Modular Dimension Subgroups

by

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Abstract

Conditions under which Lazard's formula for modular dimension subgroups is valid are investigated; for example, we show that it holds for groups with torsion free lower central factors, a generalization of Lazard's result which states this for the free group only. Validity of this formula is connected to the problem of understanding how Lie commutator laws of an algebra influence the commutator laws of its group of units. This connection is explored and some new results on the problem of transfer of commutator laws are obtained. In particular, we resolved some questions of V.Shpilrain and of Sharma and Srivastava. Lazard's formula is modified and it is shown that the new formula holds true in dimensions less than $2p$ for finitely generated metabelian groups. The result parallels N.Gupta's solution of the integral dimension subgroup problem for metabelian odd p-groups. This identification puts the Moran-Sandling counter-examples to Lazard's formula in a new perspective (it follows in particular that those are the only possible counter-examples in the class of metabelian groups). Most of the resluts are obtained using the free group ring methods.

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Chapter I
Introduction

The principal theme in this work is the application of free group ring methods to problems concerning modular dimension subgroups. These methods are most effective when applied to metabelian groups, cf. the solution of the integral dimension subgroup problem for metabelian odd p-groups. The results presented here parallel the development of the solution of the integral problem to some extent. However, in certain situations we were led outside of the realm of the metabelian groups, and into questions related to Lie identities and their relation to commutator identities in groups.

Chapter II provides the basic notation and preliminary results which we shall quote elsewhere in the text; section 2.1 in particular provides a very brief sketch of the results on dimension subgroups. Our intention was to list the theorems we would later need, rather than offer a survey of the subject.

Chapter III is centered around Lazard's formula for modular dimension subgroups. We give a new proof of Lazard's theorem for free groups, and generalize it to show that the formula is indeed valid for groups with torsion-free lower central factors. This leads to certain questions about the influence of the Lie identities of an algebra on the commutator identities of its group of units. We answer, in the negative, some questions of Sharma and Srivastava, namely whether unit groups of Lie soluble algebras must be soluble and whether the unit groups of Lie centre-by-metabelian algebras must be centre-by-metabelian. We also show that the centre-by-metabelian law does not transfer from algebras to their unit groups even for Q-algebras — thus answering a question of Professor V.Shpilrain. In the positive direction, however, we show that finitely generated Lie soluble algebras over fields of characteristic other than 2 have nilpotent-by-abelian unit groups. We have also answered the question of whether algebras with the law $((x_1, \ldots, x_c), (y_1, \ldots, y_d)) = 0$ have nilpotent-by-abelian unit groups (we show that this is the case for finitely generated algebras over fields). Also, we show that polynilpotent laws transfer to unit groups in the class of Q-algebras (this has been announced by V.Shpilrain).

In Chapter IV we deal with dimension subgroups modulo p^e, of dimension at most $p+2$; we give a new and more elementary proof of Moran's theorem. The results from this chapter give a complete description of modular dimension subgroups of dimension at most $p+2$ for metabelian groups. It follows from these results that the examples of Moran and Sandling to show that Lazard's formula can fail for the dimension $p+1$ are the only possible such — in the class of metabelian groups. We also discuss the fourth modular dimension subgroup and how it relates to the Rips counter-example.

Chapter V begins with a version of Sjogren's lemma adjusted to our needs; then Prof. N.Gupta's solution of the integral dimension subgroup problem for metabelian groups is altered to compute the commutator structure of modular dimension subgroups of metabelian groups (odd prime powers) of dimensions less than $2p$. Finally, a description of modular dimension subgroups of metabelian groups for dimensions less than $2p$ and odd prime powers is given in the last section.

In Chapter VI we indicate some of the intrinsic limitations of the methods we have used, and list several questions that seem worth exploring.

Chapter II

Notation and Preliminaries

§2.0. The notation

The notation we use throughout this thesis is mostly standard and consistent with Magnus, Karrass, Solitar [1976] and Kargapolov, Merzljakov [1979]. However, for the sake of completeness, we shall list some of the frequently used terms and symbols.

If S is a subset of G then $\langle S \rangle$ denotes the subgroup generated by S; the commutators are defined as $[x, y] = x^{-1}y^{-1}xy$. In dealing with complex commutators, we shall assume that they are left-normed, i.e. $[x_1, \ldots, x_{m+1}] = [[x_1, \ldots, x_m], x_{m+1}]$. The commutator of two subgroups is defined to be $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$; in particular, let $G' = [G, G]$, the commutator subgroup. Inductively we define the mth derived subgroup as $G^{(m)} = [G^{(m-1)}, G^{(m-1)}]$. Let $\gamma_1(G) = G$ and $\gamma_m(G) = [\gamma_{m-1}, G]$; this is the mth term of the lower central series. G is said to be soluble of class $\leq m$ if $G^{(m)} = 1$ for some m; if this m is 2, we call such groups metabelian. G is nilpotent of class $\leq m$ if $\gamma_{m+1}(G) = 1$. Let $\sqrt{H}$ be the subgroup $\langle x \mid \text{ for some } m, x^m \in H \rangle$. The general linear group over a ring R is denoted by $GL_n(R)$; its subgroup of unitriangular matrices is denoted by $UT_n(R)$.

All rings and algebras will be assumed to have an identity element (an exception to this will occur when, in Chapter III, we briefly talk of nil algebras in the context of a question of A.E. Zalesskii); the group of units of the ring R will be denoted by $U(R)$. If $S \subseteq R$ then $\langle S \rangle$ will denote the ideal generated by S in R. The Lie commutators are defined thus: $(x, y) = xy - yx$; we shall use the left-normed convention here too, i.e. $(x_1, \ldots, x_{m+1}) = ((x_1, \ldots, x_m), x_{m+1})$. If A and B are subsets of a ring, we let (A, B) be the additive subgroup generated by the elements of the form (a, b) with $a \in A$, $b \in B$. The ring of $n \times n$ matrices over R is denoted by $M_n(R)$ and its subset of zero-trace matrices by $T_n(R)$. The field of rationals is denoted by Q, and the finite prime fields by $GF(p)$ or Z_p; the ring of integers modulo m is denoted by Z_m. The Jacobson radical of R is denoted by $J(R)$. The augmentation ideal Δ of the group ring RG is the ideal generated by the elements of the form $g - 1$, for $g \in G$.

Other notation will be introduced in the remainder of this chapter, which is divided in different sections each providing definitions and preliminaries necessary for the rest of the work.

§2.1. Dimension Subgroups

For a group G and a ring R the nth dimension subgroup of G over R is defined thus:

$$D_n(G, R) = G \cap (1 + \Delta^n)$$

where Δ is the augmentation ideal of the group ring RG, i.e. the kernel of the natural map $RG \to R$ which collapses G to 1. These subgroups occur in various contexts and perhaps the most natural way of motivating their study is is through group representations. By a representation over a ring R we mean a homomorphism $\rho : G \to Aut(M)$ of G into the automorphism group of an R-module M. Dimension subgroups of G then correspond to the kernels of certain special representations, which resemble representations by unitriangular matrices. Specifically, suppose there is a sequence of submodules $0 = M_n \subseteq \ldots \subseteq M_0 = M$ such that $\rho(G)$ acts trivially on each factor M_j/M_{j+1} (such a representation is called n-stable); it is not difficult to see that $\ker \rho = D_n(G, R)$. In this way the study of dimension subgroups is connected to the study of faithful n-stable representations over R.

However, the investigation of dimension subgroups seems to have been initiated by purely combinatorial considerations: one of the early results on dimension subgroups stems from the following result of Magnus:

Theorem 2.1.0. *(Magnus) Let F be the free group generated by $x_1, x_2, \dots$ and let $Z[[a_1, a_2, \dots]]$ be the algebra of formal power series. Then the mapping induced by assigning $x_i \to 1 + a_i$ is an embedding.*

In the algebra of formal power series, we can look at the ideal Δ generated by the elements $a_1, a_2 \dots$. Then, considering the free group F a subgroup of the group of units of the algebra, $F \cap (1 + \Delta^n)$ is a subgroup of F, and the question of identifying this subgroup in group-theoretic terms arises. Clearly, using the identity

$$[x, y] = x^{-1}y^{-1}(xy - yx)$$

repeatedly, it follows that the subgroup $F \cap (1 + \Delta^n)$ contains $\gamma_n(F)$, the nth term of the lower central series of F. In fact, Magnus showed that the subgroup is equal to the nth term of the lower central series. He was further able to conclude the following statement:

Theorem 2.1.1. *(Magnus) $D_n(F, Z) = \gamma_n(F)$.*

The formula used above actually does not depend on the group or the ring and it follows that $\gamma_n(G) \subseteq D_n(G, R)$ for all G and R.

Theorem 2.1.2. *$\gamma_n(G) \subseteq D_n(G, R)$ for all G and R.*

In the case $R = Z$, there are no obvious elements of G, other than those of $\gamma_n(G)$, that should belong to $D_n(G, Z)$. Magnus raised the following, by now well-known, problem:

Problem. *Is it true that $D_n(G, Z) = \gamma_n(G)$ for all G?*

This problem, the integral dimension subgroup problem, has induced much research; most of the work was concerned with obtaining results in the positive direction. It was solved in the negative by Rips [1972] who gave an example of a finite 2-group G such that $D_4(G, Z) \neq \gamma_4(G)$. This has later been extended by Gupta [1990] who constructed, for each $n \geq 5$ a finite metabelian 2-group G_n such that $D_n(G_n, Z) \neq \gamma_n(G_n)$.

Although the problem has negative solution, we give a brief survey of the results in this area — to illustrate the the development and complexity of the subject, and also because we shall require many of those results later. We shall, however, return to the study of groups without the dimension property in Chapter IV.

The natural first case to consider is the case of dimension subgroups over fields. In this situation the first result was obtained by Zassenhaus for free groups:

Theorem 2.1.3. *(Zassenhaus [1940]) Let F be free; then*

$$D_n(F, Z_p) = \prod_{ip^j \geq n} \gamma_i(F)^{p^j}.$$

This has been generalized by Lazard [1954], who showed that the identification above is actually valid for all groups. We have:

Theorem 2.1.4. *(Lazard [1954]) For any group G,*

$$D_n(G, Z_p) = \prod_{ip^j \geq n} \gamma_i(G)^{p^j}.$$

REMARK. A related result (which we shall not require) can be found in Jennings [1941].

The situation over the field of rationals is somewhat different. We shall prove the following result in Chapter IV:

Theorem 2.1.5. *(Jennings [1955]) For any group G, $D_n(G, Q) = \sqrt{\gamma_n(G)}$.*

One important and immediate consequence of this theorem is

Corollary 2.1.6. *(Hall-Jennings) Let G be a group with torsion-free lower central factors; then $D_n(G, Z) = \gamma_n(G)$.*

In the case of dimension subgroups over arbitrary rings, there are reduction results which enable us to express $D_n(G, R)$ in terms of $D_n(G, Z_{p^e})$ and $D_n(G, Z)$ for certain set of integers e and primes p; specifically:

Theorem 2.1.7. *(Sandling [1972]) If characteristic of R is 0, then*

$$D_n(G, R) = \prod_{p \in \sigma(R)} \{T_p(G/D_n(G, Z)) \cap D_n(G, Z_{p^e})\}$$

where T_p denotes the inverse image in G of the p-torsion of the factor, $\sigma(R) = \{p \mid p \text{ is a prime and } p^n R = p^{n+1} R \text{ for some } n\}$ and for $p \in \sigma(R)$, p^e is the smallest power of p for which $p^e R = p^{e+1} R$. If characteristic of R is $r > 0$ then

$$D_n(G, R) = D_n(R, Z_r) = \bigcap_i D_n(G, Z_{p_i^{e_i}})$$

where $r = \prod_i p_i^{e_i}$ is the prime factorization or r.

This reduction allows us to restrict our attention to the rings of integers modulo a prime power — which we shall henceforth do without further notice.

One of the early results on dimension subgroups modulo a prime power is Lazard's result which generalizes 2.1.3.

Theorem 2.1.8. *(Lazard [1954]) Let F be a free group; then*

$$D_n(F, Z_{p^e}) = \prod_{ip^{(j-e+1)^+} \geq n} \gamma_i(F)^{p^j}$$

where $k^+ = \max\{k, 0\}$.

In Chapter III we shall give a proof and a generalization of this theorem; let us mention that if the formula from 2.1.8 (we shall call it Lazard's formula) turned out to be valid for arbitrary groups — the integral dimension subgroup would have an affirmative solution. However, if Lazard's formula is false, we cannot conclude, in general, that integral dimension subgroup problem has negative solution. Indeed, there are examples which show that this formula is not valid for arbitrary groups: this buries hopes to settle the integral dimension subgroup problem via the modular reduction.

Theorem 2.1.9. *(Moran [1970]) Let G be any group. Then*

$$D_n(G, p^e) = G^{p^e}\gamma_n(G) \text{ for } n \leq p.$$

However, if $G = \langle x, y \mid x^{p^{e+1}} = y^{p^2} = 1, x^{p^{e-1}} = [x, y]\rangle$ with $e > 1$ and $p > 2$ then $x^{p^e} \neq 1$ in G and $x^{p^e} \in D_{p+1}(G, Z_{p^e})$ but $G^{p^{e+1}}G'^{p^e}\gamma_{p+1}(G) = 1$.

Thus, Lazard's formula for dimension subgroups modulo p^e is valid for $n \leq p$ but fails at $n = p + 1$. Additional examples to this effect were given by Sandling [1972]; unfortunately these examples do not give any idea on how to go about constructing an example that would solve the integral problem in the negative. Rather, they show that there is a simple property of modular dimension subgroups which is not reflected in Lazard's formula. In a way this tells us that we should adjust the formula and try to see if the new subgroup is identical to the dimension subgroup. Further, a quick look at the Moran-Sandling examples reveals that the groups are metabelian; thus it is natural to study modular dimension subgroups of metabelian groups first.

In the integral case, we have the following general result:

Theorem 2.1.10. *(Gupta [1991]) Let $p > 2$ be a prime, and let G be a finitely generated metabelian p-group. Then $D_n(G, Z) = \gamma_n(G)$ for all n.*

Such behavior of dimension subgroups of metabelian groups is to some extent reflected in the modular case: we shall show in Chapter V that for $n < 2p$ and p odd, the modified Lazard's formula is valid for finitely generated metabelian groups.

In general, the best known result is the following:

Theorem 2.1.11. *(Sjogren [1979])* $D_n(G,Z)/\gamma_n(G)$ *has exponent dividing* $b(1)^{\binom{n-2}{1}} b(2)^{\binom{n-2}{2}} \cdots b(n-2)^{\binom{n-2}{n-2}}$, *where* $b(k) = lcm\{1,2,\ldots,k\}$.

Sjogren's techniques are intrinsically different from our approach, although in Chapter V we do develop a minor part of the modular analogue of his machinery, as much of it as we need to.

So far we have only spoken of dimension subgroups themselves; there are various applications of these results; some dealing with representations, some with the power structure of p-groups, etc. We shall conclude this section by quoting two theorems which involve applications of dimension subgroups.

Theorem 2.1.12. *(Scoppola and Shalev [1991]) Let* G *be a* p*-group of class* c *(*p *odd), and let* $k = \lceil \log_p(\frac{c+1}{p-1}) \rceil$. *Then for all* i*, any product of* p^{i+k}*th powers in* G *is a* p^i*th power.*

Theorem 2.1.13. *(Wehrfritz [1987]) Let* G *be a group,* n *a positive integer and* p *a prime with* $p \geq n$*. The following are equivalent:*
(a) $\gamma_i(G)^{p^j} = 1$ *for all* $ip^j \geq n$.
(b) There is a division Z_p*-algebra* D *such that* G *embeds into the group of* $n \times n$ *unitriangular matrices over* D.

Further applications and references can be found in Shalev [1].

§2.2. Lie theory

Lie algebras and Lie-theoretic methods will be used frequently throughout this work; in this section we give the basic definitions and results; the general reference most appropriate for our purposes is Bakhturin [1985].

Let A be an associative algebra. Then under the Lie commutation A forms a Lie algebra denoted by A^*. Let L be a Lie algebra. A universal enveloping algebra of L is an algebra U such that there is a universal homomorphism $\varepsilon : L \rightarrow U^*$

i.e. if $\varphi : L \longrightarrow A^*$ is a homomorphism then there is an algebra homomorphism $\psi : U \longrightarrow A$ such that $\psi^*\varepsilon = \varphi$. The universal property assures uniqueness of U up to isomorphism; its existence needs to be proved.

Theorem 2.2.1. *Let $Lie_R(X)$ be the free Lie algebra (over a commutative ring R) generated by the set X. The universal enveloping algebra of $Lie_R(X)$ is the free associative algebra $R[X]$, on the generators X.*

Any Lie algebra can be represented in the form $Lie_R(X)/J$ where J is an ideal of $Lie_R(X)$; by the theorem above, the universal homomorphism carries J into a set of elements of the free associative algebra; we shall denote this set by J too. Thus we can look at the associative ideal generated by J in $R[X]$, and look at the factor $R[X]/\langle J\rangle$.

Theorem 2.2.2. *Let J be an ideal in $Lie_R(X)$; then $R[X]/\langle J\rangle$ is the universal enveloping algebra of $Lie_R(X)/J$.*

This establishes the existence of the universal envelope. A very fundamental result on universal envelopes is the following theorem which we shall apply on several occasions:

Theorem 2.2.3. *(Poincaré-Birkhoff-Witt) Let L be a Lie algebra over a commutative ring R and let $\varepsilon : L \longrightarrow U$ be the universal enveloping map. If L is a free R-module with ordered basis B then ε is injective and U is a free R-module with basis consisting of 1 and the ordered monomials*

$$\varepsilon(b_1)\varepsilon(b_2)\cdots\varepsilon(b_m) \text{ where } b_1 \geq b_2 \geq \cdots \geq b_m, m \geq 1.$$

Consider the Lie algebra associated with the free associative algebra; by theorem 2.2.1 this is a free Lie algebra. There are, however, different bases for this free Lie algebra. We shall not go into the construction of a basis; rather, we refer the reader to Bakhturin [1985], where two bases are explicitly given (the Širšov basis, and the Hall basis). Thus we have the following instance of the Poincaré-Birkhoff-Witt theorem, which we shall use frequently:

Theorem 2.2.4. *In $R[X]$ there is sequence of homogeneous Lie elements $\zeta_1, \zeta_2 \dots$ with nondecreasing degrees in the free generators of $R[X]$, such that*
(a) this sequence is a basis for the Lie elements of $R[X]$;
(b) the products $\zeta_{i_1}^{e_1}\zeta_{i_2}^{e_2}\dots\zeta_{i_m}^{e_m}$ with $1 \le i_1 < \cdots < i_m, m \ge 1$ and $e_k > 0$, together with the identity element form a linear basis for $R[X]$.

We shall henceforth assume that the sequence from theorem 2.2.4 is the sequence of Hall commutators. Considering these commutators as group commutators in the free group on X, we get a sequence of group elements which we shall call Hall's basic commutators. They have the following fundamental property:

Theorem 2.2.5. *(P.Hall) In a free group F there is a sequence of commutators $C_1, C_2, \dots$ of nondecreasing weights, such that for any n and any element w of F,*

$$w = C_1^{e_1} C_2^{e_2} \cdots C_{k(n)}^{e_{k(n)}} v_{n+1}$$

where $v_{n+1} \in \gamma_{n+1}(F)$ and v_{n+1} and the integers e_j are uniquely determined by w.

The connexion between Hall's basic group commutators and basic Lie elements is explored in detail in Magnus, Karrass, Solitar [1976] and we refer the reader to that monograph for a more complete account.

We now turn to Lie algebras arising from groups.

Definition. *A sequence $\{H_n\}_{n\ge1}$ of subgroups such that $H_{n+1} \subseteq H_n$ and $[H_n, H_m] \subseteq H_{m+n}$ is called a Lazard series.*

For example, the series of lower central subgroups and the series of dimension subgroups are Lazard series.

If $\{H_n\}_{n\ge1}$ is a Lazard series, the factors H_n/H_{n+1} are abelian and we can form the direct sum of abelian groups $\oplus_{n\ge1}(H_n/H_{n+1})$; on this set we define addition by

$$\sum_n x_n H_{n+1} + \sum_n y_n H_{n+1} = \sum_n x_n y_n H_{n+1}$$

and we define

$$(x_n H_{n+1}, y_m H_{m+1}) = [x_n, y_m] H_{n+m+1}$$

and define the bracket operation for arbitrary elements by linearity. With these operations, the direct sum becomes a Lie algebra, the Lie algebra associated with Lazard's series $\{H_n\}_{n\geq 1}$.

Definition. *The Lie algebra associated in this way with Lazard's series $\{H_n\}$ will be denoted by $L(\{H_n\}_{n\geq 1})$. The Lie algebra associated with the lower central series of the group G will be denoted by $L(G)$.*

Conversely, groups can be associated with certain Lie algebras. Let $Lie(X)$ be the free Lie algebra on the set X of free generators. We can look at it as a subalgebra of the algebra of formal power series $Q[[X]]$; let $\widehat{Lie(X)}$ denote the completion of $Lie(X)$, i.e. the Lie subalgebra of $Q[[X]]$ consisting of power series $\sum_i \lambda_i$ where each λ_i is an element of $Lie(X)$. For $x, y \in Lie(X)$ we let $x \circ y$ be the unique element of $Q[[X]]$ which satisfies the equation

$$e^x e^y = e^{x \circ y}$$

where $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$.

Theorem 2.2.6. *(Baker-Hausdorff) For $x, y \in Lie(X)$, the series $x \circ y$ which satisfies the identity $e^x e^y = e^{x\circ y}$ is a well defined element of $\widehat{Lie_Q(X)}$, and can be explicitly computed.*

We are not interested in the formula itself (it can be found in Bakhturin [1985]); of interest to us is the fact that this formula can be used to induce a group operation on certain Lie algebras.

Theorem 2.2.7. *Let A be a residually nilpotent Lie Q-algebra. Then the Baker-Hausdorff formula defines a binary operation $\circ$ on A, such that $(A, \circ)$ is a group.*

This is not the only case when Baker-Hausdorff multiplication induces a group operation on a Lie ring; a more general statement is given in Lazard [1954], Théorème 4.2. The connexion between the two constructions was given by Lazard [1954]; it turns out that under certain restrictions one can relate the structure of the group $L(\{H_n\}_{n>0}, \circ)$ to the structure of the group G in which the series $\{H_n\}_{n>0}$ lies. We refer to Lazard [1954] for the details; for our purposes we only need to mention the following special case of Lazard's result, which was used by Moran [1970]:

Theorem 2.2.8. *(Lazard [1954]) Let G be a p-group of class less than p. Then $(L(G), \circ) \cong G$.*

We have chosen to list these results in order to indicate to the reader our feeling that Moran's theorem should have a proof that does not make use of such fine tools as these. The Baker-Hausdorff formula seems to have some influence on Sjogren's results as well; however, in our development of a small part of his theory — we shall not require it at all.

§2.3. Free Group Rings

The central theme of this thesis will be the application of the free group ring to problems related to modular dimension subgroups. The detailed account of the methods, and of many other applications, is given in the monograph Gupta [1987], and we refer the reader to it for any further information required.

By the free group ring technique we mean the following: let G be a group given by its presentation $G = F/R$. Then there is a natural homomorphism $ZF \to ZG$; the key idea is to establish combinatorial properties of the inverse image in ZF of the object under consideration in G, for combinatorial tools are more powerful in ZF than in ZG, then map back into G and interpret the information obtained. This works particularly well with the investigation of the subgroups of G defined by means of ideals in ZG, or in our case $Z_{p^e}G$.

The notation we shall generally use is as follows: let $G = F/R$; then the kernel of the natural map $ZF \to ZG$ will be denoted by $\mathbf{r}$. This ideal, the relative augmentation ideal of R is generated by the elements $w - 1$ with $w \in R$. This convention of denoting the relative augmentation ideal of a subgroup by the same letter will be valid throughout for all subgroups except for the derived subgroups $F^{(n)}$ of F whose relative augmentation ideals will be denoted by $\mathbf{a}_n$. For simplicity, we shall write $\mathbf{a}$ for the relative augmentation ideal of the commutator subgroup F'.

Thus, for example, the inverse image in ZF of the nth modular dimension subgroup of $G = F/R$ is

$$F \cap (1 + \mathbf{r} + \mathbf{f}^n + p^e ZF).$$

Here $\mathbf{f}$ is the augmentation ideal of ZF according to our convention. If, for instance, we should want to prove that Lazard's formula is valid for G — we would have to establish the following relation in ZF:

$$F \cap (1 + \mathbf{r} + \mathbf{f}^n + p^e ZF) \subseteq \prod_{ip^{(j-e+1)^+} \geq n} \gamma_i(F)^{p^j} R;$$

for then, mapping $ZF \to ZG$ we would get the desired relation

$$D_n(G, Z_{p^e}) = \prod_{ip^{(j-e+1)^+} \geq n} \gamma_i(G)^{p^j}.$$

The relative augmentation ideals have the following fundamental property:

Theorem 2.3.1. *Let R be a normal subgroup of the free group F, and let $\{b_1, b_2, \ldots\}$ be the free generating set for R. Then $\mathbf{r}$ is a free ZF-module with the free generators $\{b_1 - 1, b_2 - 1, \ldots\}$. In particular $\mathbf{f}$ is a free ZF-module with the basis $\{x_1 - 1, x_2 - 1, \ldots\}$.*

This result allows us to perform cancellations in certain situations, a tool we shall use repeatedly.

Another notation we shall frequently use is this: for an ideal I of ZF, let

$$D_n(I, p^e) = F \cap (1 + I + \mathbf{f}^n + p^e\mathbf{f}).$$

Thus in particular the inverse image in ZF of the nth modular dimension subgroup will be denoted by $D_n(\mathbf{r}, p^e)$.

We shall sometimes use the partial derivatives of elements of ZF considered as rational functions; these partial derivatives will be denoted by ∂_k, with the understanding that ∂_k is the derivative with respect to x_k.

Finally, let us mention a reduction we shall often make; suppose $w - 1 \in \mathbf{r} + I$ and we want to show that $w \in HR$ for some subgroup H of F. Then for some $u \in R$ we have $w - 1 \equiv u - 1$ modulo $\mathbf{fr} + I$ and hence $wu^{-1} - 1 \in \mathbf{fr} + I$. Thus if we demonstrate that $F \cap (1 + \mathbf{fr} + I) \subseteq HR$ it will follow that $F \cap (1 + \mathbf{r} + I) \subseteq HR$. This reduction from $\mathbf{r}$ to $\mathbf{fr}$ will be of great importance in what follows.

Also, suppose $w - 1 \in I + p^e ZF$ where $I \subseteq \mathbf{f}$; applying the augmentation map, it follows that the constant term of $p^e ZF$ must be zero, so that actually $w - 1 \in I + p^e\mathbf{f}$. This reduction will be tacitly made henceforth, at our convenience.

It will generally be assumed, when dealing with dimension subgroups modulo p^e, that the group G is a p-group given by the presentation $G = F/R$ of the form

$$G = \langle x_1, \ldots, x_m \mid x_1^{e_1}\xi_1^{-1}, \ldots, x_m^{e_m}\xi_m^{-1}, \xi_{m+1}, \ldots, \xi_t \rangle$$

where $e_m \leq \cdots \leq e_1$ are p-powers. Let $\mathbf{f}$, $\mathbf{a}$, $\mathbf{r}$, and $\mathbf{s}$, denote the ideals of ZF generated by $F - 1$, $F' - 1$, $R - 1$, and $S - 1$ respectively, where $S = F'R$. Note that we can also asume, if we need to, that the presentation is positive, i.e. there are no appearances of the inverses of the generators.

We shall frequently use the results above without a special mention. More specific notation and results will be developed in each section — when they are needed.

Chapter III

Lazard's Formula and Related Topics

§3.1. The free group: Lazard's Theorem

The group-theoretic description of modular dimension subgroups over fields, due to Lazard, first appeared in the case of the free group (this is due to Zassenhaus): see theorems 2.1.3 and 2.1.4. Thus, it would seem feasible to expect that if we obtain a description of the dimension subgroups modulo prime powers of the free group — it might turn out to be true for all groups. Especially appealing was the possibility to settle, in this way, the integral dimension subgroup conjecture (cf. discussion in Chapter II). In his thesis, Lazard [1954] generalized the result of Zassenhaus; he identified the nth dimension subgroup modulo p^e of the free group F as

$$\prod_{ip^{(j-e+1)^+}\geq n} \gamma_i(F)^{p^j}$$

where $k^+ = \max\{k, 0\}$. We shall denote this group by F_{n,p^e}, and refer to this identification as Lazard's formula. In fact, for any group G let us define the subgroup G_{n,p^e} to be the image of F_{n,p^e} where F is free preimage of G: then it is easy to verify the following statement:

Theorem 3.1.0. *Let G be any group. Then G_{n,p^e} is contained in the nth dimension subgroup modulo p^e.*

Lazard's theorem asserts that equality holds if G is free. The proof of this important result was obtained using, essentially, combinatorial arguments with the remark that some topological terminology made its way in; we believe that the free group ring approach yields a clearer proof of the theorem. Furthermore, the methods we use to prove the statement are applicable, it turns out, in more general situations, for instance — identification of modular dimension subgroups of groups whose lower central factors are torsion-free. The computations we perform are most clearly visible in the case of the free group — thus we shall first give a proof of Lazard's theorem, and then adjust the argument for the proof of the generalized result.

Theorem 3.1.1. *(Lazard [1954]) If F is a free group then nth dimension subgroup mod p^e equals F_{n,p^e}.*

PROOF. Theorem 3.1.0 tells us that, for all groups, the subgroup G_{n,p^e} is contained in the nth dimension subgroup modulo p^e. Therefore it will suffice to demonstrate the other inclusion only. Suppose $w - 1 \in \mathbf{f}^n + p^e\mathbf{f}$. Let us write $w = \prod C_{ij}^{b_{ij}}$ where C_{ij} are the Hall basic commutators in their order, indexed in such a way that C_{ij} is of weight i. Repeated application of the standard identities

$$uv - 1 = (u-1)(v-1) + (u-1) + (v-1)$$

$$u^m - 1 = \sum_{j\geq 1} \binom{m}{j} (u-1)^j$$

yields the following expansion of $w - 1$:

$$w - 1 = \sum_{k=2}^{n-1} \sum \binom{b_{i_1,j_1}}{l_1} \cdots \binom{b_{i_s,j_1}}{l_s} (C_{i_1,j_1} - 1)^{l_1} \cdots (C_{i_s,j_s} - 1)^{l_s} \equiv 0$$

modulo $\mathbf{f}^n + p^e\mathbf{f}$ where the inner sum is being taken over the sequences (i_r, j_r), l_r such that $(1,1) \leq (i_1, j_1) < \cdots < (i_s, j_s)$ in the lexicographic order and $i_1 l_1 + \cdots + i_s l_s = k$. If $i_1 l_1 + \cdots + i_s l_s = k$, then modulo $\mathbf{f}^{k+1}$ the product

$$(C_{i_1,j_1} - 1)^{l_1} \cdots (C_{i_s,j_s} - 1)^{l_s}$$

is congruent to $\zeta_{i_1,j_1}^{l_1} \cdots \zeta_{i_s,j_s}^{l_s}$, where $\zeta_{i,j}$ are the Lie commutators which correspond to the basic Hall sequence; since the ordered products of these are linearly independent by the combinatorial version of the Poincaré-Birkhoff-Witt

Theorem, looking at the given relation successively modulo $\mathbf{f}^{k+1} + p^e\mathbf{f}$ for $k = 1, \ldots, n-1$ we conclude that the binomial coefficients which occur with different products must be divisible by p^e. Therefore, we have the following congruences:

$$\binom{b_{i_1,j_1}}{l_1} \cdots \binom{b_{i_s,j_s}}{l_s} \equiv 0 \bmod p^e$$

for each $k < n$, and all sequences as described in the relation above. However, if any of the integers $l_r (r > 1)$ is nonzero, then $i_1 l_1 < i_1 l_1 + \cdots + i_s l_s = k$ and hence we would have obtained the divisibility of $\binom{b_{i_1,j_1}}{l_1}$ by p^e when we dealt with $k = i_1 l_1$. Therefore such terms vanish modulo p^e and we are left with the relations

$$\binom{b_{i,j}}{l} \equiv 0 \bmod p^e$$

for each $il = k < n$. Let $b_{i,j} = p^\alpha m$ where m is prime to p. Without loss of generality we can assume that $i < n$; in this case α cannot be less than e since that would violate the conditions above with $l = 1$. Hence $\alpha \geq e$ and thus $(\alpha - e + 1)^+ = \alpha - e + 1$ so that it suffices to show that $ip^\alpha \geq np^{e-1}$. Now let β be the largest integer with the property $ip^\beta < n$ and put $l = p^\beta$; then as p^e divides $\binom{b_{i,j}}{p^\beta}$ it follows that $b_{i,j}$ is divisible by $p^{\beta+e}$ (cf. Passman [1977], p.482). On the other hand, by the definition of β, $ip^{\beta+e} \geq p^{e-1}n$ and hence $ip^\alpha \geq p^{e-1}n$. Consequently, $C_{i,j}^{b_{i,j}} \in F_{n,p^e}$. This proves the theorem. □

Theorem 3.1.1 has an important consequence regarding arbitrary groups. Traditionally the following result is proved in a somewhat sophisticated manner — applying Dark's Theorem (see e.g. Passi [1979]).

Corollary 3.1.2. *In any group G, $\{G_{n,p^e}\}_{n\geq 1}$ is a Lazard series.*

Remark. Note that in the proof of Lazard's Theorem at one point we use an inductive argument of the form: 'looking at this relation successively modulo $\mathbf{f}^{k+1} + p^e\mathbf{f}$ for $k = 1, \ldots, n-1$'; this means that we are in effect performing induction in $gr(ZF)$ rather than in ZF itself. This observation will be of importance in generalizing Lazard's result.

§3.2. A generalization

In our proof of Lazard's Theorem, a very prominent position belongs to the linear independence, in the graded ring $gr(ZF)$, of the ordered products of basic Lie commutators; actually, the other parts of the proof — such as the expansion of $w-1$ using the standard identities referred to above — do not depend on the fact that the group is free. Thus, to be able to generalize Lazard's Theorem, we require a variant of the Poincaré-Birkhoff-Witt Theorem. Such a result has been obtained: see Quillen [1968]. Some notation is required to state the theorem. Recall, from section 2.2, that with any Lazard series in a group we can associate a Lie algebra; in particular, since rational dimension subgroups form a Lazard series, the sum

$$L(\{D_{n,Q}\}) = \sum_n \frac{D_{n,Q}(G)}{D_{n+1,Q}(G)}$$

becomes a Lie algebra. Let $L_Q(G) = L(\{D_{n,Q}\}) \otimes Q$.

Theorem 3.2.1. *(Quillen [1968]) For any group G, $gr(QG)$ is the universal graded enveloping algebra of the Lie algebra $L_Q(G)$.*

Similarly, we can associate a Lie algebra $L(G) = L(G, \{\gamma_n\})$ with the lower central series of the group G. The following observation will be of import in our further considerations:

Theorem 3.2.2. *Let G be a group with torsion-free lower central factors. Then $gr(QG)$ is the universal enveloping algebra of the Lie algebra $L(G) \otimes Q$.*

PROOF. By Corollary 2.1.6, $D_{n,Q}(G) = \gamma_n(G)$; hence, the Lie algebras associated with the two series — the series of dimension subgroups and the lower central series — are the same. Therefore, $L_Q(G) \cong L(G) \otimes Q$. Applying Quillen's theorem yields the required statement. □

We can now prove the generalization of Lazard's theorem.

Theorem 3.2.3. *Lazard's formula is valid for the groups with torsion-free lower central factors. In other words, if G is such a group, then*

$$D_{n,p^e}(G) = \prod_{ip^{(j-e+1)^+} \geq n} \gamma_i(G)^{p^j}.$$

PROOF. Let $G = F/R$ and let $\{z_{k,j}\}_{k,j \geq 1}$ be the sequence of commutators in F such that $\{z_{k,j}R\}_{j \geq 1}$ is a basis for $\gamma_k(G)/\gamma_{k+1}(G)$. In the language of free group rings, we must show that if $w - 1 \in \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f}$ then $w \in F_{n,p^e}R$. Certainly w can be written as $w = u \prod z_{k,j}^{b_{k,j}}$ for some $u \in R$. Thus, assuming $w - 1 \in \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f}$, we have the equivalent relation

$$\prod z_{k,j}^{b_{k,j}} - 1 \in \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f}.$$

As in the proof of Theorem 3.2.1, repeated application of the standard identities

$$uv - 1 = (u-1)(v-1) + (u-1) + (v-1)$$

$$u^m - 1 = \sum_{j \geq 1} \binom{m}{j} (u-1)^j$$

yields

$$w - 1 = \sum_{k=1}^{n-1} \sum \binom{b_{i_1,j_1}}{l_1} \cdots \binom{b_{i_s,j_1}}{l_s} (z_{i_1,j_1} - 1)^{l_1} \cdots (z_{i_s,j_s} - 1)^{l_s} \equiv 0$$

modulo $\mathbf{f}^n + p^e\mathbf{f}$ where the inner sum is being taken over the sequences (i_r, j_r), l_r such that $(1,1) \leq (i_1, j_1) < \cdots < (i_s, j_s)$ lexicographically and $i_1 l_1 + \cdots + i_s l_s = k$. Looking at the given relation successively modulo $\mathbf{r} + \mathbf{f}^{k+1} + p^e\mathbf{f}$ for $k = 1, \ldots, n-1$ we obtain a set of relations among the ordered monomials in the elements $z_{i,j} - 1$

$$(z_{i_1,j_1} - 1)^{l_1} \cdots (z_{i_s,j_s} - 1)^{l_s}$$

in the graded ring $QG \cong QF/\mathbf{r}$; observe that these are ordered monomials in basic elements of the Lie algebra $L(G) \otimes Q$ so that by Theorem 3.2.2 and the

Poincaré-Birkhoff-Witt Theorem, the coefficients which occur with different ordered products must be divisible by p^e. Therefore, we have the following congruences:

$$\binom{b_{i_1,j_1}}{l_1} \cdots \binom{b_{i_s,j_s}}{l_s} \equiv 0 \bmod p^e$$

for each $k < n$, and all sequences as described in the relation above. The remainder of the proof is the same as that of Theorem 3.1.1. □

Corollary 3.2.4. *Lazard's formula is valid for free polynilpotent groups.*

§3.3. Related topics

In the preceding paragraph we have generalized Lazard's Theorem. Formally speaking, the statement of our Theorem 3.2.3 implies Jennings' result as well (Corollary 2.1.6); however, this is not really true — as we made use of Jennings' Theorem at one point in the proof. This has prompted us to attempt to find a proof which makes no reference to this particular theorem and thus bypass it completely (which would be more satisfactory) or at least try to provide an independent proof of the result (thus gaining the right to consider our proof self-contained). As it turned out, we have been able to provide an independent proof of Jennings' Theorem (see Corollary 4.1.2); but in trying to avoid referring to it, while not achieving that — we did come across some related results we think are worth mentioning. It turns out that in certain situations validity of Lazard's formula in a relatively free group is implied by a property of commutator identities which we call transferability. This reduction provides an alternative to the use of Jennings' Theorem. However, to describe the transferable laws successfully, we had to resort that very theorem again — which seems to point out that this result is indeed in the very heart of our methods and should not be avoided.

Suppose a set Σ of outer group-commutators is given; the set of corresponding Lie commutators is denoted by Σ'. Let F_Σ denote the relatively free group determined by the identities $\Sigma = 1$, and let $Lie_{\Sigma'}$ be the relatively free Lie algebra given by $\Sigma' = 0$. Σ is said to be torsion free if $Lie_{\Sigma'}$ is torsion free. If F denotes the free group with as many generators, then recall that $gr(QF/I_{\Sigma'})$ contains $Lie_{\Sigma'}$ as a subalgebra, where $I_{\Sigma'}$ is the ideal of QF generated by the set $\{\sigma'(w_1, \ldots, w_m) \mid \sigma' \in \Sigma', w_i \in F\}$. Finally, we shall require the following result:

Theorem 3.3.1. *(Kuz'min and Shapiro [1987])*

$$L(F_\Sigma) \otimes Q \cong Lie_{\Sigma'} \otimes Q.$$

In particular, if $Lie_{\Sigma'}$ is torsion free then $L(F_\Sigma) \cong Lie_{\Sigma'}$.

Because of this result, and the remarks above, we can assert that $L(F_\Sigma) \otimes Q$ is embedded in $gr(QF/I_{\Sigma'})$. Call this embedding β. Also, from Quillen's Theorem (3.2.1.) we see that $L_Q(F_\Sigma)$ is embedded in $gr(QF_\Sigma)$. Call this embedding γ. If α denotes the natural map from $L(F_\Sigma) \otimes Q$ to $L_Q(F_\Sigma)$, we have the following diagram.

$$\begin{array}{ccc} L(F_\Sigma) \otimes Q & \xrightarrow{\alpha} & L_Q(F_\Sigma) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ gr(QF/I_{\Sigma'}) & & gr(QF_\Sigma) \end{array}$$

In the proof of the generalization of Lazard's Theorem we have relied on Jennings' result to ensure that the mapping α above is injective. Indeed, a look at the proof of Theorem 3.2.2 will reveal that injectivity of this map will suffice to pull the argument through. Thus, to avoid the use of Jennings' Theorem we ought to look for conditions which will make α one-to-one. One such condition is the existence of a map δ which would make the following diagram commute.

$$\begin{array}{ccc} L(F_\Sigma) \otimes Q & \xrightarrow{\alpha} & L_Q(F_\Sigma) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ gr(QF/I_{\Sigma'}) & \xleftarrow{\delta} & gr(QF_\Sigma) \end{array}$$

For then, $\delta\gamma\alpha = \beta$ is injective and hence so is α. However, following the definitions of the mappings α, β, γ — the existence of such a δ is equivalent to the following relation in QF:

$$\Sigma - 1 \subseteq I_{\Sigma'} + \Delta^n \text{ for all } n$$

where Δ is the augmentation ideal of QF. Conversely, if α is one-to-one then it is actually an isomorphism, so that the diagram can be completed by setting δ to be the extension of $\beta\alpha^{-1}$, using the universal property of γ. Thus we have

Observation. *α is an isomorphism iff $\Sigma - 1 \subseteq I_{\Sigma'} + \Delta^n$ for all n.*

This requirement will certainly be fulfilled if Σ is such that if the algebra satisfies the Lie identities $\Sigma' = 0$ then its group of units satisfies the laws $\Sigma = 1$. We shall call such identities transferable; and sets commutators which satisfy the propoerty from the observation above will be called residually transferable. Because of the observation made, it seemed worth pursuing the problem of identifying the (residually) transferable commutators.

Several results have been obtained along these lines:

Theorem 3.3.2. *(Gupta and Levin [1983]) If an algebra is Lie nilpotent then its unit group is nilpotent (of at most the same class).*

Theorem 3.3.3. *(Smirnov [1988]) Let A be an algebra over a field of characteristic 2, the field having more than two elements. If A is Lie centre-by-metabelian and satisfies $x^4 = 0$ then its circle group is centre-by-metabelian.*

Theorem 3.3.4. *(Sharma and Srivastava [1]) If an algebra is Lie metabelian then its group of units is metabelian.*

PROOF. Let $v = [x, y]$ and $u = [z, w]$; since $[u, v] - 1 = u^{-1}v^{-1}(u, v)$, it suffices to show that (u, v) is in the ideal generated by the Lie commutators $((a, b), (c, d))$. By the standard expansion $(a, bc) = b(a, c) + (a, b)c$ we have

$$(u, x^{-1}y^{-1}(x, y)) = x^{-1}y^{-1}(u, (x, y)) + x^{-1}(u, y^{-1})(x, y) + (u, x^{-1})y^{-1}(x, y).$$

Therefore by the identity $y^{-1}(x,y) = -(x,y^{-1})y$ it follows that the left-hand side is equal to

$$x^{-1}y^{-1}(u,(x,y)) - x^{-1}(u,y^{-1})(y,x) - (u,x^{-1})(x,y^{-1})y$$

so that we can apply the identity

$$(u,x^{-1})(x,y) = (x^{-1}y,x,u) - x^{-1}(y,x,u)$$

to express $(u,x^{-1}y^{-1}(x,y))$ as a combination of elements of the form $((a,b),u)$; then the procedure can be repeated to yield the required result. □

REMARK. This proof is due to Professor Narain Gupta. A similar proof has been found independently by Krasil'nikov [1].

We now aim to describe the identities which are transferable in all finitely generated Q-algebras.

Theorem 3.3.5. *A commutator identity σ is residually transferable in all Q-algebras if and only if $L(F_\sigma)$ is torsion free; that is, if and only if the lower central factors of F_σ are torsion free.*

PROOF. By Jennings' theorem (2.1.5.) we have $D_n(F_\sigma, Q) = \sqrt{\gamma_n(F_\sigma)}$ so that α in effect maps $x\gamma_n(F_\sigma) \to x\sqrt{\gamma_n(F_\sigma)}$ whence it follows that it is injective iff the lower central factors are torsion free. Thus α is an isomorphism iff $L(F_\sigma)$ is torsion free. On the other hand, by the observation above, this is the case precisely when $\sigma - 1 \in I_{\sigma'} + \Delta^n$ for all n. □

Thus, we see how problems of transfer of identities and validity of Lazard's formula are linked via Jennings' theorem (2.1.5.) and the property of having torsion free lower central factors. It would be nice if the centre-by-metabelian law would transfer in Q-algebras: for then, Lazard's formula would be valid for the free centre-by-metabelian group which would generalize the result of C.K.Gupta and Levin [1986]. However, this law does not transfer (not even resudually) since it is well known that certain lower central factors of the free centre-by-metabelian group are not torsion free (see C.K.Gupta [1973]).

Corollary 3.3.6. *Polynilpotent commutators transfer in all Q-algebras. The centre-by-metabelian identity is not residually transferable in Q-algebras; in particular, there exists a Lie centre-by-metabelian Q-algebra whose group of units is not centre-by-metabelian.*

PROOF. We only have to prove the first statement. By 3.3.5 and the well-known results which state that the free polynilpotent groups have torsion free lower central factors, any polynilpotent commutator will transfer residually. Let σ be the polynilpotency law under consideration. Then $\sigma - 1$ belongs to the intersection $\cap_{n=1}^{\infty}(I_{\sigma'} + \Delta^n)$ which is a homomorphic image of $\cap_{n=1}^{\infty}\Delta^n \subseteq QF_\sigma$, because according to our discussion above $QF/I_{\sigma'}$ is a homomorphic image of QF_σ; since the intesection of powers of the augmentation ideal of QF_σ is zero (F_σ being free polynilpotent group) it follows that σ transfers in Q-algebras. □
REMARK. Professor V.Shpilrain has communicated to me that he has announced the first of these statements at a conference in USSR; he has asked (personal communication) in this context whether the centre-by-metabelian law is transferable in Q-algebras. Sharma and Srivastava [1] have asked if furthermore the centre-by-metabelian law transfers in all algebras. Corollary 3.3.6 already answers the first of these in the negative, but we shall also give counter-examples in positive characteristic, for other reasons.

The property of being transferable can be relaxed a little. For instance, in the article Sharma and Srivastava [1], the following question was raised:

(1) Is the unit group of a Lie soluble algebra soluble (perhaps of large class)?

We shall see in what follows that the answer to this question is in the negative, although it admits an affirmative answer under certain restrictions. The following simple example demonstrates, incidentally, just how far from solubility can the unit groups of Lie soluble algebras be.

Theorem 3.3.7. *The unit group of a Lie centre-by-metabelian algebra may contain free subgroups.*

PROOF. Let $K = GF(2)(x,y)$. Then by Lemma 3.3.14 below, the ring of 2×2 matrices over K is Lie centre-by-metabelian; we shall demonstrate that the subgroup of $GL_2(K)$ generated by the matrices

$$a = \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}, b = \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix}$$

is freely generated by them. Let $a^{m_1}b^{n_1}\cdots a^{m_r}b^{n_r}$ be a word in a, b, for some $r \geq 1$. Elementary matrix computations establish the following identities:

$$a^n = \begin{pmatrix} y^n & p(n)x \\ 0 & y^{-n} \end{pmatrix}, b^m = \begin{pmatrix} y^m & 0 \\ p(m)x & y^{-m} \end{pmatrix}$$

where $p(n)$ is a Laurent polynomial in y, of degree precisely $|n|-1$, given by

$$p(n) = n + \sum_{2k<|n|} y^{|n|-1-2k} + y^{-(|n|-1-2k)}.$$

Observe that $p(n) \neq 0$ unless $n = 0$. We have

$$a^n b^m = \begin{pmatrix} y^{n+m} + p(n)p(m)x^2 & y^{-m}p(n)x \\ y^{-n}p(m)x & y^{-n-m} \end{pmatrix}$$

and in general the entry $(1,1)$ of the matrix $a^{m_1}b^{n_1}\cdots a^{m_r}b^{n_r}$ has the form

$$p(n_1)p(m_1)\cdots p(n_r)p(m_r)x^{2r} + \{\text{ terms of lower degree in } x\}.$$

Thus if such a word equals a^n or b^n, the coefficient with x^{2r} must be zero, that is $p(n_1)p(m_1)\cdots p(n_r)p(m_r) = 0$ and hence $p(n_i) = 0$ or $p(m_i) = 0$ for some i, which is only possible if $n_i = 0$ or $m_i = 0$. Thus we conclude that

$$\langle a, b\rangle = \langle a\rangle * \langle b\rangle.$$

Since $\langle a\rangle$ and $\langle b\rangle$ are infinite cyclic, by the formulas for a^n and b^n above, the proof is complete. $\square$

REMARK. After having proved that the two matrices above generate a free group, we have found out that this had already been done by Platonov [1967].

Also, it may be that the result we prove, that $GL_2(GF(2)(x,y))$ contains free subgroups, could have been obtained using Tits' Alternative; however, we have preferred to have an elementary proof.

In a discussion with Professor A.E. Zalesskii, he has suggested that it would be of importance to determine whether the relation

$$[[x,y],[z,u],v]-1 \in I_{((x,y),(z,u),v)} + \Delta^n$$

holds for all n over finite fields of characteristic 2. We shall next present an example which shows that this need not be the case.

Example 3.3.8. Let $K = Z_2[[x,y,z]]$ and let $\Delta = \langle x,y,z\rangle$, the fundamental ideal of K. Under the operation of Lie commutation K becomes a Lie algebra K^*; elements of the Lie subalgebra of K^* generated by $\{x,y,z\}$ are called special Lie elements of K. By a special Lie element of weight n we mean a special Lie element which is homogeneous of degree n in K. Let $\Delta_n(K)$ be the ideal of K generated by the special Lie elements of weight at least n.

The example is the algebra $R = K/J$ where J is the ideal of K generated by:

(i) $\Delta^8 + \Delta_4(K) \cap \Delta^7 + \Delta\Delta_3(K)\Delta_3(K)$;

(ii) $((K,K),(K,K),K)$.

Clearly (because of the generators of type (ii)) R is Lie centre-by-metabelian; we shall prove that $U(R)$ is not centre-by-metabelian.

We want to show that $[[\bar{x},\bar{y}],[\bar{x},\bar{z}],\bar{x}] \neq 1$ in R. (Here $\bar{w} = 1+w$ for $w \in \{x,y,z\}$. Note that $\bar{x},\bar{y},\bar{z}$ are in $U(R)$.) The central place in proving that statement is occupied by the following Lemmas:

Lemma 1. $[[\bar{x},\bar{y}],[\bar{x},\bar{z}],\bar{x}]-1 \in J$ *if and only if* $(y,x)^2(z,x,x)$ *and* $(z,x)^2(y,x,x)$ *are in* J.

Lemma 2. *Let* θ *be the endomorphism of* K *given by* $x^\theta = x$, $y^\theta = y$, $z^\theta = y$. *Then* $(y,x)^2(y,x,x) \notin J^\theta$.

Assuming we have proved the lemmas, the argument is as follows: suppose $[[\bar{x},\bar{y}],[\bar{x},\bar{z}],\bar{x}]-1 \in J$. Then, by Lemma 1, $(y,x)^2(z,x,x) \in J$ and therefore $(y,x)^2(y,x,x)$ is in J^θ. However, this contradicts Lemma 2. This establishes the result: $[[\bar{x},\bar{y}],[\bar{x},\bar{z}],\bar{x}]-1 \notin J$.
We now proceed to prove the lemmas.

PROOF OF LEMMA 1. Let $u=[[\bar{x},\bar{y}],[\bar{x},\bar{z}]]$; $[u,\bar{x}]-1=u^{-1}\bar{x}^{-1}(u,\bar{x})$ so that

$$[u,\bar{x}]-1=(u^{-1}-1)\bar{x}^{-1}(u,\bar{x})+\bar{x}^{-1}(u,\bar{x}) \equiv \bar{x}^{-1}(u,\bar{x})$$

since the first summand is zero modulo Δ^8. Therefore $[u,\bar{x}]-1 \in J$ iff $(u,\bar{x}) \in J$. Let $v=[\bar{x},\bar{y}], w=[\bar{x},\bar{z}]$; then $u=[v,w]$ and using the identity

$$(rs,t)=r(s,t)+(r,t)s \tag{$*$}$$

we obtain that $(u,\bar{x})=(v^{-1}w^{-1}(v,w),\bar{x})$ equals

$$(v^{-1}w^{-1}-1)(v,w,\bar{x})+(v,w,\bar{x})+(v^{-1}w^{-1},\bar{x})(v,w).$$

The third summand is in Δ^7 so that modulo Δ^8 it takes the form $a((x,y),(x,z))$, and hence is in J. Similarly, the first summand is in Δ^7, so that modulo Δ^8 it becomes a multiple of $((x,y),(x,z),x)$ and thus belongs to J. The second summand, upon further expansion of v,w, and using $\bar{a}^{-1}=1+a+a^2+\cdots$, is congruent modulo Δ^8 to:

$$((x,y),(x,z),x)+$$

$$+\sum_{i+j=1}^{3}(x^i(x,y),x^j(x,z),x)+\sum_{i+j=1}^{3}(x^i(x,y),z^j(x,z),x)+$$

$$+\sum_{i+j=1}^{3}(y^i(x,y),x^j(x,z),x)+\sum_{i+j=1}^{3}(y^i(x,y),z^j(x,z),x)+$$

$$+(xy(x,y),(x,z),x)+((x,y),xz(x,z),x).$$

A typical element in the first three lines of this sum would be $(x^i(x,y), z^j(x,z), x)$; since $x^i(x,y) = (x^{i+1}, y) - (x^i, yx)$ by $(*)$ we see that these elements are actually in $((K,K),(K,K),K)$. Therefore

$$(u, \bar{x}) \equiv (xy(x,y),(x,z),x) + ((x,y), xz(x,z), x) \text{ mod } J.$$

Finally, repeatedly applying $(*)$, we obtain

$$\begin{aligned}(xy(x,y),(x,z),x) &= (xy((x,y),(x,z)),x) + ((x,z,xy)(x,y),x) \\ &\equiv (x,z,xy)(x,y,x) + (x,z,xy,x)(x,y) \\ &\equiv (x(x,z,y),x)(x,y) + ((x,z,x)y,x)(x,y) \\ &\equiv (y,x)^2(z,x,x)\end{aligned}$$

modulo J. Similarly, $((x,y), xz(x,z), x) \equiv (z,x)^2(y,x,x)$. Thus, we have proved that $[[\bar{x},\bar{y}],[\bar{x},\bar{z}],\bar{x}] - 1 \in J$ if and only if
$(y,x)^2(z,x,x) + (z,x)^2(y,x,x) \in J$.

Each monomial has its frequency pattern (the triple (i,j,k) telling that x occurs i times, y occurs j times and z occurs k times in that monomial); and to each triple there is an additive map of K which leaves monomials of that frequency pattern invariant and annihilates all other monomials. Since J is invariant under these frequency maps, and the two summands of $(y,x)^2(z,x,x)+$ $(z,x)^2(y,x,x)$ have different frequency patterns, it follows that
$(y,x)^2(z,x,x)+(z,x)^2(y,x,x) \in J$ iff $(y,x)^2(z,x,x)$ and $(z,x)^2(y,x,x)$ are in J. This completes the proof of Lemma 1.

For the proof of Lemma 2 we shall need the following:

Lemma 3. $K((K,K),(K,K),K)K \cap \Delta^7 \subseteq ((\Delta^3,\Delta),\Delta_2,\Delta) + ((\Delta^2,\Delta^2),\Delta_2,\Delta)$ $+((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta) + \Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8$. *Furthermore, each of* $((\Delta\Delta_2,\Delta),\Delta_2,\Delta)$, $((\Delta_2,\Delta^2),\Delta_2,\Delta)$, $((\Delta_2,\Delta),(\Delta^2,\Delta),\Delta)$ *is contained in the ideal* $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$.

PROOF. Clearly, modulo $\Delta_4 \cap \Delta^7 + \Delta^8$, $K((K,K),(K,K),K)K \cap \Delta^7$ is congruent to

$$((\Delta^2,\Delta),\Delta_2,\Delta,\Delta) + \Delta((\Delta^2,\Delta),\Delta_2,\Delta)+ \\ + ((\Delta^2,\Delta),\Delta_3,\Delta) + ((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta)+ \\ + ((\Delta^2,\Delta^2),\Delta_2,\Delta) + ((\Delta^3,\Delta),\Delta_2,\Delta).$$

Consider $((\Delta^2,\Delta),\Delta_2,\Delta,\Delta)$; we have, using $(*)$,

$$\begin{aligned} ((\Delta^2,\Delta),\Delta_2,\Delta,\Delta) &\subseteq (\Delta\Delta_2,\Delta_2,\Delta,\Delta) + (\Delta_2\Delta,\Delta_2,\Delta,\Delta) \\ &\subseteq (\Delta\Delta_4,\Delta,\Delta) + (\Delta_3\Delta_2,\Delta,\Delta)+ \\ &+ (\Delta_4\Delta,\Delta,\Delta) + (\Delta_2\Delta_3,\Delta,\Delta); \end{aligned}$$

Another application of $(*)$ shows that every summand is in $\Delta_4 \cap \Delta^7$.
In a similar way we have by $(*)$:

$$\begin{aligned} \Delta((\Delta^2,\Delta),\Delta_2,\Delta) &\subseteq \Delta(\Delta\Delta_2,\Delta_2,\Delta) + \Delta(\Delta_2\Delta,\Delta_2,\Delta) \\ &\subseteq \Delta(\Delta\Delta_4,\Delta) + \Delta(\Delta_3\Delta_2,\Delta)+ \\ &+ \Delta(\Delta_4\Delta,\Delta) + \Delta(\Delta_2\Delta_3,\Delta). \end{aligned}$$

Expanding one more time we see that each of the summands is contained in the ideal $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Furthermore, $((\Delta^2,\Delta),\Delta_3,\Delta) \subseteq ((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta)$. Hence modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8$, $K((K,K),(K,K),K)K) \cap \Delta^7$ is contained in

$$((\Delta^3,\Delta),\Delta_2,\Delta) + ((\Delta^2,\Delta^2),\Delta_2,\Delta) + ((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta).$$

The second statement of the lemma follows in much the same way.

PROOF OF LEMMA 2. Suppose $(y,x)^2(y,x,x) \in J^\theta$. By Lemma 3, J^θ is contained in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8 + ((\Delta^3,\Delta),\Delta_2,\Delta)^\theta + ((\Delta^2,\Delta^2),\Delta_2,\Delta)^\theta + ((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta)^\theta$. Hence, modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3 + \Delta^8$, $(y,x)^2(y,x,x)$ is a linear combination of elements from $((\Delta^3,\Delta),\Delta_2,\Delta)^\theta$, $((\Delta^2,\Delta^2),\Delta_2,\Delta)^\theta$ and $((\Delta^2,\Delta),(\Delta^2,\Delta),\Delta)^\theta$. Furthermore, we may assume that the summands in this linear combination are all of the frequency pattern (4,3,0). We shall show that such elements are actually in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Let us look at $((abc, d), (e, f), g) \in ((\Delta^3, \Delta), \Delta_2, \Delta)^\theta$ and assume that its frequency pattern is (4,3,0). Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, $((abc, d), (e, f), g) \equiv ((a^\pi b^\pi c^\pi, d), (e, f), g)$ for all permutations π of the set $\{a, b, c\}$—by Lemma 3; hence we only have four possibilities: $((y^2x, x), (x, y), x)$, $((x^2y, x), (x, y), y)$, $((x^2y, y), (x, y), x)$ and $((x^3, y), (x, y), y)$. Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$,

$$\begin{aligned} ((x^2y, x), (x, y), y) &= ((x^2(y, x), (x, y)), y) \\ &\equiv ((x, y, x^2)(x, y), y) \\ &\equiv (x(x, y, x)(x, y), y) + ((x, y, x)x(x, y), y) \\ &\equiv 2(x, y)(x, y)(x, y, x) = 0. \end{aligned}$$

Similar calculations will prove that the other elements are in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, too. Consider $((ab, cd), (e, f), g) \in ((\Delta^2, \Delta^2), \Delta_2, \Delta)^\theta$, having frequency pattern (4,3,0); modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, and using Lemma 3, the nontrivial choices are $((x^2, y^2), (x, y), x)$ and $((x^2, xy), (x, y), y)$. Repeated application of $(*)$ will show that each of these is in $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. Finally, let us look at a generator $((ab, c), (de, f), g)$ of $((\Delta^2, \Delta), (\Delta^2, \Delta), \Delta)^\theta$. Modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$, we may assume that x precedes y in the expressions ab and de (by Lemma 3); hence, the generator is congruent to one of: $((x^2, y), (xy, y), x)$, $((x^2, y), (xy, x), y)$, $((x^2, y), (xy, x), x)$, $((xy, y), (xy, x), x)$, $((xy, x), (y^2, x), x)$. (Using, of course, that the frequency pattern is (4,3,0).) Expanding these by $(*)$ will show that each of them is zero modulo $\Delta_4 \cap \Delta^7 + \Delta\Delta_3\Delta_3$. For example,

$$\begin{aligned} ((xy, y), (xy, x), x) &= ((y, x)y, x(y, x), x) \\ &= ((y, x)(y, x(y, x)), x) + (((y, x), x(y, x))y, x) \\ &\equiv ((y, x)^3, x) + ((y, x, x)(y, x)y, x) \\ &\equiv 3(y, x)^2(y, x, x) + (y, x)^2(y, x, x) = 0. \end{aligned}$$

Therefore, we are forced to conclude that modulo Δ^8, $(y, x)^2(y, x, x)$ is a linear combination of elements from $\Delta_4 \cap \Delta^7$ and $\Delta\Delta_3\Delta_3$. But this yields a nontrivial relation between the basic products of degree 7—which is impossible (Theorem 2.2.4). Lemma 2 is thus proved.

Thus we have the following result, which is related to a problem of A.E. Zalesskii; by the circle group of an algebra we mean the group given by the operation $x \circ y = xy + x + y$. Zalesskii has asked if a nil Lie centre-by-metabelian algebra over $GF(2)$ has soluble circle group (and in particular is it centre-by-metabelian). Our example is relevant, for if A is the algebra in question and G its circle group, then if B is the algebra arising from A by adjoining a formal identity of characteristic 2 — we have that G is naturally isomorphic to the group $1 + A \subseteq B$ under multiplication. In our example we can take $A = \Delta/J$ and $B = R$, obtaining the following (cf. 3.3.3):

Theorem 3.3.9. *The unit group of a Lie centre-by-metabelian Lie nilpotent algebra need not be centre-by-metabelian. The circle group of a nilpotent (of exponent 8) Lie centre-by-metabelian algebra over $GF(2)$ need not be center-by-metabelian.*

We shall return to Zalesskii's problem later.

Note that our counter-example for question (1) may be assumed to be a finitely generated algebra over a field. Hence it is significant to restrict our attention this special case. It turns out that characteristic 2 is exceptional here: for we have the following result which provides a complement to our answer to question (1).

Theorem 3.3.10. *The unit group of a finitely generated Lie soluble algebra over a field of characteristic other than 2 is nilpotent-by-abelian.*

To prove this statement we shall require several auxiliary results.

Theorem 3.3.11. *If A is a semiprimitive algebra with a multilinear identity, then A has the same multilinear identities as a product of matrix algebras over fields.*

PROOF. This statement can be extracted from Rowen [1980], the argument being as follows: a semiprimitive algebra has the same multilinear identities as a product of closed primitive algebras (Rowen [1980, 1.6.4]); but a closed primitive P.I. algebra is isomorphic to a matrix algebra over a field (Rowen [1980, 1.5.13]). □

Theorem 3.3.12. *(Braun [1984]) If A is a finitely generated P.I. algebra over a field, then its Jacobson radical is nilpotent.*

Lemma 3.3.13. *(cf. Sehgal [1978] V.2.1) Let R be a commutative algebra over a field of characteristic p. Then $(T_n(R), M_n(R)) = T_n(R)$ and, unless $n = p = 2$, we also have $(T_n(R), T_n(R)) = T_n(R)$.*

Lemma 3.3.14. *Let R be a commutative algebra over a field of characteristic p and let $n > 1$. $M_n(R)$ is Lie soluble if and only if $n = p = 2$, in which case it is Lie centre-by-metabelian.*

We shall omit the proof of Lemma 3.3.13 as the proof of Lemma V.2.2 in Sehgal [1978] actually establishes our statement, although Sehgal did not state it explicitly.

PROOF OF 3.3.10. By Lemma 3.3.11, $A/J(A)$ has the same multilinear laws as $\prod_{i \in I} M_{n_i}(F_i)$, a product of matrix algebras over fields (of characteristic other than 2). Thus each $M_{n_i}(F_i)$ is Lie soluble. By Lemma 3.3.14, this can only take place if $n_i = 1$ (for all i), since the characteristic is not 2. It follows that each $M_{n_i}(F_i)$ is commutative, and hence $A/J(A)$ is commutative, too; therefore $(A, A) \subseteq J(A)$. Since $[x, y] - 1 = x^{-1}y^{-1}(x, y)$, we have $U(A)' - 1 \subseteq J(A)$. By repeated use of these relations it follows without difficulty that

$$\gamma_k(U(A)') - 1 \subseteq J(A)^k.$$

However, the Jacobson radical of a finitely generated P.I. algebra over a field is nilpotent by Lemma 3.3.12, and therefore $\gamma_k(U(A)') = 1$ for some k. This is what we wanted to prove. □

REMARK. The proof relies, very substantially, on the idea of looking at the quotient of the algebra by its Jacobson radical and then using 3.3.11; I thank Professor S.A. Amitsur for mentioning this approach in a discussion at the University of Warwick.

The same method can be used to obtain the following result.

Theorem 3.3.15. *Let A be a finitely generated algebra over a field. If A obeys the law $((x_1, \ldots, x_c), (y_1, \ldots, y_d)) = 0$ then its group of units is nilpotent-by-abelian.*

PROOF. Since A is a finitely generated P.I. algebra, it follows by Lemma 3.3.11 that $A/J(A)$ the same multilinear laws as a product $\prod_{i \in I} M_{n_i}(F_i)$ of matrix algebras over fields. Hence, each $M_{n_i}(F_i)$ satisfies the identity

$$((x_1, \ldots, x_c), (y_1, \ldots, y_d)) = 0;$$

however, it is easy to see (using Lemma 3.3.13) that matrix algebras do not obey that identity unless in fact the matrices are 1×1. Therefore $n_i = 1$ for all i, so the algebras $M_{n_i}(F_i)$ are commutative, and hence it follows that $A/J(A)$ is commutative, too. Thus $(A, A) \subseteq J(A)$. The remainder of the proof is the same as in Theorem 3.3.10. □

Braun's theorem can be applied to obtain a result regarding the Zalesskii problem; it shows that in the finitely generated case Zalesskii's question has a surprising answer.

Theorem 3.3.16. *Let A be a finitely generated nil algebra over a field. If A satisfies a Lie identity, then it is nilpotent and hence its circle group is nilpotent.*

PROOF. Let B be the algebra obtained from A by formally adjoining the identity element of the appropriate characteristic. Then B is a finitely generated algebra over a field and it satisfies the same Lie identities as A. A is nil, so it must be contained in the Jacobson radical $J(B)$ which is nilpotent by Braun's Theorem.

Hence, the unit group of B is nilpotent and so is its subgroup $1 + A$, which is isomorphic to the circle group of A.

The method of factoring out the Jacobson radical and then studying the matrix identities was used in a similar context by Shalev to obtain the following result:

Theorem 3.3.17. *(Shalev [2]) If A is a finitely generated associative m-Engel algebra over a field of positive characteristic, then A is Lie nilpotent. Consequently, the unit group of an associative m-Engel algebra over a field of positive characteristic is locally nilpotent and n-Engel for some n.*

The situation for Engel algebras over a field of characteristic zero is different due to the remarkable result of Zel'manov:

Theorem 3.3.18. *(Zel'manov [1988]) If a Lie algebra over a field of characteristic zero satisfies the n-Engel condition, it is nilpotent.*

This has an immediate corollary:

Theorem 3.3.19. *If an algebra over a field of characteristic zero is n-Engel, then its unit group is nilpotent.*

Regarding Shalev's result, let us mention that it provides a partial answer, in the affirmative, to the following open problem: is every n-Engel Lie algebra over a field of characteristic p locally nilpotent? Kostrikin's theorem asserts that this is indeed true if $n \leq p$, and it was improved by Braun [1974] to $n = p + 1$. But such considerations take us into the realm of the restricted Burnside problem, which is beyond the scope of this work.

Chapter IV

Small Modular Dimension Subgroups of Metabelian Groups

§4.1. Moran's Theorem

Moran [1970] has demonstrated that Lazard's formula for modular dimension subgroups mod p^e holds for dimensions up to p: $D_n(G, p^e) = G_{n,p^e} = G^{p^e}\gamma_n(G)$ for $n \leq p$. In this section, we shall demonstrate the theorem with the aid of free group ring techniques; we had promised to do so in the preceding chapter with the goal of using it to provide an independent proof of Jennings' Theorem, but we also believe that the proof we present is more natural and elementary than the original one. For example, Moran's proof allows for Lazard's result which asserts the isomorphism of a finite p-group and its associated Lie ring under the Baker-Hausdorff multiplication (Theorem 2.2.8).

We shall assume that the group G is a p-group given by the presentation $G = F/R$ of the form

$$G = \langle x_1, \ldots, x_m \mid x_1^{e_1}\xi_1^{-1}, \ldots, x_m^{e_m}\xi_m^{-1}, \xi_{m+1}, \ldots, \xi_t \rangle$$

where $e_m \leq \cdots \leq e_1$ are p-powers. Let $\mathbf{f}$, $\mathbf{a}$, $\mathbf{r}$, and $\mathbf{s}$, denote the ideals of ZF generated by $F-1$, $F'-1$, $R-1$, and $S-1$ respectively, where $S = F'R$. In the language of free group rings, the statement of the theorem assumes the form:

Theorem 4.1.1. *(Moran [1970]) If $n \leq p$, $F \cap (1 + \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f}) = F^{p^e}\gamma_n(F)R$.*

PROOF. It is easy to verify that $F^{p^e}\gamma_n(F)R \subseteq F \cap (1 + \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f})$; hence it will suffice to prove that $F \cap (1+\mathbf{r}+\mathbf{f}^n+p^e\mathbf{f}) \subseteq F^{p^e}\gamma_n(F)R$. Suppose $w-1 \in \mathbf{r}+\mathbf{f}^n+p^e\mathbf{f}$; write $w = x_1^{b_{11}}\cdots x_m^{b_{1m}}\prod C_{ij}^{b_{ij}}$ where C_{ij} are Hall's commutators of weight $i \geq 2$ (Theorem 2.2.4). Since $\mathbf{r} \subseteq \mathbf{s}$ we certainly have $w-1 \in \mathbf{s}+\mathbf{f}^n+p^e\mathbf{f}$; on applying the endomorphism $x_j \longrightarrow x_j^{\delta_{ij}}$ we obtain the relation $x_i^{b_{1i}}-1 \in \mathbf{s}+\mathbf{f}^n+p^e\mathbf{f}$ whence it follows that, for some integers a_i, $x_i^{b_{1i}}-1 \equiv x_i^{a_ie_i}-1$ modulo $\mathbf{fs}+\mathbf{f}^n+p^e\mathbf{f}$ or, equivalently, $x_i^{b_{1i}-a_ie_i}-1 \in \mathbf{fs}+\mathbf{f}^n+p^e\mathbf{f}$. Expanding, we obtain

$$\sum_{k=1}^{n-1}\binom{b_{1i}-a_ie_i}{k}(x_i-1)^k \in \mathbf{fs}+\mathbf{f}^n+p^e\mathbf{f}.$$

The ideal $\mathbf{f}$ is a free ZF-module on the generators x_j-1, $j=1\ldots m$, so we can cancel x_i-1 in the relation above and see that p^e divides $b_{1i}-a_ie_i$ i.e. $b_{1i} \in e_iZ+p^eZ$. Therefore, because the presentation of the group is such that $x_i^{e_i}=\xi_i$ we have that w is a commutator modulo $F^{p^e}R$; i.e. we can assume that b_{1i} are divisible by p^e. Since $F^{p^e}-1 \subseteq \mathbf{f}^n+p^e\mathbf{f}$ for $n \leq p$ we may assume that $w=\prod_{k\geq 2}C_{kj}^{b_{kj}}$ and $w-1 \in \mathbf{r}+\mathbf{f}^n+p^e\mathbf{f}$. This is the initial step of the induction. Consider now the relation

$$\prod_{i\geq k} C_{ij}^{b_{ij}}-1 \in \mathbf{r}+\mathbf{f}^k+p^e\mathbf{f}$$

for $k<n$; modulo $\mathbf{f}^{k+1}+p^e\mathbf{f}$ this translates into

$$\sum_j b_{kj}\zeta_{kj}-\sum_i a_i(x_i^{e_i}\xi_i-1)b_i \equiv 0$$

where ζ_{kj} is the Lie commutator which corresponds to C_{kj}, and $a_i, b_i \in ZF$. In particular, it follows that

$$\sum_i a_i(x_i^{e_i}\xi_i-1)b_i \in \mathbf{f}^k+p^e\mathbf{f}$$

so that

$$\sum_i a_i(x_i^{e_i}\xi_i-1)b_i \equiv \sum_i a_ie_i(x_i-1)b_i \bmod (\mathbf{f}^{k+1}+p^e\mathbf{f})$$

and consequently modulo $\mathbf{f}^{k+1} + p^e\mathbf{f}$

$$\begin{aligned}\sum_j b_{kj}\zeta_{kj} &\equiv \sum_i a_i e_i (x_i - 1) b_i \\ &\equiv \sum_s \beta_s e_{i_s} \pi_s\end{aligned}$$

where $\beta_s \in Z$ and π_s are ordered products of degree k in the basic Lie commutators ζ. Therefore, by the independence of such ordered products (see Theorem 2.2.4), it follows that $b_{kj} \equiv \alpha_i e_i \bmod p^e$ for some i such that x_i occurs in the commutator C_{kj}. However, modulo $\gamma_{k+1}(F)R$ we have

$$C_{kj}^{e_i} = [\ldots x_i \ldots]^{e_i} \equiv [\ldots x_i^{e_i} \ldots] \equiv [\ldots \xi_i \ldots] \equiv 1$$

and thus it follows that $w \in F^{p^e}\gamma_{k+1}(F)R$; i.e. we can assume (arguing as above) that $w = \prod_{i \geq k+1} C_{ij}^{b_{ij}}$ and $w - 1 \in \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f}$. The induction just established shows that $F \cap (1 + \mathbf{r} + \mathbf{f}^n + p^e\mathbf{f})$ is contained in $F^{p^e}\gamma_n(F)R$. This proves Moran's Theorem. $\square$

An immediate consequence of this result the following:

Corollary 4.1.2. *Let G be a p-group; then $D_n(G, Z) = \gamma_n(G)$ for $n \leq p$.*

Also, we can now prove Jennings' Theorem. We shall require the following classical result:

Theorem. *(Mal'cev) Every finitely generated torsion-free nilpotent group embeds into $GL_n(Z)$; consequently, such groups are residually finite p for all p.*

Corollary 4.1.3. *(Jennings [1955]) For all G, $D_n(G, Q) = \sqrt{\gamma_n(G)}$.*

PROOF. Let $n \leq p$. Without loss of generality we can assume that G is finitely generated and $\sqrt{\gamma_n(G)} = 1$; i.e. G is finitely generated torsion-free nilpotent group. Suppose $y \in D_n(G,Q)$ and $y \neq 1$; it follows that $y^m \in D_n(G,Z)$ for some m. G is torsion-free so $y^m \neq 1$; we put $x = y^m$ and thus get $1 \neq x \in D_n(G,Z)$. G is residually finite p for all p (being finitely generated torsion-free nilpotent), so there is a homomorphism φ from G onto a finite p-group H with $p \geq n$ and such that $\varphi(x) \neq 1$ in H. Clearly, $x \in D_n(G,Z)$ yields $\varphi(x) \in D_n(H,Z)$. But by Corollary 4.1.2 $D_n(H,Z) = 1$ and hence $\varphi(x) = 1$, which is a contradiction.□

In the article Moran [1970] it is shown that Lazard's formula fails in general for $n = p+1$; another set of examples to this effect has been constructed by Sandling [1972]. The examples are as follows: let $p > 2$, $e > 1$ and put

$$G = \langle x, y \mid x^{p^{e+1}} = y^{p^2} = 1, [x,y] = x^{p^{e-1}} \rangle.$$

Then $x^{p^e} \neq 1$ and it belongs to $D_{p+1}(G,p^e)$. It is interesting to note that these examples are actually metacyclic groups, and that they are based on a very simple property that is not captured in Lazard's formula, namely the following.

Observation. *If $n \leq p$ and $x^{p^{e-1}}$ belongs to $D_n(G,p^e)$, then x^{p^e} belongs to $D_{n+p-1}(G,p^e)$.*

PROOF. Expanding $x^{p^{e-1}} - 1$ modulo $\mathbf{f}^n$ yields the relation $p^{e-1}(x-1) \equiv 0 \bmod \mathbf{f}^n$. Multiplying by $(x-1)^{p-1}$ it follows that $p^{e-1}(x-1)^p \equiv 0$ modulo $\mathbf{f}^{n+p-1}$. On the other hand, we have that $x^{p^e} - 1 \equiv p^{e-1}(x-1)^p a$ modulo $\mathbf{f}^{n+p-1}$ for some $a \in Z_{p^e}G$. This proves the assertion.

This has prompted us to try to obtain a description of modular dimension subgroups of dimension $p+1$ for metabelian groups; the subgroup we would consider should of course include elements arising from the observation above, as well as the subgroup G_{n,p^e}. This would eliminate the Moran-Sandling examples, but perhaps there are other simple properties that are not captured in our subgroup? However, it turns out that the counter-examples given by Moran and Sandling are indeed the only possible kind—for metabelian groups. We give, in the next two sections a complete description of $D_{p+1}(G,p^e)$ for metabelian G; furthermore, some refinements of these calculations enabled us to give a description of $D_{p+2}(G,p^e)$ as well. The cases $p=2$ and $p>2$ are slightly different and we shall therefore treat them in separate sections although some statements will not depend on the parity of p.

§4.2. $p = 2$ and $n \leq 4$

The computations we perform are an adjustment to the modular case of some of what Professor Narain Gupta has done in his work on integral dimension subgroups. A remark on the notation: recall that we are dealing with a p-group given by its pre-abelian presentation with notation being as indicated in the previous section. The polynomial $\frac{x_i^{e_i}-1}{x_i-1}$ will be denoted by $t(x_i)$; this is not completely precise (we should denote it by $t(x_i,e_i)$), but there will be no room for confusion.

Lemma 4.2.1. *(cf. Gupta [1987], IV 3.2.) Let* $w=\prod_{i<j}[x_i,x_j]^{d_{ij}} \in 1+\mathbf{fs}+\mathbf{f}^{n+2}+p^e\mathbf{f}$*; then* $d_{ij}\equiv t(x_i)a_{ij}(x_i,\ldots,x_m)\equiv t(x_j)b_{ij}(x_i,\ldots,x_m)$ *mod* $\mathbf{s}+\mathbf{f}^n+p^eZF$*. Conversely, if* d_{ij} *are such, then* $[x_i,x_j]^{d_{ij}}\in D_{n+2}(\mathbf{fs},p^e)$.

PROOF. Modulo F'', using Jacobi's identity we can assume that d_{ij} depend only on $\{x_i,\ldots,x_m\}$; and $w\equiv w_1\cdots w_{m-1}$ where $w_i=\prod_{j=i+1}^{m}[x_i,x_j]^{d_{ij}}$; as $\mathbf{s}$, $\mathbf{f}$ are invariant under endomorphisms of ZF given by $x_j\to x_j^{\delta_{ij}}$ the assumption $w-1\in 1+\mathbf{fs}+\mathbf{f}^{n+2}+p^e\mathbf{f}$ yields $w_i-1\in 1+\mathbf{fs}+\mathbf{f}^{n+2}+p^e\mathbf{f}$ for all i. Thus modulo $\mathbf{fs}+\mathbf{f}^{n+2}+p^e\mathbf{f}$

$$
\begin{aligned}
0 &\equiv w_i - 1 \\
&\equiv \sum_{j=i+1}^{m} ([x_i, x_j] - 1)d_{ij} \\
&\equiv \sum_{j=i+1}^{m} ((x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1))d_{ij} \\
&\equiv (x_i - 1) \sum_{j=i+1}^{m} (x_j - 1)d_{ij} - \sum_{j=i+1}^{m} (x_j - 1)(x_i - 1)d_{ij}.
\end{aligned}
$$

Since $\mathbf{f}$ is free with basis $\{x_1 - 1, \ldots, x_m - 1\}$ it follows that

$$
(x_i - 1)d_{ij} \equiv 0 \bmod \mathbf{s} + \mathbf{f}^{n+1} + p^e\mathbf{f} \tag{1}
$$

$$
\sum_{j=i+1}^{m} (x_j - 1)d_{ij} \equiv 0 \bmod \mathbf{s} + \mathbf{f}^{n+1} + p^e\mathbf{f}. \tag{2}
$$

But $x_i - 1$ is regular in $Z_{p^e}(F/F') \cong ZF/(\mathbf{a} + p^e ZF)$ so that using the relation

$$
\mathbf{s} + \mathbf{f}^{n+1} + p^e ZF = \sum_{k=1}^{m} (x_k - 1)(t(x_k)ZF + \mathbf{f}^n) + \mathbf{a} + p^e ZF
$$

we deduce, from (1), that

$$
d_{ij} \in t(x_i)ZF + \mathbf{f}^n + \mathbf{a} + p^e ZF.
$$

In a similar way (2) implies that

$$
d_{ij} \in t(x_j)ZF + \mathbf{f}^n + \mathbf{s} + p^e ZF.
$$

The second statement of the lemma is clear. □

This lemma alone yields a description of the third dimension subgroup mod 2^e; this is of course well-known, but the new description we obtain seems nicer since it is given in terms of the presentation of the abelianization of the group (it is easy to verify that it matches with the result of Sandling).

Theorem 4.2.2. $D_3(\mathbf{r}, 2^e) = \langle x_i^{2^e} \mid e_i < 2^e \rangle F^{2^{e+1}} \gamma_2^{2^e} \gamma_3 R$.

PROOF. We first observe that the subgroup from the statement of the Theorem is contained in $D_3(\mathbf{r}, 2^e)$. If $w - 1 \in \mathbf{r} + \mathbf{f}^3 + 2^e\mathbf{f}$ then for some $u \in R$, $w - 1 \equiv u - 1 \mod \mathbf{fr} + \mathbf{f}^3 + 2^e\mathbf{f}$ so $wu^{-1} - 1 \in \mathbf{fr} + \mathbf{f}^3 + 2^e\mathbf{f}$. Therefore it will suffice to demonstrate that $D_3(\mathbf{fr}, 2^e)$ is contained in the alleged subgroup. Suppose

$$\prod_{i=1}^{m} x_i^{\alpha_i} \prod_{i<j} [x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fr} + \mathbf{f}^3 + 2^e\mathbf{f} \subseteq \mathbf{fs} + \mathbf{f}^3 + 2^e\mathbf{f}.$$

Applying the endomorphism $\varphi_i : x_j \longrightarrow x_j^{\delta_{ij}}$ of ZF to the relation above we obtain $x_i^{\alpha_i} - 1 \in \mathbf{fs} + \mathbf{f}^3 + 2^e\mathbf{f}$. Therefore

$$\alpha_i(x_i - 1) + \binom{\alpha_i}{2}(x_i - 1)^2 \in \mathbf{fs} + \mathbf{f}^3 + 2^e\mathbf{f}$$

so $\alpha_i + \binom{\alpha_i}{2}(x_i - 1) \in \mathbf{s} + \mathbf{f}^2 + 2^e ZF$, which implies that $\alpha_i \in 2^e Z$ and then from $\binom{\alpha_i}{2}(x_i - 1) \in \mathbf{s} + \mathbf{f}^2 + 2^e ZF$ we conclude that $\binom{\alpha_i}{2} \in \langle e_i, 2^e \rangle$; i.e. α_i is in $\langle 2e_i, 2^{e+1} \rangle \cap 2^e Z$. (Thus if $e_i < 2^e$, then $2^e \mid \alpha_i$.) Also, if this condition is satisfied, then $x_i^{\alpha_i} - 1 \in \mathbf{fr} + \mathbf{f}^3 + 2^e\mathbf{f}$ so that the initial relation now reduces to

$$\prod_{i<j} [x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fs} + \mathbf{f}^3 + 2^e\mathbf{f}.$$

This, by Lemma 4.2.1, yields $d_{ij} \in t(x_i)ZF + \mathbf{f} + 2^e ZF$ so that

$$[x_i, x_j]^{d_{ij}} \equiv [x_i^{e_i}, x_j] \equiv 1 \mod \gamma_2^{2^e} \gamma_3 R.$$

The proof is now complete. □

However, to deal with the dimensions higher than three, we shall need to refine Lemma 4.2.1. The first step in this direction is the following consequence of Lemma 4.2.1.

Corollary 4.2.3. *(cf. Lemma 2.1, Gupta and Tahara [1985]) Let* $w = \prod_{i<j}[x_i, x_j]^{d_{ij}} \in D_{n+2}(\mathbf{fs}, p^e)$ *with* $n \leq p$. *Then* $d_{ij} \equiv t(x_i)a_{ij} \equiv t(x_j)b_{ij}(x_i, \ldots, x_m)$ *where* $a_{ij} \in Z$*; moreover, if* $e_i = e_j$*, then* $b_{ij} \in Z$*.*

PROOF. By Lemma 4.2.1 we only have to show that $a_{ij} \in Z$. The relation $x_k^{e_k} - 1 \equiv e_k(x_k - 1)$ modulo $e_k(x_k-1)^2 ZF + \mathbf{f}^p$ shows that $e_k(x_k - 1) \in \mathbf{s} + \mathbf{f}^p$. Thus $t(x_i) \equiv e_i + \binom{e_i}{p}(x_i - 1)^{p-1}$ modulo $\mathbf{s} + \mathbf{f}^p$ and hence if $k \geq i$ we have (because $n \leq p$)

$$t(x_i)(x_k - 1) \equiv e_i(x_k - 1) \equiv e_k(x_k - 1)e_i/e_k \equiv 0 \bmod \mathbf{s} + \mathbf{f}^n.$$

Therefore $a_{ij} \in Z$; similarly, if $e_i = e_j$ it follows that $b_{ij} \in Z$. □

Observe that, by Lemma 4.2.1, elements of $F' \cap D_{n+2}(\mathbf{fs}, p^e)$ have the form $w = \prod_{i<j}[x_i, x_j]^{d_{ij}} \prod_{i=1}^{m}[x_i^{e_i}, \eta_i]\xi$ where $\eta_i \in F'$, ξ belongs to $F''\gamma_2^{p^e}\gamma_{n+2}$, and d_{ij} are as in Lemma 4.2.1. This motivates the next result.

Lemma 4.2.4. *(cf. IV 4.4, Gupta [1987]) Let* $w = \prod_{i<j}[x_i, x_j]^{d_{ij}} \prod_{i=1}^{m}[x_i^{e_i}, \eta_i]\xi$ *where* $\eta_i \in F'$*,* ξ *belongs to* $F''\gamma_2^{p^e}\gamma_{n+2}$ *and* d_{ij} *are as in Lemma 4.2.1; then modulo* $\mathbf{f}^2\mathbf{s} + \mathbf{f}^{n+2} + p^e\mathbf{f}$,

$$w - 1 \equiv \sum_{k=1}^{m}(x_k - 1)(y_k z_k^{-1}\eta_k^{e_k} - 1)$$

where $y_k = \prod_{i<k} x_i^{-e_i a_{ik}} \prod_{k<j} x_j^{e_j b_{kj}}$, $z_k = \prod_{i<j, i\leq k}[x_i, x_j]^{x_k \partial_k(d_{ij})}$.

PROOF. By Lemma 4.2.1 we can write:

$$w = \prod_{i<j}[x_i, x_j]^{d_{ij}} \prod_{k=1}^{m}[x_k^{e_k}, \eta_k]\xi$$

where d_{ij} are as in Lemma 4.2.1, $\eta_k \in F', \xi \in F''\gamma_2^{p^e}\gamma_{n+2}$. Modulo $\mathbf{f}^2\mathbf{s}$ we have

$$[x_i, x_j]^{x_i^{\beta_i} \cdots x_m^{\beta_m}} - 1 \equiv x_m^{-\beta_m} \cdots x_i^{-\beta_i}([x_i, x_j] - 1)x_i^{\beta_i} \cdots x_m^{\beta_m}$$

$$\equiv ([x_i,x_j]-1)x_i^{\beta_i}\cdots x_m^{\beta_m} - \sum_{k=i}^{m}\beta_k(x_k-1)([x_i,x_j]-1)x_i^{\beta_i}\cdots x_m^{\beta_m}$$

$$\equiv ([x_i,x_j]-1)x_i^{\beta_i}\cdots x_m^{\beta_m} - \sum_{k=i}^{m}(x_k-1)([x_i,x_j]^{x_k\partial_k(x_i^{\beta_i}\cdots x_m^{\beta_m})}-1).$$

Hence, modulo **ffs**,

$$[x_i,x_j]^{d_{ij}} \equiv ([x_i,x_j]-1)d_{ij} - \sum_{k=i}^{m}(x_k-1)([x_i,x_j]^{x_k\partial_k(d_{ij})}-1).$$

Also, $[x_k^{e_k},\eta_k]-1 \equiv (x_k^{e_k}-1)(\eta_k-1) \equiv (x_k-1)(\eta_k^{e_k}-1)$ mod **ffs**; thence, mod $\mathbf{ffs}+\mathbf{f}^{n+2}+p^e\mathbf{f}$:

$$\begin{aligned}
w-1 \equiv & \sum_{i<j}((x_i-1)(x_j-1)-(x_j-1)(x_i-1))d_{ij}- \\
& -\sum_{i<j}\sum_{k=i}^{m}(x_k-1)([x_i,x_j]^{x_k\partial_k(d_{ij})}-1)+\sum_{k=i}^{m}(x_k-1)(\eta_k^{e_k}-1) \\
\equiv & \sum_{i<j}(x_i-1)(x_j^{e_jb_{ij}}-1)-\sum_{i<j}(x_j-1)(x_i^{e_ia_{ij}}-1)- \\
& -\sum_{i<j}\sum_{k=i}^{m}(x_k-1)([x_i,x_j]^{x_k\partial_k(d_{ij})}-1)+\sum_{k=i}^{m}(x_k-1)(\eta_k^{e_k}-1) \\
\equiv & \sum_{k=1}^{m}(x_k-1)\sum_{k<j}(x_j^{e_jb_{kj}}-1)+\sum_{k=1}^{m}(x_k-1)\sum_{i<k}(x_i^{-e_ia_{ik}}-1)- \\
& -\sum_{i<j}\sum_{k=i}^{m}(x_k-1)([x_i,x_j]^{x_k\partial_k(d_{ij})}-1)+\sum_{k=i}^{m}(x_k-1)(\eta_k^{e_k}-1) \\
\equiv & \sum_{k=1}^{m}(x_k-1)((y_k-1)-(z_k-1)+(\eta_k^{e_k}-1)) \\
\equiv & \sum_{k=1}^{m}(x_k-1)(y_kz_k^{-1}\eta_k^{e_k}-1)
\end{aligned}$$

where y_k, z_k are as in the statement of the Lemma. □

Observe that the elements y_k from the statement of Lemma 4.2.4 are connected with the original element $\prod_{i<j}[x_i,x_j]^{d_{ij}}$ by the relation

$$(\prod_{i<j}[x_i,x_j]^{d_{ij}})^2 \equiv \prod_{k=1}^{m}[x_k,y_k] \bmod [F',S]\gamma_2^{p^e}\gamma_{n+2}. \tag{3}$$

In the spirit of the lore, 2 is an exceptional prime in this context, too: the description of the fourth dimension subgroup mod 2^e to which we now turn, is different, in both proof and appearance, from the corresponding results with odd prime powers. Its complexity stems from the fact it must account for the counter-examples to the integral dimension subgroup conjecture, given by Rips, and N. Gupta. The result parallels Theorem 5.1 of Chapter IV in Gupta [1987].

Theorem 4.2.5. *$D_4(\mathbf{r}, 2^e) = KF^{2^{e+1}}\gamma_2^{2^e}\gamma_4 R$ where K is the subgroup consisting of elements $w = \prod_{i=1}^m x_i^{\alpha_i} \prod_{i<j}[x_i^{e_i}, x_j]^{a_{ij}}$ such that*
(i) $e_j \mid \binom{e_i}{2}a_{ij}$ mod 2^e, $\alpha_i \in \langle 4e_i, 2^{e+1}\rangle \cap \langle 2^e \rangle$;
(ii) $x_k^{\binom{\alpha_k}{2}} \prod_{i<k} x_i^{-e_i a_{ik}} \prod_{k<j} x_j^{e_j b_{kj}} \in \gamma_2^{e_k}\langle x_i^{2^e} \mid e_i < 2^e\rangle F^{2^{e+1}}\gamma_2^{2^e}\gamma_3 R$
$b_{kj} = e_k a_{kj}/e_j + \binom{e_k}{2} a_{kj}/e_j(x_k - 1)$.

PROOF. As noted in Chapter II, to deduce that the dimension subgroup is contained in the proposed subgroup it will suffice to prove that $D_4(\mathbf{fr}, 2^e)$ is contained in it. So let

$$\prod_{i=1}^m x_i^{\alpha_i} \prod_{i<j}[x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fr} + \mathbf{f}^4 + 2^e\mathbf{f} \subseteq \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}.$$

It follows that

$$\sum_{i=1}^m (x_i^{\alpha_i} - 1) + \prod_{i<j}[x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}.$$

As in the proof of Theorem 4.2.2 we have $x_i^{\alpha_i} - 1 \in \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}$. Therefore $\zeta - 1 = \prod_{i<j}[x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}$ which by Lemma 4.2.1 yields

$$\zeta = \prod_{i<j}[x_i^{e_i}, x_j]^{a_{ij}} \prod_{k=1}^m [x_k^{e_k}, \eta_k]\xi$$

with $\eta_k \in F', \xi \in \gamma_2^{2^e}\gamma_4, t(x_i)a_{ij} \equiv t(x_j)b_{ij}$ modulo $\mathbf{s} + \mathbf{f}^2 + 2^e ZF$ with $a_{ij} \in Z$, $b_{ij} = b_{ij,1} + b_{ij,2}(x_i - 1)$.

$$t(x_i)a_{ij} - t(x_j)(b_{ij,1} + b_{ij,2}(x_i - 1)) \in \mathbf{s} + \mathbf{f}^2 + 2^e ZF \qquad \text{so}$$

$$b_{ij} \equiv e_i a_{ij}/e_j + \binom{e_i}{2} a_{ij}/e_j(x_i - 1) \bmod 2^e ZF. \tag{4}$$

Thus (i) is satisfied and b_{ij} is as in (ii).

Also, $x_i^{\alpha_i} - 1 \in \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}$ implies that

$$\alpha_i + \binom{\alpha_i}{2}(x_i - 1) + \binom{\alpha_i}{3}(x_i - 1)^2 \in \mathbf{s} + \mathbf{f}^3 + 2^e ZF \subseteq \mathbf{s}^* + \mathbf{a} + \mathbf{f}^3 + 2^e ZF.$$

Therefore $\alpha_i \in 2^e Z$ and the relation above reduces to

$$\binom{\alpha_i}{2}(x_i - 1) + \binom{\alpha_i}{3}(x_i - 1)^2 \in \mathbf{s}^* + \mathbf{a} + \mathbf{f}^3 + 2^e ZF.$$

However, $x_i - 1$ is regular in $Z_{2^e}(F/F') \cong ZF/(\mathbf{a} + 2^e ZF)$, so

$$\binom{\alpha_i}{2} + \binom{\alpha_i}{3}(x_i - 1) \in t(x_i)ZF + \mathbf{f}^2 + 2^e ZF.$$

Thus $\binom{\alpha_i}{2}, \binom{\alpha_i}{3} \in \langle e_i, 2^e \rangle$ i.e. $\alpha_i \in \langle 2e_i, 2^{e+1} \rangle$. We shall henceforth assume $e_i < 2^e$ since otherwise $x_i^{\alpha_i} \in F^{2^{e+1}}$. $\alpha_i \in \langle 2e_i, 2^{e+1} \rangle$ implies

$$\binom{\alpha_i}{3}(x_i - 1) \in \mathbf{s} + \mathbf{f}^2 + 2^e\mathbf{f} = \mathbf{r} + \mathbf{f}^2 + 2^e\mathbf{f}$$

so that $\binom{\alpha_i}{3}(x_i - 1)^2 \in \mathbf{r} + \mathbf{f}^3 + 2^e\mathbf{f}$ whence, further, $\binom{\alpha_i}{2}(x_i - 1) \in \mathbf{s} + \mathbf{f}^3 + 2^e ZF$. It follows that $\beta_i \equiv 2e_i k \bmod 2^e$ where β_i denotes $\binom{\alpha_i}{2}$ and $k \in Z$. Observe that $\binom{2e_i k}{2}(x_i - 1)^2 \in \mathbf{r} + \mathbf{f}^3 + 2^e ZF$ and so modulo this ideal we have $2e_i k(x_i - 1) \equiv x_i^{2e_i k} - 1$. Observe, further, that $x_i^{2e_i k} - 1 \equiv x_i^{\beta_i} - 1$ modulo the same ideal (because $e_i < 2^e$, so $x_i^{2^e} - 1 \in \mathbf{r} + \mathbf{f}^3 + 2^e\mathbf{f}$ by Theorem 4.2.2); thus we see that modulo $\mathbf{r} + \mathbf{f}^3 + 2^e\mathbf{f}$, $\beta_i(x_i - 1) \equiv x_i^{\beta_i} - 1$. Therefore in particular

$$(x_i - 1)(x_i^{\beta_i} - 1) \equiv \beta_i(x_i - 1)^2 \bmod \mathbf{fr} + \mathbf{f}^4 + 2^e\mathbf{f}.$$

Since $x_i^{\alpha_i} - 1 \in \mathbf{fs} + \mathbf{f}^4 + 2^e\mathbf{f}$ it follows that $\prod_{i<j}[x_i, x_j]^{d_{ij}} - 1$ belongs to that ideal, too, which by Lemma 4.2.1 and Lemma 4.2.4 implies that modulo $\mathbf{ffs}$ this element assumes the form

$$\sum_{k=1}^{m}(x_k - 1)(y_k z_k^{-1} \eta_k^{e_k} - 1)$$

with y_k, z_k, η_k as in Lemma 4.2.4. Therefore we have

$$\sum_{k=1}^{m}(x_k-1)(x_k^{\beta_k}y_kz_k^{-1}\eta_k^{e_k}-1)\in \mathbf{fr}+\mathbf{ffs}+\mathbf{f}^4+2^e\mathbf{f}.$$

Hence $x_k^{\beta_k}y_kz_k^{-1}\eta_k^{e_k}-1\in \mathbf{r}+\mathbf{fs}+\mathbf{f}^3+2^e\mathbf{f}$. Therefore, as $\mathbf{fr}\subseteq\mathbf{fs}$, there exist $u_k\in R$ such that

$$x_k^{\beta_k}y_kz_k^{-1}\eta_k^{e_k}u_k-1\in \mathbf{fs}+\mathbf{f}^3+2^e\mathbf{f}\subseteq \mathbf{r}+\mathbf{f}^3+2^e\mathbf{f}.$$

Hence $x_k^{\beta_k}y_kz_k^{-1}\eta_k^{e_k}\in D_3(\mathbf{r},2^e)$ i.e. $x_k^{\beta_k}y_kz_k^{-1}\in\gamma_2^{e_k}D_3(\mathbf{r},2^e)$. Since $\partial_k(\mathbf{s}+\mathbf{f}^2+2^e\mathbf{f})\subseteq e_kZF+\mathbf{f}+2^eZF$ the relations between d_{ij}, a_{ij}, b_{ij} imply

$$\partial_k(d_{ij})\equiv\begin{cases} t(x_i)\partial_k a_{ij} & i\neq k\\ t(x_j)\partial_k b_{ij} & i=k\end{cases}\mod e_kZF+\mathbf{f}+2^eZF.$$

Therefore,

$$[x_i,x_j]^{\partial_k(d_{ij})}\equiv\begin{cases}[x_i,x_j]^{t(x_i)\partial_k(a_{ij})} & i\neq k\\ [x_i,x_j]^{t(x_j)\partial_k(b_{ij})} & i=k\end{cases}\mod \gamma_2^{e_k}\gamma_2^{2^e}\gamma_3.$$

It follows that these elements are congruent to 1 modulo $\gamma_2^{e_k}\gamma_2^{2^e}\gamma_3R$; consequently, z_k, being a product of such elements, is in $\gamma_2^{e_k}\gamma_2^{2^e}\gamma_3R$. Hence $x_k^{\beta_k}y_k\in\gamma_2^{e_k}D_3(\mathbf{r},2^e)$. We may assume that b_{ij} is actually equal to the expression in (ii), not only congruent to it modulo 2^e as in (4), since $x_j^{e_j2^e}\in\gamma_2^{2^e}R\subseteq D_3(\mathbf{r},2^e)$.

Conversely, let $w=\prod_{i=1}^m x_i^{\alpha_i}\zeta$, where $\zeta=\prod_{i<j}[x_i^{e_i},x_j]^{a_{ij}}$, be an element satisfying the conditions from the statement of the theorem. These conditions immediately imply that $\zeta-1\in\mathbf{fs}+\mathbf{f}^4+2^e\mathbf{f}$. Further, the requirements on the α_i's assure that $x_i^{\alpha_i}-1\equiv(x_i-1)(x_i^{\beta_i}-1)$ modulo $\mathbf{r}+\mathbf{f}^4+2^e\mathbf{f}$, where as before β_i stands for $\binom{\alpha_i}{2}$. Therefore by Lemma 4.2.4, and since $\mathbf{ffs}\subseteq\mathbf{r}+\mathbf{f}^4+2^e\mathbf{f}$, we obtain

$$w-1\equiv\sum_{i=1}^{m}(x_i-1)(x_i^{\beta_i}y_iz_i^{-1}\eta_i^{e_i}-1)\mod \mathbf{r}+\mathbf{f}^4+2^e\mathbf{f}.$$

Observe that $(x_i-1)(\gamma_2^{e_i}-1)\equiv(x_i^{e_i}-1)(\gamma_2-1)\equiv 0$ modulo $\mathbf{r}+\mathbf{f}^4+2^e\mathbf{f}$. We saw above that $z_i\in\gamma_2^{e_i}\gamma_2^{2^e}\gamma_3R$, so by the preceding remark it follows that

$(x_i - 1)(z_i^{-1} - 1) \in \mathbf{r} + \mathbf{f}^4 + 2^e\mathbf{f}$. Finally, by the hypothesis $x_i^{\beta_i} y_i \in \gamma_2^{e_i} D_3(\mathbf{r}, 2^e)$ hence $(x_i - 1)(x_i^{\beta_i} y_i - 1) \in \mathbf{r} + \mathbf{f}^4 + 2^e\mathbf{f}$, which proves that $w - 1$ belongs to $\mathbf{r} + \mathbf{f}^4 + 2^e\mathbf{f}$. The proof is now complete. □

The description given in Theorem 4.2.5 does not seem practical; however, a look at the conditions of the theorem reveals that they could be simplified. We have the following:

Corollary 4.2.6. *$D_4(R, 2^e) = KF_{4,2^e}R$ where K is the subgroup consisting of the elements $w = \prod_{i=1}^m x_i^{\alpha_i} \prod_{i<j} [x_i^{e_i}, x_j]^{a_{ij}}$ such that*
(i) $e_j \mid \binom{e_i}{2} a_{ij} \bmod 2^e$, $\alpha_i \in \langle 4e_i, 2^{e+1} \rangle \cap \langle 2^e \rangle$;
(ii) $x_k^{\binom{\alpha_k}{2}} \prod_{i<k} x_i^{-e_i a_{ik}} \prod_{k<j} x_j^{e_k a_{kj}} \in \gamma_2^{e_k} \langle x_i^{2^e} \mid e_i < 2^e \rangle F^{2^{e+1}} \gamma_2^{2^e} \gamma_3 R$.

PROOF. $x_j^{e_j b_{kj}} \equiv x_j^{e_k a_{kj}}$, modulo $F'^{e_k} D_3(R, 2^e)$. □

Observe that if we let e tend to infinity, the exponent of the group under consideration being fixed, the dimension subgroup mod 2^e will be equal to the integral dimension subgroup. Thus this description should reflect the existence of a counter-example to the integral dimension subgroup conjecture. And in fact we can use it to narrow the search for presentations of such an example. Suppose that in the pre-abelian presentation of a 4-generated group G, we have $x_i^{e_i} = \xi_i$ where $\xi_i = \prod_{j<k} [x_j, x_k]^{t_{ijk}}$. Suppose that we are looking for a presentation of a class 3 group such that in that group element

$$\prod_{1<i<j} [x_i^{e_i}, x_j]^{a_{ij}}$$

is nontrivial and belongs to the fourth dimension subgroup. The corollary tells us that the following conditions should be fulfilled:

$$\text{if } e_i = e_j, \text{ then } a_{ij} \text{ is even };$$

$$x_3^{e_2 a_{23}} x_4^{e_2 a_{24}} \in F'^{e_2} \gamma_3(F) R;$$

$$x_2^{-e_2 a_{23}} x_4^{e_3 a_{34}} \in F'^{e_3} \gamma_3(F) R;$$

$$x_2^{-e_2 a_{24}} x_3^{-e_3 a_{34}} \in F'^{e_4}\gamma_3(F)R.$$

The conditions can be reformulated as follows:

$$e_i = e_j \Rightarrow 2 \mid a_{ij};$$

$$\xi_3^{e_2 a_{23}/e_3} \xi_4^{e_2 a_{24}/e_4} \in F'^{e_2}\gamma_3(F)R;$$

$$\xi_2^{-a_{23}} x_4^{e_3 a_{34}/e_4} \in F'^{e_3}\gamma_3(F)R;$$

$$\xi_2^{-a_{24}} \xi_3^{-a_{34}} \in F'^{e_4}\gamma_3(F)R.$$

These relations can be read in terms of the t_{ijk}; thus we obtain systems of congruences:

$$t_{3jk} e_2 a_{23}/e_3 + t_{4jk} e_2 a_{24}/e_4 \equiv 0 \bmod e_2;$$

$$-t_{2jk} a_{23} + t_{4jk} e_3 a_{34}/e_4 \equiv 0 \bmod e_3;$$

$$-t_{2jk} a_{24} - t_{3jk} a_{34} \equiv 0 \bmod e_4.$$

Let us impose the relations $\gamma_4 = 1$ and $[x_i, x_j, x_k] = 1$ for all i, j, k such that they are not all distinct, on the group. There are enough coefficients t_{ijk}, so that we can try to simplify the systems above by putting $t_{ijk} = 0$ if $j \neq 1$. Further, because of the relations imposed, we can also choose that $t_{i1i} = 0$. The systems above will then read:

$$\begin{aligned} t_{312} e_2 a_{23}/e_3 + t_{412} e_2 a_{24}/e_4 &\equiv 0 \bmod e_2 \\ t_{412} e_3 a_{34}/e_4 &\equiv 0 \bmod e_3 \\ -t_{312} a_{34} &\equiv 0 \bmod e_4 \end{aligned}$$

$$\begin{aligned} t_{413} e_2 a_{24}/e_4 &\equiv 0 \bmod e_2 \\ -t_{213} a_{23} + t_{413} e_3 a_{34}/e_4 &\equiv 0 \bmod e_3 \\ -t_{213} a_{24} &\equiv 0 \bmod e_4 \end{aligned}$$

$$t_{314}e_2a_{23}/e_3 \equiv 0 \bmod e_2$$
$$-t_{214}a_{23} \equiv 0 \bmod e_3$$
$$-t_{214}a_{24} - t_{314}a_{34} \equiv 0 \bmod e_4$$

From these congruences it will follow that the element specified above belongs to the fourth dimension subgroup; however, we have to make sure that it is not trivial. A necessary condition for this is that e_3 does not divide t_{312} or t_{213}.

Theorem 4.2.7. *Let G be the four-generated group given by the following relations:*
(1) $x_i^{e_i} = \prod_{k\neq i}[x_1, x_k]^{t_{i1k}}$; (2) $[x_i, x_j, x_k] = 1$ if i, j, k are distinct; (3) $\gamma_4 = 1$.
If $g = [x_2^{e_2}, x_3]^{a_{23}}[x_2^{e_2}, x_4]^{a_{24}}[x_3^{e_3}, x_4]^{a_{34}}$ and the conditions above on the t_{ijk} and a_{ij} are satisfied, then $g \in D_4(G, p^e)$. In G, $g \neq 1$ only if e_3 does not divide $t_{312}a_{23}$ or $t_{213}a_{23}$.

Thus the search for the presentation of the counter-example has been narrowed down quite a bit. It is possible to produce counter-examples of the Rips type using these remarks.

The results above yield a description of the fourth modular dimension subgroup. Such as it is, it is indeed the best possible; for, the Rips example must be accounted for. However, it is known (Passi) that 2-generated groups will not fail the integral dimension subgroup conjecture for $n = 4$. One might hope for such tameness in the modular case, too. But this is not the case. Yet, in the 2-generated case the description obtained assumes a significantly nicer form (a reflexion, certainly, of the absence of the integral component). We shall now concentrate on the 2-generated case. The basic observation here is the following:

Observation. *In the notation of Theorem 4.2.5, $[x, y]^{e_1 a_{12}} \equiv 1$ modulo $F_{4,2^e}R$.*

To see this note that if G is 2-generated then $F'^{e_i} \subseteq \gamma_3(F)R$ so that the condition (ii) of Theorem 2 will read $y^{\binom{\beta}{2}}x^{-e_1a_{12}} \in KF_{3,2^e}$ where $K = \langle x_i^{2^e} \mid$

$e_i < 2^e\rangle$. (Here of course $x_1 = x$, $x_2 = y$.) Therefore

$$[x^{e_1 a_{12}}, y] \in [K, y] F_{4,2^e};$$

but if $e_1 \geq 2^e$ then $K \subseteq \langle y \rangle$, and if e_1 divides 2^{e-1} then

$$[x^{2^e}, y] = [x, y]^{2^e} [x, y, x]^{2^{e-1} a} \in F_{4,2^e}.$$

Hence, Theorem 4.2.5 implies

Corollary 4.2.8. *Let F be free of rank 2 and $G = F/R$. Then*

$$D_4(R, 2^e) = \langle x_i^{2^e} \mid e_i < 2^{e-1} \text{ and } x_i^{2^{e-1}} \in \langle x_1^{2^e}, x_2^{2^e} \rangle F_{3,2^e} R \rangle F_{4,2^e} R.$$

Note that if $e_i = 1$ then repeated application of the relation $x_i = \xi_i$ will show that $x_i \in \gamma_4$ so since we work modulo γ_4 we may drop the x_i from the presentation. Consequently we may assume that all e_is are at least 2. This immediately leads to the conclusion that for $e = 2$ we have $D_4(G, 4) = G_{4,4}$ if G is 2-generated.

REMARK. Note that our description makes it clear that $D_3(G, 4) = G_{3,4}$ for 2-generated groups: for if there is to be an example like that, it must have the relation $x^2 = [x, y]^a$; but then $x^4 = 1$ and similarly for y, so that $G_{3,4} = 1 \Rightarrow D_3(G, 4) = 1$. That is why the examples given by Sandling to show that $D_3(G, 2^e) \neq G_{3,2^e}$ become 3-generated for $e = 2$. In terms of the description of the third modular dimension subgroup it is not difficult come up with such an example: let $G = \langle x, y, z \mid x^2 = [y, z], y^4 = z^4 = \gamma_3 = 1 \rangle$.

Suppose G is 2-generated and $e = 3$; then if there is a counter-example it must have relations of the form $x^{e_1} = [x, y]^a \xi_1$, $y^{e_2} = [x, y]^b \xi_2$, where $\xi_1, \xi_2 \in \gamma_3$, and at least one of e_i is 2. Say $e_2 = 2$. Then clearly $y^8 \in \gamma_4$ and hence such a group is not really a counter-example. Similar considerations will show that such a conclusion is valid also for the fourth dimension subgroup modulo 16. Thus we have

Corollary 4.2.9. *For a 2-generated group G, $D_4(G, 2^e) = G_{4,2^e}$ provided $e < 5$.*

However if $e = 5$ there exist groups which fail the statement. For example, let $G = \langle x, y \mid x^8 = [x, y, x], y^{16} = [x, y, y], F_{4,32} = 1\rangle$. In this group $x^{32} \neq 1$ but $x^{32} \in D_4$ by Corollary 4.2.8 since $x^8 \in \gamma_3 R$. To demonstrate that $x^{32} = [x, y, x]^4$ is not equal to 1 in G, look at the group U of units of the ring $Z[x, y]/J$ where J is the ideal generated by Δ^4, $32\Delta^2$, 64Δ, and polynomials $(1+x)^8 - (x, y, x) - 1$ and $(1 + y)^{16} - (x, y, y) - 1$. U clearly satisfies the relations of G. But in U, $(1 + x)^{32} - 1 \equiv 4(x, y, x)$ so it will suffice to show that $4(x, y, x) \not\equiv 0$ modulo J. Suppose $4(x, y, x) \in J$. Then $4(x, y, x) \equiv$

$$\sum_i a_i((1+x)^8 - (x, y, x) - 1)b_i + \sum_i c_i((1+y)^{16} - (x, y, y) - 1)d_i$$

modulo $\Delta^4 + 32\Delta^2 + 64\Delta$. We can look at this relation modulo Δ^3; this tells us that the sum of the terms of degree one is in 64Δ; and that the degree two terms add up to an element of $32\Delta^2$. Consequently the relation above reduces to

$$4(x, y, x) \equiv a(\binom{8}{3}x^3 - (x, y, x) + b(\binom{16}{3}y^3 - (x, y, y)) \bmod \Delta^4 + 32\Delta^2 + 64\Delta$$

where a and b are integers. Then $a \equiv 0$ modulo 8 and $a + 4$ must be divisible by 32; but these conditions are contradictory, proving that $4(x, y, x)$ is not an element of J.

Thus we see a clear distinction between the behavior of the fourth dimension subgroups in the modular as opposed to the integral case even for 2-generated groups.

§4.3. $p > 2$ and $n \leq p + 2$

We can now concentrate on odd primes. Some further observations are required as the dimension grows: the two lemmas that follow will be very important in the final arguments. (They should be compared with the computations carried out in Gupta and Tahara [1985].)

Lemma 4.3.1. *Suppose* $d_{ij} \equiv t(x_i)a_{ij} \equiv t(x_j)b_{ij}$ *modulo* $\mathbf{s} + \mathbf{f}^n + p^e ZF$ *where* $n \le p$, $a_{ij} \in Z$ *and* $b_{ij} \in Z$ *if* $e_i = e_j$. *Then for* $k \ge i$

$$[x_k, [x_i, x_j]^{x_k \partial_k(d_{ij})}] \in \gamma_2^{p^e} \gamma_{n+2} R.$$

PROOF. We need to establish some properties of $\partial_k(d_{ij})$. Since $\partial_k(\mathbf{s}) \subseteq \mathbf{s} + e_k ZF$, $\partial_k(\mathbf{f}^n) \subseteq \mathbf{f}^{n-1}$, it follows that if $k > i$ or $e_i = e_j$ then $\partial_k(d_{ij}) \equiv 0$ modulo $\mathbf{s} + \mathbf{f}^{n-1} + e_i ZF + p^e ZF$, because $\partial_k(t(x_i)a_{ij}) = 0$ if $k > i$ and $\partial_i(t(x_j)b_{ij}) = 0$ if $e_i = e_j$. Further, by the relations used in the proof of Corollary 4.2.3, we see that modulo $\mathbf{s} + \mathbf{f}^{n-1} + e_i ZF + p^e ZF$, $\partial_i(t(x_i)a_{ij}) \equiv a_{ij}\binom{e_i}{p}(p-1)(x_i - 1)^{p-2}$. Therefore $\partial_i(d_{ij}) \in \mathbf{s} + \mathbf{f}^{n-1} + e_i ZF + p^e ZF + (e_i/p)(x_i - 1)^{p-2} ZF$.
Now if $k > i$ or $e_i = e_j$ then modulo $\gamma_2^{p^e}\gamma_{n+2} R$

$$\begin{aligned}[x_k, [x_i, x_j]^{x_k \partial_k(d_{ij})}] &\equiv [x_k, [x_i, x_j]^{e_k x_k v}] \\ &\equiv [x_k^{e_k}, [x_i, x_j]^{x_k v}] \equiv 1.\end{aligned}$$

And if $e_i > e_j$ then modulo the same subgroup

$$\begin{aligned}[x_i, [x_i, x_j]^{x_i \partial_i(d_{ij})}] &\equiv [x_i, [x_i, x_j]^{(e_i/p)(x_i-1)^{p-2}u + e_i x_i v}] \\ &\equiv [x_i, [x_i, x_j]^{(e_i/p)(x_i-1)^{p-2}u}] \\ &\equiv [[x_j, x_i]^{-(e_i/p)(x_i-1)^{p-2}u}, x_i]^{-1} \\ &\equiv [[x_j, \underbrace{x_i, \ldots, x_i}_{p-1}]^{-e_i u/p}, x_i]^{-1} \\ &\equiv [[x_j^{e_j}, \underbrace{x_i, \ldots, x_i}_{p-1}]^{-e_i u/(e_j p)}, x_i]^{-1} \\ &\equiv [x_j^{e_j}, \underbrace{x_i, \ldots, x_i}_{p}]^{u'} \\ &\equiv [\xi_j, \underbrace{x_i, \ldots, x_i}_{p}]^{u'} \equiv 1\end{aligned}$$

because $n + 2 \le p + 2$. □

Lemma 4.3.2. *If $n \le p-1$ then $[D_{n+2}(\mathbf{fs}, p^e), F] \subseteq [F', S]\gamma_2^{p^e}\gamma_{n+3}$.*

PROOF. Let $x_1^{\alpha_1}\cdots x_m^{\alpha_m}\zeta - 1 \in \mathbf{fs} + \mathbf{f}^{n+2} + p^e\mathbf{f}$ where $\zeta = \prod_{i<j}[x_i, x_j]^{d_{ij}}$. Then using endomorphisms as in the proof of Theorem 4.2.2, we obtain that $p^e \mid \alpha_i$ (also $\alpha_i \in \langle pe_i, p^{e+1}\rangle$ if $n = p-1$) and $\zeta - 1 \in \mathbf{fs} + \mathbf{f}^{n+2} + p^e\mathbf{f}$. By relations (1) and (2) of Lemma 4.2.1 it follows that

$$\begin{aligned}
[[x_i, x_j]^{d_{ij}}, x_k] &\equiv [x_i, x_j, x_k]^{d_{ij}} \\
&\equiv [x_k, x_i]^{(x_j-1)d_{ij}}[x_j, x_k]^{(x_i-1)d_{ij}} \\
&\equiv 1
\end{aligned}$$

modulo $[F', S]\gamma_2^{p^e}\gamma_{n+3}$. So $[\zeta, x_k]$ is in this subgroup. The conditions on the α's assure that $[x_i, x_k]^{\binom{\alpha_i}{j}(x_i-1)^{j-1}} \in \gamma_2^{p^e}$ if $j < p$; if $j = p$ then $n = p-1$ and

$$\begin{aligned}
[x_i, x_k]^{\binom{\alpha_i}{p}(x_i-1)^{p-1}} &\equiv [x_k, \underbrace{x_i, \ldots, x_i}_{p}]^{e_i a} \\
&\equiv [x_k, \underbrace{x_i, \ldots, x_i}_{p-1}, x_i^{e_i}]^a \equiv 1
\end{aligned}$$

modulo $[F', S]\gamma_2^{p^e}\gamma_{p+2}$. In any case, these observations show that $[x_i^{\alpha_i}, x_k]$ belongs to $[F', S]\gamma_2^{p^e}\gamma_{n+3}$. □

We are now ready to describe the modular dimension subgroups of dimension at most $p+2$.

Theorem 4.3.3. *If p is odd, $D_{p+1}(\mathbf{r}, p^e) = \langle x_i^{p^e} \mid e_i < p^e\rangle F^{p^{e+1}}\gamma_2^{p^e}\gamma_{p+1}R$.*

PROOF. Suppose $x_1^{\alpha_1}\cdots x_m^{\alpha_m}\prod_{i<j}[x_i, x_j]^{d_{ij}} - 1 \in D_{p+1}(\mathbf{fr}, p^e)$. Then $x_i^{\alpha_i} - 1 \in D_{p+1}(\mathbf{fs}, p^e)$ and hence

$$\alpha_i + \binom{\alpha_i}{2}(x_i - 1) + \cdots + \binom{\alpha_i}{p}(x_i - 1)^{p-1} \in \mathbf{s} + \mathbf{f}^p + p^e ZF,$$

so that $\alpha_i \in \langle p^e\rangle$ and the relation reduces to $\binom{\alpha_i}{p}(x_i - 1)^{p-1} \in \mathbf{s} + \mathbf{f}^p + p^e ZF$. This yields $\binom{\alpha_i}{p} \in \langle e_i, p^e\rangle$. Thus $\alpha_i \in \langle p^e\rangle \cap \langle pe_i, p^{e+1}\rangle$, so $x_i^{\alpha_i}$ belongs to $\langle x_j^{p^e} \mid$

$e_j < p^e\rangle F^{p^{e+1}}$. We conclude that $\prod_{i<j}[x_i, x_j]^{d_{ij}} - 1 \in \mathbf{fr} + \mathbf{f}^{p+1} + p^e\mathbf{f}$. By Lemma 4.2.1 and Lemma 4.2.4 we obtain

$$\sum_{k=1}^{m}(x_k - 1)(y_k z_k^{-1} - 1) \in \mathbf{fr} + \mathbf{ffs} + \mathbf{f}^{p+1} + p^e\mathbf{f}$$

whence $y_k z_k^{-1} - 1 \in \mathbf{r} + \mathbf{fs} + \mathbf{f}^p + p^e\mathbf{f}$. So there exist $u_k \in R$ such that $y_k z_k^{-1} u_k \in D_p(\mathbf{fs}, p^e)$, and then $[x_k, y_k z_k^{-1} u_k] \in [F', S]\gamma_2^{p^e}\gamma_{p+1}$ by Lemma 4.3.2, so that $[x_k, y_k z_k^{-1}]$ belongs to $\gamma_2^{p^e}\gamma_{p+1}R$. But Lemma 4.3.1 assures that $[x_k, z_k]$ is also in that subgroup, and thus the preceding relation implies $[x_k, y_k] \in \gamma_2^{p^e}\gamma_{p+1}R$. We now have by (4)

$$(\prod_{i<j}[x_i, x_j]^{d_{ij}})^2 \equiv \prod_{k=1}^{m}[x_k, y_k] \equiv 1 \bmod \gamma_2^{p^e}\gamma_{p+1}R$$

which, since p is odd, shows that $\prod_{i<j}[x_i, x_j]^{d_{ij}}$ belongs to $\gamma_2^{p^e}\gamma_{p+1}R$. Therefore, $D_{p+1}(\mathbf{fr}, p^e)$ is contained in the subgroup from the statement of the theorem. But then by the argument from the proof of Theorem 4.2.2, it follows that $D_{p+1}(\mathbf{r}, p^e)$ is contained in that subgroup, too. This is all we need to show—the converse inclusion being well-known. □

Theorem 4.3.4. *For odd p,*

$$D_{p+2}(\mathbf{r}, p^e) = \langle x_i^{p^e} \mid x_i^{p^{e-1}} \in F^{p^e}\gamma_3 R\rangle F^{p^{e+1}}\gamma_2^{p^e}\gamma_{p+2}R.$$

PROOF. Suppose $x_1^{\alpha_1} \cdots x_m^{\alpha_m}\zeta - 1 \in \mathbf{fr} + \mathbf{f}^{p+2} + p^e\mathbf{f}$ where $\zeta = \prod_{i<j}[x_i, x_j]^{d_{ij}}$. Then as before $p^e \mid \alpha_i$ and $\binom{\alpha_i}{p}$, $\binom{\alpha_i}{p+1}$ belong to $\langle e_i, p^e\rangle$, whence $\binom{\alpha_i}{p+1}(x_i - 1)^{p+1} \in \mathbf{fr} + \mathbf{f}^{p+2} + p^e\mathbf{f}$ so that modulo this ideal $x_i^{\alpha_i} - 1 \equiv \binom{\alpha_i}{p}(x_i - 1)^p$. Since $\zeta \in D_{p+2}(\mathbf{fs}, p^e)$, Lemma 4.2.4 and the observations we just made show that (in the notation of Lemma 4.2.4, and with $\beta_k = \binom{\alpha_k}{p}$):

$$\sum_{k=1}^{m}\beta_k(x_k - 1)^p + \sum_{k=1}^{m}(x_k - 1)(y_k z_k{-1} - 1) \in \mathbf{fr} + \mathbf{ffs} + \mathbf{f}^{p+2} + p^e\mathbf{f}.$$

Therefore $\beta_i(x_i-1)^{p-1}+y_i z_i^{-1} \in \mathbf{r}+\mathbf{fs}+\mathbf{f}^{p+1}+p^e\mathbf{f}$. However, $\beta_i(x_i-1)$ belongs to $\mathbf{s}+\mathbf{f}^3+p^e\mathbf{f}$ by the conditions above so that $\beta_i(x_i-1)^{p-1} \in \mathbf{fs}+\mathbf{f}^{p+1}+p^e\mathbf{f}$ and it follows that $y_i z_i^{-1}-1 \in \mathbf{r}+\mathbf{fs}+\mathbf{f}^{p+1}+p^e\mathbf{f}$. Just as in the proof of Theorem 4.3.3, using Lemma 4.3.1 and Lemma 4.3.2, we see that $\zeta \in \gamma_2^{p^e}\gamma_{p+2}R$. Therefore we are left with the relation

$$\beta_i(x_i-1)^{p-1} \in \mathbf{r}+\mathbf{f}^{p+1}+p^e\mathbf{f}.$$

This is equivalent to that element belonging to the ideal $\mathbf{r}\cap\mathbf{f}^{p-1}+\mathbf{f}^{p+1}+p^e\mathbf{f}$. Consequently,

$$\beta_i(x_i-1)^{p-1} \equiv \sum_{j=1}^{m} a_j(x_j^{e_j}\xi_j-1)b_j \mod \mathbf{f}^{p+1}+p^e\mathbf{f}$$

with $a_j b_j \in \mathbf{f}^{p-2}$. Canceling from left and right we obtain

$$\beta_i(x_i-1) \equiv \sum_{j=1}^{m} p_j(x_j^{e_j}\xi_j-1)q_j \mod \mathbf{f}^3+p^e\mathbf{f}$$

that is, $\beta_i(x_i-1) \in \mathbf{r}+\mathbf{f}^3+p^e\mathbf{f}$. It is straightforward to check that $\binom{\beta_i}{2}(x_i-1)^2$ is in the same ideal (because $\beta_i \in \langle e_i, p^e\rangle$) and therefore $x_i^{\beta_i}$ belongs to $D_3(\mathbf{r},p^e) = F^{p^e}\gamma_3 R$. Since $p^{e-1} \mid \beta_i$ it follows that $x_i^{\alpha_i} \in \langle x_j^{p^e} \mid x_j^{p^{e-1}} \in F^{p^e}\gamma_3 R\rangle F^{p^{e+1}}$. Hence, $D_{p+2}(\mathbf{r},p^e)$ is contained in the subgroup from the statement of the theorem; since the converse inclusion is evident—Theorem 4.3.4 is proved. □

Chapter V

Dimensions less than 2p

The integral dimension subgroup problem has been solved in the affirmative for metabelian odd p-groups by Professor Narain Gupta. While this result does not settle the problem of describing the dimension subgroups modulo odd prime powers of metabelian groups — its proof can be modified so as to yield at least a partial solution to that problem. The modular result will give the solution to the integral problem, and thus may be thought its generalization.

§5.1. A modification of Sjögren's lemma

We shall prove a special case of Sjogren's lemma which will suffice for the application we intend to make. The proof is in this case a little simpler and that is why we shall only deal with this situation.

We need to develop some notation for this section; let $A = A_0 \oplus \cdots \oplus A_n$ be the (n+1)-truncated free ring on the set of generators $\{a_1, a_2 \ldots\}$, that is $A = Z[a_1, a_2 \ldots]/\Delta^{n+1}$. Also we define A^* to be the (n+1)-truncated free ring on the set $\{a_1, a_2 \ldots, b_1, b_2 \ldots\}$ of free generators. Let $S_k = A_k \oplus \ldots \oplus A_n$ and let S_k^* be defined analogously.

Consider the element

$$\frac{(-1)^{l+1}}{l}((a_1+1)\cdots(a_m+1)-1)^l \qquad (*)$$

of $A \otimes Q$; we define

$$\psi_l(a_1 a_2 \cdots a_m) = [\frac{(-1)^{l+1}}{l}((a_1+1)\cdots(a_m+1)-1)^l]$$

where $[\mu]$ denotes the component of μ which has positive degree in each a_i for $i = 1, \dots, m$ (the component with full support). We extend ψ_l to all of A^* by linearity and

$$\psi_l(c_1 c_2 \cdots c_m) = \psi_l(a_1 a_2 \cdots a_m)|_{a_1=c_1,\dots,a_m=c_m}.$$

Finally, we let $\phi_l = \psi_1 + \cdots + \psi_l$.

These maps are defined with the following motivation: we seek functions ϕ_k such that the statement of Lemma 5.1.4 is valid; i.e. ϕs should have a certain commutativity property crucial for the proof of the result we require. It turns out that this commutativity property determines that the maps we seek should look like those defined above (G.E. Wall, personal communication to Professor Narain Gupta).

Further, let us define:

$$\mathbf{r}(k) = \sum_{i+j=k-1} \mathbf{f}^i \mathbf{r} \mathbf{f}^j$$

$$R(1) = R, R(k+1) = [R(k), F]$$

$$D(k, m, p^e) = F' \cap (1 + \mathbf{r}(k) + \mathbf{a}^2 + \mathbf{f}^m + p^e \mathbf{f}).$$

This section's principal result is the following:

Lemma 5.1.1. *$F'^{p^e} \gamma_{q+2}(F) \cap D(2, q+3, p^e) \subseteq D(3, q+3, p^e) R(2) F''$, for $q \geq 1$.*

The proof of this lemma will rely on several auxiliary statements.

Lemma 5.1.2. *If $u \in S_2^*$ then $\phi_2(u)$ is a Lie element modulo S_3^*.*

PROOF. Clearly $\psi_1(a_1 a_2) = a_1 a_2$; $\psi_2(a_1 a_2) = -\frac{1}{2}[(a_1 + a_2 + a_1 a_2)^2] \equiv -\frac{1}{2}(a_1 a_2 + a_2 a_1)$ modulo $-\frac{1}{2} S_3^*$. Therefore $\phi_2(a_1 a_2) \equiv -\frac{1}{2}(a_1, a_2)$ modulo $-\frac{1}{2} S_3^* = S_3^*$ because 2^{-1} exists in Z_{p^e}. □

Lemma 5.1.3. *If $u \in A^*_m$ is a Lie element, then $\phi_2(u) \equiv u \bmod S^*_{m+1}$. In particular, $\phi_2(u) = u$ whenever u is a Lie element of degree n.*

PROOF. Modulo S^*_{m+1}, Lie elements of degree m are generated by the left-normed Lie commutators; hence it suffices to prove the statement for these elements. Since $\psi_1(a_1 \cdots a_m) = a_1 \cdots a_m$, it suffices to prove that $\psi_2(u) \equiv 0$ modulo S^*_{m+1} whenever $u \in A_m$ is a left-normed Lie commutator. We use induction on m. The initial step, $m = 2$, is done since in Lemma 5.1.2 we showed that $\psi(a_1 a_2) \equiv -2^{-1}(a_1 a_2 + a_2 a_1) \bmod S^*_3$. The result for $m = 2$ obviously follows from this. Consider now $(a_1, \ldots, a_{m+1}) = (a_1, \ldots, a_m)a_{m+1} - a_{m+1}(a_1, \ldots, a_m)$. Let $c_1 \cdots c_m$ be a monomial from the linear expansion of $(a_1, \ldots, a_m)$; then:

$$\begin{aligned}\psi_2(c_1 \cdots c_m a_{m+1}) &= -2^{-1}[((1+c_1)\cdots(1+c_m)(1+a_{m+1}) - 1)^2] \\ &= -2^{-1}[((1+c_1)\cdots(1+c_m) - 1)(1+a_{m+1}) + a_{m+1})^2] \\ &= -2^{-1}[(c + ca + a)^2]\end{aligned}$$

where we have abbreviated $(1+c_1)\cdots(1+c_m) - 1$ to c and a_{m+1} to a. By symmetry, $\psi_2(ac_1 \cdots c_m) = -2^{-1}[(c + ac + a)^2]$. Thus, upon expanding the squares we obtain that $\psi_2((c_1 \cdots c_m, a)) = -2^{-1}[cca + caca + caa - acc - acac - aac]$. However, $[caa] \equiv 0$ modulo S^*_{m+2} since it must have full support (hence degree at least $m+1$) and a_{m+1} occurs twice. Similarly, the other components in which a_{m+1} appears twice are zero modulo S^*_{m+2}. By definition, $-2^{-1}c^2 a = \psi_2(c_1 \cdots c_m)a_{m+1}$ and $-2^{-1}(ac^2) = a_{m+1}\psi_2(c_1 \cdots c_m)$ so that modulo S^*_{m+2}

$$\psi_2((c_1 \cdots c_m, a_{m+1})) \equiv (\psi_2(c_1 \cdots c_m), a_{m+1})$$

and hence

$$\psi_2((a_1, \ldots a_m, a_{m+1})) \equiv (\psi_2((a_1, \ldots, a_m)), a_{m+1}).$$

Consequently, by the induction hypothesis, $\psi_2((a_1, \ldots, a_{m+1})) \equiv 0 \bmod S^*_{m+2}$. □

Lemma 5.1.4. *Let $u \in A_m^*$ be a linear sum of monomials of the form*

$$a_{i_1} \cdots a_{i_{t-1}} b_{i_t} a_{i_{t+1}} \cdots a_{i_m},$$

where $1 \le t \le m$, $a_{i_j} \in \{a_1, a_2, \ldots\}$ and $b_{i_t} \in \{b_1, b_2, \ldots\}$. Let $\delta : A^* \to A$ be the homomorphism given by $a_i^\delta = a_i$, $b_j^\delta = (1 + a_{j_1}) \cdots (1 + a_{j_{q_j}}) - 1$. Then $\delta\phi_2(u) = \phi_2\delta(u)$.

For the proof of this lemma, we observe that the proof given by Cliff and Hartley [1985] for the integral case actually applies to the modular case as well. We shall need the following result, too. Its proof very similar to that of the corresponding statement in the integral case (see Lemma 1.4 in Gupta [1987]).

Lemma 5.1.5. *Let $\mathbf{I}$ be the ideal of A^* generated by the elements (a_i, a_j). Then $\mathbf{I}^2$ is invariant under ϕ_2.*

Let F be the free group on $\{x_1, x_2, \ldots\}$ and let F^* be the free group on $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$. Let the augmentation ideals of the group rings ZF and ZF^* be denoted by $\mathbf{f}$ and $\mathbf{f}^*$ respectively. Then we have the homomorphisms $\theta : ZF/(\mathbf{f}^{n+1} + p^e ZF) \longrightarrow A$ and $\theta^* : ZF^*/(\mathbf{f}^{*n+1} + p^e ZF^*) \longrightarrow A^*$ induced by the bijections $x_i \to a_i$ and $y_i \to b_i$. We shall assume that the group G is given by the positive presentation $G = \langle x_1, x_2, \ldots \mid r_1, r_2, \ldots \rangle$. Define now the homomorphism $\beta : ZF^*/(\mathbf{f}^{*n+1} + p^e ZF^*) \longrightarrow ZF/(\mathbf{f}^{n+1} + p^e ZF)$ to be induced by the maps $x_i \to x_i$, $y_j \to r_j$; and finally define the homomorphism $\delta : A^* \longrightarrow A$ by $\delta(a_i) = a_i$ and $\delta(b_j) = \theta(r_j) - 1$. Then clearly we have the following commutative diagram.

$$\begin{array}{ccc}
ZF^*/\mathbf{f}^{*n+1} + p^e ZF^* & \xrightarrow{\beta} & ZF/\mathbf{f}^{n+1} + p^e ZF \\
\big\downarrow{\scriptstyle \theta^*} & & \big\downarrow{\scriptstyle \theta} \\
A^* & \xrightarrow{\delta} & A
\end{array}$$

PROOF OF LEMMA 5.1.1. Suppose that $w \in F_{n,p^e}$ is such that $w - 1 \in \mathbf{r}(2) + \mathbf{a}^2 + \mathbf{f}^{n+1} + p^e\mathbf{f}$, i.e. $w - 1 \equiv u + \Xi \bmod \mathbf{f}^{n+1} + p^e\mathbf{f}$ for some $u \in \mathbf{r}(2)$

and $\Xi \in \mathbf{a}^2$. Since $F^{p^{e+1}}\gamma_2^{p^e} \subseteq D_n(\mathbf{r}, p^e)$ we may assume that actually $w \in \gamma_n$. Modulo $\mathbf{f}^{n+1} + p^e\mathbf{f}$ we have

$$u \equiv u_2 + u_3 + \cdots + u_n$$

where u_q is linear sum of elements of the form

$$(z_1 - 1)\cdots(z_{t-1} - 1)(r - 1)(z_{t+1} - 1)\cdots(z_q - 1)$$

such that $z_i \in \{x_1, x_2, \ldots\}$, $r \in \{r_1, r_2, \ldots\}$. Let, for $z \in ZF$, $\bar{z}$ denote the coset $z + \mathbf{f}^{n+1} + p^e\mathbf{f}$. Then

$$\begin{aligned} \theta(\overline{w-1}) &= \theta(\bar{u}_2) + \cdots + \theta(\bar{u}_n) + \theta(\bar{x}) \\ &= \delta(v_2) + \cdots + \delta(v_n) + \delta(y) \end{aligned}$$

where $v_q \in A_q^*$ is a linear sum of monomials of the form $c_1 \cdots c_{t-1} b c_{t+1} \cdots c_q$ with $c_i \in \{a_1, a_2, \ldots\}$, $b \in \{b_1, b_2, \ldots\}$, and $y \in \mathbf{I}^2$. Since $\theta(\overline{w-1})$ is a Lie element in A_n, by Lemma 5.1.3 we have $\phi_2(\theta(\overline{w-1})) = \theta(\overline{w-1})$. Therefore lemmas 5.1.4 and 5.1.5 imply

$$\begin{aligned} \theta(\overline{w-1}) &= \phi_2\delta(v_2) + \cdots + \phi_2\delta(v_n) + \phi_2(\delta(y)) \\ &= \delta\phi_2(v_2) + \cdots + \delta\phi_2(v_n) + \delta(y') \end{aligned}$$

where $y' \in \mathbf{I}^2$. Then by Lemma 5.1.2 $\phi_2(v_2) = v_2' + v_2''$ where v_2' is a Lie element of A_3^* with one b-entry in each component and $v_2'' \in S_3^*$ is a sum of monomials with at least one b-entry. For $q \geq 3$, $\phi_2(v_q) \in S_3^*$ is a sum of such monomials. Therefore

$$\theta(\overline{w-1}) = \theta(\overline{f_2 - 1}) + \theta(\overline{u'}) + \theta(\Xi')$$

for some $f_2 \in R(2)$, $u' \in \mathbf{r}(2) + \mathbf{f}^{n+1} + p^e\mathbf{f}$ and $\Xi' \in \mathbf{a}^2$. Consequently,

$$w - 1 \equiv f_2 - 1 \bmod \mathbf{r}(3) + \mathbf{a}^2 + \mathbf{f}^{n+1} + p^e\mathbf{f}.$$

This proves Lemma 5.1.1. □

§5.2. The commutator structure

In this section we shall modify the method used to obtain the solution to the integral dimension subgroup problem for metabelian odd p-groups, to investigate the commutator structure of dimension subgroups modulo odd prime powers of metabelian groups. The principal result of this paragraph is the following:

Lemma 5.2.1. *Let $n > k > 2$. Then*

$$F' \cap (1 + \mathbf{r}(k) + \mathbf{ffa} + \mathbf{f}^n + p^e\mathbf{f}) = R(k)F''F'^{p^e}\gamma_n(F).$$

Before we proceed to prove this statement, we need some notation: let $\mathcal{R}(k) = ZF(R(k) - 1)$ and $\mathcal{R}^*(k) = Z(R(k) - 1)$. Let $\mathbf{x}(i) = (x_i^{e_i} - 1)ZF$ and $\mathbf{y}(i,j) = (x_i - 1)\mathbf{x}(i) + \mathbf{x}(j) + \mathbf{x}(j+1) + \cdots + \mathbf{x}(m)$. The following statement can be obtained from Lemma 2.3 in Gupta [1991], by relativizing the proof modulo p^e:

Lemma 5.2.2. *Let u be any element of $\mathbf{r}(k)$, $k > 1$. Then modulo $\mathbf{fa} + \mathcal{R}^*(k) + \mathbf{f}^n + p^e\mathbf{f}$ u has the form*

$$u = \sum_{i<j} v_{ij} + (g_{ij}^{-1} - 1) + y_{ij}$$

where $v_{ij} = (x_i^{e_i} - 1)(x_j - 1)p_{ij}$, $g_{ij} = [x_i^{e_i}, x_j]^{p_{ij}}$, with $p_{ij} \in \mathbf{f}^{k-2} + p^eZF$ and $y_{ij} \in \mathbf{y}(i,j) + \mathbf{a} + \mathbf{f}^n + p^e\mathbf{f}$. Conversely, modulo $\mathbf{fa} + \mathcal{R}^*(k) + \mathbf{f}^n + p^e\mathbf{f} + \mathbf{y}(i,j)$ such element s belong to $\mathbf{r}(k)$.

Now we can establish the parallel of Theorem 2.4 of Gupta [1991]:

Lemma 5.2.3. *Let $k > 1$. Then $F' \cap (1 + \mathbf{r}(k) + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$ is equal to $G(k)R(k)F''F'^{p^e}\gamma_n(F)$, where $G(k)$ is the subgroup generated by the elements of the form*

$$g(v) = \prod_{i<j} g_{ij} = \prod_{i<j} [x_i^{e_i}, x_j]^{p_{ij}},$$

where $p_{ij} \in \mathbf{f}^{k-2} + p^e ZF$ *and with* v_{ij} *as in 5.2.2,*

$$v = \sum_{i<j} v_{ij} + y_{ij} \equiv 0 \ mod\ \mathbf{a} + \mathbf{f}^n + p^e \mathbf{f}$$

for some $y_{ij} \in \mathbf{y}(i,j)$.

PROOF. Suppose $w \in F' \cap (1 + \mathbf{r}(k) + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$; by Lemma 5.2.2,

$$w - 1 \equiv v + (g(v) - 1) \bmod \mathbf{fa} + \mathcal{R}^*(k) + \mathbf{f}^n + p^e\mathbf{f}.$$

Looking at this relation modulo $\mathbf{a}$ it follows that $v \equiv 0 \bmod \mathbf{f}^n + p^e\mathbf{f}$ so that

$$w - 1 \equiv g(v) - 1 \bmod \mathbf{a} + \mathbf{f}^n + p^e\mathbf{f};$$

therefore there is a u_k in $R(k)$ such that

$$wg(v)^{-1}u_k - 1 \in \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f}.$$

However, $F' \cap (1 + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f}) = F'^{p^e}\gamma_n(F)F''$ by Theorem 6.1.3, so that $w \in G(k)R(k)F''F'^{p^e}\gamma_n(F)$, as desired. The reverse inclusion is clear. □

Corollary 5.2.4. *Let* $m = 2$ *and* $k > 2$; *then* $F' \cap (1 + \mathbf{r}(k) + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$ *equals* $R(k)F'^{p^e}F''\gamma_n(F)$.

PROOF. By the lemma above, we can consider an element $g = [x_1^{e_1}, x_2]^{p_{12}}$ where $p_{12} = p(x_2) = (x_2 - 1)p^*$ satisfies

$$(x_1^{e_1} - 1)(x_2 - 1)^2 p^* \equiv 0 \bmod (x_1 - 1)\mathbf{x}(1) + \mathbf{x}(2) + \mathbf{a} + \mathbf{f}^n + p^e\mathbf{f}.$$

Canceling, we get

$$t(x_1)(x_2 - 1)^2 p^* \in \mathbf{x}(1) + \mathbf{x}(2) + \mathbf{a} + \mathbf{f}^{n-1} + p^e\mathbf{f},$$

and canceling again it follows that

$$t(x_1)(x_2 - 1)p^* \in \mathbf{x}(1) + t(x_2)ZF + \mathbf{a} + \mathbf{f}^{n-2} + p^e ZF;$$

but the left-hand side is still divisible by $x_2 - 1$ so that in fact it belongs to $\mathbf{x}(1) + \mathbf{x}(2) + \mathbf{a} + \mathbf{f}^{n-2} + p^e ZF$. That is, $t(x_1)p_{12}$ belongs to this ideal; then modulo $R(k)F'^{p^e}F''\gamma_n(F)$, $g(v) \equiv [x_1, x_2]^{t(x_1)p_{12}} \equiv 1$. □

The following is an obvious consequence of the discussion preceding Theorem C in Gupta [1991]:

Lemma 5.2.5.

$$\mathbf{r}(k) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e\mathbf{f} = \mathbf{fr}(k-1) + \mathcal{R}^*(k) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e\mathbf{f}.$$

Consequently, Lemma 5.2.1 will be established once we prove the following:

Lemma 5.2.6. *Let $G(k)$ be as in 5.2.3. Then for $k > 2$ and $m > 2$,*

$$G(k) \cap (1 + \mathbf{fr}(k-1) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e\mathbf{f}) \subseteq R(k)F'^{p^e}F''\gamma_n(F).$$

PROOF. Suppose $g(v) - 1 \in \mathbf{fr}(k-1) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e\mathbf{f}$. Then $g(v) = \prod_{i<j} g_{ij}$ where g_{ij}, $v = \sum_{i<j} v_{ij} + y_{ij}$ are as in 5.2.2. Using endomorphisms and 5.2.4, we can assume that p_{ij} are such that $m > j$ and that $x_m - 1$ actually divides p_{ij}; put

$$p_{ij} = (x_m - 1)p_1(i,j) + (x_m - 1)^2 p_2(i,j)$$

where $x_m - 1$ does not divide $p_1(i,j)$. We can write $g(v) - 1$ in the form $\sum_i (x_i - 1)u_i$ for some $u_i \in \mathbf{r}(k-1) + \mathbf{fa} + \mathbf{f}^{n-1} + p^e ZF$. Expanding $g(v) - 1$ modulo $\mathbf{r}(k) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e ZF$ as in Lemma 4.2.4, we see that $u_m = g_m - 1$ where

$$g_m = \prod_{i<j} [x_i^{e_i}, x_j]^{x_m d_m(p_{ij})}.$$

$g_m \in F' \cap (1 + \mathbf{r}(k-1) + \mathbf{fa} + \mathbf{f}^{n-1} + p^e\mathbf{f})$, so by 5.2.3 it follows that

$$\sum_{i<j} (x_i^{e_i} - 1)(x_j - 1)x_m d_m(p_{ij}) + y'_{ij} \equiv 0$$

modulo $\mathbf{a} + \mathbf{f}^{n-1} + p^e ZF$, for some $y'_{ij} \in \mathbf{y}(i,j)$. Applying endomorphisms, we obtain the relations

$$(x_i^{e_i} - 1)(x_j - 1)p_1(i,j) + y'_{ij} \equiv 0$$

modulo $\mathbf{a} + \mathbf{f}^{n-1} + p^e ZF$. Hence,

$$t(x_i)(x_j - 1)p_1(i,j) \in \mathbf{x}(i) + \mathbf{x}(j) + \cdots + \mathbf{x}(m) + \mathbf{a} + \mathbf{f}^{n-2} + p^e ZF.$$

Consequently, modulo $R(k)F'^{p^e}F''\gamma_n(F)$,

$$[x_i^{e_i}, x_j]^{(x_m-1)p_1(i,j)} \equiv [x_i, x_m]^{t(x_i)(x_j-1)p_1(i,j)} \equiv 1,$$

and we can assume that $p_{ij} = (x_m - 1)^2 p_2(i,j)$. But in this case we can proceed just as in the proof of Corollary 5.2.4. This completes the proof. □

§5.3. The description

It remains only to state the description that follows from the results obtained in the preceding sections.

Theorem 5.3.1. *Let $n < 2p$; then*

$$D_n(\mathbf{r}, p^e) = \langle x_i^{p^e} \mid x_i^{p^{e-1}} \in D_{n-p+1}(\mathbf{r}, p^e)\rangle F_{n,p^e}.$$

PROOF. We shall prove the following statement by induction on n: if

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m}\zeta - 1 \in \mathbf{fr} + \mathbf{f}^n + p^e\mathbf{f}$$

where $\zeta \in F'$, then $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ belongs to $\langle x_i^{p^e} \mid x_i^{p^{e-1}} \in D_{n-p+1}(R, p^e)\rangle$, and $\zeta \in \gamma_n(F)F'^{p^e}F''R(2)$. The initial case of the induction is easy to extract from the results of Chapter IV, so we shall only prove the inductive step. Suppose $x_1^{\alpha_1} \cdots x_m^{\alpha_m}\zeta - 1 \in \mathbf{fr} + \mathbf{f}^n + p^e\mathbf{f}$ where $\zeta \in F'$. Then as we have seen many times, it follows that α_i are divisible by p^e, and thus $x_i^{\alpha_i} - 1 \equiv \beta_i(x_i - 1)^p$ where $\beta_i = \binom{\alpha_i}{p}$. Certainly then

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m}\zeta - 1 \in \mathbf{fr} + \mathbf{f}^{n-1} + p^e\mathbf{f}$$

whence by induction it follows that $\zeta \in \gamma_{n-1}(F)F'^{p^e}F''R(2)$ and consequently $\zeta - 1 \in \gamma_{n-1}(F)$ modulo $\mathbf{r}(2) + \mathbf{f}^n + p^e\mathbf{f}$. Therefore we have the relation

$$\sum_i \beta_i(x_i - 1)^p + \zeta - 1 \in \mathbf{r}(2) + \mathbf{f}^n + p^e\mathbf{f}.$$

But the summands other than $\zeta - 1$ involve only one variable and hence this relation can hold only if $\beta_i(x_i - 1)^p \in \mathbf{r}(2) + \mathbf{f}^n + p^e\mathbf{f}$ and $\zeta - 1 \in \mathbf{r}(2) + \mathbf{f}^n + p^e\mathbf{f}$. Hence, $\zeta \in F' \cap D(2, n, p^e)$ and thus by Lemma 5.1.1 it follows that ζ belongs to $F' \cap (1 + \mathbf{r}(3) + \mathbf{f}^2\mathbf{a} + \mathbf{f}^n + p^e\mathbf{f})R(2)$. Then by Lemma 5.2.1 it follows that ζ is an element of $R(2)F''F'^{p^e}\gamma_n(F)$. Therefore we are left with the relation

$$\beta_i(x_i - 1)^{p-1} \in \mathbf{r} + \mathbf{f}^{n-1} + p^e\mathbf{f}.$$

This is equivalent to that element belonging to the ideal $\mathbf{r} \cap \mathbf{f}^{p-1} + \mathbf{f}^{n-1} + p^e\mathbf{f}$. Consequently,

$$\beta_i(x_i - 1)^{p-1} \equiv \sum_{j=1}^{m} a_j(x_j^{e_j}\xi_j - 1)b_j \bmod \mathbf{f}^{n-1} + p^e\mathbf{f}$$

with $a_j b_j \in \mathbf{f}^{p-2}$. Canceling from left and right we obtain

$$\beta_i(x_i - 1) \equiv \sum_{j=1}^{m} p_j(x_j^{e_j}\xi_j - 1)q_j \bmod \mathbf{f}^3 + p^e\mathbf{f}$$

that is, $\beta_i(x_i - 1) \in \mathbf{r} + \mathbf{f}^{n-p+1} + p^e\mathbf{f}$. It is straightforward to check that $\binom{\beta_i}{j}(x_i - 1)^j$ is in the same ideal for all $j < p$ (because β_i and $\binom{\beta_i}{j}$ have the same divisibility properties mod p^e for $j < p$) and therefore $x_i^{\beta_i}$ belongs to $D_{n-p+1}(\mathbf{r}, p^e) = F^{p^e}\gamma_{n-p+1}R$. Since $p^{e-1} \mid \beta_i$ it follows that $x_i^{\alpha_i} \in \langle x_j^{p^e} \mid x_j^{p^{e-1}} \in F^{p^e}\gamma_{n-p+1}(F)R\rangle F^{p^{e+1}}$. Hence, $D_n(\mathbf{r}, p^e)$ is contained in the subgroup from the statement of the theorem; since the converse inclusion is evident—Theorem 5.3.1 is proved. □

Chapter VI

Concluding Remarks and Problems

The reader will notice that our refinement (obtained in Chapter V) of the solution of the integral dimension subgroup problem for metabelian odd p-groups only establishes the validity of modified Lazard formula for dimensions less than $2p$. This is due to an intrinsic limitation of the method we have used — peculiar to the modular case. It arises because of the following easy observation:

$$F'^{p^e} - 1 \subseteq \mathbf{a}^2 + \mathbf{f}^n + p^e\mathbf{f}.$$

For if $w \in F'$ then $w^{p^e} - 1 \equiv p^e(w-1)$ modulo $\mathbf{a}^2$, and this is in turn congruent to zero as we are working modulo p^e. Thus the proofs given in Chapter V will work in the modular case for as long as F'^{p^e} is contained in F_{n,p^e}. But, this is the case only if $n < 2p$. This simple remark illustrates a crucial difference between the integral and modular cases. Essentially, this phenomenon is reflected in the following results:

Theorem 6.1.1. *(Enright [1968]) Let R and S be normal subgroups of F. Then $F \cap (1 + \mathbf{rs}) = [R \cap S, R \cap S]$. In particular $F \cap (1 + \mathbf{fa}) = F''$.*

Theorem 6.1.2. *(Bergman and Dicks [1975]) Let R and S be normal subgroups of F. Then*

$$F \cap (1 + \mathbf{fs} + p^e ZF) = (R \cap S)^{p^e}[R \cap S, R \cap S].$$

In particular $F \cap (1 + \mathbf{fa} + p^e ZF) = F'^{p^e} F''$.

For, while in the integral case it does not matter whether we work modulo the augmentation ideal of the second derived subgroup or modulo the ideal $\mathbf{fa}$

(or $\mathbf{a}^2$) by Theorem 6.1.1, in the modular case it indeed makes a difference; since, as Theorem 6.1.2 shows, a possibly unwanted factor of F'^{p^e} will appear as soon as we choose to work modulo $\mathbf{a}^2$ rather than $\mathbf{a}_2$.

Thus, while in the integral case the reduction from $\mathbf{a}_2$ to $\mathbf{fa}$ and the identification

$$F \cap (1 + \mathbf{fa} + \mathbf{f}^n) = \gamma_n(F)F''$$

imply that the integral dimension subgroup conjecture is valid for the free metabelian group, this will not work in the modular case; for we have the following:

Theorem 6.1.3. *The subgroups $F \cap (1 + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$, $F \cap (1 + \mathbf{af} + \mathbf{f}^n + p^e\mathbf{f})$ and $F \cap (1 + \mathbf{a}^2 + \mathbf{f}^n + p^e\mathbf{f})$ are equal to the subgroup $F^{p^{a(n)}} F'^{p^e} \gamma_n(F)F''$, where $a(n)$ is the least integer such that $p^{a(n)} \geq np^{e-1}$.*

PROOF. Clearly the subgroup $F^{p^{a(n)}} F'^{p^e} \gamma_n(F)F''$ is contained in the other subgroups; since $F \cap (1 + \mathbf{af} + \mathbf{f}^n + p^e\mathbf{f})$ and $F \cap (1 + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$ contain $F \cap (1 + \mathbf{a}^2 + \mathbf{f}^n + p^e\mathbf{f})$, it will suffice to show that they are contained in $F^{p^{a(n)}} F'^{p^e} \gamma_n(F)F''$ for the theorem to follow. Suppose $w - 1$ is an element of $\mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f}$ and let $w = x_1^{b_1} \cdots x_m^{b_m} \prod_{i>1} C_{ij}^{b_{ij}}$ where C_{ij} are Hall's basic commutators. Applying the endomorphism of ZF which fixes x_i and sends $x_j \to 1$ for $i \neq j$, we obtain the relation

$$x_i^{b_i} - 1 \in \mathbf{f}^n + p^e\mathbf{f}$$

because this endomorphism sends $\mathbf{a}$ to zero; but Lazard's formula is valid for the free group, so it follows that $x_i^{b_i} \in F^{p^{a(n)}}$. Therefore we have that

$$\prod_{i>1} C_{ij}^{b_{ij}} - 1 \in \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f}.$$

But modulo $\mathbf{fa}$, $C^b - 1 \equiv b(C-1)$ and $(C_1C_2 - 1) \equiv (C_1 - 1) + (C_2 - 1)$, hence

$$\sum_{i>1} b_{ij}(C_{ij} - 1) \in \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f}. \tag{1}$$

Look at this relation modulo $\mathbf{fa} + \mathbf{f}^3 + p^e\mathbf{f}$; then we have

$$\sum_j b_{2j}\zeta_{2j} \in \mathbf{f}^3 + p^e\mathbf{f}$$

and by the independence of the basic Lie elements it follows that b_{2j} are zero modulo p^e. Suppose now, by the way of induction, that we have shown b_{ij} to be divisible by p^e for $i < k$; then reading relation (1) for $n = k + 1$ we get

$$\sum_j b_{kj}\zeta_{kj} \equiv u \bmod \mathbf{f}^{k+1} + p^e\mathbf{f}$$

for some $u \in \mathbf{fa}$. This element is then of augmentation degree k and has the form $\sum_{i,j} a_{ij}(x_i - 1)\pi_j$ where π_j is a basic product of the $\zeta_{i,j}$s for $i < k$; therefore, by the linear independence of basic products, no part of the left-hand side can cancel into u and we must conclude that the b_{kj} are divisible by p^e, too. This completes the induction to show that $F \cap (1 + \mathbf{fa} + \mathbf{f}^n + p^e\mathbf{f})$ is contained in $F^{p^{a(n)}} F'^{p^e} \gamma_n(F) F''$; similarly, $F \cap (1 + \mathbf{af} + \mathbf{f}^n + p^e\mathbf{f})$ is contained in that subgroup, which is what we wanted to demonstrate. □

Let us recall from Chapter III that Lazard's formula is indeed valid for free metabelian groups — a consequence of our generalization of Lazard's theorem (see Corollary 3.2.4).

Having made these remarks, we propose the following:

Problem. *Identify dimension subgroups mod odd prime powers of metabelian groups.*

This is certainly the most important question we have not been able to resolve; the methods we used were not powerful enough to reach beyond the dimension $2p$, and solving this problem will very likely require a substantially different approach.

There are several other questions that seem interesting; for instance, in Chapter III we have investigated the relation between the Lie identities of an algebra and the commutator laws in its group of units. Recall that our examples to show that the unit group of a Lie centre-by-metabelian algebra need not be centre-by-metabelian are algebras of characteristic 2. Given the special significance of the even prime for the centre-by-metabelian groups, we pose this

Problem. *Is the group of units of a Lie centre-by-metabelian algebra over a field of characteristic other than 2 centre-by-metabelian?*

Another point of interest is that our theorems 3.3.9 and 3.3.14 which assert that a certain group is nilpotent-by-abelian do not give any bound on the nilpotency class involved; it would be nice to have this information available.

Finally, recall that in Chapter IV we gave necessary conditions for a group to fail the integral dimension subgroup conjecture for $n = 4$;

Problem. *Are the conditions of Theorem 4.2.7 necessary and sufficient?*

We hope these questions will be answered soon.

References

Bakhturin, Yu.A.

[1985] *Identities in Lie Algebras*, Nauka, Moscow. (Russian)

Bergman, George M. and Dicks, Warren

[1975] On Universal Derivations, *J.Algebra* **36**, No.2, 193–211.

Braun, Amiram

[1974] Lie rings and the Engel condition, *J.Algebra* **31**, 287–292.

[1984] The nilpotency of the radical in a finitely generated P.I.-ring, *J.Algebra* **89**,375–396.

Cliff, G. and Hartley, B.

[1985] Sjogren's theorem on dimension subgroups, *J.Pure Appl.Algebra* **47**, No.3, 231–242.

Enright, Dennis E.

[1968] *Triangular matrices over group rings*, Doctoral Thesis, New York University.

Gupta, Chander Kanta

[1973] The free centre-by-metabelian groups, *J.Austral.Math.Soc.* **16**, 294–299.

Gupta, Chander Kanta and Levin, Frank

[1986] Dimension subgroups of free centre-by-metabelian groups, *Ill.J.Math.* **30**, 258–273.

Gupta, Narain

[1987] *Free Group Rings*, Contemporary Math. **66**, AMS, Rhode Island.

[1990] The dimension subgroup conjecture, *Bull.London Math.Soc.* **22**, 453–456.

[1991] A solution of the dimension subgroup problem, *J.Algebra* **138**, No.2, 479–490.

Gupta, Narain and Levin, Frank

[1983] On the Lie ideals in a ring, *J.Algebra* **81**, No.1, 225–231.

Gupta, Narain and Tahara, Ken-Ichi

[1985] Dimension and lower central subgroups of metabelian p-groups, *Nagoya Math. J.* **100**, 127–133.

Jennings, S.A.

[1941] The structure of the group ring of a p-group over a modular field, *Trans. Amer. Math. Soc.* **50**, 175–185.

[1955] The group ring of a class of infinite nilpotent groups, *Canad. J. Math.* **7**, 169–187.

Kargapolov, M.I. and Merzljakov, Ju.I.

[1979] *Fundamentals of the theory of groups*, GTM 62, Springer, New York.

Krasil'nikov, A.N.

[1] On unit groups of Lie metabelian rings, (preprint, Russian, not dated)

Kuz'min, Yu.V. and Shapiro, M.Z.

[1987] The connection between the varieties of groups and varieties of Lie rings, *Sibirsk.J.Math.* **28**, No. 5, 100–101.

Lazard, Michel

[1954] Sur les groupes nilpotentes et les anneaux de Lie, *Ann.Sci. Ecole Norm. Sup.* (3) **71**,101–190.

Magnus, Wilhelm

[1935] Bezeihungen swischen Gruppen und Idealen in einem speziellen Ring, *Math. Ann.* **111**, 259–280.

Magnus, Wilhelm; Karrass, Abraham; Solitar, Donald

[1976] *Combinatorial group theory*, 2nd ed, Dover, New York.

Moran, Siegfried

[1970] Dimension subgroups mod n, *Proc.Cambridge Phil.Soc.* **68**, 579–582.

Passi, Inder Bir S.

[1979] *Group Rings and their Augmentation Ideals*, LNM 715, Springer-Verlag.

Passi, Inder Bir S. and Sharma, Sneh

[1974] The third dimension subgroup mod n, *J.London Math. Soc.* (2) **9**, 176–182.

Passman, Donald S.

[1977] *The Algebraic Structure of Group Rings*, John Wiley & Sons, Inc., New York.

Platonov, V.P.

[1967] Linear groups with identical relations, *Dokl.Akad. Nauk BSSR* **11**, 581–582.

Quillen, Daniel G.

[1968] On the associated graded ring of a group ring, *J.Algebra* **10**, 411–418.

Rips, E.

[1972] On the fourth integer dimension subgroup, *Israel J.Math* **12**, 342–346.

Rowen, Louis Halle

[1978] *Polynomial identities in ring theory*, Academic Press, New York.

Sandling, Robert

[1972] Dimension subgroups over arbitrary coefficient rings, *J.Algebra* **21**, 250–265.

Scoppola, Carlo M. and Shalev, Aner

[1991] Applications of dimension subgroups to the power structure of p-groups, *Israel J. Math.* **73**, No.1, 45–56.

Sehgal, Sudarshan K.

[1978] *Topics in Group Rings*, Marcel Dekker, New York.

Shalev, Aner

[1] Applications of dimension and Lie dimension subgroups to modular group algebras, *Proc. Amitsur Conference*, manuscript (1989).

[2] On associative algebras satisfying the Engel condition, (manuscript).

Sharma, R.K. and Srivastava, J.B.

[1] Lie Centrally Metabelian Group Rings, *J.Algebra* (to appear).

Sjogren, J.A.

[1979] Dimension and lower central subgroups, *J.Pure Appl. Algebra* **14**, 175–194.

Smirnov, M.B.

[1988] On Associated Groups of Nilalgebras of Exponent 4 over Fields of characteristic 2, *Vestsi Akad. Nauk BSSR* ser. fiz.-mat, No. 5, 8–13, Minsk. (Russian)

Tahara, Ken-Ichi

[1977] On the structure of $Q_3(G)$ and the fourth dimension subgroup, *Japan J. Math.* (N.S.) **3**,381–396.

Wehrfritz, B.A.F.

[1973] *Infinite Linear Groups*, Springer, Berlin.

[1987] Unipotent representations of nilpotent groups, *Bull. London Math. Soc.* **19**, 320–324.

Zassenhaus, H.

[1935] Ein Verfahren, jeder endlichen p-Gruppe einen Lie-ring mit der Charakteristik p zuzuordnen, *Abh. Mat. Sem. Hamburg* **13**, 200–207.

Zel'manov, E.I.

[1988] On Engel Lie algebras, *Sib.J.Math.* **29**, No. 5, 112–117.