THE VALIDATION OF A MODEL CONCERNING PRECURSORS OF ALGEBRA

by

Gerhard Alfred Ameis

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ABSTRACT

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The purpose of the study was to explore the large question of what skills, understandings, and intuitions are precursors to learning algebra. For that purpose, a comprehensive a priori model of precursors of algebra was created that consisted of 18 categories. The model was subjected to both formal and exploratory tests of hypotheses. Linear regression methods were used for those purposes.

The results of the formal tests suggested 11 central notions that may be precursors of algebra. They may be placed into four clusters. One cluster of notions concerns the ability to reason deductively and inductively, and to draw analogies. A second cluster concerns the arithmetic operators and functional principles of arithmetic. A third cluster concerns the meanings and roles that can be attached to symbols. A fourth cluster concerns the hierarchy for computation and the structure of arithmetic expressions. Those results have strong implications for elementary and middle years mathematics curricula and instructional practices.

The results of the exploratory tests suggested that there are relationships between gender, achievement in algebra, and styles of algebra teachers. Paying attention to those relationships may enhance the learning of algebra.

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CHAPTER 1

INTRODUCTION

The Purpose of the Study

This study explores the large question of what set of skills, understandings, and intuitions are precursors to learning algebra. The results of the study suggest modifications to elementary school curricula, provide a rationale for the redesign of the middle years curriculum, and provide a basis for diagnosis and remediation in algebra courses.

Rationale for the Study

It was commonplace in the 1800's that one child in ten attended high school and algebra was not a required course until the first year of high school (Kieran and Wagner, 1989). The mathematical needs of that era were largely satisfied by graduates with the ability to compute. But in an information and microprocessor age, many desired mathematical skills depend directly or indirectly on algebra (NCTM, 1989). As a result, the study of algebra can now begin as early as in grade six. Algebra forms a major part of the secondary university entrance mathematics curriculum and university entrance mathematics (appropriately or not) serves as a screen for determining entry into many professions. Consequently, success or failure in algebra is often an important factor in determining students' career options.

Assessment studies in the United States (Carpenter, et al, 1981; Fey, 1989) and in Manitoba (Curriculum Branch, 1987) indicate that algebra has become a stumbling block for many students. The comment of an articulate and accelerated seventh-grade student cited by House (1988) is typical.

> Algebra . . . is quite hard, and although very educational, it is very frustrating ninety percent of the time. It means hours of instruction that you don't even come close to understanding. (p. 1)

It seems that there are at least three possible explanations for students' difficulties in learning algebra; the subject may be inherently difficult; teaching may be at fault; or we may not have identified the precursors of algebra which may be viewed as a set of understandings, skills and intuitions that students should acquire prior to learning algebra.

There could be some validity to either or both of the first two explanations. Some students may not be able to manage the level of mathematical abstractions required by algebra and the subject may therefore be inherently difficult. Some teachers may not understand the progression of abstractions involved in algebra sufficiently well to teach them. But both of these inferences invite the curriculum builder to abandon students to the vagaries of time and place. They run counter to a central tenet of twentieth-century pedagogy; if there is a lack of understanding, there is a way to remedy it. From this point of view those inferences should be last resorts, even if the search for some other explanation may be supported by only an act of faith. The accomplishments of curriculum builders since the early 1800's suggest that this is a prudent stance; ways have often been found. Considering all of this, a reasonable conclusion is that identifying the precursors of algebra may be the most promising line of inquiry concerning students' difficulties in algebra.

Precursors of Algebra

From the point of view of the mathematical skills and concepts that students should already have acquired, ritual algorithms and a smattering of problem solving skills have often been thought to be a sufficient basis for learning algebra. They may not be enough. If ritual arithmetic and problem solving skills are, in fact, not enough and it is possible to identify a better set, algebra may turn out not to be inherently difficult for the bulk of students and the teacher who is aware of that set may be more successful in teaching it.

In view of the central role of algebra in the secondary curriculum and its importance to the career aspirations of students, it would be reasonable to expect that a well-formulated model of precursors of algebra would have been developed. However, this is not the case. The literature does not include any coherent and comprehensive model of precursors.

At most, one finds reference to two clusters of notions that may be thought of as precursors. The first cluster concerns prerequisites. One must generally infer peoples' (implicit) precursors from their list of prerequisites. Those lists tend to include skill in fraction and integer arithmetic, the recall of basic facts, and strategies for problem solving restricted to a search for an appropriate algorithm (Smorodin, 1985; Boston Public Schools curriculum, 1983). The second cluster concerns factors and processes that may underlie the learning of complex mathematical tasks (Kieran and Wagner, 1989). The literature discusses such factors and processes as the structure of algebra (Kieran, 1989), cognitive obstacles in learning algebra (Herscovics, 1989), and the role of problem representation in algebra (Larkin, 1989). Those notions tend to be conceived as distinct and are not organized into a comprehensive model of precursors.

While psychology and learning theory cannot be expected to provide specific rosters of precursors, learning theories reinforce the need to determine what they are and provide some guidance. Gestalt theorists might suggest that there are insufficient organizing principles embedded in most contemporary algebra teaching and that such principles are essential. Content structuralists might suggest that the content has not been structured and has therefore not been presented appropriately. Developmental theorists might suggest that children may be at the concrete operational stage, and as such, are not cognitively ready for learning that presupposes the formal operational stage of development, or that the disequilibrium between the children's schemes concerning arithmetic and the structure of the algebraic knowledge being taught may be too great, making the development of new and nonconflicting schemes unlikely. Cumulative learning theorists might suggest that the precursors of algebra have not been learned well, in an inappropriate order, or not at all. Cognitive and information-processing theorists might suggest that the structure of the algebra being taught is not suitably attached to children's internal knowledge representation of arithmetic.

Each of these views may be thought of as a different lens tending to the same operational conclusion. Seventh-grade students and others are commonly not "ready" for algebra because they lack conceptually laden experiences, intuitions, and insights that are important for learning algebra. In other words, they have not mastered a sufficient number of precursors of algebra.

The position taken in this study is that it may be possible to determine the set of important precursors from the entire range of concepts, principles, and relationships that inhabit arithmetic and the transition between arithmetic and algebra and that ritual arithmetic and algorithm-oriented problem solving skills per se are not likely to be important precursors. Support for this position is found in the literature. The comment of Booth (1989) serves as a summary.

> Students' difficulties in algebra, it has been generally assumed, are largely difficulties in learning the syntax. Over the past decade, however, research evidence has been accumulating to indicate

> that many students have a poor understanding of the relations and mathematical structures that are the basis of algebraic representation. . . . . Indeed, a major part of students' difficulties stems precisely from their lack of understanding of arithmetical relationships. (p. 58)

Performance in ritual skills per se may have some capacity to predict performance in algebra. However, it seems most likely that any such capacity can be attributed to them being indirect measures of students' understandings of the notions that underlie or accompany these skills than to performance in ritual skills per se. For that reason measures of ritual skills are excluded from this study.

Co-requisites of learning algebra such as teacher clarity and student motivation are also excluded as the focus of this study is identifying potential precursors of algebra. However, broadly defined teacher style and gender were included in an ancillary way to broaden the scope of the study.

The Creation of a Model

The purposes of this study, then, are:

(1) To create a generic model which might be able to account for the range of possible precursors of algebra and algebraic problem solving.

(2) To subject both formal and exploratory hypotheses concerning the relationship between the components of that model and later algebraic performance to statistical test.

The generic or a priori model compresses 63 individual elements collected into 18 tentative categories. In this preliminary discussion, a 'precursor' may be taken to be either an individual element or one of the categories.

Statement of the Problem

The specific questions are:

1. What categories of the model are precursors of algebra and algebraic problem solving?
2. What elements of the model are precursors of algebra?
3. Are the precursors independent of gender?
4. Are the precursors independent of teacher style of teaching algebra?
5. What relationships are there between reasoning as defined by the model; and achievement, gender, and teacher style?

Questions 1 and 2 are addressed by using conventional conservative statistical tests. Questions 3, 4, and 5 are addressed using less conservative exploratory techniques. In both cases, linear regression analysis methods are used.

Overview of the Model concerning Precursors of Algebra

Since the precise nature of the precursors is not well-defined, two approaches were taken to testing the generic model - by identifying broadly defined notions and by identifying more detailed notions. To that purpose, the model includes both categories (broad notions) and elements (fine notions). The categories, as clusters of elements, provide cohesiveness to those elements while the elements themselves help define the categories.

The model is organized into 18 categories. They are; (1) mathematical representation using symbols, (2) the passive interpretation of symbols, (3) the structural role of symbols, (4) the replacement role of symbols, (5) the multiple meanings of symbols, (6) binary and unary arithmetical operators, (7) do and undo pairings of arithmetical operators (inverse operations), (8) the locking role of arithmetical operators, (9) the alteration of the structure of arithmetical expressions, (10) the context independence of arithmetical operators, (11) unit attachment to arithmetical operations, (12) the visual order of arithmetical computation, (13) functional principles of arithmetic, (14) inductive reasoning, (15) deductive reasoning, (16) isomorphic reasoning, (17) the relationship between language and arithmetic, and (18) template recognition in arithmetical problem solving.

Each category consists of 3 or more elements that help define it. In all, there are 63 elements or detailed precursors. No attempt has been made to relate these categories and elements in a hierarchical way as such an endeavor would be premature.

Three sources were used to develop the model; the researcher's experiences while teaching mathematics; the literature; and logical analysis. These increase the likelihood that the model includes the most useful categories.

However, there are at least two areas of concern with respect to the model. First, there are places where the domains of categories seem to overlap. This seems to be unavoidable when a model is created in this way.

Second, on account of pragmatic considerations concerning the students' concentration spans for completing instruments and the time available for testing, some categories are not as well defined as might be desired. However, that is not a serious impediment. The model is sufficiently developed for purposes of identifying what are likely the most important broadly defined precursors.

The model may be useful for algebra instruction but there are potential limits to its usefulness. Algebra is a diverse field in mathematics, including such subdivisions as matrix and Boolean algebras. The model only concerns the algebra of real numbers, the algebra that constitutes the major portion of the universityoriented mathematics curriculum taught in middle years and high school. It is uncertain to what extent the model may be applicable to learning other algebras.

Overview of the Methodology

An overview of critical aspects of the methodology is helpful at this point. In the study, four instruments were constructed to collect data used for testing the two types of hypotheses, a priori and post hoc.

A model knowledge instrument was constructed that reflects the a priori model of suggested precursors above. Its purpose is to measure students' knowledge of that model. The instrument went through several iterations with teachers and experts in mathematics education to establish face validity with the a priori model of precursors.

Two achievement tests were constructed, one concerning algebra, and the other algebraic problem solving. Both tests measure achievement consistent with the way that it is done in schools. An instrument was constructed to assess the pedagogical styles of algebra teachers. Its purpose is to categorize students' algebra teachers into three broad styles.

Two types of hypotheses are tested, a priori and post hoc. The purpose of a priori hypotheses is to identify precursors of algebra. For that purpose, precursors are identified in two ways - by considering categories as precursors and by considering elements as precursors. Employing two levels of analysis allows for the identification of both general precursors (categories) and specific precursors (elements) and increases the likelihood of identifying important precursors. However, for purposes of the category level analyses, categories are determined, not from those of the model, but from those obtained from factor analyses. The resulting categories are therefore less subjectively determined than those of the original model.

A priori hypotheses are tested using conventional conservative statistical tests. Linear regression methods involving correlations between achievement scores and scores for items of the model knowledge instrument are used to identify precursors.

The purpose of post hoc hypotheses is to investigate additional relationships, ones that concern gender, teacher style, achievement, and identified precursors. The hypotheses are tested using less conservative exploratory tests with a variety of methods.

CHAPTER 2

REVIEW OF THE LITERATURE

This chapter reviews the literature relevant to the model of precursors of algebra and a subsidiary model of the pedagogical styles of algebra teachers.

Precursors of Algebra

The Conceptualization of Algebra for the Study

Arithmetic, as usually taught in elementary school, is concerned with numbers, rules for operations, algorithms, and situations in daily life where numbers and operations are applied. Algebra inhabits a larger domain of more abstract conceptualizations. These conceptualizations are; (1) generalized arithmetic, (2) procedures for solving problems, (3) the relationships between quantities, and (4) structures (Usiskin, 1988). For purposes of this study, algebra is a combination of generalized arithmetic and procedures for solving problems. This conceptualization matches most grade 9 algebra curricula.

Pre-requisites as Precursors of Algebra

The teaching of algebra has not changed much in the last 50 years (Thorpe, 1989). Most apparent changes have been more cosmetic than substantial. In the same way, only labels have changed in attributions of the causes of difficulties in learning algebra. Any difficulties students have in learning algebra are usually attributed to "not being smart enough" and weak arithmetical skills.

The first explanation has been expressed as "there are math types and there are nonmath types" (Davis and Hersh, 1981). Seventy years ago Thorndike et al (1923) concluded that measures of numerical patterning, geometric patterning, sentence completion, and word matching were the best available for predicting success in algebra because such measures of 'abstract ability' were most likely to predict success in algebra.

The explanation pertaining to "weak arithmetic" remains pervasive. A statement in the Manitoba mathematics curriculum guide (Manitoba Education, 1979) serves as an exemplar of this point of view.

> Arithmetic operations must be reviewed and maintained.
>
> . . . It is important that students operate proficiently with integers prior to a detailed study of algebra. Students whose arithmetic skills are good tend to do well in a study of algebra. (p. 135)

A survey of various sources strongly suggests that precursors of algebra are conceptualized as a list of pre-requisites. We must generally infer the precursors that are implicit from such lists. These pre-requisites, largely computational skills, are taken to be the conventional set of precursors.

Smorodin (1985), in a report of skills to be tested for the New Jersey State Department of Education, defined skill in integer arithmetic and skill in exponent notation and operations as the pre-requisites of algebra. The Boston Public Schools curriculum (1983) lists the pre-requisites of algebra as being knowledge of numbers and numeration; computation; fractions; decimals and percent; exponents; estimating; graphs; and the function machine. The Louisiana State Department of Education (1987) considers a good algebra foundation to be a knowledge of whole number, fraction, and decimal arithmetic; number theory; ratio and percent; and integer arithmetic.

Research on Pre-algebra

Published research does not explicitly supply a useful model for the study of precursors of algebra. However, the new field of pre-algebra research implicitly pays attention to potential precursors. That literature contains four explanations as to why students have difficulties in learning algebra. Those four explanations concern; (1) dissonance between algebra and arithmetic, (2) well-structured knowledge, (3) problem solving, and (4) reasoning processes (Herscovics, 1989; Kieran, 1989; Chaiklin, 1989; Davis, 1989). These explanations helped shape the model used in this study.

Dissonance between algebra and arithmetic.

A dissonance between algebra as taught in schools and students' perceptions of arithmetic appears to be significant. Lee and Wheeler (1989) investigated the extent to which students in grade 10 relate the worlds of arithmetic and algebra. The evidence suggested to them that there is a large degree of disassociation even among students who are successful at algebraic tasks. They found "the track leading from arithmetic to algebra to be littered with procedural, linguistic, conceptual, and epistemological obstacles" (Lee and Wheeler, 1990, p. 53).

It would be difficult to dispute the inevitability of students utilizing an arithmetic framework when learning algebra. However, the literature suggests that the arithmetic experiences of students encourage the establishment of cognitive frameworks that are inappropriate for learning algebra and that place cognitive obstacles in the way (Herscovics, 1989). In other words, many of the difficulties students have in learning algebra may be attributable to their misconceptions about arithmetic or their reliance upon inappropriate informal or formal procedures. Some students seem to be well aware of the dilemma. Chalouh and Herscovics, Collis, and Davis (cited in Kieran, 1989) noted that the students they interviewed were aware that the conventions they used in algebra seem to be different from those they used in arithmetic.

A British research project (University of Bath, 1982: cited in Booth, 1988) involving students from grade 8 to grade 10 found that students made similar errors at each grade level, independent of age and experience in algebra. In it, errors are traced to four incongruencies between algebra and arithmetic; (1) the nature of acceptable answers, (2) the use of convention and notation, (3) the notion of a variable, and (4) the kinds of relationships and methods used. Kieran (1989) suggests a fifth source of error, incongruency concerning the roles of structure.

One source of students' difficulties in algebra is incongruency concerning the nature of acceptable answers. In arithmetic, the intent is usually to find a specific numerical answer. In algebra this is often not the case. Students seem to have difficulty in making that transition (Booth, 1988). They tend to see answers in algebra as "not proper" and are unable to hold unevaluated expressions (such as 'x + y') in suspension. Both Collis and Davis (cited in Kieran, 1989) note that novice algebra students tend not to view algebraic expressions as legitimate answers; they are somehow incomplete. Chalouh and Herscovics (1988) refer to it as an unwillingness to accept a lack of closure.

A second source of difficulty is incongruency concerning the uses of notation and convention. Matz (cited in Herscovics, 1989) suggested that notational incongruencies create cognitive obstacles to algebra. Students will substitute '2' for 'a' in '3a' by concatenating the 2 and obtain '32'. They attribute this error to the arithmetical convention that concatenation implies addition, not multiplication. Two-digit numerals are sometimes not replaced by a single letter because of the same conflict between notational conventions in arithmetic and algebra (Booth 1988). Another source of difficulty is differing interpretations of how to use parentheses

(Booth, 1988). In arithmetic, they tend to be interpreted as "do me first" indicators, an interpretation that is largely unworkable in algebra. MacPherson and Rousseau (1988), and Chalouh and Herscovics (1988) noted that students tend not to see the distinction between the active and passive use of notations such as '+' and '='. In arithmetic, expressions such as '2 + 3' tend to represent actions linked to an instruction to add. When confronted with algebraic expressions, students try to view them in an active sense, a perception that is inappropriate and usually results in the application of incorrect strategies.

A third source of difficulty is incongruency concerning the meanings of variables. One meaning concerns variables as replacement symbols. Usiskin (1988) found that algebra students tend to believe that a variable is always a letter or psuedo-letter. In other words, such symbols as 'x' and 'Δ' can indicate variables while such arithmetical symbols as '_' and '?' cannot. This belief suggests that students have not acquired an understanding of the replacement role of variables. Collis and Kuchemann (cited in Kieran, 1989) observed that students tend to view a variable in a number sentence (such as 'Δ + 5 = 8') as an unknown whose value is to be figured out. Students tend to substitute one specific value after which they stop substituting. This suggests that students interpret a variable as a replacement symbol that can be replaced by only one number. Booth (1988) found that there is a strong tendency for students to regard a variable as a symbol that represents a unique number and that different variables in a number sentence (such as '$\Delta + ? = 10$') must be replaced by different numbers. This arithmetic-derived conception of a variable can be referred to as "the secret name for a number" conception. It is incongruous with conceptions in algebra. Usiskin (1988) suggested that the arithmetical view of a variable is often inadequate and misleading in algebra. In algebra, a variable may be a replacement symbol, a parameter, an arbitrary element of some structure, or a symbol which may be manipulated. Booth (cited in Kieran, 1989) summarizes this dissonance by saying that students often have difficulty in interpreting letters as generalized numbers. Booth (1988) pointed out that that can lead to difficulties when comparing equivalence, for example, many students consider that '$x + y + z$' can never equal '$x + p + z$'. To them, different letters (in this case, the 'y' and 'p') always mean different replacements. Sutherland (1987) found that LOGO programming experiences can provide students with a conceptual basis for variables that improves their understanding of algebra procedures and variables. That result suggests that there is a way of overcoming the negative effects of operating from an inappropriate arithmetical view of a variable, a way that may guide the development of some precursors of algebra.

Students also tend to confuse variables and labels. In arithmetic, letters are usually interpreted as labels. For example, in the statement '3 m = 300 cm', 'm' represents a label not a number. This interpretation often leads to difficulties in understanding the meaning of mathematical language in algebra. For example, Booth (1988) found that students often interpret variables such as 'y' in '8y' as '8 yams' or '8 yachts'.

Further difficulties related to variables occur when students translate written language to mathematical language. Researchers have found that some students, all the way to first courses in engineering, translate the written statement 'There are six times as many students as professors at this university.' into the incorrect algebraic statement '6S = P' (cited in Lochhead and Mestre, 1988). Clement, Lochhead, and Monk (cited in Lochhead and Mestre, 1988) proposed an explanation for this phenomenon. They suggest that it stems from misconceptions about the structure of algebraic statements, the interpretation of variables contained in algebraic statements, and the relationship between written language and algebraic language involving variables. These misconceptions may arise from students' experiences in arithmetic where the conceptual issues involved in the variable-label confusion are not addressed or are treated in a way that is incongruous with algebra.

A fourth source of difficulty is incongruency concerning the kinds of relationships and methods used. Kieran (1989) noted that students tend to believe that the left-to-right written sequence of operations in arithmetic determines the order in which computation is to be performed. In algebra, this proclivity leads to errors in writing equations that represent relationships in algebraic word problems. Students tend to assign numbers to variables according to the order of occurrence of numbers and words (Lochhead and Mestre, 1988). Kieran (cited in Booth, 1988) concluded that many students believe that the value of an arithmetical expression remains unchanged if the order of computation is altered as long as the written order is not changed. Further, students typically do not use brackets because they believe that the written order of operations determines the value of an expression. These beliefs partly explain algebraic errors that occur when students manipulate or write expressions involving brackets.

Further to relationships and methods, students tend to learn and to use informal, intuitive methods in arithmetic (Booth,1988). These informal arithmetical methods can limit their ability to understand or produce general statements in algebra. For example, Ekenstam and Nilsson (cited in Booth, 1988) found that using informal procedures to solve equations limits students' success in seemingly similar situations. Furthermore, in arithmetic equivalency is normally determined by calculating. Cauzinille-Marmeche, Mathieu, and Resnick (cited in Kieran, 1989) found that when students rely on informal methods, they make more errors when determining equivalency in algebra. These results suggest that, in algebra, equivalency is best determined by applying a set of principles or rules.

A fifth source of difficulty is incongruency concerning the roles of structure. For these purposes, structure is the arrangements of terms and operations, and the constraints on the order of processing. The role of structure is largely ignored in arithmetic. Yet in algebra, students are required to recognize and use structure. That dissonance may be an important cause of students' difficulties in algebra. Lochhead and Mestre (1988) commented that students who are not proficient in processing algebraic expressions seem largely unable to see any consistent structures in these expressions. Further to this, Davis, Matz, Greeno, Rugg and Clark, and Breslich (cited in Kieran, 1989) noted that beginning students in algebra have great difficulty imposing structure on algebraic expressions. This failure to detect structure is suggested as one explanation for the parsing errors that students make. Larkin (1989) supports this conclusion. She found that many students see an algebraic expression as an unstructured string with the rules of algebra acting on arbitrary parts of the string. This may explain why they easily misapply these rules when manipulating expressions. Lochhead and Mestre (1988) summarize this in saying that students do not learn to read and write mathematical symbol strings in arithmetic. This leaves them at a disadvantage in learning the manipulation rules of algebra.

Cumulatively, the literature strongly suggests that, as much as assisting the learning of algebra, the conventional framework of arithmetic can create cognitive obstacles. On the other hand, some parts of arithmetic are necessary for learning algebra. The solution to that dilemma may lie in shaping an arithmetical framework that is more appropriate to learning algebra; a framework within which desired notions are embedded in arithmetic.

Those desired notions consisting of principles, concepts, and reasoning skills form the set of precursors of algebra. Furthermore, rote performances of algorithms, which have often driven the design of conventional arithmetical curricula would not belong in that set of precursors.

Well-structured knowledge.

Current practice in teaching arithmetic and algebra tends not to encourage well-structured knowledge or well-connected evolving knowledge of mathematics (Kieran and Wagner, 1989). Schoenfeld and Whitney (cited in Thorpe, 1989) commented that students most often see both arithmetic and algebra as largely a collection of tricks - a trick for this and a trick for that. That perception is likely a cognitive barrier to learning algebra. The fourth NAEP mathematics assessment (Brown et al., 1988) found that a large majority of students feel that mathematics is rule-based, with about half of them reporting that learning mathematics is mostly memorizing. Furthermore, students do not seem to understand many of the structures underlying mathematical concepts and skills. Those findings, along with low assessment results (Brown et al., 1988), suggest that students' perceptions of mathematics as disconnected bits of rules, facts, and procedures may be partly responsible for difficulties in learning algebra. Further evidence supporting that conclusion comes from Manitoba mathematics assessment results (Curriculum Branch, 1987) which indicate that students' achievement in mathematics declines significantly from grade 6 to grade 9. The mean score of all mathematical curriculum topics in grade 6 is 59.8; the mean score for grade 9 is 53.7. The lower mean score in grade 9 is pertinent as that is when students first encounter some substantial algebra. One explanation for the decline in overall mathematics achievement is that the volume of disconnected understandings and rote skills may overwhelm some students. They no longer can assemble those learned in arithmetic with those being learned in algebra. All of this suggests that any set of precursors arising out of arithmetic must provide a coherent and connected framework for learning algebra.

Problem solving.

Rachlin (1986) suggested that a successful study of algebra may require problem solving processes that are usually not developed in arithmetic. MacPherson and Rousseau (1988), and Kieran (1989), in sharpening what this may mean, have suggested that the solution of routine word problems that are encountered in arithmetic classrooms may best serve algebra if the approach involves a search for the structure of the problem rather than a search for the correct algorithm to apply. Krutetskii (cited in Rachlin, 1986) supports this position in concluding that one of the characteristics of good problem solvers is that they seek the structure of a problem rather than focus on its specifics. Vergnaud, Benhadj, and Dussouet (cited in Kieran, 1989) exhibit the tension between problem solving in arithmetic and in algebra using the following problem.

> In an existing forest, 425 trees were planted. A few years later, the 217 oldest trees were cut. The forest then contained 1063 trees. How many trees were there before the new trees were planted? (p. 37)

The structural approach to solving this problem would begin with an expression like, '? + 425 - 217 = 1063' followed by decisions as to the algorithms to employ. The expression reflects what 'happened'. They found that many teachers suggest a series of computations like; '1063 + 217 = 1280, 1280 - 425 = 855'. This sequence of algorithms does not match what 'happened' (for example, 217 trees were removed, not added, to the forest). At best, those teachers have left the creation of templates implicit, moving on to the appropriate arithmetic for obtaining answers. In doing so, it is likely that only the "bright" students can see the templates and perhaps only intuitively.

The literature has not extensively addressed the question of how well a structural approach best serves the learning of algebra, but it does suggest that there are implications for judging equivalency. Kieran (cited in Kieran, 1989) found that students have difficulty in judging equivalent equations. For example, some students believe that 'x + 37 = 150' is equivalent to 'x = 37 + 150'. Kieran (1989) suggests that such errors arise as a result of their not understanding the structural relationship between addition and subtraction. A structural approach to problem solving in arithmetic may help establish that understanding as the approach can involve transforming number sentences such as '? + 23 = 45' to '? = 45 - 23'. Further to equivalency, Greeno (cited in Kieran, 1989) found that many students do not seem to be aware that an incorrect solution, when substituted into the original equation, will yield different values for the two sides of the equation. In addition, students often do not realize that it is only the correct solution which will yield equivalent values for the resulting left and right expressions in a chain of transformations of an equation. Whitman (cited in Kieran, 1989) suggests that this lack of awareness may best be addressed by having students first learn intuitive processes for solving equations. These processes are related to the structural approach to solving arithmetical word problems.

Reasoning processes.

The literature has little to say concerning the specific reasoning abilities that students require as they begin to learn algebra. The "not smart enough" school of thought vaguely suggests that an ability in abstract reasoning may be important to learning algebra, but this is of little help in determining a set of precursors.

Freudenthal (1973) has noted that while arithmetic is intuitive and close to reality, algebra is characterized by its formal symbolic methods and therefore demands greater attention to thinking strategies. He suggests six thinking strategies that may be important to learning algebra; (1) schematizing, that is sensing regularities, (2) detecting transitive relationships between objects of thought, (3) seeking necessary and sufficient conditions, (4) using indirect proofs, (5) making analogies and using analogies to gain insights, and (6) detecting and using the 'if --> then' structure. Freudenthal does not elaborate as to how these strategies can be fostered but developmental psychology may provide some guidance. Piaget has suggested that processes of reasoning which ultimately may seem self-evident must, in the beginning, be checked against the evidence of what one finds through doing (Donaldson, 1984). That strongly suggests that any reasoning abilities or strategies that may be precursors of algebra be developed initially in a domain that requires less abstract symbolism than does algebra. The natural domain for this purpose is arithmetic. It would seem that the activation and fostering of reasoning strategies that are important to algebra must occur while students learn the concepts, principles, and procedures of arithmetic. This suggests teaching practice that focuses on understandings rather than rituals.

Pedagogical Styles of Algebra Teachers

For purposes of this study, pedagogical style is the manner in which teachers conduct lessons in algebra and algebraic problem solving. While there does not appear to be any literature specific to the pedagogical styles of algebra teachers, there is considerable literature on the ways mathematics is taught.

Mathematics is taught in ways that fall along a continuum. Davis and Hersh (1981) allude to one end of that continuum.

> As a teacher I am constantly confronted by problem after problem that has nothing to do with math. What I try to do is sell math to kids on the basis that it's fun. In this way I get through the week. (p. 274)

The findings of recent studies in mathematics suggest what may be the other end of the continuum. Goodlad (1983) and Tobin, Espinet, and Byrd (1987) reported that the dominant teaching procedure in mathematics was lecturing. Furthermore, that style of teaching consistently lacked student-teacher interactions, small group work, or alternative approaches to the development of topics. The emphasis in such mathematics classes was on recall - learning facts and memorizing algorithms without necessarily understanding why the algorithm works (Doyle, 1983; Tobin, Espinet, & Byrd, 1987). The Canfield Teaching Style inventory (cited in Raines,1976) which measures general teaching styles provides a view of the entire continuum. The styles of teaching measured are: (1) straight lecture, (2) lecture with summary notes, (3) teacher questions students, (4) student presentations, (5) small group discussion (teacher led or student led), (6) demonstrations, (7) practice exercises, (8) simulation/games/problem solving, and (9) collaboration.

It was not possible to attempt to place teachers along this whole continuum, if that is what it is, in this study. Three broad categories were selected - two polar styles and a blend of them.

Much of the commentary in the literature suggests that there are two distinct styles of teaching mathematics. They served as the basis for developing the pedagogical style of teacher model for this study.

Bach (1981) provides two icons of mathematics teachers - alphas and betas. Alphas strictly follow the text and never make mistakes in class. Betas don't rely on texts, teach concepts, make mistakes, and excite and confuse their students. Furthermore, Bach (1981) and others view the styles that fall into the alpha category as inappropriate paradigms for teaching mathematics and beta styles as appropriate paradigms. This study begins with those two icons as polar styles of mathematics teaching.

The literature describes the characteristics of the alpha category in various ways. Davis and Hersh (1981) discuss authoritarian or dogmatic teaching which may occur in classrooms or in texts. It can be exemplified by the statement, "Look, I tell you this is the way it is". Tobin (1989) described one general style, lecturing (or chalk and talk) which is characterized by whole class teaching, seat work consisting of repetitive practice of skills and algorithms, and an emphasis on recall. There is little small group work, alternate approaches, or emphasis on higher level cognitive outcomes.

The beta category is taken to be the antithesis of the alpha category. Davis and Hersh (1981) discuss a teaching style that allows students to "fiddle around" mathematically so that they may learn something of the strategies and insights that lie behind mathematics. Tobin (1989) suggests that good teachers of mathematics do not lecture but encourage students to participate in learning activities and encourage meaningful learning through principles and concepts. Pirie and Schwarzenberger (1988) describe a mathematical discussion style which is characterized by purposeful talk that has well-defined objectives and by pupils' genuine contributions in the form of inputs that assist in laying out the learning process. The NCTM standards (NCTM, 1989) for improving the quality of mathematics instruction involve a more open teaching style where students are encouraged to invent symbols, use trial and error, use imagination, conjecture, predict, verify, make decisions, and work in groups.

CHAPTER 3

MODELS DEVELOPED FOR THE STUDY

Two models were developed for use in the study; (1) a model concerning precursors of algebra and (2) a model concerning pedagogical style of algebra teachers.

A Model concerning Precursors of Algebra

Background to the Study

The researcher's interest in the precursors of algebra was stimulated by teaching algebra-related mathematics courses to re-entry adults, many of whom were returning to formal schooling after a lengthy absence.

An analysis of their difficulties in algebra suggested that the conventional set of precursors is inadequate. For example, they were unable to clearly identify structure-related aspects of an algebraic statement and they did not realize that mathematical statements need not be decoded left-to-right. Such missing notions and misconceptions seemed to partly explain some processing errors. The researcher next attempted to create a model including these and other non-conventional precursors of algebra and to study the effects of using it with re-entry adults. A perception of considerable relationship between achievement and predicting based on this preliminary version of the model encouraged continuing to a more formal version of the model and tests of these relationships.

The literature and logical analysis provided further possible categories and elements that might comprise a set of precursors of algebra.

Overview of the Model

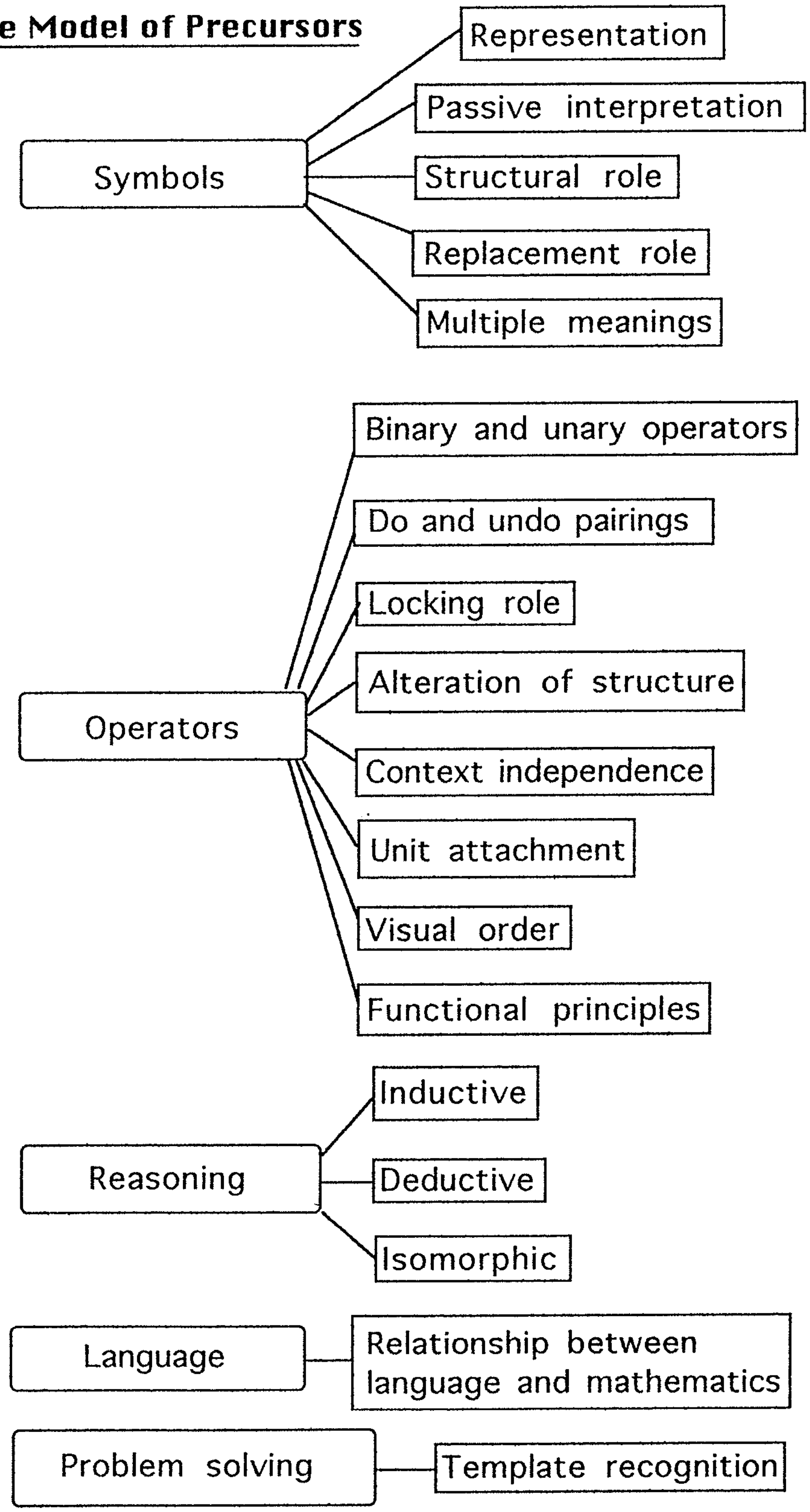
The model contains categories (broadly defined notions) and elements (more detailed notions) that are suggested precursors of algebra. The categories, as clusters of elements, provide cohesiveness to those elements while the elements themselves help define the categories. The model was used as the basis for constructing the model knowledge instrument.

The model is organized into five dimensions or clusters of categories. They are; (1) symbols, (2) operators, (3) reasoning, (4) language, and (5) problem solving. The dimensions encompass 18 categories. Each category consists of 3 or more elements. In all, there are 63 elements or detailed precursors.

Some categories overlap. However, the categories differ in the specificity of emphasis or in the generality of expression of a notion.

The following chart provides an overview of the a priori model of precursors of algebra.

The Model of Precursors



Dimension 1: Symbols

Children encounter symbols when doing arithmetic. The way they view these symbols may be important for learning algebra. Five categories show promise; (1) representation, (2) passive interpretation, (3) structural role, (4) replacement role, and (5) multiple meanings.

Category: representation.

Representation concerns symbols in relation to representing notions. The three elements of this category are; (1) different symbols may be used to represent the same notion, (2) symbols are arbitrary creations, and (3) particular symbols are used for reasons related to utility. See Appendix A - items 10, 24, and 33.

Category: passive interpretation.

Passive interpretation concerns 'another name for' (for example, '2 + 3' is another name for '5'). The three elements of this category are; (1) questions and answers may be interchanged in arithmetic (6 can be a question and 2x3 can be an answer), (2) different names can name the same number, and (3) the set of different names for some number is large. See Appendix A - items 21, 39, and 48.

Category: structural role.

Structure may be defined as the internal organization of mathematical expressions; it concerns how the parts are bound together and separated. Students' structural intuitions are rooted in arithmetic and are derived from the hierarchy of arithmetical operations. While there are far from uniform conventions in arithmetic (consider 3 1/2 = 3 + 1/2 and 34 = 30 + 4), most often the separating symbols are additive (+ and -) and the binding symbols are multiplicative (x, divide, exponentiation, root of, and the fraction indicator '/').

The structural role of symbols is expressed in the formation of chunks or terms. It involves the ability to chunk an expression, to view an expression according to the terms that comprise it. The three elements of this category are; (1) structure helps determine what is to be done in an expression, (2) certain operators bind chunks, and (3) certain operators separate chunks. See Appendix A - items 7, 55, and 63.

Category: replacement role.

The replacement role concerns variables, viewed as empty slots waiting to be filled with numbers. The empty slot can be denoted by a letter of the alphabet or by more "primitive" replacement symbols such as 'Δ' or '?'.

The six elements of this category are; (1) the replacement set is not restricted to whole numbers, (2) replacement symbols may be manipulated as though they were numbers, (3) the same replacement may be used for different replacement symbols in an expression, (4) replacement symbols may force a halt to processing (for example, processing stops for 'Δ + 5' until 'Δ' is replaced by a number), (5) each occurrence of a particular variable in an expression requires the same replacement, and (6) the set of possible replacements for a particular replacement symbol is unlimited. See Appendix A - items 5, 11, 19, 26, 34, and 37.

Category: multiple meanings.

Operation symbols can have more than one meaning. In particular, each of the symbols '+' and '-' can be interpreted in at least 3 ways. The symbol '+' can indicate the binary operator 'add' (put together), a direction (for example, '+' can mean to rotate clockwise), or it can be part of a label (as in '+5'). The symbol '-' can have more meanings. It can indicate the unary operator 'opposite of', a direction that is opposite to the direction indicated by the '+' symbol, be part of a label (as in '-5'), and the binary operator 'subtract' which can itself have at least 3 interpretations - 'take away', 'compare', and 'change in' (in relation to measuring).

The elements in this category sample the above possible meanings. The differing interpretations concerning the binary operator meaning of '-' and the 'put together' interpretation for '+' are excluded because they overlap the template recognition category below. They are best dealt with in that category. Accordingly, only the three renaming meanings of '+' and '-' are included. The symbols, '+' and '-' may indicate; (1) a direction, (2) a part of a label, or (3) a unary operation. See Appendix A - items 6, 16, and 32.

Dimension 2: Operators

While operators are implicitly and explicitly included in some of the categories of the 'symbols' dimension, they warrant separate consideration. Operators in arithmetic indicate that some transformation of numbers is to take place. Eight categories show promise; (1) binary and unary operators, (2) do and undo pairings, (3) locking role, (4) alteration of structure, (5) context independence, (6) unit attachment, (7) visual order, and (8) functional principles.

Category: binary and unary operators.

Operators can be classified according to the number of inputs. Addition, subtraction, multiplication, and division are binary operators. Square root and squaring (in general, 'finding the root of' and exponentiation), and 'opposite of' are unary operators. The elements of this category are based on the identification and discrimination of those classifications. The more general operators 'finding the root of' and exponentiation are excluded. Accordingly, the three elements are; (1) 'opposite of' - an unary operator, (2) 'square root of' - an unary operator, and (3) addition, subtraction, multiplication, and division - binary operators. See Appendix A - items 2, 25, and 49.

Category: do and undo pairings.

Do and undo pairings concern the notion of an inverse operation in the sense that an operation is undone by its inverse. For example, the effect of adding 5 to something will be undone by subtracting 5. This is a functional approach to inverses. Addition and subtraction;

multiplication and division; and 'finding the root of' and exponentiation are inverse operations (do/undo pairings). 'Opposite of' is its own inverse. The elements of this category are based on those pairings except that 'finding the root of' and exponentiation are not included. Only the specific instances of square root and squaring are. Accordingly, the three elements are; (1) multiply and divide - a do/undo pairing, (2) addition/subtraction and squaring/square root - do/undo pairings, and (3) 'opposite of' - its own inverse. See Appendix A - items 18, 54, and 57.

Category: locking role.

Over and above the other roles of operations in algebra and arithmetic, they may be thought of as having locking and unlocking roles. These roles provide a way of conceptualizing the order of computation in an expression. For example, in the expression '2 x 3 + 4', '2 x 3' is locked in relation to adding 4; before addition can proceed '2 x 3' must be unlocked. There are at least two ways to do that; by replacing '2 x 3' with '3 + 3' or with '6'. Similarly, for '3 x $(5)^2$', '$(5)^2$' is locked and must be unlocked before multiplication by 3 is possible.

There is overlap between the 'locking roles' category and the structural role (of symbols) category. The notion of locking roles, which is derived from the hierarchy of operators, conceptualizes hierarchical processing in a way that is readily integrated into the structural features of an expression. This capacity to link structure with hierarchical processing seems important to algebra for at least two reasons. First, the literature strongly suggests that a failure to detect structure is an important explanation for some algebraic errors (Lochhead and Mestre, 1988; Larkin, 1989; Davis, Matz, Greeno, Rugg and Clark, and Breslich: cited in Kieran, 1989). Second, it suggests that the lack of coherent and connected knowledge is also an explanation for difficulties in algebra (Kieran and Wagner, 1989; Schoenfeld and Whitney: cited in Thorpe, 1989).

Further, expressions can be viewed as having locked and unlocked chunks (and locked portions within chunks) that provide structure to expressions and determine the order of computation. Before processing can continue, locked chunks must be unlocked using valid procedures. Locks are hierarchical. Exponentiation and 'finding the root of' (unary operators) are more powerful locks than binary operators. The unary operator, 'opposite of' is an exception; it is on par with the binary operators, multiply and divide. The exception can be explained by viewing 'opposite of' as multiplying by '-1'. Multiplication and division (as well as 'opposite of') form more powerful locks than addition and subtraction.

While brackets are not operators, they also play locking and structural roles. Because brackets visually "capture" numbers and operators in arithmetic, students (and teachers) tend to interpret them as "do me first" indicators. That point of view may be inappropriate for algebra (Kieran; cited in Booth, 1988). It seems more appropriate to view brackets as having the capacity to disrupt the normal determination of locks and chunks. For example, the expression, '2 x 3 + 5', contains two chunks; a locked chunk '2 x 3' and an unlocked chunk '5'. Processing must be done by first unlocking '2 x 3'. If brackets are placed in that expression, its structure and processing priorities may change. For example, there is only one chunk in '2 x (3 + 5)', the entire expression. That result follows from viewing it in an overall way; '2 x (3 + 5)' can be interpreted as '2 x some number'. Processing order is also affected by the inclusion of the brackets in this case. Processing can proceed in at least two ways; add then multiply by '2' or use the distributive principle. Neither of these ways has hierarchical priority.

The elements of this category are selected from what seems to be the most cogent locking roles. The three elements are; (1) locking role of brackets, (2) locking role of additive processes, and (3) locking role of multiplicative processes. See Appendix A - items 3, 9, and 36.

Category: alteration of structure.

Alteration of structure concerns a change in the structure of an expression. The three elements of this category are; (1) the complexity of processing is related to the complexity of structure, (2) the structure of an expression changes as processing proceeds, and (3) the numerical value for a given expression is invariant as the structure changes through computation.

For example, the expression '2 x 3 + 5 x 6 - 4 x (7 + 2 x 5)' has a more complex structure than the expression '2 + 5'. See Appendix A - items 8, 15, and 23.

Category: context independence.

Context independence concerns sensing arithmetical statements as being independent of the contexts from which they are derived. For example, the two contexts, 'Two ducks were swimming in the water; along came 3 more. Now there are 5 ducks in the water.' and 'Two thoughts were in my head. Someone gave me 3 more. Now I have 5 thoughts.', both lead to the statement, '2 + 3 = 5', which can be viewed independently of the contexts that determined it. For this category, the elements are restricted to additive (+/-) and multiplicative (x/÷) examples. The three elements are; (1) a statement involving an additive operation is independent of context, (2) a statement involving a multiplicative operation is independent of context, and (3) a statement involving a combination of additive and multiplicative operations is independent of context. See Appendix A - items 4, 12, and 52.

Category: unit attachment.

Unit attachment concerns the various ways units can be attached to the numbers involved in operations. Again, the elements are restricted to additive (+/-) and multiplicative (x/÷) operations. Multiplicative operations involve no restrictions on units. For example, 'hours' can be multiplied by 'people' obtaining 'people-hours'. Additive operations require that numbers have the same units attached to them. For example, 'two apples plus 3 nails' is not allowed unless the units are subsumed under some more general definition. Furthermore, units themselves are not added (or subtracted); rather, 'counts of' are added. For example, in '2 pears plus 3 pears', the pears are not added; rather, 2 and 3 are added and these numbers concern counts of pears. The three elements of this category are; (1) additive operators require addition or subtraction of counts of units, (2) additive operators require identical units, and (3) multiplicative operators do not require identical units. See Appendix A - items 13, 44, and 60.

Category: visual order.

The order of occurrence does not determine the order of processing in an arithmetic expression. In particular, the left-to-right order that is normally employed in decoding language need not be employed in processing arithmetic expressions. The three elements of this category are; (1) computations involving additive operations can be done in many directions, (2) computations involving multiplicative operations can be done in many directions, and (3) computations involving both additive and multiplicative operations can be done in many directions. See Appendix A - items 30, 45, and 62.

Category: functional principles.

The functional principles are analogues of the principles of algebra that can serve functional purposes in students' computational procedures. That is to say, functional principles guide students in doing arithmetic in ways that enable or simplify computational tasks. Functional principles are useful in simplifying computational tasks, and can also provide alternative procedures or justify procedures. The five elements of this category are; (1) the associative principle in additive and multiplicative processes, (2) the commutative principle in additive and multiplicative processes, (3) the distributive principle, (4) the equal factors principle (that it is possible to multiply both parts of a division expression by the same non-zero number), and (5) the equal addends principle (that it is possible to add the same number to both parts of a subtraction expression). See Appendix A - items 14, 17, 27, 42, and 61.

Functional principles are ultimately used under the aegis of the notion that 'If you don't like the way something looks, change it to a more useful or convenient form.'. That notion underlies more advanced algebraic processing, but it is not tested here.

Dimension 3: Reasoning

Three categories of reasoning show promise; (1) inductive reasoning, (2) deductive reasoning, and (3) isomorphic reasoning.

Category: inductive reasoning.

Inductive reasoning concerns probable induction, the faith that an observed regularity or pattern will persist. For the purposes of this study, the domain is restricted to arithmetical patterns in parallel number sequences, in individual number sequences, and in results derived from arithmetic operations. The three elements of this category are; (1) sensing a joint pattern in two sets of numbers, (2) sensing a pattern in a sequence of numbers, and (3) sensing a pattern in arithmetic results. See Appendix A - items 28, 50, and 58.

Category: deductive reasoning.

Deductive reasoning concerns some of the components of conventional logical reasoning. Seven of what seem to be the most useful subcategorizations of logic have been chosen for purposes of the study. The seven elements of this category are; (1) modus poens, (2) converse, (3) transitivity, (4) modus poens in numerical relationships, (5) noncontradiction, (6) negation, and (7) contrapositive.

Modus poens concerns the 'IF --> THEN' relationship. If the first part is true, then the second part is also true. Two types of relationships are included for modus poens - non-arithmetic and arithmetic.

Converse entails reversing an 'IF --> THEN' relationship. The converse of an assertion is not necessarily true.

Transitivity concerns concatenated IF --> THEN relationships; if a -> b, and if b -> c, then a -> c.

'Noncontradiction' as used here is the 'law of the excluded middle'. An assertion is true or it is not. There are no other possibilities.

Negation concerns a 'not' notion. Both negation and noncontradiction involve two state logic, a yes/no form of reasoning.

Contrapositive concerns the negation of a conclusion; if p -> q, then not q -> not p.

See Appendix A - items 1, 20, 41, 43, 46, 56, and 59.

<u>Category: isomorphic reasoning.</u>

Isomorphic reasoning concerns analogies between systems. Whether or not two systems have superficial features in common, they can have common features at underlying levels that make them analogous systems. In mathematics, it is common to use such analogies in the sense that "working here" is like "working there".

Practice sometimes requires students to employ that sense in connecting manipulatives (such as Dienes blocks) and pseudo-concretes to more abstract mathematical symbols and notions. Students are expected to draw analogies by recognizing common features in a familiar everyday world and in the less familiar world of mathematical ideas. Out of this arises the lowest level of isomorphic reasoning; concluding that working in a world of concretes or pseudo-concretes can be supposed to be like working in a world of mathematics.

It is also possible to search for common features that underlie differing systems. For example, addition can be compared to multiplication, union can be compared to addition, arithmetic can be compared to algebra, grammar can be compared to the hierarchy of operators, and so on. This points to a higher level of isomorphic reasoning; concluding that working in one abstract system can be just like working in different one.

In summary, the categories of isomorphic reasoning and inductive reasoning overlap in that sensing a pattern is similar to recognizing a common feature. However, isomorphic reasoning embraces a more general notion of sensing a pattern, one that may be intimately related to transfer of learning.

The three elements of this category are drawing analogies between; (1) pseudo-concrete representations and mathematical representations, (2) differing general symbolic representations, and (3) differing mathematical representations. See Appendix A - items 35, 40, and 53.

Dimension 4: Language

There is one category in this dimension.

Category: relationship between language and mathematics.

Students must simultaneously deal with two symbolic systems, language and mathematics. It may be useful for students to see them as related.

Only a small subset of the many underlying relationships are included in this category. In particular, variables occur in both. For example, words like 'she', and 'somebody', are, in effect, variables in language. As well, mathematics and language both can be used to describe events that occur in daily life. The student must translate mathematical notions into language and vice versa. The three elements of this category are; (1) recognizing analogous roles concerning replacement, (2) expressing arithmetic operations by means of language, and (3) expressing number sentences by means of language. See Appendix A - items 22, 31, and 47.

Dimension 5: Problem Solving

There is one category in this dimension.

Category: template recognition.

Template recognition concerns attaching arithmetic templates to word problems. While this can be done using number sentences and by at least two kinds of diagrammatic representation, only number sentences (such as '12 - ? = 5') are considered for purposes of this study. These templates are derived from the relationships between arithmetical expressions and the contexts to which they are attached. That is to say, a template represents a mathematical way of representing some action or state in the concrete world.

Viewing problem solving as beginning with a search for templates differs from the viewpoint suggested in much current theory (But see, for example, MacPherson & Rousseau (1988), Carpenter & Moser (1982), Vergnaud (1982)). From this point of view, a student first searches for the structure of a problem (the template). If one is found, the student next selects an algorithm suited to calculating the answer. The algorithm may or may not involve the same arithmetic operation as the template. For example, for the word problem, 'Mary had 3 cookies. Her friend gave her some more. Now Mary has 11 cookies. How many cookies did Mary's friend give her?', the template approach involves identifying the structure of the problem and writing the corresponding number sentence, '3 + ? = 11'. This template reflects what 'happened' according to the word problem. Next, the answer may be found by using the open addition algorithm or by transforming the addition number sentence into a subtraction one and subtracting.

Only three templates have been included in the model for purposes of this study. A model of precursors based solely on templates would require an extensive study of the relationship between template recognition and algebra. The three elements of this category are; (1) sensing a subtraction template (comparison), (2) sensing a multiplication template (comparison), and (3) sensing an addition template (put together). See Appendix A - items 29, 38, and 51.

A Model concerning Pedagogical Style of Algebra Teachers

There are a large number of ways teachers' styles can be categorized. For purposes of this study, only two polar styles and a blend of them are identified. These styles are; (1) procedural, (2) blend of procedural and exploratory, and (3) exploratory

Category 1: procedural.

A teacher whose style falls into this category is one who most of the time presents students with a finished package of algorithms that are to be mastered. Interactive questioning or exploration are discouraged. Mathematics is a set of 'things you do'.

Classes conducted in this manner are characterized by whole class teaching and individual seat work. Algorithms do not evolve. They are delivered entire, and followed by repetitive practice. The emphasis is on mechanical processing, not understandings. The teacher may ask questions, but they tend to be rhetorical or at the recall level.

Category 3: exploratory.

A teacher whose style falls into this category stresses understanding by developing principles and concepts in conjunction with processing skills. Inductive and deductive reasoning are encouraged by drawing inferences from explorations and by exploring the consequences of premises. Students may be encouraged to devise and present notions or symbols that may be appropriate to the topic, to use trial and error, to predict and verify, to develop and operate from a set of principles and concepts, to conjecture about relationships, to pursue and develop alternate approaches, and to see external analogies. In short, the teacher and students act in concert in a collaborative model of learning that emphasizes exploration and the creation of knowledge.

Category 2: blend of procedural and exploratory.

This category encompasses the large zone that lies between the two categories, procedural and exploratory. A teacher whose style falls into category 2 exhibits a blend of the characteristics of categories 1 and 3.

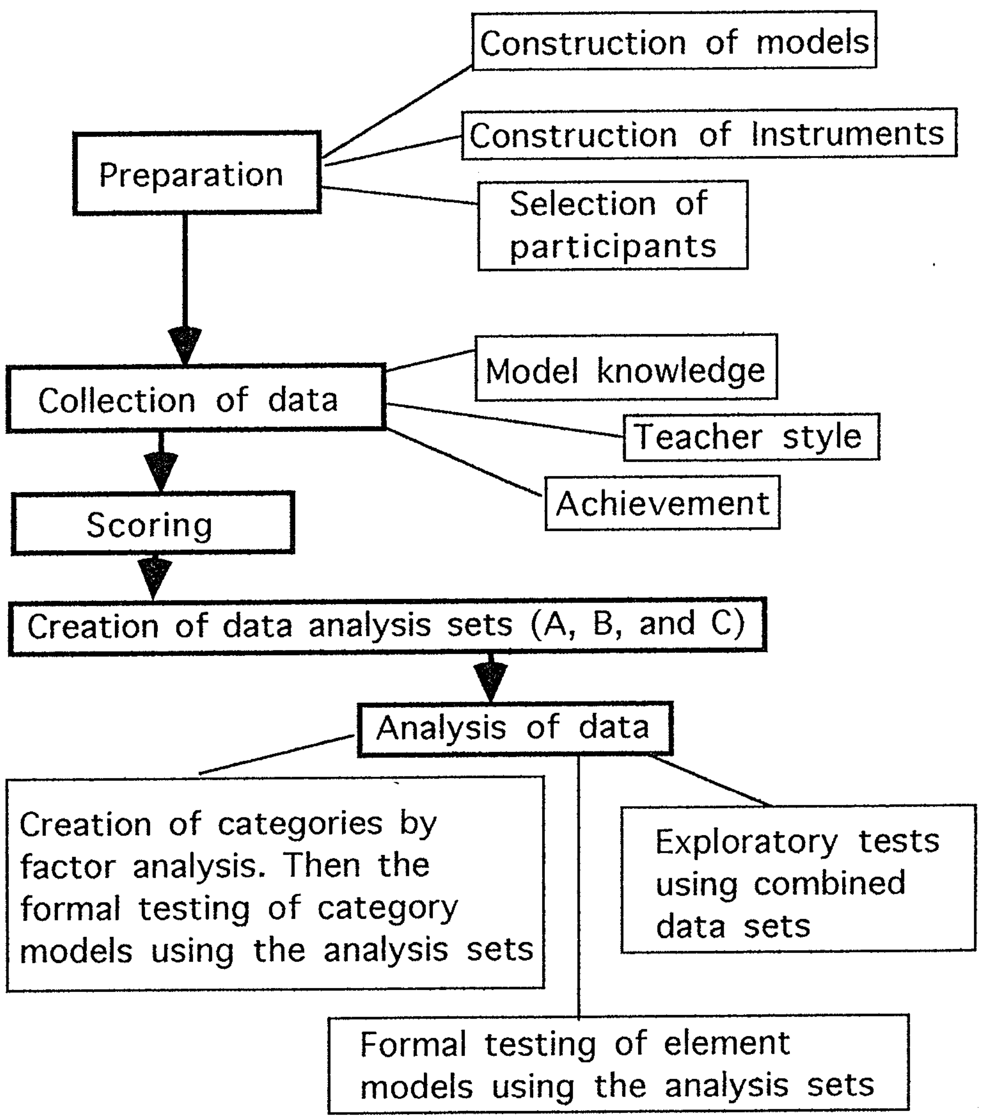
CHAPTER 4

THE METHODOLOGY OF THE STUDY

Overview of the Methodology

The study consisted of five phases. In order, they are; (1) preparation, (2) collection of data, (3) scoring, (4) creation of data analysis sets, and (5) analysis of data.

The following flowchart provides an overview of those phases.



Preparation

There were four stages. They were; (1) construction of a model concerning precursors in algebra, (2) construction of a model concerning pedagogical styles of algebra teachers, (3) construction of instruments, (4) selection of participants.

Construction of a Model concerning Precursors in Algebra

Preliminary thinking on the categories and elements that might be included in the model was stimulated while teaching mathematics to adults over the span of several years prior to 1990 and by further analysis of the literature. The model was constructed in 1990 by logical analysis. The categories and elements of the model are described in chapter 3.

Construction of a Model concerning Pedagogical Styles of Algebra Teachers

After a search of the literature, the model was constructed in 1990.

Construction of Instruments

Four instruments were constructed; (1) model knowledge, (2) achievement in algebra, (3) achievement in algebraic problem solving, and (4) pedagogical style of algebra teacher. Validity was addressed by subjecting the instruments and items to external scrutiny and modifying them accordingly.

The model knowledge instrument.

A model knowledge instrument was constructed that reflects the generic model of suggested precursors. The purpose of this instrument is to measure students' knowledge of the model. Each item of the instrument operationalizes one element (a detailed precursor) of the model. The instrument (Appendix A) consists of 63 multiple choice items with five choices per item including 'I DON'T KNOW'. See Appendix E, Table E-1, for details concerning the relationship between the items and the categories of the model.

To address the content validity of items, the first draft of the instrument was administered to a group of students taking a university level mathematics course. Each item was discussed with that group. A second draft was then administered to sixteen grade nine students to assess readability and item difficulty. A third draft was reviewed by mathematics education specialists to further assess the content validity of the items. A fourth draft was administered to three intact grade nine classes so as to further assess readability, item difficulty, and completion time. A fifth draft was again reviewed by mathematics education specialists. The sixth and final draft was used in the study to measure the model knowledge of students.

The algebra achievement instrument.

The instrument concerning achievement in algebra (Appendix B) consists of items that reflect curricular objectives. For most topics, three items (low difficulty, medium difficulty, and high difficulty) were used to measure knowledge. An independent panel of nineteen teachers scrutinized the instrument to assess content validity, difficulty level, and probable completion time. Modifications were made as necessary.

A comparison with mid-term achievement data on a randomly selected subset of students (n = 109) involved in the study suggests a reasonably close relationship between results on the algebra achievement instrument and those of the teacher-made tests. The correlation between the two was .71.

<u>The algebraic problem solving achievement instrument.</u>

The achievement in algebraic problem solving instrument (Appendix C) consists of items that reflect curricular objectives. An independent panel of nineteen teachers assessed content validity, difficulty level, and probable completion time. Modifications were made as necessary. The sixteen algebra teachers participating in the study provided mid-term achievement data for problem solving on a randomly selected subset of students (n = 109) involved in the study. The correlation between scores on the teacher-made problem solving tests and scores on the algebraic problem solving achievement instrument was .61. It should be noted that in this case the teacher-made tests tapped a broader range of understandings and skills that did this set of items.

<u>The pedagogical style of algebra teacher instrument.</u>

The design of the instrument used to assess the pedagogical styles of algebra teachers (Appendix D) was inferred from a search of the literature. An independent panel of nineteen teachers assessed the instrument for item consistency and content validity. Modifications were made as necessary.

Selection of Participants

The participants in the study were selected from four school divisions. Schools within these divisions were selected so as to maximize variations in socio-economic status and variations of involvement in French Immersion. The principal of each school selected one or two intact grade 9 classes for participation in the study, again so as to maximize diversity in mathematics achievement. The overall effect was to create a diverse sample of students. The sample consisted of 375 students and 16 algebra teachers in eighteen grade nine classes.

Collection of Data

Four sets of data were collected. In order, they were; (1) data on model knowledge, (2) data on teacher style, (3) data on mathematics achievement based on teacher-made tests, and (4) data on achievement in algebra and algebraic problem solving.

Collection of Data on Model Knowledge

Student knowledge of the model was measured over a span of two weeks in October, 1990 while teachers were reviewing arithmetical topics. Students were given 55 minutes to complete the instrument.

Collection of Data on Teacher Style

The pedagogical style of algebra teacher instrument was sent to teachers in February, 1991 (during the school midterm). All of the teachers involved completed the instrument and returned it within two weeks.

Collection of Data on Achievement (teacher-made tests)

Teachers provided data on students' achievement in mathematics based on teacher-made midterm tests. Those data were collected from a random sample of students (n =109) in February, 1991.

Collection of Achievement Data (the study's instruments)

Algebra achievement.

The algebra achievement instrument was administered over a period of three weeks in late May, 1991. Students were allowed 40 minutes to complete it. Students were given a five minute break after completing the instrument.

Algebraic Problem Solving Achievement.

The algebraic problem solving instrument was administered after the algebra achievement instrument. Students were allowed 30 minutes to complete it.

Scoring

There were four components; (1) scoring the model knowledge test, (2) categorizing the pedagogical style of teachers, (3) scoring algebraic problem solving achievement, and (4) scoring algebra achievement.

Scoring the Model Knowledge Test

Each item was scored either '0' (the response was incorrect) or '1' (the response was correct). See Appendix H for the answer key and some discussion of items. A random sample of 20 completed instruments was rescored so as to detect any possible variations in scoring standards (see Appendix F, Table F-1, for details).

Categorizing the Pedagogical Style of Teachers

The instrument of 14 items, including two distractors (items 10 and 14), was used to categorize teacher style. A Likert scale of five response levels was employed. Values were attached to the Likert scale response levels:

Very rarely ->1

Sometimes -> 3

Half the time -> 6

Frequently -> 9,

Almost always -> 11

For purposes of this study, items were classified as positive or negative. Items 1, 4, 5, 7, 8, and 9 are considered to be negative. For those items, negative values were assigned to the Likert scale response levels. Items 2, 3, 6, 11, 12, and 13 are considered to be positive. For them, positive values were assigned to the response levels. The resulting range of values assigned to the responses was from -11 to +11.

A score for the instrument was obtained by summing the integers that were assigned to the 12 items. The resulting scores ranged from -15 to +12. Those scores were used to categorize the teachers into three pedagogical styles. The criterion for inclusion into the exploratory pedagogical style (coded '3') was a score greater than +5. The criterion for inclusion into the procedural pedagogical style (coded '1') was a score less than -5. The criterion for inclusion into the blend of procedural and exploratory pedagogical style (coded '2') was a score in the range -5 to +5 inclusive.

Scoring Algebraic Problem Solving Achievement

Items on the algebraic problem solving achievement instrument were subjectively weighted so as to reflect the complexity of the problems. A score on the problem solving achievement instrument was obtained by summing those point values.

So as to minimize any confounding of algebraic achievement and problem solving achievement, algebraic errors in processing were not penalized. The intent was to isolate the problem solving component of achievement. However, processing errors (.5 points per error) were summed and are referred to as the algebra adjustment. The algebra adjustment was included in the assessment of achievement in algebra.

A class-stratified random sample of 34 completed instruments was rescored so as to discover any possible variations in scoring standards (see Appendix F, Table F-2, for details).

Scoring Algebra Achievement

Items on the algebra achievement instrument were subjectively weighted so as to reflect the complexity of the questions. An initial score on the algebra achievement instrument was obtained by summing achieved point values for items. For purposes of this study, achievement in algebra was taken to comprise both that initial score and the algebraic processing component of the problem solving test. Accordingly, the final score in algebra was obtained by subtracting the algebra adjustment score from the initial score.

A class-stratified random sample of 34 completed instruments was rescored so as to discover any possible variations in scoring standards (see Appendix F, Table F-3, for details).

Creation of Data Analysis Sets

Attrition

The initial sample size was 375. Natural attrition removed 33 students who were absent for the achievement tests. Eleven further students whose data were suspect for a variety of reasons were also removed. The final sample size was 331, a loss of 44 students.

Data Analysis Sets

The formal test of linear regression hypotheses requires at least two separate independent sets of data - an exploratory set in which a regression model can be formulated and a second set for the formal testing of that model. For this study, it proved useful to create a third set of data for purposes of independently testing a further formal hypothesis.

Accordingly, the 331 students were randomly assigned to three data sets (A, B, and C). The assignments were stratified by class. Data set A (n = 126) was to be used to explore and develop regression models (hypotheses) and to assess item consistency. Data set B (n = 105) was reserved for the formal testing of the hypothesis concerning achievement in algebraic problem solving. Once that purpose was served, data set B was also used for exploring and developing regression models concerning algebra achievement. Data set C (n = 100) was reserved for the formal testing of hypotheses concerning algebra achievement.

While the stratified random assignment of students' scores to sets A, B, and C can be taken to establish the equivalence of those sets of scores, two broad indicators were employed to test for that equivalence. They are; (1) the means and standard deviations of scores for model knowledge and achievement and (2) gender composition. The results of those tests are provided in chapter 5.

Although the model knowledge instrument scarcely qualifies as a one factor test, a lower limit of item consistency was estimated by correlating each item with the total score and by calculating a split-half consistency coefficient using odd-numbered items and even-numbered items. For the algebra and algebraic problem solving achievement instruments, a lower limit of item consistency was estimated in the same way. The results of those tests are provided in chapter 5.

Analysis of Data

Overview of the Analysis

There were two components of the analysis of data; (1) the formal testing of hypotheses concerning achievement and (2) the exploration of other hypotheses.

Formal tests.

The formal tests are discussed in this chapter. For this purpose, data sets A, B, and C were used separately and independently for developing regression models and for formal testing.

As is suggested in chapter 1 and is outlined in the following flowchart (see page 63), regression models of the precursors of algebra were developed and tested in two ways; using the elements as predictors and using clusters of elements, here called categories, as predictors. For that purpose, the items of the model knowledge instrument served to operationalize the elements of the generic model of precursors of algebra. Employing two levels of analysis allows for the identification of both general precursors (categories) and specific precursors (elements) and increases the likelihood of identifying important ones.

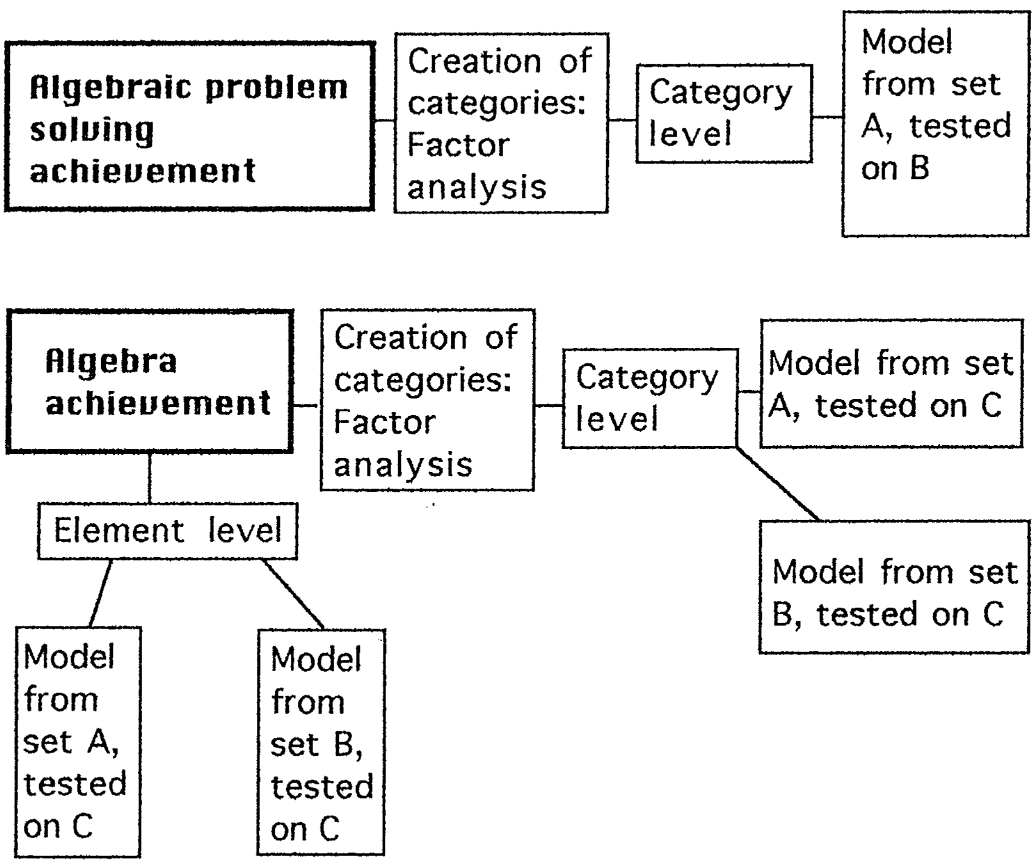
However, only one regression model concerning algebraic problem solving was developed. It was tested in one way, using categories as predictors.

Informal (exploratory) tests.

The details of those tests and the results are discussed in chapter 6. In this component, hypotheses concerning gender, teacher style, and the reasoning dimension of the model were explored. Data sets A, B, and C were combined for that purpose.

The Formal Testing of Hypotheses

The following flowchart provides an overview of the a priori hypotheses tested in this study.



Dependent Variable: Achievement in Algebraic Problem Solving

Overview of the analysis.

The formal hypothesis concerning problem solving was tested only at the category level. No analysis at the element level was done because problem solving was an ancillary concern of this study, and doing so would have called for a partition of α that would attenuate the likelihood of obtaining significant results.

The formal test of the hypothesis ($\alpha = .05$) was conducted on a regression model that consisted of categories derived from a factor analysis, rather than from the initial categories of the generic model of precursors. Abandoning the initial categories resulted in more objectively-determined categories for testing the hypothesis.

The regression model was determined from data set A. The formal test was done using data set B. The hypothesis concerning problem solving was tested first so that data set B would be available for exploring and developing additional regression models concerning algebra achievement.

Creation of the category model and the formal testing of it.

There were three steps involved in determining the regression model. First, a set of possibly useful regressors was obtained by considering the capacity of each item, taken separately, to predict achievement. An item was included in the set if its contribution to R^2 was greater than or equal to .05. Next, the selected set of items were factor analyzed (using principal components) to create categories. The resulting categories were then subjected to forward stepwise regression for purposes of determining the regression model. The criterion for inclusion was set at a level of significance of .1.

The regression model for data set A obtained from the stepwise procedure was formally tested on data set B. The formal test involved three steps. First, the regression equation derived above was used to calculate predicted values of the dependent variable in data set B and then those predicted values were correlated with the actual values. The resulting correlation coefficient was used to calculate an *F*-statistic using the equation (Hays, 1988):

$$F = \frac{\frac{R^2}{K}}{\frac{1 - R^2}{N - K - 1}} \qquad (4\text{-}1)$$

where 'R^2' is the square of that correlation coefficient, 'K' is the number of regressors, and 'N' is the number of achievement scores. The calculated *F*-statistic was compared to the criterion *F*-statistic (α = .05).

Dependent Variable: Achievement in Algebra

Overview of the analysis.

The analysis was done at both element and category levels. Four hypotheses were tested, two at the element level and two at the category level at a family-wise α_{FW} of .20. That family-wise error was partitioned with each test conducted at $\alpha = .05$.

The researcher felt that since this is an exploratory study the identification of any potential precursors of algebra is of sufficient importance to justify the additional risk of obtaining fortuitous results that can arise from adopting a generous family-wise criterion of significance.

Overview of the element level analyses.

The regression model for the test of the formal hypotheses at the element level was generated twice, once using set A and again using set B after it had been used to test the formal hypothesis concerning problem solving.

In each case, the regression model was generated from the individual items and the interactions of items in the model knowledge instrument.

While they were generated independently, both models were tested in set C, mandating the partition of overall significance referred to above. Each regression model was tested at $\alpha = .05$.

Creation of the element models and the formal testing of them.

There were two steps involved in creating each regression model.

First, a set of probably significant regressors was obtained. For that purpose, the capacity of each item and interaction, taken individually, to predict achievement was determined. An item or interaction was included if its contribution to R^2 was greater than or equal to .05. Only interactions generated from items in a category of the generic model were considered.

Second, the resulting set of regressors was subjected to a forward stepwise regression. The criterion for inclusion was set at a level of significance of .1.

Both regression models were formally tested on data set C in the same way. Three steps were involved in the formal test. First, the regression equation derived above was used to calculate predicted values of the dependent variable in data set C and then those predicted values were correlated with the actual values. The resulting correlation coefficient was used in equation 4.1 to calculate an F-statistic. The calculated F-statistic was compared to the criterion F-statistic ($\alpha = .05$).

There is a substantial risk that the two element level regression models may have few or no regressors in common. It is possible that a good number of items may account for about the same variance in algebra achievement and, because of random fluctuations, promote differing sets of them for each model (Neter, Wasserman, & Kutner, 1985). The likelihood of that occurring is further increased by these regressors being dichotomous.

Overview of the category level analyses.

The regression model for the test of the formal hypotheses at the category level was generated twice, once using set A and again using set B after it had been used to test the formal hypothesis concerning problem solving.

In each case, the regression model consisted of categories that were determined by a factor analysis, rather than from the initial categories of the generic model. Abandoning the initial categories resulted in more objectively-determined categories for testing the hypotheses. This approach is consistent with that used for testing the category level hypothesis concerning problem solving.

While they were generated independently, both models were tested in set C, again mandating the partition of overall significance. Each regression model was tested at $\alpha = .05$.

Creation of the category models and the formal testing of them.

There were three steps involved in creating each regression model.

First, a set of probably significant regressors was obtained. For that purpose, the capacity of each item, taken individually, to predict achievement was determined. An item was included if its contribution to R^2 was greater than or equal to .05. Next, the selected set of items were factor analyzed (using principal components) to create categories. The resulting categories were then subjected to forward stepwise regression for purposes of determining the regression model. The criterion for inclusion was set at a level of significance of .1.

Both regression models were formally tested on data set C in the same way. The formal test was the same as that used for testing the element level regression models. The stepwise-derived equation was used to calculate predicted values of the dependent variable in data set C and those predicted values were correlated with the actual values. The resulting correlation coefficient was used in equation 4.1 to calculate an *F*-statistic. The calculated *F*-statistic was compared to the criterion *F*-statistic (α = .05).

CHAPTER 5

RESULTS AND CONCLUSIONS - The Formal Tests

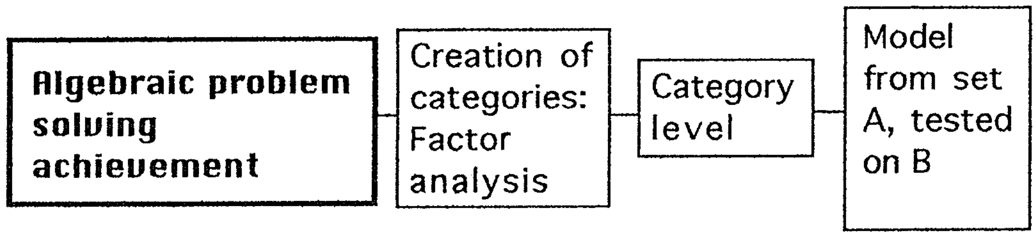
This chapter contains four sets of results and conclusions. They concern; (1) formal tests of a priori hypotheses, (2) informal observations, (3) item consistency of instruments, and (4) equivalency of data sets.

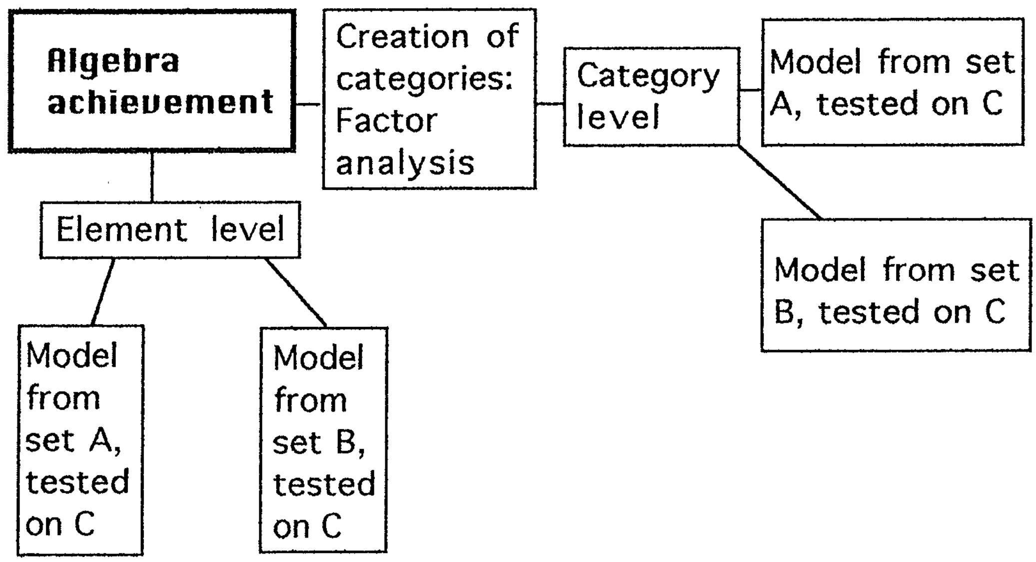
Formal Tests of A Priori Hypotheses

Overview of the Formal Tests

For all formal tests of hypotheses, the initial regression models and the descriptive statistics were determined using the Macintosh statistical software, JMP (SAS,1989). The factor analyses and the stepwise regression analyses were done using the Macintosh statistical software, Statview II (Abacus Concepts,1987).

Two levels of analysis were involved in the formal tests, an element level and a category level. The following chart provides an overview of all formal tests of hypotheses.





Five formal tests were conducted. For those tests, the dependent variable was either algebraic problem solving achievement or algebra achievement. Furthermore, for each formal test in this study the regression model derived from one set of data was tested on a fresh set.

There was one test concerning algebraic problem solving. That category level test was conducted at $\alpha = .05$.

Four tests concerned algebra, two at the element level and two at the category level at a family-wise α_{FW} of .20. That family-wise error was partitioned with each test conducted at $\alpha = .05$.

For purposes of category level analyses, categories are derived from factor analyses (using principal components) and consist of at least one element. Those categories or elements were used as independent variables in regression models with achievement as the dependent variable. For purposes of element level analyses, each element is taken as an independent variable. For both levels of analysis, the optimum predictors of achievement were obtained using stepwise regression.

The element level analyses were done to identify any elements as precursors whose importance may have been overlooked in the category level analyses, to provide fine detail on the broadly defined precusors that emerged from the category level analyses, to provide additional information concerning the importance of identified precursors, and to provide a way of considering the interactions of elements as precursors.

The data allowed for three sets of formal tests of hypotheses. They are; (1) a category level test on achievement in algebraic problem solving, (2) two category level tests on achievement in algebra, and (3) two item level tests on achievement in algebra. Each formal test is based on the correlation between the predicted values of the dependent variable derived from a regression model and the actual values of the dependent variable. Both predicted and actual values are derived from a data set reserved for formal testing.

The rejection of the null hypothesis for a regression model resulted in those regressors, whether categories or elements, being interpreted as likely precursors of algebra. Individual elements were interpreted as single precursors.

There were complications with interpreting the categories as precursors. For purposes of this study, it was desirable to determine a nominal notion for each category that best represents the element(s) that comprise the category. That notion serves as a more broadly defined precursor. To that end, it would have been preferable if the elements comprising a category derived from factor analysis were drawn from one category of the a priori model of precursors.

That did not happen very often. More often, the elements comprising a factor analysis-derived category came from two or more a priori categories.

SInce that was the case, nominal notions of categories were determined by logical analysis, weighted by considering the R^2 contribution of the elements comprising the category.

The following discussion will provide the information concerning the five formal tests, identify the categories from the category level analyses (or the elements from the element level analyses), and provide a likely explanation for what the categories (or elements) mean.

<u>Category Level Test: Achievement in Algebraic Problem Solving</u>

Data set A was used to derive a category level regression model. A factor analysis produced 16 factors. Following a stepwise regression, eight categories remained.

The regression model consisting of those eight categories was formally tested at $\alpha = .05$ using the data of set B (see Table 1).

Table 1

<u>Formal Test: Category Level Regression Model derived from Set A, Dependent Variable - Achievement in Algebraic Problem Solving</u>

Test Parameter	Value
R (predicted / actual)	.568
R^2 (predicted / actual)	.322
K	8
N	105
Degrees of freedom	8 / 96
α	.05
F - criterion	2.05
F - calculated	5.7

The null hypothesis is rejected.

It appears that the categories used in the model may be precursors of algebraic problem solving.

Once the formal test was completed, an informal test was conducted that optimized on chance by allowing the regression model to determine the ß coefficients independently in each of three data sets; set A; set B; and sets A, B, and C combined. The results concerning the contribution to R^2 of the model in each of those sets support the conclusion of the formal test (see Table 2).

Table 2

Contribution to R^2 of the Set A-derived Category Level Regression Model used independently.

Dependent Variable, Achievement in Algebraic Problem Solving

Data set	Set A	Set B	Sets A, B, C combined
R^2 contribution	.456	.403	.366

Discussion of the suggested precursors of problem solving.

Eight categories emerged as suggested precursors of algebraic problem solving. Nominal names for those categories were determined using the approach mentioned above. The resulting precursors in order of categories are; (1) the locking role of arithmetic operators, (2) functional principles of arithmetic, (3) the locking role of arithmetic operators, (4) the alteration of structure, (5) inductive reasoning, (6) inductive reasoning, (7) inverse operations, and (8) deductive reasoning.

The elements/items comprising those eight categories are shown in Table 3. Items in the outline font in Table 3 are common to

Tables 6 and 7. Those tables concern the element organization of the category level models for algebra achievement.

Table 3 also provides the R^2 contributions of the individual categories using data sets A, B, and C combined.

Table 3

Problem solving - Category level Regression Model: Element organization and R^2 contributions of Categories taken individually in the combined sets A, B, and C

Category	Elements comprising the category: items of the model knowledge instrument	R^2 contribution of the category
1	3	.017
2	18, 42	.033
3	32, 9, 27	.083
4	23	.089
5	49, 58	.091
6	16, 50	.120
7	7, 62, 46, 56	.151
8	54, 63	.152

The categories are discussed in order of their contributions to R^2 beginning with the category having the greatest contribution.

The items comprising those categories are provided in Appendix A.

Category 8 consists of items 63 and 54. Item 63 concerns the structural role of symbols, additive operators separate an arithmetic expression into chunks (terms). Item 54 concerns do and undo pairings of operators, addition and subtraction are inverse operations. As item 63 also has a do/undo component in its design, it is reasonable to assume that category 7 concerns inverse operations (do/undo pairings of operators). That notion may play a role when students manipulate equations while solving problems.

Category 7 consists of items 7, 62, 46, and 56. Item 7 concerns the structural role of symbols; multiplicative operators bind chunks (terms) together. Item 62 concerns visual order in relation to computation; multiplication does not have to be done in a left-to-right order. Item 46 concerns deductive reasoning, if p implies q then not q implies not p. Item 56 concerns deductive reasoning, the notion of negation or not. Items 46 and 56 account for 90% of the variance within this category. It is reasonable to assume that category 8 concerns deductive reasoning. It may be that deductive reasoning plays a role when students consider the various relationships that might be involved in a problem.

Category 6 consists of items 16 and 50. Item 16 concerns multiple meanings of symbols; the symbol '-' can indicate the unary operator, opposite of. Item 50 concerns inductive reasoning, sensing a regularity in a number sequence. As item 50 accounts for 83% of the variance within this category, it is reasonable to assume that category 6 concerns inductive reasoning.

Category 5 consists of items 49 and 58. Item 49 concerns binary and unary operators; square root is a unary operator. It is unclear how this notion relates to problem solving as square root is not involved in the achievement test. Item 58 concerns inductive reasoning, sensing a regularity with respect to computational results. Item 58 accounts for about 70% of the variance within this category. It is reasonable to assume that category 5, like category 6, concerns inductive reasoning.

There seem to be two categories which involve inductive reasoning but the reason is not clear. This suggests that inductive reasoning, whether or not it has two components, is an important precursor of algebraic problem solving. As inductive reasoning tends to involve trial and error in making judgments, it may influence students' attempts (or checks) when solving problems. Those students who are competent in inductive reasoning will likely be more proficient with using trial and error as a strategy for problem solving and will likely be more successful at solving problems.

Category 4 consists of item 23. It concerns the alteration of structure; the numerical value of an arithmetic expression is invariant as the structure of the expression changes through computation. The notion may be important when students first consider the composition of the algebraic expression or equation that could represent a problem. It may be that the generation and selection of appropriate expressions is influenced by an understanding that the structure of an expression may change but its value remains constant.

Category 3 consists of items 27, 32, and 9. They appear to be unrelated. Item 27 concerns functional principles, the commutative principle. Item 32 concerns multiple meanings; the symbols '+' and '-' can indicate opposite directions. Item 9 concerns the locking role of operators; square root is a more powerful lock than addition. SInce item 9 accounts for 77% of the variance within this category. It is reasonable to assume that this category principally concerns the locking role of operators in arithmetic, a notion related to the hierarchy of operators. An understanding of that notion may assist students when first considering expressions or equations that represent problems.

Category 2 consists of items 42 and 18. Both items concern functional principles - understanding principles such as the distributive principle in a way that directly supports computation. Item 42 is included in the functional principle category in the research model. Item 18 which concerns do and undo pairings in relation to multiplication and division can be interpreted as a functional principle as well. It is not clear how the notion of a functional principle in arithmetic is related to problem solving in algebra.

Category 1 consists of item 3. It concerns the locking role of operators; when an arithmetic expression contains brackets, it does not necessarily imply that whatever is in the brackets must be done first. The second occurrence of this notion as a precursor supports the conclusion that attention should be paid to students understanding priorities of computation. It may play a role in the way that students organize aspects of problems before they represent them as expressions or equations.

Three of the eight categories, inductive reasoning (occurs twice) and deductive reasoning, are part of the dimension of reasoning in the a priori model. Furthermore, those three categories have relatively large contributions to R^2 . The "not too smart" school of thought concerning difficulties in algebra may interpret that as an indication that intelligence is a factor in algebraic problem solving. Even if it is the case, the results suggest which components of intelligence are relevant. They appear to be teachable.

Categories 1 and 2 have lower contributions to R^2 concerning achievement in algebraic problem solving than do the other categories. It may be that the notions contained in those categories play some part when students translate word problems into appropriate algebraic expressions or equations. If the precise mechanisms by which those notions assist problem solving cannot be found, it would be difficult to incorporate them into a curriculum that pays attention to the precursors of algebraic problem solving.

It is interesting that template recognition, category 18 of the the a priori model, does not appear to be a predictor of algebraic problem solving. This result is unexpected considering that the templates which are included in the model knowledge instrument are well represented in the problem solving achievement instrument.

There are a variety of possible explanations but this provocative result certainly suggests that the role of template recognition in algebraic problem solving requires additional study.

Overview of the Formal Tests concerning Algebra

The researcher felt that since this is an exploratory study the identification of any potential precursors of algebra is of sufficient importance to justify the greater risk of obtaining fortuitous results that may arise from additional tests of hypotheses. Accordingly, four tests were conducted, two at the category level and two at the element level.

Each formal test was done at $\alpha = .05$. The α was obtained by partitioning the family-wise α_{FW} of .20 into four equal parts.

Results based on the "one shot" determination of R^2 invariably optimize on chance. It must be restated therefore that for each formal test in this study the regression model derived from one set of data was tested on a fresh set. For that reason, the results can be safely generalized to the population from which this sample came.

Category Level Tests: Achievement in Algebra

Two formal tests were conducted. They involve regression models for which the regressors are categories. One regression model is derived from data set A, the other from set B. The results are discussed separately.

Regression model derived from data set A.

Data set A was used to derive a category level regression model.

Naturally, the same factor analysis as before is used but since we are now predicting algebra achievement, we obtain different regression models.

A factor analysis produced 18 factors. Following a stepwise regression, nine categories remained.

The regression model consisting of those categories was formally tested at α = .05 using the data of set C (see Table 4).

Table 4

Formal Test: Category level Regression Model derived from Set A, Dependent Variable - Achievement in Algebra

Test Parameter	Value
R (predicted / actual)	.575
R^2 (predicted / actual)	.331
K	9
N	100
Degrees of freedom	9 / 91
α	.05
F - criterion	2.00
F - calculated	5.00

The null hypothesis is rejected.

It appears that the categories used in the model may be precursors of algebra.

Once the formal test was completed, an informal test was conducted that optimized on chance by allowing the regression model to determine the ß coefficients independently in each of three data sets; set A; set C; and sets A, B, and C combined. The results concerning the contribution to R^2 of the model in each of those sets support the conclusion of the formal test (see Table 5).

Table 5

Contribution to $\underline{R}^2$ of the Set A-derived Category Level Regression Model used independently.

Dependent Variable, Achievement in Algebra

Data set	Set A	Set C	Sets A, B, C combined
R^2 contribution	.598	.416	.437

Discussion of the suggested precursors of algebra.

Nominal names for the nine categories that emerged as suggested precursors of algebra were obtained using the approach discussed on page 72 of this chapter. The resulting precursors in order of categories are; (1) functional principles of arithmetic, (2) functional principles of arithmetic, (3) the replacement role of symbols, (4) two interpretations for this category: (a) a rich understanding of the multiple interpretations of '-' and (b) isomorphic reasoning (making analogies), (5) the alteration of structure, (6) the locking role of operators, (7) inductive reasoning, (8) inductive reasoning, and (9) inverse operations.

The elements/items comprising those nine categories are shown in Table 6. Items in the outline font in Table 6 are common to Table 7 which concerns the element organization of the category level model derived from set B for algebra achievement.

Table 6 also provides the R^2 contributions of the individual categories using data sets A, B, and C combined.

Table 6

Algebra - Category level Regression Model derived from data set A: Element organization and R^2 contributions of Categories taken individually in the combined sets A, B, and C

Category	Elements comprising the category: items of the model knowledge instrument	R^2 contribution of the category
1	14	.053
2	18, 42	.060
3	37	.063
4	61, 40, 51, 6	.084
5	11, 23	.091
6	32, 9	.102
7	49, 58	.129
8	50, 16	.136
9	63, 54, 36, 47	.222

The categories are discussed in order of their contributions to R^2 beginning with the category having the greatest contribution.

The items comprising those categories are provided in Appendix A.

Category 9 consists of items 63, 54, 36, 47. Item 63 concerns the structural role of symbols; additive operators separate an arithmetic expression into chunks (terms). However, item 63 also includes a do/undo component. Item 54 concerns do and undo pairings of operators; addition and subtraction are inverse operations. Item 36 concerns the locking role of multiplicative operators, a notion related to the hierarchy of operators. Item 47 concerns the relationship between mathematics and language, expressing mathematical symbols by means of language. Items 54 and 63 account for 78% of the variance within this category with item 54 accounting for 56% of it.

It is reasonable to assume that category 9 concerns the notion of do and undo relationships between operators. It appears that if students understand that notion then their achievement in algebra is likely to be greater. As category 9 accounts for 51% of the R^2 contribution of the regression model, the notion of do and undo concerning operators (inverse operations) seems to be an important precursor of algebra. It may play a role when students solve equations and manipulate expressions.

Category 8 consists of items 50 and 16. Item 50 concerns inductive reasoning, sensing a regularity in a number sequence and it accounts for 84% of the variance within this category. Item 16 concerns the multiple meanings of symbols; the symbol '-' can indicate the unary operator, opposite of. It is reasonable to assume that category 8 concerns inductive reasoning.

Category 7 consists of items 49 and 58. Item 49 concerns binary and unary operators; square root is a unary operator. Item 58 concerns inductive reasoning, sensing a regularity with respect to computational results. Item 58 accounts for 72% of the variance within this category. It is reasonable to assume that category 7 concerns inductive reasoning.

Inductive reasoning is the central notion in categories 7 and 8 of the regression model. This strongly suggests that it is important to learning algebra. Inductive reasoning may play a role when students learn concepts and principles in algebra by relating them to number patterns derived from arithmetic. As well, inductive reasoning tends to involve trial and error. It may be that trial and error strategies may be important to learning algebra. They may play a role when students determine ways to manipulate expressions or to solve equations.

Category 6 consists of items 32 and 9. Item 32 concerns the multiple meanings of symbols; the symbols '+' and '-' can indicate opposite directions. Item 9 concerns the locking role of operators; square root is a more powerful lock than addition. Item 9 accounts for 80% of the variance within this category. It is reasonable to assume that category 6 concerns the locking role of operators in arithmetic, a notion related to the hierarchy of operators. It may play a role when students make decisions concerning the various ways to manipulate an expression.

Category 5 consists of items 23 and 11. Item 23 concerns the alteration of structure; the numerical value of an arithmetic expression is invariant as the structure of the expression changes through computation. It accounts for 65% of the variance within this category. Item 11 concerns the replacement role of symbols;

replacement symbols (variables) may be manipulated. The design of item 11 includes a component that is related to the invariance of value as well. It is reasonable to assume that category 5 concerns the alteration of structure. The notion that structure changes during computation may play a role when students try to make sense of the alternate forms of algebraic expressions that occur as a result of manipulation.

Category 4 consists of items 40, 61, 51, and 6. Item 40 concerns isomorphic reasoning, making analogies between systems of representation. Item 61 concerns functional principles; adding the same number to both parts of a subtraction is a legitimate strategy. Item 51 concerns template recognition, sensing an addition template. Item 6 concerns multiple meanings of the symbol '-', part of the label for a position on the number line. There appear to be at least two ways to interpret this category.

One interpretation of category 4 concerns the many notions that can be attached to the symbol '-'. All four items of category 4 incorporate some notion related to '-'; whether explicitly or implicitly. Item 40 incorporates an analogy concerning direction. Item 61 incorporates subtraction in relation to a functional principle. Item 51 incorporates subtraction in the guise of open addition. Item 6 involves the symbol '-' in relation to part of the label for a position on the number line. That commonalty suggests that a rich understanding of '-' is a precursor of algebra. It may play a role when students manipulate expressions and solve equations.

A second interpretation of category 4 concerns the notion contained in item 40, isomorphic reasoning - making analogies between systems. Item 51 can also be interpreted in that way, in this case, making analogies between language and mathematics. As items 40 and 51 account for 76% of the variance within category 4, it may be reasonable to assume that it concerns isomorphic reasoning. From this point of view, the important notion may be that students are able to transfer concepts and principles of arithmetic to algebra. It implies that teachers teach arithmetic in a way that supports algebra. As discussed in the review of the literature, many students tend to view arithmetic and algebra in incongurent ways. The second interpretation of category 4 suggests that that dissonance must be addressed in order to improve algebra instruction.

Category 3 consists of item 37. Item 37 concerns the replacement role of symbols; the set of replacements for a particular replacement symbol is large. Item 37 can also be interpreted as concerning the notion that replacements can be selected from number sets other than the set of whole numbers, in particular from the set of rational numbers. Both interpretations involve understanding variables as symbols that indicate replacement by numbers. That notion may be important when students try to make sense of the various forms of algebra expressions or when they manipulate expressions.

Category 2 consists of items 42 and 18. The central notion of this category concerns functional principles of arithmetic. Item 42 concerns functional principles. Item 18 which concerns the do and undo relationship between multiplication and division includes as well a component that is related to functional principles. The notion of a functional principle of arithmetic may play a role in algebra when students solve equations or manipulate expressions.

Category 1 consists of item 14. It concerns functional principles, the associative, distributive, and commutative principles. While category 2 also concerns functional principles, item 14 may not have been included because it incorporates a broader range of that notion. The second occurrence of functional principles as the central notion suggests that it may be a more important precursor of algebra than is indicated by the low contributions to R^2 of categories 1 and 2 taken individually.

Regression model derived from data set B.

Since this is an exploratory study, the identification of any potential precursors of algebra is of sufficient importance to warrant an additional test of hypothesis at the category level. For that reason, a second category level regression model concerning algebra was created but it was derived from a different data set than that used for creating the regression model above.

Data set B was used to derive a category level regression model. A factor analysis produced 16 factors. Following a stepwise regression, seven categories remained.

The regression model consisting of those categories was formally tested at α = .05 using the data of set C.

The null hypothesis is rejected (see Appendix G, Table G-1).

It appears that the categories used in the model may be precursors of algebra.

Once the formal test was completed, an informal test was conducted that optimized on chance by allowing the regression model to determine the ß coefficients independently in each of three data sets; set B; set C; and sets A, B, and C combined. The results concerning the contribution to R^2 of the model in each of those sets support the conclusion of the formal test (see Appendix G, Table G-2).

Discussion of the suggested precursors of algebra.

Seven categories emerged from the formal testing as suggested precursors of algebra. Nominal names for those categories were determined using the approach discussed on page 72 of this chapter. The resulting precursors in order of categories are; (1) deductive reasoning, (2) the structural role of symbols, (3) the replacement role of symbols, (4) binary and unary operators, (5) the structural role of symbols, (6) deductive reasoning, and (7) inductive reasoning.

The elements/items comprising those seven categories are shown in Table 7. Items in the outline font in Table 7 are common to Table 6 which concerns the element organization of the category level model derived from set A for algebra achievement.

Table 7 also provides the R^2 contributions of the individual categories using data sets A, B, and C combined.

Table 7

Algebra - Category level Regression Model derived from data set B: Element organization and R^2 contributions of Categories taken individually in the combined sets A. B, and C

Category	Elements comprising the category: items of the model knowledge instrument	R^2 contribution of the category
1	59	.010
2	55	.047
3	51, 11	.067
4	49, 40	.101
5	41, 7, 63	.112
6	8, 56	.138
7	36, 42, 50	.237

Naturally, it is preferable that the precursors of algebra obtained from the test of a second hypothesis would be the same as those obtained from the first test. That was the case to some extent.

There seem to be common precursors identified in the two category level regression models concerning algebra. The confirmation of precursors in some instances is direct and in other instances it is indirect.

There are two instances of the direct confirmation of precursors.

First, inductive reasoning has been identified in both models as a precusor in both models. It accounts for about the same variance in the models when the double occurrence of inductive reasoning in the first model is taken into consideration. The sum of the R^2 contribution for each occurrence provides an upper estimate of the capacity of inductive reasoning to predict algebra achievement in the first regression model. That sum compares favourably with the R^2 associated with inductive reasoning in the second model (.129 + .136 compared to .237).

Second, the replacement role of symbols has been identified as a precursor in both models. That notion accounts for about the same variance in algebra achievement in the models (.063 compared to .047).

There is one instance of the indirect confirmation of precursors.

For the first regression model, categories 5 and 6, the alteration of structure and the locking role of operators, have a common underlying notion - the intuitive understanding of the structure of arithmetic expressions. That notion is reflected as well in categories 2 and 5 of the second model which both concern the structural role of symbols. Further, using the sum of the R^2

contributions as an upper estimate, the variance accounted for by categories 5 and 6 of the first regression model is about the same as that accounted for by categories 2 and 5 of the second model (.091 + .102 compared to .047 + .112).

The second regression model provided additional insight concerning precursors of algebra.

The ability to reason deductively was identified twice as a precusor (categories 1 and 6). As an upper estimate, it accounts for 14.8% of the variance in algebra achievement (.010 + .138). It is not clear how deductive reasoning is important to learning algebra. One explanation concerns intelligence. Deductive reasoning may be perceived as a factor of intelligence and therefore in some way intelligence may be important to learning algebra. Another explanation concerns using deductive reasoning to determine the best course of action for an algebraic purpose. The latter explanation seems to have the most promise for teaching purposes.

The ability to discriminate between unary and binary operators was identified as well as a precursor. It accounts for 10.1% of the variance in achievement. That ability may play a role when students manipulate expressions.

Four of the nine categories of the first category level regression model concerning algebra are found in the second regression model. That result strongly suggests that being aware of structure, understanding the replacement role of symbols, and being able to sense patterns are precursors of algebra.

The models also contain categories that are not common to both. However, this result should not be taken as evidence that those categories are not likely precursors. The viewpoint of this study is that the incongruence of findings suggests precursors that may have been missed in either of the models.

A summary of the category level analyses concerning algebra.

The two category level analyses have identified 11 central notions as important to learning algebra. They may be placed into four clusters.

One cluster of notions concerns reasoning skills. Inductive reasoning, deductive reasoning, and isomorphic reasoning seem to be important predictors of achievement in algebra. Those notions suggest that detecting patterns, using trial and error strategies, reasoning logically to determine the best course of action for an algebraic purpose, and making analogies between systems of thought are precursors of algebra. Furthermore, these abilities must be developed in arithmetic curricula.

A second cluster of notions concerns arithmetic operators. They suggest that an understanding of the inverse relationship between certain arithmetic operations, an understanding of the binary or unary nature of arithmetic operations, and an understanding of arithmetic principles as functional principles for doing computation are precursors of algebra.

A third cluster of notions concerns symbols. They suggest that an understanding of variables as replacement indicators and a rich understanding of the symbol '-' are precursors of algebra.

A fourth cluster of notions is related to the hierarchy for computation. They suggest that the abilities to detect terms, to do appropriate computations in complex arithmetic expressions, and to recognize that the structure of an arithmetic expression changes during computation are precursors of algebra.

There is considerable overlap between the suggested precursors of algebra and the suggested precursors of algebraic problem solving. All of the seven notions that have been identified as precursors of algebraic problem solving have also been identified as precursors of algebra. This suggests that arithmetic curricula that incorporate the suggested precursors of algebra will likely have a positive effect on problem solving achievement as well.

Seven categories of the a priori model of precursors were not identified as possible precursors of algebra.

They were; (1) representation (mathematical symbols are arbitrary creations), (2) passive interpretation (two equivalent arithmetical expressions are another name for each other), (3) context independence (arithmetical statements are independent of the context from which they are derived), (4) unit attachment (the various ways units can be attached to the numbers involved in arithmetic operations), (5) visual order (the order of occurrence does not determine the order of processing), (6) the relationship between language and mathematics, and (7) template recognition (attaching arithmetic templates to word problems).

The lack of the identification of those seven categories as precursors in this study suggests that they do not play a role in learning algebra. However, for purposes of this study achievement was measured according to proficiency in the performance of largely automatic processes (algorithms and problem types). It may be that if achievement were measured according to proficiency in the performance of non-automatic processes (such as creating alternate methods or evaluating methods) then the above seven categories may turn out to be likely precursors of algebra as well.

Overview of the Element Level Tests concerning Algebra

Since this is an exploratory study, the identification of any potential precursors of algebra is of sufficient importance to justify the greater risk of obtaining fortuitous results that may arise from additional formal tests of hypotheses. Accordingly, two element level tests concerning algebra were conducted.

For those tests, each regression model was derived from a different set of data. It was tested on the same fresh set mandating the partition of the family-wise α_{FW} of .20 into four equal parts, two for the category level tests and two for the element level tests.

The element level analyses were done to identify any elements as precursors whose importance may have been overlooked in the category level analyses, to provide fine detail on the broadly defined precusors that emerged from the category level analyses, to provide additional information concerning the importance of identified precursors, and to provide a way of considering the interactions of elements as precursors.

Element Level Tests: Achievement in Algebra

Two formal tests were conducted. They involve regression models for which the regressors are elements or interactions of elements. One regression model is derived from data set A, the other from set B. The results are discussed separately.

Regression model derived from data set A.

Data set A was used to derive a regression model consisting of individual elements and interactions of elements. Following a stepwise regression, seven elements/interactions remained.

The regression model consisting of those elements/interactions was formally tested at $\alpha = .05$ using the data of set C (see Table 8).

Table 8

Formal Test: Element Level Regression Model derived from Set A, Dependent Variable - Achievement in Algebra

Test Parameter	Value
R (predicted / actual)	.497
R^2 (predicted / actual)	.247
K	7
N	100
Degrees of freedom	7 / 92
α	.05
F - criterion	2.06
F - calculated	4.31

The null hypothesis is rejected.

It appears that the elements/interactions used in the model may be precursors of algebra.

Once the formal test was completed, an informal test was conducted that optimized on chance by allowing the regression model to determine the ß coefficients independently in each of three data sets; set A; set C; and sets A, B, and C combined. The results concerning the contribution to R^2 of the model in each of those sets support the conclusion of the formal test (see Table 9).

Table 9

Contribution to R^2 of the Set A-derived Element Level Regression Model used independently.

Dependent Variable, Achievement in Algebra

Data set	Set A	Set C	Sets A, B, C combined
R^2 contribution	.56	.294	.363

Discussion of the suggested precursors of algebra.

Seven elements/interactions emerged from the formal testing as suggested precursors of algebra. They suggest specific precursors in contrast to the more broadly defined ones that emerged from the category level analyses.

The R^2 contributions of the individual elements and interactions of elements using data sets A, B, and C combined is provided in Table 10. All the items in Table 10 also appear in the category level analyses concerning algebra.

Table 10

R^2 contributions of Elements/interactions taken individually in the combined sets A, B, and C

Element/interaction	R^2 contribution
18 x 54	.034
16	.051
23	.060
36	.082
9	.082
58	.094
50	.123

The elements and interactions of elements are discussed in order of their contributions to R^2 beginning with the one having the greatest contribution.

The items are provided in Appendix A.

Item 50 concerns sensing a regularity in a number sequence. It suggests that the ability to detect patterns in number sequences and to continue them is a precursor of algebra.

Item 58 concerns sensing a regularity with respect to computational results. It involves more complex detection of regularities than does item 50. Item 58 suggests that the ability to detect a pattern in a complex computational context is a precursor of algebra.

The skills suggested as precursors by items 50 and 58 may play a role in learning algebraic procedures that are derived from arithmetic patterns or that involve algebraic patterning in their justification or development.

Items 50 and 58 were identified in the category level analyses as pertaining to inductive reasoning. From the perspective of a category level analysis, items 50 and 58 are subsumed under that broadly defined notion. Their inclusion suggests only that a general ability to detect patterns might be important to learning algebra. From the perspective of an element level analysis, items 50 and 58 suggest specific kinds of learning experiences concerning pattern detection that students should encounter in arithmetic curricula.

Item 9 concerns the notion that square root is a more powerful lock than addition. It suggests that the ability to discriminate between the computational priorities of square root and addition is a precursor. Item 9 may also suggest that the understanding of square root itself is a precursor. It is likely that both possiblilites are appropriate to learning algebra. They may play a role in justifying and selecting appropriate algebraic processing strategies.

Item 9 was included in a category level analysis as pertaining to the locking role of operators.

Item 36 concerns the locking role of multiplicative operators. It suggests that the ability to discriminate between the computational priorities of addition and mutiplication or of multiplication and squaring or of addition and squaring is important to learning algebra. It is not clear which of the above is the precursor. It may be that all of them are. They may play a role in legitimizing or deciding algebraic processing strategies.

Item 36 was identified in both category level analyses, but was not sufficiently significant to play a role in defining the categories. It may be that the importance of item 36 was overlooked. Its inclusion in this element level analysis suggests that achievement in algebra is related to understanding a particular processing hierarchy concerning multiplicative and additive operators.

Item 23 concerns the notion that the numerical value of an arithmetic expression is invariant as the structure of the expression changes with computation. It suggests that the ability to recognize that computational steps, while altering the appearance of arithmetic expressions, do not change the values of expressions.

Item 23 was included in a category level analysis as pertaining to the alteration of structure of an arithmetic expression.

Item 16 concerns the notion that the symbol '-' can indicate the unary operator, opposite of. It suggests that the ability to interpret '-' as a unary operator is useful to learning algebra.

Item 16 was identified in a category level analysis, but was not instrumental in defining the nominal notion of a category. It may be that its importance as a precursor of algebra was overlooked.

The interaction of items 18 and 54 provides additional insight. Item 18 concerns the inverse relationship between multiplication and division. Item 54 concerns the inverse relationship between addition and subtraction. The identification of the interaction as a significant precursor suggests that a set of understandings concerning inverse operations may be important to learning algebra. It may be that students benefit more from understanding both of those inverse relationships than from either alone.

Both items 54 and 18 were identified in the category level analyses. Item 54 pertained to inverse operations while item 18 was subsumed under functional principles.

Regression model derived from data set B.

Since this is an exploratory study, the identification of any potential precursors of algebra is of sufficient importance to warrant an additional test of hypothesis at the element level. For that reason, a second element level regression model concerning algebra was created but it was derived from a different data set than that used for creating the regression model above.

Data set B was used to derive a regression model consisting of elements and interactions of elements. Following a stepwise regression, eight elements/interactions remained.

The regression model consisting of those elements/interactions was formally tested at $\alpha = .05$ using the data of set C.

The null hypothesis is rejected (see Appendix G, Table G-3).

It appears that the elements/interactions used in the model may be precursors of algebra.

Once the formal test was completed, an informal test was conducted that optimized on chance by allowing the regression model to determine the ß coefficients independently in each of three data sets; set B; set C; and sets A, B, and C combined. The results concerning the contribution to R^2 of the model in each of those sets support the conclusion of the formal test (see Appendix G, Table G-4).

Some caution is warranted concerning the importance of the regressors derived from data set B. The family-wise error ($\alpha \leq .20$) for the formal tests concerning achievement in algebra suggests that there is more than a minimal chance that at least one of the formal tests will involve an incorrect rejection of the null hypothesis. In the case of the regression model derived from data set B, the closeness in values of the calculated F - statistic and the criterion F - statistic (2.09 compared to 2.07) suggests that the rejection of the null hypothesis may be suspect.

Discussion of the suggested precursors of algebra.

Eight elements/interactions emerged from the formal testing as suggested precursors of algebra. Naturally, it is preferable that the likely precursors of algebra obtained from the test of a second hypothesis would be the same as those obtained from the first test. Those results were not obtained.

The first and second element level regression models do not have any elements or interactions of elements in common, with one exception. Items 50 and 58 which appear as an interaction in the second element level regression model appear as single elements in the first element level model.

As suggested earlier, the lack of common regressors in these two analyses is most likely that many of the items in the model knowledge instrument likely accounted for about the same variance in algebra achievement and, because of random fluctuations in those items, the selected set of regressors for each model differed. That likelihood is increased by the dichotomous scoring of items.

The R^2 contributions of the individual elements and interactions of elements using the data sets A, B, and C combined is provided in Table 11. Highlighted items in Table 11 also appear in the category level analyses concerning algebra.

Table 11

R^2 contributions of Elements/interactions taken individually in the combined sets A, B, and C

Element/interaction	R^2 contribution
13	.024
22 x 31	.025
8	.035
49	.065
11 x 37	.075
50 x 58	.094
55 x 63	.108
56	.125

The elements and interactions of elements are discussed in order of their contributions to R^2 beginning with the one having the highest contribution.

The items are provided in Appendix A.

Item 56 concerns deductive reasoning in relation to negation or the notion of not. It suggests that the ability to determine the negation of a proposition is important to learning algebra. As negation is related to the notion of negative or opposite, it may play a role when students manipulate expressions involving negative numbers and subtraction.

Item 56 was identified in a category level analysis as pertaining to deductive reasoning.

The interaction of 55 and 63 provides additional insight. Both items fall into the same category in the a priori model - the structural role of symbols, a role related to the hierarchy of operators. Item 55 concerns the notion that structure helps determine what is to be done in an expression. Item 63 concerns the notion that additive operators separate an arithmetic expression into chunks (terms). The identification of the interaction as a significant precursor suggests that the identification of terms or structure is important to the successful performance of algebraic procedures.

Both items were identified in category level analyses as pertaining to the structural role of symbols. However, they were included in different categories.

The identification of the interaction of items 50 and 58 as a significant precursor in this regression model and their identification as individual precursors in the first model supports the importance of inductive reasoning for learning algebra. The interaction suggests that the ability to detect and continue a pattern in both simple and complex arithmetic contexts is a precursor of algebra.

The interaction of items 11 and 37 provides additional insight. Both items fall into the same category in the research model - the replacement role of symbols. Item 11 concerns the notion that replacement symbols can be manipulated. Item 37 concerns the notion that the set of replacements for a particular replacement symbol is large. The identification of the interaction as a significant precursor suggests that achievement in algebra is related to the understanding that the largeness of the replacement set applies as well when replacement symbols are manipulated.

Both items were identified in category level analyses as pertaining to the replacement role of symbols. However, they were included in different categories.

Item 49 concerns the notion that square root is a unary operator. It suggests that the knowledge that square root is a unary operator is a precursor of algebra. That knowledge may play a role in processing expressions involving square root.

Item 49 was identified in both category level analyses. It was significant in one category as pertaining to binary and unary operators; it was not significant in the other category.

Item 8 concerns the notion that the complexity of computation is related to the complexity of the structure of an expression. It suggests that the ability to relate processing complexity to structural complexity is important to learning algebra. That ability may play a role in determining the strategies and procedures for processing algebraic expressions.

Item 8 was identified in a category level analysis but it was not instrumental in determining the nominal notion. It may be that its importance as a precursor of algebra was overlooked.

The interaction of items 22 and 31 provides additional insight. they both fall into the category, the relationship between mathematics and language in the a priori model. Item 22 concerns recognizing analogous roles concerning replacement. Item 31 concerns expressing arithmetic operations by means of language. The identification of the interaction as a precursor suggests that a more comprehensive set of understandings concerning the relationship between mathematics and language must be in place in order to facilitate the learning of algebra.

Item 13 concerns the notion that addition involves adding counts of objects. It suggests that the understanding that counts are added, not objects is important to learning algebra. That understanding may play a role in simplifying expressions such as '2x + 3x' to 5x'. Success at simplifying such expressions may depend on students' knowledge of the justification for such simplifications, one of which can be based on the notion that addition involves adding counts of objects.

Item 13 was not identified in a category level analysis. It may be that its significance as a precursor of algebra was overlooked.

<u>A summary of the element level analyses concerning algebra.</u>

The two element level regression models do not have elements or interactions of elements in common. Again, that result may be expected considering that many items of the model knowledge instrument likely account for about the same variance in algebra achievement and, because of random fluctuations in those items, the selected set of regressors for each model differed.

The lack of common regressors in the two models generated suggests that neither element level analysis is to be taken as definitive, but the position taken in this study is that all elements identified by the two element level tests should be considered to be potential precursors of algebra subject to the constraints imposed by the family-wise error.

That position may be related to the possibility that the model knowledge instrument has an underlying common factor. If that is the case, random fluctuations in what are likely equivalent dichotomous variables (the items of model knowledge instrument) can result in the creation of different element level regression models. The high split-half consistency coefficient for the model knowledge instrument (.61) discussed on page 118 of this chapter also suggests that the instrument may tap an underlying general factor. A possible nominal name for that factor might be mathematical ability. Again, the possibilty of such a factor being present is not investigated in this study.

It is important to reiterate the purposes for employing the element level analyses in this study. They were done to identify any elements as precursors whose importance may have been overlooked in the category level analyses, to provide fine detail on the broadly defined precusors that emerged from the category level analyses, to provide additional information concerning the importance of identified precursors, and to provide a way of considering the interactions of elements as precursors. The discussion contained in the element level analyses suggests that those purposes appear to have been fulfilled.

Summary concerning Formal Tests of A Priori Hypotheses

The results of the study suggest that the a priori model contains elements and categories that may be precursors of algebra. That conclusion is strengthened by some of the characteristics of the participants in the study. Participants were selected over a range of socio-economic and school-related classifications. The resulting diverse nature of the population for the study tends to broaden the zone of generalization of conclusions.

The regression models obtained in this study (both category and element levels) seem to be strong predictors of algebra achievement. Reasonable estimates of their capacities to predict achievement can be determined using the contributions to R^2 obtained from the informal tests that used the data of all three data sets combined (see pages 82, 90, 98, 104). Those estimates are impressive. For the two category level models, the average contribution to R^2 is .434 (see Table 5 and Appendix G, Table G-2). For the two element level models, the average contribution to R^2 is .351 (see Table 9 and Appendix G, Table G-4). These results strongly suggest that the categories and elements/interactions that were identified in this study are likely and important precursors of algebra.

The two category level analyses identified 11 central notions that are important to learning algebra. The two element level analyses identified ten elements and five interactions that are important. Those elements come from 11 categories of the a priori model concerning precursors in algebra.

There is extensive overlap of elements/items in both levels of analyses and that supports the conclusion that the general or specific notions identified in this study may be precursors of algebra.

Furthermore, the results from the element and category levels of analyses also suggest that the a priori model may be a comprehensive vehicle for investigating the notions and skills that may be important to learning algebra.

The results from the category level analyses and the element level analyses may be placed into four clusters.

One cluster of notions concerns the ability to reason deductively and inductively, and to draw analogies. A second cluster of notions concerns the relationship between and the types of arithmetic operators, and the functional application to computation of principles of arithmetic. A third cluster concerns the meanings and roles that can be attached to symbols. A fourth cluster concerns the hierarchy for computation and the structure of expressions.

The literature strongly supports one of the findings of this study - the fourth cluster of precursors that concern hierarchy and structure. Kieran (1992) comments that the general conclusion that emerges from the research on algebra learning is that many students do not acquire a sense of the structural aspects of algebra and that lack appears to be a major source of difficulty for them in learning algebra. The findings of this study concerning that suggest that the remedy may involve sensing the structural aspects of arithmetic as a precursor of algebra.

Implications of the Findings for Mathematics Instruction

The items of the model knowledge instrument that are the important predictors of algebraic achievement identified in this study could be used to guide students' placements in mathematics courses. However, this study is more concerned with the implications of the findings for curriculum and instruction in mathematics. Given the exploratory nature of this study, it is reasonable to speculate about those implications.

The findings may have major implications for the curricula and instructional practices of the grades prior to those in which algebra first becomes a significant curricular topic. Generally, those grades can be partitioned into the elementary grades and grades 7 and 8. The implications for instruction may be about the same for both, varying according to the specifics of curricular topics.

The findings concerning reasoning may imply that a major emphasis of instruction should be hypothesizing and validating mathematical ideas concerning arithmetic (natural number or integer). Further, it may be that problem contexts should be used to establish the rationales for learning necessary concepts and algorithms. In both, students may benefit from opportunities to observe mathematical patterns and to conjecture and generalize about them; to relate suppositions and conclusions; to explore possibilities; and to use mathematics for modelling real events and circumstances and to apply those models to other contexts.

The findings concerning operators, principles, symbols, and structure may imply that learning mathematics should be similar to learning a natural language, in a holistic way, with its symbols, rules, and structure. It may be that mathematical instruction concerning arithmetic should involve explicit and frequent discussions of the creation of and relationships between arithmetical concepts, symbols, operators, principles, and structure. In doing so, students may be less likely to see those notions as unrelated objects to be memorized, and be less likely to see arithmetic and later algebra as two closed and separate systems. Rather, they may begin to see mathematics (in this case arithmetic) as a language and may be better able to use it to model situations, to express ideas, and to formulate arguments, abilities that may better prepare them for algebra.

As discussed earlier, the literature strongly supports the finding of this study concerning structure. It may be useful to speculate further on the implications of that finding for curriculum and instruction . Again, for purposes of this study, structure concerns "chunking" arithmetic expressions and the locking role of chunks. Those notions are intimately related to the hierarchy of arithmetic operators and to the binary and unary distinctions of operators.

The Manitoba mathematics curriculum guide for grades 7 to 9 (Manitoba Education, 1979) is pertinent to that speculation. In it, there is minimal discussion (as order of operations) concerning the hierarchy of arithmetic operations. On the other hand, there is extensive discussion of proficiency in computational algorithms. That bias towards algorithmic performance does not reflect the finding of this study that understanding structure is an important precursor of algebra.

Furthermore, the pedagogical development of the order of operations largely concerns presents students with unconnected examples and arbitrary rules. Often, students end up memorizing the acronym 'BDMAS' (brackets, division, multiplication, addition, subtraction) and use it as the protocol for determining what to do when confronted with the simplification of expressions. In contrast, the finding in this study suggests that a deep rather than a superficial understanding of the hierarchy of operations (order of operations) is important to learning algebra.

It seems then that at least one implication of the finding concerning structure is that, before students begin to learn algebra, they should be provided with substantial and well-connected arithmetical experiences that develop deep understandings and extensive computational abilities concerning the hierarchy of arithmetic operations. To that end, it seems that some of the core objectives of the grade 7 and 8 mathematics curriculum might be reformulated and that more appropriate pedagogical strategies for teaching those objectives might be developed.

The above implications, both general and specific, of this study's findings for mathematics (largely arithmetic) instruction run counter to much of current practice. For most students, reasoning about situations and seeking ways to validate that thinking is seldom associated with learning mathematics (Lappan and Schram, 1989). Conventional instruction seems to encourage the belief that mathematics consists of getting answers to computational tasks obtained from text books or work sheets. Furthermore, students seem to acquire the belief that there is only one way to get those answers.

There is a considerable gap between much of current mathematics instruction and the kind of instruction that may be implied by the precursors of algebra identified in this study. It is clear that any changes in mathematics curricula and instructional practices, if warranted, cannot be implemented in haste nor without considering the attitudes and skills of the teachers presently teaching arithmetic. However, Faculties of Education can begin to promote, in the training and educating of pre-service teachers, skills and understandings that may support and encourage any such changes.

Limitations of the Study

Two limitations of the study should be addressed.

The first limitation concerns the limited way in which the elements sample the areas of concern. Since each such area is sampled by only one item in the model knowledge instrument, it is possible that some areas of concern are not well represented in that instrument.

There are at least two ways to address that limitation. First, the notions that have been identified here as being important to the learning of algebra could form the basis for constructing new instruments that tap the important categories more thoroughly. Second, the tentative conclusions from the study could be examined in field studies that investigate more directly the importance of the identified notions in settings where algebra is being learned.

A second limitation concerns the measurement of achievement. The algebra instrument reflects current practice. That is to say, it measures skills that are most often ritual manipulations. Students have practiced types of questions while learning algebra and are then expected to replicate them on an achievement instrument. Only 7% of the algebra achievement instrument measures abilities that can be clearly identified to be non-ritual manipulations. The problem solving instrument measures fewer ritual skills, but much of it can also be seen as reflecting the replication of question types.

It is slightly surprising that elements and categories that tap understanding, as in this study, should so effectively predict performances with such a large ritual component. It can be supposed that they would be even more effective in predicting performances that have a substantial understanding component.

The second limitation concerns future algebra curricula more than it concerns current curricula. Societal pressure and recent assessment results will likely bring about changes in mathematics instruction. The current curricula that largely focus on automatic processes may give way to curricula that focus on reflective as well as automatic processes (Hiebert, 1990). This study has focussed on achievement largely in relation to the replication of automatic processes (that reflect current practice). A suggested list of precursors of algebra has emerged. But that list may not sufficiently address the needs of curricula that focus on reflective processes in which understanding and meaning tend to be more the themes of instruction.

One way to provide a sufficient set of precursors for those curricula is to investigate the relationship between the a priori model of precursors and achievement in algebra using other tests of achievement. For that purpose, achievement might be measured in a way that reflects students' understandings of the justifications and applications of automatic processes. The researcher suspects that if achievement were to be measured in that way a larger number of the elements and categories of the a priori model of this study would emerge as precursors of algebra for curricula that focus on reflective processes.

Informal Observations

Overview of Scores on the Instruments

Some general observations concerning the scores on the model knowledge and achievement instruments are useful as they allow for some informal conclusions. Table 12 provides information on the model knowledge and achievement scores obtained from the 331 students participating in the study.

Table 12

Mean Raw and Percent Scores, and Standard Deviation for all Data

Statistic	Mean score	Standard deviation	Maximum score	Minimum score
Model knowledge	25.11 (39.9%)	7.45 (11.8%)	46 (73%)	6 (9.5%)
Achievement in algebra	36.90 (48.5%)	16.04 (21.1%)	67.5 (88.8%)	3 (3.9%)
Achievement in algebraic problem solving	22.89 (46.7%)	11.00 (22.5%)	43.5 (88.8%)	.5 (1%)

The model knowledge scores and the achievement scores (see Table 12) are all sufficiently distributed in the mid-range of possible scores to encourage their use in statistical tests.

Informal Comparison of Model Knowledge and Achievement

Once the a priori tests were done, model knowledge was compared with algebra and problem solving achievement in an informal way. That comparison was done in two ways; (1) using correlations between scores on the three instruments and (2) using all the individual items of the model knowledge instrument to predict achievement. The data from all 331 students was used for both purposes.

Correlations between students' scores on the three instruments were determined. Table 13 provides that information.

Table 13

Correlations (and R^2) between Total Scores on Instruments

Instrument	Model knowledge	Algebra	Problem solving
Model knowledge	1		
Algebra	.62 (.38)	1	
Problem solving	.59 (.35)	.67 (.45)	1

The high correlations between scores on the model knowledge instrument and scores on the achievement instruments (see Table 13) suggest that there is a relationship between knowledge of the research model, and achievement in algebra and algebraic problem solving.

Second, all of the individual items of the model knowledge instrument were used as predictors of achievement in two regression models, one for achievement in algebra and the other for achievement in problem solving. Table 14 provides information on the R^2 contribution of those items in both regression models.

Table 14

<u>R^2 contribution of all Items in the Model Knowledge Instrument with Achievement as the Independent Variable</u>

	Achievement in algebra	Achievement in Problem solving
R^2	.58	.51

The R^2 contribution of the items of the model knowledge instrument taken individually is considerably higher than the R^2 contribution of the total scores on the model knowledge instrument and scores on the achievement instruments. This higher R^2 provides an upper estimate of the capacity of the items to predict achievement.

The strength of the relationship between model knowledge and algebra achievement was expected, and suggests that the notions measured in the model may be important to learning algebra. The strength of the relationship between model knowledge and achievement in problem solving is higher than expected, as the model was constructed so as to identify a possible set of precursors of algebra, not algebraic problem solving.

Item Consistency of Instruments

Data set A (n = 126) was used to estimate the item consistencies of instruments. Tables F-4, F-5, and F-6 (see Appendix F) provide information on item correlations and on split-half consistency coefficients.

The split-half consistency coefficient for the model knowledge instrument is .61. Although that instrument does not qualify as a one factor test, the high split-half coefficient suggests that the model knowledge instrument has an underlying common factor. That possibility was not investigated in this study.

Equivalence of Data Sets

An informal determination of the equivalence of the three data sets, A, B, and C, used for creating and testing the regression models helps strengthen the results and conclusions that emerged from the formal tests of hypotheses. Two broad indicators were employed for that purpose; (1) the means and standard deviations of scores for model knowledge and achievement and (2) gender composition.

Tables 15, 16, and 17 provide information on the mean scores and standard deviations for the model knowledge and achievement instruments.

Table 15

Mean and Standard Deviation of Raw Scores on the Model Knowledge Instrument

Data set	Set A	Set B	Set C
Mean score	25.28	24.90	25.11
Standard deviation	7.23	7.81	7.39

Table 16

Mean and Standard Deviation of Raw Scores on the Achievement in Algebra Instrument

Data set	Set A	Set B	Set C
Mean score	36.46	36.49	37.88
Standard deviation	16.58	15.96	15.54

Table 17

Mean and Standard Deviation of Raw Scores on the Achievement in Algebraic Problem Solving Instrument

Data set	Set A	Set B	Set C
Mean score	23.04	22.60	23.02
Standard deviation	11.10	11.44	10.50

Table 18 provides information concerning the composition by gender of the data sets. The results suggest that females may have a slight advantage in number across the three sets.

Table 18

Composition of Data Sets by Gender

Set		Set A	Set B	Set C	Total
Male	n	61	50	47	158
	n%	(48.4%)	(47.6%)	(47%)	(47.7%)
Female	n	65	55	53	173
	n%	(51.5%)	(52.3%)	(53%)	(52.3%)
Total	n	126	105	100	331

CHAPTER 6

EXPLORATORY TESTS

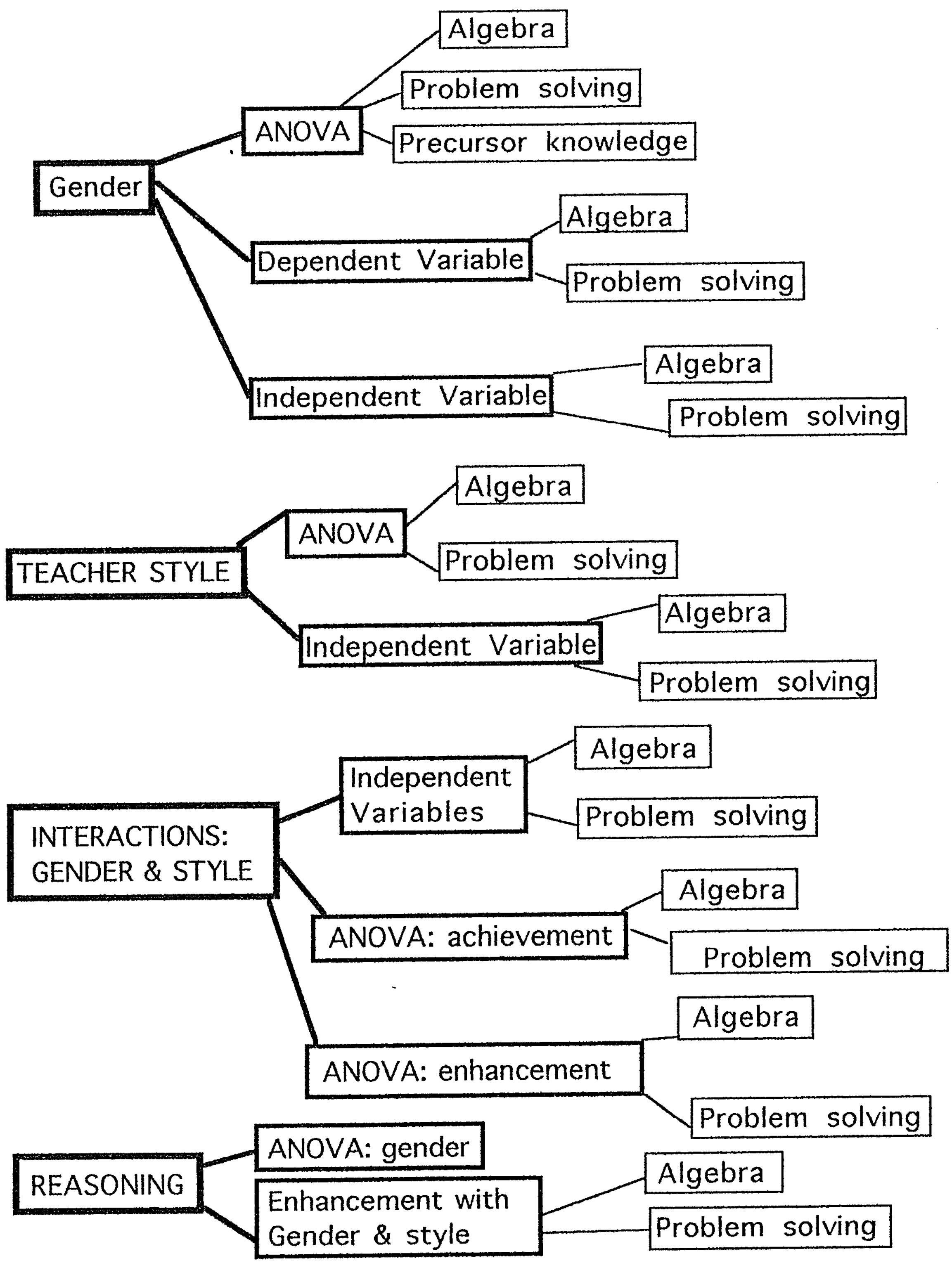
Four sets of exploratory questions were investigated. They concern; (1) gender, (2) teacher style, (3) interactions between gender and teacher style, and (4) reasoning ability (as defined by the a priori model). The data sets A, B, and C were most often combined for investigating those questions. As all hypotheses in this chapter are exploratory and their tests may be confounded with the tests of the formal hypotheses, no tests of statistical significance are valid. Accordingly, test statistics and associated probability values are provided but all conclusions are tentative.

All exploratory tests were done using the Macintosh statistical software JMP (SAS,1989) and Statview II (Abacus Concepts,1987).

Since a large number of exploratory and separate tests are included in this chapter, to avoid numerous references to appendices there appears to be no alternative but to include all appropriate tables here.

The following diagram provides an overview of the tests of the exploratory hypotheses.

Exploratory Hypotheses



Gender

The possible effects of gender were explored in three ways; (1) by an analysis of variance, (2) as a dependent variable in a regression model, and (3) as an independent variable in a regression model.

As well, gender is later investigated in other ways. Since gender is a current concern in the literature, it is worth exploring even though there may be a risk of over-investigating relationships between gender and other constructs.

Analysis of Variance

Three questions were asked:

(1) Is achievement in algebra related to gender?

(2) Is achievement in algebraic problem solving related to gender?

(3) Is knowledge concerning precursors of algebra related to gender?

Gender and achievement in algebra.

An analysis of variance was used to investigate the relationship between achievement in algebra and gender. For this purpose, achievement in algebra is the dependent variable.

The results suggest that the algebra scores of females are higher than those of males (see Tables 1 and 2). The difference in favour of females seems inconsistent with some other results concerning gender in the literature. It has commonly been found that, in courses and careers, females are not as much involved in mathematics and the related fields of science and technology as are males. If what is found here is true in general, factors other than achievement in grade nine algebra must determine females' choices concerning mathematics, science, and technology.

Table 1

Gender: Dependent Variable, Achievement in Algebra

Gender	Male	Female
Mean	45.5%	51.3%
Standard deviation	21.8	20.1
Number of subjects	158	173

Table 2

ANOVA by Gender: Dependent Variable, Achievement in Algebra

Source	Df	Sum of squares	mean squares	*F* - test
Between	1	2791.4	2791.4	6.37
Within	329	144213.1	438.3	$p = .01$
Total	330	147004.5		

Gender and achievement in algebraic problem solving.

An analysis of variance was used to investigate the relationship between achievement in algebraic problem solving and gender. Achievement in problem solving is again the dependent variable.

The results suggest that the problem solving scores of females are also higher than those of males (see Tables 3 and 4). This further supports the tentative conclusion that factors other than achievement in grade nine mathematics affect females' subsequent choices concerning mathematics, science, and technology.

Table 3

Gender: Dependent Variable, Achievement in Algebraic Problem Solving

Gender	Male	Female
Mean	44.0%	49.2%
Standard deviation	23.1	21.6
Number of subjects	158	173

Table 4

ANOVA by Gender:

Dependent Variable, Achievement in Algebraic Problem Solving

Source	Df	Sum of squares	mean squares	*F* - test
Between	1	2207.7	2207.7	4.43
Within	329	164146.8	498.9	$p = .04$
Total	330	166354.5		

Gender and precursor knowledge.

An analysis of variance was used to investigate the relationship between knowledge of precursors of algebra and gender. For this purpose, precursor knowledge is the dependent variable.

Precursor knowledge is defined in this case as knowledge of only those precursors that were identified in both category level regression models as being significantly related to achievement in algebra (see Appendix I, Table I-1). Scores for precursor knowledge were obtained by summing the scores for the items that comprised each of the identified categories. Since the regression models contained categories that had common items, such items were included only once to obtain the sum.

The results suggest that scores of females are higher than scores of males (see Tables 5 and 6). This initial difference may explain, at least in part, the higher achievement scores of females (see Tables 1, 2, 3, and 4). Given the strong relationship between precursor knowledge and achievement found in the formal tests, this result would be anticipated.

Table 5

Gender: Dependent Variable, Precursor Knowledge

Gender	Male	Female
Mean	45.8%	49.0%
Standard deviation	17.4%	14.9%
Number of subjects	158	173

Table 6

ANOVA by Gender: Dependent Variable, Precursor Knowledge

Source	Df	Sum of squares	mean squares	*F* - test
Between	1	876.5	876.5	3.37
Within	329	85504.5	259.9	p = .07
Total	330	86381.0		

Gender as the Dependent Variable in a Regression Model

Two questions were investigated. They were:

(1) What is the relationship between precursors of algebra and gender?

(2) What is the relationship between precursors of algebraic problem solving and gender?

To address those questions, gender rather than achievement was used as the dependent variable in a regression model. The regression models used for that purpose are the ones obtained from the category level analyses (see Chapter 5, formal tests on achievement in algebra and algebraic problem solving, category level).

Gender and precusors of algebra.

The category level regression model derived from data set A (with 9 regressors) accounts for 3.5% of the variance in gender. The category level regression model derived from set B (with 7 regressors) accounts for 5.8% of the variance in gender. Neither model seems to be an important predictor of gender. This suggests that there are no strong sex-related qualities of any categories of the identified precursors of algebra.

Gender and precursors of algebraic problem solving.

The category level regression model derived from data set A (with 8 regressors) accounts for 4.8% of the variance in gender. The model does not seem to be an important predictor of gender. This suggests, again, that there are no strong sex-related qualities of any categories of the identified precursors of algebraic problem solving.

Gender as an Independent Variable in a Regression Model

Two questions were investigated:

(1) When included as a precursor, does gender add significantly to our ability to account for achievement in algebra?

(2) When included as a precursor, does gender add significantly to our ability to account for achievement in algebraic problem solving?

To address these questions, gender was included as a regressor in the regression models obtained from the category level analyses (see Chapter 5, formal tests on achievement in algebra and algebraic problem solving, category level). The effect of including gender is measured by the change in R^2 and the associated F-statistic calculated by the equation (Hays, 1988):

$$F = \frac{\left(R_+^2 - R^2\right) \times \left(N - K - 1\right)}{1 - R_+^2} \qquad (6\text{-}1)$$

where 'R_+^2' is the amount of variance accounted for by the regression model having a particular regressor included in that model, 'R^2' is the amount of variance accounted for without that particular regressor, 'N' is the number of scores of the dependent variable, and 'K' is the number of regressors in the model.

Gender and achievement in algebra.

The results indicate that including gender has little effect on our ability to predict achievement in algebra (see Table 7). Even though a formal test at $\alpha = .05$, would suggest that it's effect is statistically significant in the model derived from set A, the effect is too small to be of practical significance.

Table 7

Effect of including Gender in Regression Models:

Dependent Variable, Achievement in Algebra

Regression model	R^2 with gender	R^2 no gender	Gender contribution	df	F
Derived from set A	.447	.437	.01	1/321	5.8 $p < .05$
Derived from set B	.438	.432	.006	1/323	3.44 $p > .05$

Gender and achievement in algebraic problem solving.

Similarily, a category level regression model was used to investigate the effect of including gender as a regressor in predicting achievement in algebraic problem solving.

The results indicate that including gender as a regressor produces a negligible change in predicting achievement in algebraic problem solving (see Table 8). This suggests that gender is not a significant predictor of algebraic problem solving.

Table 8

Effect of including Gender in Regression Models:

Dependent Variable, Achievement in Algebraic Problem Solving

Regression model	R^2 with gender	R^2 no gender	Gender contribution	df	F
Derived from set A	.368	.366	.002	1/322	1.02 $p > .05$

Teacher Style

Teacher style was explored in two ways; (1) by an analysis of variance and (2) as an independent variable in a regression model.

Additional caution is warranted concerning the results associated with teacher style since the instrument used to measure the three styles employed teachers' assessments of themselves.

Analysis of Variance

Two questions were investigated. They were:

(1) What is the relationship between teacher style and achievement in algebra?

(2) What is the relationship between teacher style and achievement in algebraic problem solving?

Teacher style and achievement in algebra.

An analysis of variance was used to investigate the relationship between teacher style and achievement in algebra. For this purpose, achievement is the dependent variable.

The results of this exploration indicate that teacher styles 1 and 3 lead to greater achievement in algebra (see Tables 9, 10, and 11). It seems that, for this test of performance in algebra, teachers who present students with closed procedures (teacher style 1) and teachers who encourage students to construct their own knowledge (teacher style 3) teach algebra about as well. It is not clear why blending the two styles seems to be less effective.

Table 9

Teacher Style: Dependent Variable, Achievement in Algebra

Teacher style	1	2	3
Mean	53.5%	44.9%	52.5%
Standard deviation	18.5	20.5	22.8
Number of subjects	62	180	89

Table 10

ANOVA Teacher Style: Dependent Variable, Achievement in Algebra

Source	Df	Sum of squares	mean squares	F - test
Between	2	5363.5	2681.7	6.21
Within	328	141641.1	431.8	$p = .002$
Total	330	147004.6		

Table 11

Comparison tests for ANOVA: Teacher Style

Scheffe Comparison test	F
1 vs 2	4.00, $p < .05$
1 vs 3	.043, $p > .05$
2 vs 3	4.02, $p < .05$

Teacher style and achievement in algebraic problem solving.

Similarily, an analysis of variance was used to investigate the relationship between teacher style and achievement in problem solving. For this purpose, problem solving achievement is the dependent variable.

In this area, there is no evidence that teacher styles affect achievement (see Tables 12 and 13). It is not clear why teacher style does not seem to be factor in teaching problem solving but yet it seems to be a factor in teaching algebra.

Table 12

Teacher Style: Dependent Variable, Achievement in Algebraic Problem Solving

Teacher style	1	2	3
Mean	46%	46.7%	47.3%
Standard deviation	20.5	23.0	23.0
Number of subjects	62	180	89

Table 13

ANOVA Teacher Style: Dependent Variable, Problem Solving

Source	Df	Sum of squares	Mean squares	*F* - test
Between	2	65.7	32.9	.07
Within	328	166288.7	507.0	$p = .94$
Total	330	166354.4		

Teacher Style as an Independent Variable

Two questions were investigated. They were:

(1) When included as precursors, does teacher style affect achievement in algebra?

(2) When included as precursors, does teacher style affect achievement in algebraic problem solving?

To address those questions, teacher style was included as a regressor in regression models. The regression models used for that purpose are those obtained from the category level analyses (see Chapter 5, formal tests on achievement in algebra and algebraic problem solving, category level). The effect of including teacher style is measured by the change in R^2 and the associated F-statistic.

Teacher style and achievement in algebra.

Both category level regression models were used to investigate the effect of teacher style on achievement in algebra.

The results indicate that teacher style may be of some significance as a precursor of algebra (see Table 14).

Table 14

Effect of including Teacher Style in Regression Models: Dependent Variable, Achievement in Algebra

Regression model	R^2 with style	R^2 no style	Style contribution	df	F
Derived from set A	.466	.437	.029	1/321	17.4 $p < .05$
Derived from set B	.447	.432	.015	1/323	8.73 $p < .05$

No attempt was made to determine what interactions or independent contributions to the variance accounted for should be explored further.

Teacher style and achievement in algebraic problem solving.

One category level regression model was used to investigate the effect of including teacher style as a precursor of algebraic problem solving.

Again, the results indicate that when teacher style is included as a precursor there is a minimal but possibly significant improvement in predicting achievement in algebraic problem solving (see Table 15).

Table 15

Effect of including Teacher Style in Regression Models:

Dependent Variable, Achievement in Algebraic problem solving

Regression model	R^2 with style	R^2 no style	Style contribution	df	F
Derived from set A	.373	.366	.007	1/322	3.58 $p < .05$

Interactions between Gender and Teacher Style

Possible interactions between gender and teacher style were explored in three ways. They were; (1) by including gender/teacher style interaction as a variable in a regression model, (2) by an analysis of variance on achievement, and (3) by an analysis of variance on precursor efficacy.

Precursor efficacy is here defined as the quotient of achievement and precursor knowledge. The precursors are those identified in the category level analyses (see Chapter 5).

Interaction Variables in a Regression Model

Two questions were investigated. They were:

(1) When included as a precursor of algebra, does the interaction of gender and teacher style significantly increase our ability to predict achievement in algebra?

(2) When included as a precursor of problem solving, does the interaction of gender and teacher style significantly increase our ability to predict achievement in algebraic problem solving?

To address these questions, the interaction of teacher style and gender was included in a regression model. The regression models used for that purpose are those obtained from the category level analyses (see Chapter 5, formal tests on achievement in algebra and algebraic problem solving, category level). The resulting regression model takes the form:

$$y = \text{cat.1} + \text{cat.2} + \ldots + \text{t.1} \times \text{gen} + \text{t.2} \times \text{gen} + \text{t.3} \times \text{gen} \qquad (6\text{-}2)$$

where 't.1' indicates style 1 (0/1), 't.2' indicates style 2 (0/1), 't.3' indicates style 3 (0/1), 'gen' indicates gender (0 - male, 1 - female), and 'cat.i' indicates a category level precursor. The effect of including interactions as regressors in the regression model is determined by the change in R^2 and the F - statistic.

<u>Teacher style and achievement in algebra.</u>

One category level regression model (derived from data set A) was used to investigate the effect of the interaction between gender and teacher style on achievement in algebra.

The results indicate that when the interactions 'teacher style 1 x gender' and 'teacher style 2 x gender' are included with precursors of algebra there is some improvement in predicting achievement (see Table 16). This suggests that there may be some value in further explorations of the interaction between gender and teacher style.

Table 16

Effect of including Gender/Style Interactions in Regression Models: Dependent Variable, Achievement in Algebra

Model	R^2	Change in R^2	F
categories only	.437	0	
categories + t.1 x gen + t.2 x gen + t.3 x gen	.475	.038	23.1 $p < .05$
categories + t.1 x gen + t.2 x gen	.469	.032	19.3 $p < .05$

Teacher style and achievement in algebraic problem solving.

One category level regression model (derived from data set A) was used to investigate the effect of the interaction between gender and teacher style on achievement in algebraic problem solving.

The results indicate that the interaction between gender and teacher style does not lead to any improvement in predicting achievement in problem solving (see Table 17) and suggest that the possible interaction between gender of students and teacher style is not a concern for the learning of algebraic problem solving. Again, it is not clear why an interaction effect observed for the teaching of algebra, albeit minimal, is not observed where problem solving is concerned.

Table 17

Effect of including Gender/Style Interactions in Regression Models: Dependent Variable, Achievement in Algebraic problem solving

Model	R^2	Change in R^2	F
categories only	.366	0	
categories + t.1 x gen + t.2 x gen + t.3 x gen	.370	.004	2.03 $p > .05$

The Effects of Gender and Teacher Style on Achievement using Analysis of Variance

Two questions were investigated. They were:

(1) Does ANOVA suggest any significant interaction between gender and teacher style in achievement in algebra?

(2) Does ANOVA suggest any significant interaction between gender and teacher style in achievement in algebraic problem solving?

Gender and teacher style on algebra achievement.

A two-way analysis of variance was used to investigate the relationship between gender and teacher style where the criterion variable is achievement in algebra. The independent variables are gender and teacher style.

As noted earlier regarding main effects, the results indicate that the algebra scores of females are higher than the scores of males and that teacher styles 1 and 3 seem to lead to greater achievement that does a blend of the two styles (see Tables 18 and 19). But there does not appear to be a significant interaction effect.

Table 18

Gender by Teacher Style:

Dependent variable, Achievement in Algebra

Style		Female	Male	Male and female
Teacher style 1	n	38	24	62
	mean	56.3%	49.2%	53.5%
Teacher style 2	n	94	86	180
(blend of 1 and 3)	mean	46.6%	43.0%	44.9%
Teacher style 3	n	41	48	89
	mean	57.7%	48.1%	52.5%
Styles 1, 2, and 3	n	173	158	331
	mean	51.3%	45.5%	48.6%

Table 19

ANOVA for Gender by Teacher Style:

Dependent variable, Achievement in Algebra

Source	df	SS	MS	*F*	*p*
Gender	1	2992.5	2992.5	7.03	.008
Teacher style	2	5216.9	2608.5	6.13	.002
Interaction	2	558.5	279.2	0.66	.520
Error	325	138335.3	425.6		

Gender and teacher style on problem solving achievement.

Similarily, a two-way analysis of variance was used to investigate the relationship between gender and teacher style where the criterion variable is achievement in algebraic problem solving. Gender and teacher style are the independent variables.

Consistent with the results obtained from the analysis of variance on each independent variable separately, these results indicate that the problem solving scores of females are higher than those of males (see Tables 20 and 21). Since there does not appear to be a significant interaction effect, no additional information is forthcoming from a two-way analysis of variance.

Table 20

Gender by teacher Style:

Dependent variable, Achievement in Algebraic Problem Solving

Style		Female	Male	Female and male
Teacher style 1	n	38	24	62
	mean	47.9%	43.1%	46.0%
Teacher style 2 (blend of 1 and 3)	n	94	86	180
	mean	48.9%	44.3%	46.7%
Teacher style 3	n	41	48	89
	mean	51.2%	44.0%	47.3%
Styles 1, 2, and 3	n	173	158	331
	mean	49.2%	44.0%	46.7%

Table 21

ANOVA for Gender by Teacher Style:

Dependent variable, Achievement in Algebraic Problem Solving

Source	df	SS	MS	*F*	*p*
Gender	1	2016.4	2016.4	4.00	.05
Teacher style	2	166.4	83.2	.17	.85
Interaction	2	104.4	52.2	.10	.90
Error	325	163875.9	504.2		

<u>Analysis of Variance on Precursor Efficacy</u>

Efficacy is defined as the quotient of achievement and precursor knowledge (the author's definition). This definition of precursor efficacy describes achievement in relation to students' knowledge of identified precursors. That is to say, precursor efficacy is the amount of achievement realized per unit of precursor knowledge.

Precursor knowledge is defined in the same way as in an earlier section of this chapter (see page 117). It is knowledge of only those precursors that were identified in the category level regression models for achievement. Scores for precursor knowledge were obtained by summing the scores for the items that comprised each of the identified categories. Since the regression models contained categories that had common items, such items were included only once in the sum.

Two questions were investigated:

(1) What is the relationship between gender and teacher style where the criterion variable is precursor efficacy in algebra?

(2) What is the relationship between gender and teacher style where the criterion variable is precursor efficacy in algebraic problem solving?

Precursor efficacy in algebra.

Scores for precursor efficacy were obtained by dividing achievement in algebra by precursor knowledge.

An analysis of variance was used to investigate the relationship between gender and teacher style. For this purpose, precursor efficacy is the dependent variable. Gender and teacher style are the independent variables.

The results indicate that teacher style 1 has the highest precursor efficacy (see Tables 22 and 23) suggesting that teachers who present students with closed procedures obtain proportionately higher achievement scores for given levels of students' understandings of the precursors of algebra.

That result seems reasonable. The algebra achievement test of this study measures principally students' ability to recall and apply procedures. For such a test, it is reasonable to expect that teachers who stress the learning of procedures (style 1) should realize greater achievement than teachers who stress understandings.

The results also indicate possible interaction effects. Teacher style 1 seems to be most effective with female students, and teacher style 3 least effective with male students.

This suggests some variables that may warrant further study of the kind suggested earlier. Further, male students may have more difficulty adjusting to the dissonance between the way algebra is taught and the way achievement in it is measured.

Table 22

Gender by teacher Style: Dependent variable, Precursor Efficacy

Style		Female	Male	Female and male
Teacher style 1	n	38	24	62
	mean	1.28	1.10	1.21
Teacher style 2 (blend of 1 and 3)	n	94	86	180
	mean	.98	1.05	1.01
Teacher style 3	n	41	48	89
	mean	1.14	.94	1.03
Styles 1,2, and 3	n	173	158	331
	mean	1.08	1.03	1.06

Table 23

ANOVA for Gender by Teacher Style:

Dependent variable, Precursor Efficacy

Source	df	SS	MS	*F*	*p*
Gender	1	.663	.663	3.11	.08
Teacher style	2	1.39	.693	3.25	.04
Interaction	2	1.45	.73	3.41	.03
Error	325	69.34	.213		

Precursor efficacy in algebraic problem solving.

Similarily, an analysis of variance was used to investigate the relationship between gender and teacher style with precursor efficacy as the dependent variable. Scores for precursor efficacy were obtained by dividing achievement in algebraic problem solving by precursor knowledge.

The results suggest that teacher style 1 is the most effective (see Tables 24 and 25) and teacher style 3 the least. This suggests that, for given levels of students' understandings of the precursors of algebraic problem solving, teachers who present students with closed procedures are more effective. Teachers who encourage students to construct their own knowledge are less effective. Again, given the test used here, this is a reasonable conclusion. Achievement in problem solving in this study is largely a measure of students' ability to recall specific problem types.

As would be expected given earlier results, there is no evidence for any interaction effects.

Table 24

Gender by teacher Style: Dependent variable, Precursor Efficacy

Style		Female	Male	Female and male
Teacher style 1	n	38	24	62
	mean	1.28	1.12	1.22
Teacher style 2 (blend of 1 and 3)	n	94	86	180
	mean	1.03	1.14	1.08
Teacher style 3	n	41	48	89
	mean	.95	.88	.91
Styles 1, 2, and 3	n	173	158	331
	mean	1.07	1.06	1.07

Table 25

ANOVA for Gender by Teacher Style:

Dependent variable, Precursor Efficacy

Source	df	SS	MS	*F*	*p*
Gender	1	.123	.123	.34	.56
Teacher style	2	3.295	1.65	4.53	.01
Interaction	2	.978	.489	1.35	.26
Error	325	118.2	.364		

Reasoning as defined by the A Priori Model

Reasoning is defined here by the elements that fall into the dimension of reasoning in the a priori model. Scores for reasoning are obtained by summing the scores on the 13 items of the model knowledge instrument that pertain to reasoning.

The reasoning dimension is explored in two ways; (1) by an analysis of variance with gender as the independent variable, and (2) by an analysis of variance on reasoning efficacy.

Reasoning efficacy is defined as the quotient of achievement and reasoning knowledge.

Analysis of Variance: Gender - the Independent variable

One question was investigated;

(1) What is the relationship between gender and reasoning?

In this analysis of variance, reasoning is the dependent variable and gender is the independent variable.

The results indicate no difference in male and female reasoning scores (see Tables 26 and 27) and that reasoning as defined here is independent of gender.

Table 26

Gender: Dependent Variable, Reasoning

Gender	Male	Female
Mean	29.6%	27.4%
Number of subjects	158	173

Table 27

ANOVA by Gender: Dependent Variable, Reasoning

Source	Df	Sum of squares	mean squares	*F* - test
Between	1	386.1	386.1	2.20
Within	329	57612.8	175.1	$p = .14$
Total	330	57998.9		

Analysis of Variance on Reasoning Efficacy

Reasoning efficacy is defined as the quotient of achievement and reasoning knowledge (the author's definition). This definition of efficacy describes achievement in relation to students' reasoning abilities as defined by the a priori model. That is to say, reasoning efficacy is the amount of achievement realized per unit of reasoning ability.

Again, reasoning is defined by the elements that fall into the dimension of reasoning in the a priori model.

Two questions were investigated. They were;

(1) What is the relationship between gender and teacher style, and reasoning efficacy in algebra?

(2) What is the relationship between gender and teacher style, and reasoning efficacy in algebraic problem solving?

Reasoning efficacy in algebra.

Scores for reasoning efficacy were obtained by dividing achievement in algebra by reasoning knowledge.

An analysis of variance was used to investigate the relationship between gender and teacher style. For this purpose, reasoning efficacy is the dependent variable. Gender and teacher style are the independent variables.

The results indicate that female students have the highest reasoning efficacy (see Tables 28 and 29) suggesting that, for a given level of reasoning abilities, females may be more efficient in learning algebra than males.

The results also indicate that teacher style 3 has the highest efficacy (see Tables 28 and 29). This suggests that teachers who encourage students to construct their own knowledge obtain proportionately higher achievement scores for given levels of students' reasoning abilities. Those results suggest that students' reasoning abilities are not important for learning closed algebraic procedures but that reasoning abilities are important for constructing knowledge about algebraic procedures.

Table 28

Gender by teacher Style: Dependent variable, Reasoning Efficacy

Styles		Female	Male	Female and male
Teacher style 1	n	38	24	62
	mean	2.37	1.82	2.15
Teacher style 2	n	94	86	180
(blend of 1 and 3)	mean	2.23	1.77	2.01
Teacher style 3	n	41	48	89
	mean	2.98	2.22	2.57
Styles 1, 2, and 3	n	173	158	331
	mean	2.43	1.91	2.18

Table 29

ANOVA for Gender by Teacher Style:

Dependent variable, Reasoning Efficacy

Source	df	SS	MS	*F*	*p*
Gender	1	22.54	22.54	7.25	.01
Teacher style	2	21.12	10.56	3.40	.03
Interaction	2	1.31	.654	.21	.81
Error	325	1010.75	3.11		

Reasoning efficacy in algebraic problem solving.

Scores for reasoning efficacy were obtained by dividing achievement in algebraic problem solving by reasoning knowledge.

Similarily, an analysis of variance was used to investigate the relationship between gender and teacher style with reasoning efficacy as the dependent variable.

The results again indicate that female students have the highest reasoning efficacy (see Tables 28 and 29) suggesting that, for a given level of reasoning abilities, females may be more efficient in learning algebraic problem solving than males.

For these tests, teacher style does not seem to affect reasoning efficacy. It is reasonable that here teacher style should, as was the case with algebra, be related to reasoning efficacy. Teachers who stress the construction of knowledge are more likely to encourage reasoning and realize proportionately greater achievement.

Table 30

Gender by teacher Style: Dependent variable, Reasoning Efficacy

		Female	Male	Female and male
Teacher style 1	n	38	24	62
	mean	2.18	1.77	2.02
Teacher style 2 (blend of 1 and 3)	n	94	86	180
	mean	2.29	1.80	2.05
Teacher style 3	n	41	48	89
	mean	2.55	2.03	2.27
Styles 1, 2, and 3	n	173	158	331
	mean	2.32	1.86	2.10

Table 31

ANOVA for Gender by Teacher Style:

Dependent variable, Reasoning Efficacy

Source	df	SS	MS	*F*	*p*
Gender	1	14.45	14.45	5.11	.02
Teacher style	2	4.53	2.27	.80	.45
Interaction	2	.088	.044	.016	.98
Error	325	919.75	2.83		

Summary of the Exploratory Tests

The exploratory tests of this study investigated relationships between achievement; and gender, teacher style, and reasoning. As all hypotheses here are exploratory, no tests of statistical significance are valid and accordingly all conclusions are tentative.

The relationship between gender and other variables was investigated in several ways. The results suggest that there are some gender effects. For both algebra and problem solving achievement, females obtained higher scores than males. The results concerning the effect of including gender as a predictor of algebra performance are inconclusive. When reasoning efficacy is considered, female students seem to be more efficient in learning algebra and problem solving than male students.

The results suggest that a teacher's style affects learning outcomes. Teachers who stress closed procedures and teachers who stress construction of knowledge both realize greater algebra achievement than teachers who blend those two styles. The results concerning problem solving are inconclusive.

Concerning precursor efficacy and teacher style, teachers who stress closed procedures are most effective teaching algebra and problem solving when they are measured, as was the case here, on routine skills. For problem solving, teachers who stress understandings are least effective. While these results may not be preferred, it could be that when the proficient performance of routine skills rather than the development of conceptual understandings is the desired goal of instruction, then the 'step one, step two, ...' style of teaching may be most effective for achieving that goal. However, when conceptual development is the goal that style may not be appropriate.

In the case of reasoning efficacy, the results support the exploratory style of teaching. Teachers who stress understandings are most effective for teaching algebra but teacher style does not seem to be pertinent to problem solving. Since it is likely that in the twenty-first century disconnected rules, theorems, and techniques will not be sufficient with respect to mathematical literacy, in the long run the exploratory teaching style may be preferable.

The results concerning interaction effects are inconclusive for achievement, gender and teacher style, However, there may be some interaction effects with respect to precursor efficacy, gender, and teacher style.

The results suggest that further attention should be paid to the interaction between teacher style and the gender of students. Teachers who stress closed procedures with female students may realize proportionately greater achievement in algebra than teachers who stress understandings with male students.

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APPENDIX A

Model Knowledge Instrument

DIRECTIONS:
All questions are multiple choice. There is only ONE CORRECT answer for each multiple choice question. Circle the letter of the correct answer from one of the choices 'a)' to 'd)'.
Please do not guess. If you are not very sure about the answer or don't know the answer, then circle the choice, 'e) I don't know.'

1. If a number is bigger than 10 then:
   a) it could be 10. b) it must equal 11.
   c) it is bigger than 12. d) it is bigger than 3.
   e) I DON'T KNOW.

2. To find the **opposite of** something means that as you do the **opposite of,** it must be done to:
   a) only one number at a time b) two numbers at a time
   c) three numbers at a time
   d) any amount of numbers at a time e) I DON'T KNOW.

3. When figuring out the answer to, 2x3 + 5x(18 - 10 + 1) + 7x2 what is inside the brackets:
   a) must be done before figuring out 2 x 3.
   b) must be done after figuring out 2 x 3.
   c) can be done after figuring out 2 x 3.
   d) must be done before figuring out 7 x 2.
   e) I DON'T KNOW.

4. The statement, 8 - 3 = 5,
   a) must always be about things.
   b) can't be about money.
   c) would look different if it were about carrots.
   d) could be about a relationship between numbers.
   e) I DON'T KNOW.

5. In the expression, $\Delta + \square$, the $\square$ means that:
   a) only whole numbers can replace it.
   b) only letters can replace it.
   c) any number can replace it.
   d) nothing.
   e) I DON'T KNOW.

6. In -(-7), the symbol, '-', that is <u>inside</u> the bracket indicates:
   a) position
   b) something to do
   c) subtraction
   d) opposite of
   e) I DON'T KNOW.

7. By a 'chunk', we mean a group of numbers that belong together more strongly to each other than to other numbers.
   How many chunks are in the expression:
   2 x 3 x 6 - 6 x 7 + 4 x 9 x 8 x 7 - 18 x 3 + 1
   a) 4
   b) 7
   c) 5
   d) 6
   e) I DON'T KNOW.

8. What is different about 0 + 0 - 0 and 0 - 0 ?
   a) the answers
   b) the number of things to do
   c) nothing
   d) the size of the numbers
   e) I DON'T KNOW.

9. The answer to, $\sqrt{25 + 4}$ is:
   a) 7
   b) 14.5
   c) about 5.4
   d) 9
   e) I DON'T KNOW.

10. The statement, four + √√√ = ●● ●●●●●, is:
    a) nonsense.
    b) impossible to work with.
    c) not true.
    d) not convenient to work with.
    e) I DON'T KNOW.

11. The statement, $\square + \square + \square - \square + 5 = 2 \times \square + 5$,
    a) cannot be true at all.
    b) is nonsense.
    c) is true.
    d) is true only for whole numbers.
    e) I DON'T KNOW.

12. Which one of the following statements is <u>FALSE</u>?
    a) $125 \div 4 = 31.25$ could be about people.
    b) Doing arithmetic is never about relationships between numbers.
    c) Doing arithmetic about horses is like doing arithmetic about radishes.
    d) $5 + 7 = 12$ could be about money.
    e) I DON'T KNOW.

13. When you figure out the answer to, 7 pears and 3 pears, you:
    a) add pears.
    b) add counts of pears.
    c) add words.
    d) add symbols.
    e) I DON'T KNOW.

14. Which one of the following is true?
    a) $2 \times (\square \times \Delta) = (2 \times \square) \times (2 \times \Delta)$
    b) $2 \div \square = \square \div 2$
    c) $2 + (\square + \Delta) = (2 + \square) + (2 + \Delta)$
    d) $15 \times \square = 9 \times \square + 6 \times \square$
    e) I DON'T KNOW.

15. As you work out the answer to an expression such as, $2 \times 3 + 4 - 18 \div 3$, the number of parts in the expression tends to:
    a) increase.
    b) stay the same.
    c) become 1.
    d) become 0.
    e) I DON'T KNOW.

16. For $-(-3) \times (-5) + 7$, the meaning of the - <u>outside</u> the brackets is?
    a) take away
    b) less
    c) subtract
    d) opposite of
    e) I DON'T KNOW.

17. There are many ways to get the right answer to the question, $53.2 \div 14.2$ Which one of the following could be done first before doing the division?
    a) Add the same number to 53.2 and to 14.2
    b) Move the decimal point only in 14.2
    c) Multiply 53.2 and 14.2 by the same number.
    d) Multiply 14.2 by some number.
    e) I DON'T KNOW.

18. A student figured out the answer to this multiplication question this way:

```
   5.34    -->  x 100    -->      534
 x  12     -->  x  10    -->    x 120
---------                    -------------
                                10680
                                53400
                             --------------
                               64080
```

The student then divided 64080 by 100 to get the final answer.

Which one of the following is true about what the student did to get the answer?

a) There is nothing wrong with what was done.
b) 64080 should have been divided by 1000 instead of by 100.
c) The student can't multiply the question parts by different numbers.
d) An error was made in multiplying.
e) I DON'T KNOW.

19. For the expression, $\Delta \times \square$, the Δ and the $\square$:

a) mean nothing.
b) must always be replaced by different numbers.
c) can never be replaced by numbers.
d) could be replaced by the same number. e) I DON'T KNOW.

20. Suppose that all geebles are woggles. It follows then that a woggle:

a) must be a geeble. b) could be a dimble.
c) can't be a geeble. d) can't be a dimble.
e) I DON'T KNOW.

21. If the names for two numbers are different looking, then the two numbers:
    a) could be equal.   b) are always unequal.
    c) must be equal.   d) are opposite in value.
    e) I DON'T KNOW.

22. If like matches with +, then the statement, She likes Bob, is best matched by:
    a) 8 + 7   b) ▯ + ?   c) 7 + ▯   d) ▯ + 7
    e) I DON'T KNOW.

23. As you figure out the answer to an expression like, 2 x 3 - 4 + 7 x 5, this changes:
    a) the answer to the expression.
    b) the numerical value of the expression.
    c) the numerical value of the answer.
    d) the number of parts in the expression.
    e) I DON'T KNOW.

24. We usually write the number fifty in mathematics as 50 rather than using the Roman numeral L because:
    a) the Roman numeral for fifty is wrong.
    b) the Roman numeral for fifty is more complicated.
    c) the Roman numeral for fifty is no longer very useful.
    d) for no good reason.   e) I DON'T KNOW.

25. When you add, how many numbers do you add together at one time?
    a) Two   b) One   c) It depends on the numbers.
    d) As many as you want.   e) I DON'T KNOW.

26. The ▯ in the expression, 5 x ▯ + 3:
    a) temporarily stops you from doing the arithmetic.
    b) has no effect on the result of doing the arithmetic.
    c) means that you can never do the arithmetic.
    d) can have only one value.   e) I DON'T KNOW.

27. Suppose that $\Delta \times \square \times ? = \pi \times \Omega \times$ ß.
If $\square =$ ß, then $? =$:

a) any number b) Ω c) Δ d) $\square$

e) I DON'T KNOW.

28. Look at the following matching of numbers:

100	-->	2
1	-->	0
1/100	-->	-2
1/1000	-->	$\square$

A likely value for $\square$ is:

a) -4 b) -1 c) -3 d) 4 e) I DON'T KNOW.

29. Consider the following story.

Some pennies are stacked (one penny on top of another) into two piles. One pile is higher than the other pile by 8 pennies. The shorter pile is 15 pennies high.

Which one of the following best matches what is going on in the story?

a) $15 - 8 = ?$ b) $? - 15 = 8$ c) $? - 8 = 15$

d) $8 + 15 = ?$ e) I DON'T KNOW.

30. Adding, $129.7823 + 9654.271 + 1680.009725 + 23987.6$,

a) must be done from left to right.

b) must be done from right to left.

c) in a different direction results in a different answer.

d) can be done from right to left. e) I DON'T KNOW.

31. Which one of the following best matches the arithmetic,
$30 \div 5 - 3$?

a) Remove 3 from what you give to each of 5 people.

b) Remove 3 from what you give to each of 30 people.

c) Remove 3 and then give what is left to each of 5 people.

d) Remove 3 people and then divide up 5 among those who are left.

e) I DON'T KNOW.

32. If the symbol, +, can be used to indicate clockwise, then a good choice for indicating counterclockwise is the symbol:

a) < b) AM c) - d) <--

e) I DON'T KNOW.

33. We write the number for counting eight things as 8 because it:

a) is true.

b) happens to be the way we do it.

c) is obvious to write it this way.

d) can't be written in any other way.

e) I DON'T KNOW.

34. For the statement, $\square + \square + \square = 15$, it is okay to:

a) replace the first □ by 2, the second □ by 6, and the third □ by 7.

b) replace the first □ by 5, the second □ by 5, and the third □ by 5.

c) replace each of the three □'s by 15.

d) remove one of the □'s. e) I DON'T KNOW.

35. Going from •••• √√ to •• √ most likely is not like:

a) $42 \div 2 = 21$ b) $42 - 21 = 21$ c) $6 - 3 = 3$

d) $(20 + 2) \div 2 = 11$ e) I DON'T KNOW.

36. The answer to, $2 \times 4^2 + 8 \div 4 \times 2 - 3^2$, is:

a) 24 b) 27 c) 59 d) 11 e) I DON'T KNOW.

37. Consider the statement, $\square \times \Delta = 12$. How many DIFFERENT PAIRS of numbers will make this statement true?

a) at least 3 pairs b) one pair c) exactly 6 pairs

d) no pairs at all e) I DON'T KNOW.

38. Consider the following story.

On Monday, Bob had some money in his wallet. By magic, it tripled overnight. Now there are 15 dollars in his wallet.

Which one of the following best matches what is going on in the story?

a) $3 \times 5 = ?$ b) $15 \div 3 = ?$ c) $? + 3 = 15$
d) $3 \times ? = 15$ e) I DON'T KNOW.

39. Which one of the following is <u>FALSE</u> about $8 \times 3 = 24$?

a) 24 can be the question and 8×3 can be the answer.
b) The answer to the question 8×3 is 24.
c) The answer to the question 8×3 must be written as 24.
d) 24 is another way to say 8×3. e) I DON'T KNOW.

40. Walking seven blocks north, eight blocks south, four blocks north, and five blocks north is like:

a) 7 PM, 8 AM, 4 PM, 5 PM
b) right 7, left 8, right 4, left 5
c) over 7, under 8, over 4, over 5
d) 7 apples, 8 oranges, 4 apples, 5 apples
e) I DON'T KNOW.

41. Mary is as tall as Helen. Helen is taller than Sandy, but Helen is shorter than Christa. Sandy is taller than Kandi. Kandi is:

a) shorter than Christa.
b) taller than Christa.
c) taller than Mary.
d) as tall as Christa. e) I DON'T KNOW.

42. Which one of the following is true?

a) $8375 \times 125 = 8375 \times 100 + 8375 \times 20 + 8375 \times 5$
b) $23 \div (8 + 7) = (23 \div 8) + (23 \div 7)$
c) $(92 \div 6) \div 8 = (92 \div 8) \div (6 \div 8)$
d) $13/9 - 52/30 = 52/30 - 13/9$
e) I DON'T KNOW.

43. Not not understanding nothing is the same as:
    a) not understanding nothing.
    b) understanding nothing.
    c) understanding everything.
    d) understanding something.
    e) I DON'T KNOW.

44. Which one of the following is okay to do?
    a) 3 dogs - 1
    b) 5 dogs + 3 rivers
    c) 7 dogs x 3 + 8
    d) 15 dogs ÷ 3 cats
    e) I DON'T KNOW.

45. The answer to, 14 x 5 - 7 x 2 + (3 + 4) ÷ 6,
    a) must be worked out from left to right.
    b) is different if worked out in a different direction.
    c) can be worked out from right to left.
    d) must be worked out from right to left.
    e) I DON'T KNOW.

46. Suppose that all veems are vooms. It follows that if something is not a voom then that something:
    a) is a veem
    b) is not a veem
    c) is not a fiim
    d) could be a veem.
    e) I DON'T KNOW.

47. Which one of the following best matches the statement, 6 x ▯ = Δ, ?
    a) ▯ is 6 times as big as Δ.
    b) Δ is 6 times as big as ▯
    c) Δ is one sixth as big as ▯.
    d) ▯ equals Δ.
    e) I DON'T KNOW.

48. Which one of the following is true?
    a) 18 is another name for 19.
    b) ●●●●● + ●● is another name for 8.
    c) The only other name for 4 x 5 is 20.
    d) There are many names for 23.
    e) I DON'T KNOW.

49. To find the square root of something means that each time you do the square root, it must be done to:
    a) only one number. b) two numbers. c) three numbers.
    d) any amount of numbers. e) I DON'T KNOW.

50. Think about the list of numbers, 1, 2, 3, 5, 8, 13, ▯.
    Which one of the following is a likely value for ▯?
    a) 21 b) 20 c) 18 d) 27
    e) I DON'T KNOW.

51. Consider the following story.

    A metre stick, its end broken off, starts at the 21 cm mark. Mary uses this meter stick to measure the length of a pencil. She places one end of the pencil at the 30 cm mark on the metre stick. The other end of the pencil is at the 47 cm mark.

Which one of the following best matches what is going on in the story?
    a) 47 - ? = 30 b) 30 + ? = 47 c) ? + 30 = 47
    d) 21 + ? = 47 e) I DON'T KNOW.

52. One of the following statements is <u>FALSE</u>.
    a) 4 x .25 = 1.00 can only be about money.
    b) 2 x 5 + 4 = 14 can be an example of a numerical relationship.
    c) 2 x 5 + 4 = 14 can be an example of a fact about money.
    d) 4 x 17 can be about ages. e) I DON'T KNOW.

53. The statement, 2 ÷ (7 + 5) --> 2 ÷ 7 + 2 ÷ 5, is most like:
    a) 2 x (7 + 5) --> 2 + 7 x 2 + 5
    b) 2 ÷ (7 x 5) --> 2 ÷ 7 + 2 ÷ 5
    c) 2 x (7 ÷ 5) --> 2 x 7 ÷ 5
    d) 2 + (7 x 5) --> 2 + 7 x 2 + 5 e) I DON'T KNOW.

54. Add is to subtract as:

a) squaring is to square root. b) minus is to subtract.

c) subtract is to divide. d) add is to multiply.

e) I DON'T KNOW.

55. In the expression, (2 x 3) + 5 + (24 x 8) + 2 x (7 + 9), we know what to do only because of the:

a) numbers. b) brackets.

c) additions, multiplications, and brackets.

d) brackets and additions. e) I DON'T NOW.

56. Tomorrow is not Monday, then today is:

a) Sunday b) Monday c) not Sunday d) not Tuesday.

e) I DON'T KNOW.

57. How can you undo the result of doing the opposite of -3?

a) Add 0. b) Subtract 3. c) Add negative 3.

d) Do the opposite of again. e) I DON'T KNOW.

58. Look at the following multiplication results carefully.

99 x 99 = 9 801
999 x 999 = 998 001
9 999 x 9 999 = 99 980 001

What should be the result for: 99 999 999 x 99 999 999 ?

a) 999 999 998 000 000 001

b) 99 999 980 000 001

c) 9 999 999 800 000 001

d) 9 999 999 980 000 001 e) I DON'T KNOW.

59. Suppose that, if whatsit then whatfor, if whatfor then whatelse, if whatelse then whatwhere.
One possible conclusion from all of this is:
if whatfor then whatwhere. This conclusion is:

a) wrong b) correct

c) depends on the nature of whatfor.

d) depends on the nature of whatfor and whatwhere.

e) I DON'T KNOW.

60. Figuring out the answer to the question, 7 apples and 4 oranges,
    a) is possible if the oranges or the apples are thought of as fruits.
    b) is possible if both the apples and oranges are thought of as things.
    c) is possible without any rethinking of the question.
    d) is possible if the apples are thought of as fruits.
    e) I DON'T KNOW.

61. There are many ways to get the right answer to the question, 2510 - 18. Which one of the following could be done first before doing the subtraction?
    a) Divide both 2510 and 18 by the same number.
    b) Add a number to 2510 but add a different number to 18.
    c) Add the same number to 2510 and to 18.
    d) Add a number only to 18.
    e) I DON'T KNOW.

62. Multiplying, $2 \times \square \times 4 \times 3 \times 5$,
    a) can be done from right to left.
    b) must be done from left to right.
    c) in a different direction results in a different answer.
    d) must be done from right to left.
    e) I DON'T KNOW.

63. By a 'chunk', we mean a group of numbers that belong together more strongly to each other than to other numbers. In the expression, $3 + 6 \times 5^2 + 12 \div 4 + 5^2 + 15$, which operation separates chunks?
    a) $\div$
    b) x
    c) +
    d) squaring
    e) I DON'T KNOW.

APPENDIX B

Algebra Achievement Instrument

Algebra: **Code:**________________________

1. Find the value of the expression, $x^2 + 3x + 4$, if x has a value of 4.
(2)

2. Simplify the expression, a + b + a + c + b + a.
(1)

3. Simplify: $(x^3)(x^4)$
(.5)

4. Simplify: $x^{10} \div x^5$

(.5)

5. Simplify: $-(3x - 2y)$
(.5)

6. Simplify: $(3a)(2b)$
(.5)

7. Write the opposite of: 3x
(.5)

8. Multiply: $3(x + 6)$
(1)

9. Simplify: $\frac{27ad}{9d}$
(1)

10. Simplify: $\frac{6y + 18}{3}$
(1)

11. Solve the equation: $3x + 15 = 17$
(1)

12. Solve the equation: $2(x + 3) = 14$
(1.5)

13. Solve the equation: $\frac{x}{2} = \frac{5}{3}$
(1)

14. Simplify: $\frac{15x}{x} + \frac{20}{2}$
(1)

15. Simplify:
$4abc + 5a^2bc + 6abc^2 - 3abc + 7a^2bc - 3abc^2 + ab^2c$
(2)

16. Simplify: $(2x^5)(3x^4)$
(1)

17. Simplify: $(6x^8) \div (2x^4)$
(1)

18. Simplify: $(2x - 4y) - (2x - 4y)$
(1)

19. Simplify: $(-6)(-2xy)$
(1)

20. Write the opposite of: $2x - 3y$
(1)

21. Simplify: $\frac{100m^2}{-10m^2}$
(1.5)

22. Simplify: $\frac{25mn - 10m}{-5m}$
(2)

23. Solve the equation: $12x + 13 = 8x - 7$
(2)

24. Solve the equation: $3(x - 3) + 1 = x + 5$
(3)

25. Solve the equation: $\frac{x}{2} + \frac{4x}{3} + 1 = 5$
(3)

26. Simplify: $\frac{x + 3}{-x - 3} + 1$
(2)

27. FInd the value of the expression, $-x^2 + 2(xy)^2$,
if $x = -3$ and $y = 2$.
(4)

28. Simplify: $25x - 2(-3x + 6)$
(3)

29. Simplify: $(3x^2 y)(5x^4 y^5)$
(1.5)

30. Simplify: $(20x^2 y^7) \div (4xy^4)$
(1.5)

31. Simplify: $9a - 3b - (-2b + 4a)$
(2)

32. Simplify: $(6ab)(-3a^2)$
(1.5)

33. Write the opposite of: $-(3x - y) + 5$
(1.5)

34. Multiply: $-5y(3y + 4xy - 5)$
(2)

35. Simplify: $\frac{(-4x)^2}{-4x}$

(2)

36. Simplify: $\frac{12xy^2 - 6x^2y + 24x^2y^2}{-3xy}$

(3)

37. Solve the equation: $8x + 14 - x = 10 + 2x + 4$

(3)

38. Solve the equation: $4(2x - 1) + x - 1 = -(x - 3) + 4$

(4.5)

39. Solve the equation: $\frac{x + 2}{2} + 1 = -\frac{x - 1}{3} + 1$

(5)

40. Simplify: $3\sqrt{a^2 + b^2} - \sqrt{a^2 + b^2}$

(3)

41. Simplify: $-(-2x)(3x) - (3x)(4x) - (-5x)(3)$

(3)

42. Simplify: $\frac{2x + 6}{x + 3}$

(2)

APPENDIX C

Algebraic Problem Solving Achievement Instrument

Problem solving: **Code:** ____________________

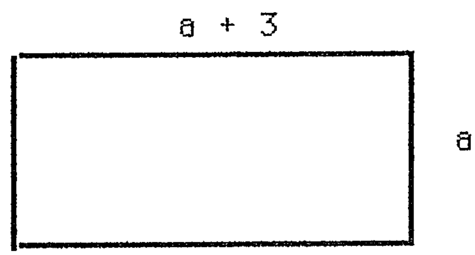
Part A: Write an algebraic expression for each of the following:

(.5) a) Total cost of 7 shirts if each shirt costs x dollars:

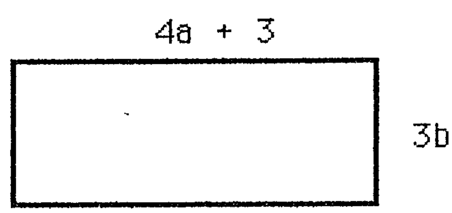
(1) b) Total cost of n cans of beans if each can costs 5t cents:

(1.5) c) Cost of one can if 45x cans cost 5y dollars:

(1.5) d) Perimeter of the rectangle

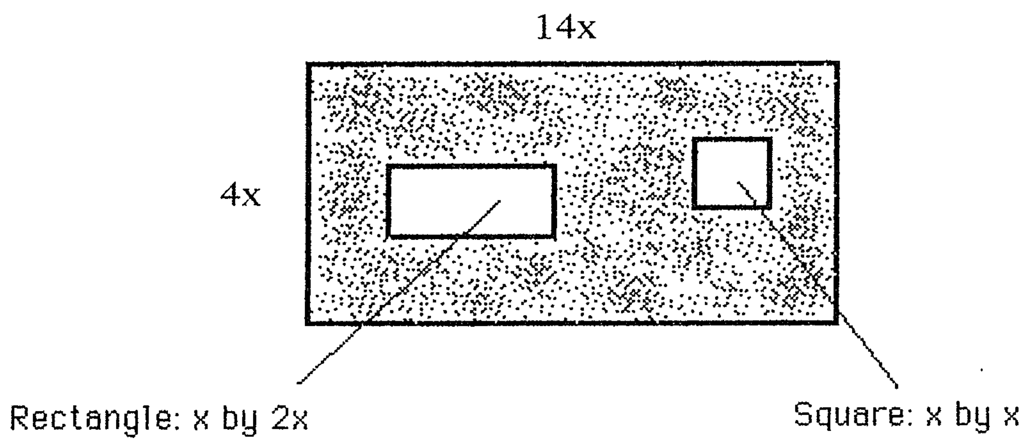


(1.5) e) Area of rectangle:



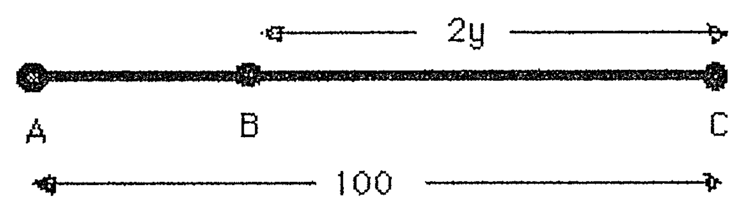
(.5) f) Five more than a number:

(3) g) Area of the shaded region:



(1.5) h) Doubling the result of subtracting 3 from a number:

(1) i) The distance from A to B:



(2) j) The mass of a copper coin; if a silver coin having a mass of (2b + 1) grams is three times the mass of the copper coin:

PART B: Solve each problem. You <u>must show</u> the equation that you used for solving the problem.

(3) a) Doubling a number and then adding 4 yields a result of 82. What is the number?

(4) b) Tony has $10 less than Bob. Mary has $17 dollars more than Bob. Altogether, the three people have $250. How much money does Bob have?

(5) c) Tom, Dick, and Harry shared a prize of $210. Dick received twice as much money as Tom. Harry received three times as much as Tom. How much money did Harry receive?

(4) d) Three consecutive odd numbers add up to 267. What are the three numbers?

(6) e) A bag contains nickels and dimes having a total value of $9.15. One more than the number of nickels is 6 times the number of dimes. How many nickels are in the bag?

(4) f) The length of a rectangular swimming pool is 17 m greater than the width of the pool. The perimeter of the pool is 154 m. What is the length of the pool?

(3) g) The difference of two numbers is 13. When the two numbers are added the result is 243. What are the two numbers?

(6) h) A piece of wood 191 cm long is cut into 4 pieces. The pieces are all of different lengths. How long is the longest piece? The pieces are cut as follows:

1 cm longer than twice the length of the piece above

1 cm longer than twice the length of the piece above

1 cm longer than twice the length of the piece above

APPENDIX D

Pedagogical Styles of Algebra Teachers Instrument

Please circle your response to each question.

1. The homework I assign consists of questions which closely resemble the examples that I have presented in class.

Very rarely Sometimes Half the time Frequently Almost always

2. When teaching algebra, I provide students with a rationale for learning algebra that involves real applications.

Very rarely Sometimes Half the time Frequently Almost always

3. In algebra class, I attempt to develop a set of principles that are applicable to the processing techniques that are part of the course.

Very rarely Sometimes Half the time Frequently Almost always

4. I insist that students do algebraic processing the way that I taught it.

Very rarely Sometimes Half the time Frequently Almost always

5. My evaluation (formal and informal) of students emphasizes processing skills.

Very rarely Sometimes Half the time Frequently Almost always

6. I encourage students to participate in the generation of principles and concepts that might be fundamental to doing algebra.

Very rarely Sometimes Half the time Frequently Almost always

7. My foremost concern in teaching algebra is that students are able to get the right answers to algebra questions rather than understanding the principles involved in getting the answers.

Very rarely Sometimes Half the time Frequently Almost always

8. I begin my algebra classes by presenting definitions and processing examples rather than by developing a principle(s) that may be applied to the processing.

Very rarely Sometimes Half the time Frequently Almost always

9. The reason that I ask questions in my algebra classes is to help me decide if students are able to do questions similar to the examples that I present (ignore reasons related to discipline matters).

Very rarely Sometimes Half the time Frequently Almost always

10. I think that the students in my algebra classes care about learning algebra.

Very rarely Sometimes Half the time Frequently Almost always

11. My evaluation (formal and informal) of students emphasizes principles and concepts.

Very rarely Sometimes Half the time Frequently Almost always

12. I encourage students to seek alternate ways to do algebra questions.

Very rarely Sometimes Half the time Frequently Almost always

13. I use a problem solving model when teaching algebra - one that encourages students to provide a major contribution to the development of the objective of the lesson.

Very rarely Sometimes Half the time Frequently Almost always

14. I approach algebra classes with a "spring in my step" and "joy in my heart".

Very rarely Sometimes Half the time Frequently Almost always

APPENDIX E

Organization of the Model Knowledge Items in the A Priori Model of Precursors

Table E-1

Item organization of the model knowledge instrument in relation to categories of the model:

Category	Item number on Instrument
1	10, 24, 33
2	21, 39, 48
3	7, 55, 63
4	5, 11, 19, 26, 34, 37
5	6, 16, 32
6	2, 25, 49
7	18, 54, 57
8	3, 9, 36
9	8, 5, 23
10	4, 12, 52
11	13, 44, 60
12	30, 45, 62
13	14, 17, 27, 42, 61
14	28, 50, 58
15	1, 20, 41, 43, 46, 56, 59
16	35, 40, 53
17	22, 31, 47
18	29, 38, 51

APPENDIX F

Instrumentation Results

Table F-1

Response Scoring Errors for Model Knowledge Instrument

Number of items scored	Number of errors in scoring
20 x 63 = 1260	5 (.4%)

Table F-2

Rescoring Data for Achievement in Problem Solving

Number of samples	34
Frequency of 0 change	14
Range of changes	- 3% to 5%
Mean change (absolute value)	1%
Standard deviation (absolute value)	1%

Table F-3

Rescoring Data for Achievement in Algebra

Number of samples	34
Frequency of 0 change	14
Range of changes	- 2% to 3%
Mean change (absolute value)	.5%
Standard deviation (absolute value)	.7%

Table F-4

Item Consistency for the Model Knowledge Instrument

Lowest correlation with total score	- .07
Highest correlation with total score	.52
Number of negative correlations	2
Split-half consistency coefficient	.61

Table F-5

Item Consistency for the Achievement in Algebra Instrument

Lowest correlation with total score	.09
Highest correlation with total score	.81
Number of negative correlations	0
Split-half consistency coefficient	.88

Table F-6

Item Consistency for the Achievement in Algebraic Problem Solving Instrument

Lowest correlation with total score	.22
Highest correlation with total score	.87
Number of negative correlations	0
Split-half consistency coefficient	.83

APPENDIX G

Results for Formal Tests

Table G-1

Formal Test: Category Level Regression Model derived from Set B, Dependent Variable - Achievement in Algebra

Test Parameter	Value
R (predicted / actual)	.556
R^2 (predicted / actual)	.309
K	7
N	100
Degrees of freedom	7 / 92
α	.05
F - criterion	2.13
F - calculated	5.88

Table G-2

Contribution to R^2 of the Set B-derived Category Level Regression Model used independently.

Dependent Variable, Achievement in Algebra

Data set	Set B	Set C	Sets A, B, C combined
R^2 contribution	.522	.364	.432

Table G-3

Formal Test: Element Level, Regression Model from Set B, Dependent Variable - Achievement in Algebra

Test Parameter	Value
R (predicted / actual)	.393
R^2 (predicted / actual)	.154
K	8
N	100
Degrees of freedom	8 / 91
α	.05
F - criterion	2.07
F - calculated	2.09

Table G-4

Contribution to R^2 of the Set B-derived Element Level Regression Model used independently.

Dependent Variable, Achievement in Algebra

Data set	Set B	Set C	Sets A, B, C combined
R^2 contribution	.500	.235	.339

APPENDIX H

Model Knowledge Instrument details

Answer Key for the Model Knowledge Instrument

1. d	2. a	3. c	4. d	5. c	6. a	7. c	8. b
9. c	10. d	11. c	12. b	13. b	14. d	15. c	16. d
17. c	18. b	19. d	20. b	21. a	22. d	23. d	24. c
25. a	26. a	27. b or d		28. c	29. b	30. d	31. a
32. c	33. b	34. b	35. c	36. b	37. a	38. d	39. c
40. c	41. a	42. a	43. b	44. d	45. c	46. b	47. b
48. d	49. a	50. a	51. b	52. a	53. d	54. a	55. c
56. c	57. d	58. c	59. b	60. a	61. c	62. a	63. c

A Discussion of some Items of the Model Knowledge Instrument

Item #16: 'Subtract' cannot be the interpretation of the symbol '-' here because of the way the expression is written.

Item #20: Choice (b) is correct by elimination. Choice (a) is not correct as IF -> THEN is not necessarily reversible. Choice (c) is not correct as a woggle might be a geeble if a geeble is a woggle. Choice (d) is not correct since there is no basis for saying that a woggle cannot be a dimble. It may help to think through the item from the point of view "If it is raining, then it is wet.".

Item #27: There is an error in the construction of this item. Both answers are acceptable. The response 'any number' is not correct. While π and Ω can be any number, once either of them has been assigned a value , then '?' must be that value. The commutative principle is the issue here.

Item #30: Students may have the vertical algorithm in mind here. That possibility makes this item somewhat flawed.

Item #35: There is an argument to be made that all of the choices are like the example given. However, choice (c) is the least like the example as subtraction requires the same objects for subtracting.

Items # 29, 38, 51: The issue in these items is not what kind of computation can be done to get the answer but rather which number sentence best models what is happening in each story.

Item #40: AM and PM are not directional.

Item #44: The issue here is that x and - require the same objects while x and ÷ do not.

Item #53: The issue here is detecting the distributive pattern (one operation applied over a different one).

Item #57: The functional use of inverse operations is the issue here (do/undo). If one begins with -3, then doing the 'opposite 'of' to it yields +3. To get back to -3 (undo), one must again do the 'opposite of' but to +3.

Item #60: Choice (b) is somewhat acceptable but the researcher wanted a sharper understanding of the issue.

APPENDIX I

Details concerning Exploratory Tests

Table I-1

Items used for defining Precursor Knowledge

Items in numerical order
6, 7, 8, 9, 11, 14, 16, 18, 23, 32, 36, 37, 40, 41, 42, 47, 49, 50, 51, 54, 55, 56, 58, 59, 61, 63