ON SINGULAR AND OTHER SPECIAL HAUSDORFF COMPACTIFICATIONS

by

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ABSTRACT

Singular compactifications of locally compact Hausdorff spaces were first introduced over a decade ago. An elegant characterization of singular compactifications is the following: A compactification αX of the locally compact Hausdorff space X is singular if and only if $\alpha X \backslash X$ is a retract of αX. In this project we provide a new representation of singular compactifications and produce various characterizations of these. These characterizations allow us to answer five open questions concerning this family of compactifications: 1) Are there compactifications which are not the supremum of a family of singular compactifications? 2) If f and g are two singular functions such that S(f) is homeomorphic to S(g) when is $X \cup_f S(f)$ equivalent to $X \cup_g S(g)$? 3) Is the supremum of all singular compactifications always βX? 4) If the family of all singular compactifications forms a lattice does this imply that βX is a singular compactification? 5) When does a space X have a largest singular compactification?

It has been previously shown that the supremum of singular compactifications need not itself be a singular compactification. Examples of this fact are easy find. We provide necessary and sufficient conditions on X for the supremum of any collection of singular compactifications to be a singular compactification. In particular we characterize those spaces X for which the supremum of the family of all singular compactifications is itself a singular compactification. In other words, we describe those spaces X which can be densely embedded in some compact space μX where $\mu X \backslash X$ is a retract of μX and where μX is the largest such space (in the lattice of compactifications of X) possessing this property. As an immediate corollary to the above result we obtain a characterization of those locally compact Hausdorff spaces for which the Stone-Čech compactification is singular.

We also show that not all compactifications of X can be expressed as the supremum of a collection of singular compactifications.

We also investigate another compactification of a space X called the *perfect* compactification. An algebraic characterization of the subring $C_\alpha(X)$ associated to a perfect compactification αX of X is known. We have provided an alternate proof to this characterization. We have also introduced a new compactification called the *pseudoperfect* compactification. Two characterizations of pseudoperfect compactifications are given. We also show that all perfect compactifications and all compactifications of pseudocompact spaces are pseudoperfect. An example of a pseudoperfect compactification which is not a perfect compactification is provided.

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CHAPTER 1

PRELIMINARY NOTIONS

All hypothesized topological spaces will be assumed to be locally compact and Hausdorff.

Two compactifications αX and γX of a space X are said to be *equivalent* if there is a homeomorphism $f : \alpha X \longrightarrow \gamma X$ from αX onto γX which fixes the points of X. This defines an equivalence relation on the family of all compactifications of X. When we will speak of a compactification αX of X it will be understood that we are referring to the equivalence class of αX. The notation $\alpha X \cong \gamma X$ will mean that αX is equivalent to γX. We will say that the compactification αX is less than or equal to the compactification γX, denoted by $\alpha X \leqq \gamma X$ if there is a continuous function $f : \gamma X \longrightarrow \alpha X$ of γX onto αX which acts as the identity on X. This defines a partial order on the family K(X) of all compactifications of X. It is well known that K(X) is a complete lattice with respect to the partial order $\leqq$(see 2.19 in [C]). Let $\mathcal{H}$ be a subfamily of K(X). If the supremum αX of $\mathcal{H}$ belongs to $\mathcal{H}$ we will say that $\mathcal{H}$ has a *largest* element, namely αX. We will say that an element γX of $\mathcal{H}$ is a *maximal* element of $\mathcal{H}$ if there does not exist an element ζX in $\mathcal{H}$ such that $\zeta X > \gamma X$. (The reader is referred to [C] for details on the subject). If αX and γX are compactifications of X such that $\alpha X \leqq \gamma X$, we will denote the projection map from γX onto αX which fixes the points of X by $\pi_{\gamma\alpha}$.

The family of compactifications studied here was first defined and discussed in [CFGM]. We introduce the object of our study in the following definitions which appear in [CFGM].

1.1 Definitions *A singular compactification induced by the function f* is constructed as follows: Let $f : X \longrightarrow K$ be a continuous function from the space X into a compact set K. Let the *singular set,* $S(f)$, of f be defined as the set $\{x \in cl_X f[X]$: for any neighbourhood U of x, $cl_X f^{\leftarrow}[U]$ is not compact$\}$. If $S(f) = K$ then f is said to be a *singular map*. It is easy to verify that $S(f)$ is closed in K and that if f is a singular map then $f[X]$ is dense in $S(f)$. If f is a singular map *the singular compactification of X induced by f*, denoted by $X \cup_f S(f)$, is the set $X \cup S(f)$ where the basic neighbourhoods of the points in X are the same as in the original space X, and the points of $S(f)$ have neighbourhoods of form $U \cup (f^{\leftarrow}[U] \setminus F)$ where U is open in $S(f)$ and F is a compact subset of X. This defines a compact Hausdorff topology on $X \cup_f S(f)$ in which X is a dense subspace. We will say that a compactification αX of X is a *singular compactification* if αX is equivalent to $X \cup_f S(f)$ for some singular map f.

Recall that a map $r : X \longrightarrow A$ sending X into a subset A of X is called a *retraction* if it is continuous and it fixes the points of A. The subset A is then called a *retract* of X.

We begin by stating some basic properties known to be possessed by singular compactifications. The following is theorem 4 in [G].

1.2 THEOREM [G] The singular compactifications of X are precisely those compactifications αX of X whose remainder $\alpha X \setminus X$ is a retract of αX.

Since a constant map is continuous, $\omega X \setminus X$ is a retract of ωX (the one-point compactification of X). So ωX is always a singular compactification.

1.3 REMARK If $r : \alpha X \longrightarrow \alpha X \setminus X$ is a retraction from αX onto $\alpha X \setminus X$ then $r|_X$ is a singular map which induces the singular compactification $X \cup_{r|X} S(r|_X)$ (since if U is open in $\alpha X \setminus X$ and $cl_X r|_X^{\leftarrow}[U]$ is compact then $(\alpha X \setminus cl_X r|_X^{\leftarrow}[U]) \cap r^{\leftarrow}[U]$ is a non-empty open subset of αX contained in $\alpha X \setminus X$). Hence $X \cup_{r|X} S(r|_X) = \alpha X$. Conversely if f is a singular map it is easily verified that its extension $f^* : X \cup_f S(f)$

$\longrightarrow S(f)$ which acts as the identity function on $S(f)$ is continuous and is a retraction map from $X \cup_f S(f)$ onto $S(f)$.

We also have the following important result from [G].

1.4 THEOREM (Theorem 7, [G]) If αX is a singular compactification and γX is any compactification of X less than αX then γX is also a singular compactification.

1.5 NOTATION For any compactification γX of X, $C_\gamma(X)$ will denote the set $\{f|_X : f \in C(\gamma X)\}$. If f is a bounded real-valued singular function, f will be regarded as a function from X into $cl_{\mathbb{R}}f[X]$, i.e. we are letting K (in our definition of singular map) be $cl_{\mathbb{R}}f[X]$.The set S_γ will denote the set of all singular maps in $C_\gamma(X)$. Thus S_β denotes the collection of all singular maps in $C^*(X)$. In order to be more specific we may sometimes use the notation $S_\gamma(X)$ instead of S_γ indicating precisely the space X under consideration. If $\mathcal{G} \subseteq C_\gamma(X)$, $\mathcal{G}^\gamma$ will denote the set of extensions f^γ to γX of the functions f in $\mathcal{G}$. The following is a generalization of theorem 1.1 of [CCF].

1.6 LEMMA Let f be a continuous function from a space X to a compact Hausdorff space Z. Let $Y = cl_Z f[X]$ and $K_X = \{F \subseteq X : F \text{ is compact}\}$. Then $S(f) = \cap\{ cl_Y f[X\backslash F] : F \in K_X \}$.

Proof: We first show that $S(f)$ is contained in $\cap \{cl_Y f[X/F] : F \in K_X\}$. Let $F \in K_X$. Suppose p belongs to $Y\backslash cl_Y f[X\backslash F]$. Then there exists an open neighbourhood U of p such that $f^{\leftarrow}[U] \subseteq F$. Hence $p \notin S(f)$. We have thus shown that $S(f) \subseteq cl_Y f[X\backslash F]$. Since F was arbitrarily chosen in K_X, it follows that $S(f) \subseteq \cap\{cl_Y f[X\backslash F] : F \in K_X \}$. Suppose now that x belongs to $\cap\{cl_Y f[X\backslash F] : f \in K_X\}$. If x belongs to

$Y\backslash S(f)$ then there exists an open neighbourhood U of x in Y such that $cl_X f^{\leftarrow}[U]$ is compact. But

$$x \in \cap\{cl_Y f[X\backslash F]: f \in K_X\}$$
$$\subseteq cl_Y f[X\backslash cl_X f^{\leftarrow}[U]] \quad (\text{since } cl_X f^{\leftarrow}[U] \in K_X)$$
$$\subseteq cl_Y f[X\backslash f^{\leftarrow}[U]]$$
$$\subseteq cl_Y f\circ f^{\leftarrow}[Y\backslash U]$$
$$= Y\backslash U.$$

This contradicts the fact that x belongs to U. Consequently $\cap\{cl_Y f[X\backslash F] : f \in K_X\} \subseteq S(f)$. The lemma follows.

QED

Proposition 1.7 is a generalization of lemma 1 in [CF].

1.7 PROPOSITION If αX is a compactification of X, K is a compact Hausdorff space and $f : X \longrightarrow K$ is a continuous function which extends to $f^\alpha : \alpha X \longrightarrow K$ then $f^\alpha[\alpha X\backslash X] = S(f)$.

Proof: We will first show that $f^\alpha[\alpha X\backslash X]$ is contained in $cl_{cl_K f[X]} f[X\backslash F]$ for all $F \in K_X$ and apply the previous lemma. Let $F \in K_X$. Then $\alpha X\backslash X \subseteq cl_{\alpha X}(X\backslash F)$. Hence $f^\alpha[\alpha X\backslash X] \subseteq f^\alpha[cl_{\alpha X}(X\backslash F)] \subseteq cl_{cl_K f[X]} f[X\backslash F]$. Since this is true for all $F \in K_X$, $f^\alpha[\alpha X\backslash X] \subseteq \cap\{cl_{cl_K f[X]} f[X\backslash F] : F \in K_X\}$. By the previous lemma $f^\alpha[\alpha X\backslash X] \subseteq S(f)$.

Let $p \in K\backslash f^\alpha[\alpha X\backslash X]$. Let U be an open neighbourhood (in K) of p such that $cl_K U$ misses $f^\alpha[\alpha X\backslash X]$. Then $cl_X f^{\leftarrow}[U] \subseteq f^{\leftarrow}[cl_{cl_K f[X]} U]$, which is a compact subset of X. This implies that p cannot belong to S(f). Hence $S(f) = f^\alpha[\alpha X\backslash X]$.

QED

1.8 COROLLARY Let $f : X \longrightarrow K$ be a continuous map into a compact Hausdorff space such that f[X] is dense in K. Let $E_f(X)$ denote the set of all compactifications αX of X such that $f : X \longrightarrow K$ extends to $f^\alpha : \alpha X \longrightarrow K$. Then

f is a singular map if and only if $f^\alpha[\alpha X\backslash X]$ contains $f[X]$ for some (equivalently for all) $\alpha X \in E_f(X)$.

Proof: (=>) If f is a singular map then $S(f) = K = cl_K f[X]$ (by definition). By 1.7, $f^\alpha[\alpha X\backslash X] = S(f) = cl_K f[X]$ for all $\alpha X \in E_f(X)$, hence $f[X]$ is contained in $f^\alpha[\alpha X\backslash X]$.

(<=) Suppose now that $f^\alpha[\alpha X\backslash X]$ contains $f[X]$ for some $\alpha X \in E_f(X)$. Since $f[X]$ is dense in K (by hypothesis) $cl_K f[X] = K$. We must show that $S(f) = K$. Let $p \in K$ and U be an open neighbourhood of p in $K = cl_K f[X]$. Then $f^{\alpha\leftarrow}[U]$ meets $\alpha X\backslash X$, hence $cl_{\alpha X} f^{\leftarrow}[U]$ meets $\alpha X\backslash X$. Since $cl_X f^{\leftarrow}[U]$ is dense in $cl_{\alpha X} f^{\leftarrow}[U]$, $cl_X f^{\leftarrow}[U]$ cannot be compact. Hence p belongs to $S(f)$. Since $K = S(f)$, f is singular.

QED

We will state now for future reference the following well known result concerning compactifications (see chapter 2 of [C]).

1.9 THEOREM Let A be a subalgebra of $C^*(X)$ that contains the constant functions, and separates the points and closed sets of X. Then,

1) there is a compactification $\gamma_A X$ of X with these properties:

 1a) For every f in A there exists an f^γ in $C(\gamma_A X)$ such that $f^\gamma|_X = f$.

 1b) Let $A^\gamma = \{f^\gamma : f \in A\}$. Then A^γ separates the points of $\gamma_A X$.

2) if αX is a compactification of X with the properties:

 2a) For every f in A there exists an f^α in $C(\alpha X)$ such that $f^\alpha|_X = f$

 2b) The family $A^\alpha = \{ f^\alpha : f \in A\}$ separates points of $\gamma_A X$,

then αX and $\gamma_A X$ are equivalent compactifications of X. In other words, $\gamma_A X$ is uniquely determined (up to equivalence) by properties 1a) and 1b).

Furthermore it is well known that if $C_\alpha(X) \subseteq C_\gamma(X)$ then $\alpha X \leqq \gamma X$ (see [F] for a discussion of this last statement).

Let $\mathscr{G} \subseteq C^*(X)$. The *evaluation map* $e_{\mathscr{G}}$ induced by $\mathscr{G}$ is the function $e_{\mathscr{G}} : X \longrightarrow \Pi\{I_g : g \in \mathscr{G}\}$ (where, for each g, I_g is a closed interval containing g[X]) defined by $e_{\mathscr{G}}(x) = \langle g(x) \rangle_{g \in \mathscr{G}}$. Note that the closure in $\Pi_{g \in \mathscr{G}} I_g$ of $e_{\mathscr{G}}[X]$ is a compact set.

If αX is a compactification of X and $\mathscr{G} \subseteq C_\alpha(X)$ then $\mathscr{G}^\alpha$ will denote the family of all extensions of the functions in $\mathscr{G}$ to αX.

By the *uniform norm topology* or *metric topology* on $C^*(X)$ we will mean the topology on $C^*(X)$ in which the closure of sets is the closure under uniform convergence. The metric on $C^*(X)$ is defined as follows: $d(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$ (see the introductory paragraph of chapter 16 of [GJ]).

The following result is the only theorem in [L]. We offer a simpler proof of that theorem here.

1.10 PROPOSITION [L] Let $\mathscr{G} \subseteq C^*(X)$. Then there exists a smallest compactification to which all functions in $\mathscr{G}$ extend.

Proof: Every function $g : X \longrightarrow \mathrm{cl}_{\mathbb{R}} g[X]$ in $\mathscr{G}$ extends to $g^\beta : \beta X \longrightarrow \mathrm{cl}_{\mathbb{R}} g[X]$. Let $T = \Pi_{g \in \mathscr{G}} I_g$ where I_g is as above. Then the evaluation map $e_{\mathscr{G}}$ induced by $\mathscr{G}$, extends to $e_{\mathscr{G}}^\beta : \beta X \longrightarrow \mathrm{cl}_T e_{\mathscr{G}}^\beta[X]$. The following collection of sets

$$\{(e_{\mathscr{G}}^\beta)^{\leftarrow}(y) \cap \beta X \backslash X : y \in (\mathrm{cl}_T e_{\mathscr{G}}^\beta[\beta X]) \backslash e_{\mathscr{G}}^\beta[X]\} \cup \{\{x\} : x \in X\}$$

is a decomposition of βX. We claim it is upper semicontinuous. Since $\beta X \backslash X$ is closed in βX we need only verify that the decomposition of $\beta X \backslash X$ induced above is upper semicontinuous (see theorem 2.4.13 of [E]). But since $e_{\mathscr{G}}^\beta|_{\beta X \backslash X}$ is continuous this decomposition of $\beta X \backslash X$ is automatically upper semicontinuous. The resulting quotient space is a compactification αX of X. It is easily seen that $\mathscr{G}^\alpha = C(\alpha X) = \{\pi_f \circ e_{\mathscr{G}}^\alpha : f \in \mathscr{G}\}$ where π_f is the projection map. Hence $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$. We claim that αX is the smallest compactification to which all functions in $\mathscr{G}$ extend. Suppose γX is a compactification which is strictly less than αX. Then $\gamma X \backslash X$ results from some upper semicontinuous decomposition of $\alpha X \backslash X$. Suppose the projection map $\pi_{\alpha\gamma} : \gamma X \longrightarrow \gamma X$ from γX to γX collapses the two points p and q in $\gamma X \backslash X$. Let $f \in C_\alpha(X)$ such that $f^\alpha(p) \neq f^\alpha(q)$. If $f \in C_\gamma(X)$ then $f^\alpha(p) = f^\gamma[\pi_{\alpha\gamma}[\{p,q\}]] = f^\alpha(q)$ (see the first paragraph of chapter 1 and 1.5 for notation). Since this is a contradiction, $f \notin C_\gamma(X)$. Hence not every function in $C_\alpha(X)$ extends to γX. This establishes the claim. We now show that αX is unique in the sense that, if γX is another "smallest" compactification to which all functions in $\mathscr{G}$ extend, then γX is equivalent to αX. Suppose γX is a compactification to which all functions in $\mathscr{G}$ extend and is such that, for any compactification ζX strictly less than γX, there is some function f in $\mathscr{G}$ which does not extend to ζX. Then $\mathscr{G}^\gamma$ must separates the points of $\gamma X \backslash X$ (for if p and q are two points in $\gamma X \backslash X$ which are not separated by $\mathscr{G}^\gamma$ we may obtain a strictly smaller compactification of X to which all functions in $\mathscr{G}$ extend by collapsing p and q to a single point). Since the one-point compactification ωX of X is less than or equal to all compactifications ηX of X then $C_\omega(X) \subseteq C_\eta(X)$ for all compactifications ηX. Hence the subalgebra $\langle C_\omega(X) \cup \mathscr{G} \rangle$ generated by $C_\omega X \cup \mathscr{G}$ is contained in both $C_\alpha(X)$ and $C_\gamma(X)$. Since $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$ and $\mathscr{G}^\gamma$ separates the points of $\gamma X \backslash X$ then $\langle C_\omega(X) \cup \mathscr{G} \rangle^\alpha$ and $\langle C_\omega(X) \cup \mathscr{G} \rangle^\gamma$ separate the points of αX and γX respectively. (This follows from the fact that $C(\omega X)$ separates the points of ωX. This implies that, 1) $C_\omega(X)$ separates the points of X, 2) $C(\omega X)$ separates the outgrowth $\omega X \backslash X$ of ωX from all points in X, hence $C_\omega(X)^\eta$ will separate the outgrowth $\eta X \backslash X$ of any compactification ηX from any point in X). We apply the Stone-Wieirstrass theorem (see 9.34 of [Ro]) to conclude that the closure in the uniform norm topology of $\langle C_\omega(X) \cup \mathscr{G} \rangle^\alpha$ equals $C(\alpha X)$. Similarly the closure in the uniform topology of $\langle C_\omega(X) \cup \mathscr{G} \rangle^\gamma$ equals $C(\gamma X)$. Hence the closure of $\langle C_\omega(X) \cup \mathscr{G} \rangle$ is simultaneously equal to $C_\alpha(X)$ and $C_\gamma(X)$. This implies that αX and γX are equivalent compactifications (see the final statement of 1.9). Hence we have shown that αX is unique (up to equivalence).

QED

The following notation was introduced in paragraph 2 of [F].

1.11 NOTATION If $\mathscr{G}$ is contained in $C^*(X)$, the symbol $\omega_{\mathscr{G}}X$ will denote the smallest compactification to which all functions in $\mathscr{G}$ extend. If f belongs to $C^*(X)$, $\omega_f X$ will denote the smallest compactification of X to which f extends.

Observe that $\omega_{\mathscr{G}}X$ is always the smallest compactification to which $e_{\mathscr{G}}$ extends. To see this, note that whenever the collection of functions $\mathscr{G}$ extends to a compactification αX of X, so does $e_{\mathscr{G}}$; hence $e_{\mathscr{G}}$ extends to $\omega_{\mathscr{G}}X$. Conversely, if $e_{\mathscr{G}}$ extends to αX we obtain an extension $f^\alpha \in C(\alpha X)$ of $f \in \mathscr{G}$ by defining f^α to be $\pi_f \circ e_{\mathscr{G}}{}^\alpha$ where π_f is the projection map. Hence the families of compactifications to which $e_{\mathscr{G}}$ and $\mathscr{G}$ extend are the same. It must then follow that they have the same smallest member and that $\omega_{\mathscr{G}}X$ is equivalent to the smallest compactification of X to which $e_{\mathscr{G}}$ extends.

1.12 PROPOSITION Let αX be a compactification of X and $\mathscr{G} \subseteq C_\alpha(X)$. Then $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$ iff $e_{\mathscr{G}}{}^\alpha (= e_{\mathscr{G}^\alpha})$ is one-to-one on $\alpha X \backslash X$.

Proof: Suppose $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$. Let p and q be distinct points in $\alpha X \backslash X$. Then there exists a function f in $\mathscr{G}$ such that $f^\alpha(p) \neq f^\alpha(q)$. Hence $e_{\mathscr{G}}{}^\alpha(p) \neq e_{\mathscr{G}}{}^\alpha(q)$. It follows that $e_{\mathscr{G}}{}^\alpha$ is one-to-one on $\alpha X \backslash X$. Conversely if $e_{\mathscr{G}}{}^\alpha$ is one-to-one on $\alpha X \backslash X$ then, if p and q are distinct points in $\alpha X \backslash X$, $e_{\mathscr{G}}{}^\alpha(p) \neq e_{\mathscr{G}}{}^\alpha(q)$; this can only happen if there exists some function f in $\mathscr{G}$ such that $f^\alpha(p) \neq f^\alpha(q)$. Hence $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$.

QED

For further reference we formally present the following easy result which appears in the proof of theorem 1 of [F].

1.13 PROPOSITION [F] Let $\mathscr{G} \subseteq C^*(X)$ and αX be a compactification of X. Then $\alpha X \cong \omega_{\mathscr{G}} X$ if and only if each function g in $\mathscr{G}$ extends to g^α in $C(\alpha X)$ and $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$.

Proof: (=>) Suppose $\alpha X \cong \omega_{\mathscr{G}} X$. Then, by definition of $\omega_{\mathscr{G}} X$, every function f in $\mathscr{G}$ extends to a function f^α in $C(\alpha X)$. Furthermore $\mathscr{G}^\alpha$ must separate the points of $\alpha X \backslash X$ for, if not, we may collapse any two points in $\alpha X / X$ which are not separated by $\mathscr{G}^\alpha$ to obtain a compactification strictly smaller than $\omega_{\mathscr{G}} X$ to which each member of $\mathscr{G}$ extends, thus obtaining a contradiction. Hence $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$.

(<=) Since every function f in $\mathscr{G}$ extends to a function f^α in $C(\alpha X)$ then $\omega_{\mathscr{G}} X$ is less than or equal to αX (by definition of $\omega_{\mathscr{G}} X$). Since $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$ then $\omega_{\mathscr{G}} X$ cannot be strictly less than αX, hence $\alpha X \cong \omega_{\mathscr{G}} X$. This proves the proposition.

QED

Compactifications of the form $\omega_{\mathscr{G}} X$ are briefly discussed in [F]. From 1.7 and 1.13 we see that $\omega_f X \backslash X$ is homeomorphic to S(f) for all $f \in C^*(X)$. (Since $\{f^{\omega_f}\}$ separates the points of $\omega_f X \backslash X$ it is one-to-one on $\omega_f X \backslash X$; hence $f^{\omega_f}|_{\omega_f X \backslash X}$ is a homeomorphism.)

Note that if αX is a compactification and $C_\alpha(X) = \{f|_X : f \in C(\alpha X)\}$ is its associated subalgebra then, since $C(\alpha X)$ separates points of $\alpha X \backslash X$, $\omega_{C_\alpha(X)} X \cong \alpha X$. Hence every compactification αX can be expressed in the form $\omega_{\mathscr{G}} X$ for some $\mathscr{G} \subseteq C^*(X)$.

We will denote the one-point compactification of X by ωX; hence $C_\omega(X) = \{g|_X : g \in C(\omega X)\}$ (see 1.5).

In paragraph 2 of [F], the author presents the following definition.

1.14 DEFINITION If f and g belong to $C^*(X)$, we will say that *f is equivalent to g*, denoted by $f \cong g$, if $f - g \in C_\omega(X)$. If $\mathcal{G}$ and $\mathcal{F}$ are subsets of $C^*(X)$, $\mathcal{G}$ is said to be *equivalent* to $\mathcal{F}$, denoted by $\mathcal{G} \cong \mathcal{F}$, if every function g in $\mathcal{G}$ is equivalent to some function f in $\mathcal{F}$ and conversely.

If $\mathcal{G}$ is contained in $C_\alpha(X)$, $<\mathcal{G}>$ will denote the subalgebra generated by $\mathcal{G}$ and $\mathrm{cl}_{C_\alpha(X)}<\mathcal{G}>$ will denote its closure in the uniform norm topology on $C_\alpha(X)$.

In corollary 1 of [F] we have the following useful proposition:

1.15 PROPOSITION [F] If $\mathcal{G} \subseteq C^*(X)$ then $C_{\omega_\mathcal{G}}(X) = \mathrm{cl}_{C_{\omega_\mathcal{G}}(X)}<C_\omega(X) \cup \mathcal{G}>$, (the closure in the uniform norm topology of the subalgebra generated by $C_\omega(X) \cup \mathcal{G}$) where $C_{\omega_\mathcal{G}}(X) = \{f|_X : f \in C(\omega_\mathcal{G}X)\}$ (as in 1.5).

Also, in theorem 1 of [CF] we have the following result:

1.16 THEOREM [CF] If αX is a compactification of X and $\mathcal{G} \subseteq S_\alpha$ then $\alpha X = \sup\{X \cup_f S(f) : f \in \mathcal{G}\}$ if and only if $\mathcal{G}^\alpha$ separates the points of $\alpha X \setminus X$.

On page 29 of [G], the author describes a method of constructing a compactification of X by using the singular set of a function f even if this function is not a singular map. The construction of this compactification is very similar to the construction of singular compactifications. We describe it here. Let $f : X \longrightarrow Y$ be a continuous map from a space X to a compact Hausdorff space Y. We define a topology on the set $X \cup S(f)$ as follows: The basic open neighbourhoods of the points in X will be the same as in the original space X. If $p \in S(f)$ we define a basic open neighbourhood of p to be any set of form $V \cup [f^{\leftarrow}[O] \setminus F]$ where O is an open neighbourhood of p in Y, $V = O \cap S(f)$ and F is a compact set in X.

1.17 NOTATION We will denote $X \cup S(f)$ equipped with the topology described above by $X \cup^* S(f)$.

It is shown in theorem 9 of [G] that $X \cup^* S(f)$ is indeed a Hausdorff compactification of X. We note that if f is a singular map then $X \cup_f S(f) \cong X \cup^* S(f)$.

We will also make use of the following previously established results.

1.18 PROPOSITION (Lemma 1, [G]) If $f : X \longrightarrow Y$ is a singular function mapping X into a closed subspace K of the compact Hausdorff space Y and $g : Y \longrightarrow Z$ is continuous so that $cl_Z(g \circ f[X]) = Z$, then $g \circ f$ is a singular function.

1.19 PROPOSITION (Corollary 3, [F]) If $\mathcal{F}$ and $\mathcal{G}$ are two equivalent subsets of $C^*(X)$ then $\omega_{\mathcal{G}}X$ is equivalent to $\omega_{\mathcal{F}}X$.

(It is also shown immediately following corollary 3 in [F] that the converse of the above statement fails).

1.20 PROPOSITION (Lemma 2, [F]) Let $\{\alpha_i X : i \in A\}$ be a family of compactifications of X and let $\alpha X = \sup\{\alpha_i X : i \in A\}$, then $C_\alpha(X) = cl_{C_\alpha(X)} \langle \cup \{C_{\alpha_i}(X) : i \in A\} \rangle$.

1.21 THEOREM [Theorem 2, [F]] If $\mathcal{G} \subseteq C^*(X)$ separates the points from the closed sets in X, then $\sup\{\omega_f X : f \in \mathcal{G}\} = \omega_{\mathcal{G}}X$.

In [SS] the authors use a special type of function from a space X into a compact Hausdorff space K to construct a compactification of X. This compactification is the closure of the graph of f in $\omega X \times K$ (where ωX is the one-point compactification of X). This construction is illustrated in the first theorem of [SS]. We state this theorem here. In what follows the point at infinity of the one-point compactification ωX of X will be denoted by ∞ and $N(\infty)$ will mean a neighbourhood of ∞ in ωX.

1.22 THEOREM [SS] Let X be locally compact and non-compact and let K be a compact Hausdorff space. If there is a continuous map $f : X \longrightarrow K$ from X into K such that $f[N(\infty) \cap X]$ is dense in K for all neighbourhoods $N(\infty)$ of ∞ in ωX then X has a compactification X^* with K as a remainder. Indeed, such an X^* is the closure of the graph of f in $\omega X \times K$.

If $f : X \longrightarrow K$ is a function from the space X into a space K let G_f denote the graph $\{(x,f(x)) : x \in X\}$ of f. In the proof of the above theorem the authors show that, for the function $f : X \longrightarrow K$ satisfying the property described in the statement, the map $h : X \longrightarrow \mathrm{cl}_{\omega X \times K}G_f$ defined by $h(x) = (x,f(x))$ embeds X densely into $\mathrm{cl}_{\omega X \times K}G_f$ and that $\mathrm{cl}_{\omega X \times K}G_f \backslash G_f = \{\infty\} \times K$ (hence $\mathrm{cl}_{\omega X \times K}G_f$ can be viewed as a compactification of X whose outgrowth is K).

The following is a generalization of the result found on page 607 of [CFGM]. Theorem 1.23 states that the singular compactifications coincide with those constructed by the method described in 1.22. The proof is practically identical to the one outlined in [CFMG]. For completeness we provide the details.

1.23 THEOREM Let X be locally compact and non-compact and let K be a compact Hausdorff space. If $f : X \longrightarrow K$ is a singular function which maps X densely into K then f maps $N(\infty) \cap X$ densely into K for any $N(\infty)$. Furthermore $X \cup_f S(f)$ is equivalent to $\mathrm{cl}_{\omega X \times K}G_f$ (in the sense that if m and h each embed X into $X \cup_f S(f)$ and $\mathrm{cl}_{\omega X \times K}G_f$ respectively then there exists a homeomorphism j from $X \cup_f S(f)$ onto $\mathrm{cl}_{\omega X \times K}G_f$ such that $j \circ m(x) = h(x)$). Hence a singular compactification induced by a singular function f is equivalent to the closure of the graph of f in $\omega X \times S(f)$. Conversely, if $f : X \longrightarrow K$ is a function from X into K

which maps $N(\infty) \cap X$ densely into K for any $N(\infty)$ then f is a singular map and $cl_{\omega X \times K}G_f$ is equivalent to $X \cup_f S(f)$ (as a compactification of $X \cong G_f$). Hence the closure of the graph of a function f (in $\omega X \times K$) satisfying the above property always yields a singular compactification of X.

Proof: (=>) Let X be locally compact and non-compact and let K be a compact Hausdorff space. Suppose $f : X \longrightarrow K$ is a singular function which maps X densely into K and let $\alpha X = X \cup_f S(f)$. Since f is singular, $S(f) = K$. Let $N(\infty)$ be some neighbourhood of ∞ in ωX. Then $X \backslash N(\infty)$ is compact; hence $\alpha X \backslash X \subseteq cl_{\alpha X}N(\infty)$. Thus $f^{\alpha}[\alpha X \backslash X] \subseteq f^{\alpha}[cl_{\alpha X}(N(\infty) \cap X)] = cl_K f[N(\infty) \cap X]$. By 1.7 we know that $f^{\alpha}[\alpha X \backslash X] = K$; hence $f[N(\infty) \cap X]$ is dense in K. By the Steiner-Steiner theorem we can construct a compactification $cl_{\omega X \times K}G_f$ of X ($\cong G_f$) such that the remainder $cl_{\omega X \times K}G_f \backslash G_f$ is K ($\cong \{\infty\} \times K$). We claim that $X \cup_f S(f) \cong cl_{\omega X \times K}G_f$. Let $j : X \cup_f S(f) \longrightarrow cl_{\omega X \times K}G_f$ be a function from $X \cup_f S(f)$ onto $cl_{\omega X \times K}G_f$ defined as follows: $j(x) = (x,f(x))$ if $x \in X$ and $j(k) = (\infty,k)$ if $k \in S(f) = K$. It will suffice to show that j pulls back open neighbourhoods of points in $cl_{\omega X \times K}G_f \backslash G_f = \{\infty\} \times K$ to open subsets of $X \cup_f S(f)$. Let (∞,k) be a point in $cl_{\omega X \times K}G_f \backslash G_f$ and $U \times V$ be an open neighbourhood of (∞,k) in $\omega X \times K$. Observe that since U is an open neighbourhood of ∞ in ωX then $X \backslash U$ is a compact subset of X. Hence $X \backslash (N(\infty) \cap U)$ is a non-empty compact subset of X. It is easily verified that $j^{\leftarrow}[U \times V] = (j^{\leftarrow}[U \times V] \cap G_f) \cup (j^{\leftarrow}[U \times V] \cap cl_{\omega X \times K}G_f \backslash G_f) = (U \cap f^{\leftarrow}[V]) \cup V$. We claim that $(U \cap f^{\leftarrow}[V]) \cup V$ is open in $X \cup_f S(f)$. Let t be a point in V ($\subseteq S(f) = K$). Then, if C is a compact subset of X, $V \cup f^{\leftarrow}[V] \backslash C$ is an open neighbourhood of t in $X \cup_f S(f)$. If we choose C to be the compact subset $X \backslash (N(\infty) \cap U)$ then $V \cup f^{\leftarrow}[V] \backslash C = V \cup f^{\leftarrow}[V] \backslash (X \backslash (N(\infty) \cap U)) = V \cup (f^{\leftarrow}[V] \cap N(\infty) \cap U) \subseteq (U \cap f^{\leftarrow}[V]) \cup V$. Hence $V \cup f^{\leftarrow}[V] \backslash C$ is an open neighbourhood of t which is contained in $(U \cap f^{\leftarrow}[V]) \cup V$. Thus $j^{\leftarrow}[U \times V] = (U \cap f^{\leftarrow}[V]) \cup V$ is open in $X \cup_f S(f)$. It follows that j is continuous and that $X \cup_f S(f) \cong cl_{\omega X \times K}G_f$ (as a compactification of X). Hence a singular compactification induced by a singular function f is equivalent to the closure of the graph of f in $\omega X \times S(f)$.

(<=) Suppose the function $f : X \longrightarrow K$ is a function from X into K which maps $N(\infty) \cap X$ densely into K for any $N(\infty)$. Then by the Steiner-Steiner theorem we can construct a compactification of $cl_{\omega X \times K}G_f$ of X ($\cong G_f$) such that the remainder $cl_{\omega X \times K}G_f \setminus G_f$ is K ($\cong \{\infty\} \times K$). We claim that f is a singular function. Let U be any non-empty open subset of K. By hypothesis f[X] is dense in K; hence $U \cap f[X]$ is non-empty. We wish to show that $cl_X f^{\leftarrow}[U]$ is not compact. Suppose $cl_X f^{\leftarrow}[U]$ is compact. Observe that $\{\infty\} \times U \subseteq \omega X \setminus cl_{\omega X} f^{\leftarrow}[U] \times U$; hence $\omega X \setminus cl_{\omega X} f^{\leftarrow}[U] \times U$ is a non-empty open subset of $\omega X \times K$ which meets $cl_{\omega X \times K}G_f$ (since $cl_{\omega X \times K}G_f \setminus G_f = \{\infty\} \times K$ and $U \subseteq K$). We claim that $(\omega X \setminus cl_{\omega X} f^{\leftarrow}[U] \times U) \cap G_f = \emptyset$. Suppose $(x,f(x))$ is a point in $(\omega X \setminus cl_{\omega X} f^{\leftarrow}[U] \times U) \cap G_f$. Then $f(x) \in U$ and $x \in \omega X \setminus cl_{\omega X} f^{\leftarrow}[U]$. But if $f(x) \in U$ then $x \in f^{\leftarrow}[U]$. This contradicts the fact that $x \in \omega X \setminus cl_{\omega X} f^{\leftarrow}[U]$. Hence $(\omega X \setminus cl_{\omega X} f^{\leftarrow}[U] \times U) \cap G_f = \emptyset$. Since G_f is dense in $cl_{\omega X \times K}G_f$ we have a contradiction. Thus $cl_X f^{\leftarrow}[U]$ cannot be compact. It follows that f is a singular map whose singular set S(f) is K (since by hypothesis f[X] is dense in K and $cl_X f^{\leftarrow}[U]$ is non-compact for any open subset U of K). In the proof of (=>) we have shown that $X \cup_f S(f) \cong cl_{\omega X \times K}G_f$ (as a compactification of X). Hence the closure of the graph of a function f (in $\omega X \times K$) satisfying the above property always yields a singular compactification of X.

QED

CHAPTER 2

PROPERTIES OF SINGULAR COMPACTIFICATIONS

In chapter one we have introduced the family of singular compactifications and described some of their basic properties. We now investigate these from a different point of view. We will show that any singular compactification $X \cup_f S(f)$ is equivalent to a singular compactification $X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$ induced by the evaluation map $e_{\mathcal{G}}$, where $\mathcal{G}$ is some subset of S_β. This new representation of singular compactifications will allow us to derive many interesting properties possessed by this family. These properties are difficult to observe when the singular compactification is expressed in its original form $X \cup_f S(f)$.

The results obtained in this chapter will not only shed light on the nature of singular compactifications but will also be used as basic tools to solve the following three open questions stated in [G]:

1) Can every compactification of X be expressed as the supremum of singular compactifications?

2) When is the supremum of a collection of singular compactifications a singular compactification?

3) Suppose $X \cup_f S(f)$ and $X \cup_g S(g)$ are two singular compactifications, and S(f) and S(g) are the same space, i.e. f and g both map X densely into the same space K. When is $X \cup_f S(f)$ equivalent to $X \cup_g S(g)$?

2.1 LEMMA Let $f : X \longrightarrow Y$ be a continuous function from the space X into a compact Hausdorff space Y. If αX is a compactification of X and f extends to f^α :

$\alpha X \longrightarrow Y$ so that f^α separates the points of $\alpha X \backslash X$, then αX is equivalent (as a compactification of X) to $X \cup^* S(f)$.

Proof: By 1.7, $f^\alpha[\alpha X \backslash X] = S(f)$. We define a function $j : \alpha X \longrightarrow X \cup^* S(f)$ as follows: $j(x) = f^\alpha(x)$ if x belongs to $\alpha X \backslash X$ and $j(x) = x$ if x belongs to X. Clearly j is one-to-one. We now verify that j is continuous. It is sufficient to verify that j pulls back open neighbourhoods of points in S(f) to open sets in αX. Recall that the open neighbourhoods of points in S(f) are of form $V \cup (f^{\leftarrow}[O] \backslash F)$ where O is an open set in Y, $V = O \cap S(f)$ and F is a compact set in X. Note that $j^{\leftarrow}[V \cup f^{\leftarrow}[O]]$

$= j^{\leftarrow}[V] \cup f^{\leftarrow}[O]$

$=(f^{\alpha\leftarrow}[V] \cap \alpha X \backslash X) \cup f^{\leftarrow}[O]$

$= f^{\alpha\leftarrow}[O]$, which is an open subset of αX.

It follows that $j^{\leftarrow}[V \cup f^{\leftarrow}[O] \backslash F]$ is open in αX, hence j is continuous. The lemma follows.

QED

The following corollary is an easy consequence of the lemma.

2.2 COROLLARY If αX is a compactification of X then αX can be expressed in the form of $X \cup^* S(f)$, i.e. αX is equivalent to $X \cup^* S(e_{C_\alpha(X)})$.

In theorem 2.1 of [Ma], the author proves the following statement: " If X is locally compact and K is a Hausdorff space, then there exists a compactification αX of X such that $\alpha X \backslash X$ is homeomorphic to K iff K is a continuous image of $\beta X \backslash X$". In corollary 2.3, we give a similar result referring specifically to *singular* compactifications.

2.3 COROLLARY Let X be a locally compact Hausdorff space and Y be a compact Hausdorff space. Then there exists a topology on the disjoint union $X \cup Y$ of X and Y such that the resulting topological space is a singular compactification of

X iff Y is homeomorphic to the singular set of some evaluation map $e_{\mathscr{G}}$ induced by a subset $\mathscr{G}$ of $C^*(X)$.

Proof: (=>) This direction follows from corollary 2.2.

(<=) This direction follows from the definition of $X \cup^* S(f)$.

QED

2.4 THEOREM a) Let $f \in C^*(X)$. Then $\omega_f X$ is equivalent to $X \cup^* S(f)$. In particular, if f is a singular map then $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$.

b) If $\mathscr{G} \subseteq C^*(X)$ and $\omega_{\mathscr{G}} X$ is a singular compactification then $t = e_{\mathscr{G}}{}^{\omega_{\mathscr{G}}} \circ r|_X$ is a singular map (where $r : \omega_{\mathscr{G}} X \longrightarrow \omega_{\mathscr{G}} X \backslash X$ is a retraction map) and $\omega_{\mathscr{G}} X$ is equivalent to $X \cup_t S(t)$.

Proof: The proof of part a) follows from the fact that f^{ω} separates the points of $\omega_f X \backslash X$ (see 1.13, 1.7 and lemma 2.1). If f is a singular map then $X \cup^* S(f)$ is equivalent to $X \cup_f S(f)$ (1.17).

We now prove part b). Let $\mathscr{G} \subseteq C^*(X)$ and suppose $\omega_{\mathscr{G}} X$ is a singular compactification. Let $r : \omega_{\mathscr{G}} X \longrightarrow \omega_{\mathscr{G}} X \backslash X$ be a retraction. Recall that $r|_X$ is a singular map (see remark 1.3). Since the composition of a continuous function with a singular function is singular (1.18), then $t = e_{\mathscr{G}}{}^{\omega_{\mathscr{G}}} \circ r|_X$ is a singular map. Since t extends continuously to $e_{\mathscr{G}}{}^{\omega_{\mathscr{G}}} \circ r$ and $e_{\mathscr{G}}{}^{\omega_{\mathscr{G}}} \circ r$ separates points of $\omega_{\mathscr{G}} X \backslash X$, it follows from lemma 2.1 that $\omega_{\mathscr{G}} X$ is equivalent to $X \cup_t S(t)$.

QED

In what follows, we will show that it is possible to express any singular compactification in a form that involves only real-valued singular maps.

We require the following lemma.

2.5 LEMMA If αX is a singular compactification, then every $f \in C_\alpha(X)$ is equivalent to some function $h \in S_\alpha$ (see 1.14).

Proof: Let αX be a singular compactification and let $f \in C_\alpha(X)$. Then there exists a retraction map r mapping αX onto $\alpha X \backslash X$. We have already seen that $r|_X$ is a singular map (see 1.3). Then $f^\alpha \circ r|_X$ is a singular map and belongs to S_α, (1.18). Let $g = f - f^\alpha \circ r|_X$. If $x \in \alpha X \backslash X$, $g^\alpha(x) = f^\alpha(x) - f^\alpha \circ r(x) = f^\alpha(x) - f^\alpha(x) = 0$.
Therefore $g^\alpha|_{\alpha X \backslash X}$ is the $\mathbf{0}$-function on $\alpha X \backslash X$. Hence $g \in C_\omega(X)$. Thus f is equivalent to $h = f^\alpha \circ r|_X$.

QED

In the second paragraph following 1.13 we showed that any compactification αX can be expressed in the form $\omega_{\mathscr{G}} X$ (where $\mathscr{G} \subseteq C_\alpha(X)$). In 2.6 we show that if αX is a singular compactification then αX can be expressed in the form $\omega_{S_\alpha} X$.

2.6 THEOREM If αX is a singular compactification then αX is equivalent to $\omega_{S_\alpha} X$. Hence every singular compactification αX of X is the supremum of the family $\{X \cup_f S(f) : f \in S_\alpha\}$ of singular compactifications.

Proof: Let $\gamma X = \omega_{S_\alpha} X$. By 1.15, $C_\gamma(X) = cl_{C_\gamma(X)} < C_\omega(X) \cup S_\alpha >$. Since $C_\omega(X) \cup S_\alpha \subseteq C_\alpha(X)$ then $\omega_{S_\alpha} X \leq \alpha X$.

Let $f \in C_\alpha(X)$. By lemma 2.5, $f \cong g$ for some $g \in S_\alpha$. It is shown in 1.19 that, if $f \cong g$ then $\omega_f X$ is equivalent to $\omega_g X$. Now $C_{\omega_f}(X) = cl_{C_{\omega_f}(X)} < C_\omega(X) \cup \{f\} >$. By 1.19, $cl_{C_{\omega_f}(X)} < C_\omega(X) \cup \{f\} > = cl_{C_{\omega_g}(X)} < C_\omega(X) \cup \{g\} > \subseteq C_\gamma(X)$ (see 1.15). It follows that $C_\alpha(X) \subseteq C_\gamma(X)$; consequently $\alpha X \leq \omega_{S_\alpha} X$. Since $\omega_{S_\alpha} X \leq \alpha X$ and $\alpha X \leq \omega_{S_\alpha} X$, αX is equivalent to $\omega_{S_\alpha} X$. By 1.13, $S_\alpha{}^\alpha$ separates the points of $\alpha X \backslash X$. By 1.16, $\omega_{S_\alpha} X$ is equivalent to $\sup\{X \cup_f S(f) : f \in S_\alpha\}$. Hence αX is the supremum of the family of singular compactifications $\{X \cup_f S(f) : f \in S_\alpha\}$ (where $X \cup_f S(f) \cong \omega_f X$ for each f in S_α, by 2.4 a)). This is the assertion of the theorem.

QED

The converse of the above theorem fails as we shall now see. Recall that S_β is the set of all real-valued singular functions. On page 20 of [G], it is shown that $\beta\mathbb{N} = \sup\{\mathbb{N} \cup_f S(f) : f \in S_\beta\}$. Since $S_\beta{}^{\omega_{S_\beta}}$ separates points of $(\omega_{S_\beta}\mathbb{N})\backslash\mathbb{N}$ (1.13), then $\omega_{S_\beta}\mathbb{N} = \sup\{\mathbb{N} \cup_f S(f) : f \in S_\beta\}$ (by 1.16). Hence $\beta\mathbb{N}$ is equivalent to $\omega_{S_\beta}\mathbb{N}$, the smallest compactification to which all real-valued singular maps extend. Since $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable, $\beta\mathbb{N}\backslash\mathbb{N}$ cannot be the continuous image of a separable space, hence $\beta\mathbb{N}$ cannot be a singular compactification. It follows that not every compactification αX of form $\omega_{S_\alpha}X$ is singular. We shall later describe, in this chapter, conditions which allow us to recognize those that are.

In [G], the author asks:

Can every compactification of X be expressed as the supremum of singular compactifications?

We give the following example which provides a negative answer to the question.

2.7 EXAMPLE Consider the two-point compactification of $\mathbb{R}$, $\alpha\mathbb{R} = \mathbb{R} \cup \{p_1, p_2\}$. We claim that $\alpha\mathbb{R}$ cannot be the supremum of singular compactifications i.e. $\alpha\mathbb{R}$ cannot be expressed in the form $\omega_{S_\alpha}\mathbb{R}$.

Proof: Suppose that $\alpha\mathbb{R} = \sup\{\gamma\mathbb{R} : \gamma\mathbb{R}$ is a singular compactification, $\gamma\mathbb{R} \leq \alpha\mathbb{R}\}$. Since every singular compactification γX can be expressed in the form $\gamma X = \omega_{S_\gamma}X = \sup\{X \cup_f S(f) : f \in S_\gamma\} = \{\omega_f X : f \in S_\gamma\}$ (see 2.6 and 2.4 a)), then $\alpha\mathbb{R} = \sup\{\sup\{\omega_f\mathbb{R} : f \in S_\gamma\} : \gamma\mathbb{R} \leq \alpha\mathbb{R}\} = \sup\{\omega_f\mathbb{R} : f \in S_\alpha\}$, (where S_α is the set of all singular real-valued maps which extend to $\alpha\mathbb{R}$). By 1.16, $S_\alpha{}^\alpha$ separates the points of $\alpha\mathbb{R}\backslash\mathbb{R}$. Let $f \in S_\alpha$ such that $f^\alpha(p_1) \neq f^\alpha(p_2)$. By 1.7, $f^\alpha[\alpha\mathbb{R}\backslash\mathbb{R}] = S(f) = \{f^\alpha(p_1), f^\alpha(p_2)\}$. But since f is a singular map, f maps $\mathbb{R}$ onto $S(f) = \{f^\alpha(p_1),$

$f^\alpha(p_2)\}$. Since $\mathbb{R}$ is connected this is clearly a contradiction. Consequently, $\gamma\mathbb{R}$ is not the supremum of singular compactifications. We have thus shown that not every compactification γX can be expressed in the form $\omega_{S_\gamma}X$.

QED

In 2.6 and 2.7 we have shown that the two-point compactification of $\mathbb{R}$ is not a singular compactification. (This is also obvious from the fact that $\mathbb{R}$ is connected and $\alpha\mathbb{R}\backslash\mathbb{R}$ is not). Note that the fact that the two-point compactification of $\mathbb{R}$ is not singular does not imply that the largest compactification of $\mathbb{R}$ which is singular must be the one-point compactification of $\mathbb{R}$. It simply implies that no compactification $\gamma\mathbb{R}$ of $\mathbb{R}$ larger than the two-point compactification of $\mathbb{R}$ is singular (by 1.4). Note that sin(x) is a singular map and that $S(\sin(x)) = [-1,1]$. Then $\mathbb{R} \cup_{\text{sine}} S(\text{sine})$ is a singular compactification of $\mathbb{R}$ which is not comparable with the two-point compactification of $\mathbb{R}$. We are however guaranteed that $\beta\mathbb{R}$ is not singular by the existence of a non-singular compactification of $\mathbb{R}$ and by 1.4. (Again, since $\beta\mathbb{R}$ is connected and $\beta\mathbb{R}\backslash\mathbb{R}$ isn't we have an even simpler reason why $\beta\mathbb{R}$ is not a singular compactification).

Let us now summarize some of the main results developed so far in this chapter.

We have shown that:

1) A singular compactification αX is of the form $\omega_{S_\alpha}X$ (2.6),

2) There exists spaces X which have at least one compactification αX which is not the supremum of a collection of singular compactifications (Example: The two point compactification of the space of the real numbers $\mathbb{R}$).

3) The supremum of a family of singular compactifications need not be a singular compactification (by the example of $\beta\mathbb{N}$).

We would now like to consider the following question:

When is the supremum of a collection of singular compactifications a singular compactification?

We begin with a brief discussion of the question. Given a family $\mathcal{A} = \{\alpha_i X : i \in A\}$ of singular compactifications of a space X, we seek ways of recognizing when the supremum, say αX, of $\mathcal{A}$ is itself a singular compactification. There are many possible approaches to this problem: one could look for a property possessed by the family $\mathcal{A}$ which will guarantee that αX is a singular compactification. But αX may be the supremum of many families of singular compactifications. Each one of these families (including the family of all singular compactifications less than or equal to αX) would have to possess this particular property. That αX is the supremum of the collection $\mathcal{A}$ tells us that αX is not a compactification such as the two-point compactification of $\mathbb{R}$ (which is not the supremum of any collection of singular compactifications (see 2.7)). After some reflection, we have chosen to study αX as the supremum of "some" family of singular compactifications rather than αX "the supremum of the collection $\mathcal{A}$ of singular compactifications". This approach has turned out to be the most fruitful. We will eventually characterize a supremum αX of singular compactifications by a property possessed by the set $S_\alpha(X)$ of all real-valued singular functions in $C_\alpha(X)$. Proposition 2.8 will show us that suprema of singular compactifications are *precisely* compactifications of the form $\omega_{\mathcal{G}}X$, where $\mathcal{G}$ is contained in S_β. Given this result, to answer our question, **it will only be necessary to characterize those compactifications of form $\omega_{\mathcal{G}}X$, where $\mathcal{G}$ is contained in S_β.**

We begin with the following proposition :

2.8 PROPOSITION Let X be a topological space. The compactification αX of X is a supremum of a collection of singular compactifications iff αX is equivalent to $\omega_{\mathscr{G}}X$ for some $\mathscr{G}$ contained in S_α.

Proof: (=>) Suppose $\mathscr{A} = \{\alpha_i X : i \in A\}$ is a collection of singular compactifications and αX is $\sup\{\alpha_i X : i \in A\}$. Then,

$\alpha X = \sup\{\omega_{S_{\alpha_i}} X : i \in A\}$ (2.6)

$= \sup\{ \sup\{X \cup_f S(f) : f \in S_{\alpha_i}\} : i \in A\}$ (by 1.13 and 1.16)

$= \sup\{X \cup_f S(f) : f \in \cup\{S_{\alpha_i} : i \in A\}\}$.

Hence, by 1.16, $(\cup S_{\alpha_i})^\alpha$ separates the points of $\alpha X \backslash X$. Thus αX is equivalent to $\omega_{\cup S_{\alpha_i}} X$. By 1.15, $\cup S_{\alpha_i}$ is contained in S_α. Hence we have shown that the supremum of a collection of singular compactifications is of form $\omega_{\mathscr{G}}X$, where $\mathscr{G}$ is a subset of S_α.

(<=) Suppose αX is equivalent to $\omega_{\mathscr{G}}X$, where $\mathscr{G}$ is contained in S_α. Then, by 1.13 and 1.16, αX is the supremum of the collection $\{X \cup_f S(f) : f \in \mathscr{G}\}$ of singular compactifications.

QED

Our question can then be reformulated as follows:

If $\mathscr{G} \subseteq S_\beta$, when is $\omega_{\mathscr{G}}X$ a singular compactification?

On page 15 of [G], we are given an example where the supremum of just two singular compactifications is not singular. In an attempt to establish a condition which precisely describes when $[X \cup_f S(f)] \vee [X \cup_g S(g)]$ is singular the author erroneously concludes that it is necessary and sufficient that the evaluation map $h = f \times g$ be a singular map (theorem 8 of [G]). We provide the following counter-example.

Consider the natural numbers $\mathbb{N}$. Let A denote the even natural numbers, B denote the odd natural numbers and $C = \{3\}$. Define the maps $f : \mathbb{N} \longrightarrow \{0, 1\}$ and $g : \mathbb{N} \longrightarrow \{0, 1\}$ as follows: $f[A \cup C] = \{0\}$ and $f[B \setminus C] = \{1\}$, $g[A] = \{1\}$ and $g[B] = \{0\}$. Clearly both f and g are singular maps and $X \cup_f S(f)$ and $X \cup_g S(g)$ each describe a two-point compactification. We verify that $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$, hence their supremum will be their common value, a singular compactification.

Let $k : X \cup_f S(f) \longrightarrow S \cup_g S(g)$ be defined as follows: $k(x) = x$ if $x \in X$ and $k(0) = 1$ and $k(1) = 0$. Now $\{1\} \cup g^{\leftarrow}(1)$ is a basic open neighbourhood of 1 in $X \cup_g S(g)$. Now $k^{\leftarrow}[\{1\} \cup g^{\leftarrow}[1]] = \{0\} \cup g^{\leftarrow}(1) = \{0\} \cup A = \{0\} \cup f^{\leftarrow}(0)\setminus\{3\}$ a basic open neighbourhood of 0 in $X \cup_f S(f)$. Consider now the set $\{0\} \cup g^{\leftarrow}(0)$ a basic open neighbourhood of 0 in $X \cup_g S(g)$. Now $k^{\leftarrow}[\{0\} \cup g^{\leftarrow}(0)] = \{1\} \cup B = \{1\} \cup (f^{\leftarrow}[1] \cup \{3\}) = (\{1\} \cup f^{\leftarrow}[1]) \cup \{3\}$ an open neighbourhood of 1 in $X \cup_f S(f)$. Thus k is continuous; hence $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$. Note that the evaluation map $h = f \times g$ maps A to $\{(0,1)\}$, $B\setminus C$ to $\{(1,0)\}$ and C to $\{(0,0)\}$. But $h^{\leftarrow}[\{(0,0)\}] = \{3\}$. Since h pulls back an open set, $\{(0,0)\}$, to an open set, $\{3\}$, whose closure in $\mathbb{N}$ is compact then h cannot be singular. Consequently the fact that $[X \cup_f S(f)] \vee [X \cup_g S(g)]$ is singular is not sufficient to imply that $h = f \times g$ is a singular map.

2.9 REMARK For further reference, we would like to emphasize an important point illustrated in the above example. In this example, $\{f, g\}$ is contained in S_β, hence, by 1.16, $\omega_{\{f, g\}}X$ is equivalent to $X \cup_f S(f) \vee X \cup_g S(g)$. We have shown that the evaluation map $e_{\{f, g\}} = f \times g$ is not a singular map even though $\omega_{\{f, g\}}X$ was proven to be a singular compactification. Hence, if $\mathcal{G}$ is an arbitrary subset of S_β, it is not sufficient that $\omega_{\mathcal{G}}X$ be a singular compactification for $e_{\mathcal{G}}$ to be a singular map, i.e. "**$\omega_{\mathcal{G}}X$ being singular does not imply that $e_{\mathcal{G}}$ is singular**". However we will show in 3.19 that, for any space X and any **subalgebra** $\mathcal{G}$ of $C^*(X)$ such that $\mathcal{G} \subseteq S_\beta$, $e_{\mathcal{G}}$ is singular iff $\omega_{\mathcal{G}}X$ is a singular compactification.

Before we pursue our goal of characterizing those spaces of form $\omega_{\mathcal{G}}X$ which are singular we need a little more preparation. In 2.6 we have shown that, if αX is a singular compactification, then αX is equivalent to $\omega_{S_\alpha}X$. Given the original definition of a singular compactification, one would naturally like to find a singular map which induces $\omega_{S_\alpha}X$ (i.e. a singular map f such that $\omega_{S_\alpha}X$ is equivalent (as a compactification of X) to $X \cup_f S(f)$). In Remark 2.9, we have shown that this singular map need not be e_{S_α}. We now describe some properties possessed by singular maps which induce a singular compactification of form $\omega_{\mathcal{G}}X$.

2.10 THEOREM Let $\mathcal{G} \subseteq S_\beta$. Then the following are equivalent:

1) $\omega_{\mathcal{G}}X$ is a singular compactification

2) There is a singular function $k : X \longrightarrow K$ mapping X densely into some compact Hausdorff space K which extends to $k^{\omega_{\mathcal{G}}} : \omega_{\mathcal{G}}X \longrightarrow K$ such that $k^{\omega_{\mathcal{G}}}$ is one-to-one on $\omega_{\mathcal{G}}X\backslash X$ (hence $\omega_{\mathcal{G}}X$ is equivalent to $X \cup_k S(k)$).

Proof: 1) => 2) Since $\omega_{\mathcal{G}}X$ is a singular compactification then there exists a retraction map $r : \omega_{\mathcal{G}}X \longrightarrow \omega_{\mathcal{G}}X\backslash X$ (by 1.2). We claim that the map $t = (e_{\mathcal{G}}{}^{\omega_{\mathcal{G}}})|_{\omega_{\mathcal{G}}X\backslash X} \circ r|_X$ is the required function. Clearly $\omega_{\mathcal{G}}X \cong X \cup_t S(t)$ (by 2.4 b)). Since $r|_X$ is a singular map (see 1.3) then, by 1.18, t is a singular map. Observe that $t^{\omega_{\mathcal{G}}}|_{\omega_{\mathcal{G}}X\backslash X} = e_{\mathcal{G}}{}^{\omega_{\mathcal{G}}}|_{\omega_{\mathcal{G}}X\backslash X}$. By 1.13, $\mathcal{G}^{\omega_{\mathcal{G}}}$ separates the points of $\omega_{\mathcal{G}}X\backslash X$ hence by 1.12 $t^{\omega_{\mathcal{G}}} = (e_{\mathcal{G}}{}^{\omega_{\mathcal{G}}})|_{\omega_{\mathcal{G}}X\backslash X} \circ r$ is one-to-one on $\omega_{\mathcal{G}}X\backslash X$.

2) => 1) Let k be a singular map such that $k^{\omega_{\mathcal{G}}}$ separates points of $\omega_{\mathcal{G}}X\backslash X$. By 2.1, $\omega_{\mathcal{G}}X \cong X \cup^* S(k) = X \cup_k S(k)$ (1.17). Hence $\omega_{\mathcal{G}}X$ is singular.

QED

The next lemma describes more specifically a singular map which induces the singular compactifications $\omega_{\mathcal{G}}X$.

2.11 THEOREM Let αX be a singular compactification of X. Let $r : \alpha X \longrightarrow \alpha X \setminus X$ be a retraction map, and define $\mathcal{F}$ to be $\{f \circ r|_X : f \in C(\alpha X)\}$. Then $\mathcal{F} \subseteq S_\alpha$, $\mathcal{F}$ is a subalgebra of $C_\alpha(X)$, $e_{\mathcal{F}}$ is a singular map, $e_{\mathcal{F}}{}^\alpha$ separates points of $\alpha X \setminus X$, and $\alpha X \cong X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}}) \cong \omega_{\mathcal{F}}X$.

Proof: Let αX, the mapping r, and the family of functions $\mathcal{F}$ be as described in the statement of the theorem. By 1.3 and 1.18, $\mathcal{F} \subseteq S_\alpha$. It is easily verified that $\mathcal{F}$ is a subalgebra of $C_\alpha(X)$. We will now show that $e_{\mathcal{F}}$ is a singular map. Let $J = \Pi_{g \in \mathcal{F}} S(g)$ and $t \in cl_J e_{\mathcal{F}}[X]$ and U be a basic open neighbourhood of t in J of the form $J \cap [\cap\{U_{f_k \circ r|_X} : k = 1 \text{ to } n\}]$ where $\{f_1,\ldots, f_n\} \subseteq C(\alpha X)$. Then $e_{\mathcal{F}}{}^{\leftarrow}(t) \subseteq e_{\mathcal{F}}{}^{\leftarrow}[U] = \cap\{(f_k \circ r|_X)^{\leftarrow}[U_{f_k \circ r|_X}]: k = 1 \text{ to } n\} = \cap\{r|_X{}^{\leftarrow} \circ f_k{}^{\leftarrow}[U_{f_k \circ r|_X}] : k = 1 \text{ to } n\}$. Suppose $cl_X \cap \{r|_X{}^{\leftarrow} \circ f_k{}^{\leftarrow}[U_{f_k \circ r|_X}] : k = 1 \text{ to } n\}$ is compact. Note that $\cap\{r|_X{}^{\leftarrow} \circ f_k{}^{\leftarrow}[U_{f_k \circ r|_X}] : k = 1 \text{ to } n\} = r|_X{}^{\leftarrow}[\cap\{f_k{}^{\leftarrow}[U_{f_k \circ r|_X}] : k = 1 \text{ to } n\}]$. Hence $cl_X r|_X{}^{\leftarrow}[\cap\{f_k{}^{\leftarrow}[U_{f_k \circ r|_X}] : k = 1 \text{ to } n\}]$ is compact. But this contradicts the fact that $r|_X$ is a singular map (see 1.3). Hence $e_{\mathcal{F}}$ is a singular map.

We now show that $\alpha X \cong X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}})$. Recall that $C_{\omega_{\mathcal{F}}}(X) = cl_{C_\alpha(X)} \langle C_\omega(X) \cup \mathcal{F} \rangle$ (1.15). Since $\mathcal{F} \subseteq S_\alpha \subseteq C_\alpha(X)$, then $\omega_{\mathcal{F}}X \leq \alpha X$. Thus every function $f^\alpha \circ r|_X$ in $\mathcal{F}$ extends to the function $f^\alpha \circ r$ on αX. Let x and y be distinct points in $\alpha X \setminus X$. Then $r(x) = x \neq y = r(y)$. Since $C(\alpha X)$ separates the points of $\alpha X \setminus X$ then there is a function f in $C(\alpha X)$ such that $f(x) \neq f(y)$. This means that $\mathcal{F}^\alpha$ separates the points of $\alpha X \setminus X$. Hence $\omega_{\mathcal{F}}X$ is equivalent to $\sup\{\omega_f X : f \in \mathcal{F}\} \cong \alpha X$ (by 1.16). Thus

$\alpha X \cong X \cup^* S(e_{\mathcal{F}})$ (2.1)

$\cong X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}})$ (by 1.17).

QED

We now proceed to answer our question: When is the supremum of singular compactifications a singular compactification? (Equivalently, when is a compactification of form $\omega_{\mathcal{G}}X$ (where $\mathcal{G}$ is contained in S_β) singular?

In what follows, we will require the following concepts. If $\mathcal{B}$ is a collection of functions in $C^*(X)$, a *maximal stationary set* of $\mathcal{B}$ is a subset of X maximal with respect to the property that every f in $\mathcal{B}$ is constant on it.

The maximal stationary sets of a subalgebra are briefly discussed in 16.31 of [GJ].

Let $\mathcal{G} \subseteq C^*(X)$, x be a point in X and $\mathcal{G}^+ = \{f - r : f \in \mathcal{G}, r \in \mathbb{R}\}$. The symbol ${}_xK_{\mathcal{G}}$ will denote the set $\cap\{Z(f) : f \in \mathcal{G}^+, x \in Z(f)\}$. Thus $y \in {}_xK_{\mathcal{G}}$ iff $f(y) = f(x)$ for each $f \in \mathcal{G}$. Suppose αX is a compactification of X such that $\mathcal{G}$ (hence $\mathcal{G}^+$) is a subset of $C_\alpha(X)$. For $x \in \alpha X$, let ${}_xK_{\mathcal{G}}{}^\alpha = \cap\{Z(f^\alpha) : f \in \mathcal{G}^+, x \in Z(f^\alpha)\}$. It is clear that the subset ${}_xK_{\mathcal{G}}$ (${}_xK_{\mathcal{G}}{}^\alpha$) is a maximal stationary set of $\mathcal{G}$ ($\mathcal{G}^\alpha$) which contains the point x. It is easily observed that, given $\mathcal{G} \subseteq C^*(X)$, the collection $\{{}_xK_{\mathcal{G}} : x \in X\}$ forms a partition of X.

2.12 THEOREM Let αX be a compactification of X. Let $\mathcal{G}$ be a subset of S_α such that the evaluation map $e_{\mathcal{G}}{}^\alpha : \alpha X \longrightarrow \Pi_{f\in\mathcal{G}}S(f)$ separates the points of $\alpha X \backslash X$. Then αX is equivalent to $\omega_{\mathcal{G}}X$. Furthermore the following are equivalent:

1) $e_{\mathcal{G}}$ is a singular map and $\omega_{\mathcal{G}}X(\cong \alpha X)$ is equivalent to the singular compactification $X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$

2) $e_{\mathcal{G}}[X] \subseteq e_{\mathcal{G}}{}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$.

3) $e_{\mathcal{G}}$ is a singular map.

4) $e_{\mathcal{F}}$ is a singular map for every finite subset $\mathcal{F}$ of $\mathcal{G}$.

5) ${}_xK_{\mathcal{G}}{}^{\omega_{\mathcal{G}}} \cap (\omega_{\mathcal{G}}X \backslash X)$ is a singleton set for every $x \in X$.

Proof: That αX is equivalent to $\omega_{\mathcal{G}}X$ follows from 1.13.

1) => 3) Obvious.

3) => 1) By 2.1, $\omega_{\mathcal{G}}X$ is equivalent to $X \cup^* S(e_{\mathcal{G}})$ (since $e_{\mathcal{G}}^{\omega_{\mathcal{G}}}$ separates the points of $\omega_{\mathcal{G}}X \backslash X$). Since $e_{\mathcal{G}}$ is singular $X \cup^* S(e_{\mathcal{G}})$ is equivalent to $X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$ (1.17).

2) <=> 3) This is a special case of 1.8.

3) => 4) Let $P = \Pi_{f \in \mathcal{G}} f[X]$. Suppose the function $e_{\mathcal{G}} : X \longrightarrow \mathrm{cl}_P e_{\mathcal{G}}[X]$ is a singular map. Define $M_{\mathcal{G},\mathcal{F}} : \Pi_{f \in \mathcal{G}} f[X] \longrightarrow \Pi_{f \in \mathcal{F}} f[X]$ by $M_{\mathcal{G},\mathcal{F}}(\langle f(x)\rangle_{f \in \mathcal{G}}) = \langle f(x)\rangle_{f \in \mathcal{F}}$. Then $e_{\mathcal{F}} = M_{\mathcal{G},\mathcal{F}} \circ e_{\mathcal{G}}$ so $e_{\mathcal{F}}$ is singular (by 1.15).

4) => 2) Let $\mathcal{G} \subseteq S_\beta$, and suppose that, for every finite subset $\mathcal{F}$ of $\mathcal{G}$, $e_{\mathcal{F}}$ is a singular map. Then, since 3) => 1), $X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}})$ is equivalent to $\omega_{\mathcal{F}}X$. We will show that $e_{\mathcal{G}}[X] \subseteq e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$ by showing that $e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X] \cap e_{\mathcal{G}}[X]$ is densely contained in $e_{\mathcal{G}}[X]$. Let $p \in e_{\mathcal{G}}[X]$. Then $p = e_{\mathcal{G}}(x) = \langle f(x)\rangle_{f \in \mathcal{G}}$ for some $x \in X$. Let U be a basic open neighbourhood of p in $\Pi_{f \in \mathcal{G}} S(f)$. We will show that $U \cap e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$ is non-empty. Let $\mathcal{F} = \{f \in \mathcal{G} : \pi_f[U] \neq S(f)\}$ where π_f is the f^{th} projection map with domain $\Pi_{f \in \mathcal{G}} S(f)$. We will denote the elements of $\mathcal{F}$ as $\{f_1, f_2, \ldots, f_n\}$ (where the indices correspond to the nontrivial components of U). Since $\omega_{\mathcal{F}}X$ is a singular compactification, $e_{\mathcal{F}}[X] \subseteq e_{\mathcal{F}}^{\omega_{\mathcal{F}}}[\omega_{\mathcal{F}}X \backslash X]$ (by 1) implies 3) implies 2)). Consequently there exists a point y in $\omega_{\mathcal{F}}X \backslash X$ such that $e_{\mathcal{F}}^{\omega_{\mathcal{F}}}(y) = (f_1^{\omega_{\mathcal{F}}}(y), f_2^{\omega_{\mathcal{F}}}(y), \ldots, f_n^{\omega_{\mathcal{F}}}(y)) = (f_1(x), \ldots, f_n(x)) = e_{\mathcal{F}}(x)$. Now $\omega_{\mathcal{F}}X \leq \omega_{\mathcal{G}}X$, hence there is a function $\pi_{\omega_{\mathcal{G}}\omega_{\mathcal{F}}} : \omega_{\mathcal{G}}X \longrightarrow \omega_{\mathcal{F}}X$ which maps $\omega_{\mathcal{G}}X$ onto $\omega_{\mathcal{F}}X$, fixing the points of X. Let $u \in \pi_{\omega_{\mathcal{G}}\omega_{\mathcal{F}}}^{\leftarrow}(y) \subseteq \omega_{\mathcal{G}}X \backslash X$. Then $f_k^{\omega_{\mathcal{G}}}(u) = f_k^{\omega_{\mathcal{F}}} \circ \pi_{\omega_{\mathcal{G}}\omega_{\mathcal{F}}}(u) = f_k^{\omega_{\mathcal{F}}}(y) = f_k(x)$ for $k = 1$ to n. Then for each $k = 1$ to n $f_k^{\omega_{\mathcal{G}}}(u) = f_k(x) \in U_k$. Therefore $f_k^{\omega_{\mathcal{G}}}(u) \in U$; hence $U \cap e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$ is non-empty. Recall that U was an arbitrary basic open neighbourhood of p in $e_{\mathcal{G}}[X]$. Hence we have shown that $e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X] \cap e_{\mathcal{G}}[X]$ is dense in $e_{\mathcal{G}}[X]$. Since $e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$ is compact, then $e_{\mathcal{G}}[X] \subseteq e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$.

2) => 5) Suppose $\mathcal{G} \subseteq S_\alpha$ and $e_{\mathcal{G}}[X] \subseteq e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \backslash X]$. We first note that, if f is a real-valued singular map then, for any $r \in \mathbb{R}$, f - **r** is a singular map (1.18). Let $x \in$

X and $K = {}_xK_{\mathscr{G}}{}^{\omega_\mathscr{G}} = \cap\{Z(f^{\omega_\mathscr{G}}) : f \in \mathscr{G}^+, x \in Z(f)\}$. By hypothesis, there exists a y $\in \omega_\mathscr{G}X\backslash X$ such that $e_\mathscr{G}(x) = e_\mathscr{G}{}^{\omega_\mathscr{G}}(y)$. Then $f^{\omega_\mathscr{G}}(y) = f(x)$ for all $f \in \mathscr{G}$. It follows that y belongs to $K \cap (\omega_\mathscr{G}X\backslash X)$. We now verify that $(\omega_\mathscr{G}X\backslash X) \cap K = \{y\}$. Suppose z belongs to $K \cap (\omega_\mathscr{G}X\backslash X)$ and $z \neq y$. Then, since $\mathscr{G}^{\omega_\mathscr{G}}$ separates the points of $\omega_\mathscr{G}X\backslash X$, there is an $f \in \mathscr{G}$ such that $f^{\omega_\mathscr{G}}(z) \neq f^{\omega_\mathscr{G}}(y)$. If $f(x) = \epsilon$, then $(f - \epsilon)(x) = 0$. Since $f - \epsilon \in \mathscr{G}^+$ and since y and z belong to K, then $(f^{\omega_\mathscr{G}} - \epsilon)(y) = 0$ and $(f^{\omega_\mathscr{G}} - \epsilon)(z) = 0$. Consequently, $f^{\omega_\mathscr{G}}(y) = f^{\omega_\mathscr{G}}(z) = \epsilon$, a contradiction. It follows that $K \cap (\omega_\mathscr{G}X\backslash X) = \{y\}$.

5) => 2) Let $\mathscr{G} \subseteq S_\beta$, $\mathscr{G}^+ = \{f - r : f \in \mathscr{G}, r \in \mathbb{R}\}$ and $K = \cap\{Z(f^{\omega_\mathscr{G}}) : f \in \mathscr{G}^+, x \in Z(f)\}$ for each $x \in X$. Suppose $K \cap (\omega_\mathscr{G}X\backslash X)$ is a singleton for each $x \in X$. Let $x_o \in X$ and suppose $K \cap (\omega_\mathscr{G}X\backslash X) = \{y_o\}$. We wish to show that $e_\mathscr{G}(x_o) \in e_\mathscr{G}{}^{\omega_\mathscr{G}}[\omega_\mathscr{G}X\backslash X]$. Suppose that for some $f \in \mathscr{G}$ $f(x) = \epsilon$. Then $f - \epsilon \in \mathscr{G}^+$ and $(f - \epsilon)(x_o) = 0$. By definition of K, $(f^{\omega_\mathscr{G}} - \epsilon)(y_o) = 0$. Hence $f^{\omega_\mathscr{G}}(y_o) = \epsilon = f(x_o)$. Consequently, since f was arbitrarily chosen in $\mathscr{G}$, $f(x_o) = f^{\omega_\mathscr{G}}(y_o)$ for all $f \in \mathscr{G}$. Hence $e_\mathscr{G}(x_o) = \langle f(x_o)\rangle_{f \in \mathscr{G}} = \langle f^{\omega_\mathscr{G}}(y_o)\rangle_{f \in \mathscr{G}} = e_\mathscr{G}{}^{\omega_\mathscr{G}}(y_o) \in e_\mathscr{G}{}^{\omega_\mathscr{G}}[\omega_\mathscr{G}X\backslash X]$. All the parts of the theorem have thus been established.

QED

We have just characterized compactifications of a space X which are the suprema of singular compactifications.

A partially ordered set $(X, \leqq)$ is said to be *upward directed* if, for any pair of elements x and y in X, there exists an element z in X such that $x \leqq z$ and $y \leqq z$. Theorem 2.12 might lead the reader to conjecture that, if $\mathscr{A}$ is an upward directed family of singular compactifications, then the supremum of $\mathscr{A}$ must be a singular compactification. This conjecture proves to be false. Let ω_1 denote the first uncountable ordinal. We will show in 3.24 that the lattice of all compactifications of the product space $X = [0,\omega_1) \times [0,\omega_1)$ contains an upward directed subfamily $\mathcal{A}$ of singular compactifications whose supremum is not a singular compactification.

Before we move to chapter three where we attack a related problem, we pause to describe some interesting consequences of the results obtained so far.

Earlier, we provided an example of a compactification αX, namely the two-point compactification of $\mathbb{R}$, which could not be expressed in the form $\omega_{S_\alpha}X$. We now present a condition which guarantees that a compactification αX is equivalent to $\omega_{S_\alpha}X$. (Recall however that this does not imply that αX is a singular compactification; see remark 2.9).

2.13 PROPOSITION If $\alpha X \backslash X$ is not totally disconnected then αX is equivalent to $\omega_{S_\alpha}X$.

Proof: If $\alpha X \backslash X$ is not totally disconnected then $\alpha X \backslash X$ has a connected component K which is not a singleton. Let p and q be distinct elements of K. Let r and s be any two distinct elements of $\alpha X \backslash X$. Then there exists an $f \in C(\alpha X)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f[\{p,r\}] = \{0\}$ and $f[\{q,s\}] = \{1\}$. Since K is connected, f maps $\alpha X \backslash X$ onto [0,1]. Since $f[\alpha X \backslash X] = S(f)$ (see 1.7), $f|_X$ maps X into S(f). Hence $f|_X$ is a singular map. Therefore we have shown that $S_\alpha{}^\alpha$ separates points of $\alpha X \backslash X$. It follows that

$$\begin{aligned} \alpha X &= \sup\{\omega_f X : f \in S_\alpha\} \quad \text{(by 1.16).} \\ &= \omega_{S_\alpha} X \quad \text{(by 1.21).} \end{aligned}$$

QED

The converse of 2.13 fails (even for connected spaces X). A space X which is almost compact noncompact (so that $\beta X \backslash X$ is simultaneously connected and totally disconnected) witnesses the failure of the converse of 2.13. In example 3.8 of chapter 3 of this dissertation we will provide a nontrivial example of a connected space X which has a compactification μX which is equivalent to $\omega_{S_\mu}X$ and whose outgrowth $\mu X \setminus X$ is totally disconnected.

The previous proposition guarantees that the compactifications of a space X whose remainders **are not** totally disconnected are the supremum of singular compactifications. A natural related question is: Which of the compactifications whose remainders are totally disconnected can be expressed as the supremum of singular compactifications? We give a partial answer here and discuss this question a bit more in the next chapter. For now let us consider the question for strongly zero-dimensional spaces X (i.e. those spaces X for which βX is zero-dimensional).

2.14 PROPOSITION Let X be a strongly zero-dimensional not almost compact space. Then βX is the supremum of the family of the two-point singular compactifications of X. Hence $\beta X = \omega_{S_\beta}X$.

Proof: Since X is strongly zero-dimensional, then βX is zero-dimensional (see 3.34 of [Wa]). Let p and q be distinct points in βX and let U be a clopen set of βX which contains p but not q. Since the singular characteristic function $f = \chi_{X \setminus (U \cap X)}$ has an extension to βX which separates p and q, then the family $\mathcal{H}$ of singular characteristic functions of X extends to $\mathcal{H}^\beta$ to separate the points of βX. By 1.16, $\beta X = \sup\{\omega_f X : f \in \mathcal{H}\}$. Since $\omega_f X$ is a singular two-point compactification of X, we are done.

QED

2.15 PROPOSITION Let K(X) denote the family of all compactifications of X. Let $\mathcal{K} = \{\alpha X \in K(X) : \alpha X \setminus X$ is homeomorphic to a closed interval of $\mathbb{R}\}$. Then $\mathcal{K}$

$\subseteq \{\omega_f X: f \in S_\beta\}$, and, if X is connected, then $\mathcal{K} = \{\omega_f X: f \in S_\beta\}$. (We will consider the singleton set $\{a\}$ in $\mathbb{R}$ as the closed interval $[a,a]$ with empty interior).

Proof: We wll first show that $\mathcal{K}$ is contained in $\{\omega_f X : f \in S_\beta\}$. Let $\alpha X \in \mathcal{K}$. Then $\alpha X \backslash X$ is homeomorphic to a closed interval of $\mathbb{R}$. Since $\alpha X \backslash X$ is an absolute retract then $\alpha X \backslash X$ is a retract of αX (see 15D4 of [Wi]). Hence αX is a singular compactification. Let $r : \alpha X \longrightarrow \alpha X \backslash X$ denote a retraction from αX onto $\alpha X \backslash X$ and $h : \alpha X \backslash X \longrightarrow \mathbb{R}$ denote a homeomorphism from $\alpha X \backslash X$ to a closed interval of $\mathbb{R}$. Since $r|_X$ is singular (see 1.3), then $h \circ r|_X$ is singular (by 1.18). That αX is equivalent to $X \cup_{h \circ r|_X} S(h \circ r|_X)$ follows from 2.1.

We now show that if X is connected then $\{\omega_f X : f \in S_\beta\}$ is contained in $\mathcal{K}$. Let $f \in S_\beta$. Recall that $\omega_f X$ is equivalent to $X \cup_f S(f)$ (by 2.4). Since $\omega_f X$ is a singular compactification, $\omega_f X \backslash X$ is the closure of the continuous image of the connected space X. This implies that $\omega_f X \backslash X$ is a connected compact subset of $\mathbb{R}$. It follows that $\omega_f X$ belongs to $\mathcal{K}$.

QED

We have shown that $\mathcal{K}$ is always contained in $\{\omega_f X : f \in S_\beta\}$. However if X is not connected then it may happen that $\mathcal{K}$ is a proper subset of $\{\omega_f X : f \in S_\beta\}$ as witnessed by the following example. Let $f \in C^*(\mathbb{N})$ be defined as follows: f maps the even numbers to $\{3\}$ and the odd numbers to $\{4\}$. Clearly f is singular. Since f maps $\omega_f \mathbb{N} \backslash \mathbb{N}$ homeomorphically onto $S(f) = \{3,4\}$ then $\omega_f \mathbb{N} \backslash \mathbb{N}$ is not connected. Consequently $\omega_f \mathbb{N} \backslash \mathbb{N}$ is not a closed interval of $\mathbb{R}$.

Given a singular compactification αX of X, we know by definition that αX is equivalent to $X \cup_f S(f)$, where $f : X \longrightarrow K$ is some singular map from X into some compact Hausdorff space K. But there may be many such maps f for which this is true. It is important to know how these maps are related to each other.

On page 35 of [G], the following question is asked:

Suppose $X \cup_f S(f)$ and $X \cup_g S(g)$ are two singular compactifications and f and g both map X densely into the same space K so that $S(f) = S(g)$. When are $X \cup_f S(f)$ and $X \cup_g S(g)$ equivalent compactifications of X?

To provide further motivation for the study of this problem, consider the two functions sine and cosine on the real numbers. Note that both are singular maps and that $[-1,1] = S(\text{sine}) = S(\text{cosine})$ is their common singular set. One would surely wonder whether the compactifications $\mathbb{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$ are equivalent. The following theorem will quickly help resolve this question.

Note that, in the following theorem, the functions f and g are not assumed to be real-valued maps. The conjecture that $X \cup_f S(f) \cong X \cup_g S(g)$ if and only if f and g agree except on a compact set, has been shown (in [G]) to be false.

We answer the question in the following theorem. For convenience we recall the following notation introduced in chapter 1: If αX and γX are compactifications of X such that $\alpha X \leqq \gamma X$, we will denote the projection map from γX onto αX which fixes the points of X by $\pi_{\gamma\alpha}$. If X and Y are two topological spaces "$X \cong Y$" will mean that X is homeomorphic to Y.

2.16 PROPOSITION Let $f : X \longrightarrow K_f$ and $g : X \longrightarrow K_g$ be two singular maps from the space X into the compact spaces K_f and K_g respectively such that $S(f) \cong S(g)$. Then the following are equivalent:

1) $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

2) The function $f : X \longrightarrow S(f)$ extends continuously to a function $f^* : X \cup_g S(g) \longrightarrow S(f)$ ($\cong S(g)$) in such a way that f^* separates the points of $S(g)$.

Proof: (1 => 2) Let $\alpha X = X \cup_f S(f)$ and $\gamma X = X \cup_g S(g)$, where $S(f) \cong S(g)$. Suppose that αX is equivalent to γX. Let $r : \alpha X \longrightarrow \alpha X \backslash X$ be defined as follows: $r(x) = x$ if x belongs to $\alpha X \backslash X$ and $r|_X = f$. Since f is singular and $\alpha X \backslash X = S(f)$ it is easily verified that r is continuous and hence is a retraction of αX onto $\alpha X \backslash X$. Let $\pi_{\gamma\alpha}$ denote the projection map from γX onto αX, (i.e. $\pi_{\gamma\alpha}$ fixes the points of X and maps $\gamma X \backslash X$ homeomorphically onto $\alpha X \backslash X$). Let $f^* : \gamma X \longrightarrow S(f)$ be defined as $f^* = r \circ \pi_{\gamma\alpha}$. Note that $f^*|_{S(g)} = r \circ \pi_{\gamma\alpha}|_{S(g)} = \pi_{\gamma\alpha}|_{S(g)}$. Clearly $f^*|_X = f$ and f^* is continuous, being the composition of two continuous functions. Hence $f : X \longrightarrow S(f)$ extends continuously to a function $f^* : X \cup_g S(g) \longrightarrow S(f)$. Furthermore, since $\pi_{\gamma\alpha}$ maps $\gamma X \backslash X$ homeomorphically onto $\alpha X \backslash X$ and r is the identity function on $\alpha X \backslash X$, f^* separates the points of $\gamma X \backslash X$. This proves that the given condition is necessary.

(2 => 1) Suppose now that the function f extends continuously to a function $f^* : X \cup_g S(g) \longrightarrow S(f)$ in such a way that f^* separates the points of $S(g)$. It must then follow from 2.1 that $X \cup_g S(g)$ is equivalent to $X \cup^* S(f)$. But since f is singular, $X \cup^* S(f)$ is equivalent to $X \cup_f S(f)$ (see 1.17). Hence $X \cup_g S(g)$ is equivalent to $X \cup_f S(f)$. This proves that the condition is sufficient.

QED

2.17 EXAMPLE The singular compactifications $\mathbb{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$ **are not** equivalent.

Proof: Suppose the singular compactifications $\mathbb{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$ are equivalent. Then, by 2.16 1) => 2) the function sine : $\mathbb{R} \longrightarrow [-1,1]$ extends to a function sine* : $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine}) \longrightarrow [-1,1]$ such that sine*$|_{S(\text{cosine})}$

is one-to-one on S(cosine) = [−1,1]. Then $\text{sine}^*|_{S(\text{cosine})}$ is a homeomorphism from [−1,1] onto [−1,1] hence is monotone (increasing or decreasing) and maps endpoints to endpoints. Suppose without loss of generality that $\text{sine}^*|_{S(\text{cosine})}$ maps −1 to −1 and 1 to 1. Let U be an open interval containing 1 such that $U \cap [-1,1] \subseteq (1/2,1]$ and $\text{sine}^*|_{S(\text{cosine})}{}^{\leftarrow}[U] = V \subseteq (1/2,1]$. Observe that $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[\sin^{\leftarrow}[U]] = \sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V]$ is empty. Since sine* is continuous on $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$, $\text{sine}^{*\leftarrow}[U] = V \cup \sin^{\leftarrow}[U]$ is open in $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$. Since $V \cup \cos^{\leftarrow}[V]$ is also open in $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$ then $(V \cup \cos^{\leftarrow}[V]) \cap \sin^{*\leftarrow}[U] = V$ is open in $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$. Since $V \subseteq S(\text{cosine})$ we have a contradiction. Hence $\mathbb{R} \cup_{\text{sine}} S(\text{sine})$ and $\mathbb{R} \cup_{\text{cosine}} S(\text{cosine})$ are not equivalent.

QED

The following corollary offers an easy method of recognizing many pairs of singular compactifications which are equivalent.

We introduce the following definition.

2.18 DEFINITION We will say that *two functions $f : X \longrightarrow K$ and $g : X \longrightarrow K$ from a space X to a space K are homeomorphically related* if there exists a homeomorphism $h : cl_K f[X] \longrightarrow cl_K g[X]$ such that $h(f(x)) = g(x)$ for all x in X.

It is clear that families of homeomorphically related functions from a space X to a space K form equivalence classes on the collection of all functions from X to K.

2.19 COROLLARY Let $f : X \longrightarrow K$ and $g : X \longrightarrow K$ be two singular maps on X such that S(f) = S(g) = K. If f and g are homeomorphically related then $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

Proof: Let $h : S(g) \longrightarrow S(f)$ be a homeomorphism from $S(g)$ onto $S(f)$ such that $h(g(x)) = f(x)$. Note that $g : X \longrightarrow S(g)$ extends continuously to the function $g^* : X \cup_g S(g) \longrightarrow S(g)$ where g^* acts as the identity function on $S(g)$ (see 1.2). Hence $h \circ g$ extends to $h \circ g^* : X \cup_g S(g) \longrightarrow S(g)$ where $(h \circ g^*)|_{S(g)} = h$, a homeomorphism from $S(g)$ onto $S(f)$. Since $f = h \circ g$ on X, f extends to $f^* : X \cup_g S(g) \longrightarrow S(g)$ such that f^* separates the points of $S(g)$. By 2.16, 2) => 1), $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

QED

2.20 EXAMPLE The singular compactifications $\mathbb{R} \cup_{sine2} S(\sin^2)$ and $\mathbb{R} \cup_{cos2} S(\cos^2)$ are equivalent.

Proof: It is easily seen that both $\sin^2$ and $\cos^2$ are singular maps on $\mathbb{R}$. Observe that if $h : [0,1] \longrightarrow [0,1]$ is the homeomorphism defined by $h(x) = 1 - x$, then $h \circ \sin^2 = \cos^2$. Hence, by 2.19, $\mathbb{R} \cup_{sine2} S(\sin^2)$ is equivalent to $\mathbb{R} \cup_{cos2} S(\cos^2)$.

QED

In chapter two we have provided solutions to the following questions proposed in [G]:

1) Can every compactification be expressed in the form of a supremum of singular compactifications? This was resolved in 2.7.

2) We have shown that suprema of singular compactifications are of form $\omega_{\mathcal{G}}X$, where $\mathcal{G}$ is contained in S_β, and that the supremum of singular compactifications need not be singular. When is a supremum of a family of singular compactifications a singular compactification? This was resolved in 2.10 and 2.12.

3) Suppose $X \cup_f S(f)$ and $X \cup_g S(g)$ are two singular compactifications, and $S(f)$ and $S(g)$ are homeomorphic. When is $X \cup_f S(f)$ equivalent to $X \cup_g S(g)$? This was resolved in 2.16.

Also in 2.13 and 2.14 we give conditions on a compactification αX which guarantee that αX is the supremum of singular compactifications.

In 2.15 we have shown that the family of compactifications of a connected space X whose outgrowth is homeomorphic to a closed subinterval of $\mathbb{R}$ is precisely the family $\{\omega_f X : f \in S_\beta\}$.

CHAPTER 3

THE LARGEST SINGULAR COMPACTIFICATION

In the last chapter we saw that every compactification αX of X could be expressed in the form $\omega_{\mathscr{G}}X$, where $\mathscr{G} \subseteq C^*(X)$. In particular αX is equivalent to $\omega_{C_\alpha(X)}(X)$. By combining theorems 2.6, 1.13 and 1.16, it was shown that, if αX is singular, then αX is equivalent to $\omega_{S_\alpha}X$, the supremum of all singular compactifications less than or equal to αX. Also, if $\mathscr{G} \subseteq S_\beta$ then, by 1.13, 1.16 and 2.4, $\omega_{\mathscr{G}}X$ is the supremum of singular compactifications (but is not necessarily a singular compactification). Propositions 2.13 and 2.14 also give conditions which imply that a compactification αX is of form $\omega_{S_\alpha}X$. We also saw that there exist spaces X which have compactifications which are not of the form $\omega_{\mathscr{G}}X$ where $\mathscr{G}$ is contained in S_β. (The two-point compactification of $\mathbb{R}$ is an example). We also noted that not all compactifications αX of form $\omega_{\mathscr{G}}X$ (where $\mathscr{G} \subseteq S_\beta$) are singular. In 2.10 and 2.12, we have characterized those compactifications of X which are the supremum of a collection of singular compactifications. By the examples of $\beta\mathbb{R}$ ($\cong \omega_{S_\beta}\mathbb{R}$, see 2.13) and $\beta\mathbb{N}$ ($\cong \omega_{S_\beta}\mathbb{N}$) we have seen that the supremum of all singular compactifications of a space X need not be a singular compactification. Hence the spaces $\mathbb{R}$ and $\mathbb{N}$ have no largest singular compactification.

This brings us to the object of our study in this chapter. We will characterize those locally compact non-compact Hausdorff spaces X which have a largest singular compactification.

Recall that the symbol K(X) denotes the family of all compactifications of the space X (see the paragraph preceding 1.1). In the introductory paragraph of chapter 1 we have explained what we mean by the largest element αX of a subfamily $\mathscr{H}$ of

K(X) (if it exists). For convenience we restate this definition for the family of all singular compactifications of X.

3.1 DEFINITIONS We will say that αX is *the largest singular compactification of X* if αX is a singular compactification and, whenever γX is a singular compactification of X, then $\gamma X \leqq \alpha X$, (i.e. X has a largest singular compactification if the supremum in $(K(X), \leq)$ of the set of all singular compactifications of X is a singular compactification). We say that the compactification γX is a *maximal* singular compactification if γX is singular and there does not exist a singular compactification ζX such that $\zeta X > \gamma X$.

Note: Recall that the family of all singular compactifications is a lower semilattice (see 1.4). Thus, to show that a locally compact Hausdorff space X has a largest singular compactification is equivalent to showing that the family of all singular compactifications of X is a complete lattice.

The following proposition will help us formulate our problem in a more succinct way.

3.2 PROPOSITION The compactification αX of X is the largest singular compactification of X if and only if $\alpha X \cong \omega_{S_\beta} X$ and $\omega_{S_\beta} X$ is singular.

Proof: (=>) Suppose αX is the largest singular compactification of the space X. Then, by 2.6, $\alpha X \cong \omega_{S_\alpha} X$. Since $\omega_{S_\alpha} X$ is the smallest compactification to which all functions in $S_\beta \cap C_\alpha(X)$ extend then $\omega_{S_\alpha} X \leqq \omega_{S_\beta} X$. Now, if $f \in S_\beta$, then, by 2.4a), $\omega_f X \cong X \cup_f S(f) \leqq \alpha X$ (since αX is the largest singular compactification). Let $\gamma X = \sup\{\omega_f X : f \in S_\beta\}$. Hence $\gamma X \leqq \alpha X$. By 1.16, $S_\beta{}^\gamma$ separates points of $\gamma X \backslash X$, consequently γX must be the smallest compactification to which the set of all functions in S_β extend, or more succinctly, $\gamma X \cong \omega_{S_\beta}X$. It must then follow that $\alpha X \cong \omega_{S_\beta}X$.

(<=) Suppose $\alpha X \cong \omega_{S_\beta}X$ and that $\omega_{S_\beta}X$ is a singular compactification. By 1.13, $S_\beta{}^\alpha$ separates the points of $\alpha X \backslash X$ and, by 1.16, $\alpha X = \sup\{X \cup_f S(f) : f \in S_\beta\}$. Since every singular compactification is of form $\omega_{\mathcal{G}}X$ for some $\mathcal{G} \subseteq S_\beta$ (by 2.6), and as

$$\begin{aligned}\omega_{\mathcal{G}}X &= \sup\{X \cup_f S(f) : f \in \mathcal{G}\} \quad \text{(by 1.4)}\\ &\leq \sup\{X \cup_f S(f) : f \in S_\beta\} \quad \text{(since } \mathcal{G} \subseteq S_\beta)\\ &= \alpha X,\end{aligned}$$

then $\omega_{S_\beta}X$ is the supremum of all singular compactifications; hence αX is the largest singular compactification of X.

QED

We can now reformulate our question as follows:

When is the compactification $\omega_{S_\beta}X$ a singular compactification?

3.3 DEFINITION The compactification $\omega_{S_\beta}X$ will be denoted by μX (whether it is singular or not). When we will speak of the *μ-compactification of X* we will mean μX.

Note that the μ-compactification of X exists for all completely regular spaces X (see remark 1.10). We know that in some cases the μ-compactification of X is equivalent to βX (both 2.13 and 2.14 describe conditions on a compactification which when applied to βX imply that $\beta X \cong \omega_{S_\beta}X \cong \mu X$). We will show that there is a multitude of spaces X whose μ-compactification μX is neither the Stone-Čech compactification nor the Freudenthal compactification. Note however that if

$\mu X < \beta X$ then $\beta X \backslash X$ is totally disconnected (since, by 2.13, if $\beta X \backslash X$ is not totally disconnected then $\beta X \cong \omega_{S_\beta} X = \mu X$). Hence if $\mu X < \beta X$ then μX cannot be the Freudenthal compactification (since if $\beta X \backslash X$ is totally disconnected βX is the Freudenthal compactification).

Before we answer the question stated above, we will develop in 3.5 to 3.7 a characterization of those spaces X such that μX is equivalent to βX. First we give an example of a space X such that μX is strictly less than βX.

3.4 EXAMPLE Let x and y be distinct points in $\beta \mathbb{R} \backslash \mathbb{R}$ and let $X = \beta \mathbb{R} \backslash \{x,y\}$ where X is equipped with the subspace topology inherited from $\beta \mathbb{R}$. If $f \in C^*(X)$, f can be extended to $cl_{\beta \mathbb{R}} X$ (via $f|_{\mathbb{R}}$), hence $\beta \mathbb{R} \cong \beta X$. Clearly X must be connected as $\mathbb{R} \subseteq X \subseteq cl_{\beta \mathbb{R}} \mathbb{R}$, and $\mathbb{R}$ is connected. It follows that $\beta X \backslash X$ is not the continuous image of X. Then the one-point compactification is the only singular compactification (since $\beta X \backslash X$ cannot be a retract of βX). Hence, by 3.2, the one-point compactification of X is μX. Hence $\mu X < \beta X$.

Note that, by 2.13 and 2.14, those spaces X such that $\mu X < \beta X$ must be amongst those spaces which are not strongly 0-dimensional and whose outgrowth $\beta X \backslash X$ is totally disconnected.

3.5 THEOREM Let X be a topological space. Then $\mu X \cong \beta X$ if and only if $S_\beta{}^\beta$ separates the points of $D \cap (\beta X \backslash X)$ for each connected component D of βX.

Proof: (=>) Suppose X is a space such that $\beta X \cong \mu X \cong \omega_{S_\beta} X$. Then $S_\beta{}^\beta$ separates the points of $\beta X \backslash X$. Hence $S_\beta{}^\beta$ separates the points of $D \cap (\beta X \backslash X)$ for each connected component D of βX.

(<=) Suppose $S_\beta{}^\beta$ separates the points of $D \cap (\beta X \backslash X)$ for each connected component D of βX. It will suffice to show that $S_\beta{}^\beta$ separates points of $\beta X \backslash X$, since

1.13 will imply that $\mu X \cong \beta X$. Let x and y be distinct points in $\beta X \backslash X$. If x and y belong to distinct components of βX, then there exists a clopen subset U of βX which contains x but not y. The restriction of the characteristic function χ_U to X is a singular map whose extension to βX separates x and y. This fact, and our hypothesis, implies that $\beta X \cong \omega_{S_\beta} X \cong \mu X$.

QED

The example of a space X such that $\mu X \not\cong \beta X$ given in 3.4 is rather trivial. We will now investigate such spaces in order to construct more complex examples of such spaces. First we develop some more theory (in 3.6 and 3.8).

3.6 THEOREM If X is a connected non-compact space which is not almost compact then the following are equivalent:

1) $\mu X \cong \beta X$.

2) There is a continuous function from $\beta X \backslash X$ onto a closed interval with non-empty interior.

3) The space X has a compactification αX whose outgrowth $\alpha X \backslash X$ is homeomorphic to a closed interval of real numbers (with non-empty interior).

4) The space X has a singular compactification which is not the one-point compactification ωX of X.

5) S_β contains a non-constant function.

Proof: We will prove the equivalence of these statements in the following order: 4 => 3 => 2 => 1 => 5 => 4.

(4 => 3) Suppose X has a singular compactification αX such that $\alpha X \backslash X$ contains more than one point. By 2.6, αX is equivalent to $\omega_{S_\alpha} X$. Let x and y be distinct points in $\alpha X \backslash X$. Since $S_\alpha{}^\alpha$ separates the points of $\alpha X \backslash X$ there is a function f in S_α such that $f^\alpha(x)$ is not equal to $f^\alpha(y)$. Since f is a singular map, the compactification

$\omega_f X$ is a singular compactification (2.4). Also, since X is connected, then by 2.15, $\omega_f X \backslash X$ is homeomorphic to a closed interval in $\mathbb{R}$. Now $\omega_f X \backslash X$ contains more than one point, hence this interval has non-empty interior.

(3 => 2) If X has a compactification αX such that $\alpha X \backslash X$ is homeomorphic to a closed interval of $\mathbb{R}$ (with nonempty interior), then the projection map, $\pi_{\beta\alpha}$, maps $\beta X \backslash X$ onto $\alpha X \backslash X$. This means $\beta X \backslash X$ can be mapped continuously onto a closed interval of $\mathbb{R}$ (with nonempty interior).

(2 => 1) Suppose there is a continuous function f from $\beta X \backslash X$ onto a closed interval [a,b] = I with nonempty interior. We must show that $\mu X \cong \beta X$. The reader will note that the connectedness of X does not play a role in the proof of 2 =>1. We will suppose that $\beta X \backslash X$ is 0-dimensional, since, if $\beta X \backslash X$ is not 0-dimensional then, by 2.13, $\mu X \cong \beta X$. Let x and y be distinct points in $\beta X \backslash X$ for which f(x) = f(y) (since $\beta X \backslash X$ is 0-dimensional and [a,b] is not this implies that f cannot be one-to-one; thus such a pair of points can be found). Let us consider the case where f(x) is a point in (a,b). (The proof for the case where f(x) is a or b will be similar) Let M = (c,d) be an open interval containing f(x) such that c is not a, and d is not b. Let U and V be disjoint clopen (in $\beta X \backslash X$) neighbourhoods of x and y respectively such that both U and V are contained in $f^{\leftarrow}(M)$. Let $f^* : \beta X \backslash X \longrightarrow \mathbb{R}$ be a function which agrees with f on $(\beta X \backslash X) \backslash (U \cup V)$ and which sends U and V to distinct points in $[a,b] \backslash M$. The function f* is continuous. Let the function $h : [a,b] \longrightarrow \mathbb{R}$ be defined as follows: h(x) = x if $x \in [a,c]$, h(x) = c if $x \in [c,d]$ and h(x) = x - (d - c) if $x \in [d,b]$. The function h is continuous and has a range which is a closed interval. Then the function $h \circ f^*$ separates the points x and y and maps $\beta X \backslash X$ onto the closed interval [a,b - (d-c)]. Let $k : \beta X \longrightarrow \mathbb{R}$ be an extension of $h \circ f^*$ to all of βX and let $g = (k \wedge \mathbf{a}) \vee \mathbf{b} - (\mathbf{d} - \mathbf{c})$. Note that g maps βX into $g[\beta X \backslash X]$ = [a,b - (d-c)]. Hence $g|_X$ is a singular function (by 1.8) which separates the

arbitrarily chosen points x and y in $\beta X\backslash X$. We have shown that $S_\beta{}^\beta$ separates the points of $\beta X\backslash X$; hence $\beta X \cong \omega_{S_\beta}X = \mu X$ (by 1.13 and the definition of μX).

(1 => 5) Suppose every function in S_β is constant. Then every function in S_β extends to ωX hence $\mu X = \omega_{S_\beta}X \cong \omega X$. As X is not almost compact and as $|\omega X\backslash X| = 1$, we have $\mu X \not\cong \beta X$.

(5 => 4) Suppose S_β contains a function f which is not a constant function. Since f is a singular function $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$ (by 2.4). Since f maps X into S(f) and f[X] contains at least two points then $\omega_f X$ is a singular compactification which is not the one-point compactification of X.

QED

We now give a general characterization of spaces X such that $\mu X \cong \beta X$.

3.7 THEOREM Let X be a locally compact space. Then the following are equivalent:

1) $\mu X \cong \beta X$.

2) At least one of the two following conditions is satisfied:

a) Any two points of $\beta X\backslash X$ are contained in distinct connected components of βX.

b) There is a continuous function from $\beta X\backslash X$ onto a closed interval with non-empty interior.

Proof: (1 => 2) Suppose the space X is such that $\mu X \cong \beta X$ and that $\beta X\backslash X$ contains a pair of points, say x and y, which both belong to the same connected component C of βX. Since $\mu X \cong \beta X$, then βX is equivalent to $\omega_{S_\beta}X$. Hence there is a function f in S_β such that f^β separates x and y (1.13). Since f is a singular map, $f^\beta[\beta X]$ is contained in $f^\beta[\beta X\backslash X]$ (1.8). Also, since C is connected and f^β separates x and y, $f^\beta[C]$ is a closed interval, say [a,b], with non-empty interior (that is, a is not equal to b). Then $f^\beta[C] = [a,b]$ is contained in $f^\beta[\beta X\backslash X]$. Let $h = (f^\beta \wedge a) \vee b$. Since h maps $\beta X\backslash X$ continuously onto [a,b], we are done.

(2 => 1) Suppose any two points in $\beta X\backslash X$ are contained in distinct connected components of βX. Let x and y be any two points in $\beta X\backslash X$ and let M and L be distinct connected components of βX such that x is in M and y is in L. Then there exists a clopen (in βX) subset U of βX which contains M but not L. If f is a characteristic map which sends U to zero and $\beta X\backslash U$ to one, then $f|_X$ is a singular function whose extension to βX separates x and y. Since x and y were arbitrarily chosen in $\beta X\backslash X$, $S_\beta{}^\beta$ separates the points of $\beta X\backslash X$, hence βX is equivalent to $\mu X = \omega_{S_\beta} X$.

We now consider the other hypothesis of 2). Suppose $\beta X\backslash X$ can be mapped by a continuous function f onto some closed interval [a,b] of $\mathbb{R}$. In 2) => 1) of 3.6 we have proven that this hypothesis implies that $\mu X \cong \beta X$ (without using the hypothesis that X is connected). The theorem follows.

QED

We now provide a method for constructing spaces X such that $\mu X \ncong \beta X$. Recall that a function $f : X \longrightarrow Y$ is called *irreducible* if f does not map any proper closed subset of X onto Y. Also recall that a topological space is a *scattered space* if it contains no nonempty dense-in-itself subset (see 30E of [Wi]).

3.8 THEOREM Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and ∞ be a point in $\beta\mathbb{R}^+\backslash\mathbb{R}^+$. Let S be an infinite compact scattered space and $Y = \beta\mathbb{R}^+\backslash\{\infty\}$. Let u and v be distinct points in S, $X = S \times Y$ and X^* be the quotient space of X obtained by collapsing to a single point the doubleton $\{(u,0),(v,0)\}$ and fixing all other points of X. Then $\mu(X^*) \ncong \beta(X^*)$.

Proof: Let S, Y, X and X^* be as described in the statement of the theorem. Then βY is $\beta\mathbb{R}^+$, the one-point compactification of Y. Since Y is pseudocompact,

$\beta X = S \times \beta Y$ (see 8.12 and 8.20 of [Wa]). It is easily verified that $\beta X^* = X^* \cup \{(x,\infty) : x \in S\}$. Then $\beta X^* \backslash X^*$ is the scattered space $S^* = \{(x,\infty) : x \in S\}$ which is homeomorphic to S itself. We claim that there is no continuous surjection from S^* onto a closed interval I with nonempty interior. For suppose, on the contrary, that f was such a function. Then S^* contains a compact subset F such that $f|_F$ is an irreducible continuous function which maps F onto I (see 3.1.C a) of [E]). By definition of a scattered space, F must contain an isolated point t. Then $f|_F$ maps t to some point u in I. Observe that $f|_F^{\leftarrow}(u)$ cannot be the singleton set $\{t\}$ since, if it is, $F\backslash\{t\}$ is compact; hence $f|_F[F\backslash\{t\}]$ must be the whole interval I. This contradicts the fact that $f|_F$ is irreducible on F. On the other hand, if $f|_F^{\leftarrow}(u)$ is not the singleton set $\{t\}$, then $f|_F$ clearly maps $F\backslash\{t\}$ onto I. This also contradicts the fact that $f|_F$ is irreducible. We must conclude that there is no continuous surjection from S^* onto a closed interval I with nonempty interior. We have just produced a completely regular non-connected Hausdorff space X^* whose outgrowth $\beta X^* \backslash X^*$ cannot be mapped continuously onto a closed interval. Note that the points (u,∞) and (v,∞) are not contained in distinct connected components of βX^*. Then, by 3.7 1) =>2), $\mu X^* \ncong \beta X^*$.

QED

The rest of this chapter (3.9 to 3.26) is devoted to solving the question (stated earlier): When is the supremum, μX ($\cong \omega_{S_\beta} X$), of all singular compactifications a singular compactification?

Recall that a subset B of X is called a *P-set* if any G_δ containing B is a neighbourhood of B.

3.9 LEMMA If D is a closed C-embedded copy of $\mathbb{N}$ in a locally compact space X then $(cl_{\beta X}D)\backslash D$ is a P-set of $\beta X\backslash X$.

Proof: Let D be a closed C-embedded copy of $\mathbb{N}$ in a space X. It suffices to show that, if $(cl_{\beta X}D)\backslash D$ is contained in a zero-set Z in $\beta X\backslash X$, then it must be contained in its $\beta X\backslash X$-interior. Let $f \in C^*(\beta X\backslash X)$ such that $(cl_{\beta X}D)\backslash D \subseteq Z(f)$. Let g be a function in $C(\beta X)$ such that $g|_{\beta X\backslash X} = f$. Since $(cl_{\beta X}D)\backslash D \subseteq Z(g)$ then, if $D = \{d_i : i \in \mathbb{N}\}$, $\{g(d_i) : i \in \mathbb{N}\}$ converges to zero. For each $i \in \mathbb{N}$, choose a neighbourhood V_i of d_i such that the closures in X of the V_i neighbourhoods form a pairwise disjoint family of compact sets and $|g(x) - g(d_i)| < 1/i$ for all x in V_i. Let $h : X \longrightarrow \mathbb{R}$ be a continuous function such that $h[d_i] = 1$ for each $i \in \mathbb{N}$ and $h[X\backslash\cup\{V_i : i \in \mathbb{N}\}] = \{0\}$. (By 9M1 of [GJ] such a function exists). Let h^β denote the extension of h to βX. Since $h^\beta[cl_{\beta X}D] = cl_{\mathbb{R}}h[D] = \{1\}$ then $h^\beta[cl_{\beta X}D\backslash D] = \{1\}$, hence $cl_{\beta X}D\backslash D \subseteq Cz(h^\beta)$. Since $X\backslash(\cup\{V_i : i \in \mathbb{N}\}) \subseteq Z(h^\beta)$, $cl_{\beta X}(X\backslash(\cup\{V_i : i \in \mathbb{N}\}) \subseteq Z(h^\beta)$. Let p be an arbitrary point in $(\beta X\backslash X) \cap Cz(h^\beta)$. Then p contains a βX-neighbourhood which misses $X\backslash\cup\{V_i : i \in \mathbb{N}\}$. Furthermore any βX-neighbourhood S of p must meet infinitely many V_i's since $cl_X V_i$ is compact for all i. Suppose $g(p) \neq 0$. Observe that $\lim_{i\longrightarrow\infty}[\sup\{|g(x)| : x \in V_i\}] = 0$ (since $|g(x) - g(d_i)| < 1/i$ for all x in V_i and $\{g(d_i) : i \in \mathbb{N}\}$ converges to zero). If $g(p) \neq 0$ then there exist an open interval T (in $\mathbb{R}$) containing g(p) such that $cl_{\mathbb{R}}T$ does not contain the point 0. But $g^{\leftarrow}[T]$ meets infinitely many V_i's. Since $\lim_{i\longrightarrow\infty}[\sup\{|g(x)| : x \in V_i\}] = 0$ the point 0 must belong to $cl_{\mathbb{R}}T$. Since this is a contradiction, $g(p) = 0 = f(p)$ (since $g|_{\beta X\backslash X} = f$). Hence $p \in Z(f)$. Since p was arbitrarily chosen in $\beta X\backslash X \cap Cz(h^\beta)$, $\beta X\backslash X \cap Cz(h^\beta) \subseteq Z(f)$. Hence Z(f) is a $\beta X\backslash X$-neighbourhood of $cl_{\beta X}D\backslash D$. Thus $cl_{\beta X}D\backslash D$ is a P-set of $\beta X\backslash X$.

QED

In 6.6 of [Wa], W.W Comfort shows (by assuming the Continuum Hypothesis) that, if βX is a singular compactification, then X must be pseudocompact. In 3.11 we have a generalization of Comfort's result. We prove it in ZFC. We begin by proving the following lemma.

3.10 LEMMA If X contains a C-embedded copy of $\mathbb{N}$ (i.e. if X is not pseudo–compact) then $\mu X \cong \beta X$.

Proof: Suppose X contains a C-embedded copy of $\mathbb{N}$. Let x and y be distinct points in $\beta X \setminus X$. We will show that there exists a singular function $t : X \longrightarrow [0,1]$ whose extension to βX separates x and y. Let u, p and z be distinct points in $\beta D \cap (\beta X \setminus X)$. If x belongs to $\beta D \setminus D$, let u = x and if y belongs to $\beta D \setminus D$, let z = y. Let U, V and M be pairwise disjoint open subsets of βX such that $u \in U$, $z \in V$ and $p \in M$. Let $f : \beta D \longrightarrow [0,1]$ be a continuous function such that f(u) = 0, f(z) = 1 and f is a bijection from $M \cap D$ onto $\mathbb{Q} \cap (0,1)$. Since the subsets U, M and V are pairwise disjoint, the subset $M \cap D$ is infinite and C-embedded in X, and the subset $\{u\} \cup \{z\} \cup cl_{\beta D}(M \cap D)$ is compact, then such a function exists. Note that $f[cl_{\beta D}(M \cap D)] = cl_{\mathbb{R}} f[M \cap D] = [0,1]$. Let $h : \beta D \cup \{x\} \cup \{y\} \longrightarrow [0,1]$ be defined as follows: h = f on βD; if x does not belong to βD, let h(x) = 0, and, if y does not belong to βD, let h(y) = 1. Observe that $\beta D \cup \{x\} \cup \{y\}$ is C-embedded in βX (since it is compact). Thus h extends to a function k on βX such that $k|_{\beta D} = f$. Let $t = \mathbf{0} \vee (k|_X \wedge \mathbf{1})$; thus $t^\beta = \mathbf{0} \vee (k \wedge \mathbf{1})$. Consequently t^β maps βX onto [0,1]. If S is an open subset of [0,1], $cl_{\beta X} t^{\leftarrow}[S]$ will meet $(cl_{\beta X} D) \setminus D = \beta D \setminus D$ since $t^\beta|_{M \cap D}$ is a bijection from $M \cap D$ onto $(0,1) \cap \mathbb{Q}$. Hence t is a singular map. Observe that the extension of the singular function t to t^β on βX separates x from y . Thus $S_\beta{}^\beta$ separates the points of $\beta X \setminus X$. By 1.13, μX is equivalent to βX. This proves the lemma.

QED

3.11 THEOREM If X has a largest singular compactification μX then X does not contain a C-embedded copy of $\mathbb{N}$ (i.e. X is pseudocompact) .

Proof: Suppose μX is singular. We will suppose that X contains a C-embedded copy D of $\mathbb{N}$ and show that this leads to a contradiction. If D is a C-embedded copy of $\mathbb{N}$ in X, then, by 6.9 of [GJ], $cl_{\beta X}D \cong \beta D$. Since D is closed in X, $\beta D \backslash D$ is contained in $\beta X \backslash X$. By 3.10, μX is equivalent to βX.

Let $r : \beta X \longrightarrow \beta X \backslash X$ be a retraction from βX onto $\beta X \backslash X$ (the retraction r will exist by 3.10 above and 1.2). The following construction will reveal a contradiction. First note that, since $r[\beta D]$ must be separable and $r[\beta D \backslash D] = \beta D \backslash D$, then $r[D] \backslash r[\beta D \backslash D]$ must contain infinitely many points. Let $T = \{s_i : i \in \mathbb{N}\}$ be an infinite discrete subset of $r[D] \backslash (\beta D \backslash D)$ and choose $k_i \in r^{\leftarrow}(s_i) \cap D$ for each $i \in \mathbb{N}$. Let $H = \{k_i : i \in \mathbb{N}\}$. Since D is C-embedded in X, then H is C-embedded in X (and hence is C*-embedded in βX). Therefore, $cl_{\beta X}H = \beta H$.

Claim: $cl_{\beta X}H \backslash H = cl_{\beta X}T \backslash T$.

Proof of claim: We first note that $r[cl_{\beta X}H] = cl_{\beta X \backslash X}r[H] = cl_{\beta X \backslash X}T$, (by definition of H).

Also,

$$r[cl_{\beta X}H] = (cl_{\beta X}H \backslash H) \cup r[H] = (cl_{\beta X}H \backslash H) \cup T.$$

Hence we have

$$\begin{aligned}(cl_{\beta X \backslash X}T) \backslash T &= r[cl_{\beta X}H] \backslash T \\ &= ((cl_{\beta X}H \backslash H) \cup T) \backslash T \\ &= cl_{\beta X}H \backslash H \qquad \text{(by the fact that } cl_{\beta X}H \backslash H \subseteq \beta D \backslash D \text{ and } T \cap (\beta D \backslash D) = \emptyset).\end{aligned}$$

Thus

$$cl_{\beta X}T \backslash T = (cl_{\beta X \backslash X}T) \backslash T = (cl_{\beta X}H) \backslash H,$$

and the claim is verified.

Define a function $g : cl_{\beta X \backslash X}T \longrightarrow \mathbb{R}$ by $g(s_i) = 1/i$ and $g[(cl_{\beta X \backslash X}T) \backslash T] = \{0\}$. Note that since T is discrete then g is continuous. Since $cl_{\beta X \backslash X}T$ is compact, g extends to a continuous function f on the space βX. Observe that, for each i, $f(s_i) = 1/i$. Hence $0 \notin f[T]$. Also $f[cl_{\beta X}H\backslash H] = g[(cl_{\beta X\backslash X}T)\backslash T] = \{0\}$.Hence Z(f) contains a copy of $cl_{\beta X}H\backslash H$.

The following argument will now produce a contradiction. By lemma 3.9, $cl_{\beta X}H\backslash H$ is a P-set of $\beta X\backslash X$. Since Z(f) contains $\beta H\backslash H$ (by the above), this implies that $cl_{\beta X}H\backslash H \subseteq int_{\beta X\backslash X}(Z(f) \cap (\beta X\backslash X))$ (since $Z(f) \cap \beta X\backslash X$ is a G_δ in $\beta X\backslash X$). Hence $Z(f) \cap \beta X\backslash X$ must have non-empty interior in $\beta X\backslash X$. Since $cl_{\beta X}H\backslash H = (cl_{\beta X\backslash X}T)\backslash T$, (by the above claim), every neighbourhood (in $\beta X\backslash X$) of a point in $cl_{\beta X}H\backslash H$ must meet T. This contradicts the fact that f does not map any points of T to zero, as seen above. Thus X cannot contain a closed C-embedded copy D of $\mathbb{N}$. Hence (by 1.21 of [GJ]) X must be pseudocompact

QED

The converse of 3.11 fails. In 8.23 of [Wa] it is shown that the product space $[0,\omega_1) \times [0,\omega_1)$ does not have a largest singular compactification even though it clearly does not contain a closed C-embedded copy D of $\mathbb{N}$. Moreover, this illustrates that countably compact spaces need not have a largest singular compactification. On the other hand the Tychonoff plank **T** is not countably compact and yet possesses a largest singular compactification $\beta\mathbf{T}$ (= ωT) induced by any constant map on **T**. (Note that **T** is almost compact noncompact hence $\beta\mathbf{T}$ is singular as clearly there is a retraction $r : \beta\mathbf{T} \longrightarrow \beta\mathbf{T}\backslash\mathbf{T}$).

The following definition leads us to a useful characterization of pseudocompact spaces.

3.12 DEFINITION The subset $C^{\#}(X)$ of C(X) is the set of all real-valued functions f such that for every maximal ideal M in C(X) there exists a real number r such that $f - \mathbf{r} \in M$.

3.13 THEOREM (1.10 of [A]) The space X is pseudocompact if and only if $C(X) = C^{\#}(X)$.

3.14 THEOREM (1.6 of [A]) The following are equivalent for f in $C^*(X)$

1) f belongs to $C^{\#}(X)$.

2) For every open subset U of βX $f[U \cap X] = f^{\beta}[U]$.

3) $Cl_{\beta X}Z(f - r) = Z(f^{\beta} - r)$ for any $r \in \mathbb{R}$.

4) f maps zero sets to closed sets.

3.15 LEMMA If X is a non-compact pseudocompact space and αX is a compactification of X then, for each $f \in S_{\alpha}$, $Z(f)$ is not compact whenever $Z(f^{\alpha})$ is non-empty. Furthermore $cl_{\alpha X}Z(f) = Z(f^{\alpha})$ for all $f \in C_{\alpha}(X)$.

Proof: Since X is pseudocompact, then $cl_{\beta X}Z(f) = Z(f^{\beta})$ for all f in $C^*(X)$ (by 3.13 and 3.14 and also by 8.8 (b) together with 8A (4) of [GJ]). Let $f \in S_{\alpha}$. Then $f[X] \subseteq S(f) = f^{\alpha}[\alpha X \backslash X]$ (1.7). Hence $Z(f^{\alpha}) \cap (\alpha X \backslash X)$ is non-empty if $Z(f)$ is non-empty. Note that $Z(f^{\alpha}) = \pi_{\beta\alpha}[Z(f^{\beta})] = \pi_{\beta\alpha}[cl_{\beta X}Z(f)] = cl_{\alpha X}\pi_{\beta\alpha}[Z(f)] = cl_{\alpha X}Z(f)$ for all $f \in C_{\alpha}(X)$. It follows that $Z(f^{\alpha}) = cl_{\alpha X}Z(f)$ for all f in $C_{\alpha}(X)$. Hence $Z(f)$ is not compact if $Z(f^{\alpha})$ is non-empty.

QED

3.16 PROPOSITION If X is pseudocompact and $\alpha X = X \cup_f S(f)$ is a singular compactification of X such that $S(f)$ is homeomorphic to a subset of $\mathbb{R}$ then $f^{\leftarrow}(x)$ is non-compact for any $x \in S(f)$ and $f[X] = S(f)$.

Proof: Suppose X is pseudocompact and $\alpha X = X \cup_f S(f)$ is a singular compactification of X such that $S(f)$ is a subset of $\mathbb{R}$. By the lemma above $Z(f^{\alpha}) = cl_{\alpha X}Z(f)$ for all f in $C_{\alpha}(X)$. Also $Z(f)$ is not compact if $Z(f^{\alpha})$ is non-empty. Hence

$f^{\leftarrow}(x) = Z(f - x)$ is not compact for any $x \in S(f)$ (since f is a singular real-valued function). By applying 3.13 and the equivalence of 3.14 (1) and 3.14 (4) we also conclude that $f[X] = S(f)$ (since $f[X]$ is dense in $S(f)$).

Suppose $S(f)$ is homeomorphic to a subset K of $\mathbb{R}$. Let $h : S(f) \longrightarrow K$ be a function which maps $S(f)$ homeomorphically onto K. By the above $(h \circ f)^{\leftarrow}(x)$ is non-compact for all x in K. Hence $f^{\leftarrow}(y)$ is non-compact for all y in $S(f)$.

QED

If X is not pseudocompact then the above proposition may fail as the following example illustrates.

3.17 EXAMPLE Let $X^* = [0,1] \times [0,1] \cup \{(-2,0)\}$ viewed as a subspace of the product space $\mathbb{R}^2$. Then X^* is a compactification of the space $X = X^* \backslash ([0,1] \times \{1\})$ and $X^* \backslash X$ is homeomorphic to to the closed interval [0,1]. Clearly X is not pseudocompact. Let us define the function $r : X^* \longrightarrow [0,1] \times \{1\}$ as follows: $r((-2,0)) = (0,1)$ and, for $a \in [0,1]$, $r((a,b)) = ((a-1)b + 1,1)$ (i.e. r linearly maps the closed interval $\{a\} \times [0,1]$ onto $[a,1] \times \{1\}$ carrying (a,1) to (a,1) and (a,0) to (1,1)). Observe that r is a well-defined continuous real-valued function and that r maps any point of $X^* \backslash X$ to itself; hence $X^* \backslash X$ is a retract of X^* and $r|_X$ is singular. Also note that (0,1) and (-2,0) are the only two points in X^* which are carried to (0,1). Hence $cl_{X^*}(X \cap r^{\leftarrow}((0,1)) = cl_{X^*}\{(-2,0)\} = (-2,0) = r|_X^{\leftarrow}((0,1))$. Thus $r|_X^{\leftarrow}((0,1))$ is compact.

3.18 LEMMA If $\{f_n : n \in \mathbb{N}\}$ is a sequence of real-valued singular functions which converges uniformly to a function f in $C^*(X)$ then f is also a singular function.

Proof: Let $x \in X$, $f(x) = r$ and U be an open interval in $\mathbb{R}$ which contains r. Let $\epsilon > 0$ such that $(r - \epsilon, r + \epsilon) \subseteq U$. Since $\{f_n : n \in \mathbb{N}\}$ converges uniformly to f there exists a number N such that for all $n \geqslant N$, $||f_n - f|| < \epsilon/3$. It follows that $|f_N(x) - f(x)| < \epsilon/3$. Let $z = f_n(x)$; then $z \in (r - \epsilon/3, r + \epsilon/3)$. Let V be an open neighbourhood of z such that $V \subseteq (r - \epsilon/3, r + \epsilon/3)$. We claim that $f_n^{\leftarrow}[V] \subseteq f^{\leftarrow}[U]$ for all $n \geqslant N$. Let $t \in f_m^{\leftarrow}[V]$ for some $m \geq N$. Then $|f_m(t) - f(t)| < \epsilon/3$; hence $f(t) \in (r - \epsilon, r + \epsilon) \subseteq U$. Thus $f[f_m^{\leftarrow}[V]] \subseteq U$. Since $f^{\leftarrow}[U] = \{x \in X : f(x) \in U\}$, $f_m^{\leftarrow}[V] \subseteq f^{\leftarrow}[U]$. This establishes the claim. Since $f_N \in S_\beta$, $cl_X f_N^{\leftarrow}[V]$ is not compact. Hence $cl_X f^{\leftarrow}[U]$ cannot be compact since $cl_X f_N^{\leftarrow}[V] \subseteq cl_X f^{\leftarrow}[U]$ (by the above claim). This implies that f is a singular map.

QED

The reader will recall the Remark 2.7 which says that for $\mathcal{G} \subseteq S_\alpha$ "$\omega_{\mathcal{G}}X$ being singular does not imply that $e_{\mathcal{G}}$ is singular". In 3.19 we show that if $\mathcal{G}$ is a subalgebra of $C^*(X)$ which is contained in S_α, then $\omega_{\mathcal{G}}X$ is singular and so is $e_{\mathcal{G}}$.

3.19 THEOREM A compactification αX of X is singular **iff** S_α contains a sub–algebra $\mathcal{G}$ of $C^*(X)$ such that $\mathcal{G}^\alpha$ separates the points of $\alpha X/X$. Furthermore if $\mathcal{G}$ is a subalgebra $C^*(X)$ which is contained in S_α such that $\mathcal{G}^\alpha$ separates the points of $\alpha X\backslash X$ then $e_{\mathcal{G}}$ is a singular map and $\alpha X \cong \omega_{\mathcal{G}}X \cong X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$ (a singular compactification).

Proof: (=>) Suppose αX is a singular compactification. Then, by 2.6, αX is equivalent to $\omega_{S_\alpha}X$. By 2.11, S_α contains a subalgebra $\mathcal{G}$ of $C^*(X)$ such that $e_{\mathcal{G}}$ is singular and $\alpha X \cong X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$. Since $e_{\mathcal{G}^\alpha}$ separates the points of $\alpha X\backslash X$ then so does $\mathcal{G}^\alpha$ (by 1.12).

(<=) Suppose αX is a compactification of X and $\mathcal{G}$ is a subalgebra of $C^*(X)$ which is contained in S_α such that $\mathcal{G}^\alpha$ separates the points of $\alpha X/X$. To obtain our result we will show that ${}_xK_{\mathscr{G}^\alpha} \cap (\alpha X\backslash X)$ is a singleton for each $x \in X$ and then apply the equivalence of 2.12 (1) and 2.12 (5).

Let k be a point in X and let $\mathscr{H} = \{Z(f^\alpha) \cap \alpha X\backslash X : f \in \mathscr{G}^+, k \in Z(f)\}$. It is easily seen that $\cap\mathscr{H} = {}_kK_{\mathscr{G}^\alpha} \cap \alpha X\backslash X$. We wish to show that $\cap\mathscr{H}$ is non-empty by verifying that $\mathscr{H}$ possesses the finite intersection property. Let $\mathscr{M} = \{Z(f_i^\alpha) \cap \alpha X\backslash X : i \in F\}$ be a finite subcollection of $\mathscr{H}$. Note that $\cap\mathscr{M} = Z(\Sigma_{i \in F}(f_i^\alpha)^2) \cap (\alpha X\backslash X)$.Since $\mathscr{G}$ is a subalgebra of $C^*(X)$ and each f_i belongs to $\mathscr{G}^+$ the function $\Sigma_{i \in F}(f_i)^2$ belongs to $\mathscr{G}^+$; hence, by 1.18, it belongs to S_α. Thus, by 1.8, $(\Sigma_{i \in F}f_i^2)[X] \subseteq (\Sigma_{i \in F}f_i^2)^\alpha[\alpha X\backslash X] = (\Sigma_{i \in F}(f_i^2)^\alpha)[\alpha X]$. As $k \in Z(\Sigma_{i \in F}(f_i^2)^\alpha)$, it follows that $\cap\mathscr{M}$ is non-empty. Hence $\mathscr{H}$ has the finite intersection property. Since $\alpha X\backslash X$ is compact $\cap\mathscr{H} = {}_kK_{\mathscr{G}^\alpha} \cap (\alpha X\backslash X)$ is non-empty. Since $\mathscr{G}^\alpha$ separates the points of $\alpha X\backslash X$, ${}_kK_{\mathscr{G}^\alpha} \cap (\alpha X\backslash X)$ is a singleton set in $\alpha X\backslash X$. By 2.12 (5) implies 2.12 (1), $e_{\mathscr{G}}$ is a singular map and $\alpha X \cong \omega_{\mathscr{G}}X \cong X \cup_{e_{\mathscr{G}}} S(e_{\mathscr{G}})$, a singular compactification.

QED

Suppose αX is a singular compactification and $r : \alpha X \longrightarrow \alpha X\backslash X$ is a retraction map. It is worth noting that the subalgebra $\mathscr{G} = \{f \circ r|_X : f \in C(\alpha X)\}$ (see 2.11) contains the constant functions (hence $\mathscr{G} = \mathscr{G}^+$). This follows from the following fact:

FACT: If $g \in S_\alpha$ is so that g^α is constant on $\alpha X\backslash X$ then g is constant

Proof: Since g is singular $g[X] \subseteq g^\alpha[\alpha X\backslash X]$, by 1.8. Since $g^\alpha[\alpha X\backslash X]$ is a singleton $g[X]$ is as well.

QED

3.20 THEOREM Let αX be a compactification of the space X. There is a one-to-one correspondence between the retraction maps from αX onto $\alpha X\backslash X$ and the subalgebras $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X \setminus X} = C(\alpha X \setminus X)$. If αX is not a singular compactification then no such retraction map r or such a subalgebra $\mathcal{G}$ exist.

Proof: If αX is not a singular compactification then, by 1.2, there does not exist a retraction map $r : \alpha X \longrightarrow \alpha X \setminus X$. Also, by 3.19, S_α does not contain a subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G}^\alpha$ separates the points of $\alpha X \setminus X$. Hence $C_\alpha(X)$ does not contain a subalgebra $\mathcal{G}$ satisfying the properties described in the statement of the theorem.

Suppose αX is a singular compactification. Then there exists a retraction map $r : \alpha X \longrightarrow \alpha X \setminus X$ from αX onto $\alpha X \setminus X$. By 2.11, the family $\mathcal{G} = \{f \circ r|_X : f \in C(\alpha X)\}$ is a subalgebra of $C_\alpha(X)$, $e_{\mathcal{G}}$ is a singular map, $e_{\mathcal{G}}{}^\alpha$ separates points of $\alpha X \setminus X$, and $\alpha X \cong X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$. Observe that $\mathcal{G}^\alpha = \{f \circ r : f \in C(\alpha X \setminus X)\}$ and that $\mathcal{G}^\alpha|_{\alpha X \setminus X} = C(\alpha X \setminus X)$ (since $r|_{\alpha X \setminus X}$ is the identity function on $\alpha X \setminus X$). We have shown that we can associate to each retraction map $r : \alpha X \longrightarrow \alpha X \setminus X$ a subalgebra $\mathcal{G} = \{f \circ r|_X : f \in C(\alpha X)\}$ of $C_\alpha(X)$ which is contained in S_α such that $\mathcal{G}^\alpha|_{\alpha X \setminus X} = C(\alpha X \setminus X)$.

Let $\mathcal{F}_r = \{f \circ r|_X : f \in C(\alpha X)\}$ and $\mathcal{F}_s = \{f \circ s|_X : f \in C(\alpha X)\}$, where $r : \alpha X \longrightarrow \alpha X \setminus X$ and $s : \alpha X \longrightarrow \alpha X \setminus X$ are retractions. We want to show that if $r \neq s$ then $\mathcal{F}_r \neq \mathcal{F}_s$ (i.e. that the map $r \longmapsto \mathcal{F}_r$ is one-to-one). If $r \neq s$ there exists $x_0 \in X$ such that $r(x_0) \neq s(x_0)$. As $C(\alpha X)$ separates the points of $\alpha X \setminus X$, there exists $f \in C_\alpha(X)$ such that $f^\alpha(r(x_0)) \neq f^\alpha(s(x_0))$, i.e. $(f^\alpha|_{\alpha X \setminus X} \circ r)(x_0) \neq (f^\alpha|_{\alpha X \setminus X} \circ s)(x_0)$. Now $f^\alpha|_{\alpha X \setminus X} \circ r \in \mathcal{F}_r$; we will show that $f^\alpha|_{\alpha X \setminus X} \circ r \notin \mathcal{F}_s$, thereby showing that $\mathcal{F}_r \neq \mathcal{F}_s$. If $f^\alpha|_{\alpha X \setminus X} \in \mathcal{F}_s$, then there exists a function $g \in C_\alpha(X)$ such that $g^\alpha|_{\alpha X \setminus X} \circ s = f^\alpha|_{\alpha X \setminus X} \circ r$. Consequently if $t \in \alpha X \setminus X$ then $s(t) = r(t) = t$ (as r and s are retractions) and $g^\alpha(t) = g^\alpha(s(t)) = f^\alpha(r(t)) = f^\alpha(t)$. Hence in particular $g^\alpha(s(x_0)) = f^\alpha(s(x_0))$. But by the above, $f^\alpha(s(x_0)) \neq f^\alpha(r(x_0))$. Thus $(g^\alpha|_{\alpha X \setminus X} \circ s)(x_0) \neq (f^\alpha|_{\alpha X \setminus X} \circ r)(x_0)$, in contradiction to the definition of g. Hence $f^\alpha|_{\alpha X \setminus X} \circ r \notin \mathcal{F}_s$, $\mathcal{F}_r \neq \mathcal{F}_s$, and $r \longmapsto \mathcal{F}_r$ is a one-to-one map.

We will now show that, for every subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X\setminus X} = C(\alpha X\setminus X)$ there exists a retraction map $r : \alpha X \longrightarrow \alpha X\setminus X$ from αX onto $\alpha X\setminus X$ such that $\mathcal{G} = \{f\circ r|_X : f \in C(\alpha X)\}$. Let $\mathcal{G}$ be a subalgebra of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X\setminus X} = C(\alpha X\setminus X)$. We have shown (in 3.19) that if $\mathcal{G}$ is a subalgebra of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$ then $e_\mathcal{G}$ is a singular map and $\alpha X \cong X \cup_{e_\mathcal{G}} S(e_\mathcal{G})$. Since $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$ then $e_\mathcal{G}{}^\alpha$ is one-to-one on $\alpha X\setminus X$; hence the function $(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha : \alpha X \longrightarrow \alpha X\setminus X$ is a retraction map (since $e_\mathcal{G}$ is singular and, by 1.7, $e_\mathcal{G}{}^\alpha[\alpha X\setminus X] = S(e_\mathcal{G}) = e_\mathcal{G}{}^\alpha[\alpha X]$; thus $(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}$ is a well-defined map whose domain is $(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})[\alpha X\setminus X]$). We claim that $\mathcal{G}^\alpha = \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. We begin by proving that $\mathcal{G}^\alpha \subseteq \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. Let $g \in \mathcal{G}$ and $x \in X$. Since $g \in \mathcal{G} \subseteq S_\alpha$, g extends to a function g^α on αX. Then $e_\mathcal{G}{}^{\alpha\leftarrow}\circ e_\mathcal{G}{}^\alpha(x)$ is a subset of αX which meets $\alpha X\setminus X$ in a singleton set, say $\{y\}$, (since, by 1.7, $e_\mathcal{G}[X] \subseteq e_\mathcal{G}{}^\alpha[\alpha X\setminus X]$ and $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$ hence $e_\mathcal{G}{}^\alpha$ is one-to-one on $\alpha X\setminus X$). Hence $e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}{}^\alpha(x) = \{y\}$. Observe that $e_\mathcal{G}{}^{\alpha\leftarrow}(e_\mathcal{G}{}^\alpha(x)) \subseteq g^{\alpha\leftarrow}(g^\alpha(x))$ (since $g^\alpha \in \mathcal{G}^\alpha$ and $e_\mathcal{G}{}^{\alpha\leftarrow}(e_\mathcal{G}{}^\alpha(x)) = \cap\{f^{\alpha\leftarrow}(f(x)) : f \in \mathcal{G}\}$). Thus $y \in g^{\alpha\leftarrow}(g(x))$. Therefore $g^\alpha(y) = g^\alpha(x) = g(x)$. We have just shown that $g^\alpha|_{\alpha X\setminus X}([e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}](x)) = g^\alpha(y) = g(x)$ for an arbitrary point x (hence for all x) in X. Thus $g^\alpha = g^\alpha|_{\alpha X\setminus X}\circ[e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] \in \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. This proves that $\mathcal{G}^\alpha \subseteq \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. We now prove $\mathcal{G}^\alpha \supseteq \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. Let $k \in \{f^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha] : f \in C_\alpha(X)\}$. Observe that if $t \in C_\alpha(X)$ such that $k = t^\alpha|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha]$ then $k|_{\alpha X\setminus X} = t^\alpha|_{\alpha X\setminus X}$ on $\alpha X\setminus X$; hence $k = k|_{\alpha X\setminus X}\circ[(e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_\mathcal{G}{}^\alpha]$. Note that $k|_{\alpha X\setminus X}$ extends to a function $g \in \mathcal{G}^\alpha$ (since $k|_{\alpha X\setminus X} \in C(\alpha X\setminus X)$ and, by hypothesis, $\mathcal{G}^\alpha|_{\alpha X\setminus X} = C(\alpha X\setminus X)$). Obviously $k|_{\alpha X\setminus X} = g|_{\alpha X\setminus X}$ on $\alpha X\setminus X$. Let $x \in X$. The argument in the proof above shows that (since $g \in \mathcal{G}^\alpha$) $g|_{\alpha X\setminus X}\circ[e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}](x) = g(x)$ (for all x in X). Hence $k(x) = k^\alpha|_{\alpha X\setminus X}\circ[e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}](x) = g|_{\alpha X\setminus X}\circ[e_\mathcal{G}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_\mathcal{G}](x) = g(x)$ (for all x in X).

Hence $k = g^\alpha \in \mathcal{G}^\alpha$. We have shown that $\mathcal{G}^\alpha \supseteq \{f^\alpha|_{\alpha X\setminus X}\circ[(e_{\mathcal{G}}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_{\mathcal{G}}{}^\alpha] : f \in C_\alpha(X)\}$. We conclude that $\mathcal{G}^\alpha = \{f\circ[(e_{\mathcal{G}}{}^\alpha|_{\alpha X\setminus X})^{\leftarrow}\circ e_{\mathcal{G}}{}^\alpha] : f \in C_\alpha(X)\}$. Hence for every subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X\setminus X} = C(\alpha X\setminus X)$ there exists a retraction map $r : \alpha X \longrightarrow \alpha X\setminus X$ from αX onto $\alpha X\setminus X$ (in this case $r = [e_{\mathcal{G}}{}^\alpha|_{\alpha X\setminus X}{}^{\leftarrow}\circ e_{\mathcal{G}}{}^\alpha]$) such that $\mathcal{G} = \{f^\alpha|_{\alpha X\setminus X}\circ r|_X : f \in C_\alpha(X)\}$.

We have thus shown that there is a one-to-one correspondence between the retraction maps r from αX onto $\alpha X\setminus X$ and the subalgebras $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X\setminus X} = C(\alpha X\setminus X)$.

QED

3.21 THEOREM The compactification αX of X is singular iff S_α contains a closed subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that the mapping $\phi : \mathcal{G} \longrightarrow C(\alpha X\setminus X)$ from $\mathcal{G}$ onto $C(\alpha X\setminus X)$ defined by $\phi(f) = f^\alpha|_{\alpha X\setminus X}$ is an isomorphism.

Proof: (=>) Suppose αX is a singular compactification. Then, by 3.19, there exists a subalgebra $\mathcal{G}$ of $C^*(X)$ contained in S_α such that $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$. As $S_\alpha \subseteq C_\alpha(X)$ clearly $\{f^\alpha|_{\alpha X\setminus X} : f \in S_\alpha\}$ is contained in $C(\alpha X\setminus X)$. As $\alpha X\setminus X$ is compact and $\mathcal{G}^\alpha$ separates points of $\alpha X\setminus X$ the collection $\mathcal{G}^\alpha|_{\alpha X\setminus X} = \{f^\alpha|_{\alpha X\setminus X} : f \in \mathcal{G}\}$ is a subalgebra of $C(\alpha X\setminus X)$ which separates the points and closed sets of $\alpha X\setminus X$. Without loss of generality we may suppose that $\mathcal{G}$ contains the constant functions since, if k is a number and $f \in S_\alpha$, $f + k$ and kf are both singular maps (by 1.18). Thus $\mathcal{G}^\alpha|_{\alpha X\setminus X}$ contains the constant functions and separates points and closed sets of $\alpha X\setminus X$. We claim that $C(\alpha X\setminus X) = (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. By the Stone-Wieirstrass theorem $(\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X} \subseteq C(\alpha X\setminus X)$. Observe that $\mathrm{cl}_{C(\alpha X\setminus X)}(\mathcal{G}^\alpha|_{\alpha X\setminus X}) = C(\alpha X\setminus X)$ (again by the Stone-Wieirstrass theorem). Hence it will suffice to show that $\mathrm{cl}_{C(\alpha X\setminus X)}(\mathcal{G}^\alpha|_{\alpha X\setminus X}) \subseteq (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. Let $f \in \mathrm{cl}_{C(\alpha X\setminus X)}(\mathcal{G}^\alpha|_{\alpha X\setminus X})$. Then we can construct a sequence $\mathcal{C} = \{f_i : i \in \mathbb{N}\}$ in $\mathcal{G}^\alpha|_{\alpha X\setminus X}$ ($\subseteq (C(\alpha X/X), \|\ \|)$) whose only cluster point is f. We wish to show that $f \in (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. Now every function f_i in $\mathcal{C}$ extends to a function f_i^* in $\mathcal{G}^\alpha$. Let $\mathcal{C}^* = \{f_i^* : i \in \mathbb{N}\} \subseteq C(\alpha X)$. Let g be a cluster point of $\mathcal{C}^*$. Then $g \in \mathrm{cl}_{C(\alpha X)}\mathcal{C}^* \subseteq \mathrm{cl}_{C(\alpha X)}(\mathcal{G}^\alpha)$. We will first show that $g|_{\alpha X\setminus X} = f$. We can construct a sequence $\mathfrak{D} = \{f_{ij} : j \in \mathbb{N}\} \subseteq \mathcal{C}^*$ whose only cluster point is g. Then for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that $\|f_{ij}^* - g\| < \varepsilon$ for every $j > N(\varepsilon)$. Thus $\|f_{ij} - g|_{\alpha X\setminus X}\| < \varepsilon$ for every $j > N(\varepsilon)$. It follows that $g|_{\alpha X\setminus X}$ is a cluster point of $\mathcal{C}$. Since $\mathcal{C}$ has only one cluster point, namely f, $g|_{\alpha X\setminus X} = f$. We will now show that $g|_{\alpha X\setminus X} \in (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. It is easily seen that $g|_X \in \mathrm{cl}_{C_\alpha(X)}\{f_i^*|_X : i \in \mathbb{N}\} \subseteq \mathrm{cl}_{C_\alpha(X)}\mathcal{G}$. Hence $g \in (\mathrm{cl}_{C_\alpha(X)}\{f_i^*|_X : i \in \mathbb{N}\})^\alpha \subseteq (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha$. Thus $g|_{\alpha X\setminus X} \in (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. Since $g|_{\alpha X\setminus X} = f$, $f \in (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. The claim is established i.e. $C(\alpha X\setminus X) = (\mathrm{cl}_{C_\alpha(X)}\mathcal{G})^\alpha|_{\alpha X\setminus X}$. By lemma 3.18, $\mathrm{cl}_{C_\alpha(X)}\mathcal{G}$ is contained in S_α. We now define the function $\phi : \mathrm{cl}_{C_\alpha(X)}\mathcal{G} \longrightarrow C(\alpha X\setminus X)$ from $\mathrm{cl}_{C_\alpha(X)}\mathcal{G}$ into $C(\alpha X\setminus X)$ as $\phi(f) = f^\alpha|_{\alpha X\setminus X}$. Clearly ϕ is a homomorphism. By the above claim ϕ is onto $C(\alpha X\setminus X)$. We now show that ϕ is one-to-one. Let f and g be two functions in $\mathrm{cl}_{C_\alpha(X)}\mathcal{G}$ such that $f^\alpha|_{\alpha X\setminus X} = g^\alpha|_{\alpha X\setminus X}$. Since f and g are both singular maps and $\mathrm{cl}_{C_\alpha(X)}\mathcal{G}$ is a subalgebra which is contained in S_α then $f - g$ is singular. Then $(f^\alpha - g^\alpha)[X] = (f^\alpha - g^\alpha)[\alpha X\setminus X] = (f^\alpha|_{\alpha X\setminus X} - g^\alpha|_{\alpha X\setminus X})[\alpha X\setminus X] = \{0\}$, (1.8). Hence $f = g$. It follows that the map ϕ is an isomorphism.

(<=) Suppose S_α contains a closed subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that the mapping $\phi : \mathcal{G} \longrightarrow C(\alpha X\setminus X)$ from $\mathcal{G}$ onto $C(\alpha X\setminus X)$ defined by $\phi(f) = f^\alpha|_{\alpha X\setminus X}$ is an isomorphism. Then clearly $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$. Hence, by 3.19, αX is a singular compactification.

QED

Let $C_\infty(X)$ denote the family of all functions f in $C^*(X)$ for which the set $\{x \in X : |f(x)| \geq 1/n\}$ is compact for all n in $\mathbb{N}$. These functions are said to "vanish at infinity", (see 7FG of [GJ]). It is easily verified that $C_\infty(X)$ is an ideal in the ring $C^*(X)$.

We now know that if αX is a singular compactification of X then S_α contains a closed subalgebra $\mathcal{G}$ of $C^*(X)$ such that $\mathcal{G}^\alpha$ separates the points of $\alpha X \backslash X$. The following theorem tells us that such a subalgebra $\mathcal{G}$ of $C_\alpha(X)$ is isomorphic to the quotient ring $\dfrac{C_\alpha(X)}{C_\infty(X)}$ under the canonical homomorphism $\sigma : \mathcal{G} \longrightarrow \dfrac{C_\alpha(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

3.22 THEOREM Let αX be a compactification of X. Then αX is a singular compactification of X iff $\dfrac{C_\alpha(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathcal{F}$ (of $C_\alpha(X)) \subseteq S_\alpha$ under the homomorphism $\sigma : \mathcal{F} \longrightarrow \dfrac{C_\alpha(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

Proof: (=>) Suppose αX is a singular compactification. Then, by 3.19, there exists a subalgebra $\mathcal{F}$ of $C^*(X)$ which is contained in S_α such that $\mathcal{F}^\alpha$ separates the points of $\alpha X \backslash X$ and such that αX is equivalent to $X \cup_{e_\mathcal{F}} S(e_\mathcal{F})$. By 3.21, the homomorphism $\phi : \mathrm{cl}_{C_\alpha(X)}\mathcal{F} \longrightarrow C(\alpha X \backslash X)$ defined by $\phi(f) = f^\alpha|_{\alpha X \backslash X}$ is a ring isomorphism. Let $\tau : C_\alpha(X) \longrightarrow C(\alpha X \backslash X)$ be the homomorphism from $C_\alpha(X)$ onto $C(\alpha X \backslash X)$ defined by $\tau(f) = f^\alpha|_{\alpha X \backslash X}$. We now define the mapping $\psi : C_\alpha(X) \longrightarrow \mathrm{cl}_{C_\alpha(X)}\mathcal{F}$ as $\psi = \phi^{\leftarrow} \circ \tau$. (Note that $\psi(f)$ is the unique $g \in \mathrm{cl}_{C_\alpha(X)}\mathcal{F}$ for which $g^\alpha|_{\alpha X \backslash X} = f^\alpha|_{\alpha X \backslash X}$). Observe that the kernel of ψ is $\psi^{\leftarrow}(0) = (\phi^{\leftarrow} \circ \tau)^{\leftarrow}(0) = \tau^{\leftarrow} \circ \phi(0) = \tau^{\leftarrow}(0^\alpha|_{\alpha X \backslash X}) = C_\infty(X)$. Hence by the Fundamental Theorem of Homomorphisms the function $\zeta : \dfrac{C_\alpha(X)}{C_\infty(X)} \longrightarrow \mathrm{cl}_{C_\alpha(X)}\mathcal{F}$ defined by $\zeta(C_\infty(X) + f) = \psi(f)$ maps $\dfrac{C_\alpha(X)}{C_\infty(X)}$ isomorphically onto the image $\mathrm{cl}_{C_\alpha(X)}\mathcal{F}$ of $C_\alpha(X)$ under ψ. Observe that, if $g \in$

$cl_{C_\alpha(X)}\mathscr{F}$, then $\psi(g) = \phi^{\leftarrow}\circ\tau(g) = \phi^{\leftarrow}(g^\alpha|_{\alpha X\setminus X}) = g$ (since ϕ is one-to-one and onto $C(\alpha X\setminus X)$). Hence, for $g \in cl_{C_\alpha(X)}\mathscr{F}$, $\zeta(C_\infty(X) + g) = \psi(g) = g$. It then follows that the canonical homomorphism $\sigma : cl_{C_\alpha(X)}\mathscr{F} \longrightarrow \frac{C_\alpha(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$ is onto $\frac{C_\alpha(X)}{C_\infty(X)}$ (since, if $g \in C_\alpha(X)$, then $C_\infty(X) + g = \zeta^{\leftarrow}(\psi(g)) = \zeta^{\leftarrow}(\psi(\psi(g)) = C_\infty(X) + \psi(g)$; hence $\sigma(\psi(g)) = C_\infty(X) + \psi(g) = C_\infty(X) + g$). Hence the canonical homomorphism σ maps $cl_{C_\alpha(X)}\mathscr{F}$ isomorphically onto $\frac{C_\alpha(X)}{C_\infty(X)}$.

(<=) Suppose now that $\frac{C_\alpha(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathscr{F}$ (of $C_\alpha(X)$) $\subseteq S_\alpha$ under the homomorphism $\sigma : \mathscr{F} \longrightarrow \frac{C_\alpha(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$. We claim that $\mathscr{F}^\alpha$ separates the points of $\alpha X\setminus X$. For any $g \in C_\alpha(X)$ there is a function $f \in \mathscr{F}$ such that $C_\infty(X) + f = C_\infty(X) + g$ (since σ maps $\mathscr{F}$ onto $\frac{C_\alpha(X)}{C_\infty(X)}$). It follows that, for every function g in $C_\alpha(X)$, there is a function f_g in $\mathscr{F}$ and a function h_g in $C_\infty(X)$ such that $g = f_g + h_g$. Observe that the function $h_g{}^\alpha$ is zero on $\alpha X\setminus X$ for each g in $C_\alpha(X)$. Since the collection $\{g^\alpha : g \in C_\alpha(X)\}$ separates the points of $\alpha X\setminus X$ then the subset $\{f_g{}^\alpha : g \in C_\alpha(X)\}$ of $\mathscr{F}^\alpha$ must separate the points of $\alpha X\setminus X$. Then, by 3.19, αX is a singular compactification.

QED

3.23 THEOREM Let αX be a compactification of X. Then αX is a singular compactification iff $C_\alpha(X) = C_\infty(X) \oplus \mathscr{G}$ (the vector space direct sum) for some closed subalgebra $\mathscr{G}$ of $C^*(X)$ contained in S_α.

Proof: (=>) Suppose αX is a singular compactification. We proceed as in the first half of the proof of 3.22. By 3.21, S_α contains a closed subalgebra $\mathscr{F}$ of

$C_\alpha(X)$ such that the mapping $\phi : \mathcal{F} \longrightarrow C(\alpha X \setminus X)$ from $\mathcal{F}$ onto $C(\alpha X\setminus X)$ defined by $\phi(f) = f^\alpha|_{\alpha X\setminus X}$ is an isomorphism. Let $\tau : C_\alpha(X) \longrightarrow C(\alpha X\setminus X)$ be the homomorphism from $C_\alpha(X)$ onto $C(\alpha X\setminus X)$ defined by $\tau(f) = f^\alpha|_{\alpha X\setminus X}$. We now define the mapping $\psi : C_\alpha(X) \longrightarrow \mathcal{F}$ as $\psi = \phi^{\leftarrow}\circ\tau$. The kernel of ψ is $\psi^{\leftarrow}(\mathbf{0}) = (\phi^{\leftarrow}\circ\tau)^{\leftarrow}(\mathbf{0}) = \tau^{\leftarrow}\circ\phi(\mathbf{0}) = \tau^{\leftarrow}(\mathbf{0}^\alpha|_{\alpha X\setminus X}) = C_\infty(X)$. Observe that, for every f in $C_\alpha(X)$, $f - \psi(f) = f_\infty$ for some f_∞ in $C_\infty(X)$. Also if $h \in \mathcal{F} \cap C_\infty(X)$ then $\tau(h) = h^\alpha|_{\alpha X\setminus X} = \mathbf{0}$ (as $h \in C_\infty(X)$). But $\tau(h) = \phi(h) = h^\alpha|_{\alpha X\setminus X}$. Consequently $\phi(h) = \mathbf{0}$. As ϕ is one-to-one, $\mathbf{h} = \mathbf{0}$. Hence $\mathcal{F} \cap C_\infty(X) = \{0\}$. Thus $C_\alpha(X) = C_\infty(X) \oplus \mathcal{F}$.

(<=) Suppose αX is a compactification of X such that $C_\alpha(X) = C_\infty(X) \oplus \mathcal{G}$, where $\mathcal{G}$ is a closed subalgebra of $C^*(X)$ which is contained in S_α. Since $f[\alpha X\setminus X] = \{0\}$ for every function f in $C_\infty(X)$ then $\mathcal{G}^\alpha$ must separate the points of $\alpha X\setminus X$. It follows that αX is equivalent to $\omega_{\mathcal{G}}X$, (by 1.13), and that $e_{\mathcal{G}^\alpha} : \alpha X \longrightarrow \Pi_{f \in \mathcal{G}}S(f)$ separates the points of $\alpha X\setminus X$. Let x be a point in X. Recall that the set ${}_xK_{\mathcal{G}} = \cap\{Z(f) : f \in \mathcal{G}^+, x \in Z(f)\}$ is the maximal stationary set of $\mathcal{G}$ which contains the point x (see the paragraph preceding 2.12). Let ${}_xK_{\mathcal{G}^\alpha} = \cap\{Z(f^\alpha) : f \in \mathcal{G}^+; x \in Z(f)\}$ be the maximal stationary set of $\mathcal{G}^\alpha$ which contains the point x. Let $\mathcal{H}_x = \{Z(f^\alpha) \cap (\alpha X\setminus X) : f \in \mathcal{G}^+, x \in Z(f)\}$. Then $\cap\mathcal{H}_x = {}_xK_{\mathcal{G}^\alpha} \cap \alpha X\setminus X$. We wish to show that $\cap\mathcal{H}_x$ is a singleton set and then apply 5) => 1) of 2.12 to obtain our result. Since $\mathcal{G}^\alpha$ separates the points of $\alpha X\setminus X$ it will suffice to show that $\cap\mathcal{H}_x$ is non-empty. In fact, since every element of $\mathcal{H}_x$ is compact, it will suffice to show that $\mathcal{H}_x$ possesses the finite intersection property. Let $\mathcal{M} = \{Z(f_i^\alpha) \cap \alpha X\setminus X : i \in F\}$ be a finite subcollection of $\mathcal{H}_x$. Note that $\cap\mathcal{M} = Z(\Sigma_{i \in F}(f_i^\alpha)^2) \cap \alpha X\setminus X$. Since $\mathcal{G}^+$ is a subalgebra, $\Sigma_{i \in F}(f_i^\alpha)^2$ is an element of $\mathcal{G}^+ \subseteq S_\beta$. Since $(\Sigma_{i \in F}(f_i^\alpha)^2)[X] \subseteq [\Sigma_{i \in F}(f_i^\alpha)^2][\alpha X\setminus X]$ (by 1.8), then $Z(\Sigma_{i \in F}(f_i^\alpha)^2) \cap \alpha X\setminus X$ is non-empty. Thus $\mathcal{H}_x$ possesses the finite intersection property. It follows that $\cap\mathcal{H}_x = {}_xK_{\mathcal{G}^\alpha} \cap \alpha X\setminus X$ is a singleton set. By 5) => 1), $e_{\mathcal{G}}$ is a singular map and $\alpha X (\cong \omega_{\mathcal{G}}X$, by 1.13) is the singular compactification $X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$ induced by $e_{\mathcal{G}}$.

QED

Recall that an upward directed partially ordered set $(X,\leqq)$ must satisfy the following condition: If a and b are elements of X, then there exists an element c of X such that c is greater than or equal to both a and b. In chapter 2 (following theorem 2.12) we mentioned that we would provide an example of an upward directed family $\mathscr{A}$ of singular compactifications whose supremum is not a singular compactification. We present this example now.

3.24 EXAMPLE Let ω_1 denote the first uncountable ordinal and $[0,\omega_1)$ be the space of all ordinals less than ω_1. Let $X = [0,\omega_1) \times [0,\omega_1)$ (equipped with the product topology). The space X is pseudocompact (see 8.21 of [Wa]). In 8.23 of [Wa], it is shown that $\beta X = [0,\omega_1] \times [0,\omega_1]$ and that βX is not a singular compactification. We will show that the lattice of all compactifications of X contains a subfamily $\mathscr{A}$ of singular compactifications which is totally ordered and whose supremum is βX. Since a totally ordered family is clearly upward directed, we will have shown that an upward directed family of singular compactifications does not necessarily have a supremum which is a singular compactification.

Let λ be a non-limit ordinal such that λ is less than ω_1. Let $\alpha_\lambda X$ be the decomposition space obtained by collapsing to a point the subset $([\lambda,\omega_1] \times \{\omega_1\}) \cup (\{\omega_1\} \times [\lambda, \omega_1])$ of βX and fixing all other points of βX. Clearly $\alpha_\lambda X$ is a compactification of X. Note that if κ is a non-limit ordinal such that $\lambda < \kappa < \omega_1$, then $\alpha_\lambda X < \alpha_\kappa X < \beta X$. Hence the family $\mathscr{A} = \{\alpha_\kappa X : 0 \leqq \kappa < \omega_1, \kappa$ a non-limit ordinal$\}$ is a totally ordered collection of compactifications of X whose supremum is βX. We now claim that every member of $\mathscr{A}$ is a singular compactification. Let $\alpha_\lambda X$ be a member of $\mathscr{A}$. Let us denote by $[\omega_1]$ the point of $\alpha_\lambda X$ which is formed by collapsing to a single point the subset $[\lambda,\omega_1] \times \{\omega_1\} \cup \{\omega_1\} \times [\lambda, \omega_1]$ of βX. If $\kappa < \lambda$, let $F_\kappa = \{\kappa\} \times [0,\omega_1)$, and $H_\kappa = [\lambda,\omega_1) \times \{\kappa\}$ (both subsets of X).

Let $K = [\lambda,\omega_1) \times [\lambda,\omega_1)$. Observe that the elements of the collection $\mathscr{D} = \{F_\kappa : \kappa < \lambda\} \cup \{H_\kappa : \kappa < \lambda\} \cup \{K\}$ of subsets are pairwise disjoint. Consider the function $r : \alpha_\lambda X \longrightarrow \alpha_\lambda X \backslash X$ defined as follows: $r[F_\kappa] = (\kappa,\omega_1)$ if $\kappa < \lambda$, $r[H_\kappa] = (\omega_1,\kappa)$ if $\kappa < \lambda$, and $r[K] = [\omega_1]$. It is easily verified that r is continuous and is a retraction map. Hence $\alpha_\lambda X$ is a singular compactification. Then $\mathscr{A}$ is an upward directed family of singular compactifications whose supremum is βX, a compactification of X which is not singular.

We summarize the main result of this chapter in the following theorem.

3.25 THEOREM If X is a locally compact and Hausdorff space then the following are equivalent:

1) The space X has a largest singular compactification (i.e. μX is a singular compactification).

2) The set S_μ contains a subalgebra $\mathscr{G}$ of $C_\mu(X)$ such that $\mathscr{G}^\mu$ separates the points of $\mu X \backslash X$.

3) The set S_μ contains a closed subalgebra $\mathscr{G}$ of $C_\mu(X)$ such that the mapping $\phi : \mathscr{G} \longrightarrow C(\mu X \backslash X)$ from $\mathscr{G}$ onto $C(\mu X \backslash X)$ defined by $\phi(f) = f^\mu|_{\mu X \backslash X}$ is an isomorphism.

4) The quotient ring $\dfrac{C_\mu(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathscr{F}$ (of $C_\mu(X)) \subseteq S_\mu$ under the homomorphism $\sigma : \mathscr{F} \longrightarrow \dfrac{C_\mu(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

5) The set $C_\mu(X) = C_\infty(X) \oplus \mathscr{G}$ (the vector space direct sum) for some closed subalgebra $\mathscr{G}$ of $C^*(X)$ contained in S_μ.

Proof: 1) <=> 2) This is 3.19.

1) <=> 3) This is 3.21.

1) <=> 4) This is 3.22.

1) <=> 5) This is 3.23.

QED

By 3.11 any one of the above five conditions on μX implies that X is pseudocompact.

Recall that a space X is said to be retractive if $\beta X \setminus X$ is a retract of βX, i.e. βX is a singular compactification. W. W. Comfort has shown using CH that retractive spaces are locally compact and pseudocompact (see 6.6 of [Wa]). A precise characterization of retractive spaces can now be given.

3.26 COROLLARY For a locally compact Hausdorff space X the following are equivalent:

1) The space X is retractive (i.e. βX is a singular compactification).

2) The set S_β contains a subalgebra $\mathscr{G}$ of $C^*(X)$ such that $\mathscr{G}^\beta$ separates the points of $\beta X \setminus X$.

3) The set S_β contains a closed subalgebra $\mathscr{G}$ of $C^*(X)$ such that the mapping $\phi : \mathscr{G} \longrightarrow C(\beta X \setminus X)$ from $\mathscr{G}$ onto $C(\beta X \setminus X)$ defined by $\phi(f) = f^\beta|_{\beta X \setminus X}$ is an isomorphism.

4) The quotient ring $\dfrac{C^*(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathscr{F}$ (of $C_\mu(X)) \subseteq S_\mu$ under the homomorphism $\sigma : \mathscr{F} \longrightarrow \dfrac{C^*(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

5) The set $C^*(X) = C_\infty(X) \oplus \mathscr{G}$ (the vector space direct sum) for some closed subalgebra $\mathscr{G}$ of $C^*(X)$ contained in S_β.

Proof: The equivalence of the statements 1 to 5 follow directly from 3.25.

QED

In the introductory paragraph of [CFV] (*Two Applications of Singular Sets to the Theory of Compactifications*, To appear) the authors state:

"The principal unresolved conjecture in the theory (of lattices of singular compactifications) is the following:

CONJECTURE: The singular compactifications of a space X forms a lattice iff βX is singular"

We will show that this conjecture fails by constructing a space X whose family of singular compactifications forms a (complete) lattice even though βX is not singular.

3.27 EXAMPLE Let Y be a locally compact connected space such that $\beta Y\backslash Y$ is finite and has more than one point. (The space $Y = (\beta\mathbb{R})\backslash F$ where F is a finite subset of $\beta\mathbb{R}\backslash\mathbb{R}$ is an example of such a space). Let $\mathbb{N}$ denote the natural numbers and $\omega\mathbb{N}$ denote its one-point compactification. Let $X = \omega\mathbb{N} \times Y$ (with the product topology). By 9D 3) of [GJ], Y is pseudocompact. By 8.12 and 8.20 of [Wa], $\beta X = \omega\mathbb{N} \times \beta Y$. We claim that $\mu X = \omega\mathbb{N} \times \omega Y$ (where ωY denotes the one-point compactification of Y). Let u and v be distinct points in $\beta Y\backslash Y$. Let $f \in S_\beta$ and x_0 be a point in $\omega\mathbb{N}$. Then f extends to the function $f^\beta : \beta X \longrightarrow \mathbb{R}$. Let $x_0 \in \omega\mathbb{N}$ and suppose f^β separates the points (x_0,u) and (x_0,v). Since f is singular, $f^\beta[\beta X] = f^\beta[\beta X\backslash X]$ (by 1.8) and $f^\beta[\{x_0\} \times \beta Y] \subseteq f^\beta[\beta X\backslash X]$, which is a totally disconnected set (since it is countable). Since f^β separates (x_0,u) and (x_0,v), then $f^\beta[\{x_0\} \times \beta Y]$ is not a singleton, hence is not connected. This contradicts the fact that $f^\beta[\{x_0\} \times \beta Y]$ is connected (being the continuous image of the connected set $\{x_0\} \times \beta Y$). Hence, for any $x \in \omega\mathbb{N}$, every singular function f in S_β has an extension f^β which is constant on $(cl_{\beta X}(\{x\} \times Y))\backslash(\{x\} \times Y)$. Thus, for each x in $\omega\mathbb{N}$, $(cl_{\mu X}(\{x\} \times$

Y))\({x} x Y) is a singleton set, (this follows from the facts that $(cl_{\mu X}(\{x\}$ x Y))\({x} x Y) is either a singleton or contains finitely many elements and the collection $S_\beta{}^\mu$ separates the points of $\mu X \backslash X$). Let x_0 and y_0 be distinct points in $\omega\mathbb{N}$. Since $\{x_0\}$ x βY and $\{y_0\}$ x βY are distinct connected components of βX then there exists a clopen subset U of βX such that $\{x_0\}$ x $\beta Y \subseteq U$ and $\{y_0\}$ x $\beta Y \subseteq \beta X \backslash U$. Let $g : \beta X \longrightarrow \{0,1\}$ denote the characteristic function with respect to U. Then the function $g|_X$ is a singular function whose extension to βX separates $\{x_0\}$ x βY and $\{y_0\}$ x βY. Hence $S_\beta{}^\beta$ separates the connected components $\{\{x\}$ x $\beta Y : x \in \omega\mathbb{N}\}$ of βX. This implies that μX is the union of the **disjoint** collection $\{cl_{\mu X}(\{x\}$ x $\beta Y) : x \in \omega\mathbb{N}\}$.The map r defined by $r[cl_{\mu X}(\{x\}$ x $\beta Y)] = cl_{\mu X}(\{x\}$ x $\beta Y) \backslash (\{x\}$ x $\beta Y)$ (where $x \in \omega\mathbb{N}$) is easily seen to be a retraction map from μX onto $\mu X \backslash X$. Thus we conclude that μX is a singular compactification. Since μX is the supremum of all singular compactifications, the collection of all singular compactifications forms a (complete) lattice (see the note following 3.1). Since $(cl_{\mu X}(\{x\}$ x Y))\({x} x Y) is a singleton for each $x \in \omega\mathbb{N}$, then μX is strictly less than βX. Hence βX is not a singular compactification.

In theorem 6 of [CF] the authors claim that "If the set of singular compactifications of a space X forms a lattice, then it forms a complete lattice". The proof of this statement supplied by the authors is flawed. This error is pointed out in [CFV].The truth or falsity of this statement remains an open question.

We consider a simpler problem. In the following example, we show that a subfamily $\mathscr{F}$ of the family of all singular compactifications of a space X may form a lattice which is not complete.

3.28 EXAMPLE A lattice of singular compactifications of a space X is not necessarily a complete lattice.

Proof: In example 3.24, we have shown that the family of all singular compactifications of the space $X = [0,\omega_1) \times \{0,\omega_1)$ contains a totally ordered lattice $\mathscr{A} = \{\alpha_\kappa X : 0 \leqq \kappa < \omega_1, \kappa$ a non-limit ordinal$\}$ of singular compactifications whose supremum is βX a compactification which is not singular.

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Observe that the family of all singular compactifications of the space $X = [0,\omega_1) \times \{0,\omega_1)$ does not form a lattice. To see this let αX be the decomposition space obtained by collapsing to a point the subset $\{\omega_1\} \times [0,\omega_1]$ of $\mu X = [0,\omega_1] \times [0,\omega_1]$ (and fixing all other points). Clearly αX is a compactification of X. Let γX be the decomposition space obtained by collapsing to a point the subset $[0,\omega_1] \times \{\omega_1\}$ of $\beta X = \mu X = [0,\omega_1] \times [0,\omega_1]$ (and fixing all other points). It is easy to verify that both αX and γX are singular compactifications. Note that the supremum of αX and γX is μX, a non-singular compactification (since $[0,\omega_1] \times [0,\omega_1]$ is not singular).

CHAPTER 4

EXAMPLES

In the last chapter we have characterized those spaces X which have a largest singular compactification μX (3.25 and 3.26). In this chapter we will construct a few examples which illustrate some of the results of the previous chapters.

We will make use of the Glicksberg theorem and another useful result found in 8.12 and 8.20 of [Wa] respectively. We state them here.

4.1 THEOREM (Glicksberg) If X and Y are infinite, then the product space X x Y is pseudocompact if and only if X x Y is C*-embedded in $\beta X \times \beta Y$, i.e. $\beta(X \times Y) = \beta X \times \beta Y$.

4.2 PROPOSITION (8.21 of [Wa]) The product of two pseudocompact spaces one of which is also locally compact is pseudocompact.

4.3 PROPOSITION The outgrowths of the singular compactifications of $\mathbb{N}$ are the compact separable spaces.

Proof: Suppose $\alpha\mathbb{N}$ is a singular compactification of $\mathbb{N}$. Then by 2.6 and 2.11, there exist a subset $\mathcal{F}$ of S_α such that $e_{\mathcal{F}}$ is a singular map and $\alpha\mathbb{N} \cong X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}})$. Since $e_{\mathcal{F}}[\mathbb{N}]$ is countable and dense in $S(e_{\mathcal{F}})$, $\alpha\mathbb{N}\setminus\mathbb{N}$ is compact and separable.

Suppose K is a compact space which contains a dense subspace $D = \{d_i : i \in N\}$. Let $\mathcal{D} = \{D_i : i \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. Let f be the (continuous) function from $\mathbb{N}$ onto D which maps D_i to d_i. Clearly f is a singular function whose singular set is $cl_K f[\mathbb{N}] = K$. Hence $\mathbb{N} \cup_f S(f)$ is a singular compactification of $\mathbb{N}$. Thus K is the remainder of a singular compactification of $\mathbb{N}$.

QED

If a space X is almost compact non-compact then its one-point compactification is βX ($\cong \mu X$), a singular compactification. Hence spaces X for which μX is a singular compactification are easy to find. Here is a non-trivial example of a space such that μX is a singular compactification.

4.4 EXAMPLE The product space $X = [0,\omega_1) \times I$ (where I is the closed unit interval) equipped with the product topology has a μ-compactification μX which is singular.

Proof: By 4.1 and 4.2 $\beta X = \beta[0,\omega_1) \times I = [0,\omega_1] \times I$. Since $\beta X \setminus X = \{\omega_1\} \times I$ is connected then, by 2.13, μX is equivalent to βX. The map r defined as $r[\{x\} \times [0,\omega_1]] = (x,\omega_1)$ is easily seen to be a retraction from μX onto $\mu X \setminus X$. Hence μX is singular.

QED

Here is an example which shows that if X and Y are two spaces, $\mu(X \times Y)$ need not be equivalent to $\mu X \times \mu Y$.

4.5 EXAMPLE Let $X = \beta\mathbb{R}\setminus\{x, y\}$ where x and y are distinct points in $\beta\mathbb{R}\setminus\mathbb{R}$. Let Y = I, where I is the closed unit interval. Then $\mu(X \times Y)$ is not equivalent to $\mu X \times \mu Y$.

Proof: Clearly $\mu Y = Y$ as Y is compact. Since the cardinality of $\beta X \setminus X$ is strictly less than 2^c, X is pseudocompact (by 9D of [GJ]). By 4.1 and 4.2, $\beta(X \times Y) = \beta X \times Y$. In example 3.4 we have seen that μX is the one-point compactification of X while βX (the two-point compactification of X) is homeomorphic to $\beta\mathbb{R}$. Hence $\beta X \times Y$ is not equivalent to $\mu X \times Y$ (as a compactification of $X \times Y$).

However since the sets $\{x\} \times Y$ and $\{y\} \times Y$ are connected in $\beta(X \times Y) = \beta X \times Y$, $\mu(X \times Y)$ is equivalent to $\beta(X \times Y)$ (by 2.13). Hence $\mu(X \times Y)$ is not equivalent to $\mu X \times Y$.

QED

We now present two examples of a space X such that $\mu X \cong \beta X$ but where μX is not singular.

4.6 EXAMPLES a) Let x and y be distinct points in $\beta\mathbb{R}\backslash\mathbb{R}$ and let $Y = \beta\mathbb{R}\backslash\{x, y\}$. Let $X = [\beta\mathbb{N}\backslash\mathbb{N}] \times Y$. Then $\mu X \cong \beta X$ and βX is not a singular compactification.

Proof: By 4.2, X is pseudocompact. By 4.1, $\beta X = [\beta\mathbb{N}\backslash\mathbb{N}] \times \beta\mathbb{R}$. Having shown in example 3.4 that μY is the one-point compactification of Y, a compactification strictly less than $cl_{\beta\mathbb{R}}Y = \beta\mathbb{R}$, one would not expect that $\mu X \cong \beta X = [\beta\mathbb{N}\backslash\mathbb{N}] \times \beta\mathbb{R}$ as we now prove it to be.

To prove that μX is equivalent to βX it suffices to show that $S_\beta{}^\beta$ separates pairs of points of form $\{(a,x),(a,y)\}$ (by 3.5). Let a be an element of $\beta\mathbb{N}\backslash\mathbb{N}$ and $u = (a,x)$ and $v = (a,y)$ be two such points in $\beta X\backslash X$. Let D_1 be a (C*-embedded) copy of $\mathbb{N}$ in $[\beta\mathbb{N}\backslash\mathbb{N}] \times \{x\}$ such that $cl_{\beta X\backslash X}D_1$ does not contain u or v (see 14N5 of [GJ]). Let a_o be an element of $\beta\mathbb{N}\backslash\mathbb{N}$ such that $\{a_o\} \times \beta\mathbb{R}$ does not contain u or v and does not meet $cl_{\beta X}D_1$. Let D_2 be a C*-embedded copy of $\mathbb{N}$ in $\{a_o\} \times \beta\mathbb{R}$. Let $Z = \{u,v\} \cup D_1 \cup D_2$. Define a function $f : \{u,v\} \cup D_1 \cup D_2 \longrightarrow \mathbb{R}$ as follows: $f(u) = 0$, $f(v) = 1$, and f maps D_1 onto $(0,1) \cap \mathbb{Q}$ and D_2 onto $(0,1) \cap \mathbb{Q}$. Since D_1 and D_2 are disjoint and both C*-embedded and $\{u,v\} \subseteq \beta X\backslash(cl_{\beta X}D_1 \cup cl_{\beta X}D_2)$, f extends continuously to $cl_{\beta X}Z$. Since the collection $\{u\}$, $\{v\}$, $cl_{\beta X}D_1$ and $cl_{\beta X}D_2$ are pairwise disjoint, f extends to a function h on βX for which $h[\beta X] = [0,1]$. Let $g = h|_X$ (hence $g^\beta = h$). The function g is easily seen to be a singular map whose extension to βX separates u and v (since $g^\beta[\beta X\backslash X] = [0,1] = g[X]$ and by applying

1.8). Hence $S_\beta{}^\beta$ separates the points of $\beta X \setminus X$. This implies βX is equivalent to μX ($= \omega_{S_\beta} X$).

We now show that μX is not a singular compactification. Suppose $r : \mu X \longrightarrow \mu X \setminus X$ is a retraction from μX onto $\mu X \setminus X$. Since r fixes the points of $\mu X \setminus X$, the points (a_o,x) and (a_o,y) belong to $r[\{a_o\} \times \beta\mathbb{R}]$. Since $\mu X \setminus X$ is totally disconnected and $\{a_o\} \times \beta\mathbb{R}$ is connected, $r[\{a_o\} \times \beta\mathbb{R}]$ cannot meet any points in $\mu X \setminus X$ other than (a_o,x) and (a_o,y). Hence $r[\{a_o\} \times \beta\mathbb{R}] = \{(a_o,x),(a_o,y)\}$. Since $\{(a_o,x),(a_o,y)\}$ is itself disconnected we have a contradiction. Hence μX is not a singular compactification.

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b) Let Y denote the space $\beta\mathbb{R}\setminus\{y\}$, where y belongs to $\beta\mathbb{R}\setminus\mathbb{R}$. Then $\beta Y = \beta\mathbb{R}$, the one-point compactification of Y. Let $X = [\beta\mathbb{N}\setminus\mathbb{N}] \times Y$. Then, by 4.1 and 4.2, $\beta X = [\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y$. The map $r : \beta X \longrightarrow \beta X \setminus X$ defined by $r[\{a\} \times \beta Y] = \{(a,y)\}$ is easily seen to be a retraction map, hence μX is a singular compactification and $\mu X \cong \beta X$. We now slightly modify the above space X. Let $Z = [\beta\mathbb{N}\setminus\mathbb{N}] \times Y] \cup (\beta\mathbb{N} \times \{0\})$, viewed as a subspace of $\beta\mathbb{N} \times Y$. We claim that μZ is not a singular compactification of Z.

Proof: The following argument shows that $\beta Z = ([\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y) \cup (\beta\mathbb{N} \times \{0\})$. By 4.1 and 4.2 $\beta([\beta\mathbb{N}\setminus\mathbb{N}] \times Y) = [\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y$.Let g be a real-valued bounded function on Z. Since $g|_{\beta\mathbb{N}\setminus\mathbb{N} \times Y}$ extends to $[\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y$ and $cl_Z(\mathbb{N} \times \{0\}) = \beta\mathbb{N} \times \{0\}$ then g extends to a function g^β on $([\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y) \cup (\beta\mathbb{N} \times \{0\})$. Hence Z is C*-embedded in $([\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y) \cup (\beta\mathbb{N} \times \{0\})$ and so $\beta Z = ([\beta\mathbb{N}\setminus\mathbb{N}] \times \beta Y) \cup (\beta\mathbb{N} \times \{0\})$. By applying 3.5 we obtain that μZ is equivalent to βZ.

We now show that βZ cannot be a singular compactification. Suppose that $r : \beta Z \longrightarrow \beta Z \setminus Z$ is a retraction map from βZ onto $\beta Z \setminus Z$. If a belongs to $[\beta\mathbb{N}\setminus\mathbb{N}]$ then, since $\{a\} \times \beta Y$ is connected and $[\beta\mathbb{N}\setminus\mathbb{N}] \times \{y\}$ is totally disconnected, r can only map $\{a\} \times \beta Y$ onto $\{(a,y)\}$. Hence r maps $[\beta\mathbb{N}\backslash\mathbb{N}] \times \{0\}$ homeomorphically onto $[\beta\mathbb{N}\backslash\mathbb{N}] \times \{y\}$. Since $\beta\mathbb{N} \times \{0\}$ is separable, $r[\beta\mathbb{N} \times \{0\}]$ is separable. Since $r[\beta\mathbb{N} \times \{0\}] = [\beta\mathbb{N}\backslash\mathbb{N}] \times \{y\}$ we have a contradiction. Hence βZ cannot be a singular compactification.

QED

In example b) of 4.6 we have considered two subspaces $X = [\beta\mathbb{N}\backslash\mathbb{N}] \times Y$ and $Z = [\beta\mathbb{N}\backslash\mathbb{N}] \times Y] \cup (\beta\mathbb{N} \times \{0\})$ of $\beta\mathbb{N} \times Y$ (where $Y = \beta\mathbb{R}\backslash\{y\}$). Note that $X \subseteq Z$ and that they have identical outgrowths $\mu X\backslash X$ and $\mu Z\backslash Z$. However there compactifications μX and μZ differ in nature since μX is singular and μZ is not.

Up to this point we have not encountered spaces X such that $\mu X \not\equiv \beta X$ and μX is not a singular compactification. Could it be that if a space X is such that $\mu X \not\equiv \beta X$ then μX is necessarily singular? The following example assures us that such a conjecture fails.

4.7 EXAMPLE Let $Y = \beta\mathbb{R}\backslash\{x,y\}$ where x and y are distinct points in $\beta\mathbb{R}\backslash\mathbb{R}$. Let $X = \mathbb{N} \oplus Y$ be the free union of $\mathbb{N}$ and Y. We claim that X is such that $\mu X \not\equiv \beta X$ and μX is not a singular compactification.

Proof: If f belongs to $S_\beta(X)$ then $f|_\mathbb{N}$ extends to $\beta\mathbb{N}$ and $f|_Y$ extends to the one-point compactification of Y. In example 3.4 we have shown that μY is the one-point compactification of Y. By 2.14, 1.16 and 1.13 $\mu\mathbb{N}$ is equivalent to $\beta\mathbb{N}$. Hence f extends to $\alpha X = \beta\mathbb{N} \oplus \mu Y$, a compactification of $\mathbb{N} + Y$ strictly less than $\beta\mathbb{N} \oplus \beta Y = \beta X$. It is shown in the proof of 2.14 that $S_\beta{}^\beta(X)$ separates the points of $\beta\mathbb{N}\backslash\mathbb{N}$. The characteristic function function $\chi_\mathbb{N}$ extends to $\chi_\mathbb{N}{}^\alpha$ separating $\beta\mathbb{N}$ from μY. Thus $S_\beta(X)$ has extensions to $\beta\mathbb{N} \oplus \mu Y$ which separate the points of $(\beta\mathbb{N} \oplus \mu Y)\backslash(\mathbb{N} \oplus Y)$. Hence μX is equivalent to $\beta\mathbb{N} \oplus \mu Y$. Clearly X is not pseudocompact . Hence by 3.11, μX cannot be a singular compactification. Since

βX is not $\beta\mathbb{N} \oplus \mu Y \cong \mu X$ we have shown that X is such that $\mu X \ncong \beta X$ and μX is not a singular compactification.

QED

CHAPTER 5

PERFECT AND PSEUDOPERFECT COMPACTIFICATIONS

In this chapter we investigate another compactification of a space X called the *perfect* compactification. This compactification was introduced in [Sk] and was further studied in [D]. In [D] the author gives an algebraic characterization of the subring $C_\alpha(X)$ associated to a perfect compactification αX of X. We provide an alternate proof to this characterization. We also introduce a new compactification called the *pseudoperfect* compactification and characterize it. We show that all perfect compactifications and compactifications of pseudocompact spaces are pseudoperfect compactifications. We also show that there are pseudoperfect compactifications which are not perfect.

A perfect compactification is defined as follows:

5.1 DEFINITION Let γX be a compactification of X and $\pi_{\beta\gamma} : \beta X \longrightarrow \gamma X$ denote the natural map from βX onto γX. We say that γX is perfect if, for every p $\in \gamma X \backslash X$, $\pi_{\beta\gamma}^{\leftarrow}(p)$ is connected.

It is clear from this definition that βX is a perfect compactification. It is also easily seen that the smallest perfect compactification is the Freudenthal compactification (the compactification of X obtained by collapsing the connected components of $\beta X \backslash X$ to points)

There are various characterizations of perfect compactifications. The proofs of the following characterizations are given in Theorems 1 and 2 and Lemma 1 of [S].

5.2 PROPOSITION Let γX be a compactification of X. For any open subset U of X, let $Ex_{\gamma X}U = \gamma X \backslash cl_{\gamma X}(X \backslash U)$, the extension of U in γX. Then the following are equivalent:

1) γX is a perfect compactification.

2) For any pair of disjoint subsets A and B of X, $cl_{\gamma X}A \cap cl_{\gamma X}B = \emptyset$ iff $cl_{\gamma X}Fr_X A \cap cl_{\gamma X}Fr_X B = \emptyset$.

3) If U and V are disjoint open subsets of X, then $Ex_{\gamma X}(U \cup V) = Ex_{\gamma X}U \cup Ex_{\gamma X}V$.

4) For any open subset U of X, $cl_{\gamma X}(Fr_X U) = Fr_{\gamma X}Ex_{\gamma X}U$.

In [D], the author relates perfect compactifications of X to *algebraic* subrings of $C^*(X)$. The following definition of an algebraic subring first appears in 16.29 of [GJ].

5.3 DEFINITION A subring $\mathcal{F}$ of $C^*(X)$ is *algebraic* if $\mathcal{F}$ contains the constant functions and those functions $f \in C^*(X)$ such that $f^2 \in \mathcal{F}$.

To establish a relationship between perfect compactifications and algebraic subrings of $C^*(X)$ we will require the following result from [GJ].

5.4 PROPOSITION (16.30 and 16.31 of [GJ]) a) The set $\mathcal{F}$ of all functions in $C^*(X)$ which are constant on a given subset S of X is an algebraic subring of $C^*(X)$ which is closed in the uniform norm topology.

b) If X is compact and $\mathcal{F}$ is an algebraic subring of $C^*(X)$ then each maximal stationary set of $\mathcal{F}$ is connected.

5.5 LEMMA Let γX be a compactification of X and $\mathcal{G} = C_\gamma(X)$. Then $\pi_{\beta\gamma} = e_{\mathcal{G}}{}^{\gamma\leftarrow} \circ e_{\mathcal{G}}{}^{\beta}$.

Proof: Observe that, for every $f \in C_\gamma(X)$, $f^\beta(X) = (f^\gamma \circ \pi_{\beta\gamma})(x)$ for all x in βX. Then $e_{\mathcal{G}}{}^{\beta}(x) = <f^\beta(x)>_{f \in C_\gamma(X)} = <(f^\gamma \circ \pi_{\beta\gamma})(x)>_{f \in C_\gamma(X)} = <(f^\gamma(\pi_{\beta\gamma}(x))>_{f \in C_\gamma(X)} = e_{\mathcal{G}}{}^{\gamma}(\pi_{\beta\gamma}(x))$. Also note that $e_{\mathcal{G}}{}^{\gamma}$ is one-to-one on γX since $\mathcal{G}^\gamma = C(\gamma X)$ separates the points of γX (1.12); hence $e_{\mathcal{G}}{}^{\gamma\leftarrow}$ is a well-defined function on $e_{\mathcal{G}}{}^{\gamma}[\gamma X] =$

$e_{\mathscr{G}}{}^{\gamma}[\pi_{\beta\gamma}[\beta X]] = e_{\mathscr{G}}{}^{\beta}[\beta X]$. Since $e_{\mathscr{G}}{}^{\beta}(x) = e_{\mathscr{G}}{}^{\gamma}(\pi_{\beta\gamma}(x))$ for all x in βX, $\pi_{\beta\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$.

QED

5.6 NOTE Let $\mathscr{G} \subseteq C_\gamma(X)$. We claim that *the subsets of X of the form $e_{\mathscr{G}}{}^{\gamma\leftarrow}(p)$ (where $p \in e_{\mathscr{G}}{}^{\gamma}[\gamma X]$) are the maximal stationary sets of $\mathscr{G}^{\gamma}$*. Let S be a maximal stationary set of $\mathscr{G}^{\gamma}$. Let $x \in S$ and $p = e_{\mathscr{G}}{}^{\gamma}(x)$. By definition S is the largest subset (of γX) containing x on which all functions in $\mathscr{G}^{\gamma}$ are constant. Since all functions in $\mathscr{G}^{\gamma}$ are constant on $e_{\mathscr{G}}{}^{\gamma\leftarrow}(p)$ and $x \in e_{\mathscr{G}}{}^{\gamma\leftarrow}(p)$ then $e_{\mathscr{G}}{}^{\gamma\leftarrow}(p) \subseteq S$. Suppose $t \in \gamma X \backslash e_{\mathscr{G}}{}^{\gamma\leftarrow}(p)$. Then there exists a function h in $\mathscr{G}^{\gamma}$ such that $\{h(t)\} \neq \{h[x]\} = h[S]$ (otherwise $e_{\mathscr{G}}{}^{\gamma}(t) = \langle f^{\gamma}(t)\rangle_{f \in C_\gamma(X)} = e_{\mathscr{G}}{}^{\gamma}(x) = p$). Then $t \notin S$. Hence $e_{\mathscr{G}}{}^{\gamma\leftarrow}(p) = S$. Thus subsets of form $e_{\mathscr{G}}{}^{\gamma\leftarrow}(p)$ (where $p \in e_{\mathscr{G}}{}^{\gamma}[\gamma X]$) are the maximal stationary sets of $\mathscr{G}^{\gamma}$.

Note that if αX is a compactification of X, then $C(\alpha X)$ and $C_\alpha(X)$ are ring isomorphic (via the map $f \longmapsto f|_X$). Hence (as being algebraic is a ring-theoretic property) $C(\alpha X)$ is algebraic iff $C_\alpha(X)$ is algebraic.

The following result is found in Theorem 2.5 (a)<=>(c) of [D]. Using 5.5 and 5.6 we provide a shorter proof of this statement.

5.7 THEOREM [D] Let γX be a compactification of X. Then γX is a perfect compactification iff $C_\gamma(X)$ is an algebraic subring of $C^*(X)$.

Proof: (<=) Suppose $C_\gamma(X)$ is an algebraic subring of $C^*(X)$. Note that if $f^2 \in C_\gamma(X)^{\beta}$, $f^2|_X = f|_X{}^2 \in C_\gamma(X)$. Since $C_\gamma(X)$ is algebraic, $f|_X \in C_\gamma(X)$, hence $f \in C_\gamma(X)^{\beta}$. This implies that $C_\gamma(X)^{\beta}$ is an algebraic subring of $C(\beta X)$. Let $\mathscr{G} = C_\gamma(X)$. By 5.5, $\pi_{\beta\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$. If $p \in \gamma X \backslash X$, $\pi_{\beta\gamma}{}^{\leftarrow}(p) = [e_{\mathscr{G}}{}^{\gamma\leftarrow}(e_{\mathscr{G}}{}^{\beta})]^{\leftarrow}(p) = e_{\mathscr{G}}{}^{\beta\leftarrow}(e_{\mathscr{G}}{}^{\gamma}(p))$, a maximal stationary set of $\mathscr{G}^{\beta}$ in βX (by 5.6). Since $\mathscr{G}^{\beta}$ is an algebraic subring of $C(\beta X)$, the maximal stationary sets of $\mathscr{G}^\beta$ are connected (by 5.4). Hence $\pi_{\beta\gamma}^{\leftarrow}(p)$ is connected. It follows that γX is a perfect compactification.

(=>) Suppose γX is a perfect compactification. Then for every $p \in \gamma X \setminus X$ $\pi_{\beta\gamma}^{\leftarrow}(p)$ is connected. Let $\mathscr{G} = C_\gamma(X)$. We claim that $\mathscr{G}^\beta$ is an algebraic subring of $C(\beta X)$ (hence $\mathscr{G}$ is an algebraic subring of $C^*(X)$). Let $p \in \gamma X$. Let $\mathscr{F}_p$ denote all functions in $C(\beta X)$ which are constant on $\pi_{\beta\gamma}^{\leftarrow}(p)$. Then $\mathscr{G}^\beta \subseteq \mathscr{F}_p$ (since $\pi_{\beta\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$). Hence $\mathscr{G}^\beta \subseteq \cap\{\mathscr{F}_p : p \in \gamma X\}$, an algebraic subring of $C(\beta X)$ (since, by 5.4, $\mathscr{F}_p$ is an algebraic subring and since the intersection of algebraic subrings is an algebraic subring). Suppose $f \in \cap\{\mathscr{F}_p : p \in \gamma X\}$. Let $g : \gamma X \longrightarrow \mathbb{R}$ be the function which maps $\pi_{\beta\gamma}(x)$ to $f^\beta(x)$ for every x in βX. Then $f|_X = g|_X \in C_\gamma(X)$ (by definition of g). Hence $\cap\{\mathscr{F}_p : p \in \gamma X\} \subseteq C_\gamma(X)$. Thus $C_\gamma(X)^\beta = \cap\{\mathscr{F}_p : p \in \gamma X\}$. We have shown that $C_\gamma(X)^\beta$ is an algebraic subring of $C(\beta X)$. It easily follows that $C_\gamma(X)$ is an algebraic subring of $C^*(X)$.

QED

A subring $\mathscr{F}$ of $C^*(X)$ containing the constant functions *determines a compactification* γX of X if $\mathscr{F} \subseteq C_\gamma(X)$ and $\mathscr{F}^\gamma$ separates points of γX.

In the following proposition we show that any algebraic subring of $C^*(X)$ determines a perfect compactification of X.

5.8 PROPOSITION If $\mathscr{F}$ is an algebraic subring of $C^*(X)$ which separates the points and closed sets of X then $\mathscr{F}$ determines a perfect compactification of X.

Proof: Suppose $\mathscr{F}$ is an algebraic subring of $C^*(X)$ which separates the points and closed sets of X. Let $\alpha X = \omega_{\mathscr{F}} X$. Then $\mathscr{F}^\alpha$ separates the points of αX (by 1.13). Hence $e_{\mathscr{F}}{}^\alpha$ is one-to-one on αX (see 1.12). Let $x \in \alpha X$ and $p = e_{\mathscr{F}}{}^\alpha(x)$. We wish to show that $\pi_{\beta\alpha}^{\leftarrow}(x)$ is a connected subset of βX. Observe that p is of the form $<f^\alpha(x)>_{f \in \mathscr{F}}$. Hence $e_{C(\alpha X)}{}^{\leftarrow}(e_{C(\alpha X)}(x)) = e_{C(\alpha X)}{}^{\leftarrow}(<f^\alpha(x)>_{f \in C_\alpha(X)}) = \cap\{f^{\alpha\leftarrow}(f^\alpha(x)) : f \in C_\alpha(X)\} \subseteq \cap\{f^{\alpha\leftarrow}(f^\alpha(x)) : f \in \mathscr{F}\} = e_{\mathscr{F}}{}^{\alpha\leftarrow}(<f^\alpha(x)>_{f \in \mathscr{F}}) = e_{\mathscr{F}}{}^{\alpha\leftarrow}(p)$.

Since $e_{C(\alpha X)}^{\leftarrow}(e_{C(\alpha X)}(x))$ is nonempty and $e_{\mathscr{F}}{}^{\alpha\leftarrow}(p)$ is a singleton set (by 1.12) $e_{C(\alpha X)}^{\leftarrow}(e_{C(\alpha X)}(x)) = e_{\mathscr{F}}{}^{\alpha\leftarrow}(p)$. Equivalently $e_{C_\alpha(X)}{}^{\beta\leftarrow}(e_{C(\alpha X)}(x)) = e_{\mathscr{F}}{}^{\beta\leftarrow}(p)$. Now since $\mathscr{F}^\alpha$ is algebraic $\mathscr{F}^\beta$ is algebraic; hence, by 5.6 and 5.4, $e_{\mathscr{F}}{}^{\beta\leftarrow}(p)$ is connected. It then follows that $e_{C_\alpha(X)}{}^{\beta\leftarrow}(e_{C(\alpha X)}(x))$ is connected. By 5.5, $\pi_{\beta\alpha} = e_{C(\alpha X)}^{\leftarrow} \circ e_{C_\alpha(X)}{}^{\beta}$. Then $\pi_{\beta\alpha}{}^{\leftarrow}(x) = (e_{C(\alpha X)}^{\leftarrow} \circ e_{C_\alpha(X)}{}^{\beta})^{\leftarrow}(x) = e_{C_\alpha(X)}{}^{\beta\leftarrow}(e_{C(\alpha X)}(x))$ is connected (as shown above). Since x was arbitrarily chosen, $\pi_{\beta\alpha}{}^{\leftarrow}(x)$ is connected for all $x \in \alpha X$. Hence αX is a perfect compactification.

QED

Besides being an algebraic subring of $C^*(X)$ the subring $C_\gamma(X)$ associated with a perfect compactification γX possesses another interesting property. This property is described in lemma 2.3 of [D].

5.9 THEOREM [D] If γX is a perfect compactification then $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

One may wonder whether the perfect compactifications are the only compactifications of X for which $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$. The following result from [D] will help us answer this question.

5.10 THEOREM (Corollary 3.6 of [D]) If X is pseudocompact then, for any compactification γX of X, $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

We now provide an example of a non-perfect compactification γX such that $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

5.11 EXAMPLE Let X be the product space $[0,\omega_1) \times [0,\omega_1)$. Then $\beta X = [0,\omega_1] \times [0,\omega_1]$ (4.1 and 4.2). Since $\beta X \setminus X$ is totally disconnected βX is the smallest perfect compactification of X. Since X is pseudocompact $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$ for all compactifications γX of X (by 5.10). Hence if γX is any compactification strictly less than βX (for example the one-point compactification of X), γX is a non-perfect compactification for which $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

We introduce the following definitions.

5.12 DEFINITION Let γX be a compactification of X and U and V be disjoint subsets of X. The sets U and V are said to be *γ-separated* if there exists a subset M of γX containing X and a real-valued function f on M with compact fibres such that $f|_X \in C^*(X)$ and such that f[U] and f[V] have disjoint closures in $\mathbb{R}$.

5.13 DEFINITION A compactification γX is said to be a *pseudoperfect compactification* if any two subsets U and V of X which are γ-separated have disjoint closures in γX.

We will show that the pseudoperfect compactifications are precisely the compactifications γX of X for which $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

The following proposition which is an easy consequence of 3.2.1 of [E] will be required.

5.14 PROPOSITION (Taimanov's Theorem) Let αX and αY be a compactification of X and Y respectively, and let f be a map from X onto Y. There is a map f^* : $\alpha X \longrightarrow \alpha Y$ extending f if and only if, for A, B $\subseteq$ Y, $cl_{\alpha Y}A \cap cl_{\alpha Y}B = \varnothing$ implies that $cl_{\alpha X}f^{\leftarrow}[A] \cap cl_{\alpha X}f^{\leftarrow}[B] = \varnothing$.

5.15 THEOREM Let γX be a compactification of X. Then the following are equivalent:

1) γX is a pseudoperfect compactification.

2) $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

3) If $f \in C^*(X)$ such that $cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2$ then $cl_{\gamma X}f^{\leftarrow}[A] \cap cl_{\gamma X}f^{\leftarrow}[B] = \varnothing$ for any subsets A and B of f[X] with disjoint closures in $\mathbb{R}$.

Proof: (1=>2) Suppose γX is pseudoperfect. We wish to show that $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$. It is always true that $C_\gamma(X) \subseteq \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$. We will prove that $\{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\} \subseteq C_\gamma(X)$. Let $g \in \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$. Let A and B be subsets of g[X] with disjoint closures in $\mathbb{R}$. By 5.14, it will suffice to show that $cl_{\gamma X}g^{\leftarrow}[A] \cap cl_{\gamma X}g^{\leftarrow}[B] = \varnothing$. Let $M = \cup\{cl_{\gamma X}Z(g - r) : r \in g[X]\}$. Let $g^* : M \longrightarrow \mathbb{R}$ be a function defined as $g^*[cl_{\gamma X}Z(g - r)] = r$ for each $r \in g[X]$. Since $g \in \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$, then g^* is well-defined, $g^*|_X = g \in C^*(X)$ and g^* has compact fibres. Also $A = g^*[g^{\leftarrow}[A]]$ and $B = g^*[g^{\leftarrow}[B]]$ have disjoint closures in $\mathbb{R}$. Then by definition $g^{\leftarrow}[A]$ and $g^{\leftarrow}[B]$ are γ-separated subsets of X. By hypothesis $cl_{\gamma X}g^{\leftarrow}[A] \cap cl_{\gamma X}g^{\leftarrow}[B] = \varnothing$. By 5.14, $g \in C_\gamma(X)$.

(2=>3) Suppose $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$. Let $f \in C^*(X)$ such that $cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2$. Let A and B be disjoint subsets of f[X] such that $cl_{\mathbb{R}}A \cap cl_{\mathbb{R}}B = \varnothing$. Since $f \in C_\gamma(X)$, $cl_{\gamma X}f^{\leftarrow}[A] \cap cl_{\gamma X}f^{\leftarrow}[B] = \varnothing$ (by 5.14).

(3=>1) Suppose that, for any $f \in C^*(X)$ such that $cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2$, $cl_{\gamma X}f^{\leftarrow}[A] \cap cl_{\gamma X}f^{\leftarrow}[B] = \varnothing$ for any subsets A and B of f[X] with disjoint closures in $\mathbb{R}$. Let U and V be γ-separated subsets of X. We wish to show that $cl_{\gamma X}U \cap cl_{\gamma X}V = \varnothing$. To say that U and V are γ-separated subsets of X means that there exists a subset M of γX containing X and a real-valued function f on M with compact fibres such that $f|_X \in C^*(X)$ and such that $cl_{\mathbb{R}}f[U] \cap cl_{\mathbb{R}}f[V] =$

$\varnothing$. Let r_1 and r_2 be distinct points in $f|_X[X]$. Since f has compact fibres $Z(f - r_1)$ and $Z(f - r_2)$ are compact; hence $cl_{\gamma X}Z(f|_X - r_1) \subseteq Z(f - r_1)$ and $cl_{\gamma X}Z(f|_X - r_2) \subseteq Z(f - r_2)$. It follows that $cl_{\gamma X}Z(f|_X - r_1) \cap cl_{\gamma X}Z(f|_X - r_2) = \varnothing$. By hypothesis $cl_{\gamma X}f|_X^{\leftarrow}[f[U]] \cap cl_{\gamma X}f|_X^{\leftarrow}[f[V]] = \varnothing$ (since $cl_{\mathbb{R}}f[U] \cap cl_{\mathbb{R}}f[V] = \varnothing$). Since $U = f|_X^{\leftarrow}[f[U]]$ and $V = f|_X^{\leftarrow}[f[V]]$, $cl_{\gamma X}U \cap cl_{\gamma X}V = \varnothing$. Thus γX is a pseudoperfect compactification.

QED

By 5.10, every perfect compactification is a pseudoperfect compactification. By 5.11, every compactification of a pseudocompact space is a pseudoperfect compactification.

We now present an example of a space X which has a compactification γX which is not pseudoperfect.

5.16 EXAMPLE The one-point compactification $\omega\mathbb{R}$ of $\mathbb{R}$ is not pseudoperfect. To see this consider a strictly increasing function $f \in C^*(\mathbb{R})$. Since f is one-to-one on $\mathbb{R}$, $cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2$, ($Z(f - r)$ being a singleton for all $r \in f[\mathbb{R}]$). However f clearly does not belong to $C_\omega(\mathbb{R})$. Hence, by 1)<=>2) of 5.15, $\omega\mathbb{R}$ is not pseudoperfect.

CHAPTER 6

CONCLUSIONS

In this thesis we have developed a better understanding of the family of singular compactifications, i.e. those compactifications of X whose outgrowth $\alpha X \setminus X$ is a retract of αX. Originally, a singular compactification was defined as the union of the space X and a set S(f) (where S(f) denotes the singular set of a continuous function $f : X \longrightarrow K$ from X into a compact Hausdorff space K) equipped with an appropriate topology (see 1.2). In 2.6, we have shown that any singular compactification αX of X can be expressed in the form $\omega_{S_\alpha} X$, the smallest compactification to which all functions in the set S_α of all singular real-valued functions in $C_\alpha(X)$ can be extended. Equivalently $\omega_{S_\alpha} X$ is shown to be the supremum of the collection of singular compactifications $\{X \cup_f S(f) : f \in S_\alpha\}$ (2.8).

In example 2.7, we have shown that there exist compactifications of X which are not the suprema of singular compactifications.

It was previously known that the supremum of a family of singular compactifications need not be a singular compactification. In 2.10, 2.11 and 2.12 we characterize those suprema of singular compactifications which are singular.

In 2.13 and 2.14 we give conditions on a compactification αX which guarantee that αX is a supremum of singular compactifications.

It may happen that two singular compactifications $X \cup_f S(f)$ and $X \cup_g S(g)$ have the same underlying set (i.e. S(f) = S(g)). In 2.16, we precisely show what conditions f and g must satisfy for these two compactifications to be equivalent.

In chapter 3, we have named the supremum of all singular compactifications the μ-compactification of X and denote it by μX. We show that μX is the smallest compactification $(\omega_{S_\beta} X)$ to which the set S_β of all real-valued singular functions extend. We have shown that μX may or may not be the same as βX. In 3.5, 3.6, 3.8 and 3.7 we have characterized those spaces X such that $\mu X \cong \beta X$ and have provided examples of spaces X which do not possess this property (see example 3.4).

Spaces X for which the supremum of all singular compactifications of X is not singular have been shown to exist. If μX is a singular compactification, we say that μX is the *largest singular compactification*. If μX is not a singular compactification then X has no largest singular compactification. The main result of chapter 3 is a characterization of those spaces X which have a largest singular compactification (3.9 to 3.26). Having done this we now have a characterization of retractive spaces.

We also show (in 3.27) that the conjecture (proposed in [CFV]) "The singular compactifications of a space X forms a lattice iff βX is a singular compactification" fails by providing a counterexample.

Finally, in [CFV], the authors wonder whether the following statement is true: "If the family of singular compactifications of a space X forms a lattice then it is a complete lattice". We prove the weaker statement: A subfamily $\mathcal{F}$ of the family of all singular compactifications of a space X may form a lattice which is not complete (see 3.28).

In chapter 4, we have constructed various examples of spaces X for which μX is a singular compactification. We also show that the remainders of the singular compactifications of $\mathbb{N}$ are the compact separable spaces.

In chapter 5 we have investigated another compactification of a space X called the *perfect* compactification. This compactification was introduced in [Sk] and further studied in [D]. In [D] the author gave an algebraic characterization of the subring $C_\alpha(X)$ associated to a perfect compactification αX of X.We have provided an alternate proof to this characterization. We have also introduced a new compactification called the *pseudoperfect* compactification. Two characterizations of pseudoperfect compactifications were given. We showed that all perfect compactifications and compactifications of pseudocompact spaces are pseudoperfect. An example of a pseudoperfect compactification which is not a perfect compactification was provided.

We propose the following questions for further study:

1) Are there spaces which have a maximal singular compactification which is not μX?

2) If the family of all singular compactifications of a space X forms a lattice, is it necessarily a complete lattice?

3) Find an internal topological characterization of spaces X for which μX is a singular compactification.

4) Find and internal topological characterization of spaces X for which $\mu X \cong \beta X$.

5) Given the singular compactifications $X \cup_f S(f)$ and $X \cup_g S(g)$ find necessary and sufficient conditions in terms of f and g so that their supremum is singular.

LIST OF SYMBOLS

SUMMARY

[NOTE: Results marked with an asterisk are new.]

CHAPTER 1

1.1 Definitions *A singular compactification induced by the function f* is constructed as follows: Let $f : X \longrightarrow K$ be a continuous function from the space X into a compact set K. Let the *singular set,* S(f), of f be defined as the set $\{x \in cl_X f[X]$: for any neighbourhood U of x, $cl_X f^{\leftarrow}[U]$ is not compact$\}$. If S(f) = K then f is said to be a *singular map.* It is easy to verify that S(f) is closed in K and that if f is a singular map then f[X] is dense in S(f). If f is a singular map *the singular compactification of X induced by f,* denoted by $X \cup_f S(f)$, is the set $X \cup S(f)$ where the basic neighbourhoods of the points in X are the same as in the original space X, and the points of S(f) have neighbourhoods of form $U \cup (f^{\leftarrow}[U] \backslash F)$ where U is open in S(f) and F is a compact subset of X. This defines a compact Hausdorff topology on $X \cup_f S(f)$ in which X is a dense subspace. We will say that a compactification αX of X is a *singular compactification* if αX is equivalent to $X \cup_f S(f)$ for some singular map f.

Recall that a map $r : X \longrightarrow A$ sending X into a subset A of X is called a *retraction* if it is continuous and it fixes the points of A. The subset A is then called a *retract* of X.

1.2 THEOREM [G] The singular compactifications of X are precisely those compactifications αX of X whose remainder $\alpha X \backslash X$ is a retract of αX.

1.3 REMARK If $r : \alpha X \longrightarrow \alpha X \backslash X$ is a retraction from αX onto $\alpha X \backslash X$ then $r|_X$ is a singular map which induces the singular compactification $X \cup_{r|X} S(r|_X)$ (since if U is open in $\alpha X \backslash X$ and $cl_X r|_X^{\leftarrow}[U]$ is compact then $(\alpha X \backslash cl_X r|_X^{\leftarrow}[U]) \cap r^{\leftarrow}[U]$ is a non-empty open subset of αX contained in $\alpha X \backslash X$). Hence $X \cup_{r|X} S(r|_X) = \alpha X$. Conversely if f is a singular map it is easily verified that its extension $f^* : X \cup_f S(f)$

$\longrightarrow S(f)$ which acts as the identity function on $S(f)$ is continuous and is a retraction map from $X \cup_f S(f)$ onto $S(f)$.

1.4 THEOREM (Theorem 7, [G]) If αX is a singular compactification and γX is any compactification of X less than αX then γX is also a singular compactification.

1.5 NOTATION For any compactification γX of X, $C_\gamma(X)$ will denote the set $\{f|_X : f \in C(\gamma X)\}$. If f is a bounded real-valued singular function, f will be regarded as a function from X into $cl_{\mathbb{R}} f[X]$, i.e. we are letting K (in our definition of singular map) be $cl_{\mathbb{R}} f[X]$.The set S_γ will denote the set of all singular maps in $C_\gamma(X)$. Thus S_β denotes the collection of all singular maps in $C^*(X)$. In order to be more specific we may sometimes use the notation $S_\gamma(X)$ instead of S_γ indicating precisely the space X under consideration. If $\mathcal{G} \subseteq C_\gamma(X)$, $\mathcal{G}^\gamma$ will denote the set of extensions f^γ to γX of the functions f in $\mathcal{G}$.

1.6 LEMMA Let f be a continuous function from a space X to a compact Hausdorff space Z. Let $Y = cl_Z f[X]$ and $K_X = \{F \subseteq X : F \text{ is compact}\}$. Then $S(f) = \bigcap\{ cl_Y f[X \setminus F] : F \in K_X \}$.

1.7 PROPOSITION If αX is a compactification of X, K is a compact Hausdorff space and $f : X \longrightarrow K$ is a continuous function which extends to $f^\alpha : \alpha X \longrightarrow K$ then $f^\alpha[\alpha X \setminus X] = S(f)$.

1.8 COROLLARY* Let $f : X \longrightarrow K$ be a continuous map into a compact Hausdorff space such that f[X] is dense in K. Let $E_f(X)$ denote the set of all compactifications αX of X such that $f : X \longrightarrow K$ extends to $f^\alpha : \alpha X \longrightarrow K$. Then f is a singular map if and only if $f^\alpha[\alpha X \setminus X]$ contains f[X] for some (equivalently for all) $\alpha X \in E_f(X)$.

1.9 THEOREM Let A be a subalgebra of C(X) that contains the constant functions, and separates the points and closed sets of X. Then,

1) there is a compactification $\gamma_A X$ of X with these properties:

1a) For every f in A there exists an f^γ in $C(\gamma_A X)$ such that $f^\gamma|_X = f$.

1b) Let $A^\gamma = \{f^\gamma : f \in A\}$. Then A^γ separates the points of $\gamma_A X$.

2) if αX is a compactification of X with the properties:

2a) For every f in A there exists an f^α in $C(\alpha X)$ such that $f^\alpha|_X = f$

2b) The family $A^\alpha = \{ f^\alpha : f \in A\}$ separates points of $\gamma_A X$,

then αX and $\gamma_A X$ are equivalent compactifications of X. In other words, $\gamma_A X$ is uniquely determined (up to equivalence) by properties 1a) and 1b).

Furthermore it is well known that if $C_\alpha(X) \subseteq C_\gamma(X)$ then $\alpha X \leqq \gamma X$.

1.10 PROPOSITION [L] Let $\mathscr{G} \subseteq C^*(X)$. Then there exists a smallest compactification to which all functions in $\mathscr{G}$ extend.

1.11 NOTATION If $\mathscr{G}$ is contained in $C^*(X)$, the symbol $\omega_{\mathscr{G}}X$ will denote the smallest compactification to which all functions in $\mathscr{G}$ extend. If f belongs to $C^*(X)$, $\omega_f X$ will denote the smallest compactification of X to which f extends.

1.12 PROPOSITION Let αX be a compactification of X and $\mathscr{G} \subseteq C_\alpha(X)$. Then $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$ iff $e_{\mathscr{G}}{}^\alpha (= e_{\mathscr{G}^\alpha})$ is one-to-one on $\alpha X \backslash X$.

1.13 PROPOSITION [F] Let $\mathscr{G} \subseteq C^*(X)$ and αX be a compactification of X. Then $\alpha X \cong \omega_{\mathscr{G}} X$ if and only if each function g in $\mathscr{G}$ extends to g^α in $C(\alpha X)$ and $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$.

1.14 DEFINITION If f and g belong to $C^*(X)$, we will say that *f is equivalent to g*, denoted by $f \cong g$, if $f - g \in C_\omega(X)$. If $\mathscr{G}$ and $\mathscr{F}$ are subsets of $C^*(X)$, $\mathscr{G}$ is said to be *equivalent* to $\mathscr{F}$, denoted by $\mathscr{G} \cong \mathscr{F}$, if every function g in $\mathscr{G}$ is equivalent to some function f in $\mathscr{F}$ and conversely.

1.15 PROPOSITION [F] If $\mathscr{G} \subseteq C^*(X)$ then $C_{\omega_{\mathscr{G}}}(X) = cl_{C_\omega(X)}\langle C_\omega(X) \cup \mathscr{G}\rangle$, (the closure in the uniform norm topology of the subalgebra generated by $C_\omega(X) \cup \mathscr{G}$) where $C_{\omega_{\mathscr{G}}}(X) = \{f|_X : f \in C(\omega_{\mathscr{G}} X)\}$ (as in 1.5).

1.16 THEOREM [CF] If αX is a compactification of X and $\mathscr{G} \subseteq S_\alpha$ then $\alpha X = \sup\{X \cup_f S(f) : f \in \mathscr{G}\}$ if and only if $\mathscr{G}^\alpha$ separates the points of $\alpha X \backslash X$.

1.17 NOTATION We will denote $X \cup S(f)$ equipped with the topology described above by $X \cup^* S(f)$.

1.18 PROPOSITION (Lemma 1, [G]) If $f : X \longrightarrow Y$ is a singular function mapping X into a closed subspace K of the compact Hausdorff space Y and $g : Y \longrightarrow Z$ is continuous so that $cl_Z(g \circ f[X]) = Z$, then $g \circ f$ is a singular function.

1.19 PROPOSITION (Corollary 3, [F]) If $\mathcal{F}$ and $\mathcal{G}$ are two equivalent subsets of $C^*(X)$ then $\omega_{\mathcal{G}}X$ is equivalent to $\omega_{\mathcal{F}}X$.

1.20 PROPOSITION (Lemma 2, [F]) Let $\{\alpha_i X : i \in A\}$ be a family of compactifications of X and let $\alpha X = \sup\{\alpha_i X : i \in A\}$, then $C_\alpha(X) = cl_{C_\alpha(X)}\langle \cup \{C_{\alpha_i}(X) : i \in A\}\rangle$.

1.21 THEOREM [Theorem 2, [F]] If $\mathcal{G} \subseteq C^*(X)$ separates the points from the closed sets in X, then $\sup\{\omega_f X : f \in \mathcal{G}\} = \omega_{\mathcal{G}}X$.

1.22 THEOREM [SS] Let X be locally compact and non-compact and let K be a compact Hausdorff space. If there is a continuous map $f : X \longrightarrow K$ from X into K such that $f[N(\infty) \cap X]$ is dense in K for all neighbourhoods $N(\infty)$ of ∞ in ωX then X has a compactification X^* with K as a remainder. Indeed, such an X^* is the closure of the graph of f in $\omega X \times K$.

1.23 THEOREM Let X be locally compact and non-compact and let K be a compact Hausdorff space. If $f : X \longrightarrow K$ is a singular function which maps X densely into K then f maps $N(\infty) \cap X$ densely into K for any $N(\infty)$. Furthermore $X \cup_f S(f)$ is equivalent to $cl_{\omega X \times K}G_f$ (in the sense that if m and h each embed X into $X \cup_f S(f)$ and $cl_{\omega X \times K}G_f$ respectively then there exists a homeomorphism j from $X \cup_f S(f)$ onto $cl_{\omega X \times K}G_f$ such that $j \circ m(x) = h(x)$). Hence a singular compactification induced by a singular function f is equivalent to the closure of the graph of f in $\omega X \times S(f)$. Conversely, if $f : X \longrightarrow K$ is a function from X into K which maps $N(\infty) \cap X$ densely into K for any $N(\infty)$ then f is a singular map and $cl_{\omega X \times K}G_f$ is equivalent to $X \cup_f S(f)$ (as a compactification of $X \cong G_f$). Hence the closure of the graph of a function f (in $\omega X \times K$) satisfying the above property always yields a singular compactification of X.

CHAPTER 2

2.1 LEMMA* Let $f : X \longrightarrow Y$ be a continuous function from the space X into a compact Hausdorff space Y. If αX is a compactification of X and f extends to $f^{\alpha} : \alpha X \longrightarrow Y$ so that f^{α} separates the points of $\alpha X \setminus X$, then αX is equivalent (as a compactification of X) to $X \cup^{*} S(f)$.

2.2 COROLLARY* If αX is a compactification of X then αX can be expressed in the form of $X \cup^{*} S(f)$, i.e. αX is equivalent to $X \cup^{*} S(e_{C_{\alpha}(X)})$.

2.3 COROLLARY* Let X be a locally compact Hausdorff space and Y be a compact Hausdorff space. Then there exists a topology on the disjoint union $X \cup Y$ of X and Y such that the resulting topological space is a singular compactification of X iff Y is homeomorphic to the singular set of some evaluation map $e_{\mathscr{G}}$ induced by a subset $\mathscr{G}$ of $C^{*}(X)$.

2.4 THEOREM* a) Let $f \in C^{*}(X)$. Then $\omega_f X$ is equivalent to $X \cup^{*} S(f)$. In particular, if f is a singular map then $\omega_f X$ is a singular compactification and $\omega_f X$ is equivalent to $X \cup_f S(f)$.

b) If $\mathscr{G} \subseteq C^{*}(X)$ and $\omega_{\mathscr{G}} X$ is a singular compactification then $t = e_{\mathscr{G}}^{\omega_{\mathscr{G}}} \circ r|_X$ is a singular map (where $r : \omega_{\mathscr{G}} X \longrightarrow \omega_{\mathscr{G}} X \setminus X$ is a retraction map) and $\omega_{\mathscr{G}} X$ is equivalent to $X \cup_t S(t)$.

2.5 LEMMA* If αX is a singular compactification, then every $f \in C_{\alpha}(X)$ is equivalent to some function $h \in S_{\alpha}$ (see 1.14).

2.6 THEOREM* If αX is a singular compactification then αX is equivalent to $\omega_{S_{\alpha}} X$. Hence every singular compactification αX of X is the supremum of the family $\{X \cup_f S(f) : f \in S_{\alpha}\}$ of singular compactifications.

2.7 EXAMPLE* Consider the two-point compactification of $\mathbb{R}$, $\alpha\mathbb{R} = \mathbb{R} \cup \{p_1, p_2\}$. We claim that $\alpha\mathbb{R}$ cannot be the supremum of singular compactifications i.e. $\alpha\mathbb{R}$ cannot be expressed in the form $\omega_{S_{\alpha}}\mathbb{R}$.

2.8 PROPOSITION* Let X be a topological space. The compactification αX of X is a supremum of a collection of singular compactifications iff αX is equivalent to $\omega_{\mathcal{G}}X$ for some $\mathcal{G}$ contained in S_α.

2.9 REMARK* For further reference, we would like to emphasize an important point illustrated in the above example. In this example, {f, g} is contained in S_β, hence, by 1.16, $\omega_{\{f,g\}}X$ is equivalent to $X \cup_f S(f) \bigvee X \cup_g S(g)$. We have shown that the evaluation map $e_{\{f,g\}} = f \times g$ is not a singular map even though $\omega_{\{f,g\}}X$ was proven to be a singular compactification. Hence, if $\mathcal{G}$ is an arbitrary subset of S_β, it is not sufficient that $\omega_{\mathcal{G}}X$ be a singular compactification for $e_{\mathcal{G}}$ to be a singular map, i.e. "**$\omega_{\mathcal{G}}X$ being singular does not imply that $e_{\mathcal{G}}$ is singular**".

2.10 THEOREM* Let $\mathcal{G} \subseteq S_\beta$. Then the following are equivalent:

1) $\omega_{\mathcal{G}}X$ is a singular compactification

2) There is a singular function $k : X \longrightarrow K$ mapping X densely into some compact Hausdorff space K which extends to $k^{\omega_{\mathcal{G}}} : \omega_{\mathcal{G}}X \longrightarrow K$ such that $k^{\omega_{\mathcal{G}}}$ is one-to-one on $\omega_{\mathcal{G}}X \setminus X$ (hence $\omega_{\mathcal{G}}X$ is equivalent to $X \cup_k S(k)$).

2.11 THEOREM* Let αX be a singular compactification of X. Let $r : \alpha X \longrightarrow \alpha X \setminus X$ be a retraction map, and define $\mathcal{F}$ to be $\{f \circ r|_X : f \in C(\alpha X)\}$. Then $\mathcal{F} \subseteq S_\alpha$, $\mathcal{F}$ is a subalgebra of $C_\alpha(X)$, $e_{\mathcal{F}}$ is a singular map, $e_{\mathcal{F}}^{\alpha}$ separates points of $\alpha X \setminus X$, and $\alpha X \cong X \cup_{e_{\mathcal{F}}} S(e_{\mathcal{F}}) \cong \omega_{\mathcal{F}}X$.

2.12 THEOREM* Let αX be a compactification of X. Let $\mathcal{G}$ be a subset of S_α such that the evaluation map $e_{\mathcal{G}}^{\alpha} : \alpha X \longrightarrow \Pi_{f \in \mathcal{G}} S(f)$ separates the points of $\alpha X \setminus X$. Then αX is equivalent to $\omega_{\mathcal{G}}X$. Furthermore the following are equivalent:

1) $e_{\mathcal{G}}$ is a singular map and $\omega_{\mathcal{G}}X (\cong \alpha X)$ is equivalent to the singular compactification $X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$

2) $e_{\mathcal{G}}[X] \subseteq e_{\mathcal{G}}^{\omega_{\mathcal{G}}}[\omega_{\mathcal{G}}X \setminus X]$.

3) $e_{\mathcal{G}}$ is a singular map.

4) $e_{\mathcal{F}}$ is a singular map for every finite subset $\mathcal{F}$ of $\mathcal{G}$.

5) ${}_x K_{\mathscr{G}}^{\omega_{\mathscr{G}}} \cap (\omega_{\mathscr{G}} X \setminus X)$ is a singleton set for every $x \in X$.

2.13 PROPOSITION* If $\alpha X \setminus X$ is not totally disconnected then αX is equivalent to $\omega_{S_\alpha} X$.

2.14 PROPOSITION Let X be a strongly zero-dimensional not almost compact space. Then βX is the supremum of the family of the two-point singular compactifications of X. Hence $\beta X = \omega_{S_\beta} X$.

2.15 PROPOSITION* Let K[X] denote the family of all compactifications of X. Let $\mathscr{K} = \{\alpha X \in K[X] : \alpha X \setminus X$ is homeomorphic to a closed interval of $\mathbb{R}\}$. Then $\mathscr{K} \subseteq \{\omega_f X : f \in S_\beta\}$, and, if X is connected, then $\mathscr{K} = \{\omega_f X : f \in S_\beta\}$. (We will consider the singleton set $\{a\}$ in $\mathbb{R}$ as the closed interval [a,a] with empty interior).

2.16 PROPOSITION* Let $f : X \longrightarrow K_f$ and $g : X \longrightarrow K_g$ be two singular maps from the space X into the compact spaces K_f and K_g respectively such that $S(f) \cong S(g)$. Then the following are equivalent:

1) $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

2) The function $f : X \longrightarrow S(f)$ extends continuously to a function $f^* : X \cup_g S(g) \longrightarrow S(f)$ ($\cong S(g)$) in such a way that f^* separates the points of S(g).

2.17 EXAMPLE* The singular compactifications $\mathbb{R} \cup_{sine} S(\text{sine})$ and $\mathbb{R} \cup_{cosine} S(\text{cosine})$ **are not** equivalent.

2.18 DEFINITION* We will say that *two functions $f : X \longrightarrow K$ and $g : X \longrightarrow K$ from a space X to a space K are homeomorphically related* if there exists a homeomorphism $h : cl_K f[X] \longrightarrow cl_K g[X]$ such that $h(f(x)) = g(x)$ for all x in X.

2.19 COROLLARY* Let $f : X \longrightarrow K$ and $g : X \longrightarrow K$ be two singular maps on X such that $S(f) = S(g) = K$. If f and g are homeomorphically related then $X \cup_f S(f)$ is equivalent to $X \cup_g S(g)$.

2.20 EXAMPLE* The singular compactifications $\mathbb{R} \cup_{sine2} S(\sin^2)$ and $\mathbb{R} \cup_{cos2} S(\cos^2)$ are equivalent.

CHAPTER 3

3.1 DEFINITIONS We will say that αX is *the largest singular compactification of X* if αX is a singular compactification and, whenever γX is a singular compactification of X, then $\gamma X \leqq \alpha X$, (i.e. X has a largest singular compactification if the supremum in $(K(X), \leq)$ of the set of all singular compactifications of X is a singular compactification). We say that the compactification γX is a *maximal* singular compactification if γX is singular and there does not exist a singular compactification ζX such that $\zeta X > \gamma X$.

3.2 PROPOSITION* The compactification αX of X is the largest singular compactification of X if and only if $\alpha X \cong \omega_{S_\beta} X$ and $\omega_{S_\beta} X$ is singular.

3.3 DEFINITION* The compactification $\omega_{S_\beta} X$ will be denoted by μX (whether it is singular or not). When we will speak of the *μ-compactification of X* we will mean μX.

3.4 EXAMPLE* A space X such that μX is strictly less than βX.

3.5 THEOREM* Let X be a topological space. Then $\mu X \cong \beta X$ if and only if $S_\beta{}^\beta$ separates the points of $D \cap (\beta X \backslash X)$ for each connected component D of βX.

3.6 THEOREM* If X is a connected non-compact space which is not almost compact then the following are equivalent:

1) $\mu X \cong \beta X$.

2) There is a continuous function from $\beta X \backslash X$ onto a closed interval with non-empty interior.

3) The space X has a compactification αX whose outgrowth $\alpha X \backslash X$ is homeomorphic to a closed interval of real numbers (with non-empty interior).

4) The space X has a singular compactification which is not the one-point compactification ωX of X.

5) S_β contains a non-constant function.

3.7 THEOREM* Let X be a locally compact space. Then the following are equivalent:

1) $\mu X \cong \beta X$.

2) At least one of the two following conditions is satisfied:

a) Any two points of $\beta X \backslash X$ are contained in distinct connected components of βX.

b) There is a continuous function from $\beta X \backslash X$ onto a closed interval with non-empty interior.

3.8 THEOREM* Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geqslant 0\}$ and ∞ be a point in $\beta\mathbb{R}^+ \backslash \mathbb{R}^+$. Let S be an infinite compact scattered space and $Y = \beta\mathbb{R}^+ \backslash \{\infty\}$. Let $X = S \times Y$ and X^* be the quotient space of X obtained by collapsing to a single point the doubleton $\{(0,0),(\omega_2,0)\}$ and fixing all other points of X. Then $\mu(X^*) \ncong \beta(X^*)$.

3.9 LEMMA* If D is a closed C-embedded copy of $\mathbb{N}$ in a locally compact space X then $(\mathrm{cl}_{\beta X}D)\backslash D$ is a P-set of $\beta X \backslash X$.

3.10 LEMMA* If X contains a C-embedded copy of $\mathbb{N}$ (i.e. if X is not pseudo–compact) then $\mu X \cong \beta X$.

3.11 THEOREM* If X has a largest singular compactification μX then X does not contain a C-embedded copy of $\mathbb{N}$ (i.e. X is pseudocompact) .

3.12 DEFINITION The subset $C^\#(X)$ of $C(X)$ is the set of all real-valued functions f such that for every maximal ideal M in C(X) there exists a real number r such that $f - \mathbf{r} \in M$.

3.13 THEOREM (1.10 of [A]) The space X is pseudocompact if and only if $C(X) = C^\#(X)$.

3.14 THEOREM (1.7 of [A]) The following are equivalent for f in $C^*(X)$

1) f belongs to $C^\#(X)$.

2) For every open subset U of βX $f[U \cap X] = f^\beta[U]$.

3) $\mathrm{Cl}_{\beta X}Z(f - \mathbf{r}) = Z(f^\beta - \mathbf{r})$ for any $r \in \mathbb{R}$.

4) f maps zero sets to closed sets.

3.15 LEMMA* If X is a non-compact pseudocompact space and αX is a compactification of X then, for each $f \in S_\alpha$, Z(f) is not compact whenever $Z(f^\alpha)$ is non-empty. Furthermore $cl_{\alpha X}Z(f) = Z(f^\alpha)$ for all $f \in C_\alpha(X)$.

3.16 PROPOSITION* If X is pseudocompact and $\alpha X = X \cup_f S(f)$ is a singular compactification of X such that S(f) is a subset of $\mathbb{R}$ (up to homeomorphism) then $f^{\leftarrow}(x)$ is non-compact for any $x \in S(f)$ and $f[X] = S(f)$.

3.17 EXAMPLE*

3.18 LEMMA* If $\{f_n : n \in \mathbb{N}\}$ is a sequence of real-valued singular functions which converges uniformly to a function f in C*(X) then f is also a singular function.

3.19 THEOREM* A compactification αX of X is singular iff S_α contains a sub–algebra $\mathcal{G}$ of C*(X) such that $\mathcal{G}^\alpha$ separates the points of $\alpha X/X$. Furthermore if $\mathcal{G}$ is a subalgebra C*(X) which is contained in S_α such that $\mathcal{G}^\alpha$ separates the points of $\alpha X \backslash X$ then $e_{\mathcal{G}}$ is a singular map and $\alpha X \cong \omega_{\mathcal{G}} X \cong X \cup_{e_{\mathcal{G}}} S(e_{\mathcal{G}})$ (a singular compactification).

3.20 THEOREM* Let αX be a compactification of the space X. There is a one-to-one correspondence between the retraction maps from αX onto $\alpha X \backslash X$ and the subalgebras $\mathcal{G}$ of $C_\alpha(X)$ such that $\mathcal{G} \subseteq S_\alpha$ and $\mathcal{G}^\alpha|_{\alpha X \backslash X} = C(\alpha X \backslash X)$. If αX is not a singular compactification then no such retraction map r or such a subalgebra $\mathcal{G}$ exist.

3.21 THEOREM* The compactification αX of X is singular iff S_α contains a closed subalgebra $\mathcal{G}$ of $C_\alpha(X)$ such that the mapping $\phi : \mathcal{G} \longrightarrow C(\alpha X \backslash X)$ from $\mathcal{G}$ onto $C(\alpha X \backslash X)$ defined by $\phi(f) = f^\alpha|_{\alpha X \backslash X}$ is an isomorphism.

3.22 THEOREM* Let αX be a compactification of X. Then αX is a singular compactification of X iff $\dfrac{C_\alpha(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathcal{F}$ (of

$C_\alpha(X)) \subseteq S_\alpha$ under the homomorphism $\sigma : \mathcal{F} \longrightarrow \dfrac{C_\alpha(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

3.23 THEOREM* Let αX be a compactification of X. Then αX is a singular compactification iff $C_\alpha(X) = C_\infty(X) \oplus \mathcal{G}$ (the vector space direct sum) for some closed subalgebra $\mathcal{G}$ of $C^*(X)$ contained in S_α.

3.24 EXAMPLE* An example of an upward directed family $\mathcal{A}$ of singular compactifications whose supremum is not a singular compactification.

3.25 THEOREM* If X is a locally compact and Hausdorff space then the following are equivalent:

1) The space X has a largest singular compactification (i.e. μX is a singular compactification).

2) The set S_μ contains a subalgebra $\mathcal{G}$ of $C_\mu(X)$ such that $\mathcal{G}^\mu$ separates the points of $\mu X \backslash X$.

3) The set S_μ contains a closed subalgebra $\mathcal{G}$ of $C_\mu(X)$ such that the mapping $\phi : \mathcal{G} \longrightarrow C(\mu X \backslash X)$ from $\mathcal{G}$ onto $C(\mu X \backslash X)$ defined by $\phi(f) = f^\mu|_{\mu X \backslash X}$ is an isomorphism.

4) The quotient ring $\dfrac{C_\mu(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathcal{F}$ (of $C_\mu(X)) \subseteq S_\mu$ under the canonical homomorphism $\sigma : \mathcal{F} \longrightarrow \dfrac{C_\mu(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

5) The set $C_\mu(X) = C_\infty(X) \oplus \mathcal{G}$ (the vector space direct sum) for some closed subalgebra $\mathcal{G}$ of $C^*(X)$ contained in S_μ.

3.26 COROLLARY* For a locally compact Hausdorff space X the following are equivalent:

1) The space X is retractive (i.e. βX is a singular compactification).

2) The set S_β contains a subalgebra $\mathcal{G}$ of $C^*(X)$ such that $\mathcal{G}^\beta$ separates the points of $\beta X \backslash X$.

3) The set S_β contains a closed subalgebra $\mathcal{G}$ of $C^*(X)$ such that the mapping $\phi : \mathcal{G} \longrightarrow C(\beta X \backslash X)$ from $\mathcal{G}$ onto $C(\beta X \backslash X)$ defined by $\phi(f) = f^\beta|_{\beta X \backslash X}$ is an isomorphism.

4) The quotient ring $\dfrac{C^*(X)}{C_\infty(X)}$ is the isomorphic image of a closed subring $\mathcal{F}$ (of $C_\mu(X)) \subseteq S_\mu$ under the canonical homomorphism $\sigma : \mathcal{F} \longrightarrow \dfrac{C^*(X)}{C_\infty(X)}$ defined by $\sigma(f) = C_\infty(X) + f$.

5) The set $C_\beta(X) = C_\infty(X) \oplus \mathcal{G}$ (the vector space direct sum) for some closed subalgebra $\mathcal{G}$ of $C^*(X)$ contained in S_β.

3.27 EXAMPLE* A space X whose family of singular compactifications forms a (complete) lattice even though βX is not singular.

3.28 EXAMPLE* A lattice of singular compactifications of a space X is not necessarily a complete lattice.

CHAPTER 4

4.1 THEOREM (Glicksberg) If X and Y are infinite, then the product space X x Y is pseudocompact if and only if X x Y is C*-embedded in $\beta X \times \beta Y$, i.e. $\beta(X \times Y) = \beta X \times \beta Y$.

4.2 PROPOSITION (8.21 of [Wa]) The product of two pseudocompact spaces one of which is also locally compact is pseudocompact.

4.3 PROPOSITION The outgrowths of the singular compactifications of $\mathbb{N}$ are the compact separable spaces.

4.4 EXAMPLE* The product space $X = [0,\omega_1) \times I$ (where I is the closed unit interval) equipped with the product topology has a μ-compactification μX which is singular.

4.5 EXAMPLE* Let $X = \beta\mathbb{R}\setminus\{x, y\}$ where x and y are distinct points in $\beta\mathbb{R}\setminus\mathbb{R}$. Let Y = I, where I is the closed unit interval. Then $\mu(X \times Y)$ is not equivalent to $\mu X \times \mu Y$.

4.6 EXAMPLES* Of two spaces X such that $\mu X \cong \beta X$ but where μX is not singular.

4.7 EXAMPLE* Of a space X such that $\mu X \ncong \beta X$ and μX is not singular.

CHAPTER 5

5.1 DEFINITION Let γX be a compactification of X and $\pi_{\beta\gamma} : \beta X \longrightarrow \gamma X$ denote the natural map from βX onto γX. We say that γX is perfect if, for every $p \in \beta X \setminus X$, $\pi_{\beta\gamma}^{\leftarrow}(p)$ is connected.

5.2 PROPOSITION Let γX be a compactification of X. For open subset U of X, let $Ex_{\gamma X}U = \gamma X \setminus cl_{\gamma X}(X \setminus U)$, the extension of U in γX. Then the following are equivalent:

1) γX is a perfect compactification.

2) For any pair of disjoint subsets A and B of X, $cl_{\gamma X}A \cap cl_{\gamma X}B = \varnothing$ iff $cl_{\gamma X}Fr_X A \cap cl_{\gamma X}Fr_X B = \varnothing$.

3) If U and V are disjoint open subsets of X, then $Ex_{\gamma X}(U \cup V) = Ex_{\gamma X}U \cup Ex_{\gamma X}V$.

4) For any open subset U of X, $cl_{\gamma X}(Fr_X U) = Fr_{\gamma X}Ex_{\gamma X}U$.

5.3 DEFINITION A subring $\mathcal{F}$ of $C^*(X)$ is *algebraic* if $\mathcal{F}$ contains the constant functions and those functions $f \in C^*(X)$ such that $f^2 \in \mathcal{F}$.

5.4 PROPOSITION (16.30 and 16.31 of [GJ]) a) The set $\mathcal{F}$ of all functions in $C^*(X)$ which are constant on a given subset S of X is an algebraic subring of $C^*(X)$ which is closed in the uniform norm topology.

b) If X is compact and $\mathcal{F}$ is an algebraic subring of $C^*(X)$ then each maximal stationary set of $\mathcal{F}$ is connected.

5.5 LEMMA* Let γX be a compactification of X and $\mathcal{G} = C_\gamma(X)$. Then $\pi_{\beta\gamma} = e_{\mathcal{G}}{}^{\gamma\leftarrow} \circ e_{\mathcal{G}}{}^{\beta}$.

5.6 NOTE* Let $\mathcal{G} \subseteq C_\gamma(X)$. We claim that *the subsets of X of the form* $e_{\mathcal{G}}{}^{\gamma\leftarrow}(p)$ *(where* $p \in e_{\mathcal{G}}{}^{\gamma}[\gamma X]$*) are the maximal stationary sets of* $\mathcal{G}^\gamma$. Let S be a maximal stationary set of $\mathcal{G}^\gamma$.

5.7 THEOREM [D] Let γX be a compactification of X. Then γX is a perfect compactification iff $C_\gamma(X)$ is an algebraic subring of $C^*(X)$.

5.8 PROPOSITION* If $\mathcal{F}$ is an algebraic subring of $C^*(X)$ which separates the points and closed sets of X then $\mathcal{F}$ determines a perfect compactification of X.

5.9 THEOREM (Lemma 2.3 of [D]) If γX is a perfect compactification then $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

5.10 THEOREM (Corollary 3.6 of [D]) If X is pseudocompact then, for any compactification γX of X, $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

5.11 EXAMPLE* Of a non-perfect compactification γX such that $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

5.12 DEFINITION* Let γX be a compactification of X and U and V be disjoint subsets of X. The sets U and V are said to be *γ-separated* if there exists a subset M of γX containing X and a real-valued function f on M with compact fibres such that $f|_X \in C^*(X)$ and such that f[U] and f[V] have disjoint closures in of $\mathbb{R}$.

5.13 DEFINITION* A compactification γX is said to be a *pseudoperfect compactification* if any two subsets U and V of X which are γ-separated have disjoint closures in γX.

5.14 PROPOSITION (Taimanov's Theorem) Let αX and αY be a compactification of X and Y respectively, and let f be a map from X onto Y. There is a map f^* : $\alpha X \longrightarrow \alpha Y$ extending f if and only if, for A, B $\subseteq$ Y, $cl_{\alpha Y}A \cap cl_{\alpha Y}B = \varnothing$ implies that $cl_{\alpha X}f^{\leftarrow}[A] \cap cl_{\alpha X}f^{\leftarrow}[B] = \varnothing$.

5.15 THEOREM* Let γX be a compactification of X. Then the following are equivalent:

1) γX is a pseudoperfect compactification.

2) $C_\gamma(X) = \{f \in C^*(X) : cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2\}$.

3) If $f \in C^*(X)$ such that $cl_{\gamma X}Z(f - r_1) \cap cl_{\gamma X}Z(f - r_2) = \varnothing$ whenever $r_1 \neq r_2$ then $cl_{\gamma X}f^{\leftarrow}[A] \cap cl_{\gamma X}f^{\leftarrow}[B] = \varnothing$ for any subsets A and B of f[X] with disjoint closures in $\mathbb{R}$.

5.16 EXAMPLE* The one-point compactification $\omega\mathbb{R}$ of $\mathbb{R}$ is not pseudo-perfect.

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