{A_i(\boldsymbol{x})} + \Pi^2_{\xi(\boldsymbol{x})} + \Pi^2_{\eta(\boldsymbol{x})}\right)\right.\\
&\quad \left. + \frac{1}{4}(F_{ij})^2 + \frac{1}{2}(\partial_i \xi - eA_i\eta)^2 + \frac{1}{2}(\partial_i \eta + eA_i \xi)^2 + U \right\} && (28)
\end{aligned}$$

in the case of scalar electrodynamics. Here G^{AB} denotes the matrix inverse of G_{AB}. The λ^α are not determined by the equations of motions, but rather are Lagrange multipliers enforcing the Gauss law constraints:

$$C_\alpha(Q,P) := \phi^B_\alpha(Q) P_B = 0. \tag{29}$$

For scalar electrodynamics these are

$$C_\alpha(Q,P) = \frac{1}{e}\partial_{x^i}\Pi_{A_i(\boldsymbol{x})} - \eta(\boldsymbol{x})\Pi_{\xi(\boldsymbol{x})} + \xi(\boldsymbol{x})\Pi_{\eta(\boldsymbol{x})}. \tag{30}$$

These secondary constraints preserve the κ_α under time evolution.

Hence we may eliminate the trivial canonical pairs $(\lambda^\alpha, \kappa_\beta)$ from the phase space. This represents a 'partial reduction' of the classical system, resulting in the following equivalent system: The extended configuration space, M, is the infinite-dimensional manifold with (global) coordinates Q^A. The extended phase space $\Gamma = T^*M$ with canonical coordinates $\Xi^I := (Q^A, P_A)$. The Poisson bracket is denoted by $\{,\}$.

The Hamiltonian in (27) canonically generates time evolution,

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}, \tag{31}$$

for any dynamical variable, i.e. function F on Γ (which may depend explicitly on time). With the λ^α undetermined the time evolution is arbitrary up to gauge transformations canonically generated by the C_α:

$$\begin{aligned} \delta_\lambda Q^A &= \{Q^A, \lambda^\alpha C_\alpha\} = \lambda^\alpha \phi^A_\alpha(Q), \\ \delta_\lambda P_A &= \{P_A, \lambda^\alpha C_\alpha\} = -\lambda^\alpha \partial_A \phi^B_\alpha(Q) P_B, \end{aligned} \tag{32}$$

for infinitesimal λ^α (here $\partial_A := \partial/\partial Q^A$). For scalar electrodynamics these read

$$\begin{array}{ll} \delta_\lambda A_i(\boldsymbol{x}) = -\frac{1}{e}\partial_{x^i}\lambda(\boldsymbol{x}) & \delta_\lambda \Pi_{A_i(\boldsymbol{x})} = 0 \\ \delta_\lambda \xi(\boldsymbol{x}) = -\lambda(\boldsymbol{x})\eta(\boldsymbol{x}) & \delta_\lambda \Pi_{\xi(\boldsymbol{x})} = -\lambda(\boldsymbol{x})\Pi_{\eta(\boldsymbol{x})} \\ \delta_\lambda \eta(\boldsymbol{x}) = \lambda(\boldsymbol{x})\xi(\boldsymbol{x}) & \delta_\lambda \Pi_{\eta(\boldsymbol{x})} = \lambda(\boldsymbol{x})\Pi_{\xi(\boldsymbol{x})} \end{array}. \tag{33}$$

The projection of these gauge transformations in Γ down to M are diffeomorphisms generated by the vector fields $\phi_\alpha = \phi^A_\alpha \partial_A$. We thus identify the ϕ_α with those discussed at the beginning of the chapter.

It might be well at this point to mention that the ϕ_α are in fact linearly independent, at least off of the $\xi = \eta = 0$ axis. This is easy to see by considering an arbitrary linear combination of the gauge vectors in (20):

$$f^\beta \phi^A_\beta(Q) \;=\; \int d^3y f(\boldsymbol{y}) \left(-\frac{1}{e}\partial_{x^i}\delta(\boldsymbol{x}-\boldsymbol{y}), -\eta(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}), \xi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\right)$$

$$= \left(-\frac{1}{e}\partial_{x^i} f(x), -\eta(x)f(x), \xi(x)f(x)\right). \tag{34}$$

For this vector to vanish we require the function f = constant, and unless $\xi = \eta = 0$, this constant must be zero.

The C_α also define a constraint surface $\Gamma_C \subset \Gamma$ of physical states via $C_\alpha \approx 0$ [see (29); henceforth $\approx$ will denote equality on restriction to Γ_C]. But Γ_C can equally well be defined by [cf (13)]

$$\mu^A P_A \approx 0 \; \forall \mu \in \mathcal{G}. \tag{35}$$

For Γ_C to be preserved under time evolution we need the C_α to be constants of the motion on Γ_C:

$$0 \approx \{C_\alpha, H\} = -\frac{1}{2}(\mathcal{L}_{\phi_\alpha} G)^{AB} P_A P_B - \lambda^\beta (\mathcal{L}_{\phi_\alpha}\phi_\beta)^A P_A - \mathcal{L}_{\phi_\alpha} V, \tag{36}$$

where $\mathcal{L}_{\phi_\alpha}$ denotes Lie derivative with respect to ϕ_α. It is easy to see that this is equivalent to the $\mu^A P_A$ being constants of the motion on Γ_C. In the spirit of our earlier discussion this means that at least these classical considerations do not depend on the additional information in the gauge algebra structure over the gauge orbit structure. Returning to (36) we see that we have three conditions relating the gauge structure and the Hamiltonian.

First, the ϕ_α must satisfy the closure relations (10) for arbitrary $f^\gamma_{\alpha\beta}$, i.e. the C_α must be first class. This means that the gauge orbits (32) of points in Γ_C remain in Γ_C and, furthermore, are surface-forming therein. These gauge orbits in Γ_C project down to the gauge orbits in M discussed earlier. The set of gauge orbits in Γ_C is called the reduced phase space, γ, and in fact $\gamma = T^*m$. This will be discussed more fully in section 3.2 on classical reduction. Of course scalar electrodynamics is an abelian gauge theory, and so our example satisfies the first condition with vanishing structure constants. This is easily verified by direct calculation using (20).

Second, the potential V in the Hamiltonian must be gauge invariant, in other words constant along the gauge orbits in M:

$$\mathcal{L}_{\phi_\alpha} V = 0. \tag{37}$$

This can also be verified for our scalar electrodynamics example (see appendix). Alternatively, we shall see this explicitly in section 3.3 when we express V as a function of gauge invariant coordinates.

Before we go on to the third condition we note that the kinetic energy in the Hamiltonian does not contain a quadratic κ_α piece (indeed $\kappa_\alpha = 0$ defines the primary constraint surface), and so it was necessary to eliminate the $(\lambda^\alpha, \kappa_\alpha)$ pair to produce a nondegenerate kinetic term, which in turn provides a natural metric G_{AB} on M. In fact, inspection of (18) reveals that M is flat for scalar electrodynamics, but we shall not make use of this fact before we address factor ordering problems in Dirac quantization. So until then we shall entertain an arbitrary metric on M.

Thus, the third condition requires that the ϕ_α satisfy a 'projected Killing equation':

$$(\mathcal{L}_{\phi_\alpha} G)^{AB} = \zeta_\alpha^{\beta(A} \phi_\beta^{B)}, \tag{38}$$

where $\zeta_\alpha^\beta = \zeta_\alpha^{\beta A} \partial_A$ are arbitrary vector fields on M, and (AB) denotes symmetrization of indices. For scalar electrodynamics it can be shown (see appendix) that the ϕ_α are in fact Killing, so the ζ_α^β are zero, but this is more restrictive than necessary, at least for our present classical considerations. But we do note that (38) is an important relation between the gauge structure and the metric structure on M, and the ζ_α^β will play a larger role in our later work.

1.2 Helix Model

We now turn to the helix model discussed in [22]. Take the extended configuration space $M = \boldsymbol{R}^3$, with Cartesian coordinates $Q^A = (X, Y, Z)$; we may imagine this corresponding to a nonrelativistic particle of unit mass in ordinary three-space. Let the one parameter translation group act on M by helical gauge transformations:

$$\begin{aligned} X(\lambda) &= X(0)\cos(\lambda) - Y(0)\sin(\lambda), \\ Y(\lambda) &= X(0)\sin(\lambda) + Y(0)\cos(\lambda), \\ Z(\lambda) &= Z(0) + \lambda. \end{aligned} \tag{39}$$

These are generated by the vector field $\phi_1 = \phi_1^A \partial_A$ (α just takes the single value, 1), where

$$\phi_1^A(Q) = \left.\frac{\partial Q^A(\lambda)}{\partial \lambda}\right|_{\lambda=0} = (-Y, X, 1). \tag{40}$$

With an extended phase space $\Gamma = T^*M$ we take the Hamiltonian to be

$$H(\lambda, Q, P) = \frac{1}{2} G^{AB}(Q) P_A P_B + \lambda^1 C_1(Q, P) + V(Q), \tag{41}$$

where λ^1 is a Lagrange multiplier enforcing the 'Gauss law' constraint

$$C_1(Q, P) := \phi_1^A(Q) P_A = P_Z - Y P_X + X P_Y = 0. \tag{42}$$

We take the potential V to be constant along the gauge orbits and, to model scalar electrodynamics, assign a flat metric $G_{AB} = \delta_{AB}$ on M. Then ϕ_1, being a combination of translation and rotation, is a Killing vector.

Clearly the helix model is not as rich as scalar electrodynamics, but every feature of the former has an analogue in the latter. For example, the translations and rotations of ϕ_1 are just copied an infinite number of times, once for each α [see

(20)], to produce, respectively, the gauge transformations of the electromagnetic potential and the corresponding phase rotation of the complex scalar field in scalar electrodynamics. On the other hand, Poincaré invariance is an important symmetry of scalar electrodynamics not shared by the helix model. Indeed, one of the initial motivations of this work was to exploit this extra symmetry to help clarify some of the issues in Dirac versus reduced quantization. This is in response to a comment made by Kuchař in [22]: "Standard quantization of scalar electrodynamics corresponds to the 'naive' [Dirac] factor ordering. One may wonder whether the 'physical' [reduced] ordering leads to higher order corrections which could be in principle detectable".

2 Symmetries on the Extended Phase Space

A classical observable in a gauge theory is a gauge invariant function of the classical states on the constraint surface. These observables form a Lie algebra with Lie product given by the Poisson bracket. Dirac [1] proposed that quantization of such a system corresponds to a representation of this algebra by operators on a Hilbert space of states. It is well known that serious difficulties such as factor ordering ambiguities and van Hove obstructions ([4]; [2] and references therein) arise in this programme, not to mention additional ambiguities inherent in constrained systems [18,19,20,21,22,23].

We discussed some of the research into these problems in the introduction, and there we argued in favour of dynamical quantization [12]: restrict the set of classical observables to a subalgebra of the full Poisson algebra associated with the generators of a symmetry group of the classical system. Although it can happen that some classical symmetries are not realized as quantum symmetries (and vice-versa), we do not anticipate problems with the Poincaré (and gauge) groups of scalar electrodynamics, which is what we shall consider in this work. Indeed, scalar electrodynamics is quantized in textbooks (e.g. [25]) with manifest Poincaré symmetry—what is novel here is that by quantizing (in the Schrödinger representation) a gauge theory with an additional symmetry (that has pieces quadratic in the momenta), we uncover a deep connection between the Dirac quantization (on the reduced state space) and other proposed quantization schemes on curved configuration manifolds, which in turn leads to some geometrical insights into the Dirac versus reduced quantization debate.

2.1 Energy-Momentum Tensor and Conserved Currents

The construction of the Poincaré generators begins with a determination of the energy-momentum tensor (see e.g. [27]). Consider a curved spacetime, $(\mathcal{M}, g)$, where g is the metric. Let ϵ be the natural volume element associated with g, and ∇ be the Levi-Civita connection. Then $\nabla\epsilon = 0$, which implies

$$\mathcal{L}_V \epsilon = \epsilon \nabla \cdot V, \tag{43}$$

where $\mathcal{L}_V$ denotes Lie derivative with respect to the vector field V on $\mathcal{M}$, and $\nabla\cdot$ denotes (Levi-Civita) divergence. In particular, then, $\int \epsilon \nabla \cdot V = 0$ (assuming no contribution from boundary integrals).

Now consider some tensor fields Φ_i on $\mathcal{M}$, where i labels the various tensors, which are the dynamical variables of our physical theory. For scalar electrodynamics these consist of the covector field A and the scalar field φ. The Lagrangian density is taken to be a scalar, $\mathcal{L}(g, \Phi, \nabla\Phi)$, which is constructed out of the metric, the fields and their derivatives. It is usually obtained from the Lagrangian density for Minkowski spacetime by replacing the flat metric η with g, and ∂ with ∇ ('minimal coupling'). For scalar electrodynamics [cf (14)] we have

$$\mathcal{L} = \frac{1}{2} g^{ab} \left(\nabla_a \varphi + ieA_a\varphi\right) \overline{\left(\nabla_b\varphi + ieA_b\varphi\right)} - U - g^{ac} g^{bd} \nabla_{[a} A_{b]} \nabla_{[c} A_{d]}, \tag{44}$$

where $[ab]$ denotes antisymmetrization of the abstract spacetime indices a, b. Note that in this case the '$\nabla\Phi$' terms in $\mathcal{L}$ do not depend on g.

Here, of course, g is not a dynamical variable—we are simply setting our theory against a curved background spacetime. The action is then

$$S = \int_{\mathcal{M}} \epsilon \mathcal{L}, \tag{45}$$

a functional of g and Φ. A local variation $\delta\Phi_i$ of the fields about the classical solution Φ_i yields the equations of motion in the usual way:

$$
\begin{aligned}
0 &= \delta_\Phi S = \int_{\mathcal{M}} \epsilon \left\{ \frac{\partial \mathcal{L}}{\partial \Phi_i} \cdot \delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \nabla\Phi_i} \cdot \delta\nabla\Phi_i \right\} = \int_{\mathcal{M}} \epsilon \left\{ \frac{\partial \mathcal{L}}{\partial \Phi_i} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla\Phi_i} \right) \right\} \cdot \delta\Phi_i \\
&\Longrightarrow \frac{\partial \mathcal{L}}{\partial \Phi_i} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla\Phi_i} \right) = 0, \qquad (46)
\end{aligned}
$$

(where '$\cdot$' denotes the appropriate tensor contractions).

The variation of S with respect to the metric does not vanish:

$$\delta_g S =: -\frac{1}{2} \int \epsilon T^{ab} \delta g_{ab}. \qquad (47)$$

This defines the tensor T^{ab}; the proportionality factor, $-1/2$, is chosen so we may later identify T^{ab} with the energy-momentum tensor. Also, with δg_{ab} symmetric we make T^{ab} unique by taking it to be symmetric too. Note that

$$\delta\epsilon = \frac{1}{2} \epsilon g^{ab} \delta g_{ab} \qquad (48)$$

and

$$\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd} \qquad (49)$$

so for scalar electrodynamics we obtain, from (44),

$$T^{ab} = \left\{ (\nabla^a \varphi + ieA^a \varphi) \overline{(\nabla^b \varphi + ieA^b \varphi)} - F^{ac} F_c^{\ b} - g^{ab} \mathcal{L} \right\}. \qquad (50)$$

Under a diffeomorphism generated by an arbitrary vector field V^a on $\mathcal{M}$,

$$
\begin{aligned}
\delta\Phi_i &= \mathcal{L}_{\zeta V} \Phi_i & (51) \\
\delta g_{ab} &= (\mathcal{L}_{\zeta V} g)_{ab} = -2\zeta \nabla_{(a} V_{b)}, & (52)
\end{aligned}
$$

where ζ is an infinitesimal parameter. Since S is a scalar it is invariant under this combined variation:

$$0 = \delta_\Phi S + \delta_g S. \qquad (53)$$

Taking the variation to be about the classical solution Φ_i eliminates the first term on the right hand side [see (46)], which leaves us with [see (47)]

$$0 = \int \epsilon T^{ab} \nabla_{(a} V_{b)} = -\int \epsilon \nabla_a (T^{ab}) V_b. \tag{54}$$

The arbitrariness of V^a then implies the conservation of T^{ab}:

$$\nabla_a T^{ab} = 0. \tag{55}$$

Now suppose that the spacetime admits a Killing vector W^a, i.e. $\nabla_{(a} W_{b)} = 0$, then the vector field $J^a := T^{ab} W_b$ is a conserved current (see, e.g. [29]):

$$\nabla_a J^a = \nabla_a (T^{ab}) W_b + T^{ab} \nabla_a W_b = 0, \tag{56}$$

using the symmetry of T^{ab}.

Flat spacetime $g = \eta$ [signature $(+---)$] admits the Poincaré group of isometries generated by a set of ten basis vectors ${}^{\mathcal{Q}}W^a$ ($\mathcal{Q}$ labelling the basis elements). Fixing an inertial frame x^μ we can choose the basis as follows: Translations

$${}^{\mathcal{P}^\mu} W^\rho := \eta^{\mu\rho}, \; ({}^{\mathcal{P}^\mu} W = {}^{\mathcal{P}^\mu} W^\rho \partial_\rho), \tag{57}$$

and Lorentz transformations

$${}^{\mathcal{J}^{\mu\nu}} W^\rho = -{}^{\mathcal{J}^{\nu\mu}} W^\rho := x^\mu \eta^{\nu\rho} - x^\nu \eta^{\mu\rho}, \; ({}^{\mathcal{J}^{\mu\nu}} W = {}^{\mathcal{J}^{\mu\nu}} W^\rho \partial_\rho). \tag{58}$$

Their algebra is

$$\left[{}^{\mathcal{P}^\mu} W, {}^{\mathcal{P}^\nu} W\right] = 0, \tag{59}$$

$$\left[{}^{\mathcal{J}^{\mu\nu}} W, {}^{\mathcal{P}^\rho} W\right] = \eta^{\nu\rho}\, {}^{\mathcal{P}^\mu} W - \eta^{\mu\rho}\, {}^{\mathcal{P}^\nu} W, \tag{60}$$

$$\left[{}^{\mathcal{J}^{\mu\nu}} W, {}^{\mathcal{J}^{\rho\sigma}} W\right] = \eta^{\mu\sigma}\, {}^{\mathcal{J}^{\nu\rho}} W - \eta^{\nu\sigma}\, {}^{\mathcal{J}^{\mu\rho}} W + \eta^{\nu\rho}\, {}^{\mathcal{J}^{\mu\sigma}} W - \eta^{\mu\rho}\, {}^{\mathcal{J}^{\nu\sigma}} W. \tag{61}$$

The corresponding currents are

$$
\begin{aligned}
{}^{\mathcal{P}^\mu} J^\rho &= T^{\rho\sigma}\,{}^{\mathcal{P}^\mu} W_\sigma = T^{\rho\mu}, && (62)\\
{}^{\mathcal{J}^{\mu\nu}} J^\rho &= T^{\rho\sigma}\,{}^{\mathcal{J}^{\mu\nu}} W_\sigma = x^\mu T^{\rho\nu} - x^\nu T^{\rho\mu}, && (63)
\end{aligned}
$$

and conserved charges

$$
\begin{aligned}
\mathcal{P}^\mu &:= \int d^3x\, {}^{\mathcal{P}^\mu} J^0 = \int d^3x\, T^{0\mu}, && (64)\\
\mathcal{J}^{\mu\nu} &:= \int d^3x\, {}^{\mathcal{J}^{\mu\nu}} J^0 = \int d^3x\, \left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right). && (65)
\end{aligned}
$$

As dynamical variables these should be constants of the motions and should canonically generate Poincaré transformations on the classical states, reproducing the same algebra as (59–61), except with respect to the Poisson bracket (at least on the constraint surface Γ_C). We now verify this, and along the way establish the geometric notation we shall find useful later.

2.2 Poincaré Generators for Scalar Electrodynamics

Specializing (50) to flat spacetime we find

$$
\begin{aligned}
T^{00} &= \frac{1}{2}\left(\Pi_{A_i}^2 + \Pi_\xi^2 + \Pi_\eta^2\right) + \mathcal{V}(A_i,\xi,\eta), && (66)\\
T^{0i} &= -\Pi_{A_j} F_{ij} - \Pi_\xi\left(\partial_i\xi - eA_i\eta\right) - \Pi_\eta\left(\partial_i\eta + eA_i\xi\right), && (67)
\end{aligned}
$$

where $\mathcal{V}(A_i,\xi,\eta)$ is the integrand in (19). Thus, as expected, the Poincaré generator of time translation is the Hamiltonian:

$$
\mathcal{P}^0 = H_0, \tag{68}
$$

with H_0 given in (28).

After an integration by parts the spatial translation generators turn out to be

$$
\begin{aligned}
\mathcal{P}^k &= \int d^3x\, \left\{-\Pi_{A_l}\partial_k A_l - \Pi_\xi \partial_k\xi - \Pi_\eta\partial_k\eta - eA_k C_\alpha\right\}\\
&=: {}^{\mathcal{P}^k} V^A(Q) P_A && (69)
\end{aligned}
$$

in the condensed notation of section 1.1; C_α is given in (30). The vector field ${}^{\mathcal{P}^k}V = {}^{\mathcal{P}^k}V^A\partial_A$ on M naturally divides into two pieces

$$ {}^{\mathcal{P}^k}V =: {}^{\mathcal{P}^k}V_0 + {}^{\mathcal{P}^k}F^\gamma\phi_\gamma, \tag{70} $$

where the vector field components

$$ {}^{\mathcal{P}^k}V_0^A(Q) = (-\partial_{x^k}A_i(\boldsymbol{x}), -\partial_{x^k}\xi(\boldsymbol{x}), -\partial_{x^k}\eta(\boldsymbol{x})) = -\partial_{x^k}Q^A, \tag{71} $$

and the scalar field

$$ {}^{\mathcal{P}^k}F^\gamma(Q) = -eA_k(\boldsymbol{z}), \tag{72} $$

and ϕ_γ are the gauge vectors defined in (20).

The six independent $\mathcal{J}^{\mu\nu}$ can be split into three rotation generators

$$ \mathcal{J}^k := \frac{1}{2}[kmn]\mathcal{J}^{mn} \tag{73} $$

and three boost generators

$$ \mathcal{K}^k := \mathcal{J}^{0k}, \tag{74} $$

where $[kmn]$ is the completely antisymmetric symbol in three dimensions, with $[123] = 1$. After an integration by parts the rotation generators take the form

$$ \begin{aligned} \mathcal{J}^k &= \int d^3x\,[kmn]\,\{x^m\,[-\Pi_{A_l}\partial_n A_l - \Pi_\xi\partial_n\xi - \Pi_\eta\partial_n\eta - eA_nC_\alpha] - \Pi_{A_m}A_n\} \\ &=: {}^{\mathcal{J}^k}V^A(Q)P_A. \end{aligned} \tag{75} $$

The vector field ${}^{\mathcal{J}^k}V$ is also naturally composed of two pieces:

$$ {}^{\mathcal{J}^k}V =: {}^{\mathcal{J}^k}V_0 + {}^{\mathcal{J}^k}F^\gamma\phi_\gamma, \tag{76} $$

where

$$ {}^{\mathcal{J}^k}V_0^A(Q) = [kmn]\,(-x^m\partial_{x^n}A_i(\boldsymbol{x}) - \delta_i^m A_n(\boldsymbol{x}), -x^m\partial_{x^n}\xi(\boldsymbol{x}), -x^m\partial_{x^n}\eta(\boldsymbol{x})) \tag{77} $$

$$ {}^{\mathcal{J}^k}F^\gamma(Q) = -e[kmn]z^mA_n(\boldsymbol{z}). \tag{78} $$

Finally, the boost generators are

$$\mathcal{K}^k = t\mathcal{P}^k - {}^0\mathcal{K}^k, \tag{79}$$

where we have split off the linear (in momenta) spatial translation piece, $t\mathcal{P}^k$, and defined the quadratic piece

$$\begin{aligned} {}^0\mathcal{K}^k &:= \int d^3x\, x^k \left\{\frac{1}{2}\left(\Pi_{A_i}^2 + \Pi_\xi^2 + \Pi_\eta^2\right) + \mathcal{V}(A_i, \xi, \eta)\right\} \\ &=: \frac{1}{2}{}^{\mathcal{K}^k}K^{AB}(Q)P_A P_B + {}^{\mathcal{K}^k}Z(Q). \end{aligned} \tag{80}$$

${}^{\mathcal{K}^k}K$ is a symmetric tensor on M whose components in the Cartesian coordinates Q^A are

$${}^{\mathcal{K}^k}K^{AB}(Q) = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{2}(x^k + y^k)\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{81}$$

Note that they are actually field independent; compare with the metric G_{AB} in (18). The scalar potential term

$${}^{\mathcal{K}^k}Z(Q) = \int d^3x\, x^k \mathcal{V}(A_i, \xi, \eta) \tag{82}$$

is analogous to the potential $V(Q)$ in the Hamiltonian.

We now establish more notation (borrowed from [7]), which will summarize our results so far and give our further analysis geometrical clarity. Let $T^{(s)}M$ be the space of real C^∞ symmetric contravariant tensor fields S on M with valence s. Define

$$\mathcal{C}_s(S) := S^{A_1 \cdots A_s}(Q)P_{A_1} \cdots P_{A_s} \tag{83}$$

as the homogeneous classical dynamical variable on Γ associated with the tensor S on M. The Poisson bracket can be expressed as

$$\{\mathcal{C}_s(S), \mathcal{C}_t(T)\} = \mathcal{C}_{s+t-1}(-[\![S, T]\!]), \tag{84}$$

where

$$\begin{aligned}
[\![S,T]\!]^{A_1\cdots A_{s+t-1}} &:= sS^{A(A_1\cdots A_{s-1}}\nabla_A T^{A_s\cdots A_{s+t-1})} - tT^{A(A_1\cdots A_{t-1}}\nabla_A S^{A_t\cdots A_{s+t-1})} \\
&\in T^{(s+t-1)}M \qquad (85)
\end{aligned}$$

is the Schouten concomitant [30]. It is antisymmetric in S, T, and independent of derivative operator ∇; usually we use ∂_A, the one associated with the coordinates Q^A. In case $S \in T^{(1)}M$ it reduces to the Lie derivative. One can list several other useful properties of this bracket [31], the most important of which is that it satisfies the Jacobi identity.

Now we can write

$$\begin{aligned}
\mathcal{P}^0 &= \mathcal{C}_2(\frac{1}{2}G^{-1}) + \mathcal{C}_0(V), &(86)\\
\mathcal{P}^k &= \mathcal{C}_1({}^{\mathcal{P}^k}V), &(87)\\
\mathcal{J}^k &= \mathcal{C}_1({}^{\mathcal{J}^k}V), &(88)\\
\mathcal{K}^k &= -\mathcal{C}_2(\frac{1}{2}{}^{\mathcal{K}^k}K) - \mathcal{C}_0({}^{\mathcal{K}^k}Z) + t\mathcal{P}^k. &(89)
\end{aligned}$$

Also, the constraints appear as

$$C_\gamma = \mathcal{C}_1(\phi_\gamma). \qquad (90)$$

Now, for a dynamical variable F to be an observable its restriction to Γ_C must be gauge invariant:

$$\{F, \mathcal{C}_1(\phi_\gamma)\} \approx 0 \ \forall\gamma. \qquad (91)$$

It is easy to see that this is equivalent to $\{F, \mathcal{C}_1(\mu)\} \approx 0 \ \forall\mu \in \mathcal{G}$ [recall (11)], so we need only concern ourselves with (91). With the Poincaré charges being a measure of field energy, momentum, angular momentum, and so on, we certainly expect them to be observables. We now verify this. The Hamiltonian was already dealt with in section 1.1 in the context of preservation of Γ_C under time evolution:

$$\left\{\mathcal{P}^0, \mathcal{C}_1(\phi_\gamma)\right\} = \mathcal{C}_2(\frac{1}{2}\mathcal{L}_{\phi_\gamma}G^{-1}) + \mathcal{C}_0(\mathcal{L}_{\phi_\gamma}V) = 0 \tag{92}$$

by virtue of (38) and (37). (Recall that ϕ_γ is actually Killing in scalar electrodynamics so we have $=$ in place of $\approx$ on the right hand side). The results for the other Poincaré charges are similar.

In fact, the Lie derivative with respect to ϕ_γ of every valence zero, one and two tensor that appears on the right hand side of (86– 89) vanishes. We defer these details to the appendix. We would just like to point out that these facts are stronger than necessary, but they certainly simplify the quantization. For example, for the classical analysis it is sufficient that

$$\left(\mathcal{L}_{\phi_\gamma}{}^{\mathcal{K}^k}K\right)^{AB} = {}^{\mathcal{K}^k}\zeta_\gamma^{\beta(A}\phi_\beta^{B)} \tag{93}$$

for arbitrary vector fields ${}^{\mathcal{K}^k}\zeta_\gamma^{\beta} = {}^{\mathcal{K}^k}\zeta_\gamma^{\beta A}\partial_A$ on M. But, as we shall see later, unless these vectors have certain properties the transference of classical observable status to quantum observable status is not straightforward. We shall encounter these issues more intensely in the case of the Poincaré algebra next.

2.3 The Poincaré Algebra

The verification of the Poincaré algebra at the classical level is a long and tedious calculation. Furthermore, as just mentioned in the previous paragraph, it is not sufficient to know that the classical algebra is realized up to terms which vanish on Γ_C—when we come to quantization we need to know exactly *how* these terms vanish on Γ_C, as well as other special properties of the tensors involved. This is where our geometric notation pays off. We shall record below a list of all Schouten concomitants encountered in the classical analysis, leaving the detailed calculations in the appendix:

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k}V, V]\!] &= 0, && (94)\\
[\![{}^{\mathcal{J}^k}V, V]\!] &= 0, && (95)\\
[\![{}^{\mathcal{K}^k}Z, {}^{\mathcal{P}^l}V]\!] &= -\delta^{kl}V, && (96)\\
[\![{}^{\mathcal{K}^k}Z, {}^{\mathcal{J}^l}V]\!] &= -[klm]^{\mathcal{K}^m}Z, && (97)\\
[\![\frac{1}{2}{}^{\mathcal{K}^k}K, V]\!]^A - [\![\frac{1}{2}G^{-1}, {}^{\mathcal{K}^k}Z]\!]^A &= -{}^{\mathcal{P}^k}V^A, && (98)\\
[\![\frac{1}{2}{}^{\mathcal{K}^k}K, {}^{\mathcal{K}^l}Z]\!]^A - [\![\frac{1}{2}{}^{\mathcal{K}^l}K, {}^{\mathcal{K}^k}Z]\!]^A &= [klm]^{\mathcal{J}^m}V^A, && (99)\\
[\![{}^{\mathcal{P}^k}V, {}^{\mathcal{P}^l}V]\!]^A &= -{}^{\{\mathcal{P}^k,\mathcal{P}^l\}}\sigma^\gamma\phi^A_\gamma, && (100)\\
\text{where } {}^{\{\mathcal{P}^k,\mathcal{P}^l\}}\sigma^\gamma &:= -eF^{kl}(\boldsymbol{z});\\
[\![{}^{\mathcal{J}^k}V, {}^{\mathcal{P}^l}V]\!]^A &= -[klm]^{\mathcal{P}^m}V^A - {}^{\{\mathcal{J}^k,\mathcal{P}^l\}}\sigma^\gamma\phi^A_\gamma, && (101)\\
\text{where } {}^{\{\mathcal{J}^k,\mathcal{P}^l\}}\sigma^\gamma &:= -e[kmn]z^mF^{nl}(\boldsymbol{z});\\
[\![{}^{\mathcal{J}^k}V, {}^{\mathcal{J}^l}V]\!]^A &= -[klm]^{\mathcal{J}^m}V^A - {}^{\{\mathcal{J}^k,\mathcal{J}^l\}}\sigma^\gamma\phi^A_\gamma, && (102)\\
\text{where } {}^{\{\mathcal{J}^k,\mathcal{J}^l\}}\sigma^\gamma &:= -e[kmn][lpq]z^mz^pF^{nq}(\boldsymbol{z});\\
[\![{}^{\mathcal{P}^k}V, \frac{1}{2}G^{-1}]\!]^{AB} &= -{}^{\{\mathcal{P}^k,\mathcal{P}^0\}}\psi^{\gamma(A}\phi^{B)}_\gamma, && (103)\\
\text{where } {}^{\{\mathcal{P}^k,\mathcal{P}^0\}}\psi^{\gamma A} &:= \nabla^A\left({}^{\mathcal{P}^k}F^\gamma\right) = -e\left(\delta^i_k\delta(\boldsymbol{z}-\boldsymbol{x}), 0, 0\right);\\
[\![{}^{\mathcal{J}^k}V, \frac{1}{2}G^{-1}]\!]^{AB} &= -{}^{\{\mathcal{J}^k,\mathcal{P}^0\}}\psi^{\gamma(A}\phi^{B)}_\gamma, && (104)\\
\text{where } {}^{\{\mathcal{J}^k,\mathcal{P}^0\}}\psi^{\gamma A} &:= \nabla^A\left({}^{\mathcal{J}^k}F^\gamma\right) = -e[kmn]\left(\delta^i_nz^m\delta(\boldsymbol{z}-\boldsymbol{x}), 0, 0\right);\\
[\![\frac{1}{2}{}^{\mathcal{K}^k}K, {}^{\mathcal{P}^l}V]\!]^{AB} &= -\delta^{kl}\frac{1}{2}G^{AB} + {}^{\{\mathcal{K}^k,\mathcal{P}^l\}}\psi^{\gamma(A}\phi^{B)}_\gamma, && (105)\\
\text{where } {}^{\{\mathcal{K}^k,\mathcal{P}^l\}}\psi^{\gamma A} &:= -e\left(\delta^i_lx^k\delta(\boldsymbol{z}-\boldsymbol{x}), 0, 0\right);\\
[\![\frac{1}{2}{}^{\mathcal{K}^k}K, {}^{\mathcal{J}^l}V]\!]^{AB} &= -[klm]\frac{1}{2}{}^{\mathcal{K}^m}K^{AB} + {}^{\{\mathcal{K}^k,\mathcal{J}^l\}}\psi^{\gamma(A}\phi^{B)}_\gamma, && (106)\\
\text{where } {}^{\{\mathcal{K}^k,\mathcal{J}^l\}}\psi^{\gamma A} &:= e[lmn]\left(\delta^i_mx^kz^n\delta(\boldsymbol{z}-\boldsymbol{x}), 0, 0\right);\\
[\![{}^{\mathcal{K}^k}K, G^{-1}]\!]^{ABC} &= 0, && (107)
\end{aligned}
$$

$$[\![{}^{\mathcal{K}^k}K, {}^{\mathcal{K}^l}K]\!]^{ABC} \quad = \quad 0. \tag{108}$$

Here $\nabla^A = G^{AB}\partial_B$ is the contravariant Levi-Civita derivative acting on scalars.

In general, using ∇ in the evaluation of the Schouten concomitant it follows immediately from the definition (85) that

$$[\![S, G^{-1}]\!]^{A_0 \cdots A_s} = -2\nabla^{(A_0} S^{A_1 \cdots A_s)}, \tag{109}$$

so that $[\![S, G^{-1}]\!] = 0$ is equivalent to S being a Killing tensor on M. Thus (107) tells us that the ${}^{\mathcal{K}^k}K$ are Killing tensors. Furthermore, by (108) we see that they are in involution. In fact, of course, this must be true for any closed algebra with elements at most quadratic in the momenta (at least on Γ_C). Indeed, since M is flat, and ${}^{\mathcal{K}^k}K^{AB}$ are field independent in the Cartesian coordinates Q^A [see (81)], we know that the ${}^{\mathcal{K}^k}K$ are covariantly constant tensors. This considerably simplifies the Dirac quantization of the Poincaré algebra.

Inspection of (103) and (104) reveals that the ${}^{\mathcal{P}^k}V$ and ${}^{\mathcal{J}^k}V$ are not Killing vectors on M. Nevertheless, they do satisfy the weaker condition of being divergence free:

$$\begin{aligned} \nabla \cdot {}^{\mathcal{P}^k}V &= -\frac{1}{2} G_{AB} \left(\mathcal{L}_{{}^{\mathcal{P}^k}V} G\right)^{AB} = {}^{\{\mathcal{P}^k, \mathcal{P}^0\}}\psi^\gamma \cdot \phi_\gamma \\ &= \int d^3x \, d^3z \, \delta(\boldsymbol{x} - \boldsymbol{z}) \partial_{x^k} \delta(\boldsymbol{x} - \boldsymbol{z}) \\ &= \operatorname{tr} \partial_{x^k} \delta(\boldsymbol{x} - \boldsymbol{z}) = 0, \end{aligned} \tag{110}$$

where tr denotes functional trace. We take this to vanish on the grounds that $\partial_x \delta(x - z)$ is antisymmetric. Similarly,

$$\begin{aligned} \nabla \cdot {}^{\mathcal{J}^k}V &= [kmn] \int d^3x \, d^3z \, z^m \delta(\boldsymbol{x} - \boldsymbol{z}) \partial_{x^n} \delta(\boldsymbol{x} - \boldsymbol{z}) \\ &= [kmn] \mathrm{tr} z^m \partial_{x^n} \delta(\boldsymbol{x} - \boldsymbol{z}). \end{aligned} \tag{111}$$

Symmetrizing on $\boldsymbol{x}, \boldsymbol{z}$ the distribution in question becomes

$$\Delta^k(\boldsymbol{x}, \boldsymbol{z}) := [kmn] \{z^m \partial_{x^n} \delta(\boldsymbol{x} - \boldsymbol{z}) + x^m \partial_{z^n} \delta(\boldsymbol{z} - \boldsymbol{x})\}, \tag{112}$$

which is zero since

$$\int d^3x\, f(\boldsymbol{x}) \Delta^k(\boldsymbol{x}, \boldsymbol{z}) = [kmn] \{-z^m \partial_n f + \partial_n (z^m f)\}_{\boldsymbol{z}} = 0 \tag{113}$$

for any smooth function f. So ${}^{\mathcal{J}^k}V$ is also divergence free.

Combining the results of (86– 89) and (94–108) we calculate:

$$\begin{aligned}
\left\{\mathcal{P}^k, \mathcal{P}^0\right\} &= -\mathcal{C}_2\left(\left[\!\left[{}^{\mathcal{P}^k}V, \frac{1}{2}G^{-1}\right]\!\right]\right) - \mathcal{C}_0\left(\left[\!\left[{}^{\mathcal{P}^k}V, V\right]\!\right]\right) \\
&= -e \int d^3z\, d^3x\, \delta^i_k \delta(\boldsymbol{z} - \boldsymbol{x}) \Pi_{A_i(\boldsymbol{x})} C_\gamma \\
&= -e \int d^3z\, F^{k0} C_\gamma && (114) \\
\left\{\mathcal{P}^k, \mathcal{P}^l\right\} &= -\mathcal{C}_1\left(\left[\!\left[{}^{\mathcal{P}^k}V, {}^{\mathcal{P}^l}V\right]\!\right]\right) \\
&= -e \int d^3z\, F^{kl} C_\gamma && (115) \\
\left\{\mathcal{J}^k, \mathcal{P}^0\right\} &= -\mathcal{C}_2\left(\left[\!\left[{}^{\mathcal{J}^k}V, \frac{1}{2}G^{-1}\right]\!\right]\right) - \mathcal{C}_0\left(\left[\!\left[{}^{\mathcal{J}^k}V, V\right]\!\right]\right) \\
&= -e[kmn] \int d^3z\, d^3x\, \delta^i_n z^m \delta(\boldsymbol{z} - \boldsymbol{x}) \Pi_{A_i(\boldsymbol{x})} C_\gamma \\
&= -e[kmn] \int d^3z\, z^m F^{n0} C_\gamma && (116) \\
\left\{\mathcal{J}^k, \mathcal{P}^l\right\} &= -\mathcal{C}_1\left(\left[\!\left[{}^{\mathcal{J}^k}V, {}^{\mathcal{P}^l}V\right]\!\right]\right) \\
&= [klm]\mathcal{P}^m - e[kmn] \int d^3z\, z^m F^{nl} C_\gamma && (117) \\
\left\{\mathcal{J}^k, \mathcal{J}^l\right\} &= -\mathcal{C}_1\left(\left[\!\left[{}^{\mathcal{J}^k}V, {}^{\mathcal{J}^l}V\right]\!\right]\right) \\
&= [klm]\mathcal{J}^m - e[kmn][lpq] \int d^3z\, z^m z^p F^{nq} C_\gamma && (118) \\
\left\{\mathcal{K}^k, \mathcal{P}^0\right\} &= \mathcal{C}_3\left(\left[\!\left[\frac{1}{2}{}^{\mathcal{K}^k}K, \frac{1}{2}G^{-1}\right]\!\right]\right) + \mathcal{C}_1\left(\left[\!\left[\frac{1}{2}{}^{\mathcal{K}^k}K, V\right]\!\right] - \left[\!\left[\frac{1}{2}G^{-1}, {}^{\mathcal{K}^k}Z\right]\!\right]\right) + t\left\{\mathcal{P}^k, \mathcal{P}^0\right\} \\
&= -\mathcal{P}^k - e \int d^3z\, t F^{k0} C_\gamma && (119)
\end{aligned}$$

$$
\begin{aligned}
\left\{\mathcal{K}^k,\mathcal{P}^l\right\} &= \mathcal{C}_2\left(\llbracket\frac{1}{2}{}^{\mathcal{K}^k}K,{}^{\mathcal{P}^l}V\rrbracket\right)+\mathcal{C}_0\left(\llbracket{}^{\mathcal{K}^k}Z,{}^{\mathcal{P}^l}V\rrbracket\right)+t\left\{\mathcal{P}^k,\mathcal{P}^l\right\} \\
&= -\delta^{kl}\mathcal{P}^0-e\int d^3z\,d^3x\,\delta^i_l x^k\delta(z-x)\Pi_{A_i(x)}C_\gamma+t\left\{\mathcal{P}^k,\mathcal{P}^l\right\} \\
&= -\delta^{kl}\mathcal{P}^0-e\int d^3z\,\left(tF^{kl}-z^kF^{0l}\right)C_\gamma \qquad (120) \\
\left\{\mathcal{K}^k,\mathcal{J}^l\right\} &= \mathcal{C}_2\left(\llbracket\frac{1}{2}{}^{\mathcal{K}^k}K,{}^{\mathcal{J}^l}V\rrbracket\right)+\mathcal{C}_0\left(\llbracket{}^{\mathcal{K}^k}Z,{}^{\mathcal{J}^l}V\rrbracket\right)+t\left\{\mathcal{P}^k,\mathcal{J}^l\right\} \\
&= [klm]\mathcal{K}^m+e[lmn]\int d^3z\,\left(\int d^3x\,\delta^i_m x^kz^n\delta(z-x)\Pi_{A_i(x)}+tz^mF^{nk}\right)C_\gamma \\
&= [klm]\mathcal{K}^m-e[lmn]\int d^3z\,z^m\left(tF^{kn}-z^kF^{0n}\right)C_\gamma \qquad (121) \\
\left\{\mathcal{K}^k,\mathcal{K}^l\right\} &= -\mathcal{C}_3\left(\llbracket\frac{1}{2}{}^{\mathcal{K}^k}K,\frac{1}{2}{}^{\mathcal{K}^l}K\rrbracket\right)-\mathcal{C}_1\left(\llbracket\frac{1}{2}{}^{\mathcal{K}^k}K,{}^{\mathcal{K}^l}Z\rrbracket-\llbracket\frac{1}{2}{}^{\mathcal{K}^l}K,{}^{\mathcal{K}^k}Z\rrbracket\right. \\
&\quad \left.-t^2\llbracket{}^{\mathcal{P}^k}V,{}^{\mathcal{P}^l}V\rrbracket\right)+t\left(\left\{\mathcal{K}^k,\mathcal{P}^l\right\}-\left\{\mathcal{K}^l,\mathcal{P}^k\right\}\right) \\
&= -[klm]\mathcal{J}^m-e\int d^3z\,t\left(tF^{kl}+z^kF^{l0}-z^lF^{k0}\right)C_\gamma. \qquad (122)
\end{aligned}
$$

Using our earlier result that the Poincaré charges are physical observables it is now trivial to verify that they are also constants of the motion. For example, using (31) and (119) we find

$$
\frac{d\mathcal{K}^k}{dt}=\{\mathcal{K}^k,H\}+\frac{\partial\mathcal{K}^k}{\partial t}\approx-\mathcal{P}^k+\mathcal{P}^k=0. \qquad (123)
$$

With $z^0:=t$ we can collect the results of (114– 122) together in the more covariant form

$$
\begin{aligned}
\{\mathcal{P}^\mu,\mathcal{P}^\nu\} &= 0-e\int d^3z\,F^{\mu\nu}C_\gamma, \qquad (124) \\
\{\mathcal{J}^{\mu\nu},\mathcal{P}^\rho\} &= \eta^{\nu\rho}\mathcal{P}^\mu-\eta^{\mu\rho}\mathcal{P}^\nu-e\int d^3z\,\left(z^\mu F^{\nu\rho}-z^\nu F^{\mu\rho}\right)C_\gamma, \qquad (125) \\
\{\mathcal{J}^{\mu\nu},\mathcal{J}^{\rho\sigma}\} &= \eta^{\mu\sigma}\mathcal{J}^{\nu\rho}-\eta^{\nu\sigma}\mathcal{J}^{\mu\rho}+\eta^{\nu\rho}\mathcal{J}^{\mu\sigma}-\eta^{\mu\rho}\mathcal{J}^{\nu\sigma} \qquad (126) \\
&\quad +e\int d^3z\,\left(z^\mu z^\sigma F^{\nu\rho}-z^\nu z^\sigma F^{\mu\rho}+z^\nu z^\rho F^{\mu\sigma}-z^\mu z^\rho F^{\nu\sigma}\right)C_\gamma.
\end{aligned}
$$

Comparison with (59–61) verifies the Poincaré algebra on Γ_C. Notice that since $F_{0k}(z)=\Pi_{A_k(z)}$, some of the terms which vanish on Γ_C are quadratic in momenta.

We are not aware of any such explicit verification of the Poincaré algebra for scalar electrodynamics in the literature.

3 Classical Reduction

We now turn our attention to the reduction of the classical system, that is, establishing a classical mechanics on the true degrees of freedom. In the course of discussion some of these points were touched on in previous sections; here we give a more comprehensive treatment. (But for simplicity our considerations will be valid only locally). To begin with we present a brief exposition of classical mechanics (following, e.g., [10]) to fix ideas and notation. We take advantage of the clarity and conciseness of modern language where possible. The gauge structure is then introduced, which leads to a reduction of the extended phase space Γ to the reduced phase space γ. The Poisson algebra on Γ induces a corresponding algebra on γ. Finally we apply this analysis to our two examples, in particular we establish the action of the Poincaré group on the reduced space.

3.1 Geometrical Analysis of Classical Reduction

Classical mechanics consists of a $2N$ dimensional (symplectic) manifold Γ, the phase space, equipped with a geometrical object called the symplectic two-form, Ω. Ω is closed ($d\Omega = 0$) and nondegenerate ($i_X\Omega = 0 \Longleftrightarrow X = 0$, where i denotes left interior product and X is a vector field). Dynamical variables $F, G, \ldots$ are functions on Γ, the space of classical states—they measure properties of these states. For example, the Poincaré charges considered in section 2.2 are a measure of the energy, momentum, angular momentum, and so on of the classical states of the fields.

When these dynamical variables come to us from classical symmetry groups (especially) we also want them to generate an action on Γ, representing the group. Thus, for example, $\mathcal{K}^i$ in (74) should map states in Γ to their boosted counterparts. To generate such diffeomorphisms on Γ requires a vector field. The only natural way to construct a vector field out of F is to construct the one-form dF and 'raise' it to a vector using the symplectic structure on Γ:

$$i_{X_F}\Omega = -dF \tag{127}$$

uniquely defines the Hamiltonian vector field X_F (since Ω is nondegenerate). This action that X_F generates is canonical in the sense that it preserves the symplectic structure:

$$\mathcal{L}_{X_F}\Omega \equiv di_{X_F}\Omega + i_{X_F}d\Omega = 0 \tag{128}$$

since $i_{X_F}\Omega$ is exact and Ω is closed. ($\equiv$ denotes an identity).

Using (128) we can examine the commutator of two actions X_F, X_G on Γ:

$$\begin{aligned} i_{[X_F,X_G]}\Omega &\equiv \mathcal{L}_{X_F}i_{X_G}\Omega - i_{X_G}\mathcal{L}_{X_F}\Omega = -\mathcal{L}_{X_F}dG \\ &\equiv -di_{X_F}dG - i_{X_F}d^2G \equiv -dX_FG. \end{aligned} \tag{129}$$

Thus $[X_F, X_G]$ is the Hamiltonian vector field associated with the function X_FG. We define the Poisson bracket of two dynamical variables F, G as

$$\{F, G\} = -\{G, F\} := -\Omega(X_F, X_G) = \Omega^{-1}(dF, dG) = -X_FG. \tag{130}$$

These identities follow from (127). The antisymmetric contravariant tensor Ω^{-1} is the inverse of Ω. Thus

$$[X_F, X_G] = -X_{\{F,G\}}. \tag{131}$$

The Jacobi identity for the Poisson bracket then easily follows. Thus the symplectic structure induces a Lie algebra structure on the space of smooth functions on Γ called the Poisson algebra.

We now introduce the gauge structure into the phase space. It appears as a set of dynamical variables C_α, $\alpha = 1 \ldots C$, which define (at least locally) a constraint surface $\Gamma_C \in \Gamma$ of physical states via $C_\alpha = 0$. Their Hamiltonian vector fields, X_{C_α}, generate gauge transformations on the classical states. These diffeomorphisms preserve Γ_C in the sense that

$$X_{C_\alpha} C_\beta = -\{C_\alpha, C_\beta\} = f^\gamma_{\alpha\beta} C_\gamma \approx 0 \tag{132}$$

(the $f^\gamma_{\alpha\beta} C_\gamma$ are the structure functions introduced in section 1, and as before, $\approx$ denotes restriction to Γ_C). In other words, $X_{C_\gamma}\big|_{\Gamma_C} \in T\Gamma_C$. Furthermore, it follows immediately from (127) and the nondegeneracy of Ω that

$$X_{FG} = F X_G + G X_F \tag{133}$$

so that

$$\left[X_{C_\alpha}, X_{C_\beta}\right] = -X_{\{C_\alpha, C_\beta\}} = X_{f^\gamma_{\alpha\beta} C_\gamma} \approx f^\gamma_{\alpha\beta} X_{C_\gamma}. \tag{134}$$

Hence, the X_{C_α} are surface forming on Γ_C, which leads to a foliation of Γ_C by gauge orbits. The space of these gauge orbits on Γ_C is called the reduced phase space, γ, and we have the projection $\pi : \Gamma_C \longrightarrow \gamma$. γ is the set of physically distinct states, and hence embodies the true classical degrees of freedom. Note that $\dim(\gamma) = 2N - 2C := 2n$.

γ can always be realized, at least locally, by a surface embedded in Γ_C, cutting through the bundle of gauge orbits, intersecting each at exactly one point. Clearly there is a great deal of freedom in the choice of this surface. Here we shall imagine selecting one such surface, called gauge fixing, but then later demonstrate that our results are independent of this choice (under appropriate conditions). Associated with the surface γ is the inclusion map $i : \gamma \longrightarrow \Gamma$. The classical mechanics on Γ consists of functions (the dynamical variables F) and a two-form (the symplectic structure Ω). Using i we can pull these back to γ:

$$f := i_* F, \tag{135}$$

$$\omega := i_* \Omega. \tag{136}$$

Before we can say that these represent a classical mechanics on γ we must establish that ω is symplectic, i.e. that γ is a symplectic manifold. Furthermore, we must demonstrate that this induced classical mechanics is independent of the gauge fixing surface we choose to represent γ. Finally, we would like to see a direct correspondence between the original Poisson algebra (restricted to Γ_C) and the induced Poisson algebra on γ.

In the first place ω is closed, since $d\omega = di_*\Omega \equiv i_* d\Omega = 0$. To show that ω is nondegenerate it is convenient to introduce an adapted coordinate system on Γ. Of course there are more elegant ways than this (see, e.g., [2]), but this method will make contact with the adapted coordinates we shall later introduce on M (and use extensively). We begin by finding a set of independent functions ξ^i,$i = 1\ldots 2n$, on Γ, which are gauge invariant on Γ_C:

$$X_{C_\alpha}\xi^i \approx 0 \;\forall\alpha, i. \tag{137}$$

These serve to label the gauge orbits, i.e. coordinatize γ, and represent the true degrees of freedom. Note that the ξ^i explicitly realize the projection $\pi : \Gamma_C \longrightarrow \gamma$. Of course in general finding such a set is notoriously difficult [16], especially in field theories, but fortunately our two simple examples will present no difficulties, so we say no more on the matter. All we need to be assured of here is the existence of such a set.

Next we select a set of independent functions F^α, $\alpha = 1\ldots C$ such that, together with the ξ^i, form a complete set of coordinates on Γ_C. Thus, defining the Faddeev-Popov matrix

$$F_\alpha^\beta := X_{C_\alpha}\, F^\beta\Big|_{\Gamma_C}, \tag{138}$$

we demand that $\det F_\alpha^\beta \neq 0$ everywhere on Γ_C. This means that any surface of constant F^α on Γ_C intersects each gauge orbit exactly once, at least locally, and so if we wish we can define γ by imposing the gauge fixing constraints $F^\alpha = 0$ on Γ_C. By making nonsingular transformations of the form

$$\begin{aligned} \xi^i &\longmapsto \tilde{\xi}^i(\xi), \\ F^\alpha &\longmapsto \tilde{F}^\alpha(\xi, F) \end{aligned} \tag{139}$$

on Γ_C we can reach any other gauge fixing surface (as well as, perhaps, changing the coordinates on γ). The F^α coordinates explicitly realize the inclusion $j : \gamma \longrightarrow \Gamma_C$.

Finally, we already have the C_α, which realize the inclusion $k : \Gamma_C \longrightarrow \Gamma$. Together the set $\Xi^{\overline{I}} := (\xi^i, F^\alpha, C_\alpha)$ are local coordinates in the neighbourhood (in Γ) of any point in γ, and establish a basis of vector fields and one-forms. [Recall that the original set of canonical coordinates were denoted $\Xi^I = (Q^A, P_A)$]. Since F^α and C_α have the same label, α, we avoid potential confusion by denoting, for example, the $F^\alpha C_\beta$ components of Ω as $\Omega_{F^\alpha C_\beta}$. This will not be necessary for the $\xi^i \xi^j$ components of ω on γ. Thus from (136)

$$\omega_{ij} = \Omega_{\xi^i \xi^j}\Big|_\gamma = \Omega(\frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j})\Big|_\gamma . \tag{140}$$

Now using the definition of the Poisson bracket (130), as well as (137), (138) and (132) we observe that

$$\Omega^{\xi^i C_\beta} = \Omega^{-1}(d\xi^i, dC_\beta) = X_{C_\beta}\xi^i \approx 0, \tag{141}$$

$$\Omega^{F^\alpha C_\beta} = \Omega^{-1}(dF^\alpha, dC_\beta) = X_{C_\beta}F^\alpha \approx F^\alpha_\beta, \tag{142}$$

$$\Omega^{C_\alpha C_\beta} = \Omega^{-1}(dC_\alpha, dC_\beta) = X_{C_\beta}C_\alpha \approx 0. \tag{143}$$

But in general,

$$\delta^{\overline{K}}_{\overline{I}} = \Omega_{\overline{I}\overline{J}}\Omega^{\overline{J}\overline{K}}, \tag{144}$$

so in particular

$$\begin{aligned} 0 &= \delta_{\xi^i}^{C_\gamma} = \Omega_{\xi^i\xi^j}\Omega^{\xi^j C_\gamma} + \Omega_{\xi^i F^\beta}\Omega^{F^\beta C_\gamma} + \Omega_{\xi^i C_\beta}\Omega^{C_\beta C_\gamma} \\ &\approx \Omega_{\xi^i F^\beta} F^\beta_\gamma. \end{aligned} \tag{145}$$

Thus, by the invertibility of the Faddeev-Popov matrix we have

$$\Omega_{\xi^i F^\beta} = \Omega(\frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial F^\beta}) \approx 0, \tag{146}$$

i.e. the physical vector fields $\partial/\partial\xi^i$ are 'orthogonal' to the gauge vector fields $\partial/\partial F^\beta$ on Γ_C. (Note that

$$X_{C_\alpha} = X_{C_\alpha}\xi^i \frac{\partial}{\partial \xi^i} + X_{C_\alpha} F^\beta \frac{\partial}{\partial F^\beta} + X_{C_\alpha} C_\beta \frac{\partial}{\partial C_\beta} \approx F^\beta_\alpha \frac{\partial}{\partial F^\beta} \tag{147}$$

so the Faddeev–Popov matrix just relates the X_{C_α} basis of gauge generators to the holonomic basis $\partial/\partial F^\beta$). Using (144) again with (141) and (146) we find

$$\delta_{\xi^i}^{\xi^k} = \Omega_{\xi^i\xi^j}\Omega^{\xi^j\xi^k} + \Omega_{\xi^i F^\beta}\Omega^{F^\beta \xi^k} + \Omega_{\xi^i C_\beta}\Omega^{C_\beta \xi^k} \approx \Omega_{\xi^i\xi^j}\Omega^{\xi^j\xi^k} \tag{148}$$

so that $\Omega_{\xi^i\xi^j}$ is invertible on Γ_C, and in particular on γ:

$$\omega^{ij} = \Omega^{\xi^i\xi^j}\Big|_\gamma = \Omega^{-1}(d\xi^i, d\xi^j)\Big|_\gamma = \left\{\xi^i, \xi^j\right\}\Big|_\gamma. \tag{149}$$

Thus ω is nondegenerate and γ is a symplectic manifold.

Given any dynamical variable f, say induced on γ from F on Γ, we can now construct its unique Hamiltonian vector field, x_f, via

$$i_{x_f} = -df, \tag{150}$$

and define a corresponding Poisson bracket on γ:

$$\{f, g\} = -\{g, f\} := -\omega(x_f, x_g) = \omega^{-1}(df, dg) = -x_f g. \tag{151}$$

In fact, of course, this is nothing but the Dirac bracket [16].

Now we ask if this induced Poisson algebra is independent of the gauge fixing surface chosen. Clearly, for $f = i_* F = F|_\gamma$ to be invariant we require $F|_{\Gamma_C}$ to be gauge invariant, in which case we say that F is an observable and f is a physical observable. Recall that we demonstrated this for the Poincaré charges in the discussion following (91). Concerning the Poisson bracket itself we note that by (128), (147) and (146)

$$\begin{aligned} \mathcal{L}_{X_{C_\alpha}}\Omega(\frac{\partial}{\partial\xi^i},\frac{\partial}{\partial\xi^j}) &\equiv (\mathcal{L}_{X_{C_\alpha}}\Omega)(\frac{\partial}{\partial\xi^i},\frac{\partial}{\partial\xi^j}) + \Omega(\mathcal{L}_{X_{C_\alpha}}\frac{\partial}{\partial\xi^i},\frac{\partial}{\partial\xi^j}) + \Omega(\frac{\partial}{\partial\xi^i},\mathcal{L}_{X_{C_\alpha}}\frac{\partial}{\partial\xi^j}) \\ &= -\frac{\partial F^\beta_\alpha}{\partial\xi^i}\Omega(\frac{\partial}{\partial F^\beta},\frac{\partial}{\partial\xi^j}) - \frac{\partial F^\beta_\alpha}{\partial\xi^j}\Omega(\frac{\partial}{\partial\xi^i},\frac{\partial}{\partial F^\beta}) \approx 0 \end{aligned} \tag{152}$$

so that the scalars $\Omega(\frac{\partial}{\partial\xi^i},\frac{\partial}{\partial\xi^j})$ are actually gauge invariant on Γ_C, and thus ω_{ij} in (140) is independent of gauge fixing surface γ. Combining this with invariance of f yields the desired result for the Poisson algebra.

Finally, it is instructive to demonstrate directly that the extended space Poisson algebra of observables (restricted to Γ_C) induce the same algebra on γ. We first prove that

$$\pi_*\left(X_F|_{\Gamma_C}\right) = x_f. \tag{153}$$

Denoting the components of X_F in the $\Xi^{\bar{I}}$ basis as $X_F^{\bar{I}}$, we note that F being an observable is equivalent to

$$X_F^{C_\alpha} = X_F C_\alpha = -X_{C_\alpha}F \approx 0, \tag{154}$$

or $X_F|_{\Gamma_C} \in T\Gamma_C$. In addition, for the push forward

$$\pi_*\left(X_F|_{\Gamma_C}\right) = X_F^{\xi^i}\Big|_\gamma \frac{\partial}{\partial\xi^i} \tag{155}$$

to be well defined we need $X_F^{\xi^i}$ to be gauge invariant on Γ_C. We find

$$\mathcal{L}_{X_{C_\alpha}}X_F^{\xi^i} = -\mathcal{L}_{X_{C_\alpha}}\left(X_{\xi^i}F\right) = -\left[X_{C_\alpha},X_{\xi^i}\right]F - X_{\xi^i}X_{C_\alpha}F \approx 0 \tag{156}$$

as required. Next we contract the left hand side of (153) into $\omega = i_*\Omega$, and show that

$$i_{\pi_*(X_F|_{\Gamma_C})} i_*\Omega = i_*\left(i_{X_F}\Omega\right). \tag{157}$$

The left hand side here is equal to

$$\left(X_F^{\xi^i}\Omega_{\xi^i\xi^j}\right)\Big|_\gamma d\xi^j, \tag{158}$$

whereas the right hand side is

$$\left(X_F^{\xi^i}\Omega_{\xi^i\xi^j} + X_F^{F^\alpha}\Omega_{F^\alpha\xi^j} + X_F^{C_\alpha}\Omega_{C_\alpha\xi^j}\right)\Big|_\gamma d\xi^j. \tag{159}$$

The latter reduces to the former when we apply (154) and (146). But the right hand side of (157) is also equal to $i_*(-dF) \equiv -di_*F = -df$, so we have shown that

$$i_{\pi_*(X_F|_{\Gamma_C})}\omega = -df. \tag{160}$$

But (150) uniquely determines x_f, so we have the desired result (153).

Using this fact we can now relate the extended and reduced Poisson algebras:

$$\begin{aligned}
\pi_*\left([X_F, X_G]|_{\Gamma_C}\right) &= \pi_*\left[X_F|_{\Gamma_C}, X_G|_{\Gamma_C}\right] \equiv \left[\pi_*\left(X_F|_{\Gamma_C}\right), \pi_*\left(X_G|_{\Gamma_C}\right)\right] \\
&= \left[x_f, x_g\right]. \qquad (161)
\end{aligned}$$

Later in section 3.3 we shall reduce the Poincaré charges, and this last result guarantees that their Poisson algebra on γ is the same as (124– 126) on the constraint surface $C_\gamma = 0$. This completes the essential content of the classical reduction process.

3.2 Phase Space Coordinates Adapted to the Gauge Structure

Let us now develop further the choice of adapted coordinates $\Xi^{\bar{I}} = (\xi^i, F^\alpha, C_\alpha)$. Recall that the original canonical coordinate on $\Gamma = T^*M$ were $\Xi^I = (Q^A, P_A)$, where Q^A,$A = 1 \ldots N$, were coordinates on M. Consider first adapted coordinates on M:

$$Q^{\overline{A}} := (q^a, F^\alpha)\,, \tag{162}$$

where q^a, $a = 1 \ldots n$, are a set of independent gauge invariant functions on M which label the gauge orbits on M:

$$\phi_\alpha q^a = 0 \;\forall \alpha, a. \tag{163}$$

These can then serve as coordinates on the reduced configuration space, m. Note that with $C_\alpha = \phi_\alpha^A P_A$ the gauge orbits on M are the projections of the gauge orbits on Γ, since in this case

$$X_{C_\alpha} = X_{C_\alpha} Q^A \frac{\partial}{\partial Q^A} + X_{C_\alpha} P_A \frac{\partial}{\partial P_A} = \phi_\alpha^A \frac{\partial}{\partial Q^A} - \frac{\partial \phi_\alpha}{\partial Q^A}\frac{\partial}{\partial P_A} \tag{164}$$

The F^α complete the coordinate system on M, and are the same F^α as in $\Xi^{\overline{I}}$; we have simply exercised our freedom to choose them to be independent of the momenta. Thus, from (138) and (164) the Faddeev-Popov matrix is

$$F_\alpha^\beta = X_{C_\alpha} F^\beta\Big|_{\Gamma_C} = \phi_\alpha F^\beta. \tag{165}$$

Then $\det F_\alpha^\beta \neq 0$ on M means that surfaces of constant F^α intersect each gauge orbit exactly once, at least locally.

Now we can define a coordinate basis of vector fields

$$\begin{aligned} \frac{\partial}{\partial q^a} &= \frac{\partial Q^A}{\partial q^a}\frac{\partial}{\partial Q^A} =: Q_a^A \frac{\partial}{\partial Q^A} =: Q_a \qquad (166)\\ \frac{\partial}{\partial F^\alpha} &= \frac{\partial Q^A}{\partial F^\alpha}\frac{\partial}{\partial Q^A} =: \Phi_\alpha^A \frac{\partial}{\partial Q^A} =: \Phi_\alpha \qquad (167) \end{aligned}$$

and one-forms

$$\begin{aligned} dq^a &= \frac{\partial q^a}{\partial Q^A} dQ^A =: Q_A^a dQ^A =: Q^a, \qquad (168)\\ dF^\alpha &= \frac{\partial F^\alpha}{\partial Q^A} dQ^A =: \Phi_A^\alpha dQ^A =: \Phi^\alpha \qquad (169) \end{aligned}$$

on M, which satisfy the usual interior product relations. We also will make use of the completeness relation:

$$\delta_A^B = Q_A^a Q_a^B + \Phi_A^\alpha \Phi_\alpha^B. \tag{170}$$

Finally, notice that Φ_α is a holonomic basis of gauge vectors, with

$$\phi_\alpha = F_\alpha^\beta \frac{\partial}{\partial F^\beta} \tag{171}$$

[refer to (147)].

The q^a serve as half of the ξ^i, and given already $C_\alpha = \phi_\alpha^A P_A$, we naturally choose the other half to be the complementary coordinates (also linear in the momenta)

$$p_a := Q_a^A P_A. \tag{172}$$

This completes the choice of adapted coordinates $\Xi^{\bar{I}}$. We need only verify that the $\xi^i = (q^a, p_a)$ are gauge invariant on Γ_C. Of course the q^a satisfy this condition; as for the p_a we have

$$X_{C_\alpha} p_a = [\phi_\alpha, Q_a]^A P_A. \tag{173}$$

Since the physical projection

$$\begin{aligned} Q_A^b [\phi_\alpha, Q_a]^A &= Q_A^b \left(\phi_\alpha^B \partial_B Q_a^A - Q_a^B \partial_B \phi_\alpha^A\right) \\ &= \phi_\alpha^B \partial_B \left(Q_A^b Q_a^A\right) - \phi_\alpha^B Q_a^A \partial_B Q_A^b - Q_a^B \partial_B \left(Q_A^b \phi_\alpha^A\right) + Q_a^B \phi_\alpha^A \partial_B Q_A^b \\ &= \phi_\alpha^A Q_a^B \left(\partial_B \partial_A q^b - \partial_A \partial_B q^b\right) = 0 \end{aligned} \tag{174}$$

we obtain $X_{C_\alpha} p_a \approx 0$, as desired. Thus the ξ^i can serve as coordinates on γ.

In fact, they are even canonical. This is obvious from their construction as a point canonical transformation. Alternatively, in view of (149) we calculate

$$\left\{q^a, q^b\right\} = 0, \tag{175}$$

$$\{q^a, p_b\} = Q_b q^a = \delta_b^a, \tag{176}$$

$$\{p_a, p_b\} = -[Q_a, Q_b]^A P_A = 0, \tag{177}$$

with the q's and p's treated as functions on Γ.

It provides useful insight to observe that since the constraints C_α are linear in the momenta, $\gamma = T^*m$: With m the space of gauge orbits on M, Tm is the space of equivalence classes of vector fields $v \in TM$, with $v \sim v + \epsilon^\alpha \phi_\alpha$ for arbitrary ϵ^α. Hence T^*m is the set of one-forms $\lambda \in T^*M$ which do not distinguish between v and $v + \epsilon^\alpha \phi_\alpha$:

$$i_v \lambda = i_{v+\epsilon^\alpha \phi_\alpha} \lambda \iff i_{\phi_\alpha} \lambda = 0 \; \forall \alpha. \tag{178}$$

Expanding in our coordinate basis of one-forms, $\lambda =: p_a Q^a + p_\alpha \Phi^\alpha$, we have $p_\alpha = 0$ and p_a are coordinates for the linear vector space $\boldsymbol{R}^n$. This is essentially the content of (172). (We discussed this from a slightly different vantage point at the beginning of section 1).

The gauge fixing condition $F^\alpha = 0$ now selects a surface embedded in M (no restrictions on momenta besides the $C_\alpha = 0$) which, at least locally, intersects each gauge orbit on M exactly once, and so can represent m. To reach other representations of m we make nonsingular transformations of the form [refer to (139)]:

$$\begin{aligned} q^a &\longmapsto \tilde{q}^a(q), \\ F^\alpha &\longmapsto \tilde{F}^\alpha(q, F). \end{aligned} \tag{179}$$

With any such representation, $\partial/\partial q^a|_m \in Tm$.

As for the reduction of observables, $f = i_* F$, recall the definition (83) of homogeneous classical dynamical variables on Γ. Using (170) and (172) we have

$$\begin{aligned} \mathcal{C}_s(S) &= S^{A_1 \cdots A_s} \left(Q^{a_1}_{A_1} Q^{B_1}_{a_1} + \Phi^{\alpha_1}_{A_1} \Phi^{B_1}_{\alpha_1} \right) P_{B_1} \cdots \left(Q^{a_s}_{A_s} Q^{B_s}_{a_s} + \Phi^{\alpha_s}_{A_s} \Phi^{B_s}_{\alpha_s} \right) P_{B_s} \\ &\approx s^{a_1 \cdots a_s} p_{a_1} \cdots p_{a_s} =: c_s(s), \end{aligned} \tag{180}$$

in a notation analogous to (83), where

$$s^{a_1 \cdots a_s}(Q) := S^{A_1 \cdots A_s}(Q) Q^{a_1}_{A_1}(Q) \cdots Q^{a_s}_{A_s}(Q). \tag{181}$$

That $\mathcal{C}_s(S)$ is an observable is thus obviously equivalent to $s^{a_1\cdots a_s}(Q) = s^{a_1\cdots a_s}(q)$, i.e. the physical, or reduced tensors must be tensors on m—their components expressible as functions of the q^a.

3.3 Adapted Coordinates for Scalar Electrodynamics and the Helix Model

We now choose adapted coordinates for our scalar electrodynamics example. Inspection of (16) and (33) suggests we take

$$q^a = (B_i(\boldsymbol{x}), \rho(\boldsymbol{x}))\,, \tag{182}$$

where $B_i(\boldsymbol{x}) := A_i(\boldsymbol{x}) + \frac{1}{e}\partial_{x^i}\theta(\boldsymbol{x})$ and $\varphi(\boldsymbol{x}) =: \rho(\boldsymbol{x})\exp i\theta(\boldsymbol{x})$ ('unitary gauge'); the q^a are easily shown to be gauge invariant. As for the remaining coordinates we select

$$F^\alpha = \theta(\boldsymbol{x}),\ 0 \leq \theta(\boldsymbol{x}) < 2\pi \tag{183}$$

so then using (20) in (165) the Faddeev-Popov matrix is

$$F^\beta_\alpha = \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{184}$$

which is nonsingular as required. Note that using these coordinates the gauge vectors can be written

$$\phi_\alpha = \frac{\delta}{\delta\theta(\boldsymbol{x})}. \tag{185}$$

To reduce tensors on M to tensors on m [see (181)] we need the gradient of the physical coordinates:

$$Q^a_B(Q) = \frac{\partial q^a}{\partial Q^B} = \begin{pmatrix} \delta^j_i & -\frac{1}{e}\partial_{x^i}\frac{\eta(\boldsymbol{x})}{\rho^2(\boldsymbol{x})} & \frac{1}{e}\partial_{x^i}\frac{\xi(\boldsymbol{x})}{\rho^2(\boldsymbol{x})} \\ 0 & \frac{\xi(\boldsymbol{x})}{\rho(\boldsymbol{x})} & \frac{\eta(\boldsymbol{x})}{\rho(\boldsymbol{x})} \end{pmatrix} \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{186}$$

Here the row index is a and the column index is B, and ∂_{x^i} acts also on $\delta(\boldsymbol{x}-\boldsymbol{y})$. Beginning with the zero order tensors in (86– 89) we find [see (19)]

$$v(q) := V(Q) = \int d^3x \left\{\frac{1}{4}(F_{ij}(B))^2 + \frac{1}{2}e^2\rho^2(B_i)^2 + \frac{1}{2}(\partial_i\rho)^2 + U(\rho)\right\}, \quad (187)$$

where $F_{ij}(B) := \partial_i B_j - \partial_j B_i$, and similarly [see (82)]

$$^{\mathcal{K}^k}z(q) := {}^{\mathcal{K}^k}Z(Q) = \int d^3x\, x^k \mathcal{V}(B_i, \rho), \quad (188)$$

where $\mathcal{V}(B_i,\rho)$ is the integrand in (187).

For the reduced spatial translation vectors we calculate [see (70)]

$$\begin{aligned} {}^{\mathcal{P}^k}v^a(q) &:= Q^a_B(Q)^{\mathcal{P}^k}V^B(Q) = Q^a_B(Q)^{\mathcal{P}^k}\overset{B}{V_0}(Q) \\ &= \int d^3y \begin{pmatrix} -\delta^j_i\partial_{y^k}A_j(\boldsymbol{y}) + \frac{1}{e}\partial_{x^i}\frac{\eta(\boldsymbol{x})}{\rho^2(\boldsymbol{x})}\partial_{y^k}\xi(\boldsymbol{y}) - \frac{1}{e}\partial_{x^i}\frac{\xi(\boldsymbol{x})}{\rho^2(\boldsymbol{x})}\partial_{y^k}\eta(\boldsymbol{y}) \\ -\frac{\xi(\boldsymbol{x})}{\rho(\boldsymbol{x})}\partial_{y^k}\xi(\boldsymbol{y}) - \frac{\eta(\boldsymbol{x})}{\rho(\boldsymbol{x})}\partial_{y^k}\eta(\boldsymbol{y}) \end{pmatrix} \delta(\boldsymbol{x}-\boldsymbol{y}) \\ &= (-\partial_{x^k}B_i(\boldsymbol{x}), -\partial_{x^k}\rho(\boldsymbol{x})) = -\partial_{x^k}q^a. \end{aligned} \quad (189)$$

(Note that when convenient we use row/column notation interchangeably if there is no chance of confusion). Similarly, for the reduced spatial rotation vectors we have

$$\begin{aligned} {}^{\mathcal{J}^k}v^a(q) &:= Q^a_B(Q)^{\mathcal{J}^k}V^B(Q) = Q^a_B(Q)^{\mathcal{J}^k}\overset{B}{V_0}(Q) \\ &= [kmn]\left(-x^m\partial_{x^n}B_i(\boldsymbol{x}) - \delta^m_i B_n(\boldsymbol{x}), -x^m\partial_{x^n}\rho(\boldsymbol{x})\right). \end{aligned} \quad (190)$$

The reduction of the inverse metric is calculated as follows:

$$\begin{aligned} g^{ab}(q) &:= Q^a_C(Q)G^{CD}Q^b_D(Q) = \int d^3z \\ &\begin{pmatrix} \delta^k_i & -\frac{1}{e}\partial_{x^i}\frac{\eta(\boldsymbol{x})}{\rho^2(\boldsymbol{x})} & \frac{1}{e}\partial_{x^i}\frac{\xi(\boldsymbol{x})}{\rho^2(\boldsymbol{x})} \\ 0 & \frac{\xi(\boldsymbol{x})}{\rho(\boldsymbol{x})} & \frac{\eta(\boldsymbol{x})}{\rho(\boldsymbol{x})} \end{pmatrix} \delta(\boldsymbol{x}-\boldsymbol{z}) \begin{pmatrix} \delta_{kj} & 0 \\ -\frac{1}{e}\partial_{y^j}\frac{\eta(\boldsymbol{y})}{\rho^2(\boldsymbol{y})} & \frac{\xi(\boldsymbol{y})}{\rho(\boldsymbol{y})} \\ \frac{1}{e}\partial_{y^j}\frac{\xi(\boldsymbol{y})}{\rho^2(\boldsymbol{y})} & \frac{\eta(\boldsymbol{y})}{\rho(\boldsymbol{y})} \end{pmatrix} \delta(\boldsymbol{z}-\boldsymbol{y}) \\ &= \begin{pmatrix} \delta_{ij} + \partial_{x^i}\partial_{y^j}\frac{1}{e^2\rho(\boldsymbol{x})\rho(\boldsymbol{y})} & 0 \\ 0 & 1 \end{pmatrix} \delta(\boldsymbol{x}-\boldsymbol{y}) =: D^{ab}(q)\delta(\boldsymbol{x}-\boldsymbol{y}), \end{aligned} \quad (191)$$

where $D^{ab}(q)$ is a matrix valued differential operator which acts on the delta function. Finally, the reduced boost tensors are determined in a similar fashion, yielding

$$^{\mathcal{K}^k}k^{ab}(q) := Q^a_C(Q)^{\mathcal{K}^k}K^{CD}Q^b_D(Q) = D^{ab}(q)\frac{1}{2}(x^k + y^k)\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{192}$$

The reduced Poincaré charges are written

$$\mathcal{P}^0 = c_2(\frac{1}{2}g^{-1}) + c_0(v), \tag{193}$$

$$\mathcal{P}^k = c_1(^{\mathcal{P}^k}v), \tag{194}$$

$$\mathcal{J}^k = c_1(^{\mathcal{J}^k}v), \tag{195}$$

$$\mathcal{K}^k = -c_2(\frac{1}{2}{}^{\mathcal{K}^k}k) - c_0(^{\mathcal{K}^k}z) + t\mathcal{P}^k. \tag{196}$$

By our previous general discussion we know that they satisfy relations analogous to (124– 126) (with $C_\gamma = 0$), and so generate the Poincaré group action on γ. Notice that this means the $^{\mathcal{P}^k}v$ and $^{\mathcal{J}^k}v$ are Killing vectors for the metric g on m (unlike in the extended space), and also the $^{\mathcal{K}^k}k$ are Killing tensors in involution with each other, as before. The question of their covariant constancy will be taken up in section 6.3.

Reduction of the helix model proceeds in an analogous fashion [22]—we simply quote the results [refer to (39– 42)]: The adapted coordinates can be chosen as

$$q^a = (B, \rho), \tag{197}$$

$$F^1 = \theta,\ 0 \leq \theta < 2\pi, \tag{198}$$

where $B := Z - \theta$ and $X + iY =: \rho \exp i\theta$. B and ρ are obviously gauge invariant, and the Faddeev-Popov matrix is

$$F^1_1 = \phi_1 F^1 = 1, \tag{199}$$

which is nonsingular on M. The gradients of the physical coordinates are

$$Q^a_B(Q) = \frac{\partial q^a}{\partial Q^B} = \begin{pmatrix} \frac{Y}{\rho^2} & -\frac{X}{\rho^2} & 1 \\ \frac{X}{\rho} & \frac{Y}{\rho} & 0 \end{pmatrix}, \tag{200}$$

analogous to (186). Since the Hamiltonian potential V was chosen to be invariant along the gauge orbits we can define $v(q) := V(Q)$. The reduced inverse metric is

$$g^{ab}(q) = Q^a_C G^{CD} Q^b_D = \begin{pmatrix} \frac{Y}{\rho^2} & -\frac{X}{\rho^2} & 1 \\ \frac{X}{\rho} & \frac{Y}{\rho} & 0 \end{pmatrix} \begin{pmatrix} \frac{Y}{\rho^2} & \frac{X}{\rho} \\ -\frac{X}{\rho^2} & \frac{Y}{\rho} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{\rho^2} & 0 \\ 0 & 1 \end{pmatrix}, \tag{201}$$

which can be compared with (191).

Note that the gauge fixing surface $F^1 =$ constant is associated with a Gribov ambiguity (see, e.g. [16]) in that a given gauge orbit intersects such a surface more than once (in fact infinitely many times). But locally it is well defined, which is all that need concern us here.

This essentially completes the usual classical analysis. But before going on to quantization we shall first examine in more detail some geometrical structures associated with gauge theories in which M is endowed with a metric. This analysis will provide useful insight into the Dirac quantization.

4 Geometrical Structures in Gauge Theories

In this chapter we would like to illuminate some of the geometrical connections between the dynamics (at least the kinetic term in the Hamiltonian) and the gauge structure of the class of theories we have been considering. This includes a discussion of the relationship between the curvature of the extended and reduced spaces.

On the extended configuration manifold M we have two natural, or geometrical structures: First is the action of the gauge group on M as diffeomorphisms generated by the basis ϕ_α of gauge vector fields. We discussed this in chapter 1, the principle relationship being given in (10). This establishes M as a fibre bundle with projection $\pi : M \to m$, m being the space of gauge orbits, or reduced configuration space. A basic assumption we make here is that the ϕ_α chosen are linearly independent, but as yet impose no restrictions on the structure functions. We saw in (34) that this assumption is valid for scalar electrodynamics (for $\rho \neq 0$).

The second structure is the kinetic energy term of the Hamiltonian, which yields a real, symmetric contravariant tensor G^{-1} on M. A fundamental assumption we make here is that the kinetic energy, and equivalently G^{-1}, are positive definite. This implies that G^{-1} is invertible (which justifies the notation), and we denote its inverse as G, which serves as a natural metric (also positive definite) on M. Even in field theories G^{-1} is usually ultralocal (contains no derivatives of δ-functions), so there is no difficulty in forming the inverse. Our two examples easily satisfy this assumption.

4.1 "Kaluza-Klein" Form of the Line Element

By letting each of these two structures act on each other in a natural way we generate two more structures: First, given a metric G on M we can form an inner product of vector fields, and are thus lead to construct

$$\gamma_{\alpha\beta} := G(\phi_\alpha, \phi_\beta), \tag{202}$$

which is real and symmetric, and is essentially a metric induced on the orbits. In fact, $\gamma_{\alpha\beta}$ is positive definite:

$$\gamma(\mu, \mu) := \gamma_{\alpha\beta}\mu^\alpha\mu^\beta = G(\mu^\alpha\phi_\alpha, \mu^\beta\phi_\beta) > 0 \ \forall\mu(\neq 0) \in \mathcal{G} \tag{203}$$

since, by the linear independence of the ϕ_α, $\mu^\alpha\phi_\alpha \neq 0$ unless all $\mu^\alpha = 0$, and G is positive definite. In particular, then, $\gamma_{\alpha\beta}$ is invertible, and we denote its inverse by $\gamma^{\alpha\beta}$. Notice that invertibility of G and linear independence of the ϕ_α are not sufficient to establish the invertibility of $\gamma_{\alpha\beta}$, a fact which is sometimes overlooked in the literature [32]. It is positive definiteness, and not invertibility, that is primary here.

For scalar electrodynamics we use (20) to compute

$$\gamma_{\alpha\beta} = \left(-\frac{1}{e^2}\partial^2 + \rho^2\right)_{\boldsymbol{x}} \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{204}$$

where $\partial^2_{\boldsymbol{x}} := \partial_{x^k}\partial_{x^k}$, and its finite (one) dimensional counterpart

$$\gamma_{\alpha\beta} = \left(1 + \rho^2\right) \tag{205}$$

in the helix model [see (40)]. Notice that the positive definiteness of $\gamma_{\alpha\beta}$ in the field theory case requires suitable boundary conditions.

Second, the projection $\pi : M \to m$ induces the push forward of G^{-1}:

$$g^{-1} := \pi_*(G^{-1}), \tag{206}$$

which is a real, symmetric, contravariant tensor on m. Of course for the push forward to be well defined, g^{-1} and G^{-1} must be π-related:

$$g^{-1}_{\pi(Q)} = \pi_*(G^{-1}_Q)\ \forall Q \in M. \tag{207}$$

This is equivalent to (38), a property of the system we have already taken for granted, namely preservation of the classical constraints under time evolution. Indeed, in general this push forward is precisely the reduction process of homogeneous observables [see (181)], and π-relatedness is just observable status.

Using arguments similar to those that established the positive definiteness of $\gamma_{\alpha\beta}$, we can readily see that g^{-1}, too is positive definite. In fact, the two statements are equivalent, a fact worth demonstrating since we shall need some of the intermediate results later anyway.

Consider the components of G and G^{-1} in adapted coordinates:

$$G_{\overline{AB}} = \begin{pmatrix} G_{ab} & G_{a\beta} \\ G_{\alpha b} & G_{\alpha\beta} \end{pmatrix} \text{ and } G^{\overline{AB}} = \begin{pmatrix} (G^{-1})^{ab} & (G^{-1})^{a\beta} \\ (G^{-1})^{\alpha b} & (G^{-1})^{\alpha\beta} \end{pmatrix}, \tag{208}$$

respectively, where

$$\begin{aligned} G_{ab} &= Q^A_a G_{AB} Q^B_b, & (209)\\ G_{a\beta} &= G_{\beta a} = Q^A_a G_{AB} \Phi^B_\beta, & (210)\\ G_{\alpha\beta} &= \Phi^A_\alpha G_{AB} \Phi^B_\beta, & (211) \end{aligned}$$

and

$$\begin{aligned} (G^{-1})^{ab} &= Q^a_A G^{AB} Q^b_B, & (212)\\ (G^{-1})^{a\beta} &= (G^{-1})^{\beta a} = Q^a_A G^{AB} \Phi^\beta_B, & (213)\\ (G^{-1})^{\alpha\beta} &= \Phi^\alpha_A G^{AB} \Phi^\beta_B. & (214) \end{aligned}$$

From (171) and (202) we note that

$$G_{\alpha\beta} = (F^{-1})^\gamma_\alpha \gamma_{\gamma\delta} (F^{-1})^\delta_\beta. \tag{215}$$

Also, $(G^{-1})^{ab} = g^{ab}$, the components of g^{-1} in the coordinates q^a on m [see (206)]. In general, the 'holonomic horizontal' subspace of TM spanned by the $\partial/\partial q^a$ need not be orthogonal to the gauge orbits:

$$G_{a\beta} = G\left(\frac{\partial}{\partial q^a}, \frac{\partial}{\partial F^\beta}\right) \neq 0. \tag{216}$$

We will elaborate on this shortly, here we simply want to point out that $(G^{-1})^{\alpha\beta}$ is not the inverse of $G_{\alpha\beta}$, nor is G_{ab} the inverse of $(G^{-1})^{ab} = g^{ab}$.

Now, $\gamma_{\alpha\beta}$ positive definite implies

$$G_{\alpha\beta}\mu^\alpha\mu^\beta = \gamma_{\gamma\delta}\left[(F^{-1})^\gamma_\alpha\mu^\alpha\right]\left[(F^{-1})^\delta_\beta\mu^\beta\right] > 0 \;\forall\mu(\neq 0) \in \mathcal{G} \tag{217}$$

since the Faddeev-Popov matrix has no null vectors; so $G_{\alpha\beta}$ is positive definite. The converse is similar.

To establish that the positive definiteness of $G_{\alpha\beta}$ is equivalent to the positive definiteness of $(G^{-1})^{ab}$ it is convenient to use matrix notation for (208):

$$G =: \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \text{ and } G^{-1} =: \begin{pmatrix} \alpha & \beta \\ \beta^T & \delta \end{pmatrix}. \tag{218}$$

Then D positive definite implies D invertible, which allows us to decompose G as follows:

$$G = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A - BD^{-1}B^T & 0 \\ D^{-1}B^T & 1 \end{pmatrix}. \tag{219}$$

Then using the positive definiteness of G we have

$$0 < (x^T\; y^T)G\begin{pmatrix} x \\ y \end{pmatrix} = x^T(A - BD^{-1}B^T)x + (y + D^{-1}B^Tx)^T D(y + D^{-1}B^Tx) \tag{220}$$

for all nonvanishing $(x^T\; y^T)$. In particular, with the restriction $y = -D^{-1}B^Tx$ we learn that $(A - BD^{-1}B^T)$ is positive definite, and hence invertible. But, inverting (219) we have

$$G^{-1} = \begin{pmatrix} (A - BD^{-1}B^T)^{-1} & 0 \\ -D^{-1}B^T(A - BD^{-1}B^T)^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix}, \tag{221}$$

from which we read off [see (218)]

$$\alpha^{-1} = A - BD^{-1}B^T \tag{222}$$

or

$$g_{ab} = G_{ab} - G_{a\alpha}G^{\alpha\beta}G_{\beta b}. \tag{223}$$

Here g_{ab} (resp. $G^{\alpha\beta}$) denotes the matrix inverse of g^{ab} (resp. $G_{\alpha\beta}$). Hence, positive definiteness of $G_{\alpha\beta}$ implies positive definiteness of g^{ab} (and g_{ab}). The converse argument is exactly reciprocal.

To summarize, we have established the positive definiteness of both metrics g_{ab} and $\gamma_{\alpha\beta}$, the equivalency of these two statements, and an explicit expression for g_{ab}. These two metrics will play a central role in the following discussions. In the case of our two examples, g^{ab} is given in (191) and (201), respectively.

Consider the length element on M, expressed in the adapted coordinate basis:

$$ds^2 = G_{ab}dq^a dq^b + G_{a\beta}dq^a dF^\beta + G_{\alpha b}dF^\alpha dq^b + G_{\alpha\beta}dF^\alpha dF^\beta. \tag{224}$$

We can bring this to block diagonal, or Kaluza-Klein form by constructing the one-forms

$$\delta F^\alpha := dF^\alpha + B^\alpha_a dq^a, \tag{225}$$

which, in general, are not exact. Using these in (224) we find that

$$B^\alpha_a = G^{\alpha\beta}G_{\beta a} \tag{226}$$

'completes the square', and we have

$$ds^2 = g_{ab}dq^a dq^b + G_{\alpha\beta}\delta F^\alpha \delta F^\beta, \tag{227}$$

where g_{ab} was given earlier in (223).

With a rescaling of the one-forms dF^α,$B^\alpha := B^\alpha_a dq^a$ by the inverse Faddeev-Popov matrix:

$$\phi^\alpha := (F^{-1})^\alpha_\beta dF^\beta, \tag{228}$$

$$A^\alpha := (F^{-1})^\alpha_\beta B^\beta, \tag{229}$$

we can define a new nonholonomic basis of one-forms on M, $e^{\overline{A}} := \{e^a, e^\alpha\}$, where

$$e^a := dq^a \tag{230}$$

$$e^\alpha := \phi^\alpha + A^\alpha. \tag{231}$$

In this basis the length element can be decomposed in terms of the two metrics g_{ab} and $\gamma_{\alpha\beta}$:

$$ds^2 = g_{ab}e^a e^b + \gamma_{\alpha\beta}e^\alpha e^\beta \tag{232}$$

[see (215)]. The corresponding dual basis of vector fields, $w_{\overline{A}} := \{w_a, w_\alpha\}$, is given by

$$w_a = \frac{\partial}{\partial q^a} - A^\alpha_a \phi_\alpha, \tag{233}$$

$$w_\alpha = \phi_\alpha, \tag{234}$$

which satisfy $\langle e^{\overline{A}}, w_{\overline{B}}\rangle = \delta^{\overline{A}}_{\overline{B}}$.

4.2 Horizontal/Vertical Nonholonomic Basis

We shall make extensive use of this basis in the coming pages, so let us examine it in more detail here. First notice that, although e^α is constructed from the coordinate one-forms dq^a and dF^α, it is actually independent of the choice of coordinates. It might be instructive to demonstrate this explicitly: Under the coordinate transformation (179) we have

$$dq^a \mapsto d\tilde{q}^a = \frac{\partial \tilde{q}^a}{\partial q^b} dq^b, \tag{235}$$

$$
\begin{aligned}
dF^\beta \quad &\mapsto \quad d\tilde{F}^\beta = \left.\frac{\partial \tilde{F}^\beta}{\partial q^a}\right|_F dq^a + \left.\frac{\partial \tilde{F}^\beta}{\partial F^\delta}\right|_q dF^\delta, && (236)\\
Q_a = \left.\frac{\partial}{\partial q^a}\right|_F \quad &\mapsto \quad \tilde{Q}_a = \left.\frac{\partial}{\partial \tilde{q}^a}\right|_{\tilde{F}} = \frac{\partial q^b}{\partial \tilde{q}^a}\left(\left.\frac{\partial}{\partial q^b}\right|_F + \left.\frac{\partial F^\gamma}{\partial q^b}\right|_{\tilde{F}} \left.\frac{\partial}{\partial F^\gamma}\right|_q\right), && (237)\\
(F^{-1})^\alpha_\beta \quad &\mapsto \quad (\tilde{F}^{-1})^\alpha_\beta = (F^{-1})^\alpha_\gamma \left.\frac{\partial F^\gamma}{\partial \tilde{F}^\beta}\right|_q . && (238)
\end{aligned}
$$

Using the equivalent form

$$
A^\alpha = \gamma^{\alpha\beta} G\left(\phi_\beta, \left.\frac{\partial}{\partial q^a}\right|_F\right) dq^a =: \gamma^{\alpha\beta}\phi_\beta \cdot Q_a dq^a \tag{239}
$$

we find

$$
\begin{aligned}
\phi^\alpha \quad &\mapsto \quad (\tilde{F}^{-1})^\alpha_\beta d\tilde{F}^\beta = \phi^\alpha + (F^{-1})^\alpha_\gamma \left.\frac{\partial F^\gamma}{\partial \tilde{F}^\beta}\right|_q \left.\frac{\partial \tilde{F}^\beta}{\partial q^a}\right|_F dq^a, && (240)\\
A^\alpha \quad &\mapsto \quad \gamma^{\alpha\beta}\phi_\beta \cdot \tilde{Q}_a d\tilde{q}^a = A^\alpha + (F^{-1})^\alpha_\gamma \left.\frac{\partial F^\gamma}{\partial q^a}\right|_{\tilde{F}} dq^a. && (241)
\end{aligned}
$$

Now recall from calculus that if x, y, and z are related by some $f(x, y, z) = 0$ then

$$
\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x = -\left.\frac{\partial x}{\partial z}\right|_y , \tag{242}
$$

so since $\tilde{F}^\alpha = \tilde{F}^\alpha(q, F)$ we have that $\phi^\alpha + A^\alpha \mapsto \phi^\alpha + A^\alpha$. Thus e^α is invariant, as is its dual, w_α. Similarly, e^a and w_a transform in an obvious fashion, dual to each other.

Alternatively, using (170) in (239) we have

$$
\begin{aligned}
A^\alpha_A &= \gamma^{\alpha\beta}\phi^B_\beta G_{BC} Q^C_a Q^a_A = \gamma^{\alpha\beta}\phi^B_\beta G_{BC}\left(\delta^C_A - \phi^C_\gamma \phi^\gamma_A\right)\\
&= \gamma^{\alpha\beta}\phi^B_\beta G_{BA} - \phi^\alpha_A, && (243)
\end{aligned}
$$

or

$$
e^\alpha_A = \gamma^{\alpha\beta} G_{AB} w^B_\beta . \tag{244}
$$

Similarly, it is easy to show that

$$w_a^A = g_{ab} G^{AB} e_B^b. \tag{245}$$

Notice that one converts from a basis element to its dual by using the various metrics to raise and lower their respective indices. The transformation properties of the nonholonomic basis are now obvious. We also identify this as the same basis used, for example, by Kuchař [21].

Since the metric (232) is block diagonal in this basis, we have the important result

$$G(w_a, w_\beta) = 0 \; \forall a, \beta, \tag{246}$$

which can be compared with (216).

Using the adapted coordinates selected in section 3.3, let us calculate e^α for our two examples. Since the Faddeev-Popov matrix is just the identity matrix in both cases [see (184) and (199)], we have simply $\phi^\alpha = dF^\alpha$, which are exact. The A^α require more work. We first need to determine the vectors $Q_a^A = \partial Q^A / \partial q^a$. Inspection of the adapted coordinates for scalar electrodynamics, see (182) and (183), yields

$$Q_c^A = \begin{pmatrix} \delta_k^i & 0 \\ 0 & \cos\theta(\boldsymbol{x}) \\ 0 & \sin\theta(\boldsymbol{x}) \end{pmatrix} \delta(\boldsymbol{x} - \boldsymbol{z}), \tag{247}$$

where A is the row index and c the column index. Using ϕ_β^A from (20), and the trivial metric $G_{AB} = \delta_{AB}$, we find

$$\phi_\beta \cdot Q_c = \left(-\frac{1}{e} \partial_{z^k} \delta(\boldsymbol{z} - \boldsymbol{y}), 0 \right). \tag{248}$$

The inverse of $\gamma_{\alpha\beta}$ in (204) is a Green function, which we denote as

$$\gamma^{\alpha\beta} =: \frac{1}{\left(-\frac{1}{e^2}\partial^2 + \rho^2 \right)_{\boldsymbol{x}}} \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{249}$$

It is a functional of the magnitude, ρ, of the scalar field. If ρ is localized, i.e. $\rho(\boldsymbol{x}) \to 0$ sufficiently rapidly a $|\boldsymbol{x}| \to \infty$, we assume that suitable boundary conditions can be chosen to determine $\gamma^{\alpha\beta}$. Using these results in (239) we find

$$\begin{aligned} A^\alpha &= \int d^3z\, d^3y \frac{1}{\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)_{\boldsymbol{x}}} \delta(\boldsymbol{x}-\boldsymbol{y}) \left(-\frac{1}{e}\partial_{z^k}\delta(\boldsymbol{z}-\boldsymbol{y}), 0\right) \begin{pmatrix} dB_k(\boldsymbol{z}) \\ d\rho(\boldsymbol{z}) \end{pmatrix} \\ &= \int d^3z \left(-\frac{1}{e}\partial_{z^k} \frac{1}{\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)_{\boldsymbol{x}}} \delta(\boldsymbol{x}-\boldsymbol{z})\right) dB_k(\boldsymbol{z}). \end{aligned} \tag{250}$$

The important point here is that the coefficient of $dB_k(\boldsymbol{z})$ depends on ρ, so the one-forms A^α are not closed (see also Kunstatter [33]). Thus $de^\alpha \neq 0$ for scalar electrodynamics, a fact which will play a role in our later discussions.

The analogous result also occurs in the helix model, but can be calculated more easily: From (197) and (198) we find

$$Q_c^A = \begin{pmatrix} 0 & \cos\theta \\ 0 & \sin\theta \\ 1 & 0 \end{pmatrix}, \tag{251}$$

so with (40) we have

$$\phi_\beta \cdot Q_c = (1, 0)\,. \tag{252}$$

The metric $\gamma_{\alpha\beta}$ in (205) is trivially inverted, yielding

$$A^\alpha = \frac{1}{1+\rho^2}(1,0)\begin{pmatrix} dB \\ d\rho \end{pmatrix} = \frac{1}{1+\rho^2} dB, \tag{253}$$

which is obviously not closed. Since the ϕ^α are exact we have

$$de^\alpha = dA^\alpha = \frac{2\rho}{(1+\rho^2)^2} dB \wedge d\rho. \tag{254}$$

We have seen that the action of the gauge group on M induces a fibre bundle structure $\pi : M \to m$. The gauge vectors ϕ_α allow us to move within the fibres, but as yet we have not identified a natural way to move in a horizontal direction, to connect one fibre with its neighbour. This is achieved using the metric G. At any point $Q \in M$ we can decompose the tangent space,

$$T_Q M \simeq V_Q M \oplus H_Q M \tag{255}$$

into a vertical subspace, $V_Q M$, spanned by the w_α, and a horizontal subspace, $H_Q M$, spanned by the w_a. Because of (246) these two spaces are orthogonal:

$$V_Q M \perp H_Q M. \tag{256}$$

We would like to emphasize that this vertical/horizontal decomposition is, of course, independent of any choice of coordinates or choice of basis for the gauge vectors ϕ_α—the gauge structure π gives us $V_Q M$ and the metric structure G gives us $H_Q M$.

4.3 Yang-Mills Connection and Curvature on the Extended Field Space

Establishing $H_Q M$ as above is not in itself sufficient to endow M with a connection (see, e.g. [26]): we further demand that the vertical/horizontal decomposition is *compatible* with the action of the gauge group on M. This means

$$\mathcal{L}_{\phi_\alpha} w_\beta = [w_\alpha, w_\beta] \in VM, \tag{257}$$

$$\mathcal{L}_{\phi_\alpha} w_b = [w_\alpha, w_b] \in HM, \tag{258}$$

i.e. the vertical and horizontal subspaces are invariant under gauge transformations generated by the ϕ_α. The first condition is obviously satisfied, being just the integrability condition for the gauge orbits; the second requires some investigation.

With an eye on (233), we first calculate

$$\mathcal{L}_{\phi_\alpha} \frac{\partial}{\partial q^a} = [F^\beta_\alpha \frac{\partial}{\partial F^\beta}, \frac{\partial}{\partial q^a}] = -(F^{-1})^\gamma_\beta \frac{\partial}{\partial q^a} F^\beta_\alpha \phi_\gamma, \tag{259}$$

where we used (171). Then

$$\begin{aligned}\mathcal{L}_{\phi_\alpha} w_a &= \mathcal{L}_{\phi_\alpha}\frac{\partial}{\partial q^a} - \mathcal{L}_{\phi_\alpha} A^\beta_a \phi_\beta - A^\beta_a \mathcal{L}_{\phi_\alpha}\phi_\beta \\ &= -\left[\mathcal{L}_{\phi_\alpha} A^\gamma_a + f^\gamma_{\alpha\beta} A^\beta_a + (F^{-1})^\gamma_\beta \frac{\partial}{\partial q^a} F^\beta_\alpha\right]\phi_\gamma. \qquad (260)\end{aligned}$$

Since this has only a vertical, or gauge piece, and the ϕ_α are linearly independent, (258) is equivalent to

$$\mathcal{L}_{\phi_\alpha} A^\gamma_a = -f^\gamma_{\alpha\beta} A^\beta_a + F^\beta_\alpha \frac{\partial}{\partial q^a}(F^{-1})^\gamma_\beta, \qquad (261)$$

which also means

$$[w_\alpha, w_b] = 0. \qquad (262)$$

Now, whether or not A^γ_a transforms as in (261) is not *a priori* obvious, and needs to be checked in any given theory. But A^γ_a arises essentially from an interplay between the gauge and metric structure on M, and so we should be able to connect this transformation law with the fundamental relation (38).

Thus, let us begin with (245) instead of (233). Since the q^a are gauge invariant,

$$\mathcal{L}_{\phi_\alpha} e^a = \mathcal{L}_{\phi_\alpha} dq^a = d\phi_\alpha q^a = 0, \qquad (263)$$

and the metric on m is also gauge invariant, so we learn from (245) and (38) that

$$\begin{aligned}(\mathcal{L}_{\phi_\alpha} w_a)^A &= g_{ab}(\mathcal{L}_{\phi_\alpha} G)^{AB} e^b_B = g_{ab}\zeta^{\beta(A}_\alpha \phi_\beta{}^{B)} e^b_B \\ &= \frac{1}{2} g_{ab}\zeta^{\beta b}_\alpha w^A_\beta, \qquad (264)\end{aligned}$$

where $\zeta^{\beta b}_\alpha := \langle e^b, \zeta^\beta_\alpha\rangle$, the horizontal, or physical projection of the vector ζ^β_α. Again, since this has only a vertical piece, (258) is equivalent to

$$\zeta^{\beta a}_\alpha = 0, \qquad (265)$$

where we used the invertibility of g_{ab} and the linear independence of the w_β.

This is an extra condition relating the metric and gauge structures, beyond the usual preservation of the classical constraints under time evolution. In the language of (38) it appears as

$$(\mathcal{L}_{\phi_\alpha} G)^{AB} = \zeta_\alpha^{\beta\gamma} \phi_\beta^{(A} \phi_\gamma^{B)}, \tag{266}$$

where $\zeta_\alpha^{\beta\gamma} := \langle e^\gamma, \zeta_\alpha^\beta \rangle$, the vertical projection of the vector ζ_α^β, is still arbitrary. Recall that in our two examples the ϕ_α are actually Killing vectors ($\zeta = 0$), so in both cases the vertical/horizontal decomposition is compatible with the action of the gauge group on M (at least with this choice of gauge vector basis—see below).

It is often suggested in the literature [21,34] that the classical symmetry under change of gauge vector basis be preserved at the quantum level. We will come back to this point again in chapter 5, and whether or not this idea is compatible with the van Hove restrictions on factor ordering we encounter in this work. But even before we get to that we want to see if this idea is compatible with the conditions (257) and (258). Of course for any $\mu, \nu \in \mathcal{G}$ [see (11)], $[\mu, \nu] \in \mathcal{G}$, so (257) is still satisfied.

As noted earlier, from (245) we see that the w_a are independent of any choice of gauge vector basis, so we are interested in

$$\begin{aligned}
(\mathcal{L}_\mu w_a)^A &= g_{ab} (\mathcal{L}_\mu G)^{AB} e_B^b \\
&= g_{ab} \left[\mu^\alpha (\mathcal{L}_{\phi_\alpha} G)^{AB} - \phi_\alpha^A G^{BC} \partial_C \mu^\alpha - \phi_\alpha^B G^{AC} \partial_C \mu^\alpha \right] e_B^b \\
&= \left[\frac{1}{2} g_{ab} \mu^\alpha \zeta_\alpha^{\beta b} - w_a \mu^\beta \right] w_\beta^A .
\end{aligned} \tag{267}$$

Assuming we have found a 'preferred' basis ϕ_α in which (265) is satisfied, the coefficients μ^α are not arbitrary, but must satisfy

$$w_a \mu^\beta = 0, \tag{268}$$

i.e. they must be constant in the horizontal direction. At the very least this tells us that the notion of full symmetry under change of gauge vector basis is not compatible with the notion of gauge invariance of the horizontal subspaces.

Another way to see this is to take the Lie derivative of (10) with respect to w_a. Assuming gauge invariance of the horizontal subspaces, (262), we have on the left hand side

$$[w_a, [w_\alpha, w_\beta]] = -[w_\beta, [w_a, w_\alpha]] - [w_\alpha, [w_\beta, w_a]] = 0, \tag{269}$$

whereas on the right hand side we find

$$w_a(f^\gamma_{\alpha\beta})w_\gamma + f^\gamma_{\alpha\beta}[w_a, w_\gamma] = w_a(f^\gamma_{\alpha\beta})w_\gamma. \tag{270}$$

Hence the structure functions are restricted to be horizontally constant:

$$w_a f^\gamma_{\alpha\beta} = 0. \tag{271}$$

But notice that the notion of gauge invariance of the horizontal subspaces *is*, of course, compatible with the more restrictive symmetry of arbitrary constant μ^α in the case of structure constants, i.e., in the case of a Lie group acting on M with linearly independent ϕ_α, as in our two examples.

An equivalent way to define this connection on M is to recognize that the e^α are components of a standard connection one-form in the sense that

$$\langle e^\alpha, w_\beta \rangle = \delta^\alpha_\beta, \tag{272}$$

and

$$\begin{aligned} \mathcal{L}_{\phi_\alpha} e^\gamma &= \langle \mathcal{L}_{\phi_\alpha} e^\gamma, w_b \rangle e^b + \langle \mathcal{L}_{\phi_\alpha} e^\gamma, w_\beta \rangle e^\beta \\ &= -\langle e^\gamma, \mathcal{L}_{\phi_\alpha} w_b \rangle e^b - \langle e^\gamma, \mathcal{L}_{\phi_\alpha} w_\beta \rangle e^\beta \\ &= -f^\gamma_{\alpha\beta} e^\beta, \end{aligned} \tag{273}$$

a result which relies upon (258). In other words, e^γ transforms according to the adjoint representation of the gauge group. Using the e^α, horizontal vectors $V \in HM$ are defined by

$$\langle e^\alpha, V\rangle = 0 \; \forall \alpha. \tag{274}$$

In fact, in the decomposition (231), the ϕ^α essentially correspond to the Maurer-Cartan form, and the A^α to a Yang-Mills field[3]. Then the condition (261) corresponds to the requirement that this field have a standard gauge transformation law. (Of course this Yang-Mills field is not to be confused with the electromagnetic potential $A_i(\boldsymbol{x})$).

Given this connection on M we can calculate its curvature:

$$\begin{aligned}
[w_a, w_b] &= [\partial_a - A_a^\alpha \phi_\alpha, \partial_b - A_b^\beta \phi_\beta] \\
&= -\left(\partial_a A_b^\gamma - \partial_b A_a^\gamma + f^\gamma_{\alpha\beta} A_a^\alpha A_b^\beta\right) w_\gamma \\
&\quad -A_a^\alpha [w_\alpha, w_b] + A_b^\beta [w_\beta, w_a] \\
&= -\mathcal{F}^\gamma{}_{ab} w_\gamma,
\end{aligned} \tag{276}$$

where we used the gauge invariance of the horizontal subspaces, (262). The curvature two-form

$$\mathcal{F}^\gamma := \frac{1}{2}\mathcal{F}^\gamma{}_{ab} dq^a \wedge dq^b, \tag{277}$$

where (with $\partial_a := \partial/\partial q^a$)

$$\mathcal{F}^\gamma{}_{ab} := \partial_a A_b^\gamma - \partial_b A_a^\gamma + f^\gamma_{\alpha\beta} A_a^\alpha A_b^\beta. \tag{278}$$

[3]Strictly speaking (see, e.g. [26]), a Yang-Mills field is a local representative of e^α obtained by defining a local section $\overline{m} : m \to M$ of M, which in our case would correspond to an embedded surface representing the reduced configuration space m, and then defining the pullback $\overline{e}^\alpha := \overline{m}_*(e^\alpha)$. If, for example, we take the gauge fixing condition $F^\alpha = 0$ to define $\overline{m}$, we get

$$\overline{e}^\alpha(q) = A^\alpha(q, F = 0). \tag{275}$$

We would like to express this in terms of de^α, a quantity we have already calculated (at least for the helix model). Recall that for any basis $e^{\overline{C}}$ and its dual $w_{\overline{C}}$ we have the identity

$$de^{\overline{C}} \equiv -\frac{1}{2}[w_{\overline{A}}, w_{\overline{B}}]^{\overline{C}} e^{\overline{A}} \wedge e^{\overline{B}}. \tag{279}$$

Thus, collecting together (10), (262), and (276) we find

$$de^\gamma = \frac{1}{2}\mathcal{F}^\gamma{}_{ab} dq^a \wedge dq^b - \frac{1}{2} f^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta, \tag{280}$$

which is just the Cartan structural equation

$$\mathcal{F}^\gamma = de^\gamma + \frac{1}{2} f^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta. \tag{281}$$

A final result we will need concerns the gauge dependence of $\mathcal{F}^\gamma{}_{ab}$. Taking the Lie derivative of both sides of (276) with respect to ϕ_α we find, on the left hand side

$$[w_\alpha, [w_a, w_b]] \equiv -[w_b, [w_\alpha, w_a]] - [w_a, [w_b, w_\alpha]] = 0$$

by the gauge invariance of the horizontal subspaces, and on the right hand side

$$-\mathcal{L}_{\phi_\alpha} \mathcal{F}^\gamma{}_{ab} w_\gamma - \mathcal{F}^\gamma{}_{ab} f^\beta_{\alpha\gamma} w_\beta.$$

Hence the curvature transforms according to the adjoint representation, as expected:

$$\mathcal{L}_{\phi_\alpha} \mathcal{F}^\gamma{}_{ab} = -f^\gamma_{\alpha\beta} \mathcal{F}^\beta{}_{ab}. \tag{282}$$

Now, in the case of the helix model we can use (254) in (281) to get

$$\mathcal{F}^\gamma = de^\gamma = \frac{2\rho}{(1+\rho^2)^2} dB \wedge d\rho. \tag{283}$$

Note that $\mathcal{F}^\gamma \neq 0$ means the distribution of horizontal subspaces $H_Q M$ is not integrable, which in our case means it is not possible, even locally, to find a surface $\overline{m}$ embedded in M, representing the reduced configuration space m, which is orthogonal to the gauge orbits.

4.4 Levi-Civita Connections on the Extended and Reduced Field Spaces

So far we have discussed the Yang-Mills connection (and curvature) associated with the induced fibre bundle structure of M. There are three other connections to discuss: the Levi-Civita connections associated with the metrics G_{AB}, g_{ab} and $\gamma_{\alpha\beta}$, on the extended configuration space M, the reduced configuration space m, and the orbits, respectively. The relationship between these connections, as well as their respective curvatures, will become important in interpreting the Dirac quantization, but are of interest in their own right as well.

We begin with the Levi-Civita connection, ∇, associated with the metric G on M. It is convenient to use the horizontal/vertical basis $e^{\overline{A}}$ and $w_{\overline{A}}$ of one-forms and vectors defined earlier. Arbitrary one-forms and vectors are written as

$$\Theta = \Theta_{\overline{A}} e^{\overline{A}} \quad \text{and} \quad V = V^{\overline{A}} w_{\overline{A}} \tag{284}$$

for scalars $\Theta_{\overline{A}}$ and $V^{\overline{A}}$, and the metric appears as

$$G = G_{\overline{AB}} e^{\overline{A}} \otimes e^{\overline{B}}, \tag{285}$$

with $G_{\overline{AB}} = \text{diagonal}(g_{ab}, \gamma_{\alpha\beta})$ [see (232)]. The action of ∇:

$$\nabla\Theta = \nabla\Theta_{\overline{C}} \otimes e^{\overline{C}} + \Theta_{\overline{C}} \nabla e^{\overline{C}}, \tag{286}$$

is defined by

$$\begin{aligned} \nabla\Theta_{\overline{C}} &:= d\Theta_{\overline{C}} && (287) \\ \nabla e^{\overline{C}} &:= -\Gamma^{\overline{C}}_{\overline{AB}} e^{\overline{A}} \otimes e^{\overline{B}}, && (288) \end{aligned}$$

where the $\Gamma^{\overline{C}}_{\overline{AB}}$ are Ricci rotation coefficients. A similar relation exists for vector fields:

$$\nabla w_{\overline{B}} := \Gamma^{\overline{C}}_{\overline{AB}} e^{\overline{A}} \otimes w_{\overline{C}}, \tag{289}$$

such that $\nabla\langle\Theta, V\rangle = \langle\nabla\Theta, V\rangle + \langle\Theta, \nabla V\rangle = d\langle\Theta, V\rangle$.

As usual, the Levi-Civita connection is uniquely determined by the conditions of no torsion and metricity. For arbitrary vector fields U and V on M, the torsion

$$\begin{aligned} T(U,V) &:= \nabla_U V - \nabla_V U - [U,V] \\ &= U^{\overline{A}} V^{\overline{B}} \left(\Gamma^{\overline{C}}_{\overline{AB}} - \Gamma^{\overline{C}}_{\overline{BA}}\right) w_{\overline{C}} + U(V^{\overline{B}}) w_{\overline{B}} - V(U^{\overline{B}}) w_{\overline{B}} - [U,V] \\ &= U^{\overline{A}} V^{\overline{B}} \left\{\left(\Gamma^{\overline{C}}_{\overline{AB}} - \Gamma^{\overline{C}}_{\overline{BA}}\right) w_{\overline{C}} - [w_{\overline{A}}, w_{\overline{B}}]\right\}. \end{aligned} \tag{290}$$

Note that since the horizontal/vertical basis is nonholonomic, no-torsion does not mean the usual condition $\Gamma^{\overline{C}}_{\overline{AB}} = \Gamma^{\overline{C}}_{\overline{BA}}$, but rather

$$\Gamma^{\overline{C}}_{\overline{AB}} - \Gamma^{\overline{C}}_{\overline{BA}} = C^{\overline{C}}_{\overline{AB}}, \tag{291}$$

where the C's are defined by the commutators

$$[w_{\overline{A}}, w_{\overline{B}}] =: C^{\overline{C}}_{\overline{AB}} w_{\overline{C}}. \tag{292}$$

Collecting the results of (276), (262) and (10) we learn that the nonvanishing C's are

$$C^{\gamma}_{ab} = -\mathcal{F}^{\gamma}{}_{ab}, \tag{293}$$

$$C^{\gamma}_{\alpha\beta} = f^{\gamma}_{\alpha\beta}. \tag{294}$$

Note that we are assuming that the horizontal subspaces are gauge invariant [see (265)].

Using (291) with the metricity condition $\nabla G = 0$ then determines, in the usual manner, the Ricci rotation coefficients:

$$\Gamma^{\overline{C}}_{\overline{AB}} = \frac{1}{2} G^{\overline{CD}} \left\{ w_{\overline{A}} G_{\overline{BD}} + w_{\overline{B}} G_{\overline{AD}} - w_{\overline{D}} G_{\overline{AB}} - C_{\overline{ABD}} - C_{\overline{BAD}} + C_{\overline{DAB}} \right\}, \tag{295}$$

where $C_{\overline{CAB}} := G_{\overline{CD}} C^{\overline{D}}_{\overline{AB}}$.

The calculation of the Γ's involves computing derivatives of components of the metric in (285). We recall that the components g_{ab} are gauge invariant—a necessary condition for the Hamiltonian to be a classical observable. The same is not necessarily true for the $\gamma_{\alpha\beta}$ components.

Using (38) and (273) we learn that

$$\phi_\gamma \gamma^{\alpha\beta} = \mathcal{L}_{\phi_\gamma}\left(G^{AB} e_A^\alpha e_B^\beta\right) = \zeta_\gamma^{\alpha\beta} - f_{\gamma\delta}^\alpha \gamma^{\delta\beta} - f_{\gamma\delta}^\beta \gamma^{\alpha\delta}. \tag{296}$$

Except for the ζ term, this looks like a transformation according to the adjoint representation on the $\alpha\beta$ indices. Tracing with $\gamma_{\alpha\beta}$ we find

$$\phi_\gamma \ln\sqrt{\gamma} = \nabla\cdot\phi_\gamma - f_{\alpha\gamma}^\alpha, \tag{297}$$

where we used the fact that the (Levi-Civita) divergence of a (gauge) vector is given by

$$\nabla\cdot\phi_\gamma = -\frac{1}{2}G_{AB}\left(\mathcal{L}_{\phi_\gamma}G\right)^{AB} = -\frac{1}{2}\zeta_\gamma^{\alpha\beta}\gamma_{\alpha\beta}. \tag{298}$$

Here $\sqrt{\gamma} := \sqrt{\det\gamma_{\alpha\beta}}$ is the volume element on the gauge orbits.

In section 5.3 we shall argue that a consistent Dirac quantization requires the existence of a basis of gauge vectors which are divergence-free. Furthermore, for a gauge theory based on a compact semi-simple Lie group the $f_{\alpha\beta}^\gamma$ are trace-free. (For more discussion see [33,28]). In any case, we shall assume

$$\nabla\cdot\phi_\gamma = 0 \;\text{ and }\; f_{\alpha\gamma}^\alpha = 0, \tag{299}$$

which is certainly borne out in our two examples. This means $\sqrt{\gamma}$ is gauge invariant, and is sufficient to carry out the Dirac quantization.

It turns out, though, that the *interpretation* of the Dirac factor ordering, which involves a lot of detailed calculation, is considerably simplified if we make the stronger assumption that $\gamma_{\alpha\beta}$, instead of just its determinant, is gauge invariant.

We remark that we are already assuming $\zeta_\gamma^{\alpha b} = 0$ in connection with the gauge invariance of the horizontal subspaces [see (265)], and that the trace $\zeta_\gamma^{\alpha\beta}\gamma_{\alpha\beta} = 0$ for $\nabla \cdot \phi_\gamma = 0$; the further assumption $\zeta_\gamma^{\alpha\beta} = 0$ (instead of just its trace) means that the ϕ_γ are Killing. It is not clear what is to be gained by not taking this last step. If we do, then

$$\phi_\gamma \gamma^{\alpha\beta} = 2f^{(\alpha\beta)}{}_\gamma. \tag{300}$$

Thus, to make the components $\gamma_{\alpha\beta}$ gauge invariant let us assume, unless otherwise noted, the following conditions:

$$\mathcal{L}_{\phi_\gamma} G = 0 \ \text{ and } \ f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma}. \tag{301}$$

The Killing condition restricts our freedom of choice of gauge vector basis to those which can be reached by linear transformations with constant coefficients, and we assume the $f_{\alpha\beta}^\gamma$ are structure constants, as is the case for a Lie group acting on M with linearly independent ϕ_γ. The antisymmetry condition makes $f_{\alpha\beta\gamma}$ antisymmetric in all pairs of indices. Of course these conditions imply (299), and are true in our two examples.

The Ricci rotation coefficients are evaluated using (295):

$$\begin{aligned}
\Gamma^c_{ab} &= \Gamma^c_{ba} = \tilde{\Gamma}^c_{ab} = \frac{1}{2} g^{cd}\left\{\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right\}, && (302)\\
\Gamma^c_{a\beta} &= \Gamma^c_{\beta a} = \frac{1}{2}\mathcal{F}_{\beta a}{}^c, && (303)\\
\Gamma^c_{\alpha\beta} &= \Gamma^c_{\beta\alpha} = -\frac{1}{2}\tilde{\nabla}^c \gamma_{\alpha\beta}, && (304)\\
\Gamma^\gamma_{ab} &= -\Gamma^\gamma_{ba} = -\frac{1}{2}\mathcal{F}^\gamma{}_{ab}, && (305)\\
\Gamma^\gamma_{a\beta} &= \Gamma^\gamma_{\beta a} = \frac{1}{2}\gamma^{\gamma\alpha}\tilde{\nabla}_a \gamma_{\alpha\beta}, && (306)\\
\Gamma^\gamma_{\alpha\beta} &= -\Gamma^\gamma_{\beta\alpha} = \frac{1}{2} f^\gamma_{\alpha\beta}. && (307)
\end{aligned}$$

Here indices are raised/lowered using the appropriate metric g_{ab} or $\gamma_{\alpha\beta}$, and a tilde indicates an object on the reduced configuration space, m, to distinguish it from the corresponding object on the extended configuration space, M. In particular, $\tilde{\nabla}$ is the Levi-Civita connection on m, with Christoffel symbol $\tilde{\Gamma}$.

We note the contractions

$$\begin{aligned} \Gamma^{c}_{c\beta} &= 0, && (308) \\ \Gamma^{\gamma}_{\gamma b} &= \tilde{\nabla}_b \ln \sqrt{\gamma}, && (309) \\ \Gamma^{\gamma}_{\gamma\beta} &= 0. && (310) \end{aligned}$$

We shall explore the relationships between the covariant derivatives ∇ and $\tilde{\nabla}$ of tensors on M and m in section 6.3, where a relevant context can be established. Let us consider here the curvatures associated with the Levi-Civita connections on M and m.

4.5 Relationships Between the Curvatures of the Various Connections

For vector fields U and V on M (defining "the parallelogram"), the curvature operator acting on a vector field W is

$$R(U,V)W := \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W. \tag{311}$$

Taking $U = w_{\overline{A}}$, $V = w_{\overline{B}}$, $W = w_{\overline{C}}$, we have

$$\nabla_{\overline{A}} \nabla_{\overline{B}} w_{\overline{C}} = \nabla_{\overline{A}} \left(\Gamma^{\overline{D}}_{\overline{BC}} w_{\overline{D}} \right) = \left(w_{\overline{A}} \Gamma^{\overline{D}}_{\overline{BC}} + \Gamma^{\overline{E}}_{\overline{BC}} \Gamma^{\overline{D}}_{\overline{AE}} \right) w_{\overline{D}}, \tag{312}$$

and on using (292) we then have

$$\begin{aligned} R(w_{\overline{A}}, w_{\overline{B}}) w_{\overline{C}} &= \left(w_{\overline{A}} \Gamma^{\overline{D}}_{\overline{BC}} - w_{\overline{B}} \Gamma^{\overline{D}}_{\overline{AC}} + \Gamma^{\overline{D}}_{\overline{AE}} \Gamma^{\overline{E}}_{\overline{BC}} - \Gamma^{\overline{D}}_{\overline{BE}} \Gamma^{\overline{E}}_{\overline{AC}} - \Gamma^{\overline{D}}_{\overline{EC}} C^{\overline{E}}_{\overline{AB}} \right) w_{\overline{D}} \\ &=: -R_{\overline{ABC}}{}^{\overline{D}} w_{\overline{D}}. && (313) \end{aligned}$$

In particular, we shall be interested in the Ricci tensor

$$\begin{aligned}\mathcal{R}_{\overline{AB}} &:= R_{\overline{ADB}}{}^{\overline{D}} \\ &= w_{\overline{D}}\Gamma^{\overline{D}}_{\overline{AB}} - w_{\overline{A}}\Gamma^{\overline{D}}_{\overline{DB}} + \Gamma^{\overline{D}}_{\overline{DE}}\Gamma^{\overline{E}}_{\overline{AB}} - \Gamma^{\overline{D}}_{\overline{AE}}\Gamma^{\overline{E}}_{\overline{DB}} - \Gamma^{\overline{D}}_{\overline{EB}}C^{\overline{E}}_{\overline{DA}}. \end{aligned} \tag{314}$$

Using (302– 307), (308– 310) and (293–294), it can be shown that the ab components are

$$\mathcal{R}_{ab} = \tilde{\mathcal{R}}_{ab} + \frac{1}{2}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}{}_{b}{}^{c} - \tilde{\nabla}_a(\tilde{\nabla}_b(\ln\sqrt{\gamma})) + \frac{1}{4}\tilde{\nabla}_a(\gamma^{\alpha\beta})\tilde{\nabla}_b(\gamma_{\alpha\beta}). \tag{315}$$

Here $\tilde{\mathcal{R}}_{ab}$ is the Ricci tensor associated with the Levi-Civita connection $\tilde{\nabla}$ on m, which has a form analogous to (314), but with no C term. We used the fact that w_a reduces to ∂_a when acting on a gauge invariant function [see (233)], as well as

$$\phi_\gamma\mathcal{F}^{\gamma}{}_{ab} = -f^{\gamma}_{\gamma\beta}\mathcal{F}^{\beta}{}_{ab} = 0 \tag{316}$$

by (282) and (299). For (at least Ricci) flat extended space M, $\mathcal{R}_{ab} = 0$, so the Kaluza-Klein-like equation (315) yields an expression for $\tilde{\mathcal{R}}_{ab}$ in terms of the square of the Yang-Mills curvature $\mathcal{F}$, as well as other objects involving the gauge orbit structure on M[4].

For example, in the helix model [see (283) and (205)] a simple calculation reveals that all terms on the right hand side of (315) contribute to $\tilde{\mathcal{R}}_{ab}$, yielding (of course: the space is two dimensional)

$$\tilde{\mathcal{R}}_{ab} = \frac{1}{2}g_{ab}\tilde{\mathcal{R}}, \tag{317}$$

[4]I would like to thank Gabor Kunstatter for suggesting this method of computing the curvature on m (see also [35]). At that time I was thinking along the lines of a Gauss-Codazzi-like analysis, relating the intrinsic and extrinsic curvature of a surface, $\overline{m}$ (representing m), with the curvature of the embedding space M. The immediate obstacle is that in order for $\overline{m}$ to inherit the correct metric from M, the vectors tangent to $\overline{m}$ must be orthogonal to the gauge orbits (with respect to the metric on M). But we know that the existence of a nontrivial Yang-Mills curvature $\mathcal{F}$ means that such a surface cannot be found. There is also the complication of multiple normal vectors. Although this approach may offer an interesting perspective, the Kaluza-Klein-like approach is more straightforward.

where the Ricci scalar

$$\tilde{\mathcal{R}} := g^{ab}\tilde{\mathcal{R}}_{ab} = 6(1+\rho^2)^{-2}, \tag{318}$$

as noted in [22].

It can also be shown that the $a\beta$ components of the Ricci tensor on M are

$$2\mathcal{R}_{a\beta} = w_b \mathcal{F}_{\beta a}{}^b + \tilde{\Gamma}^b_{bc}\mathcal{F}_{\beta a}{}^c - \tilde{\Gamma}^c_{ba}\mathcal{F}_{\beta c}{}^b + \tilde{\nabla}_b(\ln\sqrt{\gamma})\mathcal{F}_{\beta a}{}^b, \tag{319}$$

where we made use of (301). The left hand side vanishes, as usual, and the right hand side has the form of a divergence of the Yang-Mills curvature $\mathcal{F}$—a source equation, but the physical interpretation is not clear. Nevertheless, it is useful in calculating $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$, for K covariantly constant on M, as we shall see in section 6.4.

Finally, the $\alpha\beta$ components are

$$\begin{aligned}\mathcal{R}_{\alpha\beta} &= -\frac{1}{4}f^{\delta}_{\gamma\alpha}f^{\gamma}_{\delta\beta} - \frac{1}{4}\mathcal{F}_{\alpha a}{}^b\mathcal{F}_{\beta b}{}^a - \frac{1}{2}\tilde{\Delta}\gamma_{\alpha\beta} \\ &\quad +\frac{1}{2}\gamma^{\gamma\delta}\tilde{\nabla}_a(\gamma_{\gamma\alpha})\tilde{\nabla}^a(\gamma_{\delta\beta}) - \frac{1}{2}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}^a(\gamma_{\alpha\beta}),\end{aligned} \tag{320}$$

where $\tilde{\Delta} := \tilde{\nabla}_a\tilde{\nabla}^a$ acting on scalars. With $\mathcal{R}_{\alpha\beta} = 0$ this is another condition on $\mathcal{F}^2$.

Having examined the Yang-Mills connection, as well as the Levi-Civita connections ∇ and $\tilde{\nabla}$, we come now to the fourth and final connection: As mentioned earlier, $\gamma_{\alpha\beta}$ is a positive definite metric in a given orbit [in the nonholonomic basis ϕ^α, with dual ϕ_α—see (228), (171) and (215)], and so we can determine the Levi-Civita connection and curvature associated with it. We restrict ourselves to a single orbit, labelled by q, and, where confusion might arise, indicate this explicitly with $|_q$. We also temporarily drop the restrictions (299) and (301).

From (228) and (171) we calculate

$$d\left(\phi^\gamma|_q\right) = \frac{\partial}{\partial F^\epsilon}\left(F^{-1}\right)^\gamma_\delta\Big|_q dF^\epsilon \wedge dF^\delta = -F^\delta_{[\alpha}\phi_{\beta]}\left(F^{-1}\right)^\gamma_\delta\Big|_q \phi^\alpha\wedge\phi^\beta. \tag{321}$$

But

$$[\phi_\alpha, \phi_\beta] = [F^\delta_\alpha \frac{\partial}{\partial F^\delta}, F^\epsilon_\beta \frac{\partial}{\partial F^\epsilon}] = 2F^\delta_{[\alpha}\phi_{\beta]}(F^{-1})^\gamma_\delta \phi_\gamma, \tag{322}$$

so

$$f^\gamma_{\alpha\beta} = 2F^\delta_{[\alpha}\phi_{\beta]}(F^{-1})^\gamma_\delta, \tag{323}$$

and hence

$$0 = d\left(\phi^\gamma|_q\right) + \frac{1}{2} f^\gamma_{\alpha\beta}\phi^\alpha \wedge \phi^\beta. \tag{324}$$

This is essentially the Maurer-Cartan equation (see, e.g. [26]), and should be contrasted with (281), where e^γ differs from ϕ^γ by the "Yang-Mills field" A^α—see (231).

The Ricci rotation coefficients in an orbit, which we denote as $\tilde{\Gamma}^\gamma_{\alpha\beta}$, are calculated exactly as was done earlier for $\Gamma^{\overline{C}}_{\overline{AB}}$:

$$\tilde{\Gamma}^\gamma_{\alpha\beta} - \tilde{\Gamma}^\gamma_{\beta\alpha} = \tilde{C}^\gamma_{\alpha\beta} \tag{325}$$

is the analogue of (291), with

$$[\phi_\alpha, \phi_\beta] = \tilde{C}^\gamma_{\alpha\beta}\phi_\gamma;\ \tilde{C}^\gamma_{\alpha\beta} = f^\gamma_{\alpha\beta} \tag{326}$$

corresponding to (292–294). Inspection of (295) reveals that $\tilde{\Gamma}^\gamma_{\alpha\beta}$ is precisely $\Gamma^\gamma_{\alpha\beta}$ [before any of the restrictions (299) or (301)]:

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2}\gamma^{\gamma\delta}\left\{\phi_\alpha\gamma_{\beta\delta} + \phi_\beta\gamma_{\alpha\delta} - \phi_\delta\gamma_{\alpha\beta} - f_{\alpha\beta\delta} - f_{\beta\alpha\delta} + f_{\delta\alpha\beta}\right\}. \tag{327}$$

The corresponding Ricci tensor

$$\tilde{\mathcal{R}}_{\alpha\beta} = \phi_\delta\tilde{\Gamma}^\delta_{\alpha\beta} - \phi_\alpha\tilde{\Gamma}^\delta_{\delta\beta} + \tilde{\Gamma}^\delta_{\delta\epsilon}\tilde{\Gamma}^\epsilon_{\alpha\beta} - \tilde{\Gamma}^\delta_{\alpha\epsilon}\tilde{\Gamma}^\epsilon_{\delta\beta} - \tilde{\Gamma}^\delta_{\epsilon\beta}\tilde{C}^\epsilon_{\delta\alpha} \tag{328}$$

is analogous to (314). It is easy to show that, in general, this $\tilde{\mathcal{R}}_{\alpha\beta}$ appears on the right hand side of (320), just as $\tilde{\mathcal{R}}_{ab}$ appears on the right hand side of (315), and when we reinstate the conditions (301), $\tilde{\Gamma}^{\gamma}_{\alpha\beta}$ reduces to $\Gamma^{\gamma}_{\alpha\beta}$, given in (307), and $\tilde{\mathcal{R}}_{\alpha\beta}$ reduces to

$$\begin{aligned}\tilde{\mathcal{R}}_{\alpha\beta} &= \frac{1}{2}\phi_\delta f^\delta_{\alpha\beta} - \frac{1}{2}\phi_\alpha f^\delta_{\delta\beta} + \frac{1}{4}f^\delta_{\delta\epsilon}f^\epsilon_{\alpha\beta} - \frac{1}{4}f^\delta_{\alpha\epsilon}f^\epsilon_{\delta\beta} - \frac{1}{2}f^\delta_{\epsilon\beta}f^\epsilon_{\delta\alpha} \\ &= -\frac{1}{4}f^\delta_{\gamma\alpha}f^\gamma_{\delta\beta}, \end{aligned} \tag{329}$$

the first term on the right hand side of (320), as promised.

Here we used the fact that, quite generally,

$$\phi_\gamma f^\gamma_{\alpha\beta} = \phi_\alpha f^\gamma_{\gamma\beta} - \phi_\beta f^\gamma_{\gamma\alpha} - f^\epsilon_{\alpha\beta}f^\gamma_{\gamma\epsilon}, \tag{330}$$

which follows from taking the Lie derivative of (10) with respect to ϕ_δ, and then contracting δ and γ. The right hand side vanishes when $f^\gamma_{\gamma\beta} = 0$.

The Ricci scalar within an orbit is

$$\gamma^{\alpha\beta}\tilde{\mathcal{R}}_{\alpha\beta} = \frac{1}{4}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta}. \tag{331}$$

With $f^\gamma_{\alpha\beta}$ structure *constants*, and $\gamma_{\alpha\beta}$ gauge invariant, this Ricci scalar is constant, but may change from orbit to orbit.

It is interesting how both $\tilde{\mathcal{R}}_{ab}$ and $\tilde{\mathcal{R}}_{\alpha\beta}$ fall out of the same Kaluza-Klein-like analysis of M. Although $\tilde{\mathcal{R}}_{ab}$ is more important to us than $\tilde{\mathcal{R}}_{\alpha\beta}$, in the sense that reduced quantization takes place on the orbit space, not within a given orbit, it will nevertheless play a nontrivial role in the interpretation of Dirac quantization.

Part III
Quantum Analysis

5 Dirac Quantization

We have completed the classical analysis, and now we come to quantization. In reduced quantization the constraints are first solved classically to yield the physical degrees of freedom, i.e. the classical reduction in chapter 3. Then one attempts to apply the 'usual' canonical quantization procedure. As emphasized before, the problem is we have observables quadratic in the momenta, as well as a curved configuration manifold, m, so in general van Hove obstructions seriously obscure the correct quantization. In Dirac's [1] method, on the other hand, one initially ignores the constraints and quantizes on M; the advantage being that M is flat. Then one applies the constraints as operators to select the physical state space. As we will explain shortly, the quantum reduction then induces a Hilbert space structure on the physical state space. We shall first apply Dirac quantization, and then discuss some implications for reduced quantization.

5.1 Symmetric Operators and Minimal Quantization

For Dirac quantization we begin by quantizing the extended classical mechanics (Γ, Ω). Since $\Gamma = T^*M$ the Schrödinger picture is a natural choice (or functional Schrödinger picture in the field theoretic case, but we shall not distinguish between these in our formal development). The state space $\mathcal{F} = C^\infty(M, C)$, i.e. all smooth complex valued functions on M. The Hilbert space $\mathcal{H} = L^2(M, E)$, consists of those elements of $\mathcal{F}$ which are square integrable on M, with inner product given by

$$(\Psi_1, \Psi_2) := \int_M \boldsymbol{E}\, \overline{\Psi}_1 \Psi_2. \tag{332}$$

The choice of volume form $\boldsymbol{E}$ is, of course, arbitrary: If for one choice of $\boldsymbol{E}$ we can find a consistent quantization (suitable factor ordering of operators $\mathcal{Q}$, etc.), then by the mapping

$$\boldsymbol{E} \longmapsto \boldsymbol{E}' = \sigma \boldsymbol{E},\ \sigma > 0, \tag{333}$$

$$\Psi \longmapsto \Psi' = \sigma^{-\frac{1}{2}} \Psi, \tag{334}$$

$$\mathcal{Q} \longmapsto \mathcal{Q}' = \sigma^{-\frac{1}{2}} \mathcal{Q} \sigma^{\frac{1}{2}}, \tag{335}$$

we can get to any other volume form $\boldsymbol{E}'$ by a suitable choice of σ. The two quantizations are obviously equivalent:

$$(\Psi_1', \mathcal{Q}' \Psi_2')' = (\Psi_1, \mathcal{Q} \Psi_2). \tag{336}$$

In the Schrödinger representation quantum operators are realized as differential operators acting on the wavefunctions $\Psi \in \mathcal{H}$. Thus we shall need to define a derivative operator, D, on M. We take D to be torsion-free, for simplicity, and to be compatible with the inner product:

$$D\boldsymbol{E} = 0. \tag{337}$$

These are the only two restrictions on D, otherwise it is arbitrary. The latter restriction defines the divergence $D \cdot V \equiv D_A V^A$ of a vector field V by

$$\mathcal{L}_V \boldsymbol{E} = \boldsymbol{E}\, D \cdot V \tag{338}$$

and, correspondingly, inner product compatibility means

$$0 = \int_M \mathcal{L}_V \boldsymbol{E} = \int_M \boldsymbol{E}\, D \cdot V \tag{339}$$

for suitable boundary conditions on V.

We now construct the quantum operator, $\mathcal{Q}_s(S)$, corresponding to the homogeneous classical dynamical variable $\mathcal{C}_s(S)$ in (83). We take as an initial ansatz

$$\begin{aligned}\mathcal{Q}_s(S) \; := \; & (-i\hbar)^s \Big\{ S^{A_1\cdots A_s} D_{A_1}\cdots D_{A_s} + S_Q^{A_1\cdots A_{s-1}} D_{A_1}\cdots D_{A_{s-1}} + \cdots \\ & + S_Q^{A_1} D_{A_1} + S_Q \Big\}, \end{aligned} \tag{340}$$

where the leading term is fixed, and the complementary lower valence tensors $S_Q^{A_1\cdots A_t}$, $t < s$, (which may be complex), are chosen to make $\mathcal{Q}_s(S)$ symmetric with respect to the inner product (332):

$$(\Psi_1, \mathcal{Q}_s(S)\Psi_2) = (\mathcal{Q}_s(S)\Psi_1, \Psi_2). \tag{341}$$

The S_Q should be linear and homogeneous in S, and without loss of generality we may take them to be symmetric tensors (because any antisymmetric part would induce a commutator of two derivatives, which could be replaced with the Riemann tensor, and thus absorbed into a lower order term).

For a scalar field X, on M, we thus have[5]

$$\mathcal{Q}_0(X) = X, \tag{342}$$

which is automatically symmetric (we assume all tensors $S^{A_1\cdots A_s}$ are real). For a vector field V we have

$$\mathcal{Q}_1(V) = -i\hbar \left\{ V^A D_A + V_Q \right\}. \tag{343}$$

[5]Notation: The valence of a tensor S is usually evident from the subscript t in an expression like $\mathcal{Q}_t(S)$. For example, this allows one to distinguish the vector $S_Q^{A_1}$ used in $\mathcal{Q}_1(S_Q)$ from the scalar S_Q used in $\mathcal{Q}_0(S_Q)$. To further help clarify the notation we adopt a convention used by Kuchař: $X, Y, \ldots$ denote scalars, $U, V, \ldots$ denote vectors, $K, L, \ldots$ denote valence two tensors, and $T, \ldots$ denote valence three tensors.

The usual elementary calculation on the left hand side of (341) yields

$$
\begin{aligned}
(\Psi_1, \mathcal{Q}_1(V)\Psi_2) &= \int_M \boldsymbol{E}\,\overline{\Psi}_1(-i\hbar)\left\{V^A D_A + V_Q\right\}\Psi_2 \\
&= \int_M \boldsymbol{E}\,\overline{\left[(-i\hbar)\left\{V^A D_A - \overline{V}_Q + D_A(V^A)\right\}\Psi_1\right]}\Psi_2 \\
&\quad -i\hbar \int_M \boldsymbol{E}\, D_A\left(\overline{\Psi}_1 \Psi_2 V^A\right),
\end{aligned} \tag{344}
$$

the last integral vanishing by (339). Comparing with the right hand side,

$$
(\mathcal{Q}_1(V)\Psi_1, \Psi_2) = \int_M \boldsymbol{E}\,\overline{[(-i\hbar)\left\{V^A D_A + V_Q\right\}\Psi_1]}\Psi_2, \tag{345}
$$

we learn that the real part of V_Q is fixed:

$$
\Re V_Q = \frac{1}{2} D_A(V^A), \tag{346}
$$

but the imaginary part is arbitrary. Hence a linear symmetric operator has the form

$$
\begin{aligned}
\mathcal{Q}_1(V) &= -i\hbar\left\{V^A D_A + \frac{1}{2} D_A(V^A)\right\} \\
&= -i\hbar\left\{V^A \partial_A + \frac{1}{2}(\partial_A V^A + V^A \partial_A \ln\Omega)\right\},
\end{aligned} \tag{347}
$$

up to the optional addition of any zero order operator of the form

$$
\hbar \mathcal{Q}_0(\Im V_Q), \tag{348}
$$

involving the imaginary part of V_Q. In the second line we have used a coordinate system Q^A, in which the volume form appears as

$$
\boldsymbol{E} = \Omega dQ^1 \wedge \cdots \wedge dQ^N \tag{349}
$$

for a strictly positive function Ω.

Similarly, a quadratic symmetric operator has the form

$$\begin{aligned} \mathcal{Q}_2(K) &= (-i\hbar)^2 \left\{ K^{AB} D_A D_B + D_A(K^{AB}) D_B \right\} = (-i\hbar)^2 D_A K^{AB} D_B \\ &= (-i\hbar)^2 \Omega^{-1} \partial_A \Omega K^{AB} \partial_B, \end{aligned} \tag{350}$$

up to the optional addition of arbitrary lower order operators of the form

$$\hbar \mathcal{Q}_1(\Im K_Q) - \hbar^2 \mathcal{Q}_0(\Re K_Q). \tag{351}$$

The second line in (350) emphasizes that no knowledge of D, beyond $DE = 0$ is needed to evaluate $\mathcal{Q}_2(K)$. Finally, for cubic operators we find[6]

$$\mathcal{Q}_3(T) = (-i\hbar)^3 \left\{ T^{ABC} D_A D_B D_C + \frac{3}{2} D_A(T^{ABC}) D_B D_C - \frac{1}{4} D_A(D_B(D_C(T^{ABC}))) \right\}, \tag{352}$$

up to the addition of

$$\hbar \mathcal{Q}_2(\Im T_Q) - \hbar^2 \mathcal{Q}_1(\Re T_Q) - \hbar^3 \mathcal{Q}_0(\Im T_Q). \tag{353}$$

The quantization scheme represented by (342), (347), (350), (352)..., without any of the optional additional terms, is the minimum required to achieve symmetric operators, and here we shall refer to it as "minimal quantization". By selecting amongst the various optional additional terms we can reach any other Schrödinger-type quantization scheme. Notice the pattern that is developing in (348), (351), and (353). Of particular interest are the schemes where all S_Q are real. In this case the minimal quantization is modified to [7]

$$\mathcal{Q}'_s(S) = \mathcal{Q}_s(S) - \hbar^2 \mathcal{Q}_{s-2}(S_Q) - \hbar^4 \mathcal{Q}_{s-4}(S_Q) - \cdots \tag{354}$$

Thus the first modification is the addition of a scalar to the quadratic operators. This is reasonable in that, besides the derivative operators themselves,

[6]Cubic operators occur in the commutator of two quadratic operators.

the only natural geometrical object in the theory is the curvature of D, which has dimensions inverse length squared, so quadratic operators would be the first place it could be used to construct S_Q. For example, K_Q may be proportional to $K^{AB}\mathcal{R}_{AB}$, where $\mathcal{R}_{AB}$ is the Ricci tensor of D. On the other hand, if there existed a dimensionless preferred scalar in the theory, one could imagine several ways of combining it with K and two derivative operators to construct a scalar.

We shall see that both of these possibilities become very relevant when we come to discuss Dirac quantization in the reduced framework. For the time being we restrict ourselves to minimal quantization, and show that it is sufficient to implement a consistent Dirac quantization of the Poincaré algebra of scalar electrodynamics.

5.2 Van Hove Anomalies

We now work out the commutators of operators in the minimal quantization scheme. The two lowest order cases are obvious:

$$\frac{1}{i\hbar}[\mathcal{Q}_0(X), \mathcal{Q}_0(Y)] = 0, \tag{355}$$

$$\frac{1}{i\hbar}[\mathcal{Q}_0(X), \mathcal{Q}_1(V)] = \mathcal{Q}_0(-[\![X, V]\!]) = \mathcal{L}_V X, \tag{356}$$

and they agree with the Dirac prescription [cf (84)]:

$$\{\,,\,\} \longrightarrow \frac{1}{i\hbar}[\,,\,]. \tag{357}$$

Recall that we may use any torsion-free derivative operator (e.g. D) in the Schouten concomitant $[\![\,,\,]\!]$.

Before we work out the higher order cases we need to introduce the curvature of D. Our conventions are as follows:

$$(D_A D_B - D_B D_A)V^D =: -R_{ABC}{}^D V^C \tag{358}$$

for the Riemann tensor, and

$$\mathcal{R}_{AC} := R_{ABC}{}^{B} \tag{359}$$

for the Ricci tensor. We shall need to know that the Ricci tensor is symmetric, but this is not *a priori* obvious—it follows from the no-torsion stipulation on D, and its compatibility with the inner product. Indeed, in the local coordinates used in (349) we can write

$$D_A V^B = \partial_A V^B + \Gamma^B_{AC} V^C. \tag{360}$$

No torsion means $\Gamma^B_{AC} = \Gamma^B_{CA}$, and (337) means the contraction

$$\Gamma^B_{BC} = \partial_C \ln \Omega. \tag{361}$$

The Ricci tensor is given by

$$\mathcal{R}_{AC} = \partial_B \Gamma^B_{AC} - \partial_A \Gamma^B_{BC} + \Gamma^D_{AC}\Gamma^B_{BD} - \Gamma^B_{DA}\Gamma^D_{BC}, \tag{362}$$

whose symmetry, in particular the second term on the right hand side, is a consequence of (361).

Using this symmetry property it is then easy to verify the desired result:

$$\frac{1}{i\hbar}[\mathcal{Q}_1(U), \mathcal{Q}_1(V)] = \mathcal{Q}_1(-[\![U, V]\!]). \tag{363}$$

From (84) we note the well known fact that the zero and first order classical dynamical variables form a (closed) subalgebra of the classical Poisson algebra, and, as we have just seen, this algebra maps homomorphically to the quantum commutator algebra under the minimal quantization scheme. With the introduction of second order variables we will see that both of these results break down.

The lowest order commutator involving a quadratic operator is easily evaluated:

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_0(Y)] = \mathcal{Q}_1(-[\![K, Y]\!]). \tag{364}$$

It might be instructive to work out the next (quadratic-linear) commutator in some detail to see what is involved. Thus we calculate

$$\begin{aligned}
\frac{1}{(-i\hbar)^3}[\mathcal{Q}_2(K), \mathcal{Q}_1(V)] &= [K^{AB}D_AD_B + D_A(K^{AB})D_B, V^CD_C + \frac{1}{2}D_C(V^C)] \\
&= 2K^{B[A}V^{C]}D_AD_BD_C \\
&\quad +\{2K^{A(B}D_A(V^{C)}) - V^AD_A(K^{BC})\}D_BD_C \\
&\quad +\{K^{AB}D_A(D_B(V^C)) + K^{BC}D_B(D_A(V^A)) \\
&\quad +D_A(K^{AB})D_B(V^C) - D_B(D_A(K^{AC}))V^B\}D_C \\
&\quad +\frac{1}{2}D_A(K^{AB}D_B(D_C(V^C))).
\end{aligned} \tag{365}$$

Using the no-torsion condition we have (acting on a scalar)

$$D_AD_BD_C - D_CD_BD_A = [D_A, D_C]D_B = R_{ACB}{}^DD_D, \tag{366}$$

so the cubic term is reduced to one linear in derivatives:

$$\text{cubic} = K^{AB}V^CR_{ACB}{}^DD_D. \tag{367}$$

From (85) we recognize the Schouten concomitant in the quadratic term:

$$\text{quadratic} = [\![K, V]\!]^{BC}D_BD_C. \tag{368}$$

Bringing the D_A to the left in the linear term:

$$\begin{aligned}
D_B(D_A(V^A)) &= D_A(D_B(V^A)) - V^A\mathcal{R}_{AB} \\
D_B(D_A(K^{AC})) &= D_A(D_B(K^{AC})) - K^{AC}\mathcal{R}_{AB} + K^{AD}R_{ABD}{}^C,
\end{aligned}$$

we find

$$\text{linear} = D_B(\llbracket K, V\rrbracket^{BC})D_C - \text{cubic}. \tag{369}$$

The commutator is thus

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_1(V)] = \mathcal{Q}_2(-\llbracket K, V\rrbracket) + \hbar^2\mathcal{Q}_0(Z), \tag{370}$$

where

$$Z := \frac{1}{2}D_A(K^{AB}D_B(D_C(V^C))). \tag{371}$$

Notice how the curvature terms have cancelled.

The $\mathcal{Q}_0$ term on the right hand side of (370) is undesirable since it violates (357), and is a manifestation of van Hove's well known result that it is impossible to realize Dirac's prescription, (357), for the entire Poisson algebra (see, e.g. [2], and references therein). Nor can it be eliminated by a different choice of factor ordering [21]. But notice that it *does* disappear if V is a symmetry of the inner product, i.e. $\mathcal{L}_V \boldsymbol{E} = 0$.

Of course such vector fields form a subalgebra of all vector fields on M (with respect to the Schouten concomitant):

$$\mathcal{L}_{[U,V]}\boldsymbol{E} = [\mathcal{L}_U, \mathcal{L}_V]\boldsymbol{E} = 0. \tag{372}$$

Thus one is led to the idea of abandoning the notion of quantizing all observables, and instead restricting to certain subalgebras of the full Schouten algebra. (The Schouten algebra refers to the Lie algebra of all symmetric contravariant tensors on M with Schouten concomitant as Lie product). This is the approach we take, the subalgebra in question being the Poincaré algebra.

In a similar, but considerably more lengthy calculation, we can show that the quadratic-quadratic commutator is

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_2(L)] = \mathcal{Q}_3(-\llbracket K, L\rrbracket) + \hbar^2\mathcal{Q}_1(W), \tag{373}$$

where

$$\begin{aligned}
W^D &:= \frac{1}{2}D_B(D_C([\![K,L]\!]^{BCD})) + D_B(A^{BD}),\\
A^{BD} &:= -\{K^{AB}L^{CD}\mathcal{R}_{AC} + D_C(K^{AB})D_A(L^{CD}) - (K \leftrightarrow L)\}\\
&\quad +\frac{1}{6}D_C(T^{BCD} - T^{DCB}),\\
T^{BCD} &:= 2K^{AB}D_A(L^{CD}) - (K \leftrightarrow L).
\end{aligned} \tag{374}$$

This time we have a linear van Hove term corresponding to the vector field W. A^{BD} is antisymmetric, so that divergence of the last term in W vanishes. Notice that W involves the Ricci tensor, among other things, which means we must give more information about the derivative operator D than simply that it is inner product compatible, at least to evaluate W in this form[7]. In the generic case W does not vanish, and represents the second, and more difficult van Hove obstruction to Dirac's prescription (see, e.g. [7]).

Note that the above analysis is germane to curved space quantization in general, or reduced quantization with a curved m (with D the Levi-Civita connection), and below we will specialize it to Dirac quantization of a gauge theory on a Ricci flat space. However, we have continued the preceding analysis elsewhere [36]. Originally, we observed the well known fact (see, e.g. [2]) that a sufficient condition for the van Hove term in (370) to disappear is for V to be a Killing vector (which is necessarily divergence-free). It is trivial to quantize the Poisson subalgebra of observables linear and/or zero order in momenta, but the interesting observables are usually quadratic ([37] and references therein); for simplicity we ignore cubic and higher order. Now, in order for a Poisson subalgebra of observables of degree less than or equal to two to close, the tensors, K_i say,

[7]Of course, since $\mathcal{Q}_2(K)$ can be expressed entirely in terms of K, ∂, and Ω [see (350)], the commutator (373) is actually independent of the particular D chosen, beyond that $DE = 0$.

associated with the quadratic pieces of these observables must be in involution:

$$[\![K_i, K_j]\!] = 0 \; \forall i, j. \tag{375}$$

Furthermore, if we agree that one of the observables is to be the Hamiltonian (with nondegenerate kinetic energy), then we take $K_0 := \frac{1}{2}G^{-1}$, and so all the K_i must be Killing tensors:

$$[\![K_i, G^{-1}]\!] = 0 \; \forall i. \tag{376}$$

So the idea was to quantize the Poisson subalgebra of all observables constructed out of Killing tensors of valence two or less. Note that because of the Jacobi identity, the Schouten concomitant of two Killing tensors is also a Killing tensor. (But $[\![K_i, V]\!]$, a tensor of valence two, may not necessarily be in involution with the other K_i for generic V: there are some additional restrictions.) We found that a similar analysis was done by Bloore *et. al.* [7], except they looked at keeping the commutator $[\mathcal{Q}_2(K), \mathcal{Q}_2(\frac{1}{2}G^{-1})]$ canonical for arbitrary K. Of course this commutator is not closed: it assumes the existence of cubic observables which, presumably, would need to be quantized 'correctly' too. They discovered that no suitable Schrödinger-like quantization scheme [see (354)] exists, unless M is of constant curvature of Ricci flat. In our case, with commutator $[\mathcal{Q}_2(K_i), \mathcal{Q}_2(K_j)]$, and K_i, K_j Killing and in involution, we derived the general conditions on the quantization scheme for this to vanish. Although we could 'almost' find a solution, it appears that with this restriction on observables no quantization scheme exists either, at least not without restricting the curvature [36].

These difficulties led us to consider symmetry subalgebras, for example the Poincaré subalgebra for scalar electrodynamics, to see if it might provide some guidance in the selection of restrictions on class of observables and curvature, sufficient for a consistent curved space quantization scheme. This, coupled with the idea of exploiting gauge theories, as described in the introduction, is what our discussion now continues with, as well as the geometrical interpretation of that quantization scheme.

5.3 Selection of the Physical State Space

So far we have been ignoring the gauge (and metric) structures on M. In the spirit of Dirac we now quantize the gauge observables on the same footing as any other observables linear in the momenta:

$$\hat{C}_\alpha = \mathcal{Q}_1(\phi_\alpha). \tag{377}$$

The physical state space, $\mathcal{F}_{\text{phys}} \subset \mathcal{F}$, is defined to be the collection of states Ψ_{phys} annihilated by the constraint operators:

$$\hat{C}_\alpha \Psi_{\text{phys}} = 0 \; \forall \alpha \iff \Psi_{\text{phys}} \in \mathcal{F}_{\text{phys}}. \tag{378}$$

We remark that if the orbits of the gauge group are not compact then the Ψ_{phys} are not square integrable, in which case $\mathcal{F}_{\text{phys}} \not\subset \mathcal{H}$. This is a well known issue, which we leave outside the scope of this work, preferring to concentrate instead on geometrical questions (see also comments in [38], and references therein). We only want to include a remark by DeWitt [28] to the effect that in order to factor out the gauge group in the path integral one must treat the gauge group formally as if it were compact, which is related to the traceless condition $f^\gamma_{\gamma\beta} = 0$. See also [23], where the gauge group is divided out of the Dirac quantization inner product, and the result is compared with reduced quantization. Alternatively, we could devoid $\mathcal{F}$ of any Hilbert space structure, as Kuchař [21] argues, and take the pragmatic view that the minimal factor ordering on the extended space is merely a means (as we shall see) to induce a particular factor ordering (and Hilbert space structure) on the reduced space which preserves the quantum Poincaré algebra.

In any case, these issues do not affect the consequences for factor ordering in the reduced space, or its geometrical interpretation.

Now, even if we are able to consistently define an $\mathcal{F}_{\text{phys}}$ this way, notice that we are working with a certain basis ϕ_α of $\mathcal{G}$. Recall that classically the constraints $C_\alpha = \mathcal{C}_1(\phi_\alpha) \approx 0$ imply $\mathcal{C}_1(\mu) \approx 0$ $\forall \mu \in \mathcal{G}$, and so the classical constraint structure is basis independent. However, in the quantum case

$$\mathcal{Q}_1(\mu)\Psi_{\text{phys}} = \{\mu^\alpha \mathcal{Q}_1(\phi_\alpha) - \frac{i\hbar}{2}(\phi_\alpha \mu^\alpha)\}\Psi_{\text{phys}} = -\frac{i\hbar}{2}(\phi_\alpha \mu^\alpha)\Psi_{\text{phys}} = 0 \qquad (379)$$

will place further restrictions on $\mathcal{F}_{\text{phys}}$ (indeed reduce $\mathcal{F}_{\text{phys}}$ to the trivial zero element), unless μ is restricted to a smaller subset $\overline{\mathcal{G}} \subset \mathcal{G}$ defined by

$$\phi_\alpha \mu^\alpha = 0 \iff \mu = \mu^\alpha \phi_\alpha \in \overline{\mathcal{G}}. \qquad (380)$$

This restriction is an inescapable consequence of demanding a Hilbert space structure on $\mathcal{F}$. By choosing a different basis we change $\overline{\mathcal{G}}$, so the question arises if there exists a *natural* choice, i.e. this motivates the notion of a preferred basis of gauge vectors [21].

Since

$$\hat{C}_\alpha \Psi_{\text{phys}} = -i\hbar \left(\phi_\alpha \Psi_{\text{phys}} + \frac{1}{2}(D \cdot \phi_\alpha)\Psi_{\text{phys}}\right), \qquad (381)$$

(378) would imply the natural result $\Psi_{\text{phys}}(Q) = \psi(q(Q))$, i.e. $\mathcal{F}_{\text{phys}}$ consists of gauge invariant complex valued functions on M, provided $D \cdot \phi_\alpha = 0$ $\forall \alpha$. Thus, a natural choice of basis is one which is 'compatible' with the Hilbert space structure [cf (338)]:

$$\mathcal{L}_{\phi_\alpha} \boldsymbol{E} = 0 \; \forall \alpha. \qquad (382)$$

We take this as a fundamental relation. It corresponds to a subset $\overline{\mathcal{G}}$ consisting of all divergence-free vector fields in $\mathcal{G}$ (with respect to $\boldsymbol{E}$). This should be compared with a similar restriction we encountered earlier in connection with van Hove obstructions to quantization [see the discussion associated with (372)], so one might argue that such a restriction to $\overline{\mathcal{G}}$ is not so unnatural.

Thus our choice of gauge vector basis is related to our choice of volume form $\boldsymbol{E}$, and we ask if there exists a combination of choices which satisfies the relation (382). First of all, for (382) to be consistent we need

$$0 = [\mathcal{L}_{\phi_\alpha}, \mathcal{L}_{\phi_\beta}]\boldsymbol{E} = \mathcal{L}_{f^\gamma_{\alpha\beta}\phi_\gamma}\boldsymbol{E} = f^\gamma_{\alpha\beta}\mathcal{L}_{\phi_\gamma}\boldsymbol{E} + (\phi_\gamma f^\gamma_{\alpha\beta})\boldsymbol{E}, \tag{383}$$

or

$$\phi_\gamma f^\gamma_{\alpha\beta} = 0 \;\forall \alpha, \beta. \tag{384}$$

This is a condition only on the choice of basis.

We can see what it means by recalling (330):

$$\phi_\gamma f^\gamma_{\alpha\beta} = \phi_\alpha f^\gamma_{\gamma\beta} - \phi_\beta f^\gamma_{\gamma\alpha} - f^\epsilon_{\alpha\beta} f^\gamma_{\gamma\epsilon}. \tag{385}$$

We recognize the right hand side as essentially an exterior derivative of the object $\omega_\beta := f^\gamma_{\gamma\beta}$, in a nonholonomic basis, so the consistency condition of (382) is equivalent (at least locally) to

$$f^\gamma_{\gamma\beta} = \phi_\beta f, \tag{386}$$

i.e. the trace of the structure functions should be the (vertical) gradient of a scalar. This is independent of any choice of $\boldsymbol{E}$.

Now, if we set

$$\boldsymbol{E} =: \overline{\Omega}\,\overline{\boldsymbol{E}}, \tag{387}$$

where

$$\overline{\boldsymbol{E}} := dq^1 \wedge \cdots \wedge dq^n \wedge dF^1 \wedge \cdots \wedge dF^C \tag{388}$$

is the adapted-coordinate volume form, and $\overline{\Omega}$ is a strictly positive function, we can show that

$$\mathcal{L}_{\phi_\alpha}\boldsymbol{E} = 0 \;\forall\alpha \iff \phi_\alpha \ln\overline{\Omega} = -\frac{\partial}{\partial F^\beta}(F^\beta_\alpha)\;\forall\alpha. \tag{389}$$

Furthermore, using (171) in (10) we can establish that

$$f^{\gamma}_{\gamma\beta} = \frac{\partial}{\partial F^{\gamma}}(F^{\gamma}_{\beta}) - \phi_{\beta} \ln F, \tag{390}$$

where $F := |\det F^{\alpha}_{\beta}| > 0$, the determinant of the Faddeev-Popov matrix. Hence

$$\mathcal{L}_{\phi_\alpha} \boldsymbol{E} = 0 \; \forall \alpha \iff f^{\gamma}_{\gamma\beta} = -\phi_{\beta} \ln(\overline{\Omega} F), \tag{391}$$

so we see what f in (386) must be for a given choice of volume form.

In our development so far in this chapter we have not, in any way, used the metric G on M. It is then natural to try $\boldsymbol{E} = \boldsymbol{E}^{(G)}$, the volume form associated with G, and then ask if a suitable choice of basis exits to satisfy (382). Inspection of (227) and (215) reveals that

$$\overline{\Omega}^{(G)} = \sqrt{g\gamma} F^{-1}, \tag{392}$$

where $g = \det g_{ab}$ and $\gamma = \det \gamma_{\alpha\beta}$ as before. Since g is gauge invariant, (391) tells us we need

$$f^{\gamma}_{\gamma\beta} = -\phi_{\beta} \ln \sqrt{\gamma} \;\; \forall \beta. \tag{393}$$

This is, of course, the same as

$$\nabla \cdot \phi_{\alpha} = D^{(G)} \cdot \phi_{\alpha} = 0 \; \forall \alpha \tag{394}$$

[cf (297); ∇ and $D^{(G)}$ are the same as far as calculating divergence is concerned].

Note that, at this point at least, we do not require that $\sqrt{\gamma}$ be gauge invariant (or $f^{\gamma}_{\gamma\beta} = 0$), but just that the ϕ_{α} are (Levi-Civita) divergence- free. (This issue comes up again in quantum reduction.) Nevertheless, DeWitt [28] points out that $f^{\gamma}_{\gamma\beta} = 0$ automatically in the case of Yang-Mills theories, and we assume as much here. Thus $\sqrt{\gamma}$ is gauge invariant, which takes us back to our earlier condition, (299).

5.4 Application to Scalar Electrodynamics and the Helix Model

Let us have a reprieve from the formal development, and see how the results so far apply in our two model systems. Beginning with scalar electrodynamics we take $\boldsymbol{E} = \boldsymbol{E}^{(G)}$ and $D = D^{(G)}$, associated with the metric in (18). The Poincaré charges given in (86 –89) contain zero-order, linear, and quadratic terms, and are quantized accordingly using the minimal quantization scheme:

$$\begin{aligned}\hat{\mathcal{P}}^0 &= \mathcal{Q}_2(\frac{1}{2}G^{-1}) + \mathcal{Q}_0(V), &(395)\\ \hat{\mathcal{P}}^k &= \mathcal{Q}_1({}^{\mathcal{P}^k}V), &(396)\\ \hat{\mathcal{J}}^k &= \mathcal{Q}_1({}^{\mathcal{J}^k}V), &(397)\\ \hat{\mathcal{K}}^k &= -\mathcal{Q}_2(\frac{1}{2}{}^{\mathcal{K}^k}K) - \mathcal{Q}_0({}^{\mathcal{K}^k}Z) + t\hat{\mathcal{P}}^k. &(398)\end{aligned}$$

The constraint operators are given by $\mathcal{Q}_1(\mu)$, with μ restricted to $\overline{\mathcal{G}}$, the set of all (Levi-Civita) divergence-free vector fields constructed out of the ϕ_α in (20). In particular, (380) holds.

An operator $\mathcal{Q}$ is called a (quantum) observable if its action on a physical state Ψ_{phys} does not knock it out of $\mathcal{F}_{\text{phys}}$, i.e. we demand

$$\frac{1}{i\hbar}[\mathcal{Q}, \mathcal{Q}_1(\mu)]\Psi_{\text{phys}} = 0 \;\forall \mu \in \overline{\mathcal{G}}. \tag{399}$$

Note the restriction on μ compared with its classical counterpart [cf (91)]:

$$\{F, \mathcal{C}_1(\mu)\} \approx 0 \;\forall \mu \in \mathcal{G}. \tag{400}$$

We want to show that the quantum Poincaré charges are observables. To do this we will take advantage of the special result we had in the classical case [recall discussion following (92)], namely that the Lie derivative with respect to ϕ_α of every valence zero, one and two tensor that appears on the right hand side of

(395–398) vanishes. We will comment on the general case during our discussion of quantum reduction.

For a scalar X, which represents the Hamiltonian or boost potentials V or ${}^{\mathcal{K}^k}Z$, we have [cf (356)]:

$$\frac{1}{i\hbar}[\mathcal{Q}_0(X), \mathcal{Q}_1(\mu)]\Psi_{\text{phys}} = (\mathcal{L}_\mu X)\Psi_{\text{phys}} = \mu^\alpha(\mathcal{L}_{\phi_\alpha} X)\Psi_{\text{phys}} = 0. \tag{401}$$

For a vector V, representing any of the spatial translation or rotation vectors ${}^{\mathcal{P}^k}V$ or ${}^{\mathcal{J}^k}V$, we have [cf (363)]:

$$\frac{1}{i\hbar}[\mathcal{Q}_1(V), \mathcal{Q}_1(\mu)]\Psi_{\text{phys}} = \mathcal{Q}_1([\![\mu, V]\!])\Psi_{\text{phys}}. \tag{402}$$

Here

$$[\![\mu, V]\!] = \mathcal{L}_\mu V = \mu^\alpha \mathcal{L}_{\phi_\alpha} V - (V\mu^\alpha)\phi_\alpha. \tag{403}$$

The first term on the right hand side vanishes, and the second term belongs to $\overline{\mathcal{G}}$:

$$\phi_\alpha V \mu^\alpha = [\phi_\alpha, V]\mu^\alpha + V\phi_\alpha \mu^\alpha = 0. \tag{404}$$

Thus, $[\![\mu, V]\!] \in \overline{\mathcal{G}}$, so as an operator annihilates Ψ_{phys}.

Finally, letting the tensor K denote either the inverse metric G^{-1} or the boost tensors ${}^{\mathcal{K}^k}K$, we find [cf (370)]:

$$\frac{1}{i\hbar}[\mathcal{Q}_2(K), \mathcal{Q}_1(\mu)]\Psi_{\text{phys}} = \left\{\mathcal{Q}_2([\![\mu, K]\!]) + \hbar^2 \mathcal{Q}_0(Z)\right\}\Psi_{\text{phys}}. \tag{405}$$

The van Hove term, Z, vanishes precisely because of the fundamental relation (382). Furthermore, we calculate

$$[\![\mu, K]\!]^{AB} = (\mathcal{L}_\mu K)^{AB} = \mu^\alpha(\mathcal{L}_{\phi_\alpha} K)^{AB} - 2\psi^{\alpha(A}\phi_\alpha^{B)}. \tag{406}$$

The first term on the right hand side vanishes, and the vector fields ψ^α are defined by

$$\psi^{\alpha A} := K^{AB} D_B^{(G)} \mu^\alpha. \tag{407}$$

Quantizing this latter, quadratic term yields an operator proportional to

$$D_A^{(G)}\psi^{\alpha A}\phi_\alpha^B D_B^{(G)} + D_A^{(G)}\phi_\alpha^A\psi^{\alpha B}D_B^{(G)}. \tag{408}$$

The first term annihilates $\Psi_{\rm phys}$, and the second is equivalent to

$$(D^{(G)}\cdot\phi_\alpha)\psi^\alpha + \phi_\alpha\psi^\alpha = [\phi_\alpha, \psi^\alpha] + \psi^\alpha\phi_\alpha. \tag{409}$$

Now the second term on the right hand side annihilates $\Psi_{\rm phys}$, and the commutator is

$$\begin{aligned}(\mathcal{L}_{\phi_\alpha}\psi^\alpha)^A &= (\mathcal{L}_{\phi_\alpha}K)^{AB}D_B^{(G)}\mu^\alpha + K^{AB}(\mathcal{L}_{\phi_\alpha}D^{(G)}\mu^\alpha)_B \\ &= K^{AB}D_B^{(G)}(\phi_\alpha\mu^\alpha) = 0. \end{aligned}\tag{410}$$

Thus, the Poincaré charges in (395–398) are quantum observables.

By a similar analysis it is easy to show that the Hamiltonian in the helix model is also a quantum observable.

Recall from (124 –126) that we had a classical realization of the Poincaré algebra on Γ_C. We now want to see if the same is true at the quantum level, using the minimal quantization scheme. Notice that even though van Hove terms do not appear until linear-quadratic commutators, classical 'off-shell' pieces, which vanish on Γ_C but not necessarily as operators on $\Psi_{\rm phys}$ could, in principle, begin appearing in the linear-linear commutators. Thus we must examine all relevant quantum commutators that can arise out of the operators on the right hand side of (395–398), beginning with the linear-linear ones. We already know that the 'core structure' of the commutation relations will correctly realize the quantum Poincaré algebra, we need only focus on the extra off-shell and van Hove terms.

The first linear-linear commutator involves the spatial translations:

$$\frac{1}{i\hbar}[\mathcal{Q}_1({}^{\mathcal{P}^k}V), \mathcal{Q}_1({}^{\mathcal{P}^l}V)] = \mathcal{Q}_1(-[\![{}^{\mathcal{P}^k}V, {}^{\mathcal{P}^l}V]\!]). \tag{411}$$

But since the ${}^{\{\mathcal{P}^k,\mathcal{P}^l\}}\sigma^\gamma$ in (100), being proportional to $F^{kl}(z)$, are gauge invariant, we certainly have $[\![{}^{\mathcal{P}^k}V, {}^{\mathcal{P}^l}V]\!] \in \overline{\mathcal{G}}$, and so the right hand side of (411) annihilates any Ψ_{phys}, as desired. Inspection of (101) and (102) reveals that the same is true of the off-shell pieces of the other linear-linear commutators.

In principle, a quadratic-zero order commutator could produce a linear off-shell piece, perhaps not even in $\overline{\mathcal{G}}$, but inspection of (98) and (99) reveals that this is not the case here. Also, there is no van Hove term [cf (364)].

Moving on to the quadratic-linear commutators, we might encounter quadratic off-shell pieces, as well as well as van Hove terms. Using (370) we have the first such commutator:

$$\frac{1}{i\hbar}[\mathcal{Q}_2(\frac{1}{2}G^{-1}), \mathcal{Q}_1({}^{\mathcal{P}^k}V)] = \mathcal{Q}_2([\![{}^{\mathcal{P}^k}V, \frac{1}{2}G^{-1}]\!]) + \hbar^2\mathcal{Q}_0(Z). \tag{412}$$

The van Hove term, Z, disappears because ${}^{\mathcal{P}^k}V$ is (Levi-Civita) divergence-free, (110). We remark that this latter condition is stronger than necessary to establish $\mathcal{P}^k$ as a classical constant of the motion, but weaker than the Killing condition—just enough for a consistent quantization. It is interesting that the ${}^{\mathcal{P}^k}V$ (and the other Poincaré vectors ${}^{\mathcal{J}^k}V$) fall exactly into this class.

We now must quantize the quadratic off-shell piece given in (103). But this is exactly the situation encountered earlier in the discussion following (407). The Lie derivative analogous to (410) is

$$(\mathcal{L}_{\phi_\beta}{}^{\{\mathcal{P}^k,\mathcal{P}^0\}}\psi^\beta)^A = -\int d^3y\, d^3z \left\{-e\delta(y-z)\frac{\delta}{\delta A_k(z)}\right\}\phi_\beta^A(Q) = 0, \tag{413}$$

since $\phi_\beta^A(Q)$ in (20) has no dependence on the field $A_k(z)$ [and ${}^{\{\mathcal{P}^k,\mathcal{P}^0\}}\psi^{\beta A}(Q)$ has no field dependence at all—see (103)]. Inspection of (104–106) reveals that similar results hold for the other quadratic-linear commutators.

Finally we come to the quadratic-quadratic commutators. Here there could in principle be a cubic off-shell piece, but from (107) and (108) we see that, fortunately, this is not the case. Using (373), the boost-Hamiltonian quadratic-quadratic commutator is

$$\frac{1}{i\hbar}[\mathcal{Q}_2({}^{\mathcal{K}^k}K), \mathcal{Q}_2(G^{-1})] = \mathcal{Q}_3(-[\![{}^{\mathcal{K}^k}K, G^{-1}]\!]) + \hbar^2\mathcal{Q}_1(W). \tag{414}$$

We remark that ${}^{\mathcal{K}^k}K$ is, of course, not simply proportional to G^{-1} (for example, just 'x^k times G^{-1}', as (81) might suggest at first glance), so this commutator is in principle not trivial. To evaluate it, recall that we may use any $D^{(G)}$ that satisfies $D^{(G)}\boldsymbol{E}^{(G)} = 0$. $D^{(G)} = \nabla$, the Levi-Civita connection, is a natural choice. Then the Ricci tensor encountered in W [see (374)] is the one associated with the Riemann curvature of M, which in our case is zero. Thus to see that the van Hove term W vanishes we use two facts: (i) the ${}^{\mathcal{K}^k}K$ are (Levi-Civita) covariantly constant, and so are necessarily Killing, too, and (ii) M is (Ricci) flat. The same goes for the boost-boost commutator, in which case the covariant constancy of the ${}^{\mathcal{K}^k}K$ implies that they are also in involution.

In conclusion, we have seen that the Poincaré algebra is, indeed, realized as quantum operators acting on $\mathcal{F}_{\text{phys}}$, by using the minimal quantization scheme.

5.5 Quantum Reduction

The quantum Poincaré charges, or any other quantum observable $\mathcal{Q}$, say, contain off-shell pieces which vanish when acting on a $\Psi_{\text{phys}}(Q) = \psi(q(Q))$, leaving a 'physical residue' operator, which we denote as $\mathfrak{Q}$:

$$\mathcal{Q}\Psi_{\text{phys}} =: \mathfrak{Q}\psi. \tag{415}$$

This quantum reduction induces quantum operators (with a specific factor ordering) and a Hilbert space structure in the coordinate representation on the reduced configuration space m. That $\mathcal{Q}$ be a quantum observable is obviously necessary for the quantum reduction to make sense, but not sufficient, as we will argue shortly.

For zero order operators we have [cf (342)]:

$$\mathcal{Q}_0(X)\Psi_{\text{phys}} = X\Psi_{\text{phys}}. \tag{416}$$

We see that $\mathcal{C}_0(X)$ being a classical observable is equivalent to $\mathcal{Q}_0(X)$ being a quantum observable: $X(Q) =: x(q(Q))$ is gauge invariant. So the quantum reduction induces the reduced quantization

$$\mathcal{Q}_0(x) := x \tag{417}$$

of scalar fields x on m, but is insufficient to establish a Hilbert space structure on $\mathcal{F}_{\text{phys}}$. These results apply to the Hamiltonian and boost potentials [see (187) and (188)], as well as, by definition, the Hamiltonian potential in the helix model.

For linear (and higher order) operators we make use of the horizontal/vertical basis introduced in section 4.1. Then

$$V^A D_A \Psi_{\text{phys}} = \{V^a w_a + V^\alpha w_\alpha\}\Psi_{\text{phys}} = V^a \partial_a \Psi_{\text{phys}}, \tag{418}$$

where $V^a := e^a_A V^A$, etc. We recall, from the discussion following (181), that for $\mathcal{C}_1(V)$ to be a classical observable we already required $V^a(Q) =: v^a(q(Q))$ [cf (189) and (190) for scalar electrodynamics]. By (347) we then have

$$\mathcal{Q}_1(V)\Psi_{\text{phys}} = -i\hbar\left\{v^a\partial_a + \frac{1}{2}D_A(V^A)\right\}\Psi_{\text{phys}}. \tag{419}$$

So for $\mathcal{Q}_1(V)$ to be a quantum observable we require, additionally, that the scalar $D \cdot V$ be gauge invariant.

Choosing $D = D^{(G)}$ (with $\boldsymbol{E} = \boldsymbol{E}^{(G)}$) we have $D^{(G)} \cdot V = \nabla \cdot V$. We can calculate the Levi-Civita divergence using results from section 4.4 [cf (308 –310)]:

$$\begin{aligned} \nabla \cdot V &= w_{\overline{A}} V^{\overline{A}} + \Gamma^{\overline{A}}_{\overline{A}\overline{B}} V^{\overline{B}} \\ &= \partial_a v^a + v^a \partial_a \ln\sqrt{g\gamma} + \phi_\alpha V^\alpha, \end{aligned} \tag{420}$$

where $\tilde{\Gamma}^a_{ab} = \partial_b \ln \sqrt{g}$, and we take $\sqrt{\gamma}$ to be gauge invariant, as discussed earlier. In this case we can write

$$\mathcal{Q}_1(V)\Psi_{\text{phys}} = -i\hbar \left\{ v^a \tilde{D}_a + \frac{1}{2}[\tilde{D}_a(v^a) + \phi_\alpha V^\alpha] \right\} \Psi_{\text{phys}}, \tag{421}$$

where $\tilde{D}$ is any torsion-free derivative operator on m which annihilates the volume form

$$e := \omega dq^1 \wedge \cdots \wedge dq^n = \sqrt{\gamma} e^{(g)}, \ \omega := \sqrt{g\gamma} \tag{422}$$

induced on m. ($e^{(g)}$ is the volume form associated with the reduced metric g). Hence, for $\mathcal{Q}_1(V)$ to be a quantum observable we require, beyond the classical condition, that $\phi_\alpha V^\alpha$ be gauge invariant. In fact, as we will now argue, it is reasonable to demand the stronger condition

$$\phi_\alpha V^\alpha = 0. \tag{423}$$

First, we remark that this means any gauge (i.e. vertical) piece of V is an element of $\overline{\mathcal{G}}$, the same restriction we were forced to place on the gauge vectors, μ, themselves [cf (380)]. If this holds, then the quantum reduction induces the reduced quantization

$$\mathcal{Q}_1(v) := -i\hbar \left\{ v^a \tilde{D}_a + \frac{1}{2}\tilde{D}_a(v^a) \right\} = -i\hbar \left\{ v^a \partial_a + \frac{1}{2}(\partial_a v^a + v^a \partial_a \ln \omega) \right\} \tag{424}$$

of vector fields v on m. This operator is symmetric with respect to the inner product

$$(\psi_1, \psi_2) := \int_m e\, \overline{\psi}_1 \psi_2, \tag{425}$$

and thus a Hilbert space structure is naturally induced on $\mathcal{F}_{\text{phys}}$: $\mathcal{H}_{\text{phys}} := L^2(m, e)$. Notice too that (424) corresponds to a *minimal* quantization scheme (with respect to the volume form e).

On the other hand, if $\phi_\alpha V^\alpha \neq 0$, it does not appear to be possible to incorporate $\phi_\alpha V^\alpha$ into a divergence of the horizontal piece, v. Nor can we take $-i\hbar\phi_\alpha V^\alpha$

as an optional zero order operator added to $\mathcal{Q}_1(v)$ in (424), because the factor i would make this piece antisymmetric with respect to (425). Finally, if $\mathcal{Q}_1(v)$ involved the vertical piece V^α, the quantization rule would require more information about the classical observables than is available in the classically reduced picture. (Of course the same point could be levelled against the object $\sqrt{\gamma}$, but arguably the latter is of a more 'universal' character. This point will resurface in later discussions.)

An alternate form of $\phi_\alpha V^\alpha$ is

$$\phi_\alpha V^\alpha = \mathcal{L}_{\phi_\alpha}\langle e^\alpha, V\rangle = -f^\alpha_{\alpha\beta}V^\beta + \langle e^\alpha, \mathcal{L}_{\phi_\alpha}V\rangle, \tag{426}$$

where we used (273), the gauge invariance of the horizontal subspaces, which is valid when (265) is satisfied. We have this situation in scalar electrodynamics, as well as $f^\alpha_{\alpha\beta} = 0$ and $\mathcal{L}_{\phi_\alpha}V = 0$, which is certainly sufficient to establish (423) for the spatial translation and rotation vectors.

The quantum reduction of quadratic observables proceeds in a similar fashion:

$$K^{AB}D_B\Psi_{\text{phys}} = \left\{K^{Ab}w_b + K^{A\beta}w_\beta\right\}\Psi_{\text{phys}} = K^{Ab}\partial_b\Psi_{\text{phys}}, \tag{427}$$

where $K^{Ab} := K^{AB}e^b_B$, etc. Again, taking $D = D^{(G)}$ and using the divergence expression (420) with (350) we have

$$\mathcal{Q}_2(K)\Psi_{\text{phys}} = (-i\hbar)^2\left\{\partial_a k^{ab}\partial_b + k^{ab}\partial_a(\ln\omega)\partial_b + (\phi_\alpha K^{\alpha b})\partial_b\right\}\Psi_{\text{phys}}, \tag{428}$$

where we used the condition for $\mathcal{C}_2(K)$ to be a classical observable: $K^{ab}(Q) =: k^{ab}(q(Q))$. [See (191) and (192) for examples from scalar electrodynamics, or (201) for the helix model.] Hence, for $\mathcal{Q}_2(K)$ to be a quantum observable we require $\phi_\alpha K^{\alpha b}$ to be gauge invariant. This is analogous to the condition on $\phi_\alpha V^\alpha$, and for similar reasons quoted there we shall demand, beyond the requirements for classical observable, that

$$\phi_\alpha K^{\alpha b} = 0\ \forall b. \tag{429}$$

For example, it makes no sense to think of $(-i\hbar)^2(\phi_\alpha K^{\alpha b})\partial_b$ as the leading term in a linear operator, $\mathcal{Q}_1$, because the factor of i is wrong, and so on.

This condition involves mixed (vertical and horizontal) components, and the geometrical interpretation is not as clear. But if it holds, then the quantum reduction induces the reduced quantization

$$\mathcal{Q}_2(k) := (-i\hbar)^2\omega^{-1}\partial_a\omega k^{ab}\partial_b = (-i\hbar)^2\tilde{D}_a k^{ab}\tilde{D}_b \tag{430}$$

of tensor fields k on m. This operator is symmetric with respect to the *same* inner product as (425), so the Hilbert space structure induced on $\mathcal{F}_{\text{phys}}$ is consistent, and also corresponds to a *minimal* quantization scheme.

So we ask under what conditions is (429) valid. Note that it is automatically true for $K = G^{-1}$ since the metric is block diagonal in the horizontal/vertical basis, but this need not be true for generic K. Analogous to (426), an equivalent form is

$$\phi_\alpha K^{\alpha b} = -f^\alpha_{\alpha\beta}K^{\beta b} + e^\alpha_A(\mathcal{L}_{\phi_\alpha}K)^{AB}e^b_B, \tag{431}$$

where, additionally, we made use of (263). Thus we see that $\phi_\alpha K^{\alpha b}$ certainly vanishes for any of the Hamiltonian or boost tensors encountered in our examples. Notice that the vanishing of the second term on the right hand side is analogous to (265) for the metric (except here α is contracted with β), and will play a role in our later discussions.

We have thus established a quantum reduction, the conditions for its consistency (and perhaps the reasonableness of these conditions), and the verification that our examples satisfy these conditions. The resulting reduced quantization is identical in structure to the extended space quantization (both 'minimal'), except one is with respect to the volume form $\boldsymbol{E}^{(G)}$ on M, and the other with respect to $\boldsymbol{e} = \sqrt{\gamma}\boldsymbol{e}^{(g)}$, instead of $\boldsymbol{e}^{(g)}$, on m. The factor $\sqrt{\gamma}$, the volume element on the gauge orbits, will play a large role in our interpretation of these results.

We can now state an important result: Since the quantum Poincaré charges in the extended (Dirac) quantization realize the Poincaré algebra on $\mathcal{F}_{\text{phys}}$, so do the corresponding induced operators in the reduced quantization. We remark that the latter involves a definite factor ordering of (in particular quadratic) operators on a curved configuration manifold that is devoid of van Hove obstructions (albeit in a small, but physically significant subalgebra of the Schouten algebra on m).

5.6 Explicit Verification of the Reduced Quantum Poincaré Algebra

It is instructive to actually verify this explicitly, before going on with more formalism. Since m is curved, we do not want to have to deal with its Ricci tensor and covariant derivatives (when evaluating quadratic-quadratic commutators), but we know that a knowledge of $\omega = \sqrt{g\gamma}$ is sufficient for anything we need to do. (In the extended quantization discussion we wanted to highlight geometrical aspects of the problem—here we just want to make the calculation tractable). So let us begin with a calculation of ω.

From (191), and with $\det \delta(\boldsymbol{x}-\boldsymbol{y}) = 1$, we have

$$g = \det{}^{-1} g^{ab} = \det{}^{-1} g^{a'b'}, \tag{432}$$

where

$$g^{a'b'} := \left(\delta_{ij} + \partial_{x^i}\partial_{y^j}\frac{1}{e^2\rho(\boldsymbol{x})\rho(\boldsymbol{y})}\right)\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{433}$$

Consider the eigenvalue problem

$$g^{a'b'}f^{b'} = \lambda f^{a'};\ f^{a'} = f^{(i,\boldsymbol{x})} =: f_i(\boldsymbol{x}). \tag{434}$$

Hence we have

$$\lambda f_i = \int d^3y \left[\left(\delta_{ij} + \partial_{x^i}\partial_{y^j}\frac{1}{e^2\rho(\boldsymbol{x})\rho(\boldsymbol{y})}\right)\delta(\boldsymbol{x}-\boldsymbol{y})\right] f_j(\boldsymbol{y})$$

$$= \left(\delta_{ij} - \partial_i \frac{1}{e^2\rho^2}\partial_j\right) f_j =: \mathcal{O}_{ij} f_j \tag{435}$$

(argument $\boldsymbol{x}$ suppressed). Decompose the 'vector field' f into transverse and longitudinal parts:

$$f_j = f_j^T + \partial_j \phi, \tag{436}$$

where $\partial_j f_j^T \equiv 0$, and ϕ is a scalar field. Note that ϕ is determined only up to an additive constant—we can eliminate this ambiguity by imposing the boundary condition that ϕ vanish at $|\boldsymbol{x}|$ infinity. Using this decomposition in (435) we have (with λ independent of $\boldsymbol{x}$, of course)

$$\lambda f_i^T + \partial_i \lambda \phi = f_i^T + \partial_i \left\{ \frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)\phi \right\};\ \partial^2 := \partial_j \partial_j. \tag{437}$$

We can equate, separately, the transverse and longitudinal parts on either side:

$$f_i^T = \lambda f_i^T, \tag{438}$$

$$\frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)\phi = \lambda\phi. \tag{439}$$

The second equality is up to an integration constant, which we take to be zero in accordance with our previously mentioned boundary condition on ϕ. There are two possibilities: (i) $f_i^T \neq 0$, in which case $\lambda = 1$, which does not affect $\det \mathcal{O}$, or (ii) $f_i^T \equiv 0$, in which case we are interested in the latter eigenvalue problem, (439). In other words,

$$\det \mathcal{O} = \det \frac{1}{\rho^2}\left(-\frac{1}{e^2}\partial^2 + \rho^2\right) = \det\left(-\frac{1}{e^2}\partial^2 + \rho^2\right)\det{}^{-2}\rho. \tag{440}$$

Inspection of (204) then yields the desired result:

$$\omega = \sqrt{g\gamma} = \sqrt{\frac{\det \gamma_{\alpha\beta}}{\det g^{ab}}} = \det \rho, \tag{441}$$

where $\det \rho$ refers to the functional determinant of the operator $\rho(x)\delta(x-y)$. We remind the reader of the positive definiteness of all the operators involved here. Finally, the analogous result for the helix model is

$$\omega = \sqrt{\frac{1+\rho^2}{1+\frac{1}{\rho^2}}} = \rho \tag{442}$$

[see (205) and (201)].

Recall from section 3.3 that, using (193 –196), the Poincaré algebra is realized classically as a subalgebra of the Poisson algebra on the reduced phase space $\gamma = T^*m$. We now want to quantize this algebra using the minimal reduced quantization scheme (417), (424), and (430). Of course now there are no gauge problems to worry about—all we have are the van Hove obstructions. In particular, we must deal with the reduced space analogues of (370) and (373) (replace D_A with $\tilde{D}_a$, etc.).

The van Hove term in (370) involves the divergence $\tilde{D}\cdot v$. But (420) is the same as

$$\tilde{D}\cdot v = \partial_a v^a + v^a \partial_a \ln \omega = \nabla\cdot V - \phi_\alpha V^\alpha, \tag{443}$$

and we have already seen that for the spatial translation and rotation vectors, both terms on the far right vanish, and so, at least formally, we do not encounter any quadratic-linear van Hove terms in the reduced quantization.

Let us calculate this explicitly for, say, the spatial translation vectors $v = {}^{p^k}v$ given in (189). The 'coordinate divergence' is

$$\partial_a\, {}^{p^k}v^a = -\delta^b_a \partial_b \partial_{x^k} q^a = -4\mathrm{tr}\, \partial_{x^k}\delta(x-y) = 0 \tag{444}$$

[cf (110)]. To calculate $v^a \partial_a \ln \omega$, we notice that ω is a functional of ρ only, so we are interested in

$$\frac{\delta}{\delta\rho(x)} \ln \omega = \frac{\delta}{\delta\rho(x)} \mathrm{tr} \ln \rho = \int d^3y\, d^3z \frac{\delta(y-z)}{\rho(y)} \frac{\delta}{\delta\rho(x)} [\rho(y)\delta(y-z)] = \frac{\delta(\mathrm{o})}{\rho(x)}. \tag{445}$$

Then

$$
\begin{aligned}
{}^{\mathcal{P}^k}v^a \partial_a \ln\omega &= \int d^3x\, {}^{\mathcal{P}^k}v^{(\rho,\boldsymbol{x})} \frac{\delta}{\delta\rho(\boldsymbol{x})} \ln\omega = \int d^3x\, [-\partial_{x^k}\rho(\boldsymbol{x})] \frac{\delta(\mathbf{o})}{\rho(\boldsymbol{x})} \\
&= -\delta(\mathbf{o}) \int d^3x\, \partial_{x^k} \ln\rho(\boldsymbol{x}) = 0 \qquad (446)
\end{aligned}
$$

for suitable boundary conditions (for example, $\rho(\boldsymbol{x}) \sim 1/r^\alpha$ as $r := |\boldsymbol{x}| \longrightarrow \infty$, for any α that makes the integral converge).

Notice the presence of $\delta(\mathbf{o})$, and the importance of boundary conditions—artifacts which are hidden in the purely formal analysis. It is clear that a careful regularization might be helpful, but at this point we are more interested in the geometrical analysis of the reduced factor ordering which we will illuminate shortly. We only remark that we have $\delta(\mathbf{o})$ times zero, which is not as severe as $\delta(\mathbf{o})$ times something finite. It would be very interesting to repeat the present analysis, but with a finite dimensional gauge theory that has a nontrivial symmetry algebra involving quadratic operators (which the helix model lacks). We leave this for future research.

Let us continue with the quadratic-quadratic commutators, and their van Hove terms. For arbitrary k, l we find, at the end of some tedious calculation,

$$
\begin{aligned}
\frac{1}{(-i\hbar)^4}[\mathcal{Q}_2(k), \mathcal{Q}_2(l)] &= [\omega^{-1}\partial_a \omega k^{ab}\partial_b, \omega^{-1}\partial_c \omega l^{cd}\partial_d] \\
&= [\![k,l]\!]^{bcd}\partial_b\partial_c\partial_d \\
&\quad + \frac{3}{2}\left\{\partial_b [\![k,l]\!]^{bcd} + \partial_b(\ln\omega)[\![k,l]\!]^{bcd}\right\}\partial_c\partial_d \\
&\quad + \left\{k^{ab}\partial_a(\partial_b(v^d)) + u^b\partial_b(v^d) - (k \leftrightarrow l)\right\}\partial_d, \quad (447)
\end{aligned}
$$

where $[\![\ ,\ ]\!]$ denotes the Schouten bracket of symmetric contravariant tensors on m, and the 'vectors'

$$
u^b := \partial_a(k^{ab}) + \partial_a(\ln\omega)k^{ab}, \qquad (448)
$$

$$
v^d := \partial_c(l^{cd}) + \partial_c(\ln\omega)l^{cd}. \qquad (449)
$$

From our general discussion of classical reduction in chapter 3 [cf (161)] we know that

$$[\![k, l]\!]^{bcd} = Q^b_B Q^c_C Q^d_D [\![K, L]\!]^{BCD}, \tag{450}$$

which can also be proven directly from the definition of the Schouten bracket when $\mathcal{C}_2(K)$ and $\mathcal{C}_2(L)$ are classical observables. Hence, from (107) and (108) we know that the ${}^{\kappa^k}k$ are killing tensors in involution with each other on m, as already pointed out earlier. Thus the cubic and quadratic terms in (447) vanish in our case, and we concentrate on the linear term [analogous to $\mathcal{Q}_1(W)$ in (373)].

For the boost-boost commutator we take $k^{ab} \equiv {}^{\kappa^m}k^{ab}$ and $l^{cd} \equiv {}^{\kappa^n}k^{cd}$ [cf (192)]. We first calculate u^b: Because k^{ab} is block diagonal, and depends only on the field ρ, $\partial_a(k^{ab}) = 0$ unless $b = \rho(\boldsymbol{y})$, in which case it also vanishes because the $\rho(\boldsymbol{x})\rho(\boldsymbol{y})$ component of ${}^{\kappa^k}k$ is field independent. Similarly, $\partial_a(\ln \omega)k^{ab} = 0$ unless $b = \rho(\boldsymbol{y})$, in which case [using (445)]

$$\partial_a(\ln \omega)k^{a\rho(\boldsymbol{y})} = \int d^3x \, \frac{\delta(\mathbf{o})}{\rho(\boldsymbol{x})}\frac{1}{2}(x^m + y^m)\delta(\boldsymbol{x} - \boldsymbol{y}) = \delta(\mathbf{o})\frac{y^m}{\rho(\boldsymbol{y})}. \tag{451}$$

Hence

$$\begin{aligned} u^b &= \left(0, \delta(\mathbf{o})\frac{y^m}{\rho(\boldsymbol{y})}\right), \\ v^d &= \left(0, \delta(\mathbf{o})\frac{w^n}{\rho(\boldsymbol{w})}\right). \end{aligned} \tag{452}$$

We can now calculate the pieces in the linear term of (447):

$$\begin{aligned} u^b\partial_b(v^d) &= \int d^3y \, \delta(\mathbf{o})\frac{y^m}{\rho(\boldsymbol{y})}\frac{\delta}{\delta\rho(\boldsymbol{y})}\left(0, \delta(\mathbf{o})\frac{w^n}{\rho(\boldsymbol{w})}\right) \\ &= [\delta(\mathbf{o})]^2\left(0, -\frac{w^m w^n}{\rho^3(\boldsymbol{w})}\right), \end{aligned} \tag{453}$$

and

$$k^{ab}\partial_a(\partial_b(v^d)) = \int d^3x \, d^3y \, \frac{1}{2}(x^m + y^m)\delta(\boldsymbol{x} - \boldsymbol{y})\frac{\delta}{\delta\rho(\boldsymbol{x})}\frac{\delta}{\delta\rho(\boldsymbol{y})}\left(0, \delta(\mathbf{o})\frac{w^n}{\rho(\boldsymbol{w})}\right)$$

$$
\begin{aligned}
&= \delta(\mathbf{o}) \int d^3x\, d^3y\, (x^m + y^m)\delta(\boldsymbol{x} - \boldsymbol{y}) \left(0, \frac{w^n}{\rho^3(\boldsymbol{w})}\delta(\boldsymbol{w} - \boldsymbol{y})\delta(\boldsymbol{w} - \boldsymbol{x})\right) \\
&= [\delta(\mathbf{o})]^2 \left(0, 2\frac{w^m w^n}{\rho^3(\boldsymbol{w})}\right). \qquad (454)
\end{aligned}
$$

Adding, we finally have

$$
\text{linear} = [\delta(\mathbf{o})]^2 \int d^3w \left(\frac{w^m w^n}{\rho^3(\boldsymbol{w})} - (m \leftrightarrow n)\right) \frac{\delta}{\delta\rho(\boldsymbol{w})} = 0. \qquad (455)
$$

Again the $\delta(\mathbf{o})$ appears, but it is multiplied by something which is identically zero. A very similar result holds for the boost-Hamiltonian commutator.

In summary, we have established, at least formally, that the minimal quantization scheme on the (curved) reduced space, with respect to the volume element $\omega = \sqrt{g\gamma}$, is sufficient to implement the important and nontrivial Poincaré subalgebra of operators without van Hove anomalies. An explicit calculation reveals the expected presence of $\delta(\mathbf{o})$ type terms, but these do not appear to be too severe, in the sense that the $\delta(\mathbf{o})$ is always multiplied by something that vanishes either identically or by boundary conditions. It is thus quite plausible that with suitable regularization this result could be established rigorously. Of course the result *is* rigorous for any finite dimensional theory on a Ricci flat M, with a symmetry whose quadratic tensors are covariantly constant (listing just the main conditions).

What is more interesting is to discover that this quantization scheme has remarkable connections with other proposed quantization schemes on curved configuration spaces, as we will demonstrate in the following chapter.

In the way of a post script, some additional remarks concerning regularization should be included here. S. Carlip has pointed out that the second equality in (440) may or may not be rigorous: while the identity

$$
\det AB = \det A \det B \qquad (456)
$$

holds for finite dimensional matrices, when A and/or B are differential operators—in our case $A = \frac{1}{\rho^2}$, $B = (-\frac{1}{e^2}\partial^2 + \rho^2)$—there may be subtleties associated with regularization. Firstly, it is not obvious that the regularized determinant of a product of operators factorizes into a product of the regularized determinants. Secondly, even if such a factorization is possible (allowing us to cancel $\det B$ as in (441)), the regularization of $\det A$ itself will likely depend on the operator B in a nontrivial way.

To see how these subtleties can arise consider the simple case $A = aI$, a constant multiple of the identity operator. When B is an N-dimensional matrix, N finite, then I is the $N \times N$ unit matrix, and $\det A = a^N$. However, when B is a differential operator, even if the factorization property holds it is not a priori obvious what to write for "$\det A$" in (456). One would expect the answer to depend on B, as well as on how $\det B$ is regularized. For instance, in zeta function regularization (see, e.g., [39] and references therein) one solves

$$Bf_\lambda = \lambda f_\lambda \tag{457}$$

for the eigenvalues λ, and then constructs a generalized zeta function

$$\zeta_B(s) := \sum_\lambda \lambda^{-s} \tag{458}$$

associated with the operator B.[8] Its derivative, $\zeta_B'(0) = -\sum_\lambda \ln \lambda$, is formally equal to $-\ln \det B$, which suggests the *definition*

$$\det B := \exp -\zeta_B'(0). \tag{459}$$

$\zeta_B(s)$ is defined by the above series in the region of the complex s-plane in which it converges, and by analytic continuation elsewhere. In particular, at $s = 0$ it

[8]Alternatively, one determines the heat kernel of B (or at least its coincidence limit), which can then be used to calculate the generalized zeta function [40].

is regular [40,28] (at least for a large class of operators). Under the constant rescaling $B \longmapsto aB$, $\lambda \longmapsto a\lambda$ and so

$$\zeta'_{aB}(0) = \frac{d}{ds}\left[\sum_{\lambda}(a\lambda)^{-s}\right]\Bigg|_{s=0} = \zeta'_B(0) - \zeta_B(0)\ln a, \tag{460}$$

and thus

$$\det aB = a^{\zeta_B(0)} \det B. \tag{461}$$

So for $A = aI$ zeta function regularization allows the factorization property (456), and furthermore yields the finite result $\det A = a^{\zeta_B(0)}$. For $B = (-\frac{1}{e^2}\partial^2 + \rho^2)$, $\zeta_B(0)$ may be a complicated functional of ρ. (Note in (458) that for B an N-dimensional matrix $\zeta_B(0)$ reduces to N, the number of eigenvalues.)

In our case $A = \frac{1}{\rho^2}$ which, although diagonal and ultra local, is not a simple multiple of the identity operator (since ρ is a function of $\boldsymbol{x}$). The eigenvalues do not simply rescale, and so (460) [and hence (461)] break down. It is not obvious whether the factorization property (456) continues to hold in this case, and even less clear what the regularized determinant of A would be.

On the other hand, (440) may be too strong of a result: actually we are interested only in the variation of $\ln\omega$ with variation in ρ [see (445)]. Thus consider

$$\begin{aligned}
\delta \ln \frac{\det AB}{\det B} &= \delta \ln\det AB - \delta\ln\det B \\
&= \delta \mathrm{tr}\ln AB - \delta\mathrm{tr}\ln B \\
&= \mathrm{tr}(AB)^{-1}\delta(AB) - \mathrm{tr}B^{-1}\delta B \\
&= \mathrm{tr}\{B^{-1}A^{-1}(\delta A\, B + A\,\delta B) - B^{-1}\delta B\} \\
&= \mathrm{tr}A^{-1}\delta A.
\end{aligned} \tag{462}$$

Hence we essentially meet up with (441) again, but by a different route which does not simply a priori assume (456). Nevertheless, the calculation still involves formal manipulations, such as cyclicity of the trace; although it might be argued that these are "safer" than (456). Unfortunately we are of course still left with the problem of regularizing the trace in the last line of (462).

At present it appears that the most promising approach to solving this problem is to calculate the heat kernels and then the generalized zeta functions of B and AB individually, and then directly evaluate the right hand side of the first line in (462). This calculation is currently in progress. It can be shown that a result having the ultra local form

$$\frac{\delta \ln \omega}{\delta \rho(\boldsymbol{x})} = F(\rho(\boldsymbol{x})), \tag{463}$$

where F is an arbitrary function, would be sufficient to allow the verification of the reduced quantum Poincaré algebra to go through as discussed earlier. However, if F depends on derivatives of ρ it is possible that there could be an interesting anomaly.

Note that such an anomaly is certainly more likely to be present in reduced quantization, where $\omega = \sqrt{g}$ manifestly involves derivatives of ρ [cf (433)], than in Dirac quantization which is, at least formally, independent of derivatives of ρ [cf (441)]. In arriving at (441), any possible dependence on derivatives of ρ comes only from regularization considerations. Indeed, from a pragmatic point of view we could just take (441) as an ansatz—motivated by Dirac quantization—and regulate $\det \rho$ in a way that is independent of derivatives of ρ (or knowledge of γ). In this way the results of this section would go through as before. In any case, none of the geometrical results outside this section are affected by these considerations.

The questions raised here are symptomatic of a generic problem in quantum field theory: the control of infinities is more complicated in the reduced space than in the relativistically covariant extended space. For example, the equivalence of various forms of the Faddeev–Popov path integral with the reduced space path integral has been shown ([54] and references therein), but the calculations involve formal manipulations such as (456).

6 Dirac versus Reduced: a Geometrical Analysis

In this final chapter we would like to analyze what, if anything, makes Dirac quantization special. The scalar electrodynamics model has provided a physically relevant, and nontrivial example in which it succeeds, and thus motivates our further work. To increase the generality of our discussion we can abstract the features which make it work for scalar electrodynamics, the most important being covariant constancy of the tensors K in quadratic observables, as well as the (Ricci) flatness of M. Also, the helix model turns out to be useful to verify rigorously that nontrivial examples exist which support some of our conclusions.

6.1 Factor Ordering Ambiguity and Other Known Results

Let us collect together the formulas for the quantum-reduced Dirac quantization scheme:

$$\mathcal{Q}_0(x) = x, \tag{464}$$

$$\mathcal{Q}_1(v) = -i\hbar\left\{v^a\tilde{D}_a + \frac{1}{2}\tilde{D}_a(v^a)\right\}, \tag{465}$$

$$\mathcal{Q}_2(k) = (-i\hbar)^2\tilde{D}_a k^{ab}\tilde{D}_b. \tag{466}$$

These operators are symmetric with respect to an inner product (425) using the volume form $\boldsymbol{e} = \sqrt{\gamma}e^{(g)}$ on m, and overcome van Hove anomalies at least when the corresponding tensors and curvature on M satisfy certain geometrical properties discussed earlier. In other words, the v and k tensors are not arbitrary, nor is the curvature on m, but these objects are nontrivial enough to seriously obscure an uninitiated attempt at direct reduced quantization. In particular, what role does $\sqrt{\gamma}$, the volume element on the gauge orbits play? This is not an object available in the reduced classical theory—or is it?

The first thing we do is transform, in the spirit of (333 –335), to an equivalent quantization scheme which uses, instead of e, the 'natural' volume form $e^{(g)}$:

$$e \longmapsto e' = \gamma^{-\frac{1}{2}} e = e^{(g)}, \tag{467}$$

$$\psi \longmapsto \psi' = \gamma^{\frac{1}{4}} \psi, \tag{468}$$

$$\mathcal{Q} \longmapsto \mathcal{Q}' = \gamma^{\frac{1}{4}} \mathcal{Q} \gamma^{-\frac{1}{4}}. \tag{469}$$

A straightforward calculation yields the result

$$\mathcal{Q}_0'(x) = \mathcal{Q}_0^{(g)}(x), \tag{470}$$

$$\mathcal{Q}_1'(v) = \mathcal{Q}_1^{(g)}(v), \tag{471}$$

$$\mathcal{Q}_2'(k) = \mathcal{Q}_2^{(g)}(k) + \hbar^2 \alpha^{(\gamma)}(k), \tag{472}$$

where the superscript (g) refers to the minimal quantization scheme with respect to the volume form $e^{(g)}$. [Replace $\tilde{D}$ with $\tilde{\nabla}$, the Levi-Civita connection of g, in (464 –466)]. Note that the quadratic operator acquires, to order $\hbar^2$, an additional scalar piece

$$\alpha^{(\gamma)}(k) := \gamma^{-\frac{1}{4}} \tilde{\nabla}_a (k^{ab} \tilde{\nabla}_b (\gamma^{\frac{1}{4}})), \tag{473}$$

which embodies the $\sqrt{\gamma}$ factor originally contained in the volume element. This term makes the Dirac-induced quantization scheme physically distinct from the minimal one (with respect to $e^{(g)}$) [19], e.g. the Hamiltonians have different spectra. (See also cautionary remarks in [20] regarding domains of operators).

A necessary condition for $\alpha^{(\gamma)}(k)$ to be nontrivial is that $\sqrt{\gamma}$ have some field dependence—pure electromagnetism would be insufficient, which is a reason for going to scalar electrodynamics. For the helix model, a simple calculation reveals (for the kinetic energy $k = \frac{1}{2}g^{-1}$)

$$\alpha^{(\gamma)}(\frac{1}{2}g^{-1}) = \frac{4-\rho^2}{8(1+\rho^2)^2}, \tag{474}$$

as in [22] (modulo a sign discrepancy).

It is also known that the presence of $\alpha^{(\gamma)}(k)$ can be understood in terms of factor ordering as follows [23]: In the minimal quantization scheme (with respect to $e^{(g)}$) the operator corresponding to the classical physical variable p_a is

$$\hat{p}_a^{(g)} = -i\hbar\left\{\partial_a + \frac{1}{2}\partial_a(\ln\sqrt{g})\right\}. \tag{475}$$

It can be shown that in terms of this operator,

$$\mathcal{Q}_2^{(g)}(k) = g^{-\frac{1}{4}}\hat{p}_a^{(g)}g^{\frac{1}{2}}k^{ab}\hat{p}_b^{(g)}g^{-\frac{1}{4}}, \tag{476}$$

whereas

$$\mathcal{Q}_2'(k) = (g\gamma)^{-\frac{1}{4}}\hat{p}_a^{(g)}(g\gamma)^{\frac{1}{2}}k^{ab}\hat{p}_b^{(g)}(g\gamma)^{-\frac{1}{4}}. \tag{477}$$

These two operators differ only in their factor ordering, and the difference involves the extraneous scalar field $\sqrt{\gamma}$— both reduce to $k^{ab}p_ap_b$ in the classical limit $\hat{p}_a^{(g)} \longrightarrow p_a$.

Also, compare (477) with

$$\mathcal{Q}_2(k) = (g\gamma)^{-\frac{1}{4}}\hat{p}_a(g\gamma)^{\frac{1}{2}}k^{ab}\hat{p}_b(g\gamma)^{-\frac{1}{4}}, \tag{478}$$

where

$$\hat{p}_a = -i\hbar\left\{\partial_a + \frac{1}{2}\partial_a(\ln\sqrt{g\gamma})\right\}. \tag{479}$$

The effect of (469) is to replace $\hat{p}_a$ with $\hat{p}_a^{(g)}$.

Although the term $\alpha^{(\gamma)}(k)$ has been known for several years in the literature [22,19], we have shown here that its presence is at least sufficient to overcome van Hove anomalies in quadratic-quadratic commutators in the nontrivial and physically important example of Poincaré symmetry in scalar electrodynamics. Even more interesting, and more universally applicable, is its interpretation by geometrical methods, which we now describe.

6.2 Curved-Space Quantization Schemes

When the classical configuration space is a generic Riemannian manifold there exist several proposed quantization schemes which go beyond minimal quantization. As mentioned earlier, all Schrödinger-type quantization schemes can be reached by adding operators of lower order to the minimal quantization scheme, the lowest order correction usually being a scalar term added to a quadratic operator [cf (354)].

For example, let $\xi = v^a p_a$ denote a generic linear homogeneous dynamical variable associated with the vector field v on m, and if we quantize, $\xi \longmapsto \hat{\xi}$ in the usual manner, then we can construct the quadratic operator $\hat{\xi}^2$. What in not *a priori* obvious is how to quantize ξ^2. Vaisman [41] argues that to "preserve as many brackets a possible" (by which he means the bracket of a quadratic with a zero order operator), $\widehat{\xi^2}$ must equal $\hat{\xi}^2$—up to an arbitrary zero order operator constructed out of v:

$$\widehat{\xi^2} - \hat{\xi}^2 =: \hbar^2 A(v). \tag{480}$$

By decomposing

$$k^{ab} = \sum_{m=1}^{r} \epsilon_m v_m^a v_m^b, \tag{481}$$

where r is the rank of k, $\epsilon_m = \pm 1$, and v_m are suitable vector fields on m, he generalizes his result to (using our notation):

$$\mathcal{Q}_2'(k) = (-i\hbar)^2 \left\{ k^{ab}\tilde{\nabla}_a\tilde{\nabla}_b + \tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b + \frac{1}{4}\left[\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) - k^{ab}\tilde{\mathcal{R}}_{ab}\right]\right\} + \hbar^2 B(k). \tag{482}$$

The first two terms on the right hand side correspond to $\mathcal{Q}_2^{(g)}(k)$ in (472), and the rest to our $\hbar^2\alpha^{(\gamma)}(k)$. As before, $\tilde{\mathcal{R}}_{ab}$ is the Ricci tensor on m.

The term $B(k)$ is largely arbitrary, but unless carefully chosen, will depend on the choice of local basis v_m [41]. Vaisman gives two examples of consistent choices: The first is simply $B(k) \equiv 0$ for all k, and the second results in a factor $\frac{1}{6}$ instead of $\frac{1}{4}$ multiplying the term in brackets (and no B term). For the kinetic energy operator, $k^{ab} = \frac{1}{2}g^{ab}$, this quantization rule yields the usual Laplace-Beltrami operator term, plus a Ricci scalar term $\frac{\hbar^2}{8}\tilde{\mathcal{R}}$ or $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$, respectively, in the two cases. Notice that the latter case corresponds to the result of geometric quantization [9], which was Vaisman's motivation for considering it.

Zhang-Ju and Min [42] present another quantization scheme based on a generalization of the λ-Weyl transformation to curved configuration spaces. For flat space (with Cartesian coordinates x^a) we have, as usual,

$$x^a \longmapsto \hat{x}^a = x^a; \; p_a \longmapsto \hat{p}_a = -i\hbar\frac{\partial}{\partial x^a}, \tag{483}$$

and the λ-Weyl quantization corresponds to

$$\widehat{x^a p_a} = \lambda \hat{p}_a \hat{x}^a + (1-\lambda)\hat{x}^a \hat{p}_a, \; \lambda \in [0,1]. \tag{484}$$

For a generic quadratic operator they obtain

$$\mathcal{Q}_2'(k) = (-i\hbar)^2 \left\{ k^{ab}\tilde{\nabla}_a\tilde{\nabla}_b + 2\lambda\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b + \lambda^2\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) - \frac{2\lambda^2+1}{6}k^{ab}\tilde{\mathcal{R}}_{ab} \right\}. \tag{485}$$

For $\lambda = 1/2$ this result agrees with Vaisman, (482), with the choice $B(k) \equiv 0$, but does not agree with the other choice of $B(k)$ for any λ[9]. For the kinetic energy

[9]It appears that this operator is not self adjoint for $\lambda \neq 1/2$, at least for generic k—it *is* in the case of the kinetic energy, though. Also, see (484), which is self adjoint only for $\lambda = 1/2$.

operator, the additional scalar is $(2\lambda^2+1)\frac{\hbar^2}{12}\tilde{\mathcal{R}}$, which means $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ for $\lambda=0$ (normal), $\frac{\hbar^2}{8}\tilde{\mathcal{R}}$ for $\lambda=1/2$ (Weyl), and $\frac{\hbar^2}{4}\tilde{\mathcal{R}}$ for $\lambda=1$ (antinormal).

As a final example, Underhill [13] also works with the Weyl transformation ($\lambda=1/2$ case only), but instead finds

$$\mathcal{Q}_2'(k)=(-i\hbar)^2\left\{k^{ab}\tilde{\nabla}_a\tilde{\nabla}_b+\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b+\frac{1}{4}\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))+\frac{1}{12}k^{ab}\tilde{\mathcal{R}}_{ab}\right\}. \quad (486)$$

This result differs from that of Zhang-Ju and Min in the numerical factor in the curvature term, resulting in $-\frac{\hbar^2}{24}\tilde{\mathcal{R}}$ in the case of the kinetic energy operator.

There are numerous other papers addressing quantization on curved configuration manifolds, but usually the only quadratic operator considered is the Hamiltonian. In the configuration space path integral category, DeWitt [43] originally obtained an ambiguous result: $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ or $\frac{\hbar^2}{6}\tilde{\mathcal{R}}$ depending on the measure chosen, or even zero if one was willing to introduce factors of $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ into the classical Lagrangian. See also [44,45,46]. Kuchař [47] reviews these, and other results, and presents a phase space path integral analysis in which he chooses the natural Liouville measure, but considers various skeletonizations. He obtains $(1-\mu)\frac{\hbar^2}{6}\tilde{\mathcal{R}}$, $\mu\in(-\infty,\infty)$, and argues that $\mu=1$ is the most natural.

Weinstein [48] applies Maslov's quasi-classical mechanics to a free particle moving on a sphere of constant curvature and suggests[10] that the correct term is $\frac{\hbar^2}{8}\tilde{\mathcal{R}}$, which agrees with Vaisman's $B(k)\equiv 0$ case and Zhang-Ju and Min's $\lambda=1/2$ (Weyl) case above. Gotay [5] discusses the geometric quantization result: $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ (see also [9,10]). By considering a free particle moving on the Riemannian manifold of a semi-simple Lie group, Dowker [49] also obtains the term $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$. As mentioned above, Vaisman too can achieve this result, as can Zhang-Ju and Min with their $\lambda=0$ (normal) case. Finally, Emch [50] also concurs, stating that $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$

[10]He actually writes down $\frac{\hbar^2}{4}\tilde{\mathcal{R}}$, but I think he means $\frac{\hbar^2}{8}\tilde{\mathcal{R}}$. The former would agree with Zhang-Ju and Min's $\lambda=1$ (antinormal) case.

"produces the best possible fit of the quantum partition function to its classical limit".

Thus we see that there is no general agreement as to which scalar terms to add to $\mathcal{Q}_2^{(g)}(k)$, but usually terms of the form $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$ and $k^{ab}\tilde{\mathcal{R}}_{ab}$ appear. These are really the only natural dimensionally correct objects that can be constructed within the theory, except perhaps $k^{ab}g_{ab}\tilde{\mathcal{R}}$[11], or even terms such as $k^{ab}\tilde{\nabla}_a(\tilde{\nabla}_b(f))$ if there happens to be a naturally occurring dimensionless scalar f in the theory.

Furthermore, none of these quantization schemes satisfy Dirac's prescription (357) for generic quadratic-quadratic commutators (even in the simplest case when the classical commutator vanishes, which is what we are dealing with here). Underhill and Taraviras [14] present a scheme in which constants of the classical and quantum free particle motion correspond, but only for spaces of constant curvature. Not only is our space not of constant curvature, this addresses only hamiltonian-boost commutators, not boost-boost. Bloore, Assimakopoulos, and Ghobrial [7] have shown that *no* quantization scheme, essentially in the class we have been considering [see (354)], exists in which the commutator between a generic quadratic operator and the kinetic energy operator is free of van Hove anomalies, unless the configuration space is of constant curvature or Ricci flat. Clearly this is a thorny problem.

The fact that (472) is a solution relies mainly on two conditions: (i) special properties of the tensors k (especially that their counterpart K on the extended space should be covariantly constant, which, by the way, does *not* mean $\tilde{\nabla}k = 0$ on m), and (ii) m, although not of constant curvature, must have a special curvature, one induced by the action of a gauge group on a (Ricci) flat M. These two

[11] With this term the numerical factor multiplying $\tilde{\mathcal{R}}$ in the kinetic energy operator would depend on the dimension of the space (like the curvature scalar term in the conformally invariant Klein-Gordon equation), and this might explain why it does not appear.

conditions play a central role in our further analysis.

6.3 Consequences of K Covariantly Constant

Now the interesting question is whether or not (472) fits in with any of the quantization schemes discussed above. In particular, can $\alpha^{(\gamma)}(k)$ be related to terms of the form $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$ or $k^{ab}\tilde{\mathcal{R}}_{ab}$, etc.

As implied above, the answer to this question will rely, among other things, on the fact that $\nabla K = 0$. It is instructive to first examine this type of condition for a vector field V: To begin with, let us suppose that V is associated with a classical observable on Γ:

$$(\mathcal{L}_{\phi_\gamma} V)^A = \xi^\alpha_\gamma \phi^A_\alpha, \tag{487}$$

for arbitrary scalar fields ξ^α_γ on M, which is equivalent to the physical projection $V^a(Q) =: v^a(q(Q))$ being gauge invariant, where $V^a := e^a_A V^A$. We note that this means

$$\begin{aligned}\phi_\gamma V^\alpha &= \mathcal{L}_{\phi_\gamma}(e^\alpha_A V^A) = (\mathcal{L}_{\phi_\gamma} e^\alpha)_A V^A + e^\alpha_A(\mathcal{L}_{\phi_\gamma} V)^A \\ &= \xi^\alpha_\gamma - f^\alpha_{\gamma\beta} V^\beta,\end{aligned} \tag{488}$$

where we made use of (273). Of course the condition, (487), has nothing to do with V being covariantly constant or not on M. Let us now assume the additional condition $\nabla V = 0$; we shall see that this fixes the scalars ξ^α_γ as linear functions of the components of V.

Recall from section 4.4 that the components of ∇V in the horizontal/vertical basis are

$$\nabla_{\overline{A}} V^{\overline{B}} = w_{\overline{A}} V^{\overline{B}} + \Gamma^{\overline{B}}_{\overline{AC}} V^{\overline{C}}. \tag{489}$$

Reinstating the conditions (301) (for the remainder of the chapter), and using the Ricci rotation coefficients in (302 –307), these read

$$\nabla_a V^b = \tilde{\nabla}_a v^b + \frac{1}{2}\mathcal{F}_{\gamma a}{}^b V^\gamma, \tag{490}$$

$$\nabla_a V^\beta = w_a V^\beta - \frac{1}{2}\mathcal{F}^\beta{}_{ac} v^c + \frac{1}{2}\gamma^{\beta\alpha}\tilde{\nabla}_a(\gamma_{\alpha\gamma})V^\gamma, \tag{491}$$

$$\nabla_\alpha V^b = \frac{1}{2}\mathcal{F}_{\alpha c}{}^b v^c - \frac{1}{2}\tilde{\nabla}^b(\gamma_{\alpha\gamma})V^\gamma, \tag{492}$$

$$\nabla_\alpha V^\beta = w_\alpha V^\beta + \frac{1}{2}\gamma^{\beta\gamma}\tilde{\nabla}_c(\gamma_{\gamma\alpha})v^c + \frac{1}{2}f^\beta_{\alpha\gamma} V^\gamma. \tag{493}$$

Taking the left hand side to vanish yields conditions on V.

For example, from the first equation we learn that, although v is *not* covariantly constant on m in the generic case:

$$\tilde{\nabla}_a v^b = -\frac{1}{2}\mathcal{F}_{\gamma a}{}^b V^\gamma, \tag{494}$$

it must be Killing:

$$\tilde{\nabla}^{(a} v^{b)} = 0 \tag{495}$$

(and of course, then, divergence free). Comparing the fourth equation with (488), we see that

$$\xi^\alpha_\gamma = -\frac{1}{2}\gamma^{\alpha\beta}\tilde{\nabla}_b(\gamma_{\beta\gamma})v^b + \frac{1}{2}f^\alpha_{\gamma\beta}V^\beta \tag{496}$$

is fixed 'algebraically' (no derivatives) as promised.

Further, recall that in connection with the quantum reduction being consistent we demanded (423):

$$0 = \phi_\gamma V^\gamma = \xi^\gamma_\gamma - f^\gamma_{\gamma\beta}V^\beta = \xi^\gamma_\gamma \tag{497}$$

by (301). Using (496), this is equivalent to

$$v^a\tilde{\nabla}_a(\ln\sqrt{\gamma}) = 0, \tag{498}$$

i.e. the volume element on the gauge orbit should be constant along the integral curves of v. These results should be compared with (420).

Let us show that such vector fields V exist by constructing one in the helix model. $\nabla V = 0$ means that the components V^A in the Cartesian coordinates X, Y, Z are constants. Using (40) we find

$$(\mathcal{L}_{\phi_\gamma} V)^A = (V^2, -V^1, 0). \tag{499}$$

If this is to equal $\xi^\alpha_\gamma \phi^A_\alpha$, as in (487), the only solution is

$$V = (0, 0, \zeta) \text{ and } \xi^\alpha_\gamma = 0, \tag{500}$$

where ζ is an arbitrary constant. In this case

$$v^a = e^a_A V^A = \begin{pmatrix} \zeta \\ 0 \end{pmatrix}, \tag{501}$$

see (200), and

$$V^\alpha = e^\alpha_A V^A = \gamma^{\alpha\beta} G_{AB} \phi^B_\beta V^A = \zeta(1 + \rho^2)^{-1}, \tag{502}$$

where we used (244) and (205).

In particular, we notice from (494), and the fact that $\mathcal{F}^\gamma{}_{ab}$ is nonzero, that v in (501) is not covariantly constant. In fact, from the identity

$$\tilde{\nabla}_a \tilde{\nabla}_b v^b - \tilde{\nabla}_b \tilde{\nabla}_a v^b = -\tilde{\mathcal{R}}_{ab} v^b \tag{503}$$

we know that if m has a nondegenerate Ricci tensor then there cannot exist a nontrivial covariantly constant vector field on m.[12] So we could have guessed this result from the nondegeneracy of $\tilde{\mathcal{R}}_{ab}$ in (317), at least for $\rho \neq 0$. Perhaps there is a connection between the existence of a nontrivial Yang-Mills curvature $\mathcal{F}^\gamma{}_{ab}$ on M and a nondegenerate Ricci tensor $\tilde{\mathcal{R}}_{ab}$ on m.

[12]This point is also relevant for the Ricci tensor $\tilde{\mathcal{R}}_{\alpha\beta}$ within an orbit, given in (329), when we note that for a semi-simple Lie group $f^\delta_{\gamma\alpha} f^\gamma_{\delta\beta}$ is nondegenerate (see, e.g. the footnote on page 171 of [27]).

Finally, since $v = \zeta \partial/\partial B$, whereas $\sqrt{\gamma}$ depends only on ρ, we see that v also satisfies the quantum reduction consistency condition (498). We will make use of this v shortly.

Moving now to the tensor case, we proceed in an exactly analogous fashion. The condition for K to be associated with a classical observable on Γ, i.e. $K^{ab}(Q) =: k^{ab}(q(Q))$, can be stated as

$$(\mathcal{L}_{\phi_\gamma} K)^{AB} = \xi_\gamma^{\beta(A} \phi_\beta^{B)}, \tag{504}$$

for arbitrary vector fields $\xi_\gamma^{\beta A}$ on M [cf (93)]. From this we find

$$\begin{aligned} \phi_\gamma K^{a\beta} &= \mathcal{L}_{\phi_\gamma}(e_A^a K^{AB} e_B^\beta) = \frac{1}{2}\xi_\gamma^{\beta a} - f_{\gamma\delta}^\beta K^{a\delta}, & (505)\\ \phi_\gamma K^{\alpha\beta} &= \mathcal{L}_{\phi_\gamma}(e_A^\alpha K^{AB} e_B^\beta) = \xi_\gamma^{\beta\alpha} - 2 f_{\gamma\delta}^{(\alpha} K^{\beta)\delta}, & (506) \end{aligned}$$

where $\xi_\gamma^{\beta a}$ and $\xi_\gamma^{\beta\alpha}$ are the horizontal and vertical components of $\xi_\gamma^{\beta A}$, and $\xi_\gamma^{\beta\alpha} = \xi_\gamma^{\alpha\beta}$ by definition.

We now demand that K be covariantly constant on M. The components of ∇K in the horizontal/vertical basis are

$$\begin{aligned} \nabla_c K^{ab} &= \tilde{\nabla}_c k^{ab} + \mathcal{F}_{\delta c}{}^{(a} K^{b)\delta}, & (507)\\ \nabla_\gamma K^{ab} &= \mathcal{F}_{\gamma d}{}^{(a} k^{b)d} - \tilde{\nabla}^{(a}(\gamma_{\gamma\delta}) K^{b)\delta}, & (508)\\ \nabla_c K^{a\beta} &= w_c K^{a\beta} + \tilde{\Gamma}^a_{cd} K^{d\beta} + \frac{1}{2}\left[\gamma^{\beta\alpha}\tilde{\nabla}_c(\gamma_{\alpha\delta}) K^{a\delta} - \mathcal{F}^\beta{}_{cd} k^{ad} + \mathcal{F}_{\delta c}{}^a K^{\delta\beta}\right], & (509)\\ \nabla_\gamma K^{a\beta} &= w_\gamma K^{a\beta} + \frac{1}{2}\left[\mathcal{F}_{\gamma d}{}^a K^{d\beta} + f^\beta_{\gamma\delta} K^{a\delta} + \gamma^{\beta\alpha}\tilde{\nabla}_d(\gamma_{\alpha\gamma}) k^{ad} - \tilde{\nabla}^a(\gamma_{\gamma\delta}) K^{\delta\beta}\right] & (510)\\ \nabla_c K^{\alpha\beta} &= w_c K^{\alpha\beta} + \gamma^{\gamma(\alpha}\tilde{\nabla}_c(\gamma_{\gamma\delta}) K^{\beta)\delta} - \mathcal{F}^{(\alpha}_{cd} K^{\beta)d}, & (511)\\ \nabla_\gamma K^{\alpha\beta} &= w_\gamma K^{\alpha\beta} + f^{(\alpha}_{\gamma\delta} K^{\beta)\delta} + \gamma^{\delta(\alpha}\tilde{\nabla}_d(\gamma_{\delta\gamma}) K^{\beta)d}. & (512) \end{aligned}$$

It is easy to show that these are identically satisfied in the case of the metric, $K = G^{-1}$.

Taking the left hand side to vanish, we learn from the first relation that, as in the vector case, k is not covariantly constant in the generic case:

$$\tilde{\nabla}_c k^{ab} = -\mathcal{F}_{\delta c}{}^{(a} K^{b)\delta}, \tag{513}$$

but must, of course, be Killing:

$$\tilde{\nabla}^{(c} k^{ab)} = 0. \tag{514}$$

Combining the fourth and sixth relations with (505 –506), we can solve for the $\xi_\gamma^{\beta A}$:

$$\begin{aligned} \xi_\gamma^{\beta a} &= -\mathcal{F}_{\gamma d}{}^a K^{d\beta} + \tilde{\nabla}^a(\gamma_{\gamma\delta}) K^{\delta\beta} - \gamma^{\beta\alpha}\tilde{\nabla}_d(\gamma_{\alpha\gamma}) k^{ad} + f^\beta_{\gamma\delta} K^{a\delta}, & (515) \\ \xi_\gamma^{\beta\alpha} &= -\gamma^{\delta(\beta}\tilde{\nabla}_d(\gamma_{\delta\gamma}) K^{\alpha)d} + f^{(\beta}_{\gamma\delta} K^{\alpha)\delta}. & (516) \end{aligned}$$

For $K = G^{-1}$ these vanish identically, in accordance with our assumption that the ϕ_γ are Killing [cf (38)].

For the quantum reduction of quadratic variables to be consistent we had (429):

$$0 = \phi_\beta K^{a\beta} = \frac{1}{2}\xi_\beta^{\beta a} - f^\beta_{\beta\delta} K^{a\delta} = \frac{1}{2}\xi_\beta^{\beta a}, \tag{517}$$

which we now see means [cf (515)]

$$k^{ab}\tilde{\nabla}_b(\ln\sqrt{\gamma}) + \frac{1}{2}\mathcal{F}_{\delta b}{}^a K^{b\delta} - \frac{1}{2}\tilde{\nabla}^a(\gamma_{\alpha\beta}) K^{\alpha\beta} = 0. \tag{518}$$

This relation turns out to produce an important simplification of our later results. Notice that $\xi_\beta^{\beta a} = 0$ is analogous to (265), a condition on the metric in order to ensure the gauge invariance of the horizontal subspaces, but not as strong.

Returning to (513), we contract c and b yielding an expression for the divergence of k:

$$\tilde{\nabla}_b(k^{ab}) = -\frac{1}{2}\mathcal{F}_{\delta b}{}^a K^{b\delta}, \tag{519}$$

which appears in (518) above. On the other hand, contracting (511) with $\gamma_{\alpha\beta}$ yields

$$0 = w_a(\gamma_{\alpha\beta}K^{\alpha\beta}) + \mathcal{F}_{\delta ba}K^{b\delta}. \tag{520}$$

We can see that the trace expression, $\gamma_{\alpha\beta}K^{\alpha\beta}$, is gauge invariant by contracting (512) with $\gamma_{\alpha\beta}$:

$$\gamma_{\alpha\beta}\phi_\gamma K^{\alpha\beta} = -\tilde{\nabla}_d(\gamma_{\beta\gamma})K^{d\beta} = 0, \tag{521}$$

the last equality following from contracting (508) with g_{ab}. Finally, combining (520) with (519) we have an alternative expression for the divergence of k:

$$\tilde{\nabla}_b(k^{ab}) = \frac{1}{2}\tilde{\nabla}^a(\gamma_{\alpha\beta}K^{\alpha\beta}), \tag{522}$$

which shows it as the gradient of a gauge invariant scalar on m. For the metric, this scalar is a constant, as expected (equal to one half the dimension of an orbit).

In fact we could have guessed this relation as follows: If K is covariantly constant on M then it is certainly Killing, which means (assuming $\mathcal{C}_2(K)$ is a classical observable) that k is Killing on m: (514) [cf (450)]. Contracting (514) with g_{ab} yields

$$\tilde{\nabla}_a(k^{ac}) = -\frac{1}{2}\tilde{\nabla}^c(g_{ab}k^{ab}). \tag{523}$$

But the trace $G_{AB}K^{AB} = g_{ab}k^{ab} + \gamma_{\alpha\beta}K^{\alpha\beta}$ is a constant, which then establishes (522).

Having verified that the various expressions for $\tilde{\nabla}_b(k^{ab})$ are, indeed, gauge invariant, we can work out the quantity we are particularly interested in:

$$\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) = w_a(\tilde{\nabla}_b(k^{ab})) + \tilde{\Gamma}^a_{ac}\tilde{\nabla}_b(k^{cb}), \tag{524}$$

where we have put w_a in place of ∂_a, with impunity. Using (519) we find

$$\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) = -\frac{1}{2}\left\{w_a(\mathcal{F}_{\delta b}{}^a K^{b\delta}) + \tilde{\Gamma}^a_{ac}\mathcal{F}_{\delta b}{}^c K^{b\delta}\right\}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left\{\left[w_a\mathcal{F}_{\delta b}{}^{a} + \tilde{\Gamma}^{a}_{ac}\mathcal{F}_{\delta b}{}^{c} - \tilde{\Gamma}^{c}_{ab}\mathcal{F}_{\delta c}{}^{a}\right]K^{b\delta}\right. \\
&\quad \left. + \left[w_a K^{b\delta} + \tilde{\Gamma}^{b}_{ac}K^{c\delta}\right]\mathcal{F}_{\delta b}{}^{a}\right\}, \qquad (525)
\end{aligned}
$$

where we added and subtracted the term $\tilde{\Gamma}^{c}_{ab}\mathcal{F}_{\delta c}{}^{a}K^{b\delta}$. Notice that this is expressed entirely in terms of cross components of K. We can eliminate the derivative term in the last line by using (509):

$$
\begin{aligned}
\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) &= -\frac{1}{2}\left[w_a\mathcal{F}_{\delta b}{}^{a} + \tilde{\Gamma}^{a}_{ac}\mathcal{F}_{\delta b}{}^{c} - \tilde{\Gamma}^{c}_{ab}\mathcal{F}_{\delta c}{}^{a}\right]K^{b\delta} \\
&\quad -\frac{1}{4}\left[\mathcal{F}^{\beta}{}_{ac}k^{cb} - \gamma^{\beta\gamma}\tilde{\nabla}_a(\gamma_{\gamma\delta})K^{b\delta} - \mathcal{F}_{\alpha a}{}^{b}K^{\alpha\beta}\right]\mathcal{F}_{\beta b}{}^{a}. \qquad (526)
\end{aligned}
$$

We remark that all of the $\nabla K = 0$ conditions in (507–512) have now been used, in one way or another, and the main result is an expression for $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$.

6.4 Consequences of M Ricci Flat

We now turn to the conditions imposed by the fact that M is (Ricci) flat [i.e. (315), (319), and (320), with vanishing left hand side], and we expect these to yield at least an expression involving the other important term: $k^{ab}\tilde{\mathcal{R}}_{ab}$. Contracting the first with k^{ab} yields

$$
k^{ab}\mathcal{R}_{ab} = k^{ab}\tilde{\mathcal{R}}_{ab} + \frac{1}{2}k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}_{\ b}{}^{c} - k^{ab}\tilde{\nabla}_a(\tilde{\nabla}_b(\ln\sqrt{\gamma})) + \frac{1}{4}k^{ab}\tilde{\nabla}_a(\gamma^{\alpha\beta})\tilde{\nabla}_b(\gamma_{\alpha\beta}). \qquad (527)
$$

The third term on the right hand side, second order in derivatives of γ, closely resembles $\alpha^{(\gamma)}(k)$ in (473). In fact,

$$
\begin{aligned}
k^{ab}\tilde{\nabla}_a(\tilde{\nabla}_b(\ln\sqrt{\gamma})) &= \tilde{\nabla}_a(k^{ab}\tilde{\nabla}_b(\ln\sqrt{\gamma})) - \tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma}) \qquad (528) \\
&= 2\alpha^{(\gamma)}(k) - \frac{1}{2}k^{ab}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}_b(\ln\sqrt{\gamma}) - \tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma}).
\end{aligned}
$$

Hence

$$
k^{ab}\mathcal{R}_{ab} = -2\alpha^{(\gamma)}(k) + k^{ab}\tilde{\mathcal{R}}_{ab} + \frac{1}{2}k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}_{\ b}{}^{c} + \frac{1}{4}k^{ab}\tilde{\nabla}_a(\gamma^{\alpha\beta})\tilde{\nabla}_b(\gamma_{\alpha\beta})
$$

$$+\frac{1}{2}k^{ab}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}_b(\ln\sqrt{\gamma})+\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma}). \tag{529}$$

Next we contract (320) with $K^{\alpha\beta}$:

$$\begin{aligned} K^{\alpha\beta}\mathcal{R}_{\alpha\beta} &= -\frac{1}{4}K^{\alpha\beta}f^{\delta}_{\gamma\alpha}f^{\gamma}_{\delta\beta}-\frac{1}{4}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a}-\frac{1}{2}K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} \\ &+\frac{1}{2}K^{\alpha\beta}\gamma^{\gamma\delta}\tilde{\nabla}_a(\gamma_{\gamma\alpha})\tilde{\nabla}^a(\gamma_{\delta\beta})-\frac{1}{2}\tilde{\nabla}_a(\ln\sqrt{\gamma})K^{\alpha\beta}\tilde{\nabla}^a(\gamma_{\alpha\beta}). \end{aligned} \tag{530}$$

Again, the third term on the right hand side, second order in derivatives of γ, can be written in terms of $\alpha^{(\gamma)}(k)$ as follows:

$$\begin{aligned} K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} &= K^{\alpha\beta}\left[w_a(\tilde{\nabla}^a(\gamma_{\alpha\beta}))+\tilde{\Gamma}^a_{ab}\tilde{\nabla}^b(\gamma_{\alpha\beta})\right] \\ &= w_a(K^{\alpha\beta}\tilde{\nabla}^a(\gamma_{\alpha\beta}))+\tilde{\Gamma}^a_{ab}K^{\alpha\beta}\tilde{\nabla}^b(\gamma_{\alpha\beta})-w_a(K^{\alpha\beta})\tilde{\nabla}^a(\gamma_{\alpha\beta}) \end{aligned} \tag{531}$$

But we know from the quantum consistency condition (518), combined with (519), that

$$K^{\alpha\beta}\tilde{\nabla}^a(\gamma_{\alpha\beta})=-2\left[\tilde{\nabla}_b(k^{ab})-k^{ab}\tilde{\nabla}_b(\ln\sqrt{\gamma})\right] \tag{532}$$

is a gauge invariant vector on m. (We remark that $K^{\alpha\beta}$ itself need not be gauge invariant in the generic case). Using this fact, as well as (511), we find

$$\begin{aligned} K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} &= -2\tilde{\nabla}_a\left[\tilde{\nabla}_b(k^{ab})-k^{ab}\tilde{\nabla}_b(\ln\sqrt{\gamma})\right] \\ &+K^{\alpha\beta}\gamma^{\gamma\delta}\tilde{\nabla}_a(\gamma_{\gamma\alpha})\tilde{\nabla}^a(\gamma_{\delta\beta})+\mathcal{F}^{\alpha}_{b}{}^{a}\tilde{\nabla}_a(\gamma_{\alpha\beta})K^{b\beta}. \end{aligned} \tag{533}$$

Then, as we did for $k^{ab}\tilde{\nabla}_a(\tilde{\nabla}_b(\ln\sqrt{\gamma}))$, it is straightforward to show that

$$\begin{aligned} K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} &= 4\alpha^{(\gamma)}(k)-2\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))-k^{ab}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}_b(\ln\sqrt{\gamma}) \\ &+K^{\alpha\beta}\gamma^{\gamma\delta}\tilde{\nabla}_a(\gamma_{\gamma\alpha})\tilde{\nabla}^a(\gamma_{\delta\beta})+\mathcal{F}^{\alpha}_{b}{}^{a}\tilde{\nabla}_a(\gamma_{\alpha\beta})K^{b\beta}. \end{aligned} \tag{534}$$

Finally, before using this in (530), we note that for the last term in the latter we can again make use of the important relation (532). The result is

$$\begin{aligned} K^{\alpha\beta}\mathcal{R}_{\alpha\beta} &= -2\alpha^{(\gamma)}(k)+\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))-\frac{1}{4}K^{\alpha\beta}f^{\delta}_{\gamma\alpha}f^{\gamma}_{\delta\beta}-\frac{1}{4}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} \\ &-\frac{1}{2}k^{ab}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}_b(\ln\sqrt{\gamma})+\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma})-\frac{1}{2}\mathcal{F}^{\alpha}_{b}{}^{a}\tilde{\nabla}_a(\gamma_{\alpha\beta})K^{b\beta}. \end{aligned} \tag{535}$$

Finally, we contract (319) with $K^{a\beta}$:

$$2K^{a\beta}\mathcal{R}_{a\beta} = \left[w_b\mathcal{F}_{\beta a}{}^{b} + \tilde{\Gamma}^{b}_{bc}\mathcal{F}_{\beta a}{}^{c} - \tilde{\Gamma}^{c}_{ba}\mathcal{F}_{\beta c}{}^{b} + \tilde{\nabla}_b(\ln\sqrt{\gamma})\mathcal{F}_{\beta a}{}^{b}\right] K^{a\beta}. \tag{536}$$

Comparing this with $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$ in (526), and using (519) again, we have the third and final curvature condition:

$$\begin{aligned} 2K^{a\beta}\mathcal{R}_{a\beta} &= -2\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) - 2\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma}) \\ &\quad -\frac{1}{2}\left[\mathcal{F}^{\beta}{}_{ac}k^{cb} - \gamma^{\beta\gamma}\tilde{\nabla}_a(\gamma_{\gamma\delta})K^{b\delta} - \mathcal{F}_{\alpha a}{}^{b}K^{\alpha\beta}\right]\mathcal{F}_{\beta b}{}^{a}. \end{aligned} \tag{537}$$

In summary, by making use of the two main conditions on the extended space: $\nabla K = 0$ and M (Ricci) flat, which allowed for a successful Dirac quantization of quadratic operators, we have arrived at the three identities (529), (537), and (535). Two of these involve the scalar term $\alpha^{(\gamma)}(k)$, which plays the central role in the nonminimal quantum-reduced Dirac quantization scheme (470–472). The left hand side of the three conditions vanishes, and so the most general (linear) way to combine them is

$$0 = k^{ab}\mathcal{R}_{ab} + \mu 2K^{a\beta}\mathcal{R}_{a\beta} + \nu K^{\alpha\beta}\mathcal{R}_{\alpha\beta} \tag{538}$$

for arbitrary μ, ν. Note that the nominal choice $\mu = \nu = 1$ corresponds to the natural contraction $K^{AB}\mathcal{R}_{AB}$, but any choice is equally valid. Solving (538) for $\alpha^{(\gamma)}(k)$ we find

$$\begin{aligned} 2(1+\nu)\alpha^{(\gamma)}(k) &= \left[k^{ab}\tilde{\mathcal{R}}_{ab} - (2\mu-\nu)\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))\right] \\ &\quad +\frac{1}{4}k^{ab}\tilde{\nabla}_a(\gamma^{\alpha\beta})\tilde{\nabla}_b(\gamma_{\alpha\beta}) + \frac{1}{4}(2\mu-\nu)K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} - \frac{\nu}{4}K^{\alpha\beta}f^{\delta}_{\gamma\alpha}f^{\gamma}_{\delta\beta} \\ &\quad +\frac{1}{2}(1-\mu)k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}_{b}{}^{c} + \frac{1}{2}(1-\nu)k^{ab}\tilde{\nabla}_a(\ln\sqrt{\gamma})\tilde{\nabla}_b(\ln\sqrt{\gamma}) \\ &\quad +[1-(2\mu-\nu)]\tilde{\nabla}_a(k^{ab})\tilde{\nabla}_b(\ln\sqrt{\gamma}) + \frac{1}{2}(\mu-\nu)\tilde{\nabla}_a(\gamma_{\alpha\beta})\mathcal{F}^{\alpha}_{b}{}^{a}K^{b\beta}. \end{aligned} \tag{539}$$

This is one of the central results of our work.

There are two immediately striking features of this result. First, the additional scalar $\alpha^{(\gamma)}(k)$ that appears in quantum-reduced Dirac quantization actually contains terms such as $k^{ab}\tilde{\mathcal{R}}_{ab}$ and $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$, in accordance with virtually all other quantization schemes on a curved configuration manifold. Second, it contains terms in addition to these, which appear to require knowledge of the gauge structure not available in the classically reduced theory.

6.5 Comparison Between Dirac Quantization and Curved-Space Quantization Schemes

The nominal choice $\mu = \nu = 1$ eliminates the last four terms in (539), and results in

$$\alpha^{(\gamma)}(k) = \frac{1}{4}\left[k^{ab}\tilde{\mathcal{R}}_{ab} - \tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))\right] + \beta^{(\gamma)}(k), \tag{540}$$

where

$$\beta^{(\gamma)}(k) := \frac{1}{16}\left[k^{ab}\tilde{\nabla}_a(\gamma^{\alpha\beta})\tilde{\nabla}_b(\gamma_{\alpha\beta}) + K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} - K^{\alpha\beta}f^{\delta}_{\gamma\alpha}f^{\gamma}_{\delta\beta}\right]. \tag{541}$$

Compare this with Vaisman's quantization scheme, (482), where his $B(k)$ corresponds to our $\beta^{(\gamma)}(k)$. The agreement between the leading terms is remarkable.

This also corresponds to (485) with $\lambda = 1/2$ (Weyl), except that, as we will soon see, $\beta^{(\gamma)}(k)$ is nonzero in general.

We can deepen the correspondence with Vaisman's quantization scheme by observing another result in his paper [41]: With reference to the decomposition, (481), his $B(k)$ has the form

$$B(k) = \sum_{m=1}^{r} \epsilon_m \left[A(v_m) + \frac{1}{4}\tilde{\nabla}_a(v^b_m)\tilde{\nabla}_b(v^a_m)\right], \tag{542}$$

with arbitrary function A defined in (480). Now suppose we construct a tensor

$$K^{AB} := \sum_{m=1}^{r} \epsilon_m V^A_m V^B_m, \tag{543}$$

$\epsilon_m = \pm 1$, out of a set of covariantly constant vector fields V_m on M, which satisfy (487) and (497) as discussed earlier. This K is covariantly constant, produces a proper physical projection:

$$k^{ab} = e^a_A K^{AB} e^b_B = \sum_{m=1}^{r} \epsilon_m v^a_m v^b_m, \tag{544}$$

and satisfies the quantum-reduction consistency condition (517):

$$\phi_\beta K^{\alpha\beta} = \phi_\beta (e^a_A K^{AB} e^b_B) = \sum_{m=1}^{r} \epsilon_m v^a_m \phi_\beta V^\beta_m = 0. \tag{545}$$

But the vector fields V_m satisfy (494), so the middle term in $\beta^{(\gamma)}(k)$ is

$$\begin{aligned}\frac{1}{16} K^{\alpha\beta} \mathcal{F}_{\alpha a}{}^{b} \mathcal{F}_{\beta b}{}^{a} &= \frac{1}{16} \sum_{m=1}^{r} \epsilon_m (V^\alpha_m \mathcal{F}_{\alpha a}{}^{b})(V^\beta_m \mathcal{F}_{\beta b}{}^{a}) \\ &= \frac{1}{4} \sum_{m=1}^{r} \epsilon_m \tilde{\nabla}_a (v^b_m) \tilde{\nabla}_b (v^a_m),\end{aligned} \tag{546}$$

exactly the term in $B(k)$! This result allows us to fix $A(v)$, which remained largely arbitrary in Vaisman's work: The Dirac quantization tells us that the difference, $\widehat{\xi^2} - \hat{\xi}^2$, is proportional to the scalar

$$A(v) = \frac{1}{16} \left[v^a v^b \tilde{\nabla}_a (\gamma^{\alpha\beta}) \tilde{\nabla}_b (\gamma_{\alpha\beta}) - V^\alpha V^\beta f^\delta_{\gamma\alpha} f^\gamma_{\delta\beta} \right] \tag{547}$$

(at least for this special class of covariantly constant V).

The first thing we notice is that $A(v)$ depends on the gauge structure—it cannot be calculated using information available only in the classical reduced theory, even in the abelian case. In view of (498), we see that $A(v)$ vanishes identically in the case of a one dimensional gauge group, as for example in the helix model, but the same argument does not apply for a generic gauge group. In the latter case the first term in $A(v)$ can be written in terms of a trace as

$$-\frac{1}{4} \mathrm{tr}\, H^2 \neq -\frac{1}{4} [\mathrm{tr}\, H]^2 = -\frac{1}{4} [v^a \tilde{\nabla}_a (\ln \sqrt{\gamma})]^2 = 0, \tag{548}$$

where the matrix

$$H^\alpha_\gamma := \frac{1}{2}\gamma^{\alpha\beta}v^a\tilde{\nabla}_a(\gamma_{\beta\gamma}). \tag{549}$$

In the generic case the inequality applies, of course, even if H is diagonal, except when H is one dimensional.

We can cast $A(v)$ in a different form by recalling that v satisfies (493) with vanishing left hand side. On squaring this equation we find that the cross terms disappear, and in fact

$$A(v) = -\frac{1}{4}w_\alpha(V^\beta)w_\beta(V^\alpha). \tag{550}$$

[Again, notice that (497) applies in the one dimensional case]. Compare this with the other term in $B(k)$ [cf (542)]: we would like to have $\tilde{\nabla}_\alpha$—the Levi-Civita connection associated with $\gamma_{\alpha\beta}$ in an orbit—in place of w_α. With regard to our discussion at the end of section 4.5 [and see (307)], we write[13]

$$\tilde{\nabla}_\alpha V^\beta = w_\alpha V^\beta + \frac{1}{2}f^\beta_{\alpha\gamma}V^\gamma. \tag{551}$$

On squaring this we find

$$A(v) = -\frac{1}{4}\tilde{\nabla}_\alpha(V^\beta)\tilde{\nabla}_\beta(V^\alpha) + \frac{1}{4}V^\alpha V^\beta \tilde{\mathcal{R}}_{\alpha\beta}, \tag{552}$$

where we used (493) again, as well as the expression (329) for the Ricci tensor $\tilde{\mathcal{R}}_{\alpha\beta}$ *within* a gauge orbit.

Applying this result to the tensor case we can write $\beta^{(\gamma)}(k)$ in the form [cf (541)]:

$$\beta^{(\gamma)}(k) = \sum_{m=1}^{r}\epsilon_m\left[-\frac{1}{4}\tilde{\nabla}_\alpha(V^\beta_m)\tilde{\nabla}_\beta(V^\alpha_m) + \frac{1}{4}\tilde{\nabla}_a(v^b_m)\tilde{\nabla}_b(v^a_m) + \frac{1}{4}V^\alpha_m V^\beta_m\tilde{\mathcal{R}}_{\alpha\beta}\right]. \tag{553}$$

It is remarkable that for every orbit space term, there appears to be a 'complementary' term within the orbits: If the first two derivative-type terms could be

[13]See the footnote on page 116 concerning non-covariant-constancy of V^α.

lumped together, in some sense, and the curvature term [which is $\frac{1}{4}K^{\alpha\beta}\tilde{\mathcal{R}}_{\alpha\beta}$, independent of the decomposition (543)] combined with $\frac{1}{4}k^{ab}\tilde{\mathcal{R}}_{ab}$ in (540), we would be tempted to conclude $A(v)$ was 'zero'. (It can even be shown that $\tilde{\nabla}_\alpha(\tilde{\nabla}_\beta(K^{\alpha\beta}))$, analogous to the term $\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))$, identically vanishes with our assumptions, which may be the only reason it does not appear like the others.)

But as Vaisman [41] points out, the case $A(v) \equiv 0$ means that $B(k)$ depends on the choice of local basis v_m. We will not address this question here, except to remark that, of course, not all covariantly constant vectors satisfy the additional properties (487) and (497), which we have demanded of the V_m.

Let us now realize this discussion in a concrete example. Recall that for the helix model we had constructed a covariantly constant vector V [see (501,502)], which is suitable to use in (543):

$$K^{AB} := \epsilon V^A V^B. \tag{554}$$

The components of this K in the horizontal/vertical basis turn out to be

$$\begin{aligned} k^{ab} &= \epsilon v^a v^b = \zeta^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & (555) \\ K^{a\beta} &= \epsilon v^a V^\beta = \epsilon\zeta^2\gamma^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & (556) \\ K^{\alpha\beta} &= \epsilon V^\alpha V^\beta = \epsilon\zeta^2\gamma^{-2}, & (557) \end{aligned}$$

where $\gamma = \det \gamma_{\alpha\beta} = 1 + \rho^2$.

Although k is Killing, it is not covariantly constant; it can be shown that

$$\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab})) = \epsilon\zeta^2(3\rho^2 - 2)\gamma^{-3}. \tag{558}$$

Furthermore, using (317) we find

$$k^{ab}\tilde{\mathcal{R}}_{ab} = \epsilon\zeta^2 3\rho^2\gamma^{-3}, \tag{559}$$

so the leading terms in $\alpha^{(\gamma)}(k)$ are

$$\frac{1}{4}\left[k^{ab}\tilde{\mathcal{R}}_{ab}-\tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))\right]=\frac{1}{2}\epsilon\zeta^2\gamma^{-3}. \tag{560}$$

As already mentioned, $A(v)$ vanishes identically for a one dimensional gauge group, so the only term that contributes to $\beta^{(\gamma)}(k)$ is

$$\frac{1}{16}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a}=-\frac{1}{2}\epsilon\zeta^2\gamma^{-3}, \tag{561}$$

where we made use of (283).

Hence, although $\beta^{(\gamma)}(k)$ is nontrivial for this k, in this model, it cancels with the leading terms, so that $\alpha^{(\gamma)}(k)$ vanishes [cf (540)]. In other words, the Dirac quantization of this k we constructed coincides with minimal quantization. We remark that, by its construction, (544), k^{ab} is a *degenerate* Killing tensor, a fact which may be relevant to this discussion, but needs more investigation. Indeed, with seemingly so many conditions on K, one might wonder if there even exists a nontrivial example with $\alpha^{(\gamma)}(k)\neq 0$: if we could just be clever enough with all the conditions maybe we could see that $\alpha^{(\gamma)}(k)$ vanishes identically. This is *not* true, as we now demonstrate for the important case of the kinetic energy: $k=\frac{1}{2}g^{-1}$. (See also [19] and (474)).

With $k^{ab}=\frac{1}{2}g^{ab}$ the Dirac quantization, (472), tells us that the kinetic energy operator

$$\hat{T}:=\mathcal{Q}_2'(\frac{1}{2}g^{-1})=-\hbar^2\left\{\frac{1}{2}\tilde{\Delta}-\frac{1}{8}\tilde{\mathcal{R}}-\beta^{(\gamma)}(\frac{1}{2}g^{-1})\right\}, \tag{562}$$

where [cf (540,541)]

$$\beta^{(\gamma)}(\frac{1}{2}g^{-1})=\frac{1}{32}\left[\tilde{\nabla}_c(\gamma_{\alpha\beta})\tilde{\nabla}^c(\gamma^{\alpha\beta})-\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab}+f_{\gamma\alpha\beta}f^{\gamma\alpha\beta}\right]. \tag{563}$$

As discussed earlier, the curvature scalar term $\frac{\hbar^2}{8}\tilde{\mathcal{R}}$ agrees, for example, with the $B(k)\equiv 0$ case of Vaisman, the $\lambda=1/2$ (Weyl) case of Zhang-Ju and Min, as well as Maslov's semi-classical theory [48], except here we have the additional term $\beta^{(\gamma)}(\frac{1}{2}g^{-1})$. This term is anomalous also in the sense that it appears to require knowledge of the gauge structure on M not available from within the classical reduced theory.

The natural question is whether or not $\beta^{(\gamma)}(\frac{1}{2}g^{-1})$ is zero, and if not, is it proportional to $\tilde{\mathcal{R}}$?

The answer to the first question is no: even in an example as simple as the helix model we find

$$\begin{aligned} \tilde{\nabla}_c(\gamma_{\alpha\beta})\tilde{\nabla}^c(\gamma^{\alpha\beta}) &= -4\rho^2(1+\rho^2)^{-2}, & (564)\\ \mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} &= 8(1+\rho^2)^{-2}, & (565)\end{aligned}$$

whose difference does not vanish. The answer to the second question is also no—but inspection of (318) suggests the identification

$$\mathcal{F}^2 \longleftrightarrow \frac{4}{3}\tilde{\mathcal{R}}, \tag{566}$$

true at least for this example. We can then naturally pull this term out of $\beta^{(\gamma)}(\frac{1}{2}g^{-1})$, yielding

$$\hat{T} = -\hbar^2\left\{\frac{1}{2}\tilde{\Delta} - \frac{1}{12}\tilde{\mathcal{R}} + \frac{1}{8}\rho^2(1+\rho^2)^{-2}\right\}. \tag{567}$$

Remarkably, this reproduces the $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ result of geometric quantization!

But we still have a term left over, which cannot be absorbed into a numerical factor in front of $\hbar^2\tilde{\mathcal{R}}$, and this term depends on the gauge structure of the unreduced theory. So even in this simple example we have an anomaly, in the sense that working strictly from within the reduced classical theory the *only* natural object with the correct dimensions that can be added to the Laplace-Beltrami operator is a multiple of $\hbar^2\tilde{\mathcal{R}}$.

Let us generalize (566). Recall the three terms (527), (530), and (536) in $K^{AB}\mathcal{R}_{AB}$. In the special case of the kinetic energy, $K = \frac{1}{2}G^{-1}$, the cross term equation is trivial, and the remaining two reduce to

$$\begin{aligned} \tilde{\mathcal{R}} &= \frac{1}{2}\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} + \tilde{\Delta}\ln\sqrt{\gamma} - \frac{1}{4}\tilde{\nabla}_c(\gamma_{\alpha\beta})\tilde{\nabla}^c(\gamma^{\alpha\beta}), && (568)\\ 0 &= \frac{1}{4}\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} - \tilde{\Delta}\ln\sqrt{\gamma} - \tilde{\nabla}_c(\ln\sqrt{\gamma})\tilde{\nabla}^c(\ln\sqrt{\gamma}) + \frac{1}{4}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta} && (569)\end{aligned}$$

Notice, for example, that $k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}{}_{b}{}^{c}$ is the same as $K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a}$ only for $K \propto G^{-1}$. It is natural to add these two equations, and thereby eliminate the term highest order in derivatives of γ, yielding

$$\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} = \frac{4}{3}\tilde{\mathcal{R}} - \frac{1}{3}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta} + \frac{1}{3}\tilde{\nabla}_c(\gamma_{\alpha\beta})\tilde{\nabla}^c(\gamma^{\alpha\beta}) + \frac{4}{3}\tilde{\nabla}_c(\ln\sqrt{\gamma})\tilde{\nabla}^c(\ln\sqrt{\gamma}). \quad (570)$$

We remark that the second term on the right hand side is actually $-\frac{4}{3}$ times the Ricci scalar, (331), *within* a given orbit; again we see 'complimentary' orbit terms [cf discussion following (553)]. Also, the last two terms identically cancel for a one dimensional gauge group [but not in the generic case: cf (548)], which explains the helix model result, (566).

Using this last result in (562), we might argue that the kinetic energy operator should be written instead as

$$\hat{T} = -\hbar^2\left\{\frac{1}{2}\tilde{\Delta} - \frac{1}{12}\tilde{\mathcal{R}} - \beta'^{(\gamma)}(\frac{1}{2}g^{-1})\right\}, \quad (571)$$

where

$$\beta'^{(\gamma)}(\frac{1}{2}g^{-1}) := \frac{1}{24}\left[\frac{1}{2}\tilde{\nabla}_c(\gamma_{\alpha\beta})\tilde{\nabla}^c(\gamma^{\alpha\beta}) - \tilde{\nabla}_c(\ln\sqrt{\gamma})\tilde{\nabla}^c(\ln\sqrt{\gamma}) + f_{\gamma\alpha\beta}f^{\gamma\alpha\beta}\right]. \quad (572)$$

Unlike its counterpart, $\beta^{(\gamma)}(\frac{1}{2}g^{-1})$, this 'anomaly term' contains no Yang-Mills curvature piece. Nevertheless, it still does not vanish even in the helix model, and requires knowledge of gauge structure not available in the classical reduced theory.

It is easy to show that this form corresponds to the choice $\mu = 0$, $\nu = 2$ in (539). Other choices of μ and ν, on the other hand, do not seem to have any special appeal.

Part IV
Conclusion

In this thesis we studied quantization of symmetries in gauge theories, in particular the Poincaré symmetry of scalar electrodynamics. This led to a more general discussion of curved-space quantization of observables quadratic in the momenta. In this context, there are several proposed quantization schemes in the literature [42,41,13,7,14], but they do not solve the problem of van Hove anomalies in quadratic-quadratic commutators[14], except when m is of constant curvature or Ricci flat, and even then the observables lie in a restricted class. This is an old and difficult problem. We discussed sufficient conditions under which Dirac quantization provides a solution to this problem—conditions which are broad enough to yield rich results (e.g. m with nonconstant curvature), and provide a new interpretation of what makes Dirac quantization special.

We briefly summarize the approach we used, which is two-fold: In the spirit of dynamical quantization [12], we restrict our observables to generators of a symmetry group of the Hamiltonian system. Next we take advantage of the fact that for gauge theories the extended configuration space M is often flat, whereas the reduced space m has nontrivial curvature. This means Dirac quantization is less troublesome than reduced, and if successful, induces a specific quantization scheme for (in particular) quadratic observables on the curved reduced space m, which is free of van Hove anomalies. Of course going to a gauge theory introduces additional subtleties associated with the redundant degrees of freedom and quantum reduction—we described these in detail, and justified the restrictions we suggested [see in particular (423) and (429) in the context of quantum reduction].

[14]Presumably geometric quantization could solve this problem provided difficulties with choice of polarization compatible with quadratic observables could be overcome [5].

We then applied this programme to the Poincaré symmetry of scalar electrodynamics, and found that is was, indeed, successful. In itself this is not surprising (see, e.g. [25]), but to our knowledge this result has not been demonstrated explicitly by the methods used here, and it provides an important physical example in support of Dirac quantization. Of course this does not rule out other possible factor orderings, such as reduced quantization, especially with a model as simple as scalar electrodynamics. It would be of interest to investigate this more fully, perhaps considering more complex and exacting models, in an attempt to find a physically relevant example for which (minimal) reduced quantization does not consistently quantize the symmetry in question, and thus definitively resolve the Dirac versus reduced controversy (at least in that example)[15]. On this point we would just like to emphasize again that virtually all proposed quantization schemes on curved configuration manifolds add some kind of curvature and/or divergence-type terms to the minimal quadratic operators in attempting to avoid van Hove anomalies, and that some of these are successful in a restricted sense when the space is of constant curvature or Ricci flat (see, e.g., [7,14]). In light of these facts, it is not likely that minimal reduced quantization is correct when m is curved.

Furthermore, we discovered deep connections between quantum-reduced Dirac quantization and these other quantization schemes, connections simply not present in minimal reduced quantization, which convinces us that there is something special about Dirac quantization. (We review this below.) What *would* have been surprising, then, is if Dirac quantization of scalar electrodynamics had *not* been

[15]We argued that the appearance of $\delta(\mathrm{o})$, common to any field theoretic example such as our scalar electrodynamics, was not a problem, in that it was always multiplied by something either identically zero, or zero by boundary conditions. However, further work in the direction suggested should really be accompanied by proper regularization. Alternatively, one might construct a suitable finite dimensional model. These directions are currently being pursued.

consistent.

Motivated by the success of scalar electrodynamics, we generalized the features of it which make it amenable to Dirac quantization in order to analyze, with some generality (and mathematical rigor in the finite dimensional case), the geometrical structure of Dirac quantization itself. These features determine a set of restrictions on class of observables and curvature, on the reduced space, that Dirac quantization is valid for. With M (Ricci) flat we found that a sufficient condition for a consistent Dirac quantization is that the tensors K associated with observables quadratic in the momenta be covariantly constant, and satisfy a certain quantum-reduction consistency condition (as well as some conditions on the vectors associated with linear observables, which are not of interest here).

We remark that although these conditions allowed us to analyze Dirac quantization as desired, with rich results, the $\nabla K = 0$ condition is probably stronger than necessary. Furthermore, considering all tensors of this type goes beyond the original Poincaré symmetry subalgebra, and one is then naturally led to the question: Given a Hamiltonian system with (Ricci) flat M, what is the largest symmetry subalgebra of the full Poisson algebra with generators of degree, say, less than or equal to two, and for which Dirac quantization and quantum reduction are consistent? This would yield a 'maximal' restricted class of observables on the reduced space. Disregarding complications inherent in gauge theories, we note that in order for the subalgebra to close, the tensors, K_i, associated with quadratic pieces of the generators must all be in involution, as we discussed earlier in a similar context [see (375)]. Taking the Hamiltonian (with nondegenerate kinetic energy) as one of the generators, we set $K_0 := \frac{1}{2}G^{-1}$, and so then all K_i's must be Killing tensors. A useful result is that if M is flat (or, more generally, of constant curvature), then the K_i must be *degenerate* Killing tensors of the form

[51]:

$$K_i^{AB} = c_0 G^{AB} + \sum_{I,J} c_{IJ} W_I^A W_J^B, \tag{573}$$

where the W_I are the Killing vectors on M (which in the flat case are just translations and rotations) and the c's are constants (with $c_{IJ} = c_{JI}$). These comments apply, for example, for the boost tensors in scalar electrodynamics.

In general, the question of the existence of quadratic first integrals for a given Hamiltonian system is addressed by Ikeda and Kimura [37]. A well known example is the Runge-Lenz observable in the Kepler problem, making $SO(4)$ the maximal symmetry group for bound states, $E < 0$ (see, e.g. [52]). When dealing with a gauge theory these issues are complicated by the fact that the symmetry algebra need only be realized when acting on physical states ("on shell") as we saw, for example, in scalar electrodynamics. But given any such maximal symmetry subalgebra one need only check that the Dirac quantization and quantum reduction are consistent, and then the same geometrical analysis of $\alpha^{(\gamma)}(k)$ we did here would go through, but with the $\nabla K = 0$ conditions replaced by some others.

We found that the quantum-reduced Dirac quantization involved the scalar $\hbar^2 \alpha^{(\gamma)}(k)$ added to the minimal quadratic operator. This term has been known for some time in the literature [19,21], and it is even known that it can be understood as a factor ordering ambiguity involving the extraneous scalar $\sqrt{\gamma}$ [23]. What we discovered here is its geometrical significance—why it makes Dirac quantization so special. A Kaluza-Klein-like analysis, within the restrictions we derived, revealed two remarkable features of the innocuous-looking $\alpha^{(\gamma)}(k)$: First, it actually *does* contain curvature and divergence terms present in virtually all other quantization schemes on curved spaces, even with the 'correct'[16] numerical factors. Second,

[16]As pointed out earlier, unlike quantum-reduced Dirac quantization, none of the quan-

despite this intimate connection, it does not exactly agree with any of them—it contains 'nonremovable' pieces which depend on knowledge of gauge structure not available in the classical reduced theory. We discuss these two points in turn.

From our analysis, the $\mu = \nu = 1$ choice in (539) appears to be the most natural: For generic k (within the class considered here) it is the unique choice that produces the leading term

$$\frac{\hbar^2}{4}\left[k^{ab}\tilde{\mathcal{R}}_{ab} - \tilde{\nabla}_a(\tilde{\nabla}_b(k^{ab}))\right], \tag{574}$$

which agrees with the $\lambda = 1/2$ (Weyl) case of Zhang-Ju and Min's [42] quantization scheme—the only case, by the way, for which their operator is self adjoint for generic k. This is also the leading term found in Vaisman's [41] completely independent approach to quantization. Even more remarkable is the further correspondence between Vaisman's $B(k)$ and our $\beta^{(\gamma)}(k)$, in particular the term

$$\frac{1}{16}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} = \frac{1}{4}\sum_{m=1}^{r}\epsilon_m\tilde{\nabla}_a(v_m^b)\tilde{\nabla}_b(v_m^a) \tag{575}$$

[see (546)], as well as the remainder term $A(v)$ in (552) which, up to a sign, seems to be merely a 'copy' of the results on m, but *within* the orbits.

Now a remarkable thing happens when we specialize to the kinetic energy operator. In this case $k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}{}_{b}{}^{c}$ is the same as $K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a}$, which allows us to (almost) identify $\mathcal{F}^2$ with $\frac{4}{3}\tilde{\mathcal{R}}$: see (570), and thus remove from $\beta^{(\gamma)}(\frac{1}{2}g^{-1})$ the dependence on the Yang- Mills curvature $\mathcal{F}^{\gamma}{}_{ab}$, in exchange for a different numerical factor on $\tilde{\mathcal{R}}$. The result, $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$, now agrees with the important case of geometric quantization [cf (6)], as well as Emch's [50] analysis. This mechanism may 'explain' why $\frac{\hbar^2}{4}k^{ab}\tilde{\mathcal{R}}_{ab}$ usually appears in generic quantization schemes,

tization schemes referred to completely avoid van Hove anomalies on spaces of nonconstant curvature, so in this sense none of them are completely correct.

whereas $\frac{\hbar^2}{12}\tilde{\mathcal{R}}$ is more common for treatments that deal only with the kinetic energy case $k^{ab} = \frac{1}{2}g^{ab}$.

The second point, that $\alpha^{(\gamma)}(k)$ cannot agree with *any* quantization scheme based on the reduced classical theory, at least not exactly, is demonstrated (rigorously) with the finite dimensional helix model: Despite the fact that we can *naturally* pull out of $\alpha^{(\gamma)}(\frac{1}{2}g^{-1})$ the term $\frac{1}{8}\tilde{\mathcal{R}}$ or $\frac{1}{12}\tilde{\mathcal{R}}$[17], as we demonstrated here, $\alpha^{(\gamma)}(\frac{1}{2}g^{-1})$ itself is not proportional to $\tilde{\mathcal{R}}$, a fact which was known to Kuchař in [22]. In other words $\beta^{(\gamma)}(k)$, or $\beta'^{(\gamma)}(\frac{1}{2}g^{-1})$, are nontrivial in generic cases [see also (561) in regard to the example tensor k we constructed]. This is an anomaly in the sense that adding a multiple of $\hbar^2\tilde{\mathcal{R}}$ is the *only* ambiguity for the kinetic energy operator, if one works strictly within the reduced theory.

In answer to this, one might try to rig a nonminimal factor ordering for the extended space quantization such that it eliminates, say, $\beta^{(\gamma)}(k)$, but it is not obvious whether or not this could be successful. On the other hand, in light of the evidence we have discussed in this work, I do not believe we can go to the other extreme of Kuchař's "principle of minimal coupling" [21], i.e. minimal quantization on the (curved) reduced space.

Another possible way out is suggested by the curious fact that the terms in $\alpha^{(\gamma)}(k)$ seem to be grouped into two types: the first type associated with objects on the orbit space, m, and the second, essentially 'complimentary' terms, associated with objects *within* the orbits. This comes to light when we use the decomposition of K in (543): compare (540) with (553). The noticeable absence of a $\tilde{\nabla}_\alpha(\tilde{\nabla}_\beta(K^{\alpha\beta}))$ term may be due only to the fact that with our assumptions it vanishes identically. The objects of the first type are *exactly* those found in Vaisman's [41] quantization scheme, and his arbitrary $A(v)$ corresponds exactly

[17] The choice depends on whether or not one wants the Yang-Mills curvature to appear in the residual β term.

to my complimentary terms. It hardly seems likely that this is mere coincidence; mathematically it is not surprising in that locally M is the product of m and the gauge group, or orbit, so a (Dirac) quantization on M should, in some sense, be a sum of quantizations on m and the gauge group, but this notion requires more investigation.

The anomalous $\beta^{(\gamma)}(k)$, or $\beta'^{(\gamma)}(\frac{1}{2}g^{-1})$ terms bring to mind an interesting paper by Jensen and Koppe [53], who consider a free particle constrained to move on a two-dimensional surface. They quantize the system as a particle in $\boldsymbol{R}^3$ confined between two parallel surfaces of separation d. In the limit $d \longrightarrow 0$ (and after subtracting off the infinite zero point energy) the equivalent system has a kinetic energy operator given (by fiat) by the usual Laplace-Beltrami term for the original curved surface, plus an additional potential depending on the curvature of this surface. Like $\beta'^{(\gamma)}(\frac{1}{2}g^{-1})$, this additional term is not related to the Ricci scalar, i.e. does not appear to depend on the *intrinsic* curvature of the surface. The analogy between the two may be no more than coincidence, but it does bring us back to our original notion of applying a Gauss-Codazzi-like analysis, as opposed to the Kaluza-Klein-like one (see footnote on page 69). It would be interesting to see if $\beta^{(\gamma)}(k)$ could be expressed in terms of *extrinsic* curvature, in some sense (provided a suitable definition of extrinsic curvature could be found, given the existence of a nonzero Yang-Mills curvature). Alternatively, since $\beta^{(\gamma)}(k)$ involves horizontal derivatives of the orbit metric, it may be a measure of the extrinsic curvature of the *orbits* (rather than m) embedded in M. Perhaps the notorious difficulty in quantizing quadratic observables on a curved configuration space stems from the fact that the quantum mechanics 'feels out' where the curvature comes from? But this is, of course, speculation, and we leave these issues for future work.

Here we focused on putting our work in the context of the literature on quantization of quadratic observables. It might also be useful to establish more connections with other quantum treatments of scalar electrodynamics, per se (e.g. [25]), paying particular attention to the Poincaré invariance of the theory.

Furthermore, in other work [54] we considered path integral quantization of gauge theories with Lagrangian of the form (15), with scalar electrodynamics as a representative example. There are basically two issues when dealing with path integrals [47]: one is choosing a skeletonization of the action integral, and the other is choosing a measure on the space on skeletonized paths. In [54] we essentially began with the covariant Faddeev path integral on the full phase space Q^A, λ^α, and conjugate momenta P_A, κ_α (refer to our notation in section 1.1). Being a phase space path integral, we naturally chose the Liouville measure, but we ignored subtleties associated with skeletonization. Instead we focused on how, by selectively integrating over certain phase space coordinates, and using the special form of the Lagrangian, (15), we could reproduce various measures[18], and thereby establish the equivalence between the various forms of path integral found in the literature: covariant and noncovariant Faddeev, covariant and noncovariant Faddeev-Popov, the reduced configuration space path integral, as well as another form derived in [55]. "Faddeev" here refers to a phase space path integral, and "Faddeev-Popov" to a configuration space path integral. "Covariant" means the Lagrange multiplier λ^α enforcing the 'Gauss Law' constraint (and its conjugate momenta κ_α) are present in the integral. (And of course "noncovariant" then means these have been integrated out.) It might be of interest to extend this work to include a careful analysis of skeletonization, perhaps using a field theoretic generalization of Kuchař's work [47], which would raise the issue of factors of $\hbar^2\tilde{\mathcal{R}}$ appearing in the Hamiltonian operator. Of course this would be very relevant in

[18]In particular, the 'controversial' factor of $\sqrt{\gamma}$ played a prominent role, which had caused some confusion in [32].

the context of the results obtained in this thesis.

Finally, it should not be difficult to extend this analysis to other gauge theories with fermionic degrees of freedom, in particular the physically more interesting QED and QCD.

7 Appendix

In this appendix we display some of the detailed calculations regarding the Poincaré symmetry of scalar electrodynamics. We divide these calculations into two groups: The first demonstrates that the classical Poincaré charges are, indeed, observables—in fact, the stronger statement, that the Lie derivative of every valence zero, one, and two tensor associated with the Poincaré charges in (86– 89), with respect to the gauge generator ϕ_γ, vanishes. The second group of calculations concerns the verification of the Poincaré algebra itself, in particular the Schouten concomitants listed in (94–108).

7.1 Poincaré Charges are Classical Observables

For the scalars in (86– 89) we calculate [refer to (20), (19), and (82)]:

$$\begin{aligned}
\mathcal{L}_{\phi_\gamma} V &= \phi_\gamma^A \partial_A V \\
&= \int d^3x \left\{ -\frac{1}{e}\partial_{x^i}\delta(\boldsymbol{x}-\boldsymbol{z})\frac{\delta}{\delta A_i(\boldsymbol{x})} - \eta(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z})\frac{\delta}{\delta\xi(\boldsymbol{x})} \right. \\
&\quad \left. + \xi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z})\frac{\delta}{\delta\eta(\boldsymbol{x})} \right\} V \\
&= \frac{1}{e}\partial_{z^k}\frac{\delta V}{\delta A_k(\boldsymbol{z})} - \eta(\boldsymbol{z})\frac{\delta V}{\delta\xi(\boldsymbol{z})} + \xi(\boldsymbol{z})\frac{\delta V}{\delta\eta(\boldsymbol{z})} \\
&= \frac{1}{e}\partial_k\left[e^2(\xi^2+\eta^2)A_k + e(\xi\partial_k\eta - \eta\partial_k\xi) + \partial_l(\partial_k A_l - \partial_l A_k)\right] \\
&\quad -\eta\left[e^2\xi A_k^2 + 2eA_k\partial_k\eta + e\eta\partial_k A_k - \partial_k^2\xi + U'\frac{\xi}{|\varphi|}\right] \\
&\quad +\xi\left[e^2\eta A_k^2 - 2eA_k\partial_k\xi - e\xi\partial_k A_k - \partial_k^2\eta + U'\frac{\eta}{|\varphi|}\right] \\
&= 0, \qquad (576)
\end{aligned}$$

and similarly one can show

$$\mathcal{L}_{\phi_\gamma}{}^{\mathcal{K}^k} Z = 0, \qquad (577)$$

except there is the added complication of the x^k in the integrand of (82), which generates additional terms when integrating by parts and differentiating. Thus, the Hamiltonian and boost potentials are constant along the gauge orbits.

For the vectors in (86– 89) we calculate [refer to (70) and (76)]:

$$(\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}V)^A = (\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}V_0)^A + (\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}F^\beta)\phi^A_\beta + {}^{\mathcal{P}^k}F^\beta(\mathcal{L}_{\phi_\gamma}\phi_\beta)^A. \tag{578}$$

The last term on the right hand side vanishes because we have an abelian basis of gauge vectors, and for the first term we find

$$\begin{aligned}
(\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}V_0)^A &= \phi^B_\gamma\partial_B{}^{\mathcal{P}^k}V_0^A - {}^{\mathcal{P}^k}V_0^B\partial_B\phi^A_\gamma \\
&= \int d^3y\left\{-\frac{1}{e}\partial_{y^j}\delta(y-z)\frac{\delta}{\delta A_j(y)} - \eta(y)\delta(y-z)\frac{\delta}{\delta\xi(y)}\right. \\
&\qquad \left. +\xi(y)\delta(y-z)\frac{\delta}{\delta\eta(y)}\right\}(-\partial_{x^k}A_i(x), -\partial_{x^k}\xi(x), -\partial_{x^k}\eta(x)) \\
&- \int d^3y\left\{-\partial_{y^k}A_j(y)\frac{\delta}{\delta A_j(y)} - \partial_{y^k}\xi(y)\frac{\delta}{\delta\xi(y)} - \partial_{y^k}\eta(y)\frac{\delta}{\delta\eta(y)}\right\} \\
&\qquad (-\frac{1}{e}\partial_{x^i}\delta(x-z), -\eta(x)\delta(x-z), \xi(x)\delta(x-z)) \\
&= \int d^3y\,(\frac{1}{e}\partial_{y^j}\delta(y-z)\partial_{x^k}\delta^j_i\delta(x-y), \eta(y)\delta(y-z)\partial_{x^k}\delta(y-x) \\
&\qquad +\partial_{y^k}\eta(y)\delta(x-y)\delta(x-z), -\xi(y)\delta(y-z)\partial_{x^k}\delta(y-x) \\
&\qquad -\partial_{y^k}\xi(y)\delta(x-y)\delta(x-z)\,) \\
&= (\,-\frac{1}{e}\partial_{z^i}\partial_{x^k}\delta(x-z), \eta(z)\partial_{x^k}\delta(x-z) - \partial_{x^k}\eta(x)\delta(x-z), \\
&\qquad -\xi(z)\partial_{x^k}\delta(x-z) + \partial_{x^k}\xi(x)\delta(x-z)\,) \\
&= \partial_{z^k}\phi^A_\gamma,
\end{aligned} \tag{579}$$

which we might have guessed since $\mathcal{P}^k$ generates spatial translations on Γ_C. Here we used the fact that the distributions

$$\partial_{z^i}\partial_{x^k}\delta(x-z) = \partial_{z^k}\partial_{x^i}\delta(x-z), \tag{580}$$

and

$$f(z)\partial_{x^k}\delta(x-z) - \partial_{x^k}f(x)\delta(x-z) = -\partial_{z^k}[f(x)\delta(x-z)], \tag{581}$$

which is easily verified by integrating against a smooth test function. Finally, the 'off-shell' piece on the right hand side of (578) is

$$\begin{aligned}
(\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}F^\beta)\phi^A_\beta &= \phi^D_\gamma\partial_D{}^{\mathcal{P}^k}F^\beta\phi^A_\beta \\
&= \int d^3w\, d^3y \left\{-\frac{1}{e}\partial_{w^l}\delta(w-z)\frac{\delta}{\delta A_l(w)} - \eta(w)\delta(w-z)\frac{\delta}{\delta\xi(w)}\right. \\
&\qquad \left. +\xi(w)\delta(w-z)\frac{\delta}{\delta\eta(w)}\right\}[-eA_k(y)]\,\phi^A_\beta \\
&= \int d^3y\, \partial_{y^k}\delta(y-z)\phi^A_\beta \\
&= -\partial_{z^k}\phi^A_\gamma,
\end{aligned} \tag{582}$$

which cancels the first term, yielding

$$\mathcal{L}_{\phi_\gamma}{}^{\mathcal{P}^k}V = 0. \tag{583}$$

Similarly, for the spatial rotation vectors one can show

$$(\mathcal{L}_{\phi_\gamma}{}^{\mathcal{J}^k}V_0)^A = [lmn]z^m\partial_{z^n}\phi^A_\gamma, \tag{584}$$

as expected, which is cancelled by the 'off-shell' piece, yielding

$$\mathcal{L}_{\phi_\gamma}{}^{\mathcal{J}^k}V = 0, \tag{585}$$

but the calculation is, again, considerably more tedious than above because of the presence of the continuous 'labels' x^m and y^m in (77) and (78).

For the valence two tensors in (86– 89) we calculate [refer to (18) and (81)]:

$$\begin{aligned}
(\mathcal{L}_{\phi_\gamma}G)^{AB} &= \phi^C_\gamma\partial_C G^{AB} - G^{CB}\partial_C\phi^A_\gamma - G^{AC}\partial_C\phi^B_\gamma \\
&= -\partial^B\phi^A_\gamma - \partial_A\phi^B_\gamma
\end{aligned}$$

$$
\begin{aligned}
&- \begin{pmatrix} -\frac{1}{e}\partial_{x^i}\delta(\boldsymbol{x}-\boldsymbol{z}) \\ -\eta(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z}) \\ \xi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z}) \end{pmatrix} \left(\frac{\overleftarrow{\delta}}{\delta A_j(\boldsymbol{y})}, \frac{\overleftarrow{\delta}}{\delta \xi(\boldsymbol{y})}, \frac{\overleftarrow{\delta}}{\delta \eta(\boldsymbol{y})} \right) \\
&- \begin{pmatrix} \delta/\delta A_i(\boldsymbol{x}) \\ \delta/\delta\xi(\boldsymbol{x}) \\ \delta/\delta\eta(\boldsymbol{x}) \end{pmatrix} \left(-\frac{1}{e}\partial_{y^j}\delta(\boldsymbol{y}-\boldsymbol{z}), -\eta(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}), \xi(\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}) \right) \\
= & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta(\boldsymbol{y}-\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z}) \\ 0 & -\delta(\boldsymbol{y}-\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{z}) & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}) \\ 0 & \delta(\boldsymbol{x}-\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z}) & 0 \end{pmatrix} \\
= & \; 0,
\end{aligned} \tag{586}
$$

so the ϕ_γ are actually Killing vectors. Similarly it can be shown that

$$
\mathcal{L}_{\phi_\gamma} {}^{\mathcal{K}^k}K = 0, \tag{587}
$$

where we use the additional fact that the distribution $x\delta(x) = 0$. Note that this last result, for example, does *not* follow from the fact that the ${}^{\mathcal{K}^k}K$ are covariantly constant, because the vectors ϕ_γ are not covariantly constant.

7.2 Schouten Concomitants in Poincaré Algebra

For (94) we calculate [refer to (70), (19), and (576)]:

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k}V, V]\!] &= \mathcal{L}_{{}^{\mathcal{P}^k}V_0} V + {}^{\mathcal{P}^k}F^\gamma \mathcal{L}_{\phi_\gamma} V = {}^{\mathcal{P}^k}V_0^A \partial_A V \\
&= \int d^3x \left\{ -\partial_{x^k} A_i(\boldsymbol{x}) \frac{\delta}{\delta A_i(\boldsymbol{x})} - \partial_{x^k}\xi(\boldsymbol{x}) \frac{\delta}{\delta\xi(\boldsymbol{x})} - \partial_{x^k}\eta(\boldsymbol{x}) \frac{\delta}{\delta\eta(\boldsymbol{x})} \right\} V \\
&= \int d^3x \left\{ -\partial_k A_i \left[e^2(\xi^2+\eta^2)A_i + e(\xi\partial_i\eta - \eta\partial_i\xi) + \partial_j(\partial_i A_j - \partial_j A_i) \right] \right. \\
&\qquad \left. -\partial_k\xi \left[e^2\xi A_i^2 + 2eA_i\partial_i\eta + e\eta\partial_i A_i - \partial_i^2\xi + U'\frac{\xi}{|\varphi|} \right] \right.
\end{aligned}
$$

$$
\begin{aligned}
& -\partial_k\eta\left[e^2\eta A_i^2 - 2eA_i\partial_i\xi - e\xi\partial_i A_i - \partial_i^2\eta + U'\frac{\eta}{|\varphi|}\right]\Big\} \\
= & \int d^3x\,\{-\partial_k\left[\frac{1}{2}e^2(\xi^2+\eta^2)A_i^2 + eA_i(\xi\partial_i\eta - \eta\partial_i\xi)\right. \\
& \left. +\frac{1}{2}(\partial_i\xi)^2 + \frac{1}{2}(\partial_i\eta)^2 + \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + U\right] \\
& +\partial_i\left[eA_i(\xi\partial_k\eta - \eta\partial_k\xi) + \partial_k\xi\partial_i\xi + \partial_k\eta\partial_i\eta\right]\}, \qquad (588)
\end{aligned}
$$

which can be verified directly by expanding the derivatives. Assuming that the fields vanish appropriately at infinity, (94) then follows. This could have been guessed since the $\mathcal{P}^k$ generate spatial translations on Γ_C, and the Hamiltonian potential V does not depend explicitly on the spatial coordinates x^k. A similar, but more tedious calculation [because of the x^m in (77)] explicitly establishes (95).

(96) and (97) are similar to the above two, except with the Hamiltonian potential V replaced with the boost potentials ${}^{\mathcal{K}^k}Z$. In this case the ${}^{\mathcal{K}^k}Z$ *do* contain the spatial coordinates x^k [see (82)], so the Schouten concomitants are nontrivial (i.e. contribute to the structure of the algebra). However, the presence of up to two factors of x^k now makes the integrations by parts a real chore, but after several pages of calculation one eventually arrives at (96) and (97).

For (98) we calculate [refer to (81), (82), and (70)]:

$$
\begin{aligned}
[\![\frac{1}{2}{}^{\mathcal{K}^k}K, V]\!]^A &= {}^{\mathcal{K}^k}K^{AB}\partial_B V \\
&= \int d^3y \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{2}(x^k+y^k)\delta(\boldsymbol{x}-\boldsymbol{y}) \begin{pmatrix} \delta/\delta A_j(\boldsymbol{y}) \\ \delta/\delta\xi(\boldsymbol{y}) \\ \delta/\delta\eta(\boldsymbol{y}) \end{pmatrix} V \\
&= x^k\partial^A V, \qquad (589)
\end{aligned}
$$

$$
[\![\frac{1}{2}G^{-1}, {}^{\mathcal{K}^k}Z]\!]^A = G^{AB}\partial_B{}^{\mathcal{K}^k}Z = \partial^A\,{}^{\mathcal{K}^k}Z = \frac{\delta}{\delta Q^A}\int d^3y\, y^k\,\mathcal{V}(Q(\boldsymbol{y}))
$$

$$
\begin{aligned}
&= \int d^3y\, y^k \left\{ \frac{\partial \mathcal{V}}{\partial \partial_{y^l} Q^B} \partial_{y^l} \delta^B_A + \frac{\partial \mathcal{V}}{\partial Q^B} \delta^B_A \right\} = x^k \partial^A V - \frac{\partial \mathcal{V}}{\partial \partial_{x^k} Q^A} \\
&= x^k \partial^A V - (\partial_k A_i - \partial_i A_k, \partial_k \xi - eA_k \eta, \partial_k \eta + eA_k \xi) \\
&= x^k \partial^A V + {}^{\mathcal{P}^k} V^A,
\end{aligned} \tag{590}
$$

and hence the result quoted. Similarly, using the above results we establish (99):

$$
\begin{aligned}
[\![\tfrac{1}{2}{}^{\mathcal{K}^k} K, {}^{\mathcal{K}^l} Z]\!]^A - [\![\tfrac{1}{2}{}^{\mathcal{K}^l} K, {}^{\mathcal{K}^k} Z]\!]^A &= x^k \partial^A\, {}^{\mathcal{K}^l} Z - (k \leftrightarrow l) \\
&= x^k x^l \partial^A V + x^k\, {}^{\mathcal{P}^l} V^A - (k \leftrightarrow l) \\
&= [klm]^{\mathcal{J}^m} V^A,
\end{aligned} \tag{591}
$$

the last equality based on a rearrangement of (76).

The commutators of the spatial translation vectors are [refer to (70)]:

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k} V, {}^{\mathcal{P}^l} V]\!]^A &= [\![{}^{\mathcal{P}^k} V_0, {}^{\mathcal{P}^l} V_0]\!]^A + [\![{}^{\mathcal{P}^k} F^\gamma \phi_\gamma, {}^{\mathcal{P}^l} F^\delta \phi_\delta]\!]^A \\
&\quad + \left([\![{}^{\mathcal{P}^k} V_0, {}^{\mathcal{P}^l} F^\gamma \phi_\gamma]\!]^A - (k \leftrightarrow l) \right).
\end{aligned} \tag{592}
$$

For the 'on-shell' piece we find

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k} V_0, {}^{\mathcal{P}^l} V_0]\!]^A &= {}^{\mathcal{P}^k} V_0^B \partial_B {}^{\mathcal{P}^l} V_0^A - (k \leftrightarrow l) \\
&= (-\partial_{y^k} Q^B) \frac{\partial}{\partial Q^B} (-\partial_{x^l} Q^A) - (k \leftrightarrow l) \\
&= \int d^3y\, \partial_{y^k} Q^{(J,\boldsymbol{y})} \partial_{x^l} \delta^I_J \delta(\boldsymbol{x} - \boldsymbol{y}) - (k \leftrightarrow l) \\
&= \partial_{x^k} \partial_{x^l} Q^A - (k \leftrightarrow l) \\
&= 0
\end{aligned} \tag{593}
$$

[see comments following (20) for notation used here]. The 'mixed' piece

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k} V_0, {}^{\mathcal{P}^l} F^\gamma \phi_\gamma]\!]^A &= [\![{}^{\mathcal{P}^k} V_0, {}^{\mathcal{P}^l} F^\gamma]\!] \phi^A_\gamma + {}^{\mathcal{P}^l} F^\gamma [\![{}^{\mathcal{P}^k} V_0, \phi_\gamma]\!]^A \\
&= {}^{\mathcal{P}^k} V_0^B \partial_B {}^{\mathcal{P}^l} F^\gamma \phi^A_\gamma + \int d^3z\, [-eA_l(\boldsymbol{z})][-\partial_{z^k} \phi^A_\gamma]
\end{aligned}
$$

$$
\begin{aligned}
&= \int d^3z\, d^3y \left\{ -\partial_{y^k} A_j(y) \frac{\delta}{\delta A_j(y)} - \partial_{y^k}\xi(y)\frac{\delta}{\delta\xi(y)} - \partial_{y^k}\eta(y)\frac{\delta}{\delta\eta(y)} \right\} \\
&\quad [-eA_l(z)]\phi^A_\gamma + \int d^3z\, [-e\partial_{z^k} A_l(z)]\phi^A_\gamma \\
&= 0,
\end{aligned} \tag{594}
$$

where we used (579) for the second concomitant on the right hand side. Finally, using (582) we determine the 'off-shell' piece:

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k}F^\gamma\phi_\gamma, {}^{\mathcal{P}^l}F^\delta\phi_\delta]\!]^A &= {}^{\mathcal{P}^k}F^\gamma[\![\phi_\gamma, {}^{\mathcal{P}^l}F^\delta]\!]\phi^A_\delta - (k \leftrightarrow l) \\
&= \int d^3z\, [-eA_k(z)][-\partial_{z^l}\phi^A_\gamma] - (k \leftrightarrow l) \\
&= e\int d^3z\, F_{kl}(z)\phi^A_\gamma,
\end{aligned} \tag{595}
$$

which establishes (100). The commutators involving the spatial rotation vectors in (101) and (102) can similarly be derived, but as usual these are more tedious and we only quote the results.

We now come to (103), which is quadratic [refer to (70)]:

$$
\begin{aligned}
[\![{}^{\mathcal{P}^k}V, \frac{1}{2}G^{-1}]\!]^{AB} &= [\![{}^{\mathcal{P}^k}V_0, \frac{1}{2}G^{-1}]\!]^{AB} + {}^{\mathcal{P}^k}F^\gamma[\![\phi_\gamma, \frac{1}{2}G^{-1}]\!]^{AB} \\
&\quad + [\![{}^{\mathcal{P}^k}F^\gamma, \frac{1}{2}G^{-1}]\!]^{(A}\phi^{B)}_\gamma.
\end{aligned} \tag{596}
$$

The first term on the right hand side is

$$
\begin{aligned}
\frac{1}{2}(\mathcal{L}_{{}^{\mathcal{P}^k}V_0}G)^{AB} &= \frac{1}{2}\,{}^{\mathcal{P}^k}V^C_0\,\partial_C G^{AB} - G^{C(A}\partial_C {}^{\mathcal{P}^k}V^{B)}_0 = -\partial^{(A}\,{}^{\mathcal{P}^k}V^{B)}_0 \\
&= \partial^{(A}\partial_{y^k}Q^{B)} = \frac{1}{2}[\partial_{y^k}\delta^{BA} + \partial_{x^k}\delta^{AB}] \\
&= 0
\end{aligned} \tag{597}
$$

because $\partial_x\delta(x-y)$ is an antisymmetric distribution. Thus the 'on-shell' piece ${}^{\mathcal{P}^k}V_0$ of the spatial translation vector is Killing. Note that M is flat, so ${}^{\mathcal{P}^k}V_0$ must be a translation and/or a rotation of the field space. (71) tells us that it is linear in the field coordinates Q^A, and so must be a *rotation*, i.e. it must have the form $M^A_B Q^B$, where $M^{AB} := \delta^{BC} M^A_C$ is antisymmetric. The antisymmetry (noted above) is accomplished by the derivative ∂_{x^k} on the coordinate labels. The second term on the right hand side of (596) vanishes since the ϕ_γ are Killing: recall (586). The last term involves the vector

$$[\![{}^{\mathcal{P}^k}F^\gamma, \frac{1}{2}G^{-1}]\!]^A = -G^{AB}\partial_B {}^{\mathcal{P}^k}F^\gamma, \tag{598}$$

which we called $-{}^{\{\mathcal{P}^k,\mathcal{P}^0\}}\psi^{\gamma A}$ in (103). We derive (104) in a similar fashion.

(105) and (106) are similar to (103) and (104), except with G^{-1} replaced with ${}^{\mathcal{K}^k}K$. Since the latter contains the labels x^k, y^k, it can be shown that instead of encountering the (zero) distribution $\partial_{y^k}\delta(\boldsymbol{y}-\boldsymbol{x}) + \partial_{x^k}\delta(\boldsymbol{x}-\boldsymbol{y})$ above [cf (597)], we encounter

$$x^k\partial_{y^l}\delta(\boldsymbol{y}-\boldsymbol{x}) + y^k\partial_{x^l}\delta(\boldsymbol{x}-\boldsymbol{y}) = \delta^{kl}\delta(\boldsymbol{x}-\boldsymbol{y}); \tag{599}$$

the last equality can be demonstrated by integrating against a smooth test function. In this way one obtains (105) and (106).

At last we come to the cubic concomitants (107) and (108). These follow trivially from the covariant constancy of the boost tensors. Thus the ${}^{\mathcal{K}^k}K$ are Killing tensors, in involution with each other.

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