COMPLEX WAVE DIGITAL NETWORKS INCLUDING THE IMPLEMENTATION OF EVEN-ORDER FILTERS

by
Gordon B. Scarth

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ABSTRACT

Fettweis first proposed that complex reference networks can be implemented in the wave digital domain with the real WD elements by requiring a restrictive property called one-realness. This thesis developes a new theory for complex wave digital filters allowing the realization of general complex reference circuits without alteration.

All quantities within a wave digital network are now allowed to be complex. The port reference impedances are defined to be complex constants, containing both resistance and constant reactance. The incident and reflected wave variables are redefined giving a new form of the reflection coefficient. The low sensitivity in the passband to variations in a complex parameter of a complex wave digital network is verified. Pseudopassivity is defined and conditions are given to ensure pseudopassivity. Equivalences are given between a delay in the complex wave digital domain and the analog domain. A delay corresponds to a capacitor in series with a constant reactance, an inversion in series with a delay corresponds to an inductor in series with a constant reactance, a complex multiplier with a magnitude less than or equal to one corresponds to a resistance in series with a constant reactance, and a wave source corresponds to a voltage source in series with a resistance and a constant reactance. The criterion for the interconnection of two wave ports is found to be the condition that the reference impedance of one port must be the complex conjugate of the other. The complex two-port adaptor is derived and the magnitude of the parameter is found to be bounded by one. The complex three-port circulator and the complex transformer are given and found to be pseudolossless. The zero-input and forced response stability of the complex wave digital network is given by pseudopassivity and incremental pseudopassivity as for the real case. Conditions on the complex nonlinear overflow and underflow operators are given, including the condition that the norm of each operator must be bounded by one.

The realization of even-order classical filters with real inputs and outputs is obtained with a single complex one-port allpass network containing complex two-port adaptors, complex three-port circulators and delays. The input-output port reference impedance is chosen to be real allowing the one-port to be externally viewed as a real network. The scattering matrix describing the non-dynamic sub-network is sparse and highly structured as it is composed of four block-submatrix types. Amplitude scaling of the filter to achieve maximum dynamic range for white Gaussian random signals is accomplished with a real

scaling transformation which leaves the frequency response invariant. The scattering matrix is quantized, or represented in finite binary form with a common denominator, by magnitude truncation of the real and imaginary parts of the scaled scattering matrix.

A computer program (listed in Appendix C) was developed on an IBM PC/XT in a version of FORTRAN-77 which generates a complex quantized scattering matrix solution for a set of design specifications. The filter types include even-order lowpass Butterworth, Chebyshev and Cauer (Elliptic) filters. The complex wave digital filters which are designed satisfy the lowpass specifications, are strictly pseudopassive, are amplitude scaled for Gaussian excitations and can be embedded in a strictly real wave digital network as a building block. Under the recursive operation of the digital filter, the response from the desired transfer function is the real part of the output.

Finally, six examples showing the satisfactory performance of the program and the validity of the theory which was developed are illustrated and discussed.

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CHAPTER I

INTRODUCTION

The theory of wave digital networks was introduced in the real domain by Fettweis [1-6][8-12] and extended by many authors. A network containing real elements in the voltage-current domain can be implemented in the real wave digital domain without delay-free directed loops; thus the digital network is said to be realizable. Advantages gained using real wave digital filters include: low sensitivity in the passband [6] to multiplier variations, zero-input and forced response stability [7], suppression of parasitic oscillations [5-7], and a continuous input-output relationship [7].

Complex reference circuits have been found useful in several applications [30-31][47]. Several new elements added to real passive networks to obtain passive complex networks are the constant reactance, or constant complex impedance jX, the complex circulator, and the complex transformer. These elements arise in the implementation of even-order classical filters [30-31], narrow-band filters [47], and non-symmetric filters where the frequency response is not symmetric about the zero-frequency axis [59].

The theory developed in the real domain cannot be used without modification to realize complex voltage-current domain networks. Fettweis has suggested a method of realization of complex reference networks that requires a property known as one-realness [26]. A general complex reference network is not one-real and therefore the complex transfer function representing the network must be altered to give one-realness, which is unacceptable in many applications. However, if a network is one-real it can be realized using the real definitions of wave digital elements with the addition of complex multipliers. This realization method does not lead to any new definitions and is restrictive, and thus it will be abandoned for a more general theory of complex wave digital networks. The purpose of this thesis is to develop a general theory of complex wave digital networks that includes the beneficial properties mentioned in the earlier paragraph.

In this thesis, complex wave digital networks are allowed to contain constant port-reference impedances, or constant resistances and reactances. New definitions for the complex incident and reflected signals follow from the recognized form of the complex allpass function [44]. The complex two-port adaptor, complex circulator and complex transformer can be defined in the complex wave digital domain. One-port elements appearing in the wave digital domain include: a complex multiplier, a delay, an inversion plus a delay and a wave source. Equivalences between these one-port elements with a constant port reference impedance and the analog domain representation follow from the definition of the wave variables. The familiar concepts of pseudopower and pseudoenergy can be defined along with the zero-input and forced-response stability arguments.

Once the theory is developed, it can be used to realize familiar reference networks.

This is accomplished by transforming the signal quantities from the voltage-current domain to the voltage-wave domain with the new definition of the wave variables. The network is then transformed into the wave digital domain with the bilinear transformation which warps the continuous frequency axis in the Laplace domain to the range $0 \leq \omega \leq \pi F$ in the wave digital domain, where F is the sampling frequency. Equivalences between elements in the analog and the complex wave digital domains can be used to derive WD networks with the beneficial properties mentioned above.

A particular filter structure that has been found to be of importance is the lattice analog structure with equal impedance terminations. The sensitivity of the wave digital equivalent of the doubly-terminated lattice structure is not of concern in the operation of the filter since the coefficients are fixed and cannot change over time. Several methods of realization of this structure have been proposed, including: three-port parallel and series adaptors with delays, cascaded unit elements, and first and second order unit element sections coupled together with a multi-port circulator.

For even-order classical filters, such as the Butterworth, Chebyshev and Cauer (Elliptic) filter types, the functions used in the lattice realization are complex allpass functions. For real inputs, the output of one complex allpass function will be the complex conjugate of the other, and thus only one complex allpass function is needed to implement the filter. The real part of the output will be the response from desired transfer function and the imaginary part will correspond to the response from the spectral complement of the transfer function.

One realization of the complex allpass function is the cascade of unit elements, which is equivalent to a cascade of circulators of different reference impedances. The complex wave digital network representing the complex allpass function will be canonic in the number of delays and it will contain complex two-port adaptors, complex three-port circulators and delays. A particular form of the structure can be chosen to maximize the decoupling of the state variables associated with the filter. The scattering matrix or state-variable matrix representation of the non-dynamic sub-network with the delay elements removed will be sparse and consist of only four complex block-submatrix types. Thus it is highly structured and it can be easily amplitude scaled with a non-singular linear transformation.

Chapter II gives an introduction to the new theory developed defining complex wave digital filters with constant complex port reference impedances. Section 2.1 summarizes the wave digital transformation and gives the new definitions of the wave variables. Section 2.2 gives an argument for low passband sensitivity with well-designed complex wave digital filters. Pseudopower and pseudoenergy are defined in Section 2.3 along with the conditions for pseudopassivity. The equivalences between the common one-port elements: a complex multiplier, a delay, a delay in series with an inversion, and a wave source, with constant port reference impedances, and the analog domain are given in Section 2.4. Complex wave digital multi-ports, such as the complex two-port adaptor, complex circulator and complex transformer are defined and investigated in Section 2.5. The criterion for port interconnection is also given. The definition of the complex scattering matrix as related to the state-variable matrix is given in Section 2.6. The scattering matrix is amplitude scaled and the frequency response of the network represented by the scattering matrix is derived. In Section 2.7 the zero-input and forced response stability of a complex wave digital network is investigated. Due to the finite number of digits representing signals within a recursive digital filter, overflow and underflow error signals will appear making the network non-linear. Conditions on the complex overflow and underflow non-linear operators are given to guarantee pseudopassivity and incremental pseudopassivity.

In Chapter III, the theory developed in Chapter II is applied in order to realize classical even-order lowpass filters, such as the Butterworth, Chebyshev and Cauer filter types. The representation of a classical filter transfer function in terms of two complex allpass functions is given in Section 3.1. It is shown that the coefficients of one complex allpass function are the complex conjugate of the coefficients of the other. The structure of the lattice realization is given in Section 3.2 along with the method of choosing the poles for the complex allpass function. In Section 3.3, the complex allpass function is realized in the complex wave digital domain with complex two-port adaptors, complex three-port circulators and delays. The method of calculating the parameters from the complex allpass function is given. The stability of all intermediate allpass functions is verified and the port reference impedances are given in terms of the complex two-port adaptor parameters. The non-dynamic sub-network of the complex two-port adaptor network is represented by a sparse scattering matrix and is scaled in Section 3.4. Also, a bound on the magnitude of the output and the states of the filter is derived. The port reference impedances are given for the delay ports after scaling in Section 3.5. An example of a complex allpass function realization is given in Section 3.6. The analog domain equivalent of the complex allpass wave digital network is given in Section 3.7.

Chapter IV gives the description of the computer program used to implement the theory derived in Chapters II and III, and is listed in Appendix C. The program generates a state-variable solution for a set of frequency specifications given for a classical even-order lowpass filter. The format of the input specification and initialization text files are given in Appendices A and B, respectively. In Section 4.1, the computer program input-output is discussed. The design specifications and the run options are described and the form of the filter solution is given. The method of calculating the complex allpass function that represents the desired transfer function which meets the frequency specifications is given in Section 4.2. The procedure for determining the optimum pole locations is discussed along with the calculation of the unimodular constant associated with the complex allpass function. The generation of the scaled scattering matrix is examined in Section 4.3. The algorithm for quantizing the scattering matrix is introduced and the pseudopassivity test is outlined as they relate to the choice of the quantized scattering matrix integer denominator.

Finally in Section 4.4, the procedure used for meeting the frequency specifications is given.

Chapter V presents six examples of complex wave digital designs of classical filters using the computer program discussed in Chapter IV.

CHAPTER II

INTRODUCTION TO COMPLEX WAVE DIGITAL FILTERS

Complex wave digital filters, CWDFs, are a generalization of real wave digital filters introduced by Fettweis [1][2][3]. Complex reference, or analog networks can be realized in the discrete domain using complex wave digital filters without restrictions on a passive transfer function. The realization process consists of transforming the complex reference transfer function into the discrete domain with the bilinear transformation, then performing a mapping into the wave digital domain with the definition of incident and reflected waves as referenced to ports with constant complex port impedances.

Constant reactances arise in circuits from several sources. They include the decomposition of even-order classical filters, including Butterworh, Chebyshev and Cauer (Elliptic) filters, giving two complex allpass functions with complex coefficients. They also include non-symmetric filters where the attenuation characteristic is not symmetric in the frequency domain about the zero frequency axis [59], and narrow-band filters [47]. These applications and others have prompted interest in complex circuits and the realization of complex transfer functions.

2.1 Complex Wave Digital Transformation

Filter design in the Laplace domain is a well-known and understood discipline, and several structures have emerged which possess beneficial properties. A digital transformation should take advantage of these properties in order to design stable digital filters with low sensitivity to changes in multiplier coefficients. An appropriate transformation is known to be the mapping from the voltage-current discrete domain to the real wave digital domain. However, this mapping was not generalized for the realization of complex reference networks and restrictions were placed on the complex transfer function [26].

A new theory is presented here which redefines wave digital networks in order to include complex reference networks. The complex wave variables are redefined giving a complex reflection function that agrees with the form suggested by Belevich [44] and several other authors [47][49][52]. The reference port impedance is still constant, but it is allowed to be complex. Concepts of pseudopassivity, pseudolosslessness and the adaptor implementation follow directly. The following development does not place restrictions on passive reference networks, and reduces, in the real case, to the theory of real wave digital filters as introduced by Fettweis [1].

2.1.1 **Definition of Complex Wave Variables**

A CWDF is derived from a complex reactance network through a transformation from the voltage-current discrete domain to the wave digital domain. All domains are assumed to be discrete and thoughout the following, voltage waves will be used exclusively. Define the following mapping from the voltage-current domain to the wave domain,

$$a(n) = v(n) + Z\, i(n) \tag{2.1}$$

$$b(n) = v(n) - Z^*\, i(n) \tag{2.2}$$

$$A(\psi) = V(\psi) + Z\, I(\psi) \tag{2.3}$$

$$B(\psi) = V(\psi) - Z^*\, I(\psi) \tag{2.4}$$

$$Z = R + jX \tag{2.5}$$

where ψ is a frequency variable given in (2.10), Z is a complex constant henceforth known as the port reference impedance, a(n) and b(n) are two complex sequences of the incident and reflected waves, and $A(\psi)$ and $B(\psi)$ are the complex steady-state incident and reflected waves, respectively. The discrete variable n represents time instants, or the actual time of a(n) and b(n) is $t = t_0 + nT$, where T is the sampling period and t_0 is a time reference. The wave quantities are shown in Figure 2.1 along with the reference port impedances.

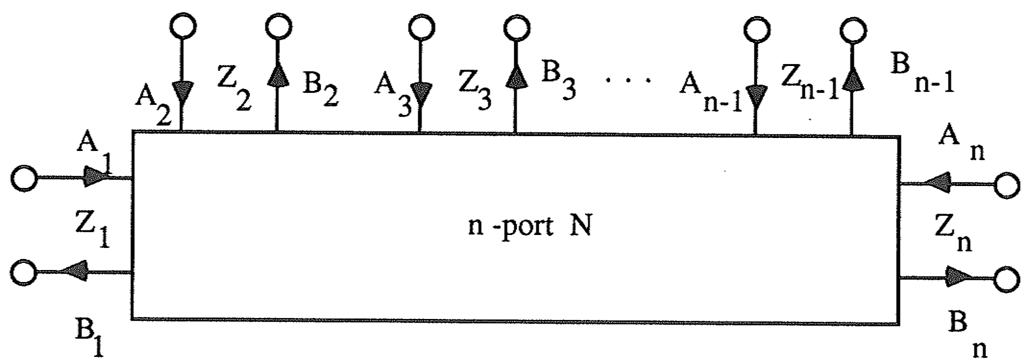

Figure 2.1: n-port complex wave digital network.

When considering a one-port WD network as shown in Figure 2.2, the incident wave A is the input while the reflected wave B is the output as referenced to the one-port with impedance Z.

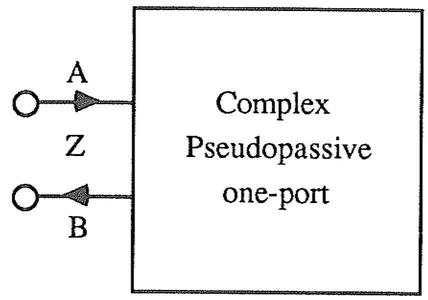

Figure 2.2: Complex one-port pseudopassive wave digital network.

For convenience, the following will be considered equivalent

$$V(\psi) = V, \quad I(\psi) = I \tag{2.6}$$

$$A(\psi) = A, \quad B(\psi) = B \tag{2.7}$$

The complex voltage and current are given in terms of the steady-state wave variables of the one-port as

$$V = \frac{Z^* A + Z B}{Z + Z^*} = \frac{Z^* A + Z B}{2 \operatorname{Re}(Z)} \tag{2.8}$$

$$I = \frac{A - B}{Z + Z^*} = \frac{A - B}{2 \operatorname{Re}(Z)} \tag{2.9}$$

2.1.2 Transfer Function Transformation

A reference transfer function in the Laplace domain is transformed through a change in the frequency variable to the digital domain with the bilinear transformation by substituting the following

$$\psi = \frac{z - 1}{z + 1} = \tanh\left(\frac{s\,T}{2}\right), \quad z = e^{s\,T} \tag{2.10}$$

$$\phi = \tan\left(\frac{\omega T}{2}\right), \quad s = j\omega, \quad \psi = j\phi \tag{2.11}$$

into the transfer function, where s is the actual Laplace transform variable, z is the digital domain variable and T is the sampling period with T=1/F. The bilinear transformation maps the Nyquist range $0 < \omega < \pi F$, in a one-to-one correspondence, onto the range in the Laplace domain $0 < \phi < \infty$. It also maps stable filters in the reference domain to stable and causal filters in the digital domain.

The Laplace domain complex transfer function of order n given in (2.12) with

Laplace domain complex zeros z^L_i, i =1 (1) m, poles p^L_i, i = 1 (1) n, and gain K_L,

$$G(\psi) = \frac{P(\psi)}{D(\psi)} = \frac{\tilde{p}^L_m \psi^m + \tilde{p}^L_{m-1}\psi^{m-1} + \ldots + \tilde{p}^L_0}{d^L_n \psi^n + d^L_{n-1}\psi^{n-1} + \ldots + d^L_0} = \frac{K_L\left(\psi - z^L_1\right)\left(\psi - z^L_2\right)\ldots\left(\psi - z^L_m\right)}{\left(\psi - p^L_1\right)\left(\psi - p^L_2\right)\ldots\left(\psi - p^L_n\right)} \tag{2.12}$$

is thus transformed into the discrete domain with z-domain zeros z_i, i = 1 (1) m, and z-domain poles p_i, i = 1 (1) n, defined in equations (2.13a) and (2.13b), with the gain K_Z,

$$z_i = \frac{1 + z^L_i}{1 - z^L_i} \quad , \quad i = 1\ (1)\ m \tag{2.13a}$$

$$p_i = \frac{1 + p^L_i}{1 - p^L_i} \quad , \quad i = 1\ (1)\ n \tag{2.13b}$$

$$K_Z = K_L \frac{\prod_{i=1}^{m}\left(1 - z^L_i\right)}{\prod_{j=1}^{n}\left(1 - p^L_i\right)} \tag{2.14}$$

giving the following z-domain complex transfer function

$$G(z) = \frac{P(z)}{D(z)} = \frac{\tilde{p}_n z^n + \tilde{p}_{n-1} z^{n-1} + \ldots + \tilde{p}_0}{d_n z^n + d_{n-1} z^{n-1} + \ldots + d_0} = \frac{K_z(z+1)^{m-n}(z - z_1)(z - z_2)\ldots(z - z_m)}{(z - p_1)(z - p_2)\ldots(z - p_n)} \tag{2.15}$$

where all polynomial coefficients, poles and zeros are allowed to be complex, or

$$\tilde{p}^L_i\ ,\ d^L_i\ ,\ z^L_i\ ,\ p^L_i\ ,\ \tilde{p}_i,\ d_i,\ z_i,\ p_i\ \varepsilon\ \mathbb{C} \qquad i = 1\ (1)\ n \tag{2.16}$$

The scattering matrix of a complex circuit can be defined using equations (2.3) and (2.4) with the familiar representation of the voltage $\mathbf{V} = \overline{\mathbf{Z}}\,\mathbf{I}$, where $\overline{\mathbf{Z}}$ is the complex impedance matrix, and $\mathbf{Z}$ is the port reference impedance matrix. Solving for $\mathbf{I}$ in both equations and equating the result yields

$$\mathbf{Z}^*\,\mathbf{A} + \mathbf{Z}\,\mathbf{B} = \overline{\mathbf{Z}}\,(\mathbf{A} - \mathbf{B}) \tag{2.17}$$

or, solving for the scattering matrix with the reflectances along the diagonal,

$$\mathbf{S} = \left(\overline{\mathbf{Z}} + \mathbf{Z}\right)^{-1}\left(\overline{\mathbf{Z}} - \mathbf{Z}^*\right) \tag{2.18}$$

An interesting form appears when $\overline{\mathbf{Z}}$ is an element, for example

$$\overline{\mathbf{Z}} = \overline{\mathrm{R}} + j\,\overline{\mathrm{X}} \tag{2.19}$$

with the reference port impedance Z given in (2.5),

$$\rho = \frac{\overline{R} + j\overline{X} - R + jX}{\overline{R} + j\overline{X} + R + jX} = \frac{(\overline{R} - R) + j(\overline{X} + X)}{(\overline{R} + R) + j(\overline{X} + X)} \qquad (2.20)$$

Clearly, when $\overline{\mathbf{Z}}$ and $\mathbf{Z}$ are real, the definition of the reflectance is the same as in the real case. From Equation (2.18), when $\rho = 0$,

$$\mathbf{Z} = \overline{\mathbf{Z}}^{\,*} \qquad (2.21)$$

or the input impedance must be a complex constant and equal to the conjugate of the reference port impedance.

2.1.3 Comparison with Wave Digital Filters Transferred to the Complex Domain

It has been suggested [26][27][28] that complex transfer functions can be implemented using real wave digital adaptors. This method requires a property known as one-realness, where the input impedance at $z = \infty$, or $\psi = 1$, must be purely real. This condition is easily verified, since a one-port must not create a delay-free loop, or the reflectance after an infinite delay must be zero. Using the definition of the real reflectance

$$\rho = \frac{\overline{Z} - R}{\overline{Z} + R} \qquad (2.22)$$

where R is the real reference port resistance and $\overline{Z}$ is the input impedance at $\psi = 1$, the condition that the reflectance must be zero states that the input impedance $\overline{Z}$ must be equal to R, or real. Since a general complex circuit will not be one-real, the transfer function representing the circuit, and thus the circuit itself, must be changed to produce one-realness. This is accomplished by multiplying the transfer function by a unimodular constant. All of the wave adaptors can then be used, with the only new elements being complex multipliers [26].

The above method is tedious and restrictive, and many realizations will require changes in the actual transfer function by at least a constant change of phase. For example, consider a general passive complex circuit consisting of resistances, constant reactances, capacitors, inductors, complex transformers, gyrators and circulators. The impedance will in general not be one-real, and therefore the transfer function of the circuit must be calculated (no matter how complicated the circuit), unless the transfer function is already known. Then the input impedance must be found, the phase constant to guarantee one-realness calculated, and a new transfer function derived in order to implement the circuit with the old (real) adaptors. This method is restrictive since if the original phase of the transfer function is critical, the circuit cannot be implemented with the real adaptors. Also, if a scattering matrix technique is to be used, no conditions guaranteeing pseudopassivity have been suggested, since all definitions apply to real, not complex, scattering matrices.

The complex WDF structure outlined in this thesis is not restrictive as it does not require the property of one-realness. This is because port reference impedances are allowed to be complex. Thus passive circuits in the analog domain with constant reactances, or constant complex impedances, can be directly transformed into the wave digital domain without alteration. Clearly, this allows the implementation of many important analog structures containing constant reactances, including those that do not allow changes in the phase characteristic.

2.2 Sensitivity of Complex Wave Digital Filters

The sensitivity of a complex wave digital network to a multiplier coefficient associated with a digital filter is very low in the passband. An argument similar to the one used by Fettweis [6] will be presented because of the importance of sensitivity.

A general complex transfer function in the z-domain describing a pseudopassive network can be viewed as a transmittance, S_{21}, from scattering matrix theory of a complex two-port as shown in Figure 2.3.

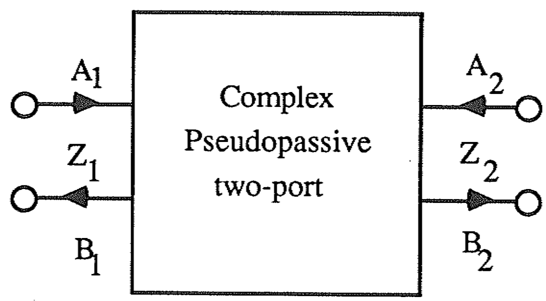

Figure 2.3: Complex pseudopassive two-port wave digital network.

The wave domain transfer function is clearly given by

$$S_{21} = \frac{B_2}{A_1} \tag{2.23}$$

where the incident wave (input) of port 1, A_1, and the reflected wave (output) of port 2, B_2, is given by

$$A_1 = V_1 + Z_1 I_1 \tag{2.24}$$

$$B_2 = V_2 - Z_2^* I_2 \tag{2.25}$$

and the complex reference port impedances for both ports are given by

$$Z_1 = R_1 + j X_1 \tag{2.26}$$

$$Z_2 = R_2 + j X_2 \tag{2.27}$$

The loss or attenuation, μ, corresponding to $S_{21}(z)$ is shown for a general case in Figure 2.4, and is defined by (2.28), where $S_{21}(z)$ is evaluated along the unit circle in the z-plane,

$$\mu = -0.5 \ln [S_{21}(z) \; S_{21}^{*}(z)] \quad , | z | = 1 \tag{2.28}$$

where μ is strictly non-negative ($\mu \geq 0$) if pseudopassivity can be maintained. Let β represent any multiplier coefficient which may in general be complex. Consider the attenuation explicitly as a real function of z and β

$$\mu = -0.5 \ln [S_{21}(z,\beta) \, S_{21}^{*}(z,\beta)] \quad , | z | = 1 \tag{2.29}$$

Let the real part of β be denoted β', and the imaginary part of β by β'',

$$\beta = \beta' + j \, \beta'' \tag{2.30}$$

then (2.29) becomes

$$\mu = -0.5 \ln [S_{21}(z,\beta' + j \, \beta'') \, S_{21}^{*}(z,\beta' + j \, \beta'')] \quad , | z | = 1 \tag{2.31}$$

Notice that the resulting term in the square braces is purely real, and thus the (real) attenuation function can be viewed as being a function of the two real, independent variables β' and β''.

The sensitivity of the attenuation to β, or equivalently to β' and β'', can be investigated by considering the paritial derivatives of μ, as given in (2.31), with respect to β' and β''. Note that since the attenuation can be considered to be a real function of two real variables β' and β'', the partial derivatives deal with real quantities, and thus the familiar definitions of differential calculus can be used. At a relative minimum of the attenuation, or at points where the response equals zero in the passband, as shown in Figure 2.4, the partial derivatives of μ with respect to β' and β'' must be zero [58].

$$\frac{\partial \mu}{\partial \beta'} = \frac{\partial \mu}{\partial \beta''} = 0 \quad \text{at} \quad \mu = 0 \tag{2.32}$$

Furthermore, at frequencies near zero attenuation, the partial derivatives must be near zero [58].

$$\frac{\partial \mu}{\partial \beta'} \approx 0 \, , \, \frac{\partial \mu}{\partial \beta''} \approx 0 \quad \text{at} \quad \mu \approx 0 \tag{2.33}$$

This suggests that the sensitivity of the attenuation in the passband with respect to β is very low. Further the sensitivity is smaller the closer the attenuation is to zero.

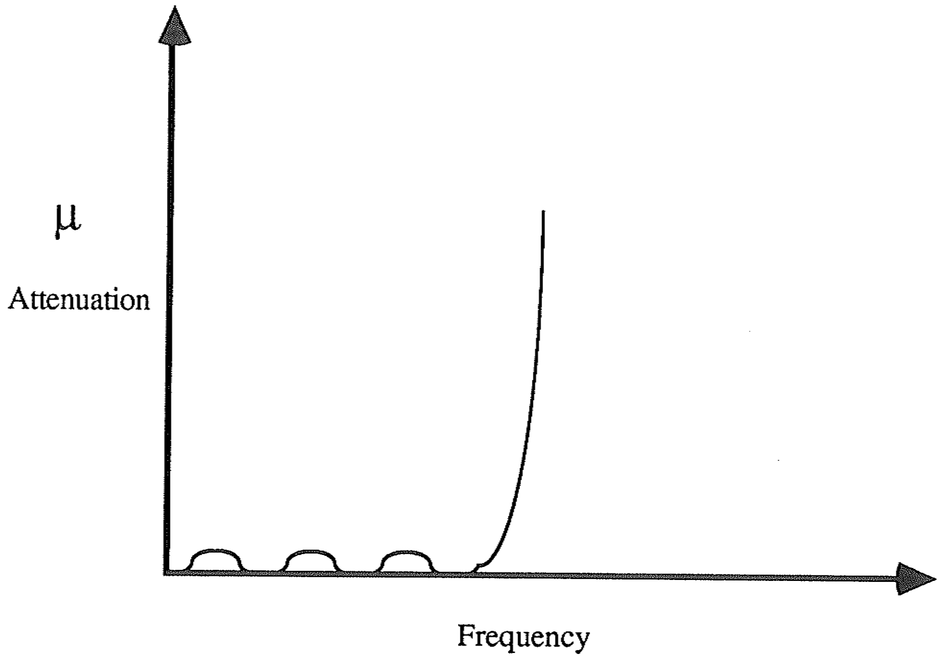

Figure 2.4: Loss response of a lowpass filter with parameter ß.

Consider a complex operator f (possible non-linear) acting on ß in such a way that it is changed from ß to ß + Δß. The operator f changes ß by changing the real part, imaginary part or both, and will clearly change the attenuation characteristic. However, if pseudopassivity can be maintained after applying f(ß), then it is guaranteed that $\mu \geq 0$. This suggests that the attenuation characteristic with f(ß) = ß + Δß must be of the form shown in Figure 2.5.

If the attenuation moves, it can only move in an upward direction because of the pseudopassivity condition. This movement is shown with arrows of different length in Figure 2.5. This process may give a better sensitivity with respect to ß than was first implied, since the total distortion of the attenuation characteristic in the passband is predominantly dictated by the differences in the upward shifts. These differences are often much smaller than the sizes of the actual shifts, and this gives low sensitivity when the passband attenuation is not near zero.

The above discussion of the sensitivity of the attenuation does not depend on how ß changes. The size of the change produced by the operator f, or whether the real part, imaginary part or both of the parameter ß are changed is clearly irrelevant, as long as pseudopassivity is maintained. Thus for a well designed filter where there is at least one region where the attenuation drops to zero and remains near zero over a range of frequencies, the sensitivity of the attenuation with respect to the change in the complex multiplier coefficient is near zero. Furthermore, the sensitivity is low throughout the passband of the filter.

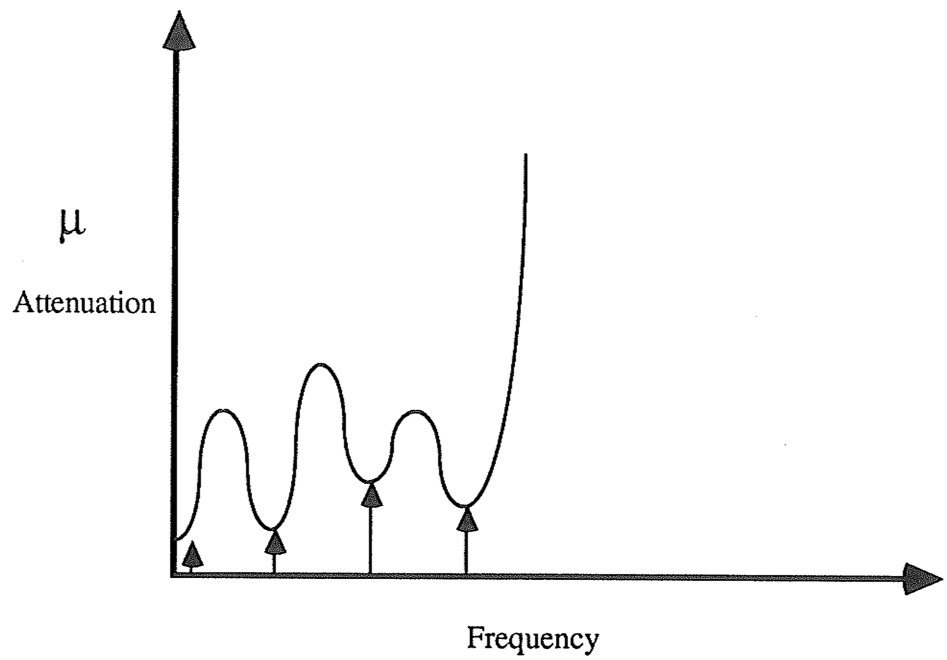

Figure 2.5: Loss response of a lowpass filter with parameter ß + Δß.

Dynamic range is defined as the ratio of the magnitude of the output signal for which overflows occur to the corresponding output quantization noise, and thus the dynamic range should ideally be large [6]. The sensitivity of a digital network has been linked to the dynamic range of the network [6][9][10], where the dynamic range is increased by a decrease in the sensitivity. The excellent sensitivity of complex wave digital filters is expected to lead to a large dynamic range as in the real case.

2.3 Pseudopower and Pseudopassivity

Pseudopower and the related topic of pseudopassivity play an important role in many of the useful properties of real wave digital filters and equivalent concepts are fundamentally necessary for the theory involving complex wave digital filters. The definition of pseudopower follows from the definition of power in the reference, or analog domain. The definition of, and conditions for, pseudopassivity and incremental pseudopassivity are developed from the theory proposed by Fettweis [3] and Meerkötter [7].

2.3.1 Definition of Pseudopower and Pseudoenergy

The steady-state pseudopower can be defined for complex wave digital filters by considering the power of a complex analog network and then transforming the power relation into the wave digital domain. Thus consider the linear time-invariant complex analog circuit shown in Figure 2.6.

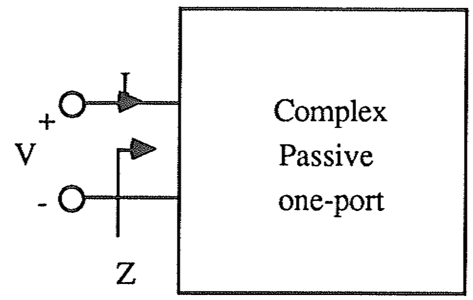

Figure 2.6: Complex passive one-port in the analog domain.

The power P is defined as the real part of the apparent power U, defined by

$$U = \mathbf{V}\,\mathbf{I}^{*T} \tag{2.34}$$

From equations (2.8) and (2.9), the voltage and current are given in terms of the incident and reflected waves, A and B, and the reference port impedance Z. For a one-port, the steady-state apparent power in the complex wave digital domain, with all quantities a function of the discrete variable z, becomes

$$U = \left(\frac{Z^*\,A + Z\,B}{2\,\mathrm{Re}(Z)}\right)\left(\frac{A^* - B^*}{2\,\mathrm{Re}(Z)}\right) \tag{2.35}$$

or

$$U = \frac{A^*\,Z^*\,A - B^*\,Z\,B + A^*\,Z\,B - B^*\,Z^*\,A}{4\,R^2} \tag{2.36}$$

Generalize to a complex n-port N with $Z_i = R_i + j\,X_i$, and define the conductance matrix

$$\tilde{Y} = \begin{bmatrix} \frac{Z_1}{R_1^2} & 0 & \dots & 0 \\ 0 & \frac{Z_2}{R_2^2} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \frac{Z_n}{R_n^2} \end{bmatrix} \tag{2.37}$$

where each Z_i is a complex constant. Also define the vectors **A** and **B** as the incident and reflected waves of the n-port. Then the steady-state apparent power becomes

$$U = \frac{\mathbf{A}^{*T}\tilde{\mathbf{Y}}^{*}\,\mathbf{A} - \mathbf{B}^{*T}\tilde{\mathbf{Y}}\,\mathbf{B} + \mathbf{A}^{*T}\tilde{\mathbf{Y}}\,\mathbf{B} - \mathbf{B}^{*T}\tilde{\mathbf{Y}}^{*}\,\mathbf{A}}{4} \tag{2.38}$$

However, from scattering matrix theory, it is known that $\mathbf{B} = \mathbf{S}\,\mathbf{A}$. Substituting

$$U = \frac{\mathbf{A}^{*T}\left(\tilde{\mathbf{Y}}^{*} - \mathbf{S}^{*T}\tilde{\mathbf{Y}}\,\mathbf{S} + \tilde{\mathbf{Y}}\,\mathbf{S} - \mathbf{S}^{*T}\tilde{\mathbf{Y}}^{*}\right)\mathbf{A}}{4} \tag{2.39}$$

$$U = \frac{\mathbf{A}^{*T}\left(\mathbf{I} - \mathbf{S}^{*T}\right)\tilde{\mathbf{Y}}\left(\mathbf{S} + \tilde{\mathbf{Y}}^{-1}\tilde{\mathbf{Y}}^{*}\right)\mathbf{A}}{4} \tag{2.40}$$

Define steady-state pseudopower P as the real part of the complex U.

$$P = \mathrm{Re}(U) = \mathrm{Re}\left(\frac{\mathbf{A}^{*T}\left(\tilde{\mathbf{Y}}^{*} - \mathbf{S}^{*T}\tilde{\mathbf{Y}}\,\mathbf{S} + \tilde{\mathbf{Y}}\,\mathbf{S} - \mathbf{S}^{*T}\tilde{\mathbf{Y}}^{*}\right)\mathbf{A}}{4}\right) \tag{2.41}$$

Identify the hermitian **H** and anti-hermitian $\mathbf{H}^A$ parts of the inner bracketted matrix in (2.41),

$$\mathbf{H} = \frac{1}{2}\left[\left(\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}^{*}\right) - \mathbf{S}^{*T}\left(\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}^{*}\right)\mathbf{S}\right] \tag{2.42}$$

$$\mathbf{H}^{A} = \frac{1}{2}\left[\left(\tilde{\mathbf{Y}}^{*} - \tilde{\mathbf{Y}}\right) - \mathbf{S}^{*T}\left(\tilde{\mathbf{Y}} - \tilde{\mathbf{Y}}^{*}\right)\mathbf{S} + 2\,\tilde{\mathbf{Y}}\,\mathbf{S} - 2\,\mathbf{S}^{*T}\tilde{\mathbf{Y}}^{*}\right] \tag{2.43}$$

The pseudopower is therefore

$$P = \frac{1}{4}\mathrm{Re}\left(\mathbf{A}^{*T}\mathbf{H}\,\mathbf{A} + \mathbf{A}^{*T}\mathbf{H}^{A}\mathbf{A}\right) = \frac{1}{4}\left[\mathrm{Re}\left(\mathbf{A}^{*T}\mathbf{H}\,\mathbf{A}\right) + \mathrm{Re}\left(\mathbf{A}^{*T}\mathbf{H}^{A}\mathbf{A}\right)\right] \tag{2.44}$$

However, since the quadratic form $(\mathbf{A}^{*T}\,\mathbf{H}\,\mathbf{A})$ of a hermitian matrix is purely real and the quadratic form of an anti-hermitian matrix is purely imaginary, the pseudopower only involves the hermitian matrix **H** as defined in (2.42). The real conductance matrix **G** can be identified from the hermitian part by considering

$$\left(\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}^{*}\right)_{i\,i} = \frac{Z_i}{R_i^2} + \frac{Z_i^*}{R_i^2} = \frac{2\,R_i}{R_i^2} = \frac{2}{R_i} = 2\,G_i \tag{2.45}$$

The real conductance matrix **G** is defined as

$$\mathbf{G} = \begin{bmatrix} G_1 & 0 & \dots & 0 \\ 0 & G_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & G_n \end{bmatrix} \tag{2.46}$$

with $G_i = 1/R_i$. Thus the steady-state pseudopower is given by (ignoring any constant mulitpliers),

$$P = \mathbf{A}^{*T}\mathbf{G}\,\mathbf{A} - \mathbf{A}^{*T}\mathbf{S}^{*T}\mathbf{G}\,\mathbf{S}\,\mathbf{A} \tag{2.47}$$

$$P = \mathbf{A}^{*T}\left(\mathbf{G} - \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S}\right)\mathbf{A} \tag{2.48}$$

When considering both the complex incident and reflected waves in the pseudopower equation,

$$P = \mathbf{A}^{*T}\mathbf{G}\,\mathbf{A} - \mathbf{B}^{*T}\mathbf{G}\,\mathbf{B} \tag{2.49}$$

$$P = \sum_{i=1}^{n} \left(|A_i|^2 - |B_i|^2 \right) G_i \tag{2.50}$$

Notice that the definition of the pseudopower is similar to the definition in the real domain, except the absolute value operators are replaced with modulus operators and the incident and reflected waves are complex.

Along with pseudopower comes the concept of pseudoenergy W(m), defined at the m^{th} time instant. The total pseudoenergy of a full synchronic complex wave digital network stored in the delays between times t_m and t_{m+1} can be defined as the sum of the pseudoenergy stored in each delay between t_m and t_{m+1}. Define the pseudoenergy stored in a delay of value T, between t_m and t_{m+1}, as

$$T\ G_i\ |a_i(m)|^2 \tag{2.51}$$

where G_i is the conductance, and $a_i(m)$ is the incident wave associated with the port i, between the m^{th} and the $m+1^{th}$ time instants. The total pseudoenergy stored in the delays is then given by

$$W(m) = T \sum_{i=1}^{n} |a_i(m)|^2 G_i \tag{2.52}$$

where n is the number of delays. Clearly, the pseudopower and pseudoenergy as defined are real quantities.

Using Equation (2.50) given above, another representation of the pseudopower is possible by using the definition of a norm as proposed by Meerkötter [7]. Define the following norm of a general n-port complex wave digital network at the m^{th} time instant,

$$||a(m)||^2 = \sum_{i=1}^{n} |a_i(m)|^2 G_i = \sum_{i=1}^{n} a_i^*(m)\, a_i(m)\, G_i \tag{2.53}$$

where a_i (m) refers to a wave associated with port i at time instant m, and G_i refers to the conductance of port i. Define the norm over m time instants, of the above norm over the n ports, as

$$\| a \|_m^2 = \sum_{j=0}^{m} \| a(j) \|^2 = \sum_{j=0}^{m} \sum_{i=1}^{n} | a_i(j) |^2 G_i = \sum_{j=0}^{m} \sum_{i=1}^{n} a_i^*(j)\, a_i(j)\, G_i \tag{2.54}$$

where the symbol 'a' inside the norm operator on the left of (2.54) refers to the sequence of wave quantities over m instants and over the n-port. The pseudopower can be represented as

$$P = \| a(m) \|^2 - \| b(m) \|^2 \tag{2.55}$$

Similarly, the pseudoenergy absorbed by the n-port until the m^{th} time instant can be represented as

$$W(m) = T \sum_{j=0}^{m} \left[\| a(m) \|^2 - \| b(m) \|^2 \right] = T\left(\| a \|_m^2 - \| b \|_m^2 \right) \tag{2.56}$$

Clearly, if $\| a \|_m^2$ is bounded, then from (2.54), we have

$$\lim_{m \Rightarrow \infty} \left[\| a(m) \|^2 \right] = 0 \tag{2.57}$$

2.3.2 Conditions for Pseudopassivity

A reference, or analog network is said to be passive if the power P is greater than or equal to zero in the closed right half Laplacian plane. Similarly, a reference network is said to be lossless if the power P is identically equal to zero along the frequency axis in the Laplacian plane. The concepts of passivity and losslessness in the analog domain lead to similar concepts in the wave digital domain. The wave digital network is pseudolossless if the steady-state pseudopower equals zero along the unit circle in the z-plane and pseudopassive if the steady-state pseudopower is non-negative outside the unit circle in the z-plane.

For a pseudolossless network, the right side of Equation (2.48) is identically zero. However, Equation (2.48) is in the quadratic form of a hermitian matrix.

Lemma: If a quadratic form of a hermitian matrix is equal to zero, the hermitian matrix must be identically equal to zero.

Proof:

The quadratic form of a hermitian matrix **H** is real and given by

$$\mathbf{A}^{*T}\, \mathbf{H}\, \mathbf{A} \tag{2.58}$$

where **A** is a complex column vector, and the result is equal to zero for all **A**. The diagonal elements of **H** are immediately seen to be zero by considering a real column vector **A** with a one in the i^{th} location. Equation (2.45) then becomes

$$(0\ 0 \ldots 1 \ldots 0\ 0)\ \mathbf{H}\ (0\ 0 \ldots 1 \ldots 0\ 0)^T = 0 \tag{2.59}$$

from which we can conclude that

$$h_{ii} = 0, \quad i=1\ (1)\ n \tag{2.60}$$

The off-diagonal elements are seen to be zero by showing that the real and imaginary parts of the off-diagonal elements must be zero. First choose a real vector **A** with a one in the i^{th} and j^{th} locations. Then perform the multiplication

$$(0\ 0 \ldots 1 \ldots 0\ 0 \ldots 1 \ldots 0\ 0)\ \mathbf{H}\ (0\ 0 \ldots 1 \ldots 0\ 0 \ldots 1 \ldots 0\ 0)^T = 0 \tag{2.61}$$

The result is

$$h_{ij} + h_{ij}^* = 0, \qquad Re(h_{ij}) = 0, \qquad i,j = 1\ (1)\ n \quad i \neq j \tag{2.62}$$

Similarly, by choosing a complex vector **A** with a one in the i^{th} location and the complex constant j in the j^{th} location and perform the multiplication

$$(0\ 0 \ldots 1 \ldots 0\ 0 \ldots j \ldots 0\ 0)^*\ \mathbf{H}\ (0\ 0 \ldots 1 \ldots 0\ 0 \ldots j \ldots 0\ 0)^T = 0 \tag{2.63}$$

The result is

$$j\,(h_{ij} - h_{ij}^*) = 0, \qquad Im(h_{ij}) = 0, \qquad i,j = 1\ (1)\ n \quad i \neq j \tag{2.64}$$

Thus if the quadratic form of a hermitian matrix is equal to zero, the hermitian matrix is also equal to zero.

Q.E.D.

Thus a condition for pseudolosslessness is

$$\mathbf{G} - \mathbf{S}^{*T}\,\mathbf{G}\,\mathbf{S} = \mathbf{0} \tag{2.65}$$

or

$$\mathbf{G} = \mathbf{S}^{*T}\,\mathbf{G}\,\mathbf{S}$$

The pseudopassive property implies from (2.48) that

$$P = \mathbf{A}^{*T}\left(\mathbf{G} - \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S}\right)\mathbf{A} \geq 0 \tag{2.66}$$

and this gives a condition on the hermitian matrix inside the brackets defined as **H**. Let the matrices **H** and **A** be given in terms of the real and imaginary parts as

$$\mathbf{H} = \mathbf{H}' + j\,\mathbf{H}'' \tag{2.67}$$

$$\mathbf{A} = \mathbf{A}' + j\,\mathbf{A}'' \tag{2.68}$$

Then (2.66) becomes

$$P = \left(\mathbf{A}'^T - j\,\mathbf{A}''^T\right)\left(\mathbf{H}' + j\,\mathbf{H}''\right)\left(\mathbf{A}' + j\,\mathbf{A}''\right) \geq 0 \tag{2.69}$$

where **A'**, **A"**, **H'**, and **H"** are real matices. From the defintion of a hermitian matrix

$$\mathbf{H}' = \mathbf{H}'^T \tag{2.70}$$

$$\mathbf{H}'' = -\,\mathbf{H}''^T \tag{2.71}$$

Multiplying the matrices in Equation (2.69) and recognizing that the imaginary part is equal to zero, we get

$$P = A'^T H' A' + A''^T H' A'' - A'^T H'' A'' + A''^T H'' A' \geq 0 \qquad (2.72)$$

Using (2.71), the last two terms of the above can be related by taking the transpose of one term, giving

$$A''^T H'' A' = - A'^T H'' A'' \qquad (2.73)$$

The steady-state pseudopassive condition after substituting the above becomes

$$P = A'^T H' A' + A''^T H' A'' - 2 A'^T H'' A'' \geq 0 \qquad (2.74)$$

and we immediately have a condition that the real part of the hermitian matrix **H**, or

$$\mathrm{Re}\{\mathbf{H}\} = \mathrm{Re}\{\mathbf{G} - \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S}\} \qquad (2.75)$$

must be positive semi-definite. A sufficient condition is given by expressing (2.74) above in matrix form where the elements of all matrices will be real

$$\begin{bmatrix} A'^T & A''^T \end{bmatrix} \begin{bmatrix} H' & -H'' \\ H'' & H' \end{bmatrix} \begin{bmatrix} A' \\ A'' \end{bmatrix} \geq 0 \qquad (2.76)$$

or

$$\begin{bmatrix} A'^T & A''^T \end{bmatrix} \begin{bmatrix} \mathrm{Re}\left(G - S^{*T} G\,S\right) & -\mathrm{Im}\left(G - S^{*T} G\,S\right) \\ \mathrm{Im}\left(G - S^{*T} G\,S\right) & \mathrm{Re}\left(G - S^{*T} G\,S\right) \end{bmatrix} \begin{bmatrix} A' \\ A'' \end{bmatrix} \geq 0 \qquad (2.77)$$

and thus the following matrix must be positive semi-definite

$$\begin{bmatrix} \mathrm{Re}\left(G - S^{*T} G\,S\right) & -\mathrm{Im}\left(G - S^{*T} G\,S\right) \\ \mathrm{Im}\left(G - S^{*T} G\,S\right) & \mathrm{Re}\left(G - S^{*T} G\,S\right) \end{bmatrix} \qquad (2.78)$$

Notice that if the scattering matrix S is purely real, the condition given by the right side of (2.75) is sufficient for guaranteeing the pseudopassivity of a wave digital network.

Thus a necessary and sufficient condition for the pseudopassivity of a general complex wave digital network is that (2.78) be positive semi-definite. Since in most analyses, the positive semi-definiteness of (2.78) is not easily verified, another method of guaranteeing pseudopassivity can be derived by considering the same criteria using the adaptor approach of implementation. Clearly, the adaptor approach is equivalent to the scattering matrix realization if none of the relavent parameters are quantized, since the scattering matrix can define and be defined by the adaptor realization; ie, there is a one-to-one correspondance between the two realizations (without quantizations).

Similar to real WDF, a connection of pseudopassive adaptors and elements forms a pseudopassive network as long as the ports are compatible for connection. Thus proving the pseudopassivity of a network is accomplished by proving the pseudopassivity of all of the building blocks that make up the network when the criteria for connecting adaptors is obeyed.

This can be verified by considering the connection of a n-port network N and a m-port network $\hat{N}$ as shown in Figure 2.7 with the connecting ports of both networks labelled 1 and $\hat{1}$.

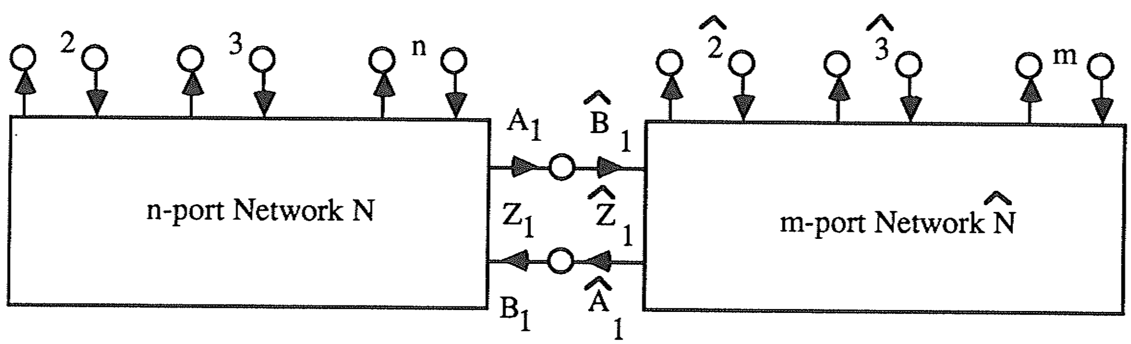

Figure 2.7: Connection of a n-port and a m-port.

The actual connection of two ports is pseudolossless since

$$A_1 = \hat{B}_1 \tag{2.79}$$

$$B_1 = \hat{A}_1 \tag{2.80}$$

and the port impedances are related by (from Section 2.5.2),

$$Z_1 = \hat{Z}_1^* \tag{2.81}$$

and certainly the port conductances are equal

$$G_1 = \hat{G}_1 \tag{2.82}$$

Thus, if networks N and $\hat{N}$ are pseudopassive, then $P \geq 0$ and $\hat{P} \geq 0$ and Equation (2.50) gives

$$P = \left(|A_1|^2 - |B_1|^2\right)G_1 + \sum_{i=2}^{n}\left(|A_i|^2 - |B_i|^2\right)G_i \geq 0 \tag{2.83}$$

$$\widehat{P} = \left(\left|\widehat{A}_1\right|^2 - \left|\widehat{B}_1\right|^2\right)\widehat{G}_1 + \sum_{i=2}^{m}\left(\left|\widehat{A}_i\right|^2 - \left|\widehat{B}_i\right|^2\right)\widehat{G}_i \geq 0 \tag{2.84}$$

Since the pseudopower of both networks is positive, their sum must also be positive. Substitute (2.79-80) and (2.82) into (2.83-84), and add the resulting expressions to derive the total pseudopower of both networks.

$$P + \widehat{P} = \sum_{i=2}^{n}\left(\left|A_i\right|^2 - \left|B_i\right|^2\right)G_i + \sum_{i=2}^{m}\left(\left|\widehat{A}_i\right|^2 - \left|\widehat{B}_i\right|^2\right)\widehat{G}_i \geq 0 \tag{2.85}$$

Thus the terms involving the connecting port of both networks cancel in the total pseudopower expression. Both summation terms of (2.85) represent the total pseudopower of each network with the first port removed. Therefore the two networks shown in Figure 2.7 can be viewed as one network with positive pseudopower from (2.85). This implies that the connection of two pseudopassive networks is itself pseudopassive.

The argument can be extended to any number of pseusopassive networks. Notice that the development did not require linearity in either network, rather the networks can contain any linear or non-linear processes as long as the pseudopassivity of each network is guaranteed and the port reference impedances are conjugates of each other (refer to Section 2.5.2).

2.3.3 Definition of Incremental Pseudopower and Incremental Pseudoenergy

Incremental pseudopassivity as introduced by Meerkötter [7] follows directly from the argument used in the real domain. The concepts involved are equivalent, and thus only a brief discussion is presented here. The definitions presented will be used in the stability argument in Section 2.7.

Let the column matrices $\mathbf{a}(m)$ and $\widehat{\mathbf{a}}(m)$ represent two complex input sequences to a stable, causal and possibly non-linear wave digital n-port network N with initial states $\mathbf{X}_0$ and $\widehat{\mathbf{X}}_0$ and with the corresponding output sequences $\mathbf{b}(m)$ and $\widehat{\mathbf{b}}(m)$. Define the incremental pseudopower, P_Δ, as

$$P_\Delta = \left\|\mathbf{a}(m) - \widehat{\mathbf{a}}(m)\right\|^2 - \left\|\mathbf{b}(m) - \widehat{\mathbf{b}}(m)\right\|^2 \tag{2.86}$$

and the incremental pseudoenergy, $W_\Delta(m)$, as

$$W_{\Delta}(m) = \sum_{j=0}^{m} \left(\left\| a(j) - \hat{a}(j) \right\|^2 - \left\| b(j) - \hat{b}(j) \right\|^2 \right) \tag{2.87}$$

2.4 One-Port Equivalences with Complex Analog Circuits

The resistor, capacitor and inductor are common elements in real analog circuits. It can be shown that passivity is ensured only if all capacitors and inductors in a circuit are real and all resistances are positive. Thus the only new elementary element that is added in the complex domain is the imaginary resistor, or constant reactance, which can be either positive or negative. An impedance made up of a positive resistor and a constant reactance is known as a constant impedance.

2.4.1 Unimodular Multiplier in the Complex Wave Digital Domain

A one-port with a constant reactance as the only element is shown in Figure 2.8. It can easily be shown that this circuit is passive in the analog domain. Consider the wave digital one-port of Figure 2.9 with a unimodular constant multiplier. This one-port only changes the phase of the incident wave. The port reference impedance is $Z = R + j\,X$, and the incident and reflected waves are as shown. The bilinear transformation is not needed since the impedance in the analog domain is constant by assumption, and the definitions of the complex wave variables can be applied directly.

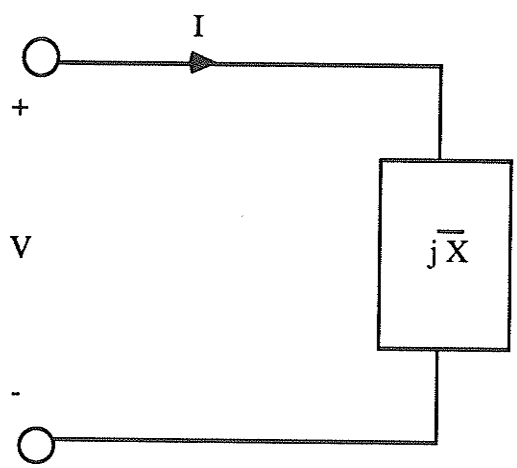

Figure 2.8: Constant reactance in the voltage-current domain.

Proceeding with the definitions of the wave variables,

$$A = V + Z\,I \tag{2.88}$$
$$B = V - Z^*\,I \tag{2.89}$$

Thus

$$\frac{B}{A} = \frac{V - Z^* I}{V + Z I} = \frac{-R + j(X + \overline{X})}{R + j(X + \overline{X})} = e^{j\theta} \tag{2.90}$$

Solving for V/I,

$$\frac{V}{I} = \frac{Z e^{j\theta} + Z^*}{1 - e^{j\theta}} = \left(\frac{1 + e^{j\theta}}{1 - e^{j\theta}}\right) R - j X \tag{2.91}$$

$$\frac{V}{I} = \left(\frac{\dfrac{\left(e^{\frac{j\theta}{2}} + e^{\frac{j\theta}{2}}\right)}{2}}{\dfrac{-j\left(e^{\frac{j\theta}{2}} - e^{\frac{j\theta}{2}}\right)}{2j}}\right) R - j X \tag{2.92}$$

$$\frac{V}{I} = j\left(\frac{\cos\left(\theta/2\right)}{\sin\left(\theta/2\right)}\right) R - j X \tag{2.93}$$

$$\frac{V}{I} = j\left[\frac{R}{\tan\left(\theta/2\right)} - X\right] \tag{2.94}$$

$$\frac{V}{I} = j\,\overline{X} \tag{2.95}$$

with the constant complex impedance defined by

$$\overline{X} = \left[\frac{R}{\tan\left(\theta/2\right)} - X\right] \tag{2.96}$$

Thus a unimodular multiplier in the wave digital domain with a port reference impedance Z = R + j X as shown in Figure 2.9 corresponds to a constant reactance in the analog domain with a value given by (2.96).

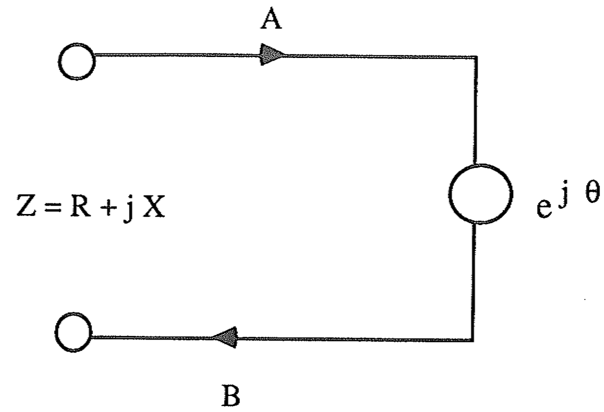

Figure 2.9: Unimodular multiplier in the complex wave digital domain.

Notice that the reverse transformation is not unique since the solution set $\{R, X, \theta\}$ is infinite when the constant reactance is given in the analog domain. Also, the wave digital one-port is pseudolossless, since the pseudopower is given by

$$P = \mathbf{A}^{*T} [\, \mathbf{G} - \mathbf{S}^{*T} \mathbf{G}\, \mathbf{S} \,]\, \mathbf{A} \tag{2.97}$$

and

$$\mathbf{G} = G = 1/R > 0 \tag{2.98}$$

$$\mathbf{S} = S = e^{j\theta} \tag{2.99}$$

Thus

$$P = A^{*T} [\, 1/R - e^{-j\theta}\, 1/R\, e^{j\theta} \,]\, A = 0 \tag{2.100}$$

2.4.2 Capacitor in Series With A Constant Reactance

Consider a real capacitor C in series with a constant reactance as shown in Figure 2.10. This one-port is clearly passive and frequency dependent.

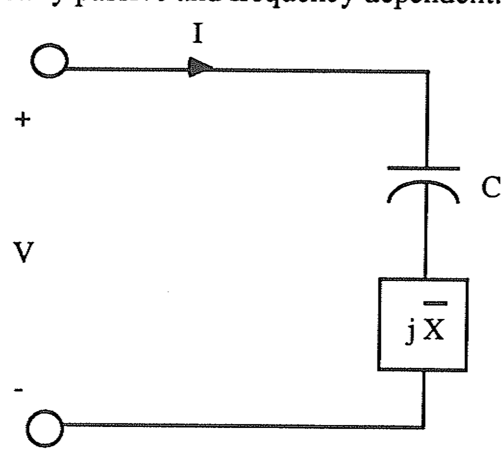

Figure 2.10: Capacitor in series with a constant reactance.

The complex voltage is given by

$$V = \left(\frac{1}{C\,\psi} + j\,\overline{X} \right) I \tag{2.101}$$

Since the input impedance is dependent upon the frequency, the bilinear transformation will be needed along with the definitions of the incident and reflected waves. Assume the wave digital port reference impedance is $Z = R + j\,X$. Substituting the equations

$$V = \frac{Z^*\,A + Z\,B}{Z + Z^*} = \frac{Z^*\,A + Z\,B}{2\,\mathrm{Re}(Z)} \tag{2.102}$$

$$I = \frac{A - B}{Z + Z^*} = \frac{A - B}{2\,\mathrm{Re}(Z)} \tag{2.103}$$

into (2.101) while delaying the substitution of the bilinear transformation, we have

$$Z^*\,A + Z\,B = \left(\frac{1}{C\,\psi} + j\,\overline{X} \right) (A - B) \tag{2.104}$$

$$\frac{B}{A} = \frac{\dfrac{1}{C\,\psi} + j\,\overline{X} - Z^*}{\dfrac{1}{C\,\psi} + j\,\overline{X} + Z} = \frac{1 + \left(j\,\overline{X} - Z^*\right) C\,\psi}{1 + \left(j\,\overline{X} + Z\right) C\,\psi} \tag{2.105}$$

Substituting the bilinear transformation

$$\psi = \frac{z - 1}{z + 1} \tag{2.106}$$

into (2.105),

$$\frac{B}{A} = \frac{z + 1 + \left(j\,\overline{X} - Z^*\right) C\,(z - 1)}{z + 1 + \left(j\,\overline{X} + Z\right) C\,(z - 1)} \tag{2.107}$$

$$\frac{B}{A} = \frac{\left[1 + \left(j\,\overline{X} - Z^*\right) C\right] + \left[1 - \left(j\,\overline{X} - Z^*\right) C\right] z^{-1}}{\left[1 + \left(j\,\overline{X} + Z\right) C\right] + \left[1 - \left(j\,\overline{X} + Z\right) C\right] z^{-1}} \tag{2.108}$$

With the correct assignment of the reference port values R and X, the above equation could represent a delay in the wave digital domain. Let the port impedance values be assigned as

$$X = -\overline{X} \tag{2.109}$$

$$R = \frac{1}{C} \tag{2.110}$$

or the reference port impedance is

$$Z = \frac{1}{C} - j\overline{X} \tag{2.111}$$

Substituting these values into Equation (2.108), the relation becomes

$$\frac{B}{A} = z^{-1} \tag{2.112}$$

Therefore a one-port delay with a reference port impedance $Z = R + j\,X$ in the wave digital domain corresponds to a capacitor $1/R$ in series with a constant reactance $-jX$. This equivalence is shown in Figure 2.11 where T is the clock period or $T = 1/F$ where F is the operating frequency of the digital filter. The resulting wave digital one-port is pseudolossless.

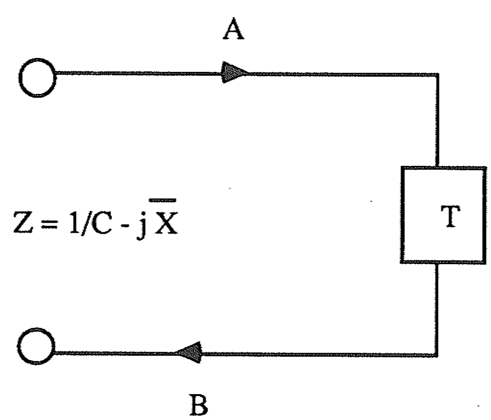

Figure 2.11: Delay in the complex wave digital domain.

Notice that the equivalence between the wave digital and analog domains is unique. Thus an analog circuit with a constant reactance in series with a capacitor can immediately be transformed into the wave digital domain as a one-port as in Figure 2.11.

Also, the equivalence reduces to the known result when the constant reactance is removed in the analog domain, and the reference port impedance becomes $Z = 1/C$. Or, a capacitor without a constant reactance transforms to a delay with a real port impedance, which is the familiar result derived in the real domain.

2.4.3 **Inductor in Series With a Constant Reactance**

A one-port consisting of a constant reactance in series with a real inductor is shown in Figure 2.12. The input impedance is complex and frequency dependent and thus the bilinear transformation will be needed when transforming to the digital domain.

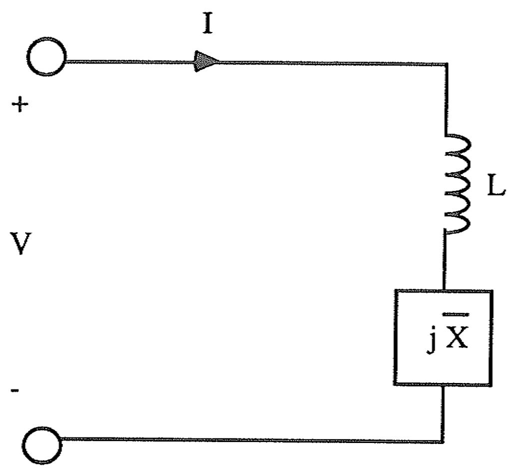

Figure 2.12: Inductor in series with a constant reactance.

The complex voltage is given by

$$V = \left(\psi L + j\overline{X}\right) I \tag{2.113}$$

Assume a reference port impedance of $Z = R + j X$ in the wave digital domain. Apply the wave variable transformation defined below, while delaying the bilinear transformation substitution for clarity,

$$V = \frac{Z^* A + Z B}{Z + Z^*} = \frac{Z^* A + Z B}{2 \operatorname{Re}(Z)} \tag{2.114}$$

$$I = \frac{A - B}{Z + Z^*} = \frac{A - B}{2 \operatorname{Re}(Z)} \tag{2.115}$$

Substituting into (2.113),

$$Z^* A + Z B = \left(\psi L + j\overline{X}\right)(A - B) \tag{2.116}$$

$$\frac{B}{A} = \frac{\psi L + j\overline{X} - Z^*}{\psi L + j\overline{X} + Z} \tag{2.117}$$

Substituting the bilinear transformation

$$\psi = \frac{z - 1}{z + 1} \tag{2.118}$$

into (2.117), the reflectance is given in the discrete domain as

$$\frac{B}{A} = \frac{L(z-1) + (j\overline{X} - Z^*)(z+1)}{L\,(z-1) + (j\overline{X} + Z)(z+1)} \tag{2.119}$$

$$\frac{B}{A} = \frac{[L + j\overline{X} - Z^*] + [j\overline{X} - Z^* - L]\,z^{-1}}{[L + j\overline{X} + Z] + [j\overline{X} + Z - L]\,z^{-1}} \tag{2.120}$$

Similar to the case with the capacitor, the above reflectance can be made equal to an inversion in series with a delay by correctly assigning the reference port impedance values R and X. By choosing

$$R = L \tag{2.121}$$

$$X = -\overline{X} \tag{2.122}$$

or

$$Z = L - j\overline{X} \tag{2.123}$$

we have the result

$$\frac{B}{A} = -\,z^{-1} \tag{2.124}$$

Thus, a delay in series with an inverter in the wave digital domain with a complex reference port impedance $Z = R + j\,X$ corresponds to an inductance $L = R$ in series with a complex reactance $-jX$. Notice that this equivalence, as shown in Figure 2.13, forms a transform pair, or the equivalence has a unique inverse. Again T is the clock period and represents a delay of one clock cycle. The resulting wave digital one-port is pseudolossless as expected.

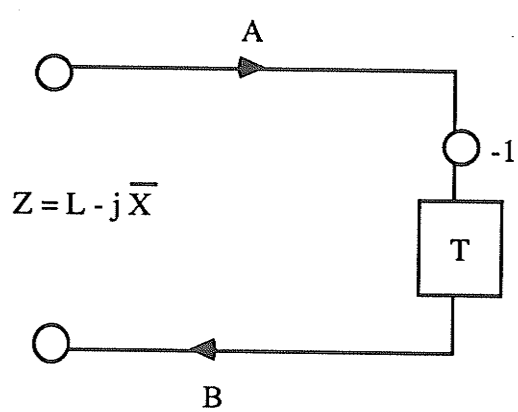

Figure 2.13: Inverting delay complex wave digital one-port.

Note that if the constant reactance in the analog domain is equal to zero, the complex reference port impedance reduces to the real quantity $Z = L$, which is the same result as derived in the real domain. Thus the equivalence shown in Figure 2.13 reduces to the familiar result of an inductor in the analog domain when the port reference impedance is real.

2.4.4 Source and Non-dynamic One-Ports

A voltage source E in series with a constant impedance $\overline{Z} = \overline{R} + j\,\overline{X}$ is shown below in Figure 2.14.

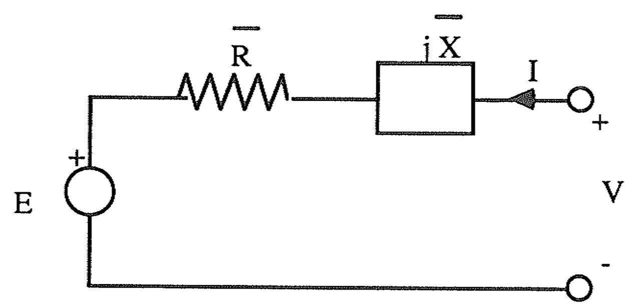

Figure 2.14: Voltage souce in series with a constant impedance.

The port voltage V is given by

$$V = E + \overline{Z}I = E + (\overline{R} + j\overline{X})I \qquad (2.125)$$

or

$$E = V - \overline{Z}I \qquad (2.126)$$

The wave digital equivalent of the voltage source in series with the constant impedance is thus recognized from the transformation

$$A = V + Z\,I \qquad (2.127)$$

$$B = V - Z^*\,I \qquad (2.128)$$

where $Z = R + j\,X$ is the reference port impedance. Comparing (2.126) and (2.128), one sees that the most convenient choice for the port reference impedance is equal to the conjugate of the series impedance of Figure 2.14, or

$$Z = \overline{Z}^* \qquad (2.129)$$

This gives the following wave digital flow diagram where the reflected wave is equal to E and the incident wave terminates in a "wave sink" as shown below.

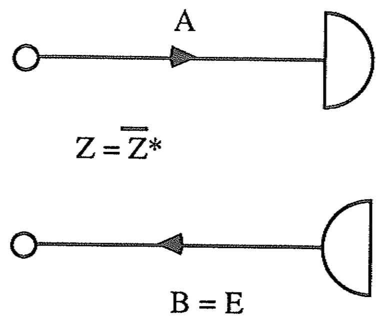

Figure 2.15: Complex wave digital equivalent of a voltage source in series with a constant impedance.

The same process can be repeated for the case where E = 0, or the one-port circuit contains a non-dynamic, or constant impedance as shown in Figure 2.16 below.

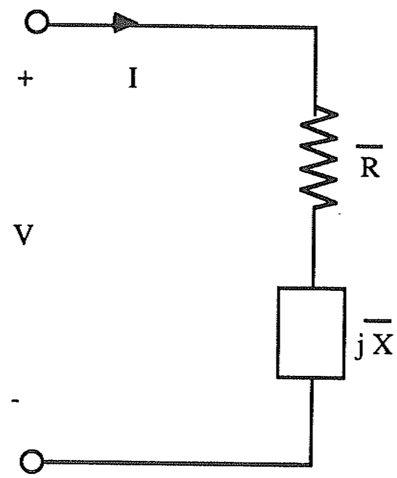

Figure 2.16: Constant one-port impedance.

The voltage is given by

$$V = \overline{Z}\, I = \left(\overline{R} + j\,\overline{X}\right) I \tag{2.130}$$

and using the wave digital transformation as given by (2.127) and (2.128), the following is derived

$$\frac{B}{A} = \frac{\overline{R} - R + j\left(X + \overline{X}\right)}{\overline{R} + R + j\left(X + \overline{X}\right)} \tag{2.131}$$

Equation (2.131) above can identify many possible wave digital equivalences, depending upon how the reference port impedance $Z = R + j\,X$ is chosen. The most straight forward choice for the reference port impedance is the conjugate of the impedance of the one-port of

Figure 2.16, or

$$Z = \overline{Z}^* \tag{2.132}$$

which gives the following wave flow diagram

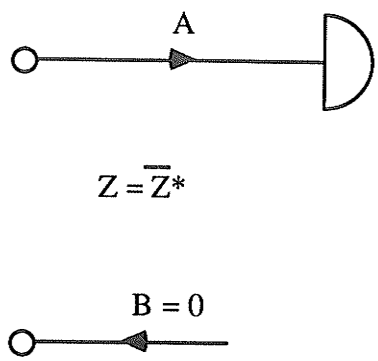

Figure 2.17: Constant impedance in the wave digital domain.

Clearly, other choices of the reference port impedance are possible. For example, in Section 2.4.1 the reflected wave from a pure constant reactance was not equal to zero and this suggests that the impedance was not matched to the port impedance. However, a port with a reference impedance equal to a pure constant reactance does not give a nonzero reflected wave, as shown in Figure 2.17, in general. A pure resistance gives the familiar result from real wave digital theory that $R = \overline{R}$, and the wave flow diagram of Figure 2.17 is given.

2.5 Definition of Complex Wave Digital Multi-Ports

A common CWDF building block is the two-port structure characterized by two complex incident and reflected waves and two constant impedances. Included in the library of two-port structures is the complex two-port adaptor and the complex transformer. The complex three-port circulator can also be defined.

2.5.1 Complex Two-port Adaptor

The two-port adaptor is useful in the interconnection of two wave ports with different reference impedances. The symbol used for a complex two-port adaptor is shown in Figure 2.18 (without the parameter characterizing the two-port), along with the incident and reflected wave variables for both ports with the corresponding complex reference port impedances.

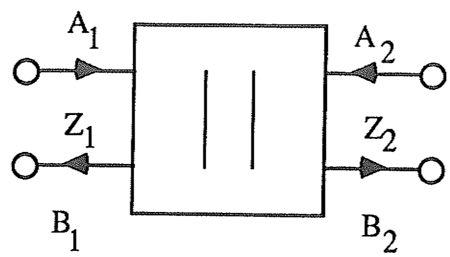

Figure 2.18: Complex two-port adaptor without the characteristic parameter.

2.5.1.1 Definition of the Complex Two-Port Adaptor

The voltage and current conditions that must be satisfied are given by

$$V_1 = V_2 \tag{2.133}$$

$$I_1 = -I_2 \tag{2.134}$$

and are shown in Figure 2.19.

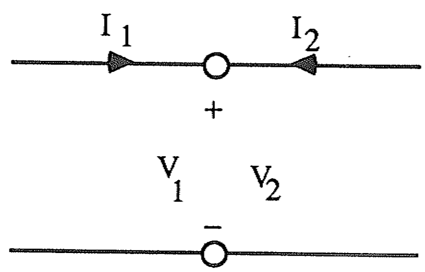

Figure 2.19: Voltages and currents for a two-port interconnection.

Assuming the complex reference port impedances are Z_1 and Z_2 for ports 1 and 2 respectively, the wave variables are defined by

$$A_1 = V_1 + Z_1 I_1 \tag{2.135}$$

$$B_1 = V_1 - Z_1^* I_1 \tag{2.136}$$

$$A_2 = V_2 + Z_2 I_2 \tag{2.137}$$

$$B_2 = V_2 - Z_2^* I_2 \tag{2.138}$$

$$Z_1 = R_1 + j X_1 \tag{2.139}$$

$$Z_2 = R_2 + j X_2 \tag{2.140}$$

Combining these equations with (2.133) and (2.134), the following are derived

$$B_1 = A_2 + \left(\frac{Z_1^* - Z_2}{Z_1 + Z_2}\right)(A_2 - A_1) \tag{2.141}$$

$$B_2 = A_1 + \left(\frac{Z_1 - Z_2^*}{Z_1 + Z_2}\right)(A_2 - A_1) \tag{2.142}$$

Notice that the above definitions for the reflected waves of the complex two-port adaptor reduce to the known definitions in the real case since the bracketted terms involving the port impedances for equations (2.141) and (2.142) become

$$\left(\frac{Z_1^* - Z_2}{Z_1 + Z_2}\right) = \left(\frac{Z_1 - Z_2^*}{Z_1 + Z_2}\right) = \left(\frac{R_1 - R_2}{R_1 + R_2}\right) \tag{2.143}$$

Define two complex parameters, α_1 and α_2, which represent the impedance terms in equations (2.141) and (2.142),

$$\alpha_1 = \left(\frac{Z_1^* - Z_2}{Z_1 + Z_2}\right) \tag{2.144}$$

$$\alpha_2 = \left(\frac{Z_1 - Z_2^*}{Z_1 + Z_2}\right) \tag{2.145}$$

Thus equations (2.141) and (2.142) become

$$B_1 = -\alpha_1 A_1 + \left(1 + \alpha_1\right) A_2 \tag{2.146}$$

$$B_2 = \left(1 - \alpha_2\right) A_1 + \alpha_2 A_2 \tag{2.147}$$

or in the form of a scattering matrix,

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 1 + \alpha_1 \\ 1 - \alpha_2 & \alpha_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{2.148}$$

Examining equations (2.144) and (2.145), one sees that α_1 and α_2 are related, and thus one parameter is redundant. It is easily verified that

$$\alpha_1 = \left(\frac{1 - \alpha_2}{1 - \alpha_2^*}\right) \alpha_2^* \tag{2.149}$$

$$\alpha_2 = \left(\frac{\alpha_1 + 1}{\alpha_1^* + 1}\right) \alpha_1^* \tag{2.150}$$

We are free to eliminate one parameter and choose the other, say α_2, as the single parameter characterizing the complex two-port adaptor. Thus define the parmeter α as follows

$$\alpha = \alpha_2 = \left(\frac{Z_1 - Z_2^*}{Z_1 + Z_2}\right) = \left(\frac{(R_1 - R_2) + j(X_1 + X_2)}{(R_1 + R_2) + j(X_1 + X_2)}\right) \tag{2.151}$$

With the removal of the redundant parameter, the scattering matrix becomes

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{1-\alpha}{1-\alpha^*}\right)\alpha^* & \left(1 + \left(\dfrac{1-\alpha}{1-\alpha^*}\right)\alpha^*\right) \\ 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{2.152}$$

All of the entries in the scattering matrix are in general complex. The magnitudes of the diagonal entries will be shown below to be bounded by one, whereas the magnitudes of the off-diagonal entries are bounded by two, which is expected from the theory of transfer functions and reflectances.

2.5.1.2 **Bound on Magitudes of the Complex Two-Port Adaptor Parameters**

The two related paramters as given by equations (2.144) and (2.145) are in general complex. A bound on their magnitudes is of interest for implementation considerations. The first parameter in terms of the resistance and reactance of the two complex port reference impedances is

$$\alpha_1 = \frac{R_1^2 - R_2^2 - (X_1 + X_2)^2 - j\,2R_1(X_1 + X_2)}{(R_1 + R_2)^2 + (X_1 + X_2)^2} \tag{2.153}$$

The magnitude of α_1 is given by

$$\left|\alpha_1\right| = \frac{\sqrt{\left(R_1^2 - R_2^2\right)^2 + 2\left(R_1^2 + R_2^2\right)(X_1 + X_2)^2 + (X_1 + X_2)^4}}{(R_1 + R_2)^2 + (X_1 + X_2)^2} \tag{2.154}$$

but it is easily seen that

$$\sqrt{\left(R_1^2 - R_2^2\right)^2 + 2\left(R_1^2 + R_2^2\right)(X_1 + X_2)^2 + (X_1 + X_2)^4} \le \left(R_1^2 + R_2^2\right) + (X_1 + X_2)^2 \tag{2.155}$$

and further that

$$\left(R_1^2 + R_2^2\right) + \left(X_1 + X_2\right)^2 \le \left(R_1 + R_2\right)^2 + \left(X_1 + X_2\right)^2 \tag{2.156}$$

and thus the magnitude of the first parameter, α_1, is bounded by one

$$|\alpha_1| \le 1 \tag{2.157}$$

where the equality holds if either of the following three alternatives are true.

$$\begin{cases} R_1 = 0 \\ R_2 = 0 \\ R_1 = R_2 = 0 \end{cases} \tag{2.158}$$

In a similar way the magnitude of α_2 can be shown to be equal to (2.154), and using the same argument as above

$$|\alpha_2| \le 1 \tag{2.159}$$

where equality holds if (2.158) is satisfied.

2.5.1.3 **Pseudolosslessness of the Complex Two-Port Adaptor**

The two-port adaptor is used when connecting two incompatible two-ports with different reference port impedances, where the compatibility condition is given in Section 2.5.2. Since this is a non-generative process, and losses are not expected, the two-port adaptor is intuitively pseudolossless under infinite precision conditions. To prove this, consider the condition for pseudolosslessness, P = 0.

$$P = \mathbf{A}^{*T}\left[\,\mathbf{G} - \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S}\,\right]\mathbf{A} = 0 \quad \Leftrightarrow \quad \mathbf{G} = \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S} \tag{2.160}$$

This condition is satisfied by the complex two-port scattering matrix as given by (2.152). This can be seen by writing (2.58) explicitly for a general two-port scattering matrix and substituting the appropriate values

$$\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = \begin{bmatrix} \left(G_1 S_{11} S_{11}^* + G_2 S_{21} S_{21}^*\right) & \left(G_1 S_{12} S_{11}^* + G_2 S_{22} S_{21}^*\right) \\ \left(G_1 S_{11} S_{12}^* + G_2 S_{21} S_{22}^*\right) & \left(G_1 S_{12} S_{12}^* + G_2 S_{22} S_{22}^*\right) \end{bmatrix} \tag{2.161}$$

For example, consider equating the upper left elements of the matrices given in (2.161) above,

$$G_1 = G_1 S_{11} S_{11}^* + G_2 S_{21} S_{21}^* \tag{2.162}$$

Substituting the values for the scattering matrix of the complex two-port adaptor,

$$G_1 = G_1\left(\alpha\,\alpha^*\right) + G_2\left(1-\alpha\right)\left(1-\alpha^*\right) \tag{2.163}$$

and with the definition of α, the right side of the above equation becomes

$$\frac{1}{R_1}\,\frac{\left(Z_1 - Z_2^*\right)\left(Z_1^* - Z_2\right)}{\left(Z_1+Z_2\right)\left(Z_1+Z_2\right)^*} + \frac{1}{R_2}\,\frac{\left(Z_2 + Z_2^*\right)\left(Z_2 + Z_2^*\right)}{\left(Z_1+Z_2\right)\left(Z_1+Z_2\right)^*} \tag{2.164a}$$

$$\frac{1}{\left(Z_1+Z_2\right)\left(Z_1+Z_2\right)^*}\left[\frac{Z_1Z_1^* + Z_2Z_2^* - Z_1Z_2 - Z_1^*Z_2^*}{R_1} + 4\,R_2\right] \tag{2.164b}$$

$$\frac{1}{\left(Z_1+Z_2\right)\left(Z_1+Z_2\right)^*}\left[\frac{\left(R_1+R_2\right)^2 + \left(X_1-X_2\right)^2}{R_1}\right] \tag{2.164c}$$

which is equal to the left side, G_1, and thus (2.162) is satisfied. The same procedure can be repeated for the remaining elements in the matrix given in (2.161), and thus the complex two-port adaptor is pseudolossless.

2.5.1.4 Flow Diagram of the Complex Two-Port Adaptor

The scattering matrix given in (2.152) can be used to identify two flow diagrams for the complex two-port adaptor. Since multiplications generally take longer to calculate than additions on a digital computer or specialized digital hardware, the flow diagrams will minimize the number of multiplications needed to calculate the reflected waves. All additions and multiplications are shown in the complex domain, and thus the equivalent real operations are not shown for clarity.

The complex two-port adaptor symbol shown in Figure 2.20 below is characterized by the parameter α which has a magnitude bounded by one.

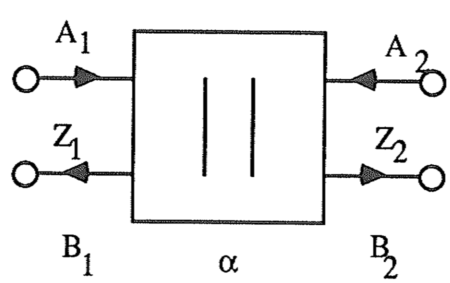

Figure 2.20: Complex two-port adaptor with parameter α.

A signal flow diagram representing the scattering matrix and expressed as a function of α is shown below in Figure 2.21.

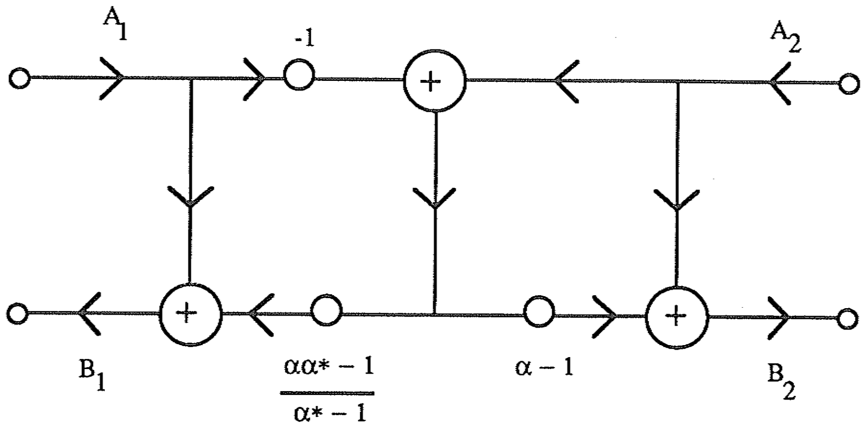

Figure 2.21: Wave flow diagram of the complex two-port adaptor.

The above signal flow diagram contains two complex multipliers, an inverter and three complex adders. The inverter can be implimented by reversing the sign of the real and imaginary parts of the signal, and thus it is not considered a multiplier. The magnitudes of the two multipliers are bounded by two.

Another signal flow diagram can be derived from the scattering matrix as shown in Figure 2.22, where the number of complex adders and multipliers remains the same, while one multiplier is replaced by a unimodular multiplier. This signal flow diagram is preferred over the above because of the unimodular multiplier, since it represents a simple change in phase of the signal.

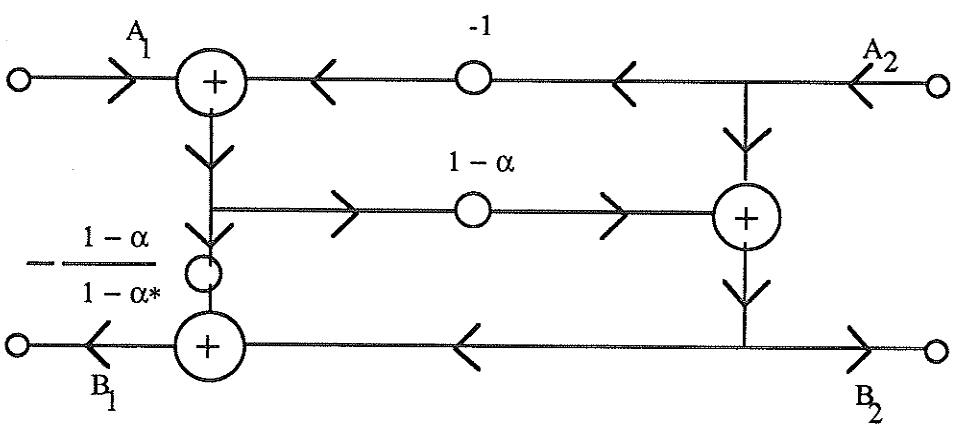

Figure 2.22: Wave flow diagram of the complex two-port adaptor with a unimodular multiplier.

2.5.2 Interconnection of Adaptors and Ports

Two complex ports must be compatible before they are connected. The compatibility criterion is derived from the conditions on the voltages and currents of the equivalent ports in the voltage-current domain, as shown in Figure 2.19, along with the conditions imposed by the flow of incident and reflected waves in the wave digital domain, as shown in Figure 2.23.

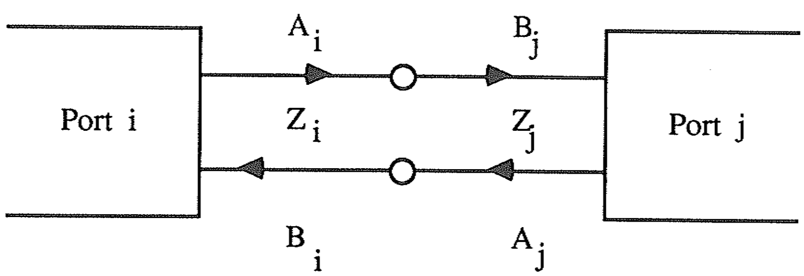

Figure 2.23: Interconnection of two ports in the voltage-current domain.

Consider ports i and j as shown above. The voltage and current conditions that must be satisfied are given by

$$V_i = V_j \tag{2.165}$$
$$I_i = -I_j \tag{2.166}$$

The complex incident and reflected waves at each port are given by

$$A_i = V_i + Z_i I_i \tag{2.167}$$
$$B_i = V_i - Z_i^* I_i \tag{2.168}$$
$$A_j = V_j + Z_j I_j \tag{2.169}$$
$$B_j = V_j - Z_j^* I_j \tag{2.170}$$

and the criteria dictated by wave-flow compatibility are given by

$$A_i = B_j \tag{2.171}$$
$$B_i = A_j \tag{2.172}$$

The above equations immediately implies the following conditions for the connection of two wave ports with constant complex reference port impedances,

$$Z_i = Z_j^* \tag{2.173}$$

Or, one port impedance must be the complex conjugate of the other. Thus if

$$Z_i = R_i + j X_i \tag{2.174}$$
$$Z_j = R_j + j X_j \tag{2.175}$$

Then the above condition gives

$$R_i = R_j \tag{2.176}$$
$$X_i = - X_j \tag{2.177}$$

2.5.3 Complex Transformer in the Complex Wave Digital Domain

The complex ideal transformer as shown in Figure 2.24 is defined with a complex turns ratio n.

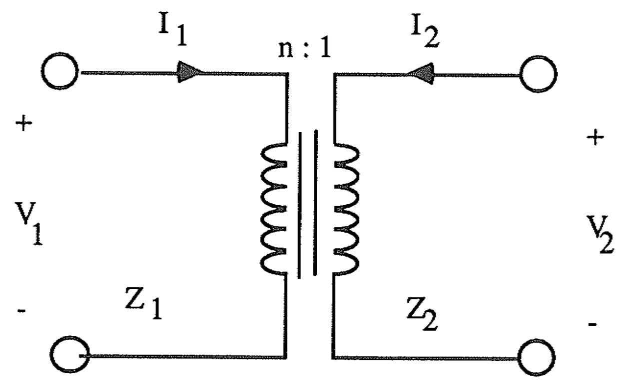

Figure 2.24: Ideal complex transformer.

The non-ideal complex transformer will not be investigated since its use in digital applications is limited. Define the primary voltage V_1 by [44]

$$V_1 = n^* V_2 \tag{2.178}$$

where V_2 is the secondary voltage. Using the lossless condition P = 0, the secondary current can be derived in terms of the turns ratio and the primary current.

$$P = V_1 I_1^* + V_2 I_2^* = 0 \tag{2.179}$$
$$V_2 (n^* I_1^* + I_2^*) = 0 \tag{2.180}$$

or

$$I_2 = - n I_1 \tag{2.181}$$

Thus the ideal complex transformer is defined by the voltage-current pair

$$V_1 = n^* V_2 \tag{2.182}$$
$$I_2 = - n I_1 \tag{2.183}$$

Transforming into the wave digital space by substituting equations (2.8) and (2.9)

$$V_k = \frac{Z_k^* A_k + Z_k B_k}{2 R_k} \tag{2.184}$$

$$I_k = \frac{A_k - B_k}{2\,R_k} \tag{2.185}$$

(repeated here for convenience) into the above equations (2.182) and (2.183),

$$B_1 = \frac{\left(n\,n^*\,Z_2 - Z_1^*\right)A_1 + 2\,n^*\,R_1 A_2}{Z_1 + n\,n^*\,Z_2} \tag{2.186}$$

$$B_2 = \frac{2\,n\,R_2 A_1 + \left(Z_1 - n\,n^*\,Z_2^*\right)A_2}{Z_1 + n\,n^*\,Z_2} \tag{2.187}$$

Or, in the form of a scattering matrix,

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \frac{1}{Z_1 + n\,n^*\,Z_2}\begin{bmatrix} n\,n^*\,Z_2 - Z_1^* & 2\,n^*\,R_1 \\ 2\,n\,R_2 & Z_1 - n\,n^*\,Z_2^* \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{2.188}$$

However, if the port impedances are related by

$$Z_1 = n\,n^*\,Z_2^* \tag{2.189}$$

Then the scattering matrix reduces to

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & n^* \\ \frac{1}{n^*} & 0 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{2.190}$$

The condition relating the impedances as given by (2.189) can be set using a complex two-port adaptor, as discussed in Section 2.5.1.1, for two general port reference impedances as shown in Figure 2.25.

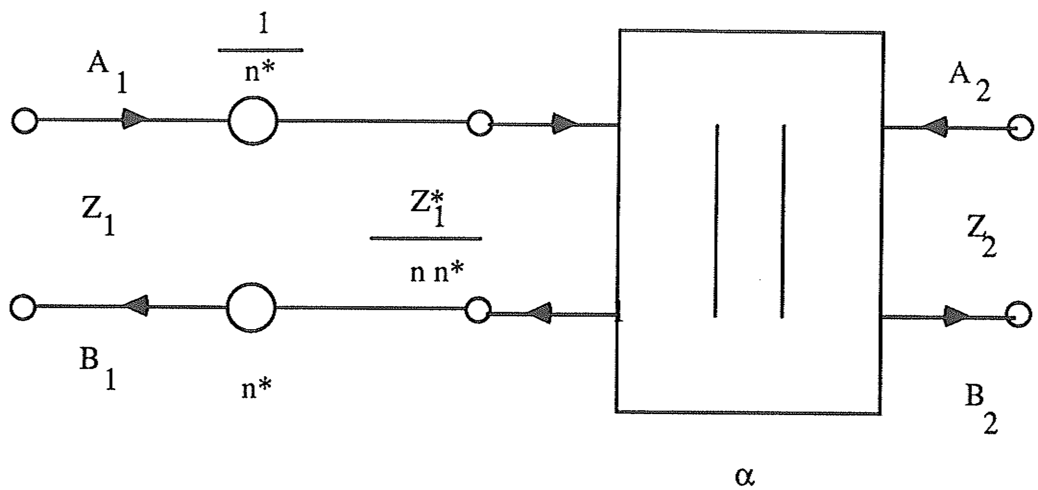

Figure 2.25: Ideal complex transformer in the wave digital domain.

The resulting network as given above is pseudolossless as expected. The port connecting the ideal complex transformer to the complex two-port adaptor has an impedance on the transformer side of

$$Z_1' = n^{-1} Z_1^* n^{-1*} \tag{2.191}$$

while the port impedance associated with the complex two-port adaptor is the conjugate of the above. Another possible definition is possible where the secondary current is defined with the complex conjugate of the primary current. This leads to an identical solution with n replaced by its complex conjugate throughout. The solution presented here is preferred since the fact that a complex turns ratio is used is obvious from the definition of the scattering matrix, and no possibility of confusion or ambiguity in this respect is possible.

2.5.4 Complex Circulator in the Complex Wave Digital Domain

The complex three-port circulator as shown in Figures 2.26 and 2.27 in the voltage-current and wave domains, respectively, is a non-reciprocal lossless device characterized by its complex impedance $\overline{Z}$. The complex circulator is defined in the wave domain as shown in Figure 2.27, and the symbol in the voltage-current domain as shown in Figure 2.26 gives the references for the voltages and currents.

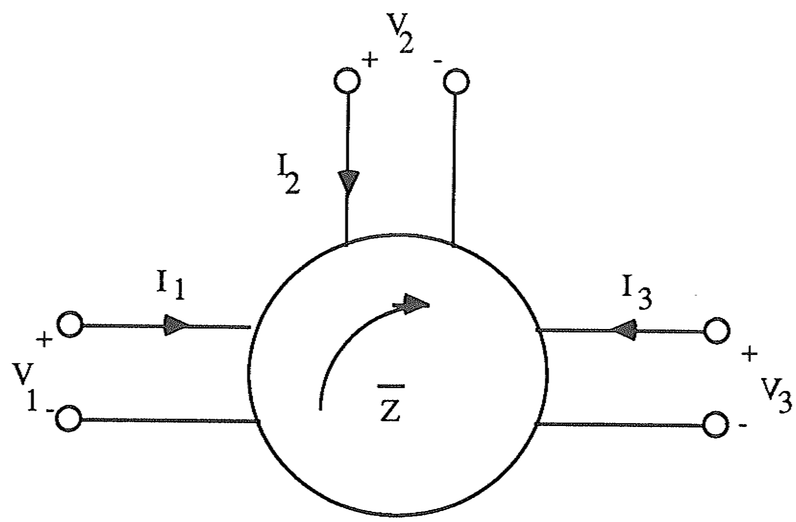

Figure 2.26: Complex circulator in the voltage-current domain.

In the wave digital domain, the complex circulator has a simple signal flow diagram as shown in Figure 2.27 below.

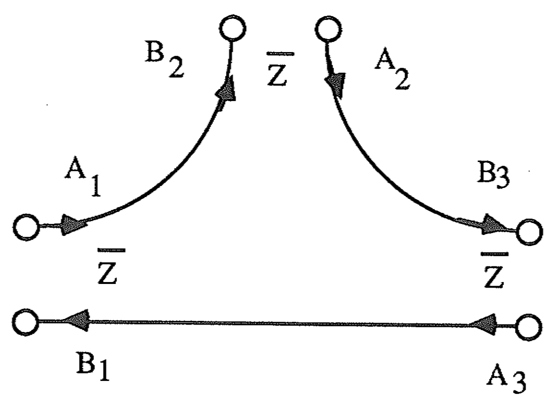

Figure 2.27: Signal flow diagram of the complex circulator.

Notice that all three ports have the same port impedance $\overline{Z}$, and the circulator simply redirects the incident and reflected waves of the ports. For this reason, the complex circulator is useful when connecting wave-digital elements in series with adaptors or other elements. For example, connecting a one-port in series with the reflected signal terminal of a two port can be accomplished with the complex circulator since the incident signal is not affected. This also implies that the complex circulator is pseudolossless, as expected.

2.6 Complex Scattering Matrix Formulation

The scattering matrix formulation method in the real domain is convenient for designing wave digital networks that can easily be expressed in this form. Such networks often display a high degree of symmetry and are implemented with common design blocks arranged in a simple order. The scaling of some common networks can be easily achieved by the scaling of the scattering matrix. It would be advantageous to be able to make use of these properties with complex wave digital networks with a complex scattering matrix.

2.6.1 Definition of the Complex Scattering Matrix

Many pseudopassive wave digital filters of order n can be designed and analyzed using adaptors and one-port elements. This method of realization can be viewed as deriving a non-dynamic (n+1)-port N, where all dynamic (memory) and non-linear elements are removed from N leaving inside N simple connections, ideal complex transformers, complex circulators and complex multipliers and adders. Consider an (n+1)-port N as shown in Figure 2.28 below,

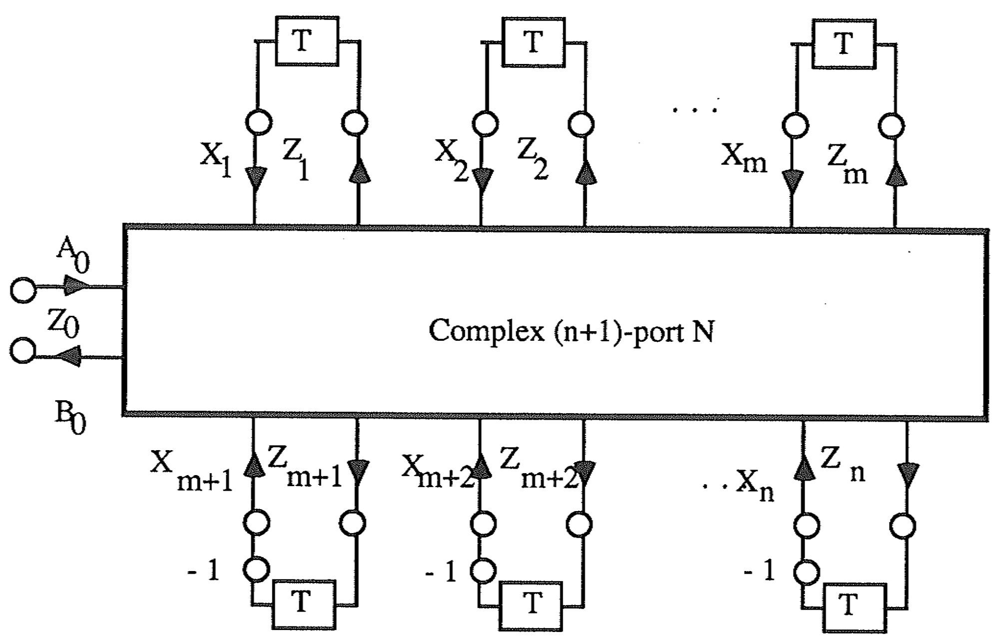

Figure 2.28: Complex (n+1)-port N with all dynamic elements removed.

where the ports 1 to m, with complex reference impedances Z_i, i=1 (1) m, are associated with capacitances in series with constant reactances, and ports m+1 to n, with complex reference impedances Z_i, i=m+1 (1) n, are equivalent to inductances in series with constant reactances, in the voltage-current domain.

Port zero with the complex reference port impedance Z_0 is the input-output port of the network, with the input being the incident wave A_0 and the output being the reflected wave B_0. The signals leaving the delays are known as the states of the network and are represented by the state-variables X_i. The output of the one-port delay with a full synchronic network is equal to the input to the delay during the previous clock cycle. In a canonic design, the order of the network equals the number of delays, and thus the number of state-variables.

The output of the network B_0 in Figure 2.28 is determined by the (n+1)-port N, the states of the network X_i, and the input A_0. Similarly, the future states, after one clock cycle, can be determined from the (n+1)-port N, the current states of the network X_i, and the input A_0. If the (n+1)-port N is linear, then the output B_0 (a scalar complex number) can be expressed as a weighted linear combination of the states X_i added to a factor of the input A_0. In a similar way, the states after a clock cycle, given by $\widehat{X}_i$, i=1 (1) n, can be expressed as a weighted linear combination of the current states added to a factor of the input A_0. Equivalently, in matrix form

$$\widehat{X} = \underline{\mathbf{A}}\, X + \underline{\mathbf{B}}\, A_0 \tag{2.192}$$

$$B_0 = \underline{C}\, X + \underline{D}\, A_0 \tag{2.193}$$

where $\underline{\mathbf{A}}$ is an (n x n) complex matrix, $\underline{\mathbf{B}}$ is an (n x 1) complex column matrix, $\underline{C}$ is an (1 x n) complex row matrix and $\underline{D}$ is a complex scalar.

Now consider the same argument using the notation associated with wave digital networks. The states X_i are the incident waves labelled A_i, and the reflected waves B_i are equal to these states during the next clock cycle, as referenced to the (n+1)-port N. Thus equations (2.192) and (2.193) above with the new notation become

$$B_0 = \underline{D}\, A_0 + \underline{C}\, \mathbf{A} \tag{2.194}$$

$$\mathbf{B} = \underline{\mathbf{B}}\, A_0 + \underline{\mathbf{A}}\, \mathbf{A} \tag{2.195}$$

where **A** is a (n x 1) complex column matrix with elements equal to the incident waves into the delay ports A_i, i = 1 (1) n, and **B** is a (n x 1) complex column matrix with elements equal to the reflected waves out of the delay ports B_i, i = 1 (1) n. The above equations define a complex scattering matrix **S** of the (n+1)-port N by writing (2.194) and (2.195) as

$$\begin{bmatrix} B_0 \\ B_1 \\ \cdot \\ B_n \end{bmatrix} = \begin{bmatrix} \underline{D} & \underline{C} \\ & \\ \underline{\mathbf{B}} & \underline{\mathbf{A}} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \cdot \\ A_n \end{bmatrix} \tag{2.196}$$

with S defined by

$$S = \begin{bmatrix} \underline{D} & \underline{C} \\ \underline{B} & \underline{A} \end{bmatrix} \tag{2.197}$$

2.6.2 Scaling the Complex Scattering Matrix

A general complex wave digital network must be scaled before it is implemented in digital hardware or software. This is necessay because digital hardware is capable of representing only a finite range and number of values, in other words a digital filter is a finite state machine. When the magnitude of a value that needs to be stored in memory exceeds the maximum range that can be represented, overflow occurs creating catastrophic errors in the output of the filter [17][54].

Ideally, the scaling of a network should be carried out in a way that no possibility of overflow exists. This criterion has been found to be excessive in practise. Another possibility for scaling is to scale the network for particular input signals that have a given maximum amplitude. This method has been adopted by many authors, and the difficulty then arises as to which input signal will be assumed.

Sinusoidal signals are one obvious input [33]. Intuitively, many signals can be represented by infinite precision sinusoids with the fourier transform, and it is well known that the output of a linear time-invariant system with sinusoidal inputs is itself sinusoidal. However, a digital filter is often non-linear as overflow and underflow errors occur. Also, it is inconvenient to represent random signals, which is usually the input to digital systems, with sinusoids.

A more appropriate signal to assume is a white wide-sense stationary gaussian random signal described by a zero-mean gaussian probability density function, as most signals encountered in practise are of this form [6][43]. The output of a linear time-invariant system with a gaussian input is itself gaussian, a useful property when considering the scaling of a system.

2.6.2.1 Scaling Transformation Applied to the Complex Scattering Matrix

A simple way of scaling a system is by scaling the complex scattering matrix characterizing the system. Thus consider the scattering matrix, or equivalently the state-variable matricies given by (2.194) and (2.195). The scaling is achieved by a linear complex non-singular diagonal transformation Γ applied to the state-variable system $\{\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{C}}, \underline{\mathbf{D}}\}$ such that the state-variable matrices are changed to

$$\left\{ \Gamma^{*-1} \underline{\mathbf{A}}\, \Gamma^{*}, \Gamma^{*-1} \underline{\mathbf{B}}, \underline{\mathbf{C}}\, \Gamma^{*}, \underline{\mathbf{D}} \right\} \tag{2.198}$$

This transformation carried out by matrix multiplication does not affect the frequency response of the system given by [41]

$$H(z) = \underline{\mathbf{D}} + \underline{\mathbf{C}}\,(\mathbf{I}\,z - \underline{\mathbf{A}})^{-1} \underline{\mathbf{B}} \tag{2.199}$$

as can be seen from

$$H(z) = \underline{\mathbf{D}} + \underline{\mathbf{C}}\,\Gamma^{*} \left(\mathbf{I}\,z - \Gamma^{*-1} \underline{\mathbf{A}}\, \Gamma^{*} \right)^{-1} \Gamma^{*-1} \underline{\mathbf{B}} \tag{2.200}$$

$$H(z) = \underline{\mathbf{D}} + \underline{\mathbf{C}}\,\Gamma^{*}\, \Gamma^{*-1} \left(\Gamma^{*}\, \mathbf{I}\, z\, \Gamma^{*-1} - \underline{\mathbf{A}} \right)^{-1} \Gamma^{*}\, \Gamma^{*-1} \underline{\mathbf{B}} \tag{2.201}$$

but since z is a scalar, it can be brought out of the matrix multiplication and the identity matrix **I** can be ignored

$$H(z) = \underline{\mathbf{D}} + \underline{\mathbf{C}}\,\Gamma^{*}\, \Gamma^{*-1} \left(z\, \Gamma^{*}\, \Gamma^{*-1} - \underline{\mathbf{A}} \right)^{-1} \Gamma^{*}\, \Gamma^{*-1} \underline{\mathbf{B}} \tag{2.202}$$

with the definition of the inverse of a non-singular matrix, the above becomes

$$H(z) = \underline{\mathbf{D}} + \underline{\mathbf{C}}\,(\mathbf{I}\,z - \underline{\mathbf{A}})^{-1} \underline{\mathbf{B}} \tag{2.203}$$

as given in (2.199) above. The diagonal elements of Γ are given by γ_{ii}.

The actual scaling is carried out by inserting ideal complex transformers between the non-dynamic network and the one-port delays as shown in Figure 2.29 below in the analog domain.

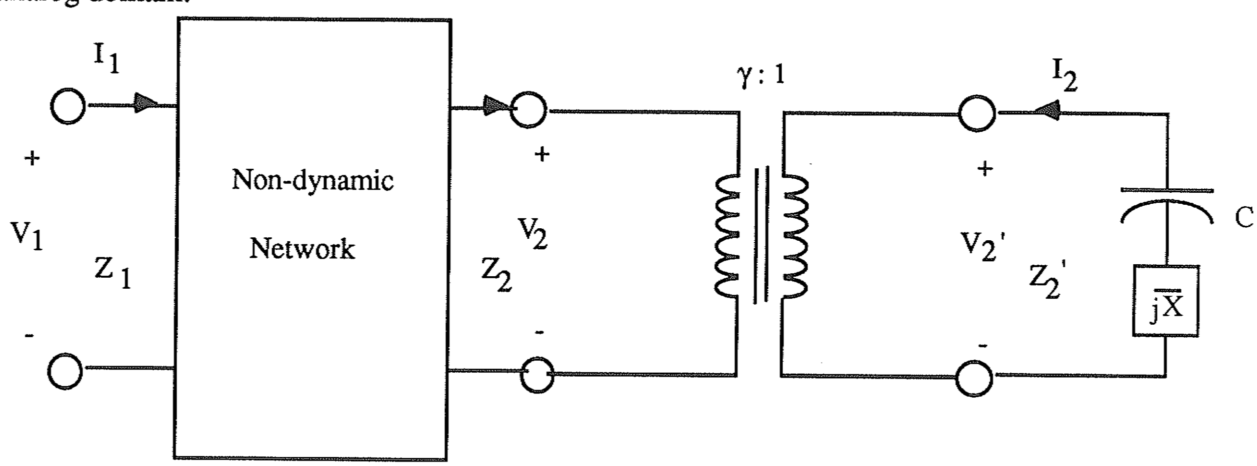

Figure 2.29: Insertion of ideal complex transformer between a non-dynamic network and a delay.

In the wave digital domain, this is equivalent to the network shown below in Figure 2.30. The resulting incident and reflected waves associated with the actual one-port delay after the ideal complex transformer are given by

$$B_D' = \gamma^{*-1}\, B_D \tag{2.204}$$

$$A_D = \gamma^{*}\, A_D' \tag{2.205}$$

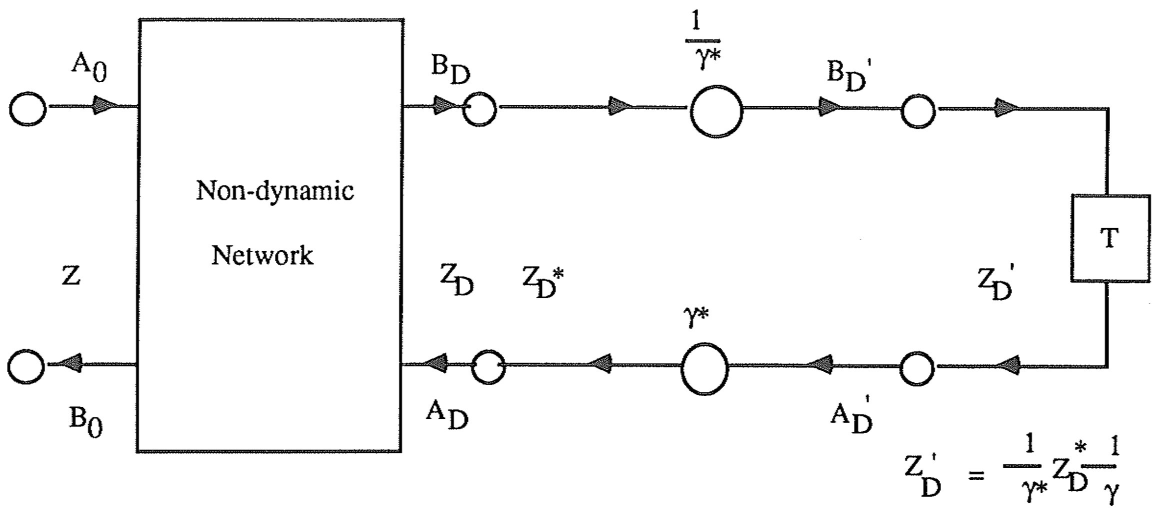

Figure 2.30: Complex wave digital domain equivalent of the insertion of a scaling transformer.

Rewrite the scattering matrix equation, for convenience, by considering the input-output port as a separate entity that does not specifically need consideration for scaling. Thus

$$\begin{bmatrix} B_0 \\ \mathbf{B}_D \end{bmatrix} = \begin{bmatrix} \underline{D} & \underline{\mathbf{C}} \\ \underline{\mathbf{B}} & \underline{\mathbf{A}} \end{bmatrix} \begin{bmatrix} A_0 \\ \mathbf{A}_D \end{bmatrix} \tag{2.206}$$

where $\mathbf{A}_D$ and $\mathbf{B}_D$ are the states of the network, or the incident and reflected waves of the one-port delays before the scaling transformer. Generalize equations (2.204) and (2.205), which apply to a single one-port, to the form of a non-dynamic network given in (2.206),

$$\begin{bmatrix} A_0 \\ \mathbf{A}_D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Gamma^* \end{bmatrix} \begin{bmatrix} A_0 \\ \mathbf{A}_D' \end{bmatrix} \tag{2.207}$$

$$\begin{bmatrix} B_0 \\ \mathbf{B}_D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Gamma^* \end{bmatrix} \begin{bmatrix} B_0 \\ \mathbf{B}_D' \end{bmatrix} \tag{2.208}$$

where $\mathbf{A}_D'$ and $\mathbf{B}_D'$ are the incident and reflected waves of the one-port delay after the scaling transformer. Thus we can rewrite (2.206) as

$$\begin{bmatrix} B_0 \\ \mathbf{B}'_D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Gamma^* \end{bmatrix}^{-1} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Gamma^* \end{bmatrix} \begin{bmatrix} A_0 \\ \mathbf{A}'_D \end{bmatrix} \tag{2.209}$$

or

$$\begin{bmatrix} B_0 \\ \mathbf{B}'_D \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12}\Gamma^* \\ \Gamma^{*-1}S_{21} & \Gamma^{*-1}S_{22}\Gamma^{*-1} \end{bmatrix} \begin{bmatrix} A_0 \\ \mathbf{A}'_D \end{bmatrix} \tag{2.210}$$

And thus the scaled scattering matrix S_S is given by

$$\mathbf{S}_s = \begin{bmatrix} S_{11} & S_{12}\Gamma^* \\ \Gamma^{*-1}\mathbf{S}_{21} & \Gamma^{*-1}\mathbf{S}_{22}\Gamma^{*-1} \end{bmatrix} \tag{2.211}$$

Equivalently, using the definition of the state-variable matrices, the scaled scattering matrix becomes

$$\mathbf{S}_s = \begin{bmatrix} \underline{D} & \underline{\mathbf{C}}\,\Gamma^* \\ \Gamma^{*-1}\underline{\mathbf{B}} & \Gamma^{*-1}\underline{\mathbf{A}}\,\Gamma^{*-1} \end{bmatrix} \tag{2.212}$$

The reference port impedance matrix of the unscaled non-dynamic network is defined by

$$\mathbf{Z} = \begin{bmatrix} z_0 & 0 \\ 0 & \mathbf{Z}_D \end{bmatrix} \tag{2.213}$$

whereas the scaled reference port impedance matrix, or the resulting impedances after the scaling transformer (which changes the reference port impedances) is given by

$$\mathbf{Z}_S = \begin{bmatrix} z_0 & 0 \\ 0 & \Gamma^{-1}\mathbf{Z}_D\Gamma^{*-1} \end{bmatrix} \tag{2.214}$$

2.6.2.2 Scaling Criteria

Define the complex impulse response at the i^{th} delay to the input as $h_i(m)$. Also define an input covariance diagonal matrix **K**, with diagonal elements equal to the inner product of the corresponding impulse response, or

$$K_{ii} = < h_i(m), h_i(m) > \tag{2.215}$$

And since the system is in a complex inner product space

$$K_{ii} = \sum_{m=0}^{\infty} h_i(m)\ h_i^*(m) \tag{2.216}$$

$$K_{ii} = \sum_{m=0}^{\infty} |\,h_i(m)\,|^2 \tag{2.217}$$

Thus the input covariance diagonal matrix is real. Using a zero-mean Gaussian input it can be shown [48] that the variance at the i^{th} delay, σ_i^2, is related to the input variance, σ_x^2, by

$$\sigma_i^2 = \left[\sum_{m=0}^{\infty} |\,h_i(m)\,|^2 \right] \sigma_x^2 \tag{2.218}$$

or

$$\sigma_i^2 = k_{ii}\,\sigma_x^2 \tag{2.219}$$

In order to provide the correct scaling at each delay, or to avoid overflow errors, the variance at the i^{th} delay should be set equal to the input variance. This defines a new covariance matrix of the scaled system, **K'**, and the above condition implies

$$K_{ii}' = 1 \quad \text{or} \quad \mathbf{K}' = \mathbf{I} \tag{2.220}$$

and this gives a condition on the scaled impulse response $h_i'(m)$,

$$1 = \sum_{m=0}^{\infty} |\,h_i'(m)\,|^2 \tag{2.221}$$

It is known [17] that the input covariance matrix is related to the state-variable matrices and the relation can be generalized to

$$\mathrm{K} = \underline{\mathbf{A}}\,\mathrm{K}\,\underline{\mathbf{A}}^{*T} + \underline{\mathbf{B}}\,\underline{\mathbf{B}}^{*T} \tag{2.222}$$

where $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are the state-variable matrices given in (2.206).

Consider a pseudolossless wave digital network with a state-variable representation. It would be convenient to find an expression involving the state-variable matrices that has the form of (2.222) above, and thus the input covariance matrix **K** of the unscaled system could be identified from the equation. Towards this end, consider the pseudolossless condition

$$\mathbf{G} = \mathbf{S}^{*T}\,\mathbf{G}\,\mathbf{S} \tag{2.223}$$

Obviously, the scattering matrix and the diagonal conductance matrix are non-singular, therefore we can take the inverse of both sides of (2.223)

$$\mathbf{G}^{-1} = \mathbf{S}^{-1}\,\mathbf{G}^{-1}\,\mathbf{S}^{*T^{-1}} \tag{2.224}$$

premultiplying by **S** and post multiplying by $\mathbf{S}^{*T}$,

$$\mathbf{S}\,\mathbf{G}^{-1}\,\mathbf{S}^{*T} = \mathbf{S}\,\mathbf{S}^{-1}\,\mathbf{G}^{-1}\,\mathbf{S}^{*T^{-1}}\,\mathbf{S}^{*T} \tag{2.225}$$

or, with the definition of the inverse of a nonsingular matrix,

$$\mathbf{G}^{-1} = \mathbf{S}\,\mathbf{G}^{-1}\,\mathbf{S}^{*T} \tag{2.226}$$

With the state-variable representation of **S** as given in (2.206), the above expression can be written

$$\begin{bmatrix} G_0^{-1} & 0 \\ 0 & \mathbf{G}_D^{-1} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{D}} & \underline{\mathbf{C}} \\ \underline{\mathbf{B}} & \mathbf{A} \end{bmatrix} \begin{bmatrix} G_0^{-1} & 0 \\ 0 & \mathbf{G}_D^{-1} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}^{*T} & \underline{\mathbf{B}}^{*T} \\ \underline{\mathbf{C}}^{*T} & \mathbf{A}^{*T} \end{bmatrix} \tag{2.227}$$

where $\mathbf{G}_D^{-1}$ is the diagonal resistance matrix of the delay ports and G_0^{-1} is the resistance of the input-output port. The above matrix equation gives four expressions relating the conductance matrix and the state-variable matrices. However, only one equation is needed, and the most appropriate choice is the last equation that would be generated,

$$\mathbf{G}_D^{-1} = \mathbf{B}\,G_0^{-1}\,\mathbf{B}^{*T} + \mathbf{A}\,\mathbf{G}_D^{-1}\,\mathbf{A}^{*T} \tag{2.228}$$

but G_0^{-1} is a scalar quantity, and thus the above can be written

$$G_0\,\mathbf{G}_D^{-1} = \mathbf{A}\,(G_0\,\mathbf{G}_D^{-1})\,\mathbf{A}^{*T} + \mathbf{B}\,\mathbf{B}^{*T} \qquad 2.229)$$

which has the exact form as (2.222). Identifying the (real) input covariance matrix **K** of the unscaled system from the above,

$$\mathbf{K} = G_0\,\mathbf{G}_D^{-1} \tag{2.230}$$

The scaled system input covariance matrix **K'** must be derived in order to relate the scaling criterion (2.220) to the relation given above. Equation (2.222), which relates the input convariance matrix of the scaled system to the state-variable matrices, can be written for the scaled system with scaled matrices $\underline{\mathbf{A}}'$ and $\underline{\mathbf{B}}'$, giving

$$\mathbf{K}' = \underline{\mathbf{A}}'\ \mathbf{K}'\ \underline{\mathbf{A}}'^{*T} + \underline{\mathbf{B}}'\ \underline{\mathbf{B}}'^{*T} \tag{2.231}$$

but we know that the non-singular linear transformation Γ relates the scaled and unscaled state matrices by

$$\underline{\mathbf{A}}' = \Gamma^{*-1}\ \underline{\mathbf{A}}\ \Gamma^{*} \tag{2.232}$$

$$\underline{\mathbf{B}}' = \Gamma^{*-1}\ \underline{\mathbf{B}} \tag{2.233}$$

and thus Equation (2.231) above becomes

$$\mathbf{K}' = (\Gamma^{*-1}\ \underline{\mathbf{A}}\ \Gamma^{*})\ \mathbf{K}'\ (\Gamma^{T}\ \underline{\mathbf{A}}^{*T}\ \Gamma^{-1T}) + \Gamma^{*-1}\ \underline{\mathbf{B}}\ \underline{\mathbf{B}}^{*T}\ \Gamma^{-1T} \tag{2.234}$$

$$\mathbf{K}' = \Gamma^{*-1}\ [\underline{\mathbf{A}}\ (\Gamma^{*}\ \mathbf{K}'\ \Gamma^{T})\ \underline{\mathbf{A}}^{*T} + \underline{\mathbf{B}}\ \underline{\mathbf{B}}^{*T}]\ \Gamma^{-1T} \tag{2.235}$$

or

$$\Gamma^{*}\ \mathbf{K}'\ \Gamma^{T} = \underline{\mathbf{A}}\ (\Gamma^{*}\ \mathbf{K}'\ \Gamma^{T})\ \underline{\mathbf{A}}^{*T} + \underline{\mathbf{B}}\ \underline{\mathbf{B}}^{*T} \tag{2.236}$$

Identifying the input covariance matrix of the unscaled system, $\mathbf{K}$, from the above we get

$$\mathbf{K} = \Gamma^{*}\ \mathbf{K}'\ \Gamma^{T} \tag{2.237}$$

which is the form of the equation that is needed. Using equations (2.220) and (2.237), a condition on the scaling matrix Γ can be found

$$\mathbf{K} = \Gamma^{*}\ \Gamma^{T} = \Gamma\ \Gamma^{*T} = |\Gamma|^{2} = G_0\ G_D^{-1} \tag{2.238}$$

which gives

$$\Gamma\ \Gamma^{*T} = G_0\ G_D^{-1} \tag{2.239}$$

or

$$|\Gamma| = G_0^{1/2}\ G_D^{-1/2} \tag{2.240}$$

The same condition can be derived by generalizing the known equation [17] to

$$\mathbf{W} = \mathbf{A}^{*T}\ \mathbf{W}\ \mathbf{A} + \mathbf{C}^{*T}\ \mathbf{C} \tag{2.241}$$

The above gives a condition on the elements of the scaling matrix Γ. Notice that since $\mathbf{G_D}$ is diagonal and if all of the diagonal elements are non-negative, the square root of the inverse of $\mathbf{G}_D$ exists and is easily calculated on a element-by-element basis. Also, since Γ is in general complex, the above condition, though necessary, does not uniquely specify the scaling matrix Γ, or a degree of freedom exists that could be taken advantage of in future applications. If the scaling matrix Γ is restricted to be real by using only real ideal transformers for scaling, (2.240) gives a unique solution for Γ (the elements of Γ could also be negative according to (2.240), but from Figure 2.30, this is equivalent to the positive definite scaling matrix because a double negative is introduced in the multipliers of the scaling transformer).

The impedance matrix associated with the one-port delays $\mathbf{Z}_D'$ after scaling is given from (2.214) by

$$\mathbf{Z}_D' = \Gamma^{*-1} \, \mathbf{Z}_D^* \, \Gamma^{-1} \tag{2.242}$$

Let Z_D' be a diagonal element of $\mathbf{Z}_D'$, Z_D be a diagonal element of $\mathbf{Z}_D$, G_D be a diagonal element of $\mathbf{G}_D$, and γ be a diagonal element of Γ. Equation (2.242) can be written on an element-by-element basis as

$$Z_D' = \gamma^{*\,-1} \, Z_D^* \, \gamma^{-1} \tag{2.243}$$

$$Z_D' = G_0^{-1} \, G_D \, Z_D^* \tag{2.244}$$

with $R_0 = 1/G_0$, and $R_D = 1/G_D$,

$$Z_D' = (R_0 \,/\, R_D) \, Z_D^* \tag{2.245}$$

2.6.3 Frequency Response of A Sytem Characterized by A Complex Scattering Matrix

The frequency response of a one-port state-variable system represented by a complex scattering matrix is often desired. From Equation (2.199) repeated here

$$H(z) = \underline{D} + \underline{C}\,(\,z\mathbf{I}\ -\ \underline{\mathbf{A}}\,)^{-1}\underline{\mathbf{B}} = \frac{N(z)}{D(z)} \tag{2.246}$$

the complex transfer function in the z-domain is given in terms of the state-variable matrices, and the discrete variable z is given by

$$z = e^{j\,\Omega} \tag{2.247}$$

where Ω is a discrete frequency in the range $0 \le \Omega < \pi$. The transfer function is rational in z with a complex numerator polynomial N(z) and denominator polynomial D(z) as given below

$$N(z) = \left[\underline{D} + \underline{C}\,(z\mathbf{I} - \underline{A})^{-1}\underline{B}\right] \det(z\mathbf{I} - \underline{A}) \tag{2.248}$$

$$D(z) = \det(z\mathbf{I} - \underline{A}) \tag{2.249}$$

The above equations are inconvenient to use for the generation of the frequency response. A preferred way can be derived by considering the matrix multiplication of two matrices, $\mathbf{X}(z)$ and $\mathbf{Y}(z)$, defined by

$$\mathbf{X}(z) = \begin{bmatrix} \mathbf{I}_n & 0 \\ \underline{C}\,(z\mathbf{I} - \underline{A})^{-1} & \underline{D} + \underline{C}\,(z\mathbf{I} - \underline{A})^{-1}\underline{B} \end{bmatrix} \tag{2.250}$$

$$\mathbf{Y}(z) = \begin{bmatrix} (z\mathbf{I} - \underline{A}) & -\underline{B} \\ 0 & \mathbf{I}_n \end{bmatrix} \tag{2.251}$$

The determinants of the above matrices can easily be shown to be

$$\det(\mathbf{X}(z)) = \underline{D} + \underline{C}\,(z\mathbf{I} - \underline{A})^{-1}\underline{B} = H(z) \tag{2.252}$$

$$\det(\mathbf{Y}(z)) = \det(z\mathbf{I} - \underline{A}) = D(z) \tag{2.253}$$

Define the matrix $\Phi(z)$ as the product of the two matrices given above.

$$\Phi(z) = \mathbf{X}(z)\,\mathbf{Y}(z) = \begin{bmatrix} (z\mathbf{I} - \underline{A}) & -\underline{B} \\ \underline{C} & \underline{D} \end{bmatrix} \tag{2.254}$$

Then the determinant of $\Phi(z)$ is the product of the determinants of $\mathbf{X}(z)$ and $\mathbf{Y}(z)$, and from (2.252) and (2.253), the determinant of $\Phi(z)$ is

$$\det(\Phi(z)) = \det(\mathbf{X}(z)\,\mathbf{Y}(z)) = \det(\mathbf{X}(z))\det(\mathbf{Y}(z)) = N(z) \tag{2.255}$$

From equations (2.253) and (2.255) above, the transfer function can be derived from the division of the determinants of two complex matrices, or

$$H(z) = \frac{\det(\Phi(z))}{\det(z\mathbf{I}_n - \underline{A})} \tag{2.256}$$

The frequency response can be found by evaluating the above for various values of z. An efficient method of evaluating the above at a specific frequency Ω_0 can be derived by calculating the determinant of the matrix $\Phi(e^{j\Omega_0})$ using the upper triangular form of $\Phi(e^{j\Omega_0})$ denoted by $\Phi^U(e^{j\Omega_0})$. Note that the size of the matrix given in (2.254) is (m x m), where m is one larger than the order of the digital transfer function being realized.

$$\Phi^U(e^{j\Omega_0}) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ 0 & \phi_{22} & \cdots & \phi_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \phi_{mm} \end{bmatrix} \tag{2.257}$$

The determinant then becomes equal to the product of the diagonal elements of the upper triangular matrix. This is because the determinant of a matrix is invariant to the addition or subtraction of a multiple of a row to or from another row, and similarly, it is invariant to the addition or subtraction of a multiple of a column to or from another column. Thus

$$\det\left(\Phi(e^{j\Omega_0})\right) = \det\left(\Phi^U(e^{j\Omega_0})\right) = \prod_{i=1}^{m} \phi_{ii} \tag{2.258}$$

From (2.254), it is clear that the value of $\det\left(z\,\mathbf{I}_n - \mathbf{A}\right)$ is a principal minor of the matrix $\Phi(e^{j\Omega_0})$. Since no rows or columns are interchanged, this determinant of the upper triangular matrix is given by

$$\det\left(z\,\mathbf{I}_n - \mathbf{A}\right) = \prod_{i=1}^{n} \phi_{ii} \tag{2.259}$$

where n is the order of the digital transfer function. From (2.256), (2.258) and (2.259), it is clear that the frequency response as given by the division of two determinants is simply

$$H\left(e^{j\Omega_0}\right) = \phi_{mm} \tag{2.260}$$

since all of the multiplication terms in (2.258) are cancelled by (2.259) except the last term.

This process can be repeated for various Ω_0 in order to generate a range of values of the frequency response. However, this process is relatively time consuming since a general (m x m) complex matrix must be upper triangularized for each desired value of the frequency. This inefficiency is a disadvantage of analysing a state-variable system by the above method.

2.7 Stability of Complex Wave Digital Networks

The stability of a digital network under zero-input, forced response and looped conditions is critical for the usefulness of the filter. It has been shown that under infinite precision conditions, when all values are calculated using floating-point values and no underflow or overflow errors can occur in the linear system, the wave digital filter can be designed to be stable for all bounded input signals. However, in a digital system values are represented with a finite number of digits and usually in binary. Thus, all signals and results of additions or multiplications are quantized and the system is no longer linear. This process creates underflow errors, which are errors in signals due to the inability to represent an infinite number of digits, and overflow errors, which develop because the magnitude of the largest number that can be represented is bounded (usually at one or two). Underflow errors create relatively small deviations in the signals of the network, whereas overflow errors create large deviations from the nominal values of the signals and thus should be avoided if possible.

The process of a complex digital system involves complex quantized additions, multiplications, and storage. A complex addition is carried out by adding the real and imaginary parts of two complex numbers, and thus requires two real additions. After each real addition, an overflow error could occur if either result becomes larger than the maximum number that can be represented. A complex multiplication requires four real multiplications and two real additions. Again, each addition could cause an overflow error to occur. Each multiplication could cause underflow errors, where the representation of the reflected wave requires more digits than the incident wave.

As suggested by several authors [43][56], the effects of overflow errors and underflow errors can be modelled in two ways; either by considering a linear system with non-linear input error signals associated with each addition or multiplication; or a non-linear system, with non-linear operators acting on the signals. Webb [43] has applied the development of Moon [56] to a diagonal conductance matrix and found that the above representations lead to conditions on the functions used for dealing with overflow and underflow errors. The same arguments can be repeated in the complex domain.

The two system models are based on the state-variable representation of the system with state-variable matrices $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$, and $\underline{\mathbf{D}}$, relating the input sequence $A_0(m)$, the output sequence $B_0(m)$, and the state sequence $\mathbf{X}(m)$, where $m \geq 0$,

$$\mathbf{X}(m+1) = \underline{\mathbf{A}}\,\mathbf{X}(m) + \underline{\mathbf{B}}\,A_0(m) \tag{2.261}$$

$$B_0(m) = \underline{\mathbf{C}}\,\mathbf{X}(m) + \underline{\mathbf{D}}\,A_0(m) \tag{2.262}$$

The linear system model with the complex overflow sequence Z(m), and the complex underflow sequence $\Delta(m)$ added to the states $\mathbf{X}(m+1)$, is shown in Figure 2.31 below.

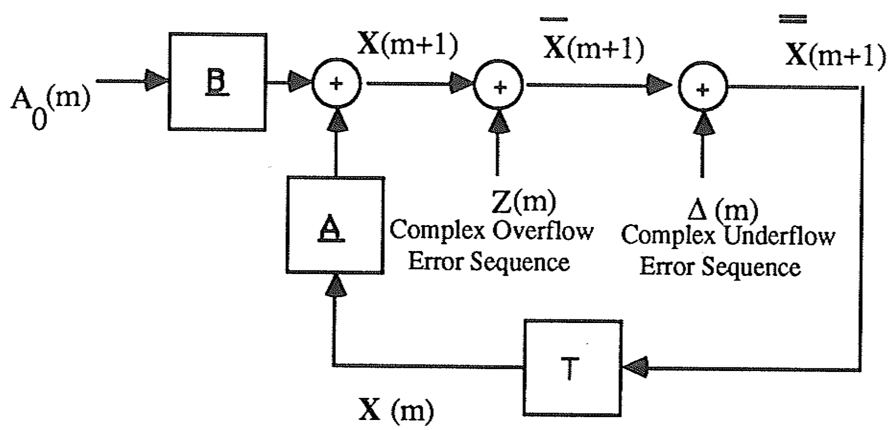

Figure 2.31: Linear system model with non-linear error input sequences at the states.

The complex overflow sequence Z(m) represents deviations due to overflow errors in the linear system and is added to the states changing the states $\mathbf{X}(m+1)$ to $\overline{\mathbf{X}}(m+1)$. Similarly, the complex underflow sequence $\Delta(m)$ represents deviations due to underflow error in the linear system and is added to $\overline{\mathbf{X}}(m+1)$ to change the sequence $\overline{\mathbf{X}}(m+1)$ to $\overline{\overline{\mathbf{X}}}(m+1)$. Obviously, the order of the summation is irrelevant as the value of the states after the delays is the needed quantity.

The non-linear system model with non-linear operators acting on the signals is shown is Figure 2.32 below.

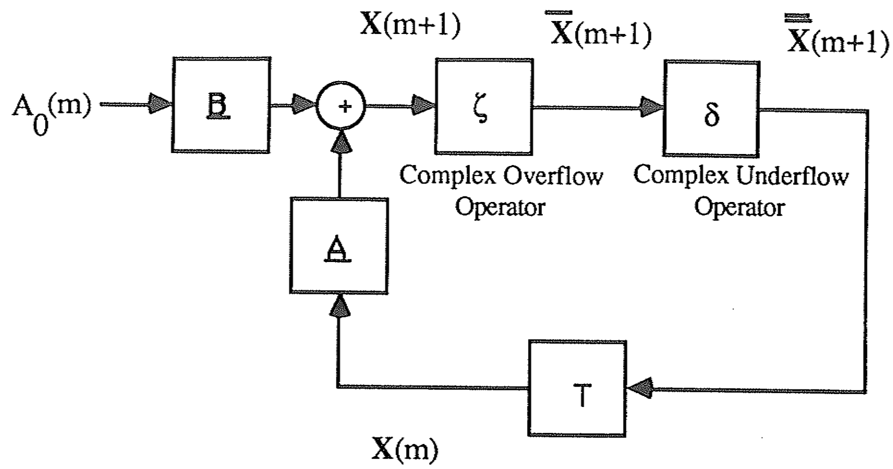

Figure 2.32: Non-linear system model with non-linear underflow and overflow operators.

Instead of considering the effects of overflow and underflow as additive complex error sequences, this model considers the non-linearities as complex operators acting on the states. Let the complex non-linear overflow operator be represented by ζ, which acts on the complex state sequence $\mathbf{X}(m+1)$ changing it to $\overline{\mathbf{X}}(m+1)$. Similarly, let the complex non-linear underflow operator be represented by δ, which acts on $\overline{\mathbf{X}}(m+1)$ changing it to $\overline{\overline{\mathbf{X}}}(m+1)$. Again, the order of the operators is irrelevant. The arguments of both operators are the real and imaginary parts of the complex input signal. Thus, in the real domain, the operators are viewed as being two-dimensional in nature; whereas in the complex domain, the operators are clearly one-dimensional. The two-dimensional view is the most useful one to adopt where the real and imaginary parts of the operators can be considered independently, ie.

$$y = y' + j\,y'' = \zeta(x) = \zeta'(x') + j\,\zeta''(x'') \tag{2.263}$$

where $x,y \in \mathbb{C}$, $y',y'',x',x'' \in \mathbb{R}$; and ζ' and ζ'' are the equivalent real operators of the complex operator ζ.

2.7.1 Zero-input Stability

A complex non-linear wave digital system under zero-input conditions, as shown in Figure 2.32 with $A_0(m) = 0$ for all m, has been shown to be stable in the real domain by Meerkötter [7] if pseudopassivity can be ensured. The arguments used can be applied directly to the complex case by redefining the norms used.

This is easily seen by considering a complex n-port N with an initial state $\mathbf{X}_0$. The initial pseudoenergy is non-negative and finite by definition, ie. $W^o(\mathbf{X}_0) \geq 0$, where the notation refers to the pseudoenergy calculated for an initial state $\mathbf{X}_0$. The n-port is pseudopassive if for every initial state X_0,

$$W(m) + W^o(X_0) \geq 0, \quad m \geq 0 \tag{2.264}$$

or using (2.56) the above becomes

$$||b||_m^2 \leq ||a||_m^2 + W^o(X_0) \tag{2.265}$$

or with a memoryless system (no delays),

$$||b||_m^2 \leq ||a||_m^2 \tag{2.266}$$

Equation (2.266) relates the complex input sequence to the output sequence and thus gives a stability requirement. Under zero input conditions with a(m) = 0 for all m greater than some finite integer M, the norm of the input sequence must be some positive finite number, say L, ie. the norm must be bounded above by L.

$$||a||_m^2 = L \tag{2.267}$$

and from (2.265) the norm of the output sequence must be bounded above by

$$||b||_m^2 \leq L + W^o(X_0) \tag{2.268}$$

Using the definition of the norm given in (2.54), the output sequence norm being bounded above implies that

$$\lim_{m \Rightarrow \infty} \left[|| \, b(m) \, ||^2 \right] = 0 \tag{2.269}$$

Clearly, this gives zero-input stability from the pseudopassivity requirement. Note that the above implies that for some m greater than an integer $\widehat{M}$, with $\widehat{M} > M$, the output must go to zero

$$\lim_{m \Rightarrow \infty} b(m) = 0, \quad m > \widehat{M} \tag{2.270}$$

which leads to the suppression of periodic oscillations under autonomous conditions.

Thus the condition of pseudopassivity gives zero-input stability and the suppession of periodic oscillations for a complex wave digital network regardless of the initial conditions of the filter as long as the initial pseudoenergy given by the initial state of the network is positive and finite.

The non-linear operators in Figure 2.32 can be defined to ensure zero-input stabiltiy. It has been shown, in the real domain [56][43], that if a real positive definite matrix **Q** can be found such that for $\overline{\overline{X}}$ a real state vector one has

$$\overline{\overline{X}}^T Q \overline{\overline{X}} - \left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right]^T Q \left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right] > 0 \tag{2.271}$$

then the output of the recursive part of the system will decay to zero in a finite amount of time. Such a **Q** is called a Lyapunov function. The argument used by Webb [43] can be repeated in the complex domain by modifying the requirement of the Lyapunov function **Q** to be

$$\overline{\overline{X}}^{*T} \mathbf{Q} \overline{\overline{X}} - \left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right]^{*T} \mathbf{Q}\left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right] > 0 \tag{2.272}$$

Using the approach adopted by Webb [43], the definition of the pseudopower given in (2.49) can be used along with the overflow and underflow operators ζ and δ to find a function of the form given above in (2.272), and the equation is given by

$$\overline{\overline{X}}^{*T} \mathbf{G}_D \overline{\overline{X}} - \left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right]^{*T} \mathbf{G}_D\left[\delta\left(\zeta\left(\overline{\overline{X}}\right)\right)\right] > 0 \tag{2.273}$$

where $\mathbf{G}_D$ is a positive definite real diagonal matrix of the conductances of the ports associated with the states, or the delay ports. The above Equation (2.273) gives a sufficient condition to guarantee the absence of zero-input limit cycles. If the norms of the operators ζ and δ as given by,

$$||\zeta|| = \sup_{x \neq 0}\left(\frac{|\zeta(x)|}{|x|}\right) \tag{2.274}$$

$$||\delta|| = \sup_{x \neq 0}\left(\frac{|\delta(x)|}{|x|}\right) \tag{2.275}$$

where |#| is the magnitude of the complex number #, are less than or equal to one,

$$||\zeta|| \leq 1 \tag{2.276}$$

$$||\delta|| \leq 1 \tag{2.277}$$

i.e. the operators are contractive, the above (2.273) is guaranteed to hold. Viewing the operators as non-linear one-ports connected in series with complex circulators at the output of the state ports, the above immediately implies that the one-ports must be pseudopassive. Thus as long as the non-linear complex one-ports are pseudopassive it is guaranteed that no zero-input limit cycles will exist, or in other words, there will be an absence of zero-input periodic oscillations.

The condition for the complex underflow operator δ given by (2.277) can be met with simple truncation of the real and imaginary parts of the signal. The characteristic ensuring (2.276) will be discussed in the following section.

2.7.2 Forced-Response Stability

A system with a non-zero input can become unstable if the amplitude at some of the signal nodes becomes larger than the maximum magnitude that can be represented. Non-linear overflow truncation errors will result because of the large signal amplitudes, and thus overflow trunction errors may not fade out when the signal amplitude is decreased. If the error signal amplitude decreases in magnitude to zero in a finite amount of time after the system is no longer overdriven, the system is forced-response stable [6].

Two complex systems showing the effects of overflow and underflow errors are shown for a complex forcing sequence A_0 in Figures 2.31 and 2.32. Using the same approach as for zero-input stability, the argument introduced by Meerkötter [7] can be repeated and extended to the complex domain.

Consider a complex n-port N with initial states X_0 and $\widehat{X}_0$. To every two complex input sequences a(m) and $\hat{a}$(m), there will correspond two complex output sequences b(m) and $\widehat{b}$(m), each derived from the corresponding input and the corresponding initial state. Let the initial incremental pseudopower of the difference in the initial states, as defined in (2.87), satisfy

$$W_\Delta{}^o(X_0 - \widehat{X}_0) \geq 0 \tag{2.278}$$

where the notation refers to the initial incremental pseudoenergy of the complex network calculated from the difference in the two initial states. The complex n-port is incrementally pseudopassive if for the difference of the initial states,

$$W_\Delta(m) + W_\Delta{}^o(X_0 - \widehat{X}_0) \geq 0, \quad m \geq 0 \tag{2.279}$$

or using the definition of incremental pseudoenergy,

$$\left|\left| b - \widehat{b} \right|\right|_m^2 \leq \left|\left| a - \hat{a} \right|\right|_m^2 + W_\Delta{}^o\left(X_0 - \widehat{X}_0\right) \tag{2.280}$$

We are free to consider any two initial states, and if the system is memoryless or if the two initial states are equal, then

$$W_\Delta{}^o(X_0 - \widehat{X}_0) = W_\Delta{}^o(0) = 0 \tag{2.281}$$

or if the initial states are not equal

$$W_\Delta{}^o(X_0 - \widehat{X}_0) = L \tag{2.282}$$

and from (2.280)

$$\left|\left| b - \widehat{b} \right|\right|_m \leq \left|\left| a - \hat{a} \right|\right|_m + L \tag{2.283}$$

where L is a real non-negative and finite value. The above, which is a consequence of incremental pseudopassivity, implies that any two complex output sequences are arbitrarily close if the corresponding complex input sequences are sufficiently close, which gives a continuous input-output relationship between the complex quantities.

Also, if the complex input sequences are equal, then

$$\left|\left| b - \hat{b} \right|\right|_m \tag{2.284}$$

is bounded above and using (2.54), this implies for any initial states,

$$\lim_{m \Rightarrow \infty} \left|\left| b(m) - \hat{b}(m) \right|\right| = 0 \tag{2.285}$$

which gives asymptotic (forced response) stability of the complex output for arbitrary input sequences independent of the initial conditions, as long as the n-port N is incrementally pseudopassive.

Forced response stability is thus given by incremental pseudopassivity. Conditions guaranteeing forced response stability have been explored by several authors [34][36][38][56]. Webb [43] has extended the arguement given by Moon [56], which includes the effects of both overflow and underflow errors, to a positive definite diagonal conductance matrix. The discussion can be directly extended to the complex domain, and will not be repeated here.

From [43], if a pseudopassive wave digital network is not overdriven for any instant after some $m \geq M$, the response due to the overflow error sequence will be determined by a zero-input recursive system. From the earlier discussion on zero-input stability, we know that this error sequence will decay to zero after a finite amount of time and thus the output will converge to the predicted output corresponding to the input signal (excect for the granular amplitude error sequence due to underflow errors). This gives forced response stability of the complex wave digital network.

Incremental pseudopassivity gives other benefitial properties of a wave digital network [6][7]. One property is given by (2.283), where small changes in the input signal cause small changes in the output signal even if overflow errors occur. Also, from [7], a change in the initial state of the digital filter does not have a lasting effect on the output signal.

Conditions similar to those in [43] are given for forced response stability. They extend to the complex case by letting the overflow truncation function, as shown below for the equivalent real and imaginary operators, ζ' and ζ'', of the complex operator ζ (the real and imaginary functions can be chosen the same), lie within the shaded area,

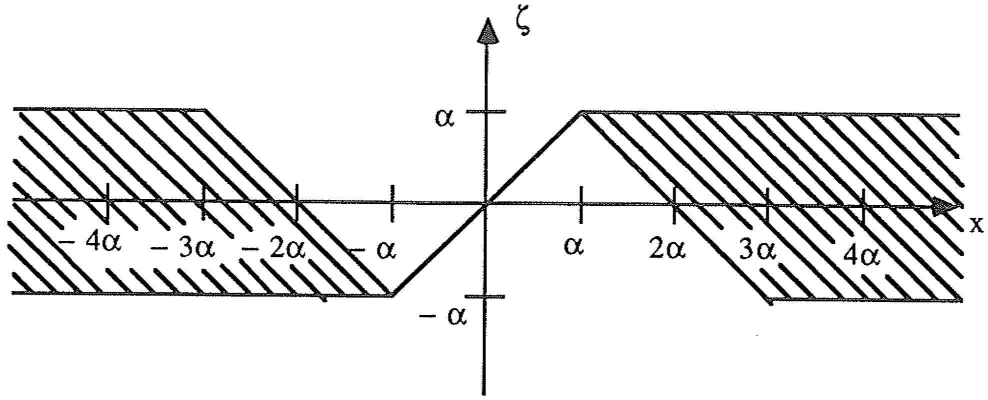

Figure 2.33: Real or imaginary equivalent of the overflow truncation function.

where α is the maximum allowable signal amplitude. This gives incremental pseudopassivity as discussed earlier. Notice that the overflow truncation function satisfies the restriction on the magnitude of the norm of the operator as given in (2.276).

The overflow function cannot have a slope greater than plus or minus one, ie. the continuous curve cannot have a slope exceeding $\pm 45^o$. Various forms of the overflow error function have been found as summarized in [43] that meet this constraint. They include the saturation function, as shown in Figure 2.34

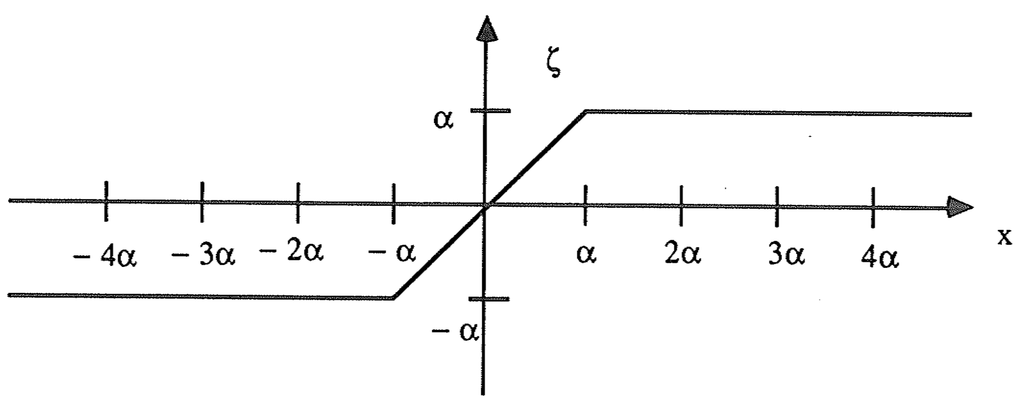

Figure 2.34: Saturation overflow function.

and the triangle function as shown in Figure 2.35.

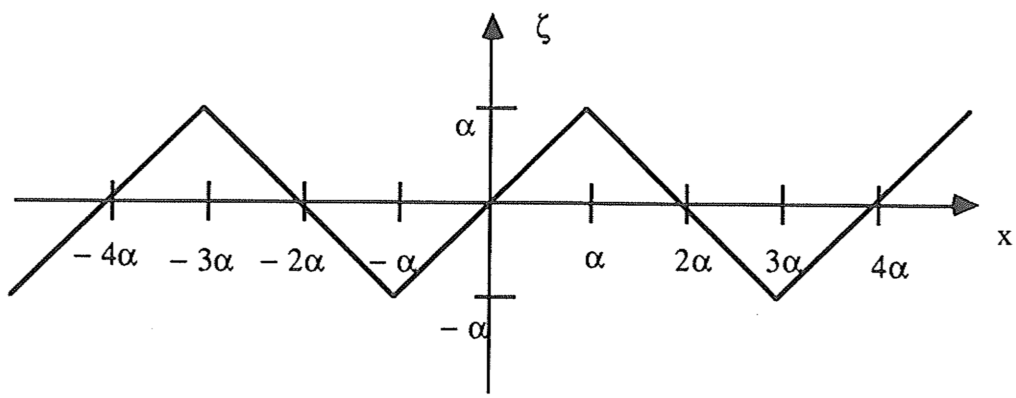

Figure 2.35: Triangle overflow function.

Both of the above functions give incremental pseudopassivity, and thus forced response stability. Notice that the triangle overflow function has a greater signal deviation error than the saturation function.

CHAPTER III

LATTICE REALIZATION OF EVEN-ORDER CLASSICAL FILTERS

The wave digital lattice configuration can be used in the realization of even order classical filters, such as: Butterworth, Chebyshev and Cauer (Elliptic) filters. For real input signals the classical filter is implemented with a single canonic complex allpass wave digital network. The network is comprised of complex two-port adaptors, complex three-port circulators, delays and scaling transformers. The scattering matrix describing the non-dynamic scaled WD sub-network is sparse, giving a high degree of decoupling of the state variables, and is only a function of the complex two-port adaptor parameters. It is also highly structured as it is composed of four complex block sub-matrices. The scattering matrix is quantized by using magnitude truncation of the real and imaginary parts of the matrix elements. If pseudopassivity of the WD network can be guaranteed, all of the advantages gained by pseudopassivity (sensitivity, stability, and suppression of parasitic oscillations) are enjoyed by this realization. By choosing the input-output port reference impedance to be purely real, i.e. a pure resistance, (with real inputs and outputs) the one-port complex allpass network as externally viewed is a real network. The analog domain equivalence of the complex wave digital network is comprised of complex circulators, capacitors and constant reactances.

3.1 Decomposition into Two Complex Allpass Functions

A classical filter, or reference filter, can be represented in the z-domain by a stable transfer function G(z) by taking the bilinear transformation of the stable transfer function in the Laplace domain. The resulting function will be rational and have the form

$$G(z) = \frac{P(z)}{D(z)} = \frac{K_z(z - z_1)(z - z_2)\dots(z - z_n)}{(z - p_1)(z - p_2)\dots(z - p_n)} \tag{3.1}$$

where z_i are the zeros, p_i are the poles and K_z is the gain of the discrete transfer function. For even-order classical filters, such as Butterworth, Chebyshev, and Cauer filters, half of the zeros (which all lie on the unit circle in the z-plane) and poles are the complex conjugate of the other half, giving a real G(z) for real z. Also, since G(z) is stable by assumption, all of the poles lie within the unit circle in the z-plane. The numerator and denominator can be thus expressed in terms of polynomials with real coefficients giving the following:

$$G(z) = \frac{P(z)}{D(z)} = \frac{\tilde{p}_n z^n + \tilde{p}_{n-1} z^{n-1} + \ldots + \tilde{p}_0}{d_n z^n + d_{n-1} z^{n-1} + \ldots + d_0} \tag{3.2}$$

where all $\tilde{p}_i$ and d_i are real. Continuing with the restriction to classical filters, it is known [30-31], that the numerator must be symmetric for even orders, or in other words

$$P(z^{-1}) = z^{-N} P(z) \tag{3.3}$$

The Feltkeller equation [44] gives the relation shown below between the transfer function and its spectral complement, H(z), along the unit circle.

$$|G(e^{j\Omega})|^2 + |H(e^{j\Omega})|^2 = 1 \quad \text{or} \quad G(e^{j\Omega})\, G(e^{-j\Omega}) + H(e^{j\Omega})\, H(e^{-j\Omega}) = 1 \tag{3.4}$$

Using analytic continuation, the above becomes

$$G(z)\, G(z^{-1}) + H(z)\, H(z^{-1}) = 1 \tag{3.5}$$

Let the spectral complement H(z) be

$$H(z) = \frac{Q(z)}{D(z)} \tag{3.6}$$

and therefore Equation (3.5) gives

$$P(z)\, P(z^{-1}) + Q(z)\, Q(z^{-1}) = D(z)\, D(z^{-1}) \tag{3.7}$$

A real transfer function, ie. a transfer function with real coefficients, with a symmetric numerator can be written in terms of two allpass functions [29-31]. When the numerator of the spectral complement is anti-symmetric, or the order of the transfer function is odd, the two allpass functions have real coefficients. However, when the numerator of both the transfer function and its spectral complement are both symmetric, or the filter is anti-metric, the allpass functions have complex coefficients.

The allpass functions can be derived in terms of the poles of the original transfer function, along with the gains at zero frequency in the Laplace domain of the transfer function and its spectral complement. Consider the transfer function expressed as a function of z^{-1} instead of z (which is always possible by multiplying the numerator and denominator by z^{-N}), and the symmetric properties

$$P(z^{-1}) = z^{-N}\, P(z) \tag{3.8a}$$

$$Q(z^{-1}) = z^{-N}\, Q(z) \tag{3.8b}$$

Substituting the above into (3.7),

$$P^2(z) + Q^2(z) = z^N D(z^{-1})\, D(z) \tag{3.9}$$

The left side of (3.9) can be factored giving

$$\left[P(z) + j\,Q(z)\right]\left[P(z) - j\,Q(z)\right] = z^N D(z^{-1}) D(z) \tag{3.10}$$

The zeros of the right side of (3.10) above can be assigned to the two factors given on the left side in a unique way dictated by the given form of the equation.

The unique assignment of the factors of $z^N D(z^{-1})D(z)$ is given by observing that both P(z) and Q(z) are polynomials with real coefficients for classical filters. Thus the two terms on the left side of (3.10) are the complex conjugates of each other. The zeros of D(z) occur in complex conjugate pairs (since D(z) has real coefficients) and this gives a general form of D(z),

$$D(z) = \prod_{k=1}^{N/2} (z - p_k)(z - p_k^*) \tag{3.11}$$

where p_k are the complex poles of the filter in the z-domain. Using this form, Equation (3.10) can be written as

$$[P(z) + j\,Q(z)][P(z) - j\,Q(z)] = z^N \prod_{k=1}^{N/2} (z - p_k)(z - p_k^*)(z^{-1} - p_k)(z^{-1} - p_k^*) \tag{3.12}$$

or

$$[P(z) + j\,Q(z)][P(z) - j\,Q(z)] = \prod_{k=1}^{N/2} (z - p_k)(z - p_k^*)(1 - z\,p_k)(1 - z\,p_k^*) \tag{3.13}$$

The right side of Equation (3.13) above gives the factors that must be assigned to the terms on the left side. But since both P(z) and Q(z) are symmetric polynomials with real coefficients by assumption, the zeros of [P(z) + j Q(z)] occur in reciprocal pairs, as do the zeros of [P(z) - j Q(z)]. This can be seen be considering a zero of [P(z) + j Q(z)], say z_0.

$$P(z_0) + j\,Q(z_0) = 0 \tag{3.14}$$

Since both real polynomials are symmetric, another zero can always be found which is the inverse of z_0, or

$$P(z_0^{-1}) + j\,Q(z_0^{-1}) = 0 \tag{3.15}$$

Also, if p_k is a zero of [P(z) + j Q(z)], then p_k^* must be a zero of [P(z) - j Q(z)] since the polynomials are real. None of the p_k are purely real since if one were, say $\widehat{p}_k$, then it has to be a zero of either [P(z) + j Q(z)] or [P(z) - j Q(z)] or both from (3.10). This implies that $P(\widehat{p}_k) = Q(\widehat{p}_k) = 0$ since P(z) and Q(z) are real polynomials. In particular, it implies a common factor of a term involving $\widehat{p}_k$ between P(z) and D(z), which is not allowed by the assumption that there are no common factors between P(z) and D(z) in Equation (3.1) or (3.2).

This leads to the assignment of factors to the terms [P(z) + j Q(z)] and [P(z) - j Q(z)] as given by

$$P(z) + j\,Q(z) = \beta \prod_{m=1}^{N/2} \left(z - p_m\right)\left(1 - z\,p_m\right) \tag{3.16}$$

$$P(z) - j\,Q(z) = \beta^* \prod_{m=1}^{N/2} \left(z - p_m^*\right)\left(1 - z\,p_m^*\right) \tag{3.17}$$

where β is a unimodular constant such that $\beta\,\beta^* = 1$. Dividing (3.16) and (3.17) by D(z) as given by (3.11), and cancelling the common factors the following is derived

$$G(z) + j\,H(z) = \beta \prod_{m=1}^{N/2} \frac{1 - z\,p_m}{z - p_m^*} \tag{3.18}$$

$$G(z) - j\,H(z) = \beta^* \prod_{m=1}^{N/2} \frac{1 - z\,p_m^*}{z - p_m} \tag{3.19}$$

The rational functions on the right side of equations (3.18) and (3.19) are recognized as the general form of a complex allpass function (an allpass function with complex coefficients), defined as

$$A_1(z) = \beta \prod_{m=1}^{N/2} \frac{1 - z\,p_m}{z - p_m^*} \tag{3.20}$$

$$A_2(z) = \beta^* \prod_{m=1}^{N/2} \frac{1 - z\,p_m^*}{z - p_m} \tag{3.21}$$

The allpass functions have a magnitude of one on the unit circle in the z-plane, and are stable since the poles of the allpass functions are a subset of the poles of the transfer function, which is stable by assumption. The transfer function G(z) is expressed in terms of the allpass functions by adding equations (3.18) and (3.19), and solving for G(z),

$$G(z) = \frac{1}{2}\left(\beta \prod_{m=1}^{N/2} \frac{1 - z\,p_m}{z - p_m^*} + \beta^* \prod_{m=1}^{N/2} \frac{1 - z\,p_m^*}{z - p_m} \right) \tag{3.22}$$

Similarly, the spectral complement of G(z) is found by subtracting (3.19) from (3.18),

$$H(z) = \frac{1}{2j}\left(\beta \prod_{m=1}^{N/2} \frac{1 - z\,p_m}{z - p_m^*} - \beta^* \prod_{m=1}^{N/2} \frac{1 - z\,p_m^*}{z - p_m} \right) \tag{3.23}$$

Or, using (3.20) and (3.21),

$$G(z) = \frac{1}{2}\left(A_1(z) + A_2(z)\right) \tag{3.24}$$

$$H(z) = \frac{1}{2j}\left(A_1(z) - A_2(z)\right) \tag{3.25}$$

Thus, the even-order real transfer function G(z) and its real spectral complement H(z) can be written in terms of, or decomposed into, two complex allpass functions. The allpass functions expressed as a function of G(z) and H(z) are

$$A_1(z) = G(z) + j\,H(z) \tag{3.26}$$
$$A_2(z) = G(z) - j\,H(z) \tag{3.27}$$

The complex coefficients of one allpass function are exactly the conjugate of the coefficients of the other allpass function. Thus, for real inputs, the output of one allpass function will be exactly the conjugate of the output of the other. This implies that for real inputs, one allpass function is superfluous and not needed. Consider implementing $A_1(z)$ only, from (3.26) given above, the real part of the output will correspond to the desired output of G(z), and the imaginary part of the output will correspond to H(z). Therefore, both of the functions G(z) and H(z) are realized with a single complex allpass function of half the order.

The unimodular constant β can be calculated by considering (3.19) at $z = 1$, or at zero frequency in the Laplace domain.

$$G(1) + j\,H(1) = \beta \prod_{m=1}^{N/2} \frac{1 - p_m}{1 - p_m^*} \tag{3.28}$$

or, solving for β,

$$\beta = \left(G(1) + j\,H(1)\right) \prod_{m=1}^{N/2} \frac{1 - p_m^*}{1 - p_m} \tag{3.29}$$

Notice that G(1) is known from the frequency specifications of the filter, and H(1) can be calculated from the Feltkeller equation given in (3.4) by restricting H(1) to be positive.

3.2 Structure of the Lattice Realization

The lattice wave digital filter structure is perhaps the most advantageous one to use for realizing classical filters [6][19][20]. It is shown below as a doubly-terminated two-port with equal impedance terminations, and thus the conductance matrix has equal entries on the diagonal [6].

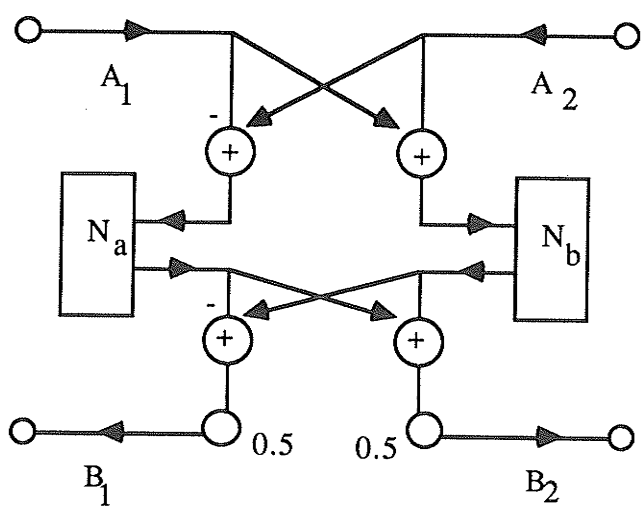

Figure 3.1: Lattice two-port structure.

S_a and S_b are allpass functions characterizing the allpass networks N_a and N_b, and are pseudolossless. However, if $A_2 = 0$ and B_1 is not of interest, then the above network can be simplified to the following one-port [6].

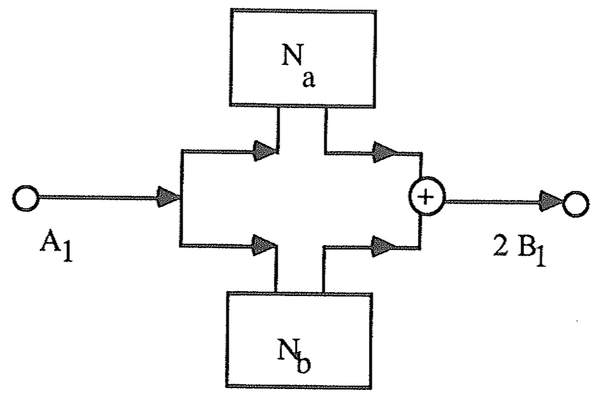

Figure 3.2: Simplified lattice one-port structure.

If the input signal is restricted to be purely real for all time, then from the earlier section one allpass network is reduntant. Therefore illiminate network N_b from Figure 3.2, leaving the single allpass one-port network N_a of Figure 3.3.

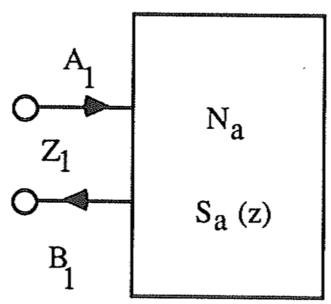

Figure 3.3: One-port complex allpass network.

The form of the above is simple which adds to the attractiveness of the realization. It should be clear that the above complex one-port allpass network was derived from a doubly-terminated lattice structure with equal impedance terminations and applies for real inputs only.

The allpass function can be derived easily from the poles of the transfer function. Thus the poles p_m of (3.20) are given by a subset of the z-domain poles of a classical filter, and $S_a = A_1$. The choice of the poles [24][31] are shown graphically below in the Laplace domain.

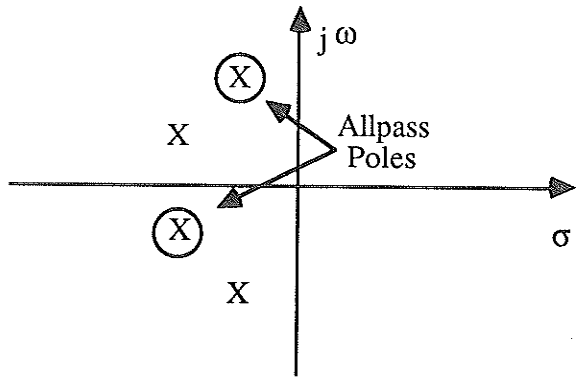

Figure 3.4: Choice of poles for the complex allpass function.

Given a set of poles from a even-order classical filter, the allpass function is defined by choosing every second pole and applying the bilinear transformation (2.10) to transform the pole into the z-domain. Clearly, there are two possible sets of poles that can be chosen, and the poles of one set are the complex conjugate of the poles of the other. Choosing one set gives $A_1(z)$ and the other gives $A_2(z)$. The set of poles chosen for the implementation of $G(z)$ is significant, since the value of β is defined in terms of the poles and the values of $G(1)$ and $H(1)$ (3.29).

3.3 Complex Allpass Realization With Adaptors

The discussion above has given a method of realizing an even-order classical, or reference filter, such as a Butterworth, a Chebyshev or a Cauer filter. The realization problem has been reduced to the implementation of a complex allpass function. Although many realization techniques have been developed, an investigation by Webb [43] has shown that the realization with circulators that have different characteristic impedances gives a solution with a minimum number of entries in the scattering matrix, which leads to the decoupling of the maximum number of state variables. The following method of realization also possesses a high degree of symmetry which gives a convenient block form of the scattering matrix and is canonic in the number of delays.

3.3.1 Complex Allpass Extraction

The general form of a stable complex allpass function, S(z), of order N is given below

$$S(z) = \beta \frac{z^N D^*\left(\frac{1}{z^*}\right)}{D(z)} = \beta \frac{d_N^* + d_{N-1}^* z + \ldots + d_1^* z^{N-1} + d_0^* z^N}{d_0 + d_1 z + \ldots + d_{N-1} z^{N-1} + d_N z^N} \qquad (3.30)$$

where the magnitude of β is unity. In the wave digital domain, the above allpass function can be viewed as the following one-port.

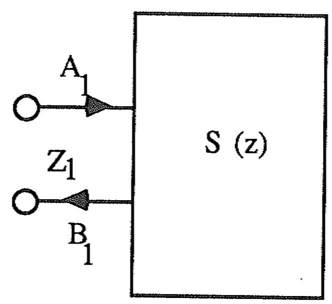

Figure 3.5: One-port realization of a complex allpass function.

Consider removing a complex two-port adaptor, as defined in Section 2.5.1, from the one-port shown above leaving a new one-port $\hat{S}(z)$. The parameter of the two-port can obviously be any complex number bounded in magnitude by unity, since the two-port simply changes the reference impedance of the new one-port $\hat{S}(z)$. However, if the parameter is chosen in a particular way, an important form of $\hat{S}(z)$ is given as shown in the following argument.

The wave digital network with the complex two-port extracted is shown below,

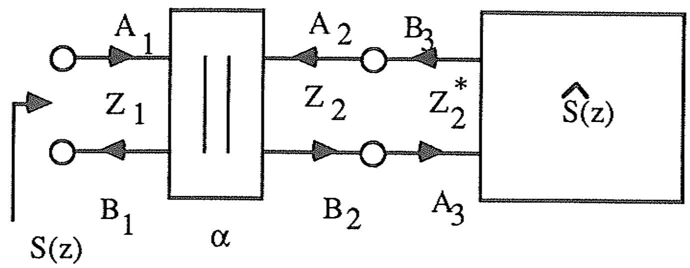

Figure 3.6: Complex two-port adaptor extracted from a complex one-port.

with the following definitions as given by (2.151) and (2.152),

$$\alpha = \frac{Z_1 - Z_2^*}{Z_1 + Z_2} \tag{3.31}$$

$$B_1 = -\left(\frac{1-\alpha}{1-\alpha^*}\right)\alpha^* A_1 + \left[1 + \left(\frac{1-\alpha}{1-\alpha^*}\right)\alpha^*\right] A_2 \tag{3.32}$$

$$B_2 = \left(1 - \alpha\right) A_1 + \alpha A_2 \tag{3.33}$$

From Figure 3.6, the definitions of the transfer functions are given by

$$S(z) = \frac{B_1}{A_1} \tag{3.34}$$

$$\hat{S}(z) = \frac{B_3}{A_3} = \frac{A_2}{B_2} \tag{3.35}$$

or

$$A_2 = \hat{S}(z)\, B_2 \tag{3.36}$$

Substituting the above into (3.33), we get

$$B_2 = \left(1 - \alpha\right) A_1 + \alpha\, \hat{S}(z)\, B_2 \tag{3.37}$$

Solving for B_2,

$$B_2 = \frac{\left(1 - \alpha\right) A_1}{1 - \hat{S}(z)\, \alpha} \tag{3.38}$$

Substituting this expression for B_2 into (3.36), and then into (3.32) in order to derive a function in terms of A_1 and B_1 only, we get

$$B_1 = -\left(\frac{1-\alpha}{1-\alpha^*}\right)\alpha^* A_1 + \left(1 + \left(\frac{1-\alpha}{1-\alpha^*}\right)\alpha^*\right)\hat{S}(z)\,\frac{(1-\alpha)A_1}{1-\hat{S}(z)\,\alpha} \tag{3.39}$$

Thus the transfer function (or allpass function) S(z), given in terms of $\hat{S}(z)$, can be found by using the above with the definition of S(z) in (3.34),

$$S(z) = \frac{B_1}{A_1} = \left(\frac{1-\alpha}{1-\alpha^*}\right)\left[\frac{\hat{S}(z)-\alpha^*}{1-\hat{S}(z)\,\alpha}\right] \tag{3.40}$$

Define

$$\gamma = \frac{1-\alpha^*}{1-\alpha}, \quad |\gamma| = 1 \tag{3.41}$$

Solving for $\hat{S}(z)$ in (3.40) we get,

$$\hat{S}(z) = \frac{\gamma S(z) + \alpha^*}{1 + \alpha\gamma S(z)} \tag{3.42}$$

The general form of S(z) as given in (3.30) is represented in a more convenient form below

$$S(z) = \beta\,\frac{z^N D^*\left(\frac{1}{z^*}\right)}{D(z)} \tag{3.43}$$

where

$$D(z) = d_0 + d_1 z + \ldots + d_{N-1} z^{N-1} + d_N z^N \tag{3.44}$$

$$D^*(z) = d_0^* + d_1^* z^* + \ldots + d_{N-1}^* (z^*)^{N-1} + d_N^* (z^*)^N \tag{3.45}$$

Substituting (3.43) into (3.42),

$$\hat{S}(z) = \frac{\gamma\beta\,\dfrac{z^N D^*\left(\frac{1}{z^*}\right)}{D(z)} + \alpha^*}{1 + \alpha\gamma\beta\,\dfrac{z^N D^*\left(\frac{1}{z^*}\right)}{D(z)}} \tag{3.46}$$

or

$$\hat{S}(z) = \frac{\gamma\beta\, z^N D^*\left(\frac{1}{z^*}\right) + \alpha^* D(z)}{D(z) + \alpha\gamma\beta\, z^N D^*\left(\frac{1}{z^*}\right)} \tag{3.47}$$

Substituting (3.44) and (3.45), we get a general form for $\hat{S}(z)$,

$$\hat{S}(z) = \frac{\gamma\beta\left(d_N^*+d_{N-1}^* z+\ldots+d_1^* z^{N-1}+d_0^* z^N\right)+\alpha^*\left(d_0+d_1 z+\ldots+d_{N-1}z^{N-1}+d_N z^N\right)}{\left(d_0+d_1 z+\ldots+d_{N-1}z^{N-1}+d_N z^N\right)+\alpha\gamma\beta\left(d_N^*+d_{N-1}^* z+\ldots+d_1^* z^{N-1}+d_0^* z^N\right)} \tag{3.48}$$

Using the defintion of γ from (3.41) and multiplying the numerator and denominator by $(1 - \alpha)$, we get

$$\hat{S}(z) = \frac{\beta\left(1-\alpha^*\right)\left(d_N^*+d_{N-1}^* z+\ldots+d_1^* z^{N-1}+d_0^* z^N\right)+\alpha^*\left(1-\alpha\right)\left(d_0+d_1 z+\ldots+d_{N-1}z^{N-1}+d_N z^N\right)}{\left(1-\alpha\right)\left(d_0+d_1 z+\ldots+d_{N-1}z^{N-1}+d_N z^N\right)+\alpha\beta\left(1-\alpha^*\right)\left(d_N^*+d_{N-1}^* z+\ldots+d_1^* z^{N-1}+d_0^* z^N\right)} \tag{3.49}$$

The above is a function of the known complex allpass function and the parameter of the complex two-port adaptor only. Rewriting (3.49) for clarity,

$$\hat{S}(z) = \frac{\left[\left(1-\alpha^*\right)\beta d_N^* + \alpha^*\left(1-\alpha\right)d_0\right]+\ldots+\left[\left(1-\alpha^*\right)\beta d_0^* + \alpha^*\left(1-\alpha\right)d_N\right]z^N}{\left[\left(1-\alpha\right)d_0 + \alpha\left(1-\alpha^*\right)\beta d_N^*\right]+\ldots+\left[\left(1-\alpha\right)d_N + \alpha\left(1-\alpha^*\right)\beta d_0^*\right]z^N} \tag{3.50}$$

The above is a complex allpass function since it is in the form of (3.30) with $\beta^*\beta = 1$, and it can also be written as

$$\hat{S}(z) = \beta\frac{\left[\left(1-\alpha^*\right)d_N^* + \alpha^*\left(1-\alpha\right)\beta^* d_0\right]+\ldots+\left[\left(1-\alpha^*\right)d_0^* + \alpha^*\left(1-\alpha\right)\beta^* d_N\right]z^N}{\left[\left(1-\alpha\right)d_0 + \alpha\left(1-\alpha^*\right)\beta d_N^*\right]+\ldots+\left[\left(1-\alpha\right)d_N + \alpha\left(1-\alpha^*\right)\beta d_0^*\right]z^N} \tag{3.51}$$

which is the general form given in (3.30). The above can represent a complex allpass function in series with a delay z^{-1} if the highest order coefficient in the numerator is set to zero,

$$\left(1-\alpha^*\right)d_0^* + \alpha^*\left(1-\alpha\right)\beta^* d_N = 0 \tag{3.52}$$

and thus the lowest order coefficient of the denominator will also be set to zero (since it is the complex conjugate of (3.52)). Solving for the complex two-port parameter α,

$$\alpha = \left(\frac{\beta d_0^* - d_N}{d_N^* - \beta^* d_0}\right)\left(\frac{\beta^* d_0}{d_N}\right) = \frac{\dfrac{d_0^*}{d_N} - \beta^*}{\dfrac{d_N^*}{d_0} - \beta^*} \tag{3.53}$$

Clearly α cannot equal zero since the numerator can never be zero for a stable allpass function. This is clear since in the general form of a allpass function as given in (3.30) and (3.20), the coefficient d_N can be set to one, and d_0 is given as the product of the poles of a stable allpass function. Thus the product must be less than one, and thus the magnitude of the first term in the numerator, d_0, is always less than one. Using the same arguement with Equation (3.53), the magnitude of α must be less than one. Substituting Equation (3.53) into (3.50) allows a delay to be extracted from $\hat{S}(z)$ reducing its order by one since the lowest order term in the denominator and the hightest order term in the numerator are set to zero, or

$$\hat{S}(z) = z^{-1}\frac{\left[\left(1-\alpha^*\right)\beta d_N^* + \alpha^*\left(1-\alpha\right)d_0\right] + \ldots + \left[\left(1-\alpha^*\right)\beta d_1^* + \alpha^*\left(1-\alpha\right)d_{N-1}\right]z^{N-1}}{\left[\left(1-\alpha\right)d_1 + \alpha\left(1-\alpha^*\right)\beta d_{N-1}^*\right] + \ldots + \left[\left(1-\alpha\right)d_N + \alpha\left(1-\alpha^*\right)\beta d_0^*\right]z^{N-1}} \tag{3.54}$$

or

$$\hat{S}(z) = z^{-1}\,\hat{\hat{S}}(z) \tag{3.55}$$

The remaining allpass function $\hat{\hat{S}}(z)$ along with the extracted complex two-port adaptor and the delay are shown below.

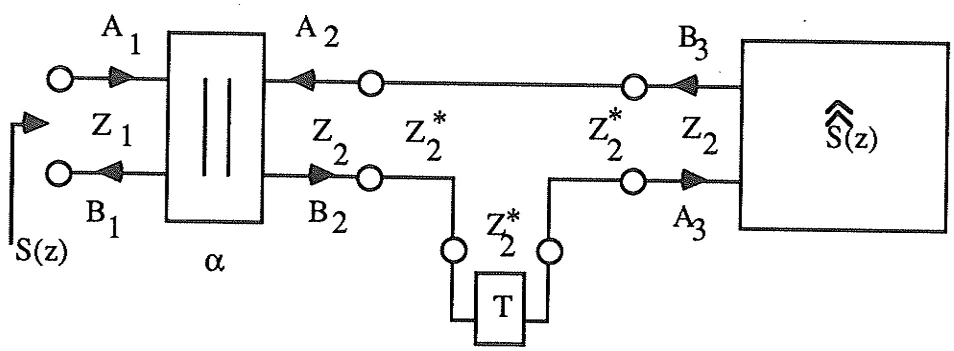

Figure 3.7: Extraction of a complex two-port adaptor with a delay.

The position of the delay vis-a-vis the two terminals is not specified by (3.55), and thus it can be positioned in the most advantageous way. The actual connection of the delay to the two-port adaptor and the remaining one-port allpass network is accomplished with an ideal complex circulator of characteristic impedance Z_2*. Also, the port impedance associated with the one-port allpass network is again Z_2 from the criterion for port connections in

Section 2.5.2.

The process of extracting a complex two-port adaptor and delay can be repeated for the complex allpass function $\hat{\hat{S}}(z)$, and all remaining allpass functions until the last term that remains is a complex scalar, which is guaranteed to be unimodular since it must be of the form of an allpass from (3.54). The number of delays (which equals the number of adaptors) will be equal to the order of the allpass function (which is half the order of the original classical filter) and thus the design is canonic.

3.3.2 Stability of the Resulting Complex Allpass Function

The form of (3.54) guarantees that $\hat{\hat{S}}(z)$ is in the form of a complex allpass function. However, for the allpass to be stable, all of the poles must lie within the unit circle in the z-plane, or the magnitude of the poles must be less than one. Equation (3.42), repeated below for clarity,

$$\hat{S}(z) = \frac{\gamma S(z) + \alpha^*}{1 + \alpha \gamma S(z)} \tag{3.56}$$

defines a nonsingular mapping between S(z) and $\hat{S}(z)$, where the magnitude of α is less than one and γ is unimodular. The poles of S(z) in the z-plane are mapped, through the above, into poles of $\hat{S}(z)$ in the $\hat{z}$-plane (analogus to the z-plane). This mapping is represented in the following figure.

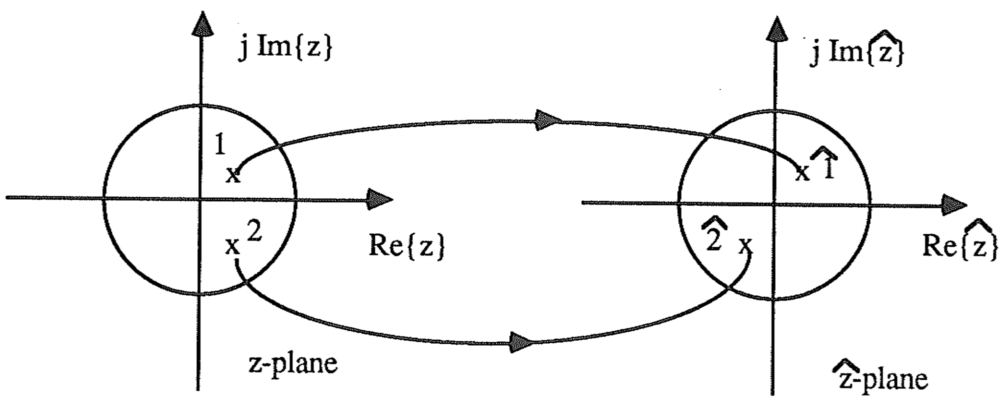

Figure 3.8: Mapping the poles of S(z) into the poles of $\hat{S}(z)$.

In a similar way the zeros of S(z) are mapped into the zeros of $\hat{S}(z)$. The following will show that the poles of $\hat{S}(z)$ all lie within the unit circle, or $\hat{S}(z)$ is a stable allpass function. Let

$$\tilde{S}(z) = \gamma\, S(z) \tag{3.57}$$

and define

$$f(z) = \frac{1}{\tilde{S}(z)} \tag{3.58}$$

Since f(z) is the reciprocal of $\tilde{S}(z)$, and $\tilde{S}(z)$ is stable by assumption, all of the zeros of f(z) lie within the unit circle and all of the poles lie outside the unit circle. Furthermore, since $\tilde{S}(z)$ is an allpass function, $| f(z) | = 1$ on the unit circle. Also, since $\tilde{S}(z)$ is a rational function, f(z) is a rational function from (3.58), and therefore f(z) is analytic within the unit circle taking its maximum on the boundary $| z | = 1$. Thus, within $| z | \leq 1$

$$| f(z) | \leq 1 \tag{3.59}$$

From (3.56-57),

$$\hat{S}(z) = \frac{\tilde{S}(z) + \alpha^*}{1 + \alpha \tilde{S}(z)} \tag{3.60}$$

Define the reciprocal of (3.60) as g(z),

$$g(z) = \frac{1}{\hat{S}(z)} \tag{3.61}$$

$$g(z) = \frac{1 + \alpha \tilde{S}(z)}{\tilde{S}(z) + \alpha^*} \tag{3.62}$$

Divide the top and bottom of (3.61) by $\tilde{S}(z)$

$$g(z) = \frac{\dfrac{1}{\tilde{S}(z)} + \alpha}{1 + \alpha^* \dfrac{1}{\tilde{S}(z)}} \tag{3.63}$$

Thus, with (3.58)

$$g(z) = \frac{f(z) + \alpha}{1 + \alpha^* f(z)} \tag{3.64}$$

and from Section 3.3.1,

$$| \alpha | < 1 \tag{3.65}$$

A function of the form (3.64) with the conditions (3.59) and (3.65) is known [61] to map the unit disk into the unit disk, or

$$|g(z)| \leq 1 \tag{3.66}$$

or from (3.61)

$$\left|\frac{1}{\widehat{S}(z)}\right| \leq 1 \tag{3.67}$$

within $|z| \leq 1$. Thus all of the poles of $\widehat{S}(z)$, which are the zeros of $g(z)$, lie within the unit circle. Since $\widehat{S}(z)$ is a complex allpass function from (3.54), the zeros must therefore lie outside of the unit circle, and $\widehat{S}(z)$ defined from the mapping (3.56) is guaranteed to be a stable allpass function.

3.3.3 Complex Allpass Realization with Complex Two-Port Adaptors

From Sections 3.3.1 and 3.3.2, a stable complex allpass function can be realized as a one-port with a chain of complex two-port adaptors and delays. All intermediate functions are stable and bounded analytic outside of the unit circle in the z-plane. Thus, the following realization of the unscaled system implementing an even-order allpass function is possible,

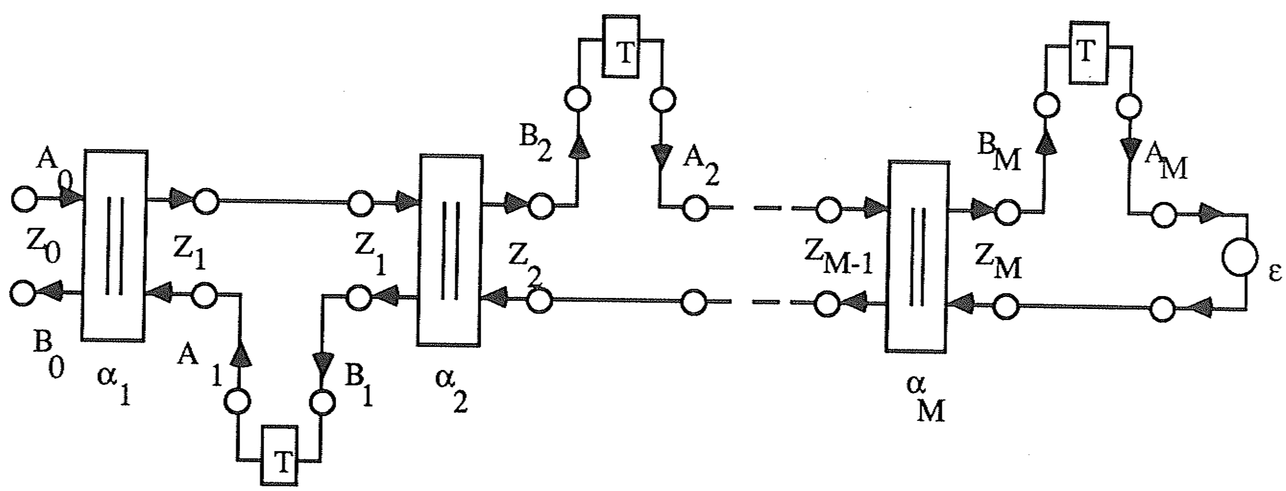

Figure 3.9: Realization of an even-order complex allpass function.

where the magnitude of ε is one, and the port reference impedances are in general complex constants. The realization for an odd-order complex allpass function is shown below.

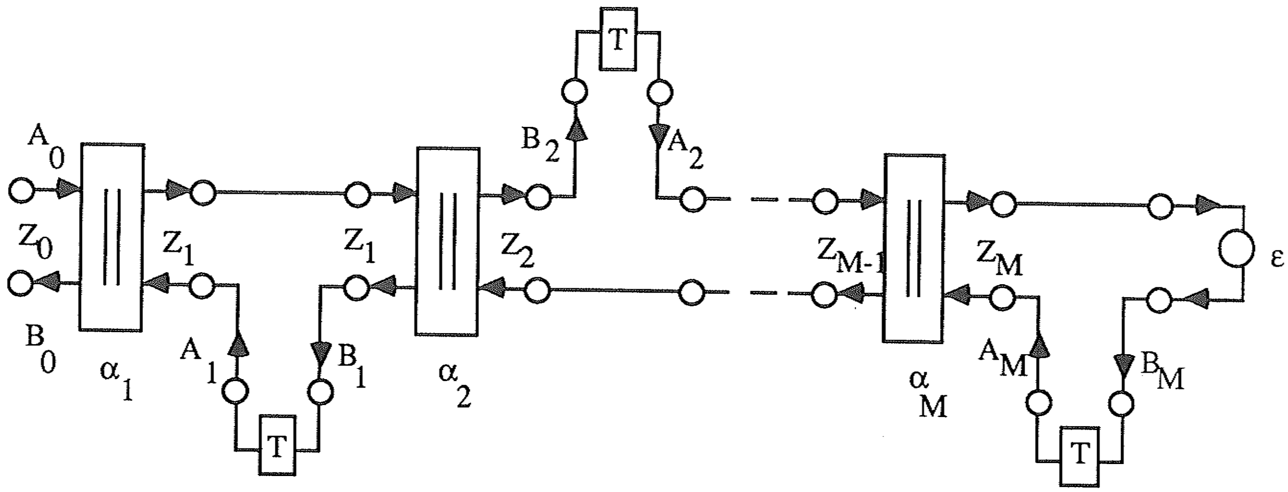

Figure 3.10: Realization of an odd-order complex allpass function.

Clearly, there are many possibilities for arranging the delays in the network. However, for reasons of decoupling of the state variables, or in order to generate a sparse scattering matrix of the system, Webb [43] has found that the above form of the realization is the most attractive (or the equivalent realization of exchanging the placement of all delays from the output terminal to the input terminal of a port).

The realization is pseudolossless if all quantities are represented by infinite precision complex numbers. This is clear since the complex two-port adaptor, complex circulator and delay ports are all pseudolossless under infinite precision conditions. Note that the above implementation of the even-order classical filter was derived from a lattice configuration and is valid only for real input signals.

3.3.4 Adaptor Port Reference Impedances

The port reference impedances are shown in Figures 3.9 and 3.10. The values of the constant impedances can be derived in terms of the complex two-port adaptor parameters. The adaptor parameters are given by

$$\alpha_i = \frac{Z_{i-1} - Z_i^*}{Z_{i-1} + Z_i} \tag{3.68}$$

with

$$Z_i = R_i + j\,X_i \tag{3.69}$$

$$Z_{i-1} = R_{i-1} + j\,X_{i-1} \tag{3.70}$$

$$\alpha_i = \alpha_i' + j\,\alpha_i'' \tag{3.71}$$

Substituting the above three equations into (3.68) and solving,

$$R_i = \left(\frac{(\alpha_i' - 1)^2 + \alpha_i''^2}{1 - |\alpha_i|^2} \right) R_{i-1} = \left(\frac{1 - (\alpha_i + \alpha_i^*) + \alpha_i \alpha_i^*}{1 - |\alpha_i|^2} \right) R_{i-1} \tag{3.72}$$

$$X_i = \frac{2\,\alpha_i''\, R_{i-1}}{1 - |\alpha_i|^2} - X_{i-1} = \frac{-j(\alpha_i - \alpha_i^*) R_{i-1}}{1 - |\alpha_i|^2} - X_{i-1} \tag{3.73}$$

and therefore the resistive part of the port reference impedance is strictly non-negative if the input-output port resistance is non-negative and all adaptor parameters are bounded by one. Note that the resistance of the i^{th} port cannot be zero if the resistance of the $(i - 1)^{th}$ port is non-zero since from Section 3.3.1 no parameter can be identically equal to one for a stable allpass function. From Section 2.5.1.2, the two-port adaptor parameter was found to have a magnitude less than or equal to one, where equality was possible only if one or both of the port resistances were zero. If the input-output port resistance is restricted to being greater than zero, then it is guaranteed that all port resistances will be positive.

The port reference resistances and constant reactances depend on the input-output port impedance. Thus, there are two degrees of freedom in assigning values to the port impedances, or we are free to choose the input-output port impedance. Since the development in this chapter deals with real input and output signals, the input-output port impedance can be chosen to be a positive resistance with zero reactance (or purely real). Thus, the one-port realization will appear, from an external view, to be a real network. The complex network can therefore be inserted into other real wave digital networks that use strictly real elements.

3.4 Scattering Matrix Realization of a Stable Complex Allpass Network

The realization of a stable complex allpass function was given using a complex wave digital network comprising of complex two-port adaptors, complex three-port circulators and delays. An equivalent realization is possible by giving the scattering matrix representation of the non-dynamic sub-network, or the network with all dynamic elements (delays) removed. This method may present some advantages over the adaptor method of implementation since the scattering matrix is easily scaled, the calculation in digital hardware or software is straight-forward since it only involves a sparse square complex matrix multiplication by a complex column matrix, and the scattering matrix representation is highly structured as it consists of simple block sub-matrices.

3.4.1 Derivation of the Scattering Matrix

The realization shown in Figures 3.9 and 3.10 can be used to generate the scattering matrix of the unscaled system as given by (2.197). Using the equations for the reflected waves of a complex two-port in (2.152), and repeated below for convenience,

$$B_1 = -\alpha^*\left(\frac{1-\alpha}{1-\alpha^*}\right)A_1 + \left(1+\alpha^*\left(\frac{1-\alpha}{1-\alpha^*}\right)\right)A_2 \tag{3.74}$$

$$B_2 = \left(1-\alpha\right)A_1 + \alpha A_2 \tag{3.75}$$

all of the reflected waves can be represented as a linear combination of the incident waves with complex multipliers. After the elimination of the intermediate variables, a system of (N/2+1) equations is formed where N is the order of the original even-order classical filter to be realized. Arranging the equations in the form of (2.196), it is found that the scattering matrix is made up four different kinds of blocks. Two of the blocks are common for both even and odd order allpass functions, and the last blocks are different for each case.

The first block S^u of the unscaled scattering matrix is only found in the upper left corner of the scattering matrix and is common for even and odd order allpass functions. It is a (1 x 2) complex sub-matrix and a function of the first two-port parameter only, and is given by

$$S^u = \left[-\alpha_1^*\left(\frac{1-\alpha_1}{1-\alpha_1^*}\right) \quad \left(\frac{1-\alpha_1\alpha_1^*}{1-\alpha_1^*}\right)\right] \tag{3.76}$$

The second block type $S^m{}_i$ is found in the inner section of the scattering matrix and is common to even and odd order allpass functions. The block is a (2 x 4) complex sub-matrix and is given by

$$S_i^m = \begin{bmatrix} -\alpha_{i+1}^*\left(1-\alpha_i\right)\left(\frac{1-\alpha_{i+1}}{1-\alpha_{i+1}^*}\right) & -\alpha_i\alpha_{i+1}^*\left(\frac{1-\alpha_{i+1}}{1-\alpha_{i+1}^*}\right) & -\alpha_{i+2}^*\left(\frac{1-\alpha_{i+1}\alpha_{i+1}^*}{1-\alpha_{i+1}^*}\right)\left(\frac{1-\alpha_{i+2}}{1-\alpha_{i+2}^*}\right) & \left(\frac{1-\alpha_{i+1}\alpha_{i+1}^*}{1-\alpha_{i+1}^*}\right)\left(\frac{1-\alpha_{i+2}\alpha_{i+2}^*}{1-\alpha_{i+2}^*}\right) \\ \left(1-\alpha_i\right)\left(1-\alpha_{i+1}\right) & \alpha_{i+1}\left(1-\alpha_{i+1}\right) & -\alpha_{i+1}\alpha_{i+2}^*\left(\frac{1-\alpha_{i+2}}{1-\alpha_{i+2}^*}\right) & \alpha_{i+1}\left(\frac{1-\alpha_{i+2}\alpha_{i+2}^*}{1-\alpha_{i+2}^*}\right) \end{bmatrix} \tag{3.77}$$

where i refers to the i^{th} reflected wave. The block is a function of the i^{th}, $(i+1)^{th}$ and $(i+2)^{th}$ two-port adaptor parameters. For example, the first block appearing will be a function of the first, second and third two-port adaptor parameters and will give the reflected signals B_1 and B_2, where B_0 is the output of the digital filter. Similarly, the last block of this type will be a function of the $(N/2-3)^{th}$, $(N/2-2)^{th}$, and $(N/2-1)^{th}$ two-port adaptor parameters (N is the order of the classical, or reference filter).

The last two types of blocks are found in the lower right corner of the scattering matrix. For an even-order allpass function, the last block $S^l{}_e$ is a complex (2 x 3) sub-matrix given by

$$S_e^l = \begin{bmatrix} -\alpha_M^*(1-\alpha_{M-1})\left(\frac{1-\alpha_M}{1-\alpha_M^*}\right) & -\alpha_M^*\alpha_{M-1}\left(\frac{1-\alpha_M}{1-\alpha_M^*}\right) & \left(\frac{1-\alpha_M\alpha_M^*}{1-\alpha_M^*}\right)\varepsilon \\ (1-\alpha_{M-1})(1-\alpha_M) & \alpha_{M-1}(1-\alpha_M) & \alpha_M\varepsilon \end{bmatrix} \tag{3.78}$$

where M=N/2, or half the order of the reference filter. This block is a function of the M^{th} and $(M-1)^{th}$ two-port adaptor parameters. The last block type, $S^l{}_o$, for odd-order allpass functions is a complex (1 x 2) sub-matrix as given below

$$S_0^l = \left[(1-\alpha_M)\varepsilon \quad \alpha_M\varepsilon\right] \tag{3.79}$$

and is a function of the M^{th} two-port adaptor parameter.

The blocks are arranged in the sparse scattering matrix as shown below,

$$S = \begin{bmatrix} S^u & 0 & \dots & 0 & 0 \\ 0 & S_1^m & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & S_i^m & 0 \\ 0 & 0 & \dots & 0 & S_e^l \end{bmatrix}$$

Figure 3.11: Block form of the complex scattering matrix for an even-order allpass function.

where all elements without a specified value are zero. The odd-order case is found by replacing $S^l{}_e$ with $S^l{}_o$. With a first order allpass, only the blocks S^u and $S^l{}_o$ are needed. A second order allpass contains the blocks S^u and $S^l{}_e$. All higher order allpass functions contain at least one $S^m{}_i$ block, the S^u block and either the $S^l{}_e$ or the $S^l{}_o$ block, depending on whether the allpass function is even or odd, respectively.

Each element in the scattering matrix is in general complex. The number of complex additions and multiplications is of interest in order to determine the relative efficiency of this design method. The number of complex additions and multiplications for several orders of complex allpass functions are given in Table 3.1 shown below.

Reference Filter Order	Complex Allpass Order	Complex Additions	Complex Multiplications
2	1	2	4
4	2	5	8
6	3	8	12
8	4	11	16
10	5	14	20
12	6	17	24
14	7	20	28
16	8	23	32

Table 3.1: Number of complex additions and multiplications given by the scattering matrix.

Clearly, the number of complex additions and multiplications from the scattering matrix realization method will not change when the system is scaled, since scaling is accomplished through a diagonal matrix multiplication which does not produce any new elements in the scattering matrix.

3.4.2 Scaling the Scattering Matrix

From Section 2.6.2, the non-singular diagonal scaling matrix is given by Equation (2.240) repeated below.

$$|\Gamma| = G_0^{1/2} G_D^{-1/2} \tag{3.80}$$

and the state variable system $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ is transformed through Γ into the scaled system

$$\left\{\Gamma^{*-1}\underline{A}\,\Gamma^*, \Gamma^{*-1}\underline{B}, \underline{C}\,\Gamma^*, \underline{D}\right\} \tag{3.81}$$

Notice from the form of the scattering matrix, as represented by the state-variable system,

$$S = \begin{bmatrix} \underline{D} & \underline{C} \\ \underline{B} & \underline{A} \end{bmatrix} \tag{3.82}$$

and from the scaling Equation (3.81), that $\underline{D}$ and the diagonal of the matrix $\underline{A}$ is invariant after the transformation.

Without loss of generalization, all elements of the scaling matrix can be assumed to be positive and real, because from Figure 2.30, the negation of a scaling element has no effect on the reflected signals from the delay ports, and the scaling criterion did not lead to the necessity of complex scaling coefficients (Section 2.6.2). The elements of the diagonal scaling matrix are given in terms of the conductances of the ports of the two-port adaptors from Equation (3.80). But from (3.72) the resistance (or the inverse of conductance) at each port can be given in terms of the parameter of the two-port adaptor associated with the port and the input-output port resistance R_0. Thus the real diagonal elements of the scaling matrix, γ_i, can be derived as

$$\gamma_i = \left(\frac{R_i}{R_0}\right)^{\frac{1}{2}} \quad , \; i = 1\,(1)\,M \tag{3.83}$$

But the i^{th} resistance, R_i, is given by successive substitution into (3.72) as

$$R_i = \prod_{j=1}^{i}\left[\frac{|1-\alpha_j|^2}{1-|\alpha_j|^2}\right]R_0 \quad , \; i = 1\,(1)\,M \tag{3.84}$$

Substituting (3.84) into (3.83),

$$\gamma_i = \prod_{j=1}^{i} \frac{|1 - \alpha_j|}{\sqrt{1 - |\alpha_j|^2}} \quad , \ i = 1\,(1)\,M \tag{3.85}$$

The expression above is real as expected, or the M number of γ_i specify M real scaling transformers. Note that the form $\gamma_i\, \gamma_k^{-1}$ will appear during the matrix multiplication (3.81) and has the value

$$\gamma_i \gamma_k^{-1} = \prod_{j=k+1}^{i} \frac{|1 - \alpha_j|}{\sqrt{1 - |\alpha_j|^2}} \quad , \ i > k \tag{3.86}$$

The actual scaling of the system through the scaling of the scattering matrix can be accomplished by performing the matrix multiplications of (3.81). The state-variable matrix $\underline{D}$ after scaling is unchanged, and given by

$$\underline{D} = -\alpha_1^* \left(\frac{1 - \alpha_1}{1 - \alpha_1^*} \right) \tag{3.87}$$

The scaled row matrix $\underline{C} = \underline{C}_i$, i = 1 (1) M, is given by

$$\underline{C}_1 = \sqrt{1 - |\alpha_1|^2} \left(\frac{1 - \alpha_1}{|1 - \alpha_1|} \right) \tag{3.88}$$

$$\underline{C}_i = 0, \quad i = 2\,(1)\,M \tag{3.89}$$

The scaled column matrix $\underline{B} = \underline{B}_i$, i = 1 (1) M, is given by

$$\underline{B}_1 = -\alpha_2^* \sqrt{1 - |\alpha_1|^2} \left(\frac{1 - \alpha_1}{|1 - \alpha_1|} \right) \left(\frac{1 - \alpha_2}{1 - \alpha_2^*} \right) \tag{3.90}$$

$$\underline{B}_2 = \sqrt{\left(1 - |\alpha_1|^2\right)\left(1 - |\alpha_2|^2\right)} \left(\frac{1 - \alpha_1}{|1 - \alpha_1|} \right) \left(\frac{1 - \alpha_2}{|1 - \alpha_2|} \right) \tag{3.91}$$

$$\underline{B}_i = 0, \quad i = 3\,(1)\,M \tag{3.92}$$

Similar to the approach adopted for the description of the blocks of the scattering matrix, a block approach can be used with the scaled $\underline{\mathbf{A}}$ state-variable matrix. Let the $\underline{\mathbf{A}}$ state-variable matrix be defined by the following block form,

$$\underline{\mathbf{A}} = \begin{bmatrix} \underline{\mathbf{A}}^{u} & 0 & \dots & 0 & 0 \\ 0 & \underline{\mathbf{A}}_{1}^{m} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \underline{\mathbf{A}}_{i}^{m} & 0 \\ 0 & 0 & \dots & 0 & \underline{\mathbf{A}}^{l} \end{bmatrix}$$

Figure 3.12: Block form of the $\underline{\mathbf{A}}$ state-variable matrix.

where $\underline{\mathbf{A}}^{u}$ is the upper block, $\underline{\mathbf{A}}^{l}$ is the lower block, and $\underline{\mathbf{A}}^{m}{}_{i}$ is the i^{th} inner block. For an even-order allpass function, $\underline{\mathbf{A}}^{l}$ is given by $\underline{\mathbf{A}}^{l}{}_{e}$, and for an odd-order allpass, $\underline{\mathbf{A}}^{l}$ is given by $\underline{\mathbf{A}}^{l}{}_{o}$. Clearly, from (3.76-79), the blocks $\underline{\mathbf{A}}^{u}$, $\underline{\mathbf{A}}^{m}{}_{i}$, $\underline{\mathbf{A}}^{l}{}_{e}$, and $\underline{\mathbf{A}}^{l}{}_{o}$ can be related to the scattering matrix blocks.

The scaled $\underline{\mathbf{A}}^{u}$ state variable sub-matrix block is a (2 x 3) complex matrix and is given by

$$\underline{\mathbf{A}}^{u} = \begin{bmatrix} -\alpha_1 \alpha_2^{*}\left(\dfrac{1-\alpha_2}{1-\alpha_2^{*}}\right) & -\alpha_3^{*}\sqrt{1-|\alpha_2|^2}\left(\dfrac{1-\alpha_3}{1-\alpha_3^{*}}\right)\left(\dfrac{1-\alpha_2}{|1-\alpha_2|}\right) & \sqrt{\left(1-|\alpha_2|^2\right)\left(1-|\alpha_3|^2\right)}\left(\dfrac{1-\alpha_2}{|1-\alpha_2|}\right)\left(\dfrac{1-\alpha_3}{|1-\alpha_3|}\right) \\ \alpha_1 \sqrt{1-|\alpha_2|^2}\left(\dfrac{1-\alpha_2}{|1-\alpha_2|}\right) & -\alpha_2 \alpha_3^{*}\left(\dfrac{1-\alpha_3}{1-\alpha_3^{*}}\right) & \alpha_2 \sqrt{1-|\alpha_3|^2}\left(\dfrac{1-\alpha_3}{|1-\alpha_3|}\right) \end{bmatrix} \tag{3.93}$$

and the scaled sub-matrix $\underline{\mathbf{A}}^{l}{}_{e}$ is a (2 x 3) complex matrix and is given by

$$\underline{A}_e^1 = \begin{bmatrix} -\alpha_M^* \sqrt{1-|\alpha_{M-1}|^2}\left(\frac{1-\alpha_M}{1-\alpha_M^*}\right)\left(\frac{1-\alpha_{M-1}}{|1-\alpha_{M-1}|}\right) & -\alpha_{M-1}\,\alpha_M^*\left(\frac{1-\alpha_M}{1-\alpha_M^*}\right) & \sqrt{1-|\alpha_M|^2}\left(\frac{1-\alpha_M}{|1-\alpha_M|}\right)\varepsilon \\ \sqrt{\left(1-|\alpha_{M-1}|^2\right)\left(1-|\alpha_M|^2\right)}\left(\frac{1-\alpha_{M-1}}{|1-\alpha_{M-1}|}\right)\left(\frac{1-\alpha_M}{|1-\alpha_M|}\right) & \alpha_{M-1}\sqrt{1-|\alpha_M|^2}\left(\frac{1-\alpha_M}{|1-\alpha_M|}\right) & \alpha_M\varepsilon \end{bmatrix} \tag{3.94}$$

Similarly, the scaled sub-matrix $\underline{A}^1{}_o$ is a (1 x 2) complex matrix and is given by

$$\underline{A}_o^1 = \left[\sqrt{1-|\alpha_M|^2}\left(\frac{1-\alpha_M}{|1-\alpha_M|}\right)\varepsilon \quad \alpha_M\varepsilon\right] \tag{3.95}$$

The scaled sub-matrix $\underline{A}^m{}_i$ is a (2 x 4) complex matrix with elements $\underline{A}^m{}_i$ (j,k), j=1,2 and k = 1 (1) 4 which are given below

$$\underline{A}_i^m(1,1) = -\alpha_{i+1}^* \sqrt{1-|\alpha_i|^2}\left(\frac{1-\alpha_{i+1}}{1-\alpha_{i+1}^*}\right)\left(\frac{1-\alpha_i}{|1-\alpha_i|}\right) \tag{3.96a}$$

$$\underline{A}_i^m(1,2) = -\alpha_i\,\alpha_{i+1}^*\left(\frac{1-\alpha_{i+1}}{1-\alpha_{i+1}^*}\right) \tag{3.96b}$$

$$\underline{A}_i^m(1,3) = -\alpha_{i+2}^* \sqrt{1-|\alpha_{i+1}|^2}\left(\frac{1-\alpha_{i+2}}{1-\alpha_{i+2}^*}\right)\left(\frac{1-\alpha_{i+1}}{|1-\alpha_{i+1}|}\right) \tag{3.96c}$$

$$\underline{A}_i^m(1,4) = \sqrt{\left(1-|\alpha_{i+1}|^2\right)\left(1-|\alpha_{i+2}|^2\right)}\left(\frac{1-\alpha_{i+1}}{|1-\alpha_{i+1}|}\right)\left(\frac{1-\alpha_{i+2}}{|1-\alpha_{i+2}|}\right) \tag{3.96d}$$

$$\underline{A}_i^m(2,1) = \sqrt{\left(1-|\alpha_i|^2\right)\left(1-|\alpha_{i+1}|^2\right)}\left(\frac{1-\alpha_i}{|1-\alpha_i|}\right)\left(\frac{1-\alpha_{i+1}}{|1-\alpha_{i+1}|}\right) \tag{3.96e}$$

$$\underline{A}_i^m(2,2) = \alpha_i \sqrt{1 - \left|\alpha_{i+1}\right|^2} \left(\frac{1 - \alpha_{i+1}}{\left|1 - \alpha_{i+1}\right|}\right) \tag{3.96f}$$

$$\underline{A}_i^m(2,3) = -\alpha_{i+1}\alpha_{i+2}^* \left(\frac{1 - \alpha_{i+2}}{1 - \alpha_{i+2}^*}\right) \tag{3.96g}$$

$$\underline{A}_i^m(2,4) = \alpha_{i+1} \sqrt{1 - \left|\alpha_{i+2}\right|^2} \left(\frac{1 - \alpha_{i+2}}{\left|1 - \alpha_{i+2}\right|}\right) \tag{3.96h}$$

The blocks appear in the $\underline{A}$ state variable matrix in the same respect as the blocks in the scattering matrix discussed earlier.

The scattering matrix is therefore scaled and only a function of the complex two-port parameters. It is a square complex matrix of order (N/2 + 1), where again N is the order of the reference filter. The modularity of the blocks appearing in the scattering matrix is a result of the symmetric structure chosen for the adaptor implementation and this gives many zero elements. This is an advantage for two reasons. First, fewer calculations (additions and multiplications) are needed with the sparse matrix. Second, there is a high degree of decoupling of the state variables, since each is a function of at most three other state variables. This is an advantage since a particular state variable can affect at most three other state variables (at most four elements in any column of the scattering matrix, and one element is the original state variable). Note that the scaled scattering matrix elements reduce, in the real domain, to the elements derived by Webb [43].

3.4.3 Bound on the Magnitude of the Output and the States

A bound on the magnitude of the output and the states as defined by the scaled scattering matrix is useful for giving an upper bound on the maximum amplitude that the states can achieve. The greatest number of nonzero elements in any row of the scaled complex scattering matrix is four, and thus a bound on the magnitude of the next state B in terms of four present states is given by

$$|B| \le |S_1||A_1| + |S_2||A_2| + |S_3||A_3| + |S_4||A_4| \tag{3.97}$$

where |#| refers to the magnitude of the complex number #, and S_i and A_i, i = 1 (1) 4, are the four nonzero elements of the scaled scattering matrix and the corresponding four states, respectively. If all the states are bounded by an overflow error function to a positive constant λ, then (3.97) becomes

$$|B| \leq \left(|S_1| + |S_2| + |S_3| + |S_4|\right) \lambda \tag{3.98}$$

or

$$|B| \leq K \lambda \tag{3.99}$$

where K is positive real and defined by

$$K = |S_1| + |S_2| + |S_3| + |S_4| \tag{3.100}$$

Thus the value of K will give a bound on the magnitudes of the output and the states. Scattering matrix elements that contain factors that have either of the forms

$$\left(\frac{1-\alpha_i}{1-\alpha_i^*}\right) \quad \text{or} \quad \left(\frac{1-\alpha_i}{|1-\alpha_i|}\right) \tag{3.101}$$

can be ignored in Equation (3.100) since the factors in (3.101) have a magnitude of one. Also, the magnitude of the complex conjugate of a complex number is clearly equal to the magnitude of the complex number.

Consider the output which is calculated with the first row of the scaled scattering matrix. The value of K from (3.100), (3.87) and (3.88-89) is

$$K = |\alpha_1| + \sqrt{1-|\alpha_1|^2} \tag{3.102}$$

and the maximum that (3.102) can attain is $\sqrt{2}$ at a magnitude of α_1 of $1/\sqrt{2}$, which is the upper bound on the magnitude of the gain of the output.

The states are defined by the scaled state-variable matricies as given in (3.90-96). Consider all state variables except the last for an odd-order allpass function, or the last two for an even-order allpass function. For the remaining state variables, assume that all of the parameters are equivalent. The first row of (3.96), or the first row of S_i^m, gives the following value of K in Equation (3.100),

$$K = 2|\alpha|\sqrt{1-|\alpha|^2} + 1 \tag{3.103}$$

The maximum value of (3.103) is 2 and occurs at a magnitude of α of $1/\sqrt{2}$. Similarly, the second row of (3.96), or the second row of S_i^m, gives the same value of K, namely 2.

For an even-order allpass function, consider the second last state variable. From (3.94) the value of K with all of the parameters equal is

$$K = |\alpha|^2 + (1 + |\alpha|)\sqrt{1-|\alpha|^2} \tag{3.104}$$

The maximum of (3.104) is bounded by 2 at a magnitude of α of approximately 0.78. The last row of the scattering matrix gives a value of K of

$$K = 1 + |\alpha| - |\alpha|^2 + |\alpha|\sqrt{1-|\alpha|^2} \tag{3.105}$$

and the maximum of (3.105) is bounded by 2 at a magnitude of α of approximately 0.64.

For an odd-order allpass function, the last row of the scattering matrix gives a value of K of

$$K = |\alpha| + \sqrt{1-|\alpha|^2} \tag{3.106}$$

and the maximum of (3.106) is $\sqrt{2}$ at a magnitude of α of $1/\sqrt{2}$.

Thus, all of the state variables are bounded in magnitude by 2 λ, where λ is the maximum allowed signal amplitude. The output is bounded in magnitude by $\sqrt{2}\ \lambda$. In an actual implementation of the digital filter, the signal amplitudes will generally be less than the maximum stated here, and thus in a well-designed filter overflow errors will not occur as often as implied earlier at the states.

3.4.4 Quantization of the Scaled Scattering Matrix

The elements of the scaled scattering matrix given by (3.87-96) are given in terms of the infinite precision complex parameters of the complex two-port adaptors. Thus the elements in the scaled scattering matrix are infinite precision complex numbers. However, the realization of the digital filter in digital hardware or software requires that all elements of the scattering matrix be quantized complex numbers, and therefore all elements must be represented as a ratio of one integer dividing another.

Many quantization schemes of the complex entries of the scattering matrix are possible. An obvious method is to represent the real and imaginary parts of all entries as the ratio of an integer over a common denominator, which is an integer power of 2. If S_{ij} is a non-zero element of the scattering matrix, with real and imaginary parts

$$S_{ij} = S_{ij}' + j\,S_{ij}'' \tag{3.107}$$

and D is an integer power of 2 (m is an integer),

$$D = 2^m \tag{3.108}$$

then S_{ij} can be represented by

$$S_{ij} \approx \frac{\tilde{S}_{ij}' + j\tilde{S}_{ij}''}{D} = \frac{\tilde{S}_{ij}' + j\tilde{S}_{ij}''}{2^m} \tag{3.109}$$

where $\tilde{S}_{ij}'$ and $\tilde{S}_{ij}''$ are integers. The values of $\tilde{S}_{ij}'$ and $\tilde{S}_{ij}''$ are determined by multiplying the infinite precision representations $S_{ij}' + j\,S_{ij}''$ by the denominator D and truncating the result.

The truncation operation reduces the magnitude of the elements of the scattering matrix. This reduction defines a new scattering matrix, and a corresponding new system with a new set of complex two-port parameters. Examination of the scaled scattering matrix elements as given in (3.87-96) shows that for each element, reducing the magnitude corresponds to increasing the magnitude of the complex two-port parameters (this is always a possible solution). Thus for a fixed $(i-1)^{th}$ port resistance with an increased parameter, from (3.72) the i^{th} port resistance will increase. Since all parameters through the truncation operation increase, all of the port resistances increase.

Consider the assumption that the truncation operation decreases the scaled scattering matrix elements in a column by a common factor. The equal reduction in magnitude of the scaled scattering matrix elements of a column can be viewed as complex multipliers with a magnitude strictly less then one in series with the states of the filter. The series connection corresponds to a complex three-port circulator with the multiplier connected to the third port. But from (2.131), a complex multiplier with a magnitude less than one in the wave digital domain is equivalent to a resistor in series with constant reactance in the analog domain as shown in Figure 2.16. Thus the operation of truncation of the scaled scattering matrix elements in the method given above corresponds to adding complex three-port circulators with a positive resistor, (from (2.131) the resistance can always be chosen to be positive), and a constant reactance to the network in the analog domain. Since adding positive resistances through a lossless complex circulator makes the analog network lossy, the equivalent wave digital network must be pseudopassive after the trunction of the elements of the scaled scattering matrix. This allows the theory developed in Chapter II to be applied, and the beneficial properties of low sensitivity, stability and the suppression of parasitic oscillations are enjoyed by the design.

3.5 Scaled Port Reference Impedance Matrix

The insertion of scaling transformers at the delay ports changes the port impedances of the delay ports. The new port impedance Z_i^D corresponding to Z_i, which is the port impedance associated with the i^{th} and $(i+1)^{th}$ complex two-port adaptors, is given by (2.244) to be

$$Z_i^D = \frac{R_0}{R_i} Z_i \tag{3.110}$$

However, the port resistance R_i is given in terms of the parameters and the input-output port resistance R_0 by (3.84), and therefore (3.110) becomes

$$Z_i^D = Z_i \prod_{j=1}^{i} \left[\frac{1 - |\alpha_j|^2}{|1 - \alpha_j|^2} \right] \tag{3.111}$$

3.6 Complex Allpass Realization Example

The development presented gives a method for realizing an even-order classical filter with a stable complex allpass network of half the order of the original transfer function in the complex wave digital domain. The allpass function was derived from the lattice configuration and the scaled scattering matrix was shown to be sparse and highly structured in sub-matrix blocks. An example showing the basic complex allpass network in the wave digital domain will demonstrate the essential realization method.

Consider a real classical LTI network of an even order N described by a transfer function and the corresponding spectral-complement function, both with a real symmetric numerator and a common denominator. The earlier development gives a realization in the complex wave digital domain as shown in Figures 3.9 and 3.10, depending upon whether the order of the complex allpass function is even or odd, respectively. The values of the complex parameters α_i, i = 1 (1) N/2 are calculated from the allpass functions described in Section 3.3.1. The port reference impedances are calculated from (3.72) and (3.73).

Consider a general 6th order classical filter of the type discussed in the above paragraph with real input signals. The following complex wave digital allpass realization is easily found,

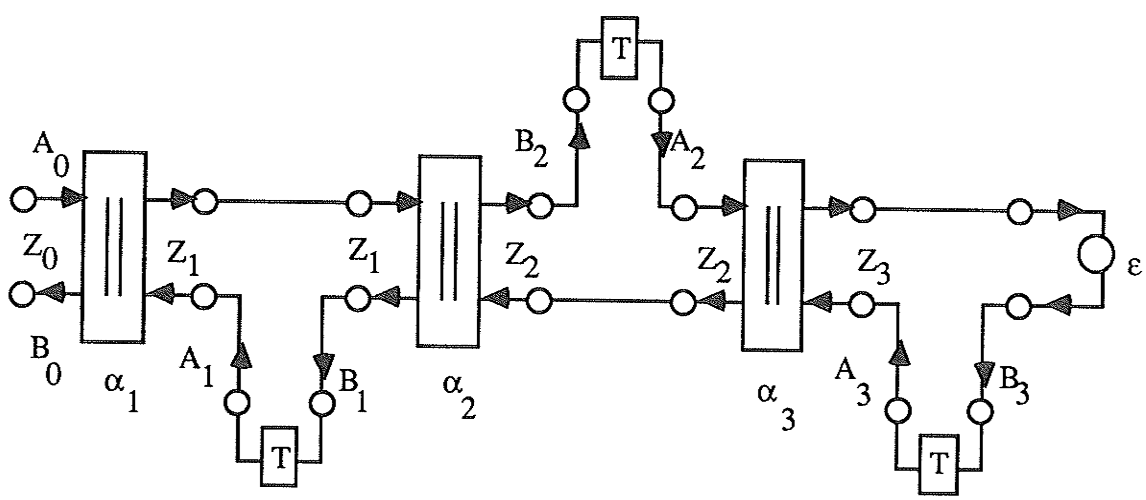

Figure 3.13: Complex wave digital allpass realization of a 6th order classical filter.

where ε is a unimodular complex number. The output of the filter is the real part of the reflected wave of the complex one-port for the response of the transfer function, or the imaginary part of the reflected wave for the response of the spectral complement of the transfer function.

The complex scaled scattering matrix as a function of the complex two-port adaptor parameters, $S = S(i,j)$, $i,j=1(1)3$, is given by

$$S(1,1) = -\alpha_1^* \left(\frac{1-\alpha_1}{1-\alpha_1^*} \right) \tag{3.112a}$$

$$S(1,2) = \sqrt{1 - |\alpha_1|^2} \left(\frac{1-\alpha_1}{|1-\alpha_1|} \right) \tag{3.112b}$$

$$S(1,3) = 0 \tag{3.112c}$$

$$S(1,4) = 0 \tag{3.112d}$$

$$S(2,1) = -\alpha_2^* \sqrt{1 - |\alpha_1|^2} \left(\frac{1-\alpha_1}{|1-\alpha_1|} \right) \left(\frac{1-\alpha_2}{1-\alpha_2^*} \right) \tag{3.112e}$$

$$S(2,2) = -\alpha_1 \alpha_2^* \left(\frac{1-\alpha_2}{1-\alpha_2^*} \right) \tag{3.112f}$$

$$S(2,3) = -\alpha_3^* \sqrt{1 - |\alpha_2|^2} \left(\frac{1-\alpha_2}{|1-\alpha_2|} \right) \left(\frac{1-\alpha_3}{1-\alpha_3^*} \right) \tag{3.112g}$$

$$S(2,4) = \sqrt{\left(1 - |\alpha_2|^2\right)\left(1 - |\alpha_3|^2\right)} \left(\frac{1-\alpha_2}{|1-\alpha_2|} \right) \left(\frac{1-\alpha_3}{|1-\alpha_3|} \right) \tag{3.112h}$$

$$S(3,1) = \sqrt{\left(1 - |\alpha_1|^2\right)\left(1 - |\alpha_2|^2\right)} \left(\frac{1-\alpha_1}{|1-\alpha_1|} \right) \left(\frac{1-\alpha_2}{|1-\alpha_2|} \right) \tag{3.112i}$$

$$S(3,2) = \alpha_1 \sqrt{1 - |\alpha_2|^2} \left(\frac{1 - \alpha_2}{|1 - \alpha_2|} \right) \tag{3.112j}$$

$$S(3,3) = -\alpha_2 \alpha_3{}^* \left(\frac{1 - \alpha_3}{1 - \alpha_3{}^*} \right) \tag{3.112k}$$

$$S(3,4) = \alpha_2 \sqrt{1 - |\alpha_3|^2} \left(\frac{1 - \alpha_3}{|1 - \alpha_3|} \right) \tag{3.112l}$$

$$S(4,1) = 0 \tag{3.112m}$$

$$S(4,2) = 0 \tag{3.112n}$$

$$S(4,3) = \sqrt{1 - |\alpha_3|^2} \left(\frac{1 - \alpha_3}{|1 - \alpha_3|} \right) \varepsilon \tag{3.112o}$$

$$S(4,4) = \alpha_3\, \varepsilon \tag{3.112p}$$

3.7 Analog Equivalence of the Complex Allpass Wave Digital Network

The analog circuit equivalent of the complex allpass realization as given in Figures 3.9 and 3.10 can be found by using the theory developed in Chapter II. The complex two-port adaptors are used for the change of the port reference impedances between the complex circulators and are not needed in the analog domain. Thus, the only devices that remain are a series connection of three-port complex circulators. In the wave digital domain, a delay with a complex port impedance was connected to the third port of a complex circulator, and from Section 2.4.2 this corresponds to a capacitor in series with a constant reactance in the analog domain as shown in Figure 2.10. The terminating unimodular multiplier ε corresponds from Section 2.4.1 to a constant reactance. This constant reactance can be combined with the constant reactance from the delay port in series with the unimodular multiplier. Thus, the equivalent analog circuit of the complex allpass wave digital network is shown below.

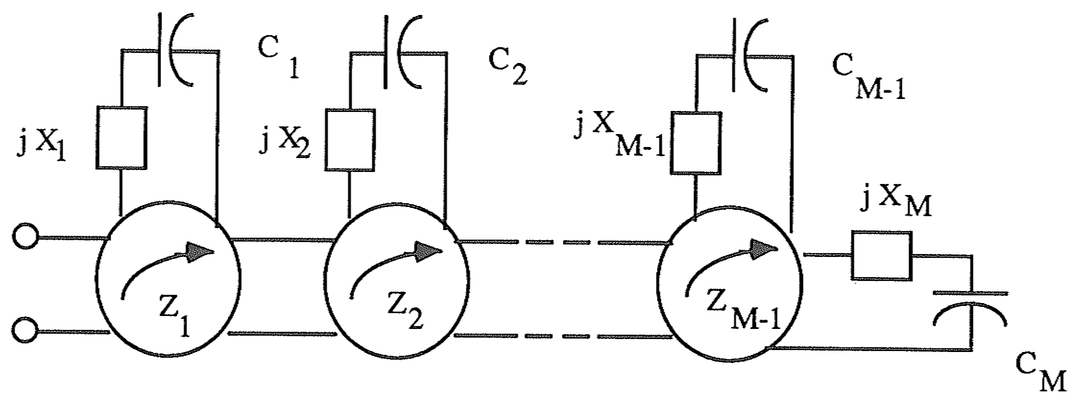

Figure 3.14: Analog circuit equivalent of the unscaled complex allpass wave digital network.

where M is half the order of the classical, or reference filter.

The value of the capacitors and constant reactances are given by (2.111) and (2.96). If Z_i is given by

$$Z_i = R_i + j\,E_i\,, \quad i = 1(1)\,M \tag{3.113}$$

then

$$C_i = 1/R_i\,, \quad i = 1\,(1)\,M \tag{3.114}$$
$$X_i = -\,E_i\,, \; i = 1\,(1)\;M\text{-}1 \tag{3.115}$$
$$X_M = -\,E_M + E_\varepsilon \tag{3.116}$$

where E_ε corresponds to the constant reactance given by the unimodular multiplier ε.

CHAPTER IV

COMPUTER IMPLEMENTATION OF EVEN-ORDER CLASSICAL FILTER DESIGN

An even order classical filter, with both the transfer function and spectral-complement function having a real symmetric numerator and a common denominator, with a real input and output, can be implemented using a single complex one-port allpass network in the complex wave digital domain. The resulting WD network contains only complex two-port adaptors, complex three-port circulators and delays. The complex scattering matrix representing the non-dynamic sub-network is generated by block sub-matrices, and is sparse giving a high degree of decoupling of the state variables. A finite binary representation can be realized by truncating the real and imaginary parts of the elements of the scaled scattering matrix (decreasing the magnitude of the elements). If pseuodopassivity (and incremental pseudopassivity) is maintained, all of the beneficial properties of wave digital filters are realized including: low sensitivity in the passband to the multiplier coefficients, zero-input and forced response stability, good dynamic range and the suppression of parasitic oscillations.

The class of reference filters designed in the program includes even-order Butterworth, Chebyshev and Cauer (Elliptic) lowpass filters. Note that highpass and bandpass filters can be found by designing the appropriate lowpass filter, then performing a transformation [48]. Butterworth and Chebyshev even-order lowpass filters ideally have zero gain at infinite frequency, while even-order Cauer filters of type A will have a finite non-zero gain at infinite frequency in the Laplace domain equal to the magnitude of the ripple in the passband.

A computer program written in a version of FORTRAN-77 has been developed by the author which includes modified versions of several of the subroutines provided by Webb [43]. The program, called CP.EXE was implemented on an IBM PC/XT micro computer and the design time required ranges between one minute and thirty minutes. The program accepts the design specifications and generates the complex quantized scaled scattering matrix that meets the specificaitions. The program listing is given in Appendix C.

The basic operation of the program is shown in the following flow chart.

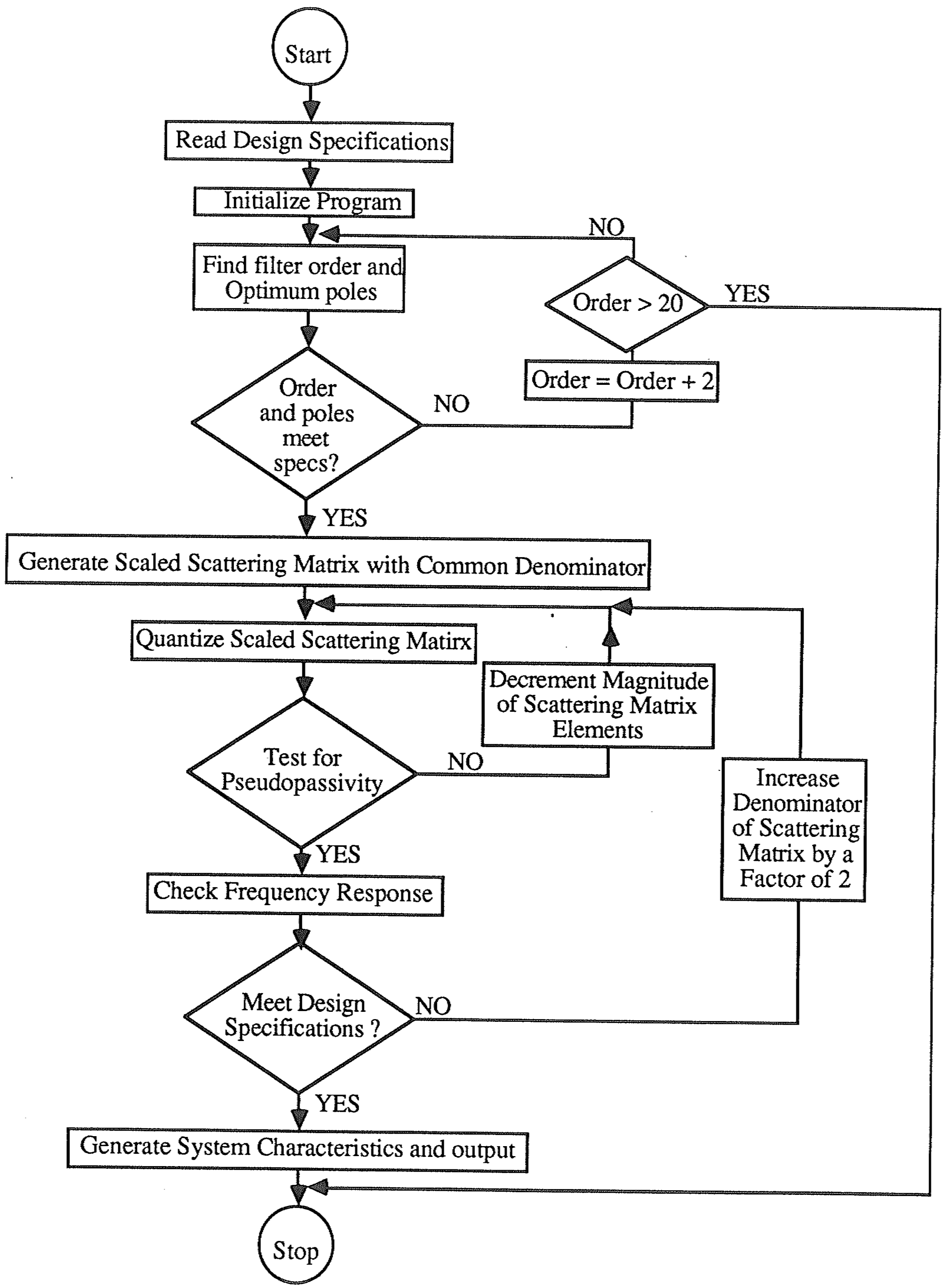

Figure 4.1: Flow chart of the operation of the lowpass filter design computer program.

4.1 Computer Program Input-Output.

The computer program accepts the design specifications given in the discrete frequency domain in the text file CP.SPC and generates the solution which meets the specifications. An example of the file CP.SPC is given in Appendix A. The output includes the complex quantized scaled scattering matrix specifying the complex allpass wave digital filter derived from the required transfer function. Also, the L_2 norm, **K** and **W** matrices of the scaled network, the filter gain at infinite frequency in the analog domain, and the frequency response are generated. All information is saved in the text file CP.DAT, except the frequency response which is saved in SPECT0.DAT.

4.1.1 Design Specifications

In general, a digital filter is needed which satisfies a set of frequency specifications. The specifications define a region of the attenuation graph that the response of the filter design is allowed to occupy. They include the passband (or corner) frequency F_p with the associated voltage gain G_p at that frequency. Also the stopband frequency F_s and the associated voltage gain G_s. The voltage gain over the passband (including the passband frequency) is allowed to be greater than or equal to the voltage gain specified G_p (but ideally less than or equal to one in the normalized case), however the voltage gain in the stopband must be less than or equal to the specified voltage gain G_s.

The meaning of the specifications change with the filter type used. For a Buttworth (maximally flat) lowpass filter, the passband voltage gain refers to the gain at the passband frequency, while for both a Chebyshev and a Cauer lowpass filter, it refers to the maximum allowable ripple width in the passband. Similarly, for both a Butterworth and a Chebyshev lowpass filter, the stopband voltage gain refers to the maximum magnitude of the gain characteristic at the stopband frequency, while for the Cauer lowpass filter, it refers to the maximum value of the ripple in the stopband.

The frequencies specified in the discrete domain F_p,F_s must first be "warped" by the tangent formula given in (2.11) in order to determine the correct frequencies ω_p,ω_s to use in the design of the reference filter.

$$\omega_p = \tan\left(\frac{F_p}{2}\right) \tag{4.1}$$

$$\omega_s = \tan\left(\frac{F_s}{2}\right) \tag{4.2}$$

An analog transfer function can then be derived using the frequencies in the range $0 < \omega < \infty$, and transformed into the discrete domain with the bilinear transformation given in

(2.10). The magnitude of the frequency spectrum of the discrete transfer function is periodic with a period of 2π along the unit circle in the z-plane.

The design specifications are often given in terms of the attenuation instead of the voltage gain. Note that if the voltage gain is given in decibels with the magnitude of the characteristic less than or equal to one, the attenuation is the negative of the gain. Thus, with a voltage gain G given in decibels as G^{dB} as

$$G^{dB} = 20 \log\{G\} \tag{4.3}$$

Then the attenuation A given in decibels is

$$A = -G^{dB} \tag{4.4}$$

For uniformity, the design specifications will be given in decibels of attenuation. The region described is shown below where the attenuation characteristic cannot enter the shaded region.

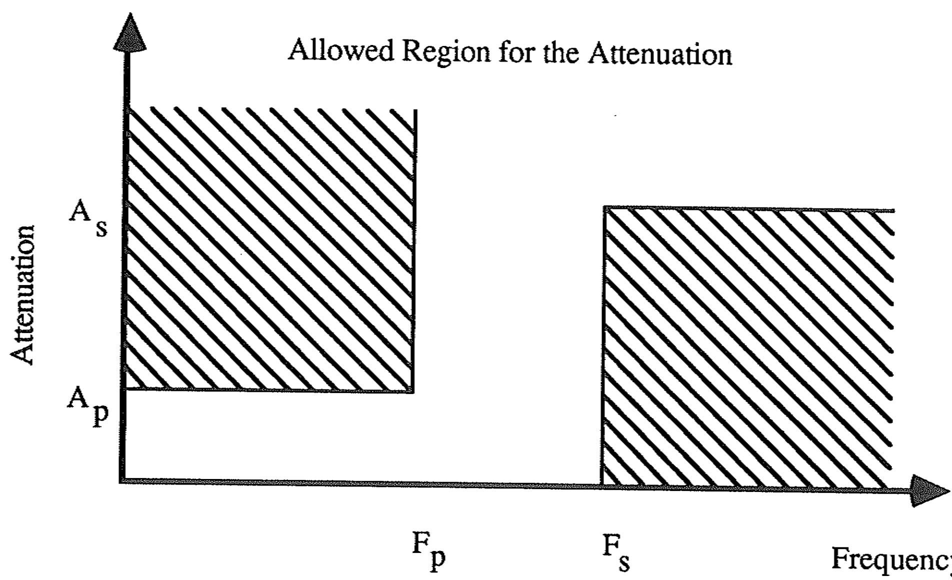

Figure 4.2: Region of attenuation characteristic defined by the design specifications.

Thus the design specifications generally include the desired filter type, the passband and stopband frequencies, and the passband and stopband attenuations in decibels. The solution attempts to find the order of the desired filter type that meets the given specifications, and the higher the order the closer the designed attenuations will approach the desired attenuations. Note that since the order can only be increased in integer increments, a design that satisfies the passband specifications may exceed the stopband specifications.

Since the six variables (filter type, filter order, F_p, F_s, A_p, and A_s) completely specify a filter design, giving any five will uniquely define the sixth. Thus, all variables except A_s may be specified in order to generate families of filters with increasing orders.

4.1.2 Run Options

As described in the earlier section, the most common way of running the program is to specify all variables except the filter order. Using this approach, several options are available for customizing the design solution and the program output. All of the options are set, or the program variables are initialized, using the text file CP.INI. The file contains integer and logical (T for true and F for false) constants only and an example of the CP.INI file is shown in Appendix B.

The decrement in the passband ripple as described in Section 4.2.1 can be illiminated by setting CHKRIP=F (false) giving the same passband ripple that was specified in the design specifications. This is generally not recommended since the denominator of the quantized scattering matrix will in general be greater without a design margin in the passband (requiring more digits for the representation) as the frequency response test described in Section 4.4.1 is sensitive to the denominator.

The frequency response of the scaled scattering matrix before quantization can be generated by setting GENFRE=T (true). The frequency, magnitude, magnitude in decibels, phase and group delay of the CWD filter and the reference filter are found in the output text file ACTSPECT.DAT. This allows the comparison of the design given by the scaled scattering matrix before quantization and the transfer function that the design is implementing. It is expected that both sets of data will be identical.

The maximum denominator of the quantized scaled scattering matrix is given with the variable MAXDNM and is defined by 2 raised to an integer power. This allows the maximum number of binary digits in the representation of the scattering matrix to be set. If this maximum denominator is exceeded, then a solution satisfying the design specifications cannot be found. An alternate solution not satisfying the design specifications can then be generated if desired by setting FIXFAL=T (true). The solution will then be generated with the denominator defined by FAILDN. Often, the values of FAILDN and MAXDNM will be set equal to generate the solution with a frequency response closest to the desired response.

The minimum denominator to be used in the representation of the quantized scattering matix can be specified by setting FIXDEN=T (true). The value of the minimum denominator is given by IDENF. Thus the denominator used in the solution will be greater than or equal to IDENF, but less than or equal to MAXDNM as long a solution in this range is possible. This option can be used for the representation of the quantized scattering matrix on specialized hardware such as digital signal processing microprocessors, or any of the general microprocessors that work with a large number of binary digits. Examples are the Intel 80386 or the Motorola 68020 that are based on a 32-bit architecture and operate at a clock frequency of 20 MHz. Generally the computation time is not decreased with these microprocessors if fewer binary digits are used in the representation since all calculations are performed with the full 32-bits, and thus there is no disadvantage in using a more accurate representation of the quantized scattering matrix.

The pseudopassive condition given by (2.66) and repeated in (4.5)

$$P = A*^{T}\left(G - S*^{T} G\, S\right) A \geq 0 \tag{4.5}$$

is ensured by the positive definiteness of the hermittian matrix S_P given in (4.6) shown below

$$S_P = G - S*^{T} G\, S \tag{4.6}$$

The actual matrix S_P can be saved to the text file CPSP.DAT by setting LOUTSP=T (true). This allows the hermitian matrix S_P to be viewed for more detailed analysis of the pseudopassive characteristic of the network given by the quantized scattering matrix.

If a specific order of the filter is desired, then the frequency specification method decribed above is inappropriate for achieving the solution. A filter order can be specified by setting FIXORD=T (true) with the filter order set by ORDERF. Thus the variables used in this case are the filter type, the filter order (given by ORDERF), the passband and stopband frequencies and the passband attenuation. The stopband atttenuation is not used in the design, but must be set at an appropriate value so that the quantized design will pass the frequency response test described in Section 4.4.1.

4.1.3 The Quantized Scaled Complex Scattering Matrix

The solution to a design problem is given in the form of a quantized scaled complex scattering matrix found in the files CP.DAT and ISS.DAT. The integer values of the real and imaginary parts of the scattering matrix are given along with the denominator used, which is equal to two raised to an integer power. Both text files contain the quantized scattering matrix given in the form of a set of columns.

The calculation of the scattering matrix is accomplished in the subroutine SSCATT and is described in Section 4.3. The scattering matrix is square and has an order equal to one greater than half of the order of the reference filter. Note that the maximum number of non-zero elements in any row or column is four giving a high degree of decoupling of the state variables of the network.

An equivalent solution is given with the two-port adaptor network as described in Chapter III. The two-port adaptor parameters along with the scaled delay-port impedances are found in the text file CP.DAT. They are calculated in the subroutine ALPHAZ based on the impedance of the input-output port which, in light of the discussion in Section 3.3.4, is purely real and set to unity in value.

4.1.4 Generation of the Actual and Quantized Frequency Responses

The normalized (unit sampling period) frequency response of a quantized complex state-variable system is generated in the subroutine RESPON based on the theory described in Section 2.6.3 by recognizing the quantized state-variable matrices from the scattering matrix as in (2.197). However, this only gives the response of one of the complex allpass functions of (3.24), and thus in order to compare the response of the quantized system with the response of the reference network, the response of the second allpass function must also be calculated by taking the complex conjugate of all of the elements in the scattering matrix.

The complex conjugate of the scattering matrix represents the second allpass function since all of the coefficients are the conjugate of the first allpass function, and thus from (3.53) all of the two-port parameters will be the conjugate of the parameters of the first allpass network, and this gives the conjugate of the scattering matrix since it is only a function of the parameters. The total response is found by adding the response from each allpass function. Note that this is necessary only for deriving the frequency response of the network represented by the allpass function, and the actual implementation involves only one allpass function.

A range of frequencies in $0 \leq \omega \leq \pi$ is generated for the normalized frequency response, with a unit sampling period, by dividing the total range into equi-spaced intervals of frequency. The real and imaginary parts of the elements of the scattering matrix are represented as real numbers instead of integers by dividing the integer numerators by the integer denominator. The matrix of (2.254) is built in the subroutine FREQRE and upper triangularized in the subroutine GENTRI. The last element on the diagonal is the response. The process is repeated for the complex conjugate of the scattering matrix and the last element on the diagonal is added to the earlier result giving the total response. The magnitude is taken (expressed also in decibels), and the phase and group delay is calculated.

The same method is used for the unquantized scaled complex scattering matrix as in subroutine ACTRES except the real and imaginary parts of the elements are already expressed in real form and thus it is not necessary to convert the elements into reals.

Each subroutine RESPON and ACTRES also calculates the magnitude (also expressed in decibels), phase and group delay of the reference filter for comparison. The responses are stored in the text files SPECT0.DAT and ACTSPECT.DAT, respectively.

4.1.5 Generation of the System Characteristics

Several characteristics of the system represented by the quantized scaled scattering matrix, such as: the input covariance matrix **K**, the output covariance matrix **W**, and the L_2 norm, are generated after a solution is found.

The L_2 norm of a network described by a complex impulse response h(m) is given by

$$L_2 = \sum_{m=0}^{\infty} | h(m) |^2 \tag{4.7}$$

The L_2 norm is calculated from the scattering matrix in the subroutine L2NORM by exciting the system with a unit impulse and recording the resulting output sequence, given by the terms of (4.7). The summation is halted when the pseudopower of the network is less than a preset constant. For a scaled pseudopassive allpass network, the L_2 norm should be less than or equal to one (ideally equal to one).

The diagonal elements, K_{ii}, of the input covariance matrix **K** are given by

$$K_{ii} = \sum_{m=0}^{\infty} | h_{in,i}(m) |^2 \tag{4.8}$$

where $h_{in,i}$ is the impulse response at the i^{th} state variable, or the i^{th} delay, to the input. The diagonal elements are clearly real and are calculated in the subroutine GENWKD using the same approach as in the calculation of the L_2 norm. For a scaled system, the diagonal elements should be approximately equal to one.

Similarly, the diagonal elements, W_{ii}, of the output covariance matrix **W** are given by

$$W_{ii} = \sum_{m=0}^{\infty} | h_{i,out}(m) |^2 \tag{4.9}$$

where $h_{i,out}$ is the impulse response at the output to the i^{th} delay. The diagonal elements are calculated in the subroutine GENWKD, again using the same approach used in calculating the L_2 norm. For a scaled system, the diagonal elements should be approximately equal to one.

The noise power gain can be calculated by summing the diagonal elements of the output covariance matrix **W** [29].

4.2 Generating the Complex Allpass Function

The scaled complex scattering matrix is a function of the complex two-port adaptor parameters, and the parameters are a function of the coefficients of the complex allpass function representing a reference filter. Thus the allpass function must be found in order to generate the digital filter solution. The first step in finding the complex allpass function is deriving the reference filter in the Laplace domain that satisfies the set of given specifications. The actual design specifications are set more stringent than the set specified by the user to allow for shifting of the attenuation characteristic after quantization. The second step is building the allpass function from a subset of the poles (chosen in a particular way) of the reference filter. The final step is calculating a unimodular constant which multiplies the allpass function and is dependent upon both the gain of the transfer function and its spectral complement at zero frequency in the Laplace domain.

4.2.1 Finding the Optimum Pole Locations

The optimum poles of the reference filter are needed to specify the allpass function. The poles are generated by the subroutines BTWHRT, CHEBRT, and ELTCRT for a Butterworth, Chebyshev and Cauer filter, respectively. The attenuation characteristic will deviate after quantization of the scattering matrix from the infinite precision solution. Thus, a "design margin" is introduced by increasing the passband frequency and decreasing the stopband frequency by 20% of the difference in the passband and stopband frequencies as suggested by Webb [43]. Thus, the design frequencies F_p^D and F_s^D are given by

$$F_p^D = F_p + 0.05\,(F_s - F_p) \tag{4.10a}$$

$$F_s^D = F_s - 0.05\,(F_s - F_p) \tag{4.10b}$$

The frequencies must be "warped" because of the bilinear transformation and thus the passband and stopband frequencies used in the generation of the transfer function are given from (2.11) by

$$\omega_p^D = \tan\left(\frac{F_p^D}{2}\right) = \tan\left(\frac{F_p + 0.05\left(F_s - F_p\right)}{2}\right) \tag{4.11}$$

$$\omega_s^D = \tan\left(\frac{F_s^D}{2}\right) = \tan\left(\frac{F_s - 0.05\left(F_s - F_p\right)}{2}\right) \tag{4.12}$$

The order of the filter can be found with the above frequencies and the given passband and stopband attenuations. The filter order is determined by starting at the minimum even order possible, which is two. The stopband attenuation is checked and if it satisfies the stopband specification the order is determined. If it does not satisfy the specification, the order is increased by two (in order to restrict the filter order to be even) and the process is repeated until there is a sufficient attenuation in the stopband, or the maximum order of 20 is reached. At that point either the order of the reference filter is determined or the program terminates with no solution possible for the given specifications.

A "design margin" is introduced into the attenuations A_p and A_s in order to allow for shifting of the attenuation characteristic after quantization. This is accomplished by decreasing the passband attenuation A_p. An investigation by Webb [43] has found that the margin should be balanced between the stopband and the passband. The same method is used here except that the value used for decrementation of the passband attenuation is restricted in size to be at most one tenth of the passband attenuation. A flow chart of the determination of the filter order and the optimal poles is given in Figure 4.4.

Once the reference filter order and the passband attenuation have been determined from the above optimization procedure, the Laplace domain poles are calculated and transformed into the z-domain with (2.14). The "design margins" are shown in the figure below, where the attenuation characteristic is not allowed to enter the shaded region.

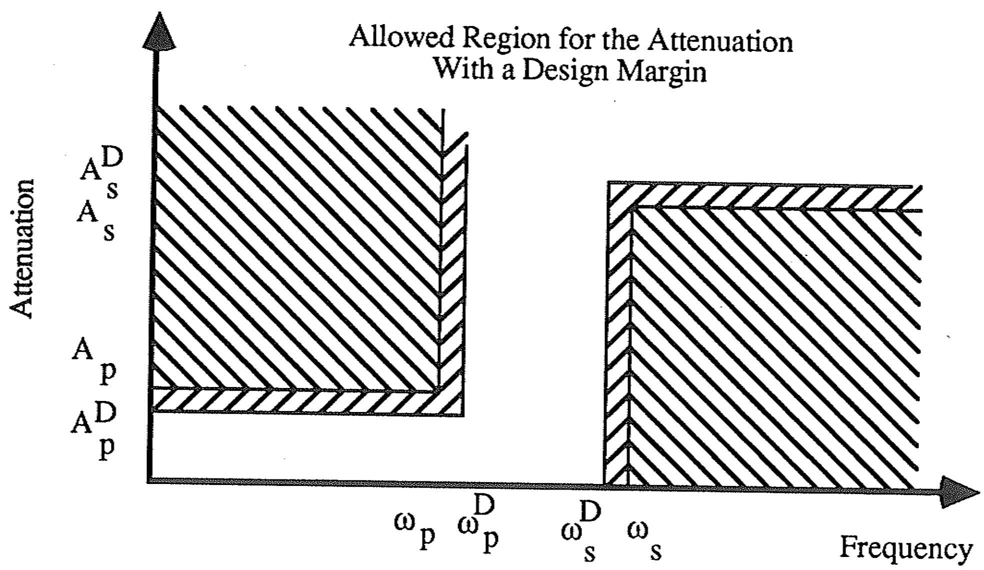

Figure 4.3: Allowed region for the attenuation characteristic with the design margins.

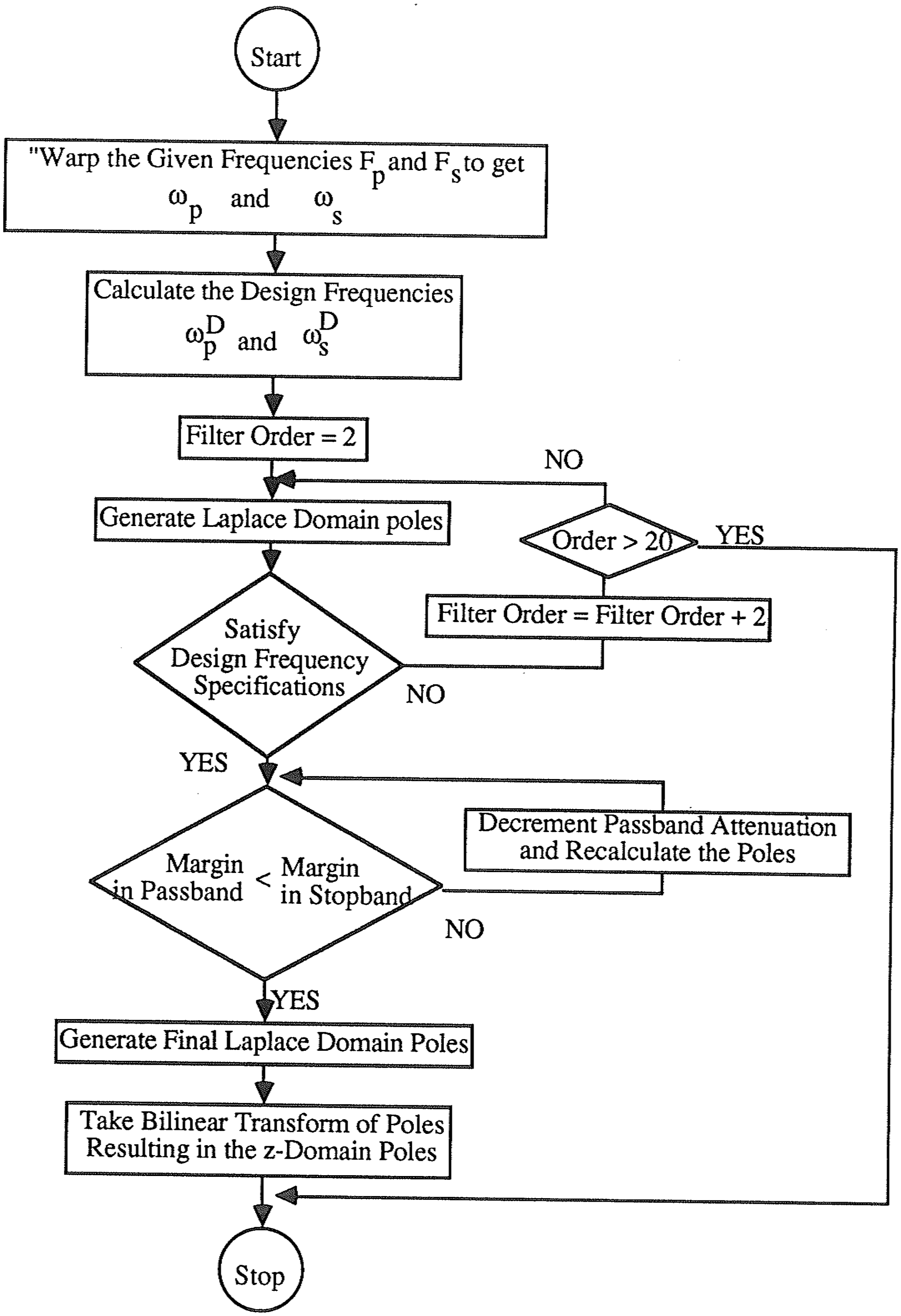

Figure 4.4: Determination of the filter order and the optimal poles.

4.2.2 Generating the Complex Allpass Function

Once the optimum poles have been found, the complex allpass function is easily determined from the discussion in Sections 3.1 and 3.2. Every second pole is chosen to specify the allpass funtion, and the unimodular constant associated with the allpass function, β, is given by (3.29). Note that G(1) is given by the new (optimum) passband attenuation of the transfer function and H(1) is calculated from the Feldkeller equation (with the restriction that the spectral complement of the transfer function should have positive gain), i.e.

$$H(1) = \sqrt{1 - G^2(1)} \tag{4.13}$$

The discrete domain transfer function and the associated spectral complement function along with the unimodular constant β are calculated in the subroutine FINDGH. The magnitude of β is checked to guarantee that it is unimodular. The complex allpass function S(z) can be generated from the chosen subset of the reference filter poles p_m, m = 1 (1) N/2 (N is the reference filter order) using

$$S(z) = \beta \prod_{m=1}^{N/2} \frac{1 - z p_m}{z - p_m^*} \tag{4.14}$$

or

$$S(z) = \frac{P_A(z)}{D_A(z)} = \beta \; \frac{d_M^* + d_{M-1}^* z + \ldots + d_1^* z^{M-1} + d_0^* z^M}{d_0 + d_1 z + \ldots + d_{M-1} z^{M-1} + d_M z^M} \tag{4.15}$$

The subroutine FINDA calculates the coefficients of $P_A(z)$ and $D_A(z)$. Once the complex allpass function has been determined, it is checked against the z-domain equivalent of the reference transfer function with the subroutine TESTG in order to avoid errors due to the accuracy of the calculations performed. This is necessary since the coefficients of the numerator of the complex allpass function are extremely sensitive to the unimodular multiplier β. This is intuitively obvious since a set of poles can determine many (equivalent) transfer functions, thus the desired transfer function is derived by "tuning" β to precisely the correct value.

4.3 Generating the Complex Scaled Scattering Matrix

The complex scaled scattering matrix is given by the paramters of the complex two-port adaptors of the wave digital realization of the complex allpass function representing a reference filter. Because of the limitations imposed by digital hardware and software, the scattering matrix must be represented in a finite binary form, or the real and imaginary parts of the elemtents must be expressed as a ratio of one integer dividing another. This is accomplished by quantizing the real and imaginary parts of the scattering matrix elements. After the quantization, the pseudopassivity of the system must be checked in order to guarantee the important properties given by pseudopassivity.

4.3.1 Finding the Complex Two-port Adaptor Parameters

The complex two-port adaptor parameters are calculated using the coefficients of the complex allpass function and (3.53) repeated below.

$$\alpha = \left(\frac{\beta\, d_0^* - d_M}{d_M^* - \beta^*\, d_0} \right) \left(\frac{\beta^*\, d_0}{d_M} \right) = \frac{\dfrac{d_0^*}{d_M} - \beta^*}{\dfrac{d_M^*}{d_0} - \beta^*} \tag{4.16}$$

Using the method outlined in Section 3.3.1, all complex parameters can be found along with the final unimodular multiplier ε. It is guaranteed that all parameters will have a magnitude less than one and no parameter will be identically equal to zero. To ensure this, both conditions are checked in the program.

Once the parameters are known the complex reference port impedances of the delay ports after scaling can be determined from (3.72), (3.73), and (3.111) using the input-output port impedance, which can be specified at a convenient value. The input-output port impedance is set at 1Ω (purely real) so that the one-port appears real to external systems. The parameters and the port reference impedances are calculated in the subroutine ALPHAZ.

4.3.2 Generating the Scaled Block Sub-Matrices

The scaled block-submatrices as given in (3.87-96) are calculated in the subroutines SSCATT, ALOWES, ALOWOS, and AIS, which use the complex two-port adaptor parameters in generating the matrix elements. The subroutine SSCATT gives the scaled $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$, and $\underline{\mathbf{D}}$ state variable matrices and the scaled $\underline{\mathbf{A}}$ state variable matrix is defined by the subroutines ALOWES, ALOWOS and AIS, giving the lower block for the even and odd allpass function cases and the middle blocks, respectively.

The subroutine SSCATT builds the scaled scattering matrix by inserting blocks from the subroutine AIS until the end of the scattering matrix is reached, when the block from either the ALOWES or the ALOWOS subroutine is finally inserted. For a first order allpass, only the block from the ALOWOS subroutine is needed and the AIS subroutine is not called. Similarly, for a second order allpass, only the block from the ALOWES subroutine is used. For allpass orders of three or higher, the AIS subroutine will be called at least once.

The resulting scaled complex scattering matrix will have floating-point quantities for the real and imaginary parts of the matrix elements. The matrix will be square and of order one greater than half of the order of the equivalent reference filter. At most four non-zero elements will exist in any row or column of the matrix.

4.3.3 Quantizing the Scattering Matrix

Since the scaled scattering matrix cannot be implemented as derived from the SSCATT subroutine, the scattering matrix elements must be quantized as explained in Section 3.4.4. The real and imaginary parts of all elements are expressed as the ratio of an integer numerator dividing a common integer denominator which is equal to two raised to an integer power. The values of the integer numerators are given by (3.109) to be equal to the truncated product of the denominator and the floating-point matrix element.

An initial guess of the denominator as recommended by Webb [43] is given by

$$D = 2^{\mathrm{INT}\left(\frac{N+6}{2}\right)} \tag{4.17}$$

where N is the order of the reference filter. With this denominator, the floating-point representation of the scaled scattering matrix is quantized. Then the quantized version of the scattering matrix is checked for pseudopassivity as discussed in Section 4.3.4. If the quantized WD system is not pseudopassive, the real and imaginary parts of the numerator of the elements are decremented in magnitude by one (decreasing all of the magnitudes) to attempt to make the WD system pseudopassive or the equivalent analog network lossy. This operation is allowed a maximum of four times. However, from practical experience the decrementation operation never had to be used with filters of order less than 12.

After the pseudopassivity check is complete, the frequency response of the WD system given by the scaled quantized scattering matrix is checked as in Section 4.4.1. If the current solution does not meet the design specifications, the denominator of the quantized scattering matrix is increased by a factor of 2. Then the operation of checking for pseudopassivity and checking the frequency response is repeated. The process continues until either the solution meets the design specifications or the maximum value of the denominator MAXDNM is reached. If MAXDNM is reached and FIXFAL is true (Section 4.1.2), then a solution with the denominator FAILDN is generated.

4.3.4 Test for Pseudopassivity

Many of the useful properties of complex WD networks arrise from the pseudopassive characteristic. The floating-point representation of the complex scattering matrix is pseudolossless. However, after the truncation operation as described in the earlier section, the matrix may no longer be pseudopassive.

The complex quantized scattering matrix is tested to determine if the equivalent complex WD network is pseudopassive in the subroutine LOSSYT. A sufficient condition for the pseudopassivity of a complex WD network is the positive definiteness of the following hermitian matrix.

$$\mathbf{S_P} = \mathbf{G} - \mathbf{S}^{*T}\mathbf{G}\,\mathbf{S} \tag{4.18}$$

The matrix S_p is generated in LOSSYT and the subroutine POSDEF determines if the network is pseudopassive by upper triangularizing the hermitian matrix and checking if the magnitude of any of the resulting diagonal elements is less than 10^{-5}.

If the quantized scattering matrix fails the pseudopassivity test, the magnitudes of the real and imaginary parts of the matrix elements are decreased by subtracting one from the magnitude of all of the numerators. Thus, consider the real or imaginary numerator of a matrix element: if the numerator is positive it is decremented, and if the numerator is negative one is added to it (decreasing the magnitude). The decrementation operation is allowed to be repeated if the test continually fails, up to a maximum of four decrementations. This process is shown in the flow chart of Figure 4.5.

4.4 Meeting the Design Specifications

The design specifications as discussed in Section 4.1.1 places restrictions on the attenuation characteristic of the complex WD network represented by the quantized scaled scattering matrix. If the attenuation characteristic enters the shaded area of Figure 4.2 then the design specifications are not met and another solution with more significant digits in the representation of the scattering matrix elements will have to be tested. Note that the attenuation characteristic of Figure 4.2 implies that the attenuation must always be positive. For a one-port scaled allpass network, this restriction can be relaxed and the attenuation is allowed to be negative as long as the magnitude of the smallest attenuation is smaller than the given passband attenuation.

The subroutine TEST checks the frequency response at 128 different frequency values given by the subroutine GENOMG.

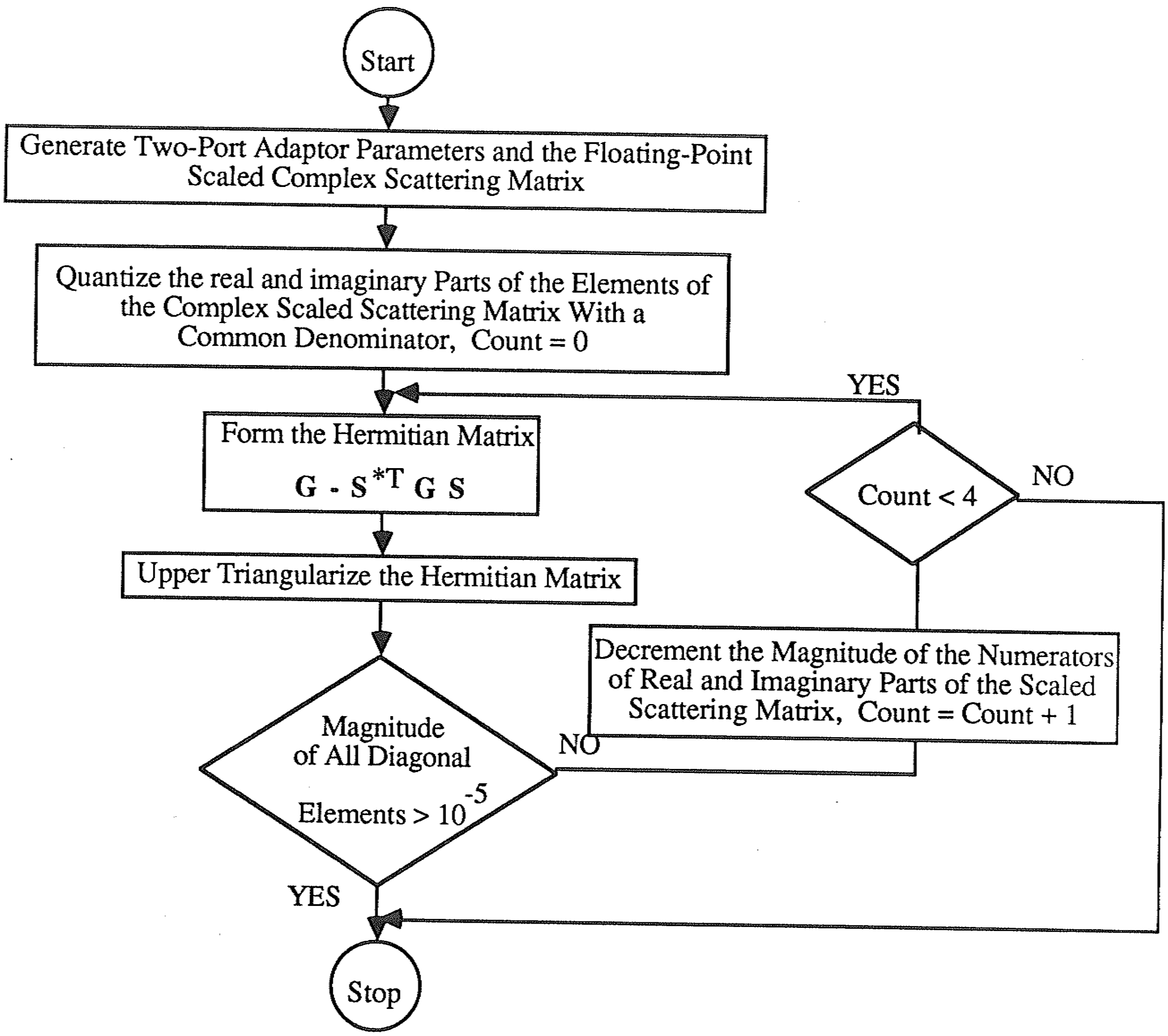

Figure 4.5: Quantized scattering matrix test for pseudopassivity.

4.4.1 Testing the Frequency Response

The frequency response of a state-variable system can be generated using the discussion in Sections 2.6.3 and 4.1.4. The subroutine TEST checkes the frequency response of the quantized scattering matrix using frequency values generated by the subroutine GENOMG. Because of the quantization process, the attenuation may become negative (a voltage gain greater than one). This ordinarily only happens in the Chebyshev and Cauer filter types with a passband ripple. Note that with these filters, the ripple width is the predominating factor, and not the actual attenuation values (as long as the magnitude of the smallest negative attenuation is small). To find the ripple width, all of the frequency response values are generated and the passband values are checked to find the maximum passband ripple width. The test fails if this ripple width is larger than the passband attenuation given in the design specifications. Also, the test fails if any value of the attenaution in the stopband becomes less than the given stopband attenuation. Note that 128 frequency values are tested in the passband and stopband, and thus there is a reasonably high probability that any portion of the attenuation characteristic that does not meet the design specifications will be identified, causing the test to fail.

For convenience, the output of the program in the text file CP.DAT will give information on whether the frequency test failed in the passband or stopband. This shows which part of the attenuation characteristic shifts out of the allowed region defined by the design specifications, after truncation. The quantized scattering matrix denominator that was used is also given.

4.4.2 Achieving the Design Specifications

The frequency response test determines if the elements of the quantized scaled scattering matrix are represented with enough significant digits, or a high enough accuracy, to meet the design specifications. If the frequency response test fails, the denominator of the quantized scattering matrix is doubled, a new representation of the scattering matrix is calculated and the pseudopassive test is repeated. Then the frequency response is checked again and the process repeats until either the quantized scattering matrix passes the test or the denominator reaches its maximum value. If it passes the test, then a solution is found and the quantized scattering matrix along with the quantized system characteristics as discussed in Section 4.1.5 are saved in the file CP.DAT. However, if the maximum denominator is reached and FIXFAL is false, no solution is generated and the program terminates. If FIXFAL is true, a solution with the denominator FAILDN is generated.

A Butterworth or Chebyshev lowpass filter by definition has zero gain at infinite frequency in the Laplace domain. Because of the limitations imposed by digital computers, the gain of the reference filter may not reach zero, but it will become extremely small (around -300 dB). However, after the quantization process, the Butterworth and Chebyshev filter types may have a finite non-zero gain (greater than -100 dB).

Thus the user has the option to increase the denominator by a factor of two, after a solution has been found that meets the design specifications. The attenuations at infinite frequency of the quantized and the "infinite-precision" representations of the system are given and the user can increase the denominator be typing a "1" or accept the current solution by typing a "0". If the user decided to increase the denominator, the user has the option of generating the frequency response of the quantized system with the old denominator, which is stored in one of the text files SPECT?.DAT, where ? is between 1 and 3. The process of increasing the denominator after a solution has been found can be repeated three times, and after each iteration the attenuation at infinite frequency may or may not be closer to the desired attenuation. This process is shown in the following flow chart.

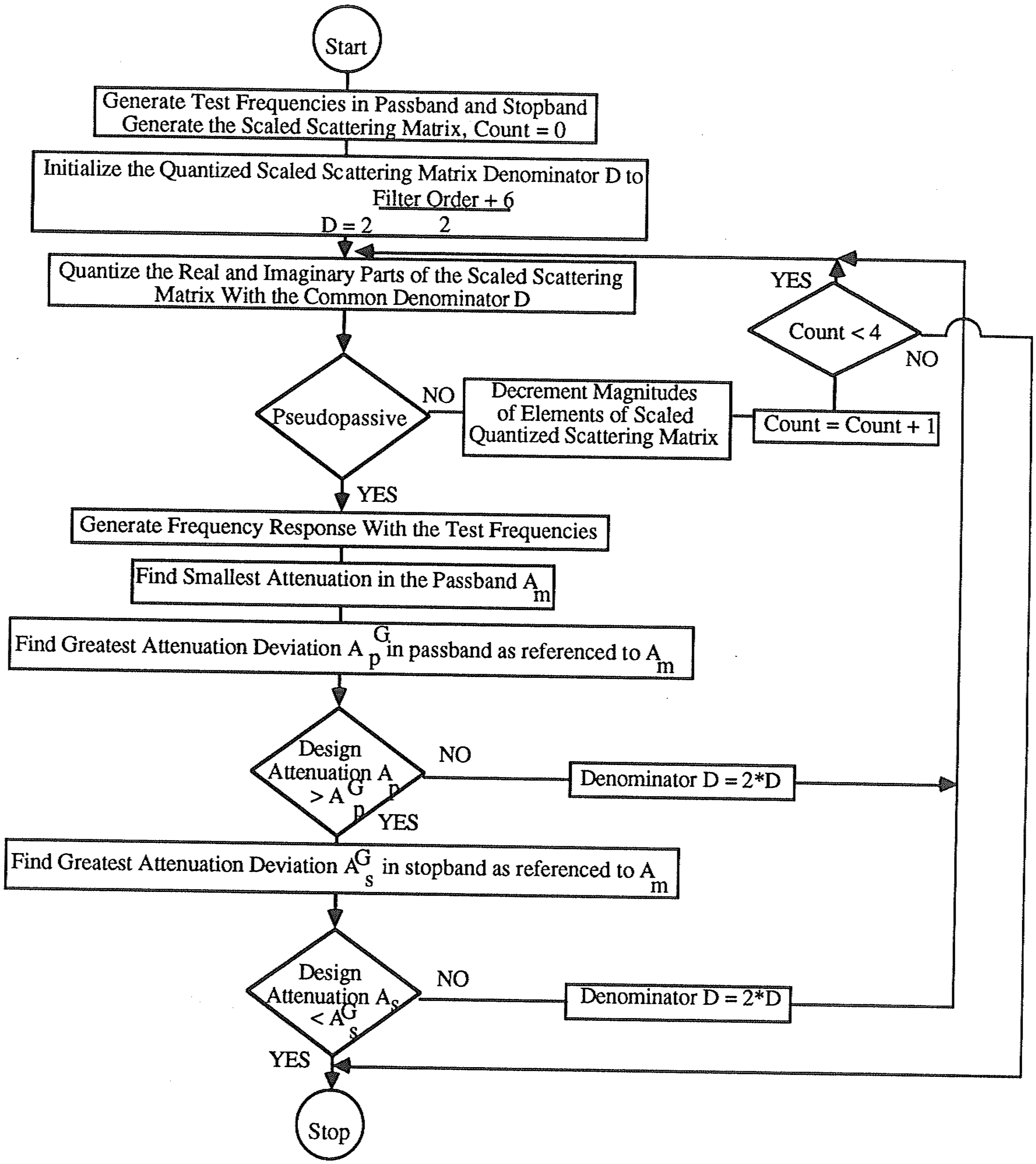

Figure 4.6: Flow chart of the frequency response test.

CHAPTER V

DESIGN EXAMPLES

The computer program discussed in Chapter IV and listed in Appendix C was used to design six even-order lowpass filters. Two Butterworth lowpass filters of orders 6 and 16, a Chebyshev lowpass filter of order 6, and two Cauer lowpass filters of orders 4 and 8 were generated. The last design example deals with a different representation of the 4th order Cauer lowpass filter for reasons of comparison. In each case the design specifications and the quantized complex scaled scattering matrix are given, along with the diagonal elements of the **K** and **W** input and output covariance matrices and the L_2 norm. The fullband and passband attenuation plots in decibels are included showing the response of both the quantized and unquantized systems, where the frequencies have been normalized to the range $0 \leq \omega \leq 0.5$.

The even-order classical filters represented by the quantized scaled scattering matrix can be implemented on a wide range of microprocessor-based hardware systems. During the recursive operation of the filter, the complex non-linear overflow and underflow operators as discussed in Section 2.7 must be implemented. The states of the filter are calculated using complex matrix multiplication by a column vector. The input to the filter must be real and the desired output corresponding to the transfer function is the real part of the output. Note that the response corresponding to the spectral complement of the transfer function is given by the imaginary part of the output.

5.1 Example #1: Butterworth Lowpass Filter of Order 6

The first example is a 6th order Butterworth lowpass digital filter with the following design specifications.

$$F_p = 0.7 \text{ rads} \tag{5.1}$$

$$F_s = 1.2 \text{ rads} \tag{5.2}$$

$$A_p = 1.0 \text{ dB} \tag{5.3}$$

$$A_s = 20.0 \text{ dB} \tag{5.4}$$

The discrete transfer function that meets the design specifications is given by

$$G(z) = \frac{\sum_{i=0}^{6} p_i z^i}{\sum_{j=0}^{6} d_j z^j} \tag{5.5}$$

where the coefficients are given by

$$p_0 = p_6 = 0.0012445724 \tag{5.6a}$$
$$p_1 = p_5 = 0.0074674344 \tag{5.6b}$$
$$p_2 = p_4 = 0.0186685860 \tag{5.6c}$$
$$p_5 = 0.0248914480 \tag{5.6d}$$

$$d_0 = 0.0386105984 \tag{5.7a}$$
$$d_1 = -0.3522726983 \tag{5.7b}$$
$$d_2 = 1.3846603313 \tag{5.7c}$$
$$d_3 = -3.0203338006 \tag{5.7d}$$
$$d_4 = 3.9042947753 \tag{5.7e}$$
$$d_5 = -2.8753065725 \tag{5.7f}$$
$$d_6 = 1.0000000000 \tag{5.7g}$$

The complex allpass function representing the transfer function is given by

$$A(z) = \frac{\sum_{i=0}^{3} a_i^n z^i}{\sum_{j=0}^{3} a_j^d z^j} \tag{5.8}$$

where the coefficients are given by

$$a_0^n = 0.4025275739 - j\,0.9154078612 \tag{5.9a}$$
$$a_1^n = -0.2477286926 + j\,1.4615732652 \tag{5.9b}$$
$$a_2^n = 0.0802984755 - j\,0.8969139490 \tag{5.9c}$$
$$a_3^n = 0.0012445724 + j\,0.1964918559 \tag{5.9d}$$

$$a_0^d = -0.1793692148 - j\,0.0802326814 \tag{5.10a}$$
$$a_1^d = 0.8533644302 + j\,0.2875267402 \tag{5.10b}$$
$$a_2^d = -1.4376532863 - j\,0.3615507479 \tag{5.10c}$$
$$a_3^d = 1.0000000000 + j\,0.0000000000 \tag{5.10d}$$

The (4 x 4) complex quantized scaled scattering matrix representing the non-dynamic CWD network that meets the design specifications is given by

$$S = \frac{1}{16}\begin{bmatrix} j3 & 15-j3 & 0 & 0 \\ 2-j10 & 2 & 4+j10 & 6-j2 \\ 11+j5 & -1+j2 & 9-j1 & -1-j5 \\ 0 & 0 & 4-j7 & 12+j7 \end{bmatrix} \tag{5.11}$$

The L_2 norm is given by

$$L_2 = 0.997388 \tag{5.12}$$

which is very close to the ideal value of one. The diagonal elements of the input and output covariance matrices **K** and **W** of the scaled system are given by

$$K_{11} = 1.053596 \tag{5.13a}$$
$$K_{22} = 1.021116 \tag{5.13b}$$
$$K_{33} = 1.148036 \tag{5.13c}$$

$$W_{11} = 0.948795 \tag{5.14a}$$
$$W_{22} = 1.018575 \tag{5.14b}$$
$$W_{33} = 0.988590 \tag{5.14c}$$

which are close to the ideal value of one for a scaled system.

The values of the passband and stopband frequencies after the normalization are

$$F_p^N = 0.111 \tag{5.15}$$
$$F_s^N = 0.191 \tag{5.16}$$

The fullband characteristic shown in Figure 5.1 clearly shows that the design passed the stopband specification as the attenuation after the stopband frequency is greater than 30 dB. Notice the pronounced peak around the stopband frequency, which indicates that the quantization process has introduced at least a pair of zeros into the transfer function. The zero pair caused a finite non-zero gain at infinite frequency in the Laplace domain.

The passband characteristic shown in Figure 5.2 demonstrates that the design passed the passband specification since the maximum width of the attenuation is less than 1.0 dB. Notice that there is a "first-order" ripple at frequencies less than the passband frequency, ie. the attenuation is not maximally-flat in the passband. This is expected from the observation that the quantization process has introduced at least a pair of zeros into the transfer function.

Thus the quantized system is not purely Butterworth in character. Also, the quantized system with a ripple in both the passband and stopband displays a Cauer-type of characteristic. A larger value of the integer denominator used in the quantized scaled scattering matrix may cause the non-zero gain at infinite frequency to decrease, or the attenuation to increase past 30 dB.

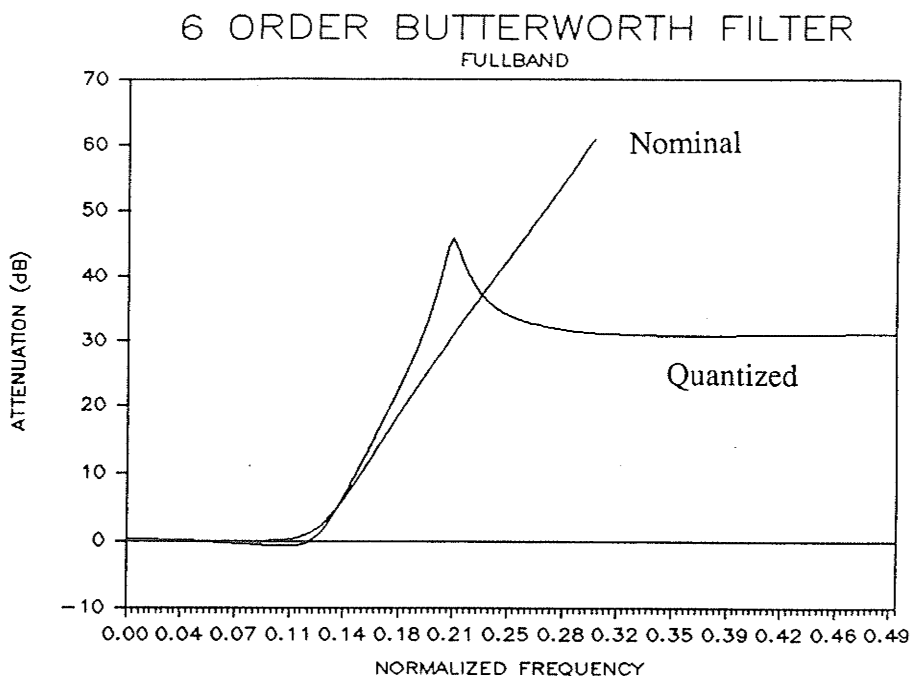

Figure 5.1: Fullband characteristic of a 6th order Butterworth lowpass filter.

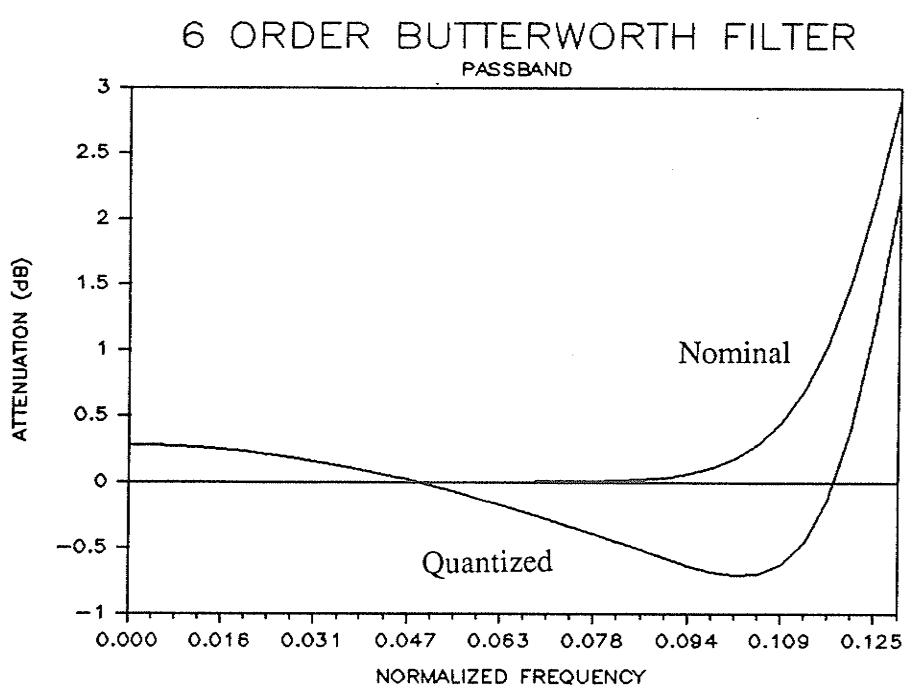

Figure 5.2: Passband characteristic of a 6th order Butterworth lowpass filter.

5.2 Example #2: Butterworth Lowpass Filter of Order 16

The second example is a 16^{th} order Butterworth lowpass filter with the following frequency specifications.

$$F_p = 0.7 \text{ rads} \tag{5.17}$$
$$F_s = 1.2 \text{ rads} \tag{5.18}$$
$$A_p = 0.5 \text{ dB} \tag{5.19}$$
$$A_s = 60.0 \text{ dB} \tag{5.20}$$

The discrete transfer function that meets the frequency specifications is given by

$$G(z) = \frac{\sum_{i=0}^{16} p_i z^i}{\sum_{j=0}^{16} d_j z^j} \tag{5.21}$$

where the coefficients are given by

$$p_0 = p_{16} = 0.0000000115 \tag{5.22a}$$
$$p_1 = p_{15} = 0.0000001845 \tag{5.22b}$$
$$p_2 = p_{14} = 0.0000013836 \tag{5.22c}$$
$$p_3 = p_{13} = 0.0000064569 \tag{5.22d}$$
$$p_4 = p_{12} = 0.0000209848 \tag{5.22e}$$
$$p_5 = p_{11} = 0.0000503635 \tag{5.22f}$$
$$p_6 = p_{10} = 0.0000923332 \tag{5.22g}$$
$$p_7 = p_9 = 0.0001319045 \tag{5.22h}$$
$$p_8 = 0.0001483926 \tag{5.22i}$$

$$d_0 = 0.0002535443 \tag{5.23a}$$
$$d_1 = -0.0060829884 \tag{5.23b}$$
$$d_2 = 0.0691193910 \tag{5.23c}$$
$$d_3 = -0.4940480722 \tag{5.23d}$$
$$d_4 = 2.4879801617 \tag{5.23e}$$
$$d_5 = -9.3671807940 \tag{5.23f}$$
$$d_6 = 27.2974518639 \tag{5.23g}$$
$$d_7 = -62.8680083325 \tag{5.23h}$$
$$d_8 = 115.7693116165 \tag{5.23i}$$
$$d_9 = -171.2389641708 \tag{5.23j}$$
$$d_{10} = 203.0720347922 \tag{5.23k}$$

$$d_{11} = -191.3829754813 \quad (5.23l)$$
$$d_{12} = 140.8150669066 \quad (5.23m)$$
$$d_{13} = -78.4014770614 \quad (5.23n)$$
$$d_{14} = 31.2564570072 \quad (5.23o)$$
$$d_{15} = -8.0081827450 \quad (5.23p)$$
$$d_{16} = 1.0000000000 \quad (5.23q)$$

The complex allpass function representing the above transfer function is given by

$$A(z) = \frac{\sum_{i=0}^{8} a_i^n z^i}{\sum_{j=0}^{8} a_j^d z^j} \quad (5.24)$$

where the complex coefficients are given by

$$a_0^n = 0.3830463139 + j\,0.9237291386 \quad (5.25a)$$
$$a_1^n = -1.2077025770 - j\,3.8339002249 \quad (5.25b)$$
$$a_2^n = 1.8150202140 + j\,7.4202741802 \quad (5.25c)$$
$$a_3^n = -1.6133275391 - j\,8.5979359035 \quad (5.25d)$$
$$a_4^n = 0.9040711608 + j\,6.4648826310 \quad (5.25e)$$
$$a_5^n = -0.3161094766 - j\,3.2104381256 \quad (5.25f)$$
$$a_6^n = 0.0634812045 + j\,1.0237444754 \quad (5.25g)$$
$$a_7^n = -0.0056201574 - j\,0.1910117399 \quad (5.25h)$$
$$a_8^n = 0.0000000115 + j\,0.0159230743 \quad (5.25i)$$

$$a_0^d = 0.0147086121 - j\,0.00060992643 \quad (5.26a)$$
$$a_1^d = -0.1785958905 + j\,0.0679748397 \quad (5.26b)$$
$$a_2^d = 0.9699788437 - j\,0.3335021093 \quad (5.26c)$$
$$a_3^d = -3.0866598139 + j\,0.9377469555 \quad (5.26d)$$
$$a_4^d = 6.3181015892 - j\,1.6412325871 \quad (5.26e)$$
$$a_5^d = -8.5601430924 + j\,1.8031299971 \quad (5.26f)$$
$$a_6^d = 7.5495602789 - j\,1.1657216141 \quad (5.26g)$$
$$a_7^d = -4.0040913725 + j\,0.3529712879 \quad (5.26h)$$
$$a_8^d = 1.0000000000 + j\,0.0000000000 \quad (5.26i)$$

The (9 x 9) complex quantized scaled scattering matrix representing the non-dynamic network that meets the design specifications is given by

$$S = \frac{1}{8192}\begin{bmatrix} j127 & 8187-j127 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14-j1039 & 14 & 307+j3239 & 7155-j2041 & 0 & 0 & 0 & 0 & 0 \\ 8071+j898 & -9+j126 & 415 & -171-j936 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 919-j4981 & 2215 & 706+j4876 & 3405-j755 & 0 & 0 & 0 \\ 0 & 0 & 5239+j1774 & -345+j2394 & 4513 & -236-j3185 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 318-j4128 & 5849+j16 & 221+j3567 & 1695-j102 & 0 \\ 0 & 0 & 0 & 0 & 2271+j258 & -125+j3227 & 6470-j167 & -76-j3076 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 502-j3211 & 6755+j1064 & 1232+j2859 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1335+j46 & -102+j2810 & 6547-j3803 \end{bmatrix} \tag{5.27}$$

The L_2 norm of the CWD one-port complex allpass network given by the above is

$$L_2 = 0.997151 \tag{5.28}$$

which is very close to the expected value of one for an allpass network. The diagonal elements of the input **K** and output **W** covariance matrices are given by

$$K_{11} = 0.988271 \tag{5.29a}$$
$$K_{22} = 0.998839 \tag{5.29b}$$
$$K_{33} = 0.988287 \tag{5.29c}$$
$$K_{44} = 0.993916 \tag{5.29d}$$
$$K_{55} = 0.985548 \tag{5.29e}$$
$$K_{66} = 0.988392 \tag{5.29f}$$
$$K_{77} = 0.983984 \tag{5.29g}$$
$$K_{88} = 0.979852 \tag{5.29h}$$

$$W_{11} = 0.999258 \tag{5.30a}$$
$$W_{22} = 0.988330 \tag{5.30b}$$
$$W_{33} = 0.996587 \tag{5.30c}$$
$$W_{44} = 0.986306 \tag{5.30d}$$
$$W_{55} = 0.990220 \tag{5.30e}$$
$$W_{66} = 0.983827 \tag{5.30f}$$
$$W_{77} = 0.986022 \tag{5.30g}$$
$$W_{88} = 0.979151 \tag{5.30h}$$

which are close to the ideal value of one for a scaled system.

The values of the passband and stopband frequencies after the normalization are

$$F_p^N = 0.111 \tag{5.31}$$
$$F_s^N = 0.191 \tag{5.32}$$

The fullband characteristic shown in Figure 5.3 clearly shows that the design satisfies the stopband specification as the minimum attenuation at frequencies greater than the stopband frequency is 65 dB. Notice the two pronounced peaks in the stopband, suggesting that at least two sets of pairs of zeros have been introduced into the transfer function by the quantization process. The gain at infinite frequency in the Laplace domain is approximately 70 dB.

Figure 5.4 shows the passband characteristic. The design meets the passband frequency specification since the design passband plot is nearly identical to the theoretical passband plot of a 16^{th} order Butterworth filter, except it is shifted upward by approximately 0.08 dB. Notice that no passband ripple exists.

Thus the attenuation characteristic is not purely Butterworth in nature. The excellent performance in the passband clearly demonstrates the passband insensitivity. A larger integer denominator used in the quantized scaled scattering matrix may cause the infinite frequency attenuation to increase.

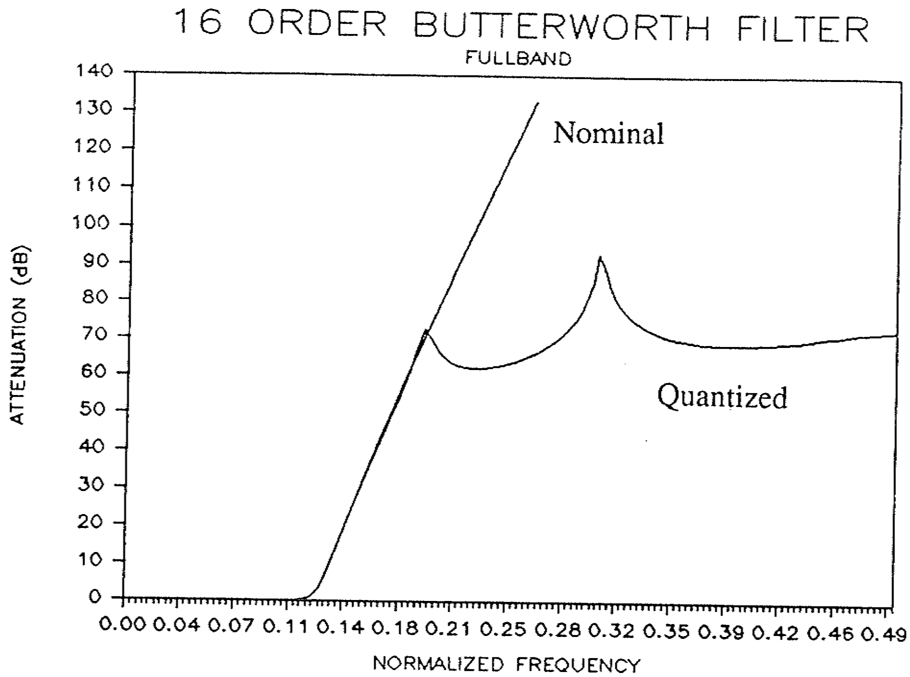

Figure 5.3: Fullband characteristic of a 16th order Butterworth lowpass filter.

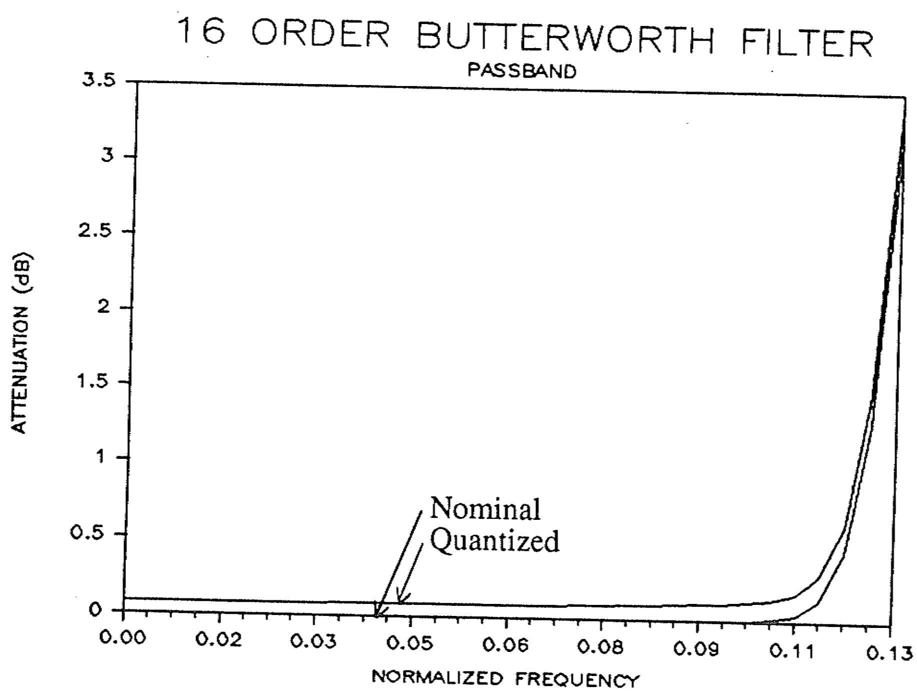

Figure 5.4: Passband characteristic of a 16th order Butterworth lowpass filter.

5.3 Example #3: Chebyshev Lowpass Filter of Order 6

The third design example is taken from Gazsi [24] and is a 6th order Chebyshev lowpass filter with the following frequency specifications.

$$F_p = 1.178097 \text{ rads} \tag{5.33}$$
$$F_s = 1.963495 \text{ rads} \tag{5.34}$$
$$A_p = 1.0 \text{ dB} \tag{5.35}$$
$$A_s = 40.0 \text{ dB} \tag{5.36}$$

The discrete transfer function that meets the design specifications is given by

$$G(z) = \frac{\sum_{i=0}^{6} p_i z^i}{\sum_{j=0}^{6} d_j z^j} \tag{5.37}$$

where the coefficients are given by

$$p_0 = p_6 = 0.0024567792 \tag{5.38a}$$
$$p_1 = p_5 = 0.0147406754 \tag{5.38b}$$
$$p_2 = p_4 = 0.0368516885 \tag{5.38c}$$
$$p_5 = 0.0491355846 \tag{5.38d}$$

$$d_0 = 0.2791373140 \tag{5.39a}$$
$$d_1 = -1.3447650307 \tag{5.39b}$$
$$d_2 = 3.2131837574 \tag{5.39c}$$
$$d_3 = -4.7640280245 \tag{5.39d}$$
$$d_4 = 4.6581881618 \tag{5.39e}$$
$$d_5 = -2.8715990167 \tag{5.39f}$$
$$d_6 = 1.0000000000 \tag{5.39g}$$

The complex allpass function representing the transfer function is given by

$$A(z) = \frac{\sum_{i=0}^{3} a_i^n z^i}{\sum_{j=0}^{3} a_j^d z^j} \tag{5.40}$$

where the complex coefficients are given by

$$a_0^n = 0.5979709138 - j\,0.8015178016 \tag{5.41a}$$
$$a_1^n = -0.4743466224 + j\,1.4374652975 \tag{5.41b}$$
$$a_2^n = 0.2715305264 - j\,1.2739219541 \tag{5.41c}$$
$$a_3^n = 0.0024567792 + j\,0.5283287596 \tag{5.41d}$$

$$a_0^d = -0.4219958234 - j\,0.3178943835 \tag{5.42a}$$
$$a_1^d = 1.1834384810 + j\,0.5441317244 \tag{5.42b}$$
$$a_2^d = -1.4357995084 - j\,0.4793651755 \tag{5.42c}$$
$$a_3^d = 1.0000000000 + j\,0.0000000000 \tag{5.42d}$$

The (4 x 4) complex scaled quantized scattering matrix representing the non-dynamic CWD network that meets the design specifications is given by

$$S = \frac{1}{1024}\begin{bmatrix} 3+j541 & 768-j407 & 0 & 0 \\ 313-j536 & 386+j18 & 23+j578 & 417-j70 \\ 586+j166 & 78+j371 & 565-j170 & -176-j394 \\ 0 & 0 & 514-j318 & 519+j643 \end{bmatrix} \tag{5.43}$$

The L_2 norm of the CWD one-port complex allpass network given by the above is

$$L_2 = 0.997297 \tag{5.44}$$

which is very close to the expected value of one for an allpass network. The diagonal elements of the input **K** and output **W** covariance matrices are given by

$$K_{11} = 0.998039 \tag{5.45a}$$
$$K_{22} = 0.999901 \tag{5.45b}$$
$$K_{33} = 0.999278 \tag{5.45c}$$

$$W_{11} = 0.999077 \tag{5.46a}$$
$$W_{22} = 0.995703 \tag{5.46b}$$
$$W_{33} = 0.995740 \tag{5.46c}$$

which are close to the ideal value of one for a scaled system.

The values of the passband and stopband frequencies after the normalization are

$$F_p^N = 0.1875 \tag{5.47}$$
$$F_s^N = 0.3125 \tag{5.48}$$

The fullband characteristic shown in Figure 5.5 clearly shows that the design passed the stopband specification as the minimum attenuation for frequencies greater than the stopband frequency is 50 dB. There is no pronounced peak in the characteristic implying that a zero-pair was not introduced into the transfer function of the quantized system, as was the case in the earlier two examples. This is a result of the large integer denominator used in the quantized scaled scattering matrix needed to pass the frequency specifications (1024 in this example as compared to 16 for the 6th order Butterworth example).

The passband characteristic is shown in Figure 5.6. The passband specifications are clearly met and the quantized design plot and the theoretical plot are nearly identical in the passband. This is a result of the large integer denominator in the quantized scaled scattering matrix.

The characteristic is not purely Chebyshev in character as there in a non-zero gain at infinite frequency. However, the quantized system is not Cauer in nature since no stopband zeros can be identified.

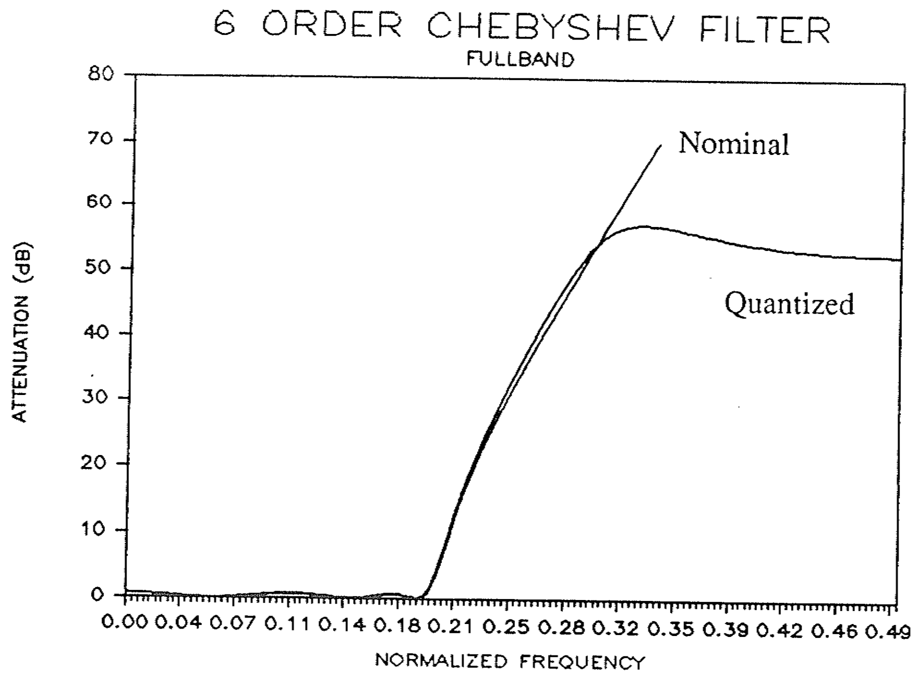

Figure 5.5: Fullband characteristic of a 6^{th} order Chebyshev lowpass filter.

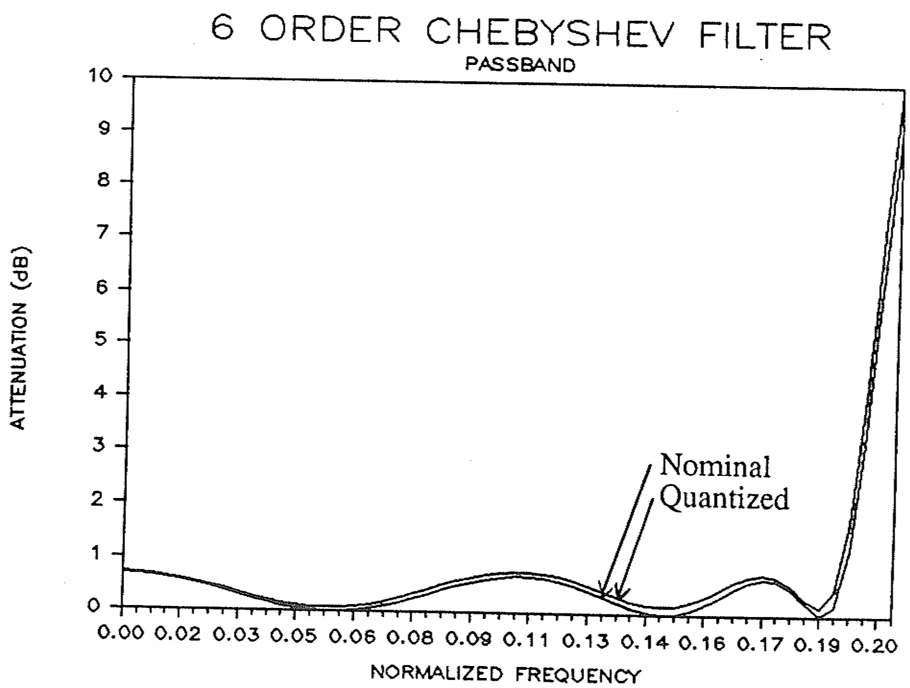

Figure 5.6: Passband characteristic of a 6^{th} order Chebyshev lowpass filter.

5.4 Example #4: Cauer Lowpass Filter of Order 4

The fourth design example is taken from [57] with the stopband attenuation changed to 25 dB giving a 4^{th} order Cauer lowpass filter with the following frequency specifications.

$$F_p = 0.667588 \text{ rads} \tag{5.49}$$
$$F_s = 0.903208 \text{ rads} \tag{5.50}$$
$$A_p = 0.3 \text{ dB} \tag{5.51}$$
$$A_s = 25.0 \text{ dB} \tag{5.52}$$

The discrete transfer function that meets the design specifications is given by

$$G(z) = \frac{\sum_{i=0}^{4} p_i z^i}{\sum_{j=0}^{4} d_j z^j} \tag{5.53}$$

where the coefficients are given by

$$p_0 = p_4 = 0.0573889274 \tag{5.54a}$$
$$p_1 = p_3 = -0.0609265102 \tag{5.54b}$$
$$p_2 = 0.1060795987 \tag{5.54c}$$

$$d_0 = 0.3921879802 \tag{5.55a}$$
$$d_1 = -1.6958691320 \tag{5.55b}$$
$$d_2 = 3.0383591303 \tag{5.55c}$$
$$d_3 = -2.6326583088 \tag{5.55d}$$
$$d_4 = 1.0000000000 \tag{5.55e}$$

The complex allpass function representing the transfer function is given by

$$A(z) = \frac{\sum_{i=0}^{2} a_i^n z^i}{\sum_{j=0}^{2} a_j^d z^j} \tag{5.56}$$

where the complex coefficients are given by

$$a_0^n = 0.3702802495 + j\,0.9289200917 \qquad (5.57a)$$
$$a_1^n = -0.1870354137 - j\,1.3424982900 \qquad (5.57b)$$
$$a_2^n = 0.0573889274 + j\,0.6236140563 \qquad (5.57c)$$

$$a_0^d = 0.6005376127 - j\,0.1776022406 \qquad (5.58a)$$
$$a_1^d = -1.3163291544 + j\,0.3233596481 \qquad (5.58b)$$
$$a_2^d = 1.0000000000 + j\,0.0000000000 \qquad (5.58c)$$

The (3 x 3) complex scaled quantized scattering matrix representing the non-dynamic CWD network that meets the design specifications is given by

$$S = \frac{1}{256}\begin{bmatrix} 15+j160 & 166-j110 & 0 \\ 123-j132 & 140+j37 & 27+j108 \\ 78+j38 & 3+j69 & 197-j120 \end{bmatrix} \qquad (5.59)$$

The L_2 norm of the CWD one-port complex allpass network given by the above is

$$L_2 = 0.997070 \qquad (5.60)$$

which is very close to the expected value of one for an allpass network. The diagonal elements of the input **K** and output **W** covariance matrices are given by

$$K_{11} = 1.007408 \qquad (5.61a)$$
$$K_{22} = 1.001022 \qquad (5.61b)$$

$$W_{11} = 0.996474 \qquad (5.62a)$$
$$W_{22} = 0.998532 \qquad (5.62b)$$

which are close to the ideal value of one for a scaled system.

The values of the passband and stopband frequencies after the normalization are

$$F_p^N = 0.1062 \qquad (5.63)$$
$$F_s^N = 0.1438 \qquad (5.64)$$

The fullband characteristic is shown in Figure 5.7. The quantized design clearly meets the stopband specification as the design and theoretical plots are nearly identical. Notice that one pair of zeros shifted slightly in frequency, while the other pair did not shift. The attenuation at infinite frequency is approximately equal to the theoretical attenuation. This is a result of the non-zero gain at infinite frequency of even-order Cauer lowpass filters.

Figure 5.8 shows the passband characteristic which clearly meets the design specification. The quantized system and theoretical plots are nearly identical except the quantized system plot is shifted downward by 0.03 dB. Notice that the integer denominator used in the quantized scaled scattering matrix, 256, is not large in comparison with the Chebyshev example.

The quantized system is Cauer in nature. The maximum deviation in the passband is 0.03 dB, while in the stopband the maximum deviation is over 10 dB, showing the passband insensitivity predicted. Note that the deviation at infinite frequency is 0.3 dB. Clearly no benefit would be gained by increasing the integer denominator used in the quantized scaled scattering matrix in this case.

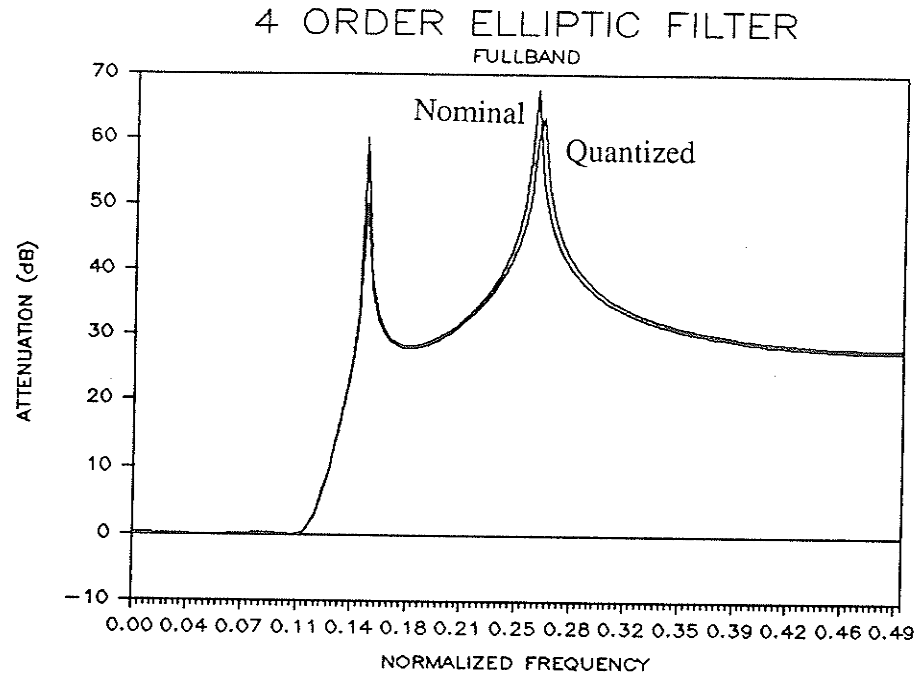

Figure 5.7: Fullband characteristic of a 4th order Cauer lowpass filter.

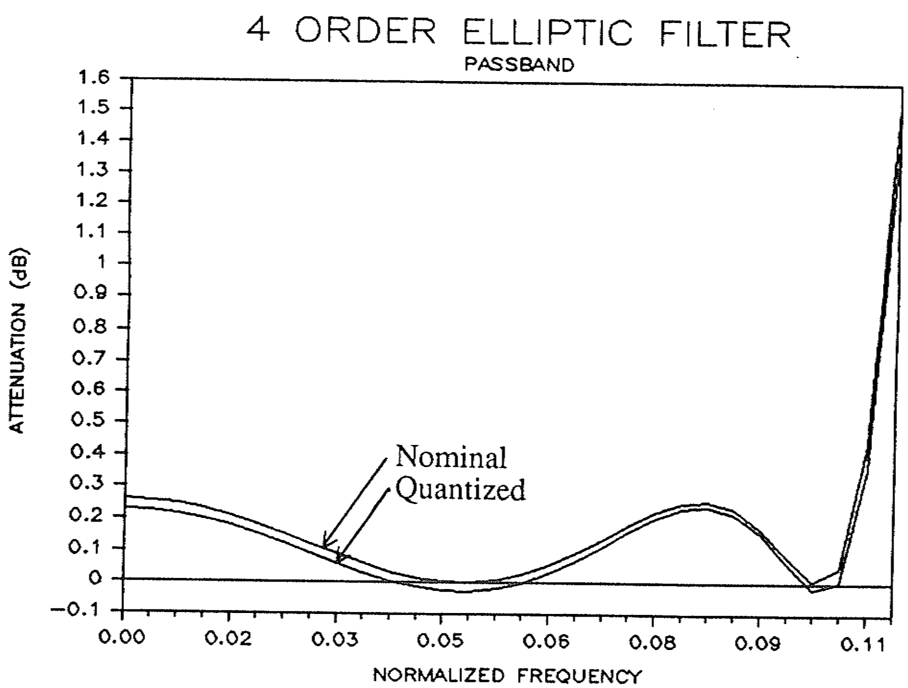

Figure 5.8: Passband characteristic of a 4th order Cauer lowpass filter.

5.5 Example #5: Cauer Lowpass Filter of Order 8

The fifth example from [30] is an 8^{th} order Cauer lowpass filter with the following frequency specifications.

$$F_p = 0.15 \text{ rads} \tag{5.65}$$
$$F_s = 0.17 \text{ rads} \tag{5.66}$$
$$A_p = 0.01 \text{ dB} \tag{5.67}$$
$$A_s = 35.0 \text{ dB} \tag{5.68}$$

The discrete transfer function that meets the design specifications is given by

$$G(z) = \frac{\sum_{i=0}^{8} p_i z^i}{\sum_{j=0}^{8} d_j z^j} \tag{5.69}$$

where the coefficients are given by

$$p_0 = p_8 = 0.0081059679 \tag{5.70a}$$
$$p_1 = p_7 = -0.0610599019 \tag{5.70b}$$
$$p_2 = p_6 = 0.2046164109 \tag{5.70c}$$
$$p_3 = p_5 = -.3986085852 \tag{5.70d}$$
$$p_4 = 0.4938923761 \tag{5.70e}$$

$$d_0 = 0.6869799462 \tag{5.71a}$$
$$d_1 = -5.7153614912 \tag{5.71b}$$
$$d_2 = 20.8472654830 \tag{5.71c}$$
$$d_3 = -43.5471011972 \tag{5.71d}$$
$$d_4 = 56.9782888451 \tag{5.71e}$$
$$d_5 = -47.8210298575 \tag{5.71f}$$
$$d_6 = 25.1425770166 \tag{5.71g}$$
$$d_7 = -7.5716185843 \tag{5.71h}$$
$$d_8 = 1.0000000000 \tag{5.71i}$$

The complex allpass function representing the transfer function is given by

$$A(z) = \frac{\sum_{i=0}^{4} a_i^n z^i}{\sum_{j=0}^{4} a_j^d z^j} \tag{5.72}$$

where the complex coefficients are given by

$$a_0^n = 0.0842381020 + j\,0.9964456544 \tag{5.73a}$$
$$a_1^n = -0.2432241374 - j\,3.7787515414 \tag{5.73b}$$
$$a_2^n = 0.2444925904 + j\,5.4008287322 \tag{5.73c}$$
$$a_3^n = -0.0933241645 - j\,3.4470490862 \tag{5.73d}$$
$$a_4^n = 0.0081059686 + j\,0.8288028955 \tag{5.73e}$$

$$a_0^d = 0.8265398749 - j\,0.0617396256 \tag{5.74a}$$
$$a_1^d = -3.4426585329 + j\,0.1973804144 \tag{5.74b}$$
$$a_2^d = 5.4022279121 - j\,0.2113319825 \tag{5.74c}$$
$$a_3^d = -3.7858092921 + j\,0.0759552230 \tag{5.74d}$$
$$a_4^d = 1.0000000000 + j\,0.0000000000 \tag{5.74e}$$

The (5 x 5) complex scaled quantized scattering matrix representing the non-dynamic CWD network that meets the design specifications is given by

$$S = \frac{1}{65536}\begin{bmatrix} 531+j54316 & 28137-j23510 & 0 & 0 & 0 \\ 23310-j27839 & 53788+j581 & 609+j9062 & 809-j43 & 0 \\ 3798+j3406 & -339+j7550 & 64606-j2139 & -113-j5765 & 0 \\ 0 & 0 & 433-j5797 & 64922+j5724 & 170+j3674 \\ 0 & 0 & 327-j6 & 22+j3663 & 64791-j9143 \end{bmatrix} \tag{5.75}$$

The L_2 norm of the CWD one-port complex allpass network given by the above is

$$L_2 = 0.997002 \tag{5.76}$$

which is very close to the expected value of one for an allpass network. The diagonal elements of the input **K** and output **W** covariance matrices are given by

$$K_{11} = 0.996832 \tag{5.77a}$$
$$K_{22} = 0.992036 \tag{5.77b}$$
$$K_{33} = 0.990928 \tag{5.77c}$$
$$K_{44} = 0.915818 \tag{5.77d}$$

$$W_{11} = 0.997825 \tag{5.78a}$$
$$W_{22} = 0.991153 \tag{5.78b}$$
$$W_{33} = 0.991821 \tag{5.78c}$$
$$W_{44} = 0.915017 \tag{5.78d}$$

which are close to the ideal value of one for a scaled system.

The values of the passband and stopband frequencies after the normalization are

$$F_p^N = 0.0239 \tag{5.79}$$
$$F_s^N = 0.0271 \tag{5.80}$$

The fullband characteristic shown in Figure 5.9 meets the stopband design specification as the quantized system and theoretical plots are vertually identical. This is a result of the extremely large integer denominator used in the quantized scaled scattering matrix needed to pass the frequency specifications.

Figure 5.10 shows the passband characteristic. The design clearly meets the passband design specification as the quantized system and theoretical plots are identical. Again, this is a result of the large integer denominator of the quantized scaled scattering matrix.

The characteristic is Cauer in nature. The maximum deviation in the passband is 0.002 dB, while in the stopband the maximum deviation is 1.7 dB, showing the passband insensitivity. The deviation at infinite frequency is 0.006 dB. From the passband plot, it is clear that the large integer denominator needed was caused by the passband specification.

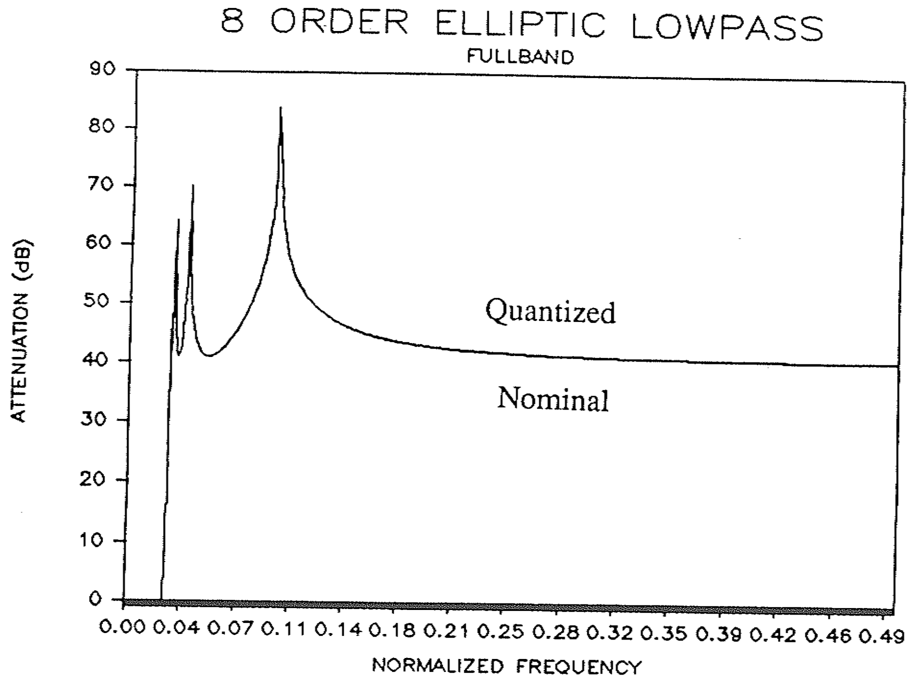

Figure 5.9: Fullband characteristic of a 8th order Cauer lowpass filter.

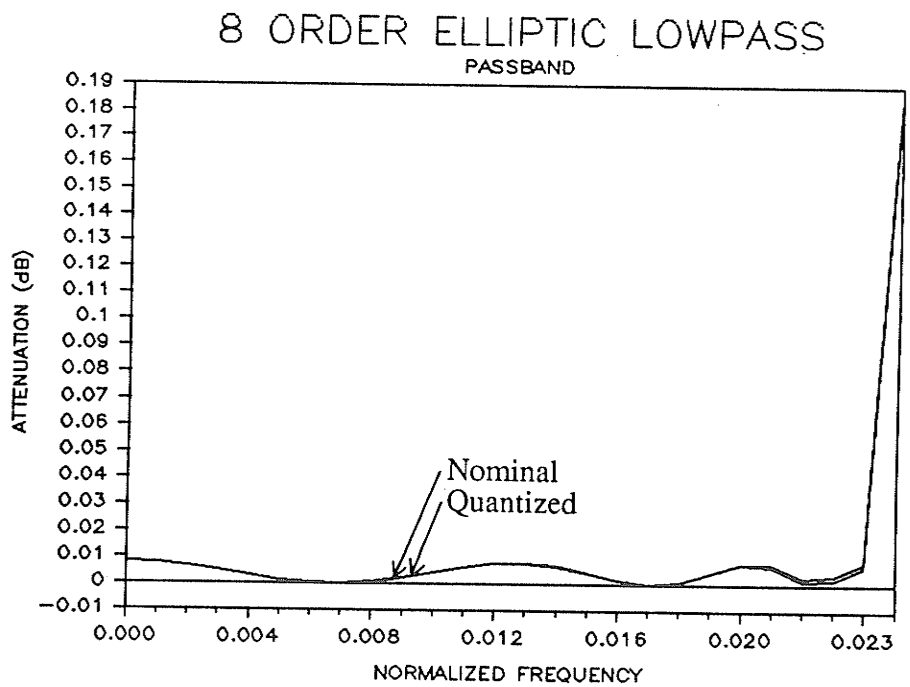

Figure 5.10: Passband characteristic of a 8th order Cauer lowpass filter.

5.6 Example #6: Alternate Representation of Example #4

An alternative representation of the quantized scaled scattering matrix of Example 4 can be derived by letting the denominator equal 64 instead of 256. This design will not meet the design specifications, however this example shows the low sensitivity of the network in the passband to the denominator of the quantized scattering matrix.

The transfer function and the complex allpass function used to derive the CWD network will not change, however, the new scattering matrix is given by

$$S = \frac{1}{64}\begin{bmatrix} 4+j40 & 42-j28 & 0 \\ 31-j33 & 35+j9 & 7+j27 \\ 19+j10 & 1+j17 & 49-j30 \end{bmatrix} \qquad (5.81)$$

The new L_2 norm of the CWD one-port complex allpass network represented by the above is

$$L_2 = 0.997070 \qquad (5.82)$$

which is very close to the expected value of one for an allpass network. The diagonal elements of the new input **K** and output **W** covariance matrices are given by

$$K_{11} = 0.996583 \qquad (5.83a)$$

$$K_{22} = 0.957284 \qquad (5.83b)$$

$$W_{11} = 1.016334 \qquad (5.84a)$$

$$W_{22} = 0.980178 \qquad (5.84b)$$

which are close to the ideal value of one for a scaled system.

The frequency response showing the attenuation of the new network represented by the new quantized complex scaled scattering matrix in decibels is shown in Figure 5.11, where the frequencies have been normalized to the range $0 \le \omega \le 1/2$. The values of the passband and stopband frequencies after the normalization are the same as the earlier case. The passband of this characteristic is shown in Figure 5.12

Clearly, the characteristic has deviated more in the stopband than the passband as expected from the arguement of low passband sensitivity. The magnitude of the maximum deviation in the passband is approximately 0.25 dB, while in the stopband it is approximately 2.5 dB. The characteristic met the design specifications in the passband, but failed in the stopband. Notice also that the stopband attenuation peaks are not as pronounced showing that the filter given by the above scattering matrix is not truly Cauer.

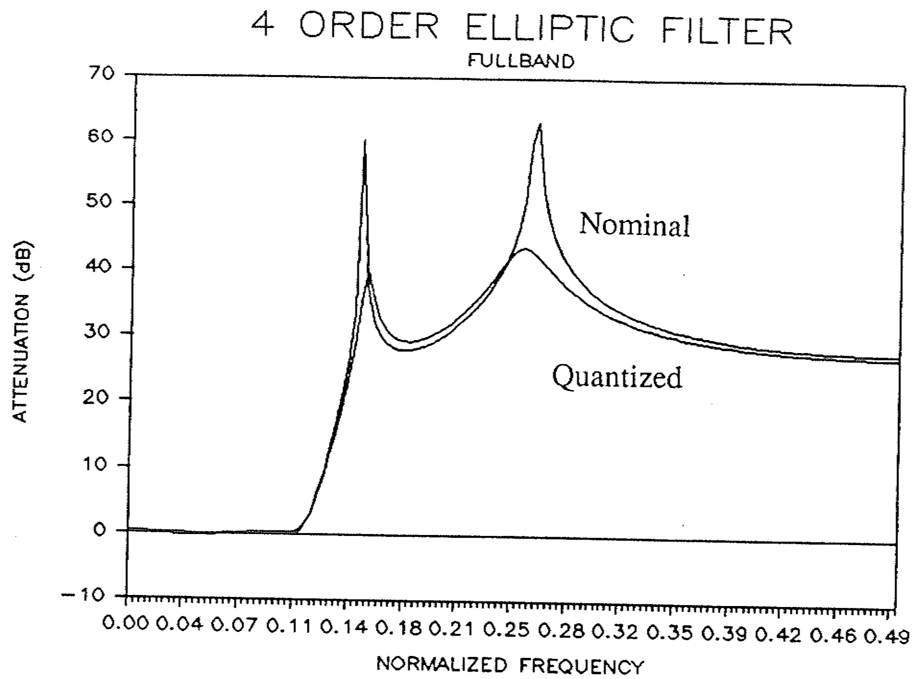

Figure 5.11: Fullband charact. of the new representation of the 4th order Cauer filter.

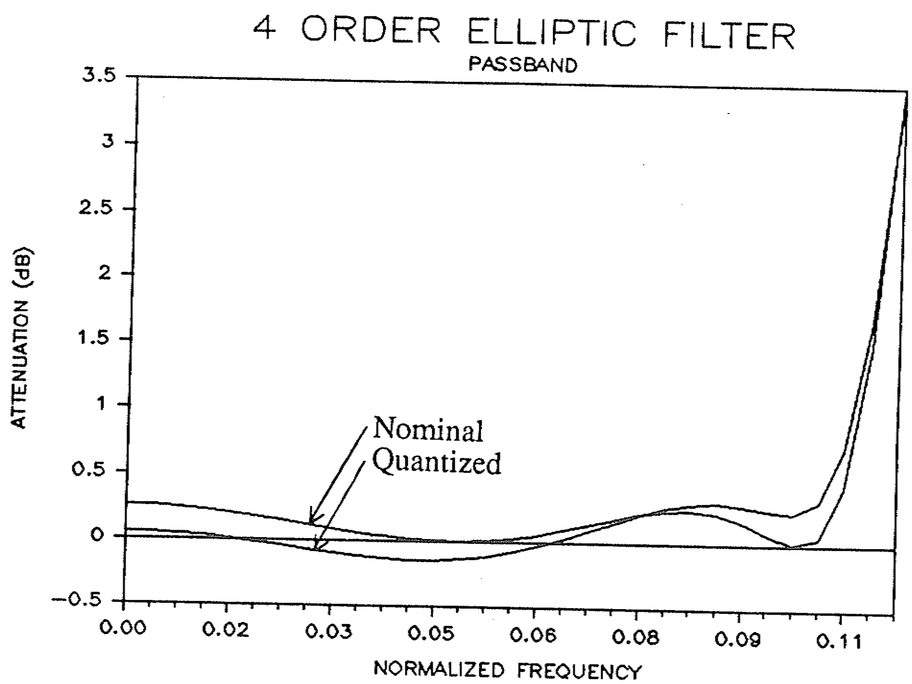

Figure 5.12: Passband charact. of the new representation of the 4th order Cauer filter.

CHAPTER VI

CONCLUSION

This thesis has developed a new theory for complex wave digital filters allowing the realization of general complex reference circuits without alteration. A comparison with the extension of real wave digital filters to the complex domain as proposed by Fettweis [26] was made, and the theory in this thesis was found to be more general. All of the theory of complex wave digital filters introduced in this thesis reduces to the known theory of real wave digital filters if all quantities are restricted to be real.

All quantities within a wave digital network are allowed to be complex. The port reference impedances are defined to be complex constants, containing both positive resistance and constant reactance. The incident and reflected wave variables are redefined giving a new form of the reflection coefficient. The low sensitivity in the passband to variations in a complex parameter of a complex wave digital network was verified. Pseudopassivity was defined and conditions were given to ensure pseudopassivity. One port equivalences were made between the complex wave digital and analog domains and the following was found:

1. A complex multiplier in the wave digital domain corresponds to a resistance in series with a constant reactance in the analog domain. Furthermore, a unimodular multiplier corresponds to a pure constant reactance in the analog domain.
2. A delay with a complex port reference impedance in the wave digital domain corresponds to a real capacitor in series with a pure constant reactance in the analog domain.
3. A delay in series with an inversion with a complex port reference impedance in the wave digital domain corresponds to a real inductor in series with a pure constant reactance in the analog domain.
4. A wave source with a complex port reference impedance in the wave digital domain corresponds to a voltage source in series with a resistance and a constant reactance in the analog domain.

The criteria for the interconnection of wave ports was found to be the condition that the reference impedance of one port must be the complex conjugate of the other. The complex two-port adaptor was derived and the parameter was found to be bounded by one in magnitude. The complex three-port circulator and the complex transformer were given and found to be pseudolossless. The complex scattering matrix was scaled with a linear non-singular complex transformation matrix and the transfer function remained invariant after the transformation. The zero-input and forced response stability of the complex wave digital network is guaranteed by pseudopassivity and incremental pseudopassivity as for the real case. Conditions on the complex non-linear overflow and underflow operators were given, including the condition that the norm of each operator must be bounded by one.

The realization of even-order classical filters with real inputs and outputs was attained with a single complex one-port allpass network containing complex two-port adaptors, complex three-port circulators and delays. The input-output port reference impedance was chosen to be real allowing the one-port to be externally viewed as a real network. The scattering matrix describing the non-dynamic sub-network of the allpass network is sparse and highly structured as it is composed of four block sub-matrix types. Amplitude scaling of the filter to achieve maximum dynamic range for white Gaussian random signals was accomplished with a real scaling transformation. The scattering matrix was quantized, i.e. represented in finite binary form with a common denominator, by magnitude truncation of the real and imaginary parts of the scaled scattering matrix.

A computer program was developed on an IBM PC/XT in a version of FORTRAN-77 which generates a complex quantized scattering matrix solution for a set of design specifications. The filter types include even-order lowpass Butterworth, Chebyshev and Cauer (Elliptic) filters. Six examples showing the satisfactory performance of the program and the validity of the theory developed were illustrated and discussed. The complex wave digital filters designed satisfy the lowpass specifications, are strictly pseudopassive, are amplitude scaled for Gaussian excitations and can be embedded in a strictly real wave digital network as a building block. Under the recursive operation of the digital filter, the response from the desired transfer function is the real part of the output.

APPENDIX A

DESIGN SPECIFICATION INPUT FILE CP.SPC

Example:

A Cauer (Elliptic) lowpass filter with normalized passband and stopband frequencies 0.366 and 0.937 rads., respectively, with a passband ripple of 0.3 dB and a stopband attenuation of 50.0 dB.

Text filte CP.SPC:

ELLIPTIC	Field 1
0000.366000000000 0000.937000000000	Field 2 Field 3
0000.300000000000 0050.000000000000	Field 4 Field 5

Data Types:

Field 1:	TYPE:	CHARACTER*12, FORMAT A12	Filter Type
Field 2:	FP:	REAL*8, FORMAT F17.12	Passband frequency (rads)
Field 3:	FS:	REAL*8, FORMAT F17.12	Stopband frequency (rads)
Field 4:	AP:	REAL*8, FORMAT F17.12	Passband attenuation (dB)
Field 5:	AS:	REAL*8, FORMAT F17.12	Stopband attenuation (dB)

APPENDIX B

PROGRAM INITIALIZATION INPUT FILE CP.INI

Example:

- do not set the minimum denominator of the quantized scaled scattering matrix
- if the minimum denominator of the quantized scaled scattering matrix is set, let the value be 65536
- do not fix the filter order
- if the filter order is fixed, set it equal to 8
- generate the frequency response of the floating-point representation of the scaled scattering matrix
- maximum denominator of the quantized scattering matrix set at 262144
- if no solution is reached with the maximum denominator, generate a solution with a denominator of 131072
- decrement the passband ripple while finding the optimum poles if necessary
- save the hermitian matrix $\mathbf{G} - \mathbf{S}^{*T} \mathbf{G} \mathbf{S}$ in the text file CPSP.DAT

Text file CP.INI:

```
F   65536          Field 1, Field 2
F       8          Field 3, Field 4
T                  Field 5
  262144           Field 6
T                  Field 7
  131072           Field 8
T                  Field 9
T                  Field 10
```

Data Types:

Field 1:	FIXDEN:	LOGICAL, FORMAT L1	Fix the minimum denominator
Field 2:	IDENF:	INTEGER, FORMAT I7	Value of minimum denominator
Field 3:	FIXORD:	LOGICAL, FORMAT L1	Fix the filter order
Field 4:	ORDERF:	INTEGER, FORMAT I2	Value of the filter order
Field 5:	GENFRE:	LOGICAL, FORMAT L1	Generate freq. resp. of floating-point SS
Field 6:	MAXDNM:	INTEGER, FORMAT I10	Maximum quantized SS denominator
Field 7:	FIXFAL:	LOGICAL, FORMAT L1	Generate a solution if freq. resp. failed
Field 8:	FAILDN:	INTEGER, FORMAT I10	Value of denominator if failed
Field 9:	CHKRIP:	LOGICAL, FORMAT L1	Decrement passband ripple to find poles
Field 10:	LOUTSP:	LOGICAL, FORMAT L1	Save the hermitian matrix

APPENDIX C

PROGRAM LISTING

Subprograms:

MAINLINE
DESIGN
RESPON
ACTRES
FREQRE
TESTAT
PRMULT
OUTFUN
ZROOT
PYMULT
OPTROT
INITIA
SPECS
ISSOUT
PRNISS
OUTA
OUTALZ
GENOMG
SSQNT
FINDG
LOSSYT
OUTSP
DCRISS
DCRVAR
POSDEF
FINDA
TESTG
ALPHAZ
SSCATT
ALOWES
ALOWOS
AHIGHS
AIS
SSOUT
TEST
TRISFR
REALSS
GENTRI
BTWHRT
CHEBRT
ARCOSH
ARSINH
INCBRT
ELTCRT
FF3
FF4
L2NORM
MULTMV
GENWKD
FINDGH

```
      PROGRAM CP
C
C  THIS PROGRAM DESIGNS A PSEUDOPASSIVE COMPLEX WAVE DIGITAL
C  FILTER WHICH MEETS THE FREQ. SPECS.  THE DESIGNED FILTER IS IN
C  THE FORM OF A COMPLEX QUANTIZED SCATTERING MATRIX OR
C  STATE-VARIABLE MATRIX FOUND IN THE TEXT FILE ISS.DAT
C
      COMPLEX*16 CS(40),CZ(40),AZ(40),BETA,P(40),D(40)
      COMPLEX*16 ALPHA(40),Z(40),Z1,SS(11,11),PA(40),DA(40)
      INTEGER ORDER,ISSR(11,11),ISSI(11,11),IDEN
      LOGICAL FAIL,LOSSY
      CHARACTER*12 TYPE
      REAL*8 FP,FS,AP,AS,G(40)
      OPEN(UNIT=5,FILE='CP.DAT',STATUS='NEW')
C
      CALL SPECS(TYPE,FP,FS,AP,AS)
      CALL DESIGN(TYPE,FP,FS,AP,AS,ORDER,ISSR,ISSI,IDEN,
     *  LOSSY,G,FAIL)
C
      CLOSE(UNIT=5)
      STOP
      END

      SUBROUTINE DESIGN(TYPE,FP,FS,AP,AS,ORDER,ISSR,ISSI,IDEN,
     * LOSSY,G,FAIL)
C
C  A FILTER DESIGN CAN BE ACCOMPLISHED IN ONE OF TWO WAYS, EITHER
C  GIVE THE FREQUENCY SPECS, OR GIVE THE ORDER AND THE PASSBAND
C  SPECS
C        1. DESIGN A PSEUDOPASSIVE LOWPASS DIGITAL FILTER OF A
C           SPECIFIED TYPE THAT MEETS THE DESIGN SPECIFICATIONS
C           GIVEN BY THE PASSBAND AND STOPBAND FREQS. FP, FS AND
C           THE PASSBAND RIPPLE AP, AND THE
C           STOPBAND ATTENUATION AS THE ORDER OF THE FILTER IS
C           THEN DETERMINED.
C        2. DESIGN A PSEUODPASSIVE LOWPASS DIGITAL FILTER OF A
C           SPECIFIED TYPE AND ORDER BY GIVING THE PASSBAND AND
C           STOPBAND FREQS, FP, FS AND THE PASSBAND RIPPLE
C
C  THE QUANTIZED SCATTERING MATRIX IS FOUND IN ISS.DAT
C  THE FREQ RESPONSE OF THE QUANTIZED SOLUTION IS GENERATED ALONG
C  WITH THE L2 NORM
C
C  RUN OPTIONS: (SET IN FILE CP.INI)
C  1. THE DIGITAL FILTER ORDER CAN BE FIXED WITH FIXORD=TRUE AND THE
C     ORDER IS EQUAL TO ORDERF
C  2. THE DENOMINATOR OF THE  COMPLEX QUANTIZED SCATTERING MATRIX CA
C     BE SET AT AN INITIAL OR STARTING VALUE WITH FIXDEN=TRUE WITH
C     THE VALUE IDENF
C  3. IF THE SPECS CANNOT BE SATISFIED WITH MAXDNM, A SOLUTION CAN BE
C     GENERATED WITH FIXFAL=TRUE WITH A DENOMINATOR FAILDN
```

```
C   4. THE PASSBAND RIPPLE CAN BE DECREASED DURING THE DESIGN PROCESS
C      IN ORDER TO AID IN MEETING THE FREQ SPECS WITH CHKRIP=TRUE
C      (RECOMMENDED)
C   5. THE COMPLEX MATRIX (G - S*T G S) USED FOR CHECKING THE
C      PSEUDOPASSIVITY OF THE DIGITAL FILTER CAN BE STORED IN FILE
C      CPSP.DAT WITH LOUTSP=TRUE
C   6. THE ACTUAL FREQUENCY RESPONSE OF THE INFINITE PRECISION
C      SCATTERING MATRIX AS COMPARED TO THE TRANSFER FUNCTION RESPONSI
C      CAN BE GENERATED IF DESIRED BY SETTING GENFRE=TRUE
C
C                       TYPE - THE TYPE OF THE LOWPASS DIGITAL FILTER,
C                              EITHER BUTTERWORTH, CHEBYSHEV OR ELLIPTIC
C                       FP - IS THE UPPER LIMIT OF THE PASSBAND (IN
C                            RADIANS).
C                       FS - IS THE LOWER LIMIT OF THE STOPBAND (IN
C                            RADIANS).
C                       AP - IS THE MAX RIPPLE IN THE PASSBAND (IN DB).
C                       AS - IS THE MIN ATTEN IN THE STOPBAND (IN DB).
C                       ORDER - IS THE ORDER OF THE FILTER.
C                       ISSR,ISSI - ARE THE REAL AND IMAG STATE VARIABLE
C                                   MATRICIES WITH RATIONAL ELEMENTS.
C                       IDEN - IS THE DENOMINATOR OF THE QUANTIZED
C                              SCATTERING MATRIX.
C                       LOSSY - IS A LOGICAL VARIABLE THAT IS TRUE IF
C                               THE FILTER IS PSEUDOPASSIVE.
C                       G - IS THE DIAGONAL CONDUCTANCE REAL MATRIX.
C                       FAIL - IS A LOGICAL VARIABLE THAT IS TRUE IF A
C                              SOLUTION CANNOT BE FOUND.
C
      INTEGER ORDER,ORDERF,ISSR(11,11),ISSI(11,11),IDEN,NUMF,NN
      INTEGER IDENF,ANS,KCOUNT,IUNIT,BANDF,FAILDN,NORDER
      CHARACTER TYPE*12,FILER*10,FILEARR(4)*10
      LOGICAL FAIL,FAIL1,PASS,LOSSY,PASSED,FIXDEN,FIXORD,GENFRE
      LOGICAL FIXFAL,FFAIL,CHKRIP,LOUTSP
      REAL*8 FP,FS,AP,AS,FCENTR,OMG(512),NORM,KOUT,SUM,G(40)
      REAL*8 WC,WA,APR,ATTNG,ATTNSS,WDIAG(11),KDIAG(11),AGAIN
      REAL*8 TRUEAS,DF,FPNEW,FSNEW,OMEGA,ASR
      COMPLEX*16 CZ(40),CS(40),CSN(40),BETA,P(40),D(40),AZ(40)
      COMPLEX*16 PA(40),DA(40),Z1,ALPHA(40),Z(40),SS(11,11)
      COMPLEX*16 Q(40),CZN(40),BETAN
      CALL INITIA(FIXORD,ORDERF,FIXDEN,IDENF,GENFRE,FILEARR,MAXDNM,
     *  FIXFAL,FAILDN,CHKRIP,LOUTSP)
      NUMF=128
      FFAIL=.FALSE.
      FAIL=.TRUE.
      FAIL1=.TRUE.
      APR=AP
      ASR=AS
      IF(FIXORD) ASR=0.0D0
      CALL OPTROT(TYPE,FP,FS,AP,AS,ORDER,NORDER,CS,CSN,CZ,CZN,FAIL1,
     *    FIXORD,ORDERF,TRUEAS,CHKRIP)
      WRITE (*,*) 'FILTER TYPE=',TYPE,' ORDER=',ORDER
```

```
      WRITE (*,30) FP,FS
      WRITE (*,31) AP,AS,TRUEAS
      WRITE (5,*) 'FILTER TYPE=',TYPE,' ORDER=',ORDER
      WRITE (5,30) FP,FS
      WRITE (5,31) AP,AS,TRUEAS
30    FORMAT(' ','PASSBAND EDGE=',F10.6,' STOPBAND EDGE=',F10.6)
31    FORMAT(' ','PASSBAND RIPPLE=',F8.4,' DESIRED STOPBAND ATTEN=',
     *  F8.4,' ACTUAL ATTEN=',F8.4)
      IF(FAIL1) GO TO 1
      CALL FINDGH(TYPE,FP,FS,AP,CS,CSN,CZ,CZN,ORDER,NORDER,BETA,AZ,
     *P,Q,D)
      CALL OUTFUN(ORDER,NORDER,BETA,CS,CSN,CZ,CZN,AZ,P,Q,D)
      CALL FINDA(AZ,ORDER,BETA,PA,DA)
      CALL OUTA(ORDER,PA,DA)
      CALL TESTG(PA,DA,P,D,ORDER,PASSED)
      IF(.NOT.PASSED) THEN
         WRITE (5,*) 'ALLPASS FUNCTION INCORRECT'
         GO TO 1
      ENDIF
      Z1=DCMPLX(1.0D0,0.0D0)
      CALL ALPHAZ(DA,PA,BETA,ORDER,Z1,ALPHA,Z)
      CALL OUTALZ(ORDER,Z,ALPHA)
      CALL FINDG(ORDER,Z,G)
      WRITE (*,*) 'GENERATE THE SCATTERING MATRIX'
      CALL SSCATT(ORDER,ALPHA,SS)
      CALL SSOUT(ORDER,SS)
      IF(GENFRE) CALL ACTRES(ORDER,SS,P,D)
C
      FCENTR=(FS-FP)/2.0+FP
      CALL GENOMG(FP,FS,NUMF,OMG)
      IF(FIXDEN) THEN
       IDEN=IDENF
      ELSE
       II1=ORDER+6
       II2=II1/2
       IDEN=2**II2
      ENDIF
      KCOUNT=0
      WRITE (*,*) 'QUANTIZE THE SCATTERING MATRIX'
5     CONTINUE
      CALL SSQNT(ORDER,SS,IDEN,ISSR,ISSI)
      ICOUNT=0
3     CONTINUE
      CALL LOSSYT(ORDER,G,ISSR,ISSI,IDEN,LOSSY,LOUTSP)
      ICOUNT=ICOUNT+1
      IF(ICOUNT.GE.4) GO TO 2
      IF(LOSSY) GO TO 2
       WRITE (*,*) '      DECREMENTING SCATTERING MATRIX'
       WRITE (5,*) '      DECREMENTING SCATTERING MATRIX'
      CALL DCRISS(ORDER,ISSR,ISSI)
      GO TO 3
2     CONTINUE
```

```
      IF(FFAIL) GO TO 7
      WRITE (*,*) 'TEST THE FREQUENCY RESPONSE'
      CALL TEST(ORDER,ISSR,ISSI,IDEN,NUMF,OMG,FCENTR,APR,ASR,PASS,NN,
     *   BANDF)
      IF(PASS) GO TO 4
       IF(BANDF.EQ.1) THEN
         WRITE (*,222) IDEN
         WRITE (5,222) IDEN
       ELSE IF(BANDF.EQ.-1) THEN
         WRITE (*,223) IDEN
         WRITE (5,223) IDEN
       ENDIF
222   FORMAT(' ',5X,'FREQ TEST FAILED WITH IDEN=',I7,' IN PASSBAND')
223   FORMAT(' ',5X,'FREQ TEST FAILED WITH IDEN=',I7,' IN STOPBAND')
      IDEN=IDEN*2
6     CONTINUE
      IF(IDEN.GT.MAXDNM) THEN
        IF(PASS) THEN
          GO TO 7
        ELSE
          FFAIL=.TRUE.
          IF(FIXFAL) THEN
            IDEN=FAILDN
      GO TO 5
          ENDIF
          WRITE (5,*)
          WRITE (5,*)
          WRITE (5,*) '*** SOLUTION DID NOT PASS THE FREQUENCY SPECS'
          IF(BANDF.EQ.-1) THEN
            WRITE (5,*) '     FAILED IN STOPBAND AT FREQ ',OMG(NN)
          ELSEIF (BANDF.EQ.1) THEN
            WRITE (5,*) '     FAILED IN PASSBAND AT FREQ ',OMG(NN)
          ELSE
            WRITE (5,*) '     DID NOT FAIL NN=',NN
          ENDIF
          GO TO 7
        ENDIF
      ENDIF
      GO TO 5
4     FAIL=.FALSE.
      OMEGA=3.1D0
      CALL TESTAT(ORDER,P,D,IDEN,ISSR,ISSI,OMEGA,ATTNG,ATTNSS)
      WRITE (*,10) OMEGA,ATTNG,ATTNSS
      WRITE (5,10) OMEGA,ATTNG,ATTNSS
10    FORMAT(' ','FREQ SPECT AT OMEGA=',F6.3,' ACT ATTN=',F10.5,
     *   ' DESIGNED ATTN=',F10.5)
      WRITE (*,*) 'INCREASE DENOMINATOR?'
      READ (*,*) ANS
      IF(ANS.EQ.1) THEN
        KCOUNT=KCOUNT+1
        IF(KCOUNT.GT.4) GO TO 8
      WRITE (*,*) 'GENERATE FREQ SPECTRUM OF SCALED SCATTERING
```

```
     *   MATRIX?'
         READ (*,*) ANS
         IF(ANS.EQ.1) CALL RESPON(ORDER,IDEN,ISSR,ISSI,P,D,KCOUNT,
     *                  FILEARR(KCOUNT))
8        IDEN=IDEN*2
         GO TO 6
       ENDIF
7      CONTINUE
       IF(ICOUNT.EQ.1) THEN
         WRITE (5,*) 'SOLUTION IS LOSSY WITHOUT DECREMENTATION'
       ELSE
         ICOUNT=ICOUNT-1
         WRITE (5,*) 'SOLUTION WAS DECREMENTED ',ICOUNT,' TIMES'
       ENDIF
       CALL ISSOUT(ORDER,IDEN,ISSR,ISSI)
       CALL PRNISS(ORDER,IDEN,ISSR,ISSI)
       CALL GENWKD(ORDER,ISSR,ISSI,IDEN,KDIAG,WDIAG,AGAIN)
       CALL L2NORM(ORDER,SS,NORM)
       M=ORDER/2
       WRITE (5,*)
       WRITE (5,*) 'NOISE AMPLITUDE BOUND=',AGAIN
       WRITE (5,*)
       WRITE (5,*) 'L2 NORM=',NORM
       WRITE (5,*)
       WRITE (5,*) 'K DIAGONAL MATRIX ELEMENTS'
       DO 20 I=1,M
         WRITE (5,25) I,KDIAG(I)
20     CONTINUE
       WRITE (5,*)
       WRITE (5,*) 'W DIAGONAL MATRIX ELEMENTS'
       DO 21 I=1,M
         WRITE (5,26) I,WDIAG(I)
21     CONTINUE
25     FORMAT(' ','I=',I3,' KDIAG=',F15.10)
26     FORMAT(' ','I=',I3,' WDIAG=',F15.10)
       FILER='SPECT0.DAT'
       IUNIT=7
       CALL RESPON(ORDER,IDEN,ISSR,ISSI,P,D,IUNIT,FILER)
1      RETURN
       END

       SUBROUTINE RESPON(ORDER,IDEN,ISSR,ISSI,P,D,IUNIT,FILER)
C
C   SUBROUTINE TO CALCULATE THE FREQ. RESPONSE OF THE QUANTIZED CWDF
C   AND THE RESPONSE OF THE ACTUAL FILTER FOR COMPARISON
C
C   THE OUTPUT FILE CONTAINS THE
C                    FREQ.
C                    QUANTIZED CWDF: MAG, DB, PHASE, GROUP DELAY
C                    ACTUAL FILTER: MAG, DB, PHASE, GROUP DELAY
C
       INTEGER ORDER,M,NUMF,MM,J,I,K1,K2,N,NN
```

```
      INTEGER  IDEN,ISSR(11,11),ISSI(11,11),IUNIT
      COMPLEX*16  SS(11,11),X1,X2,ST(11,11)
      COMPLEX*16  SSC(11,11),STC(11,11)
      COMPLEX*16  T1,T2,P(40),D(40),SUMP,SUMD
      REAL*8  PI,OME(128),XX,DEL,MT1,MT2,DEN,MT3,MT4
      REAL*8  PH1,PH2,DELPHI,PHILS1,PHILS2,GD1,GD2
      CHARACTER  FILER*10
      OPEN(UNIT=IUNIT,FILE=FILER,STATUS='NEW')
      DATA  PI/3.141592653589793D0/
      NUMF=128
      DEL=PI/DBLE(NUMF)
      M=ORDER/2+1
      MM=ORDER/2
      N=ORDER+1
      NN=ORDER
      DEN=DBLE(IDEN)
      PHILS1=0.0D0
      PHILS2=0.0D0
      DO  10  I=1,M
         DO  10  J=1,M
          SS(I,J)=DCMPLX(DBLE(ISSR(I,J))/DEN,DBLE(ISSI(I,J))/DEN)
10        SSC(I,J)=DCONJG(SS(I,J))
      DO  1  I=1,NUMF
          XX=DBLE(I-1)
          OME(I)=XX*DEL
          X1=DCMPLX(0.0D0,OME(I))
          X2=CDEXP(X1)
          CALL  FREQRE(ORDER,SS,ST,X2)
          CALL  FREQRE(ORDER,SSC,STC,X2)
          T1=(ST(M,M)+STC(M,M))/2.0D0
          MT1=CDABS(T1)
          MT2=-20.0D0*DLOG10(MT1+1.0D-15)
          PH1=DATAN2(DIMAG(T1),DREAL(T1))
          DELPHI=PH1-PHILS1
          PHILS1=PH1
          IF(DELPHI.GT.PI)  DELPHI=DELPHI-2.0D0*PI
          IF(DELPHI.LT.-PI)  DELPHI=DELPHI+2.0D0*PI
          GD1=-1.0D0*DELPHI/DEL
C
          SUMD=D(N)
          SUMP=P(N)
          DO  11  J=1,NN
              K=N-J
              SUMP=X2*SUMP+P(K)
              SUMD=X2*SUMD+D(K)
11        CONTINUE
          T2=SUMP/SUMD
          MT3=CDABS(T2)
          MT4=-20.0D0*DLOG10(MT3+1.0D-15)
          PH2=DATAN2(DIMAG(T2),DREAL(T2))
          DELPHI=PH2-PHILS2
          PHILS2=PH2
```

```
         IF(DELPHI.GT.PI) DELPHI=DELPHI-2.0D0*PI
         IF(DELPHI.LT.-PI) DELPHI=DELPHI+2.0D0*PI
         GD2=-1.0D0*DELPHI/DEL
         WRITE (*,*) I
         PH1=180.0D0*PH1/PI
         PH2=180.0D0*PH2/PI
         WRITE (IUNIT,5) OME(I),MT1,MT2,PH1,GD1,MT3,MT4,PH2,GD2
1     CONTINUE
5     FORMAT(F6.4,1X,F8.4,1X,F8.4,1X,F8.2,1X,F8.3,1X,F8.4,1X,F8.4,
     *   1X,F8.2,1X,F8.3)
      CLOSE(UNIT=IUNIT)
      RETURN
      END

      SUBROUTINE ACTRES(ORDER,SS,P,D)
C
C    SUBROUTINE TO CALCULATE THE FREQ. RESPONSE OF THE INFINITE
C    PRECISION CWDF AND THE RESPONSE OF THE ACTUAL FILTER FOR
C    COMPARISON
C
C    THE OUTPUT FILE CONTAINS THE
C                 FREQ.
C                 INFINITE PRECISION CWDF: MAG, DB, PHASE, GROUP DELAY
C                 ACTUAL FILTER: MAG, DB, PHASE, GROUP DELAY
C
      INTEGER ORDER,M,NUMF,MM,J,I,K1,K2,N,NN
      COMPLEX*16 SS(11,11),X1,X2,ST(11,11)
      COMPLEX*16 SSC(11,11),STC(11,11)
      COMPLEX*16 T1,T2,P(40),D(40),SUMP,SUMD
      REAL*8 PI,OME(128),XX,DEL,MT1,MT2,DEN,MT3,MT4
      REAL*8 PH1,PH2,DELPHI,PHILS1,PHILS2,GD1,GD2
      OPEN(UNIT=12,FILE='ACTSPECT.DAT',STATUS='NEW')
      DATA PI/3.141592653589793D0/
      NUMF=128
      DEL=PI/DBLE(NUMF)
      M=ORDER/2+1
      MM=ORDER/2
      N=ORDER+1
      NN=ORDER
      DO 10 I=1,M
         DO 10 J=1,M
10         SSC(I,J)=DCONJG(SS(I,J))
      DO 1 I=1,NUMF
         XX=DBLE(I-1)
         OME(I)=XX*DEL
         X1=DCMPLX(0.0D0,OME(I))
         X2=CDEXP(X1)
         CALL FREQRE(ORDER,SS,ST,X2)
         CALL FREQRE(ORDER,SSC,STC,X2)
         T1=(ST(M,M)+STC(M,M))/2.0D0
         MT1=CDABS(T1)
         MT2=-20.0D0*DLOG10(MT1+1.0D-15)
```

```
         PH1=DATAN2(DIMAG(T1),DREAL(T1))
         DELPHI=PH1-PHILS1
         PHILS1=PH1
         IF(DELPHI.GT.PI)  DELPHI=DELPHI-2.0D0*PI
         IF(DELPHI.LT.-PI)  DELPHI=DELPHI+2.0D0*PI
         GD1=-1.0D0*DELPHI/DEL
C
         SUMD=D(N)
         SUMP=P(N)
         DO  11  J=1,NN
             K=N-J
             SUMP=X2*SUMP+P(K)
             SUMD=X2*SUMD+D(K)
11       CONTINUE
         T2=SUMP/SUMD
         MT3=CDABS(T2)
         MT4=-20.0D0*DLOG10(MT3+1.0D-15)
         PH2=DATAN2(DIMAG(T2),DREAL(T2))
         DELPHI=PH2-PHILS2
         PHILS2=PH2
         IF(DELPHI.GT.PI)  DELPHI=DELPHI-2.0D0*PI
         IF(DELPHI.LT.-PI)  DELPHI=DELPHI+2.0D0*PI
         GD2=-1.0D0*DELPHI/DEL
         WRITE  (*,*)  I
         PH1=180.0D0*PH1/PI
         PH2=180.0D0*PH2/PI
         WRITE  (12,5)  OME(I),MT1,MT2,PH1,GD1,MT3,MT4,PH2,GD2
1     CONTINUE
5     FORMAT(F6.4,1X,F8.4,1X,F8.4,1X,F8.2,1X,F8.3,1X,F8.4,1X,F8.4,
     *   1X,F8.2,1X,F8.3)
      CLOSE(UNIT=12)
      RETURN
      END

      SUBROUTINE  FREQRE(ORDER,SS,ST,X2)
C
C   A  SUBROUTINE  TO  GENERATE  THE  MATRIX  FOR  FINDING  THE  FREQ  RESPONSE
C   AT  A  CERTAIN  FREQ  AND  THEN  UPPER  TRIANGULARIZING  THE  MATRIX
C
      INTEGER  ORDER,L,LL,MM,M
      COMPLEX*16  SS(11,11),ST(11,11),X,X2
      M=ORDER/2+1
      MM=ORDER/2
      ST(M,M)=SS(1,1)
      DO  3  L=1,MM
          ST(M,L)=SS(1,L+1)
          ST(L,M)=-1.0D0*SS(L+1,1)
          DO  4  LL=1,MM
              X=-1.0D0*SS(L+1,LL+1)
              IF(L.EQ.LL)  THEN
                  ST(L,LL)=X+X2
              ELSE
```

```
               ST(L,LL)=X
            ENDIF
4         CONTINUE
3      CONTINUE
       CALL GENTRI(M,ST)
       RETURN
       END

       SUBROUTINE TESTAT(ORDER,P,D,IDEN,ISSR,ISSI,OMEGA,ATTNG,ATTNSS)
C
C     SUBROUTINE TO TEST THE ATTENUATION OF THE QUANTIZED SCATTERING
C     MATRIX AS COMPARED WITH THE ATTENUATION OF THE ACTUAL TRANSFER
C     FUNCTION AT A CERTAIN FREQUENCY (LARGE)
C
       INTEGER ORDER,ISSR(11,11),ISSI(11,11),IDEN
       COMPLEX*16 P(40),D(40),X1,X2,SUMD,SUMP,T1,T2
       COMPLEX*16 SS(11,11),SSC(11,11),ST(11,11),STC(11,11)
       REAL*8 ATTNG,ATTNSS,OMEGA,DEN,MT1,MT2,MT3,MT4
       X1=DCMPLX(0.0D0,OMEGA)
       X2=CDEXP(X1)
       M=ORDER/2+1
       MM=ORDER/2
       N=ORDER+1
       NN=ORDER
       DEN=DBLE(IDEN)
       DO 10 I=1,M
          DO 10 J=1,M
            SS(I,J)=DCMPLX(DBLE(ISSR(I,J))/DEN,DBLE(ISSI(I,J))/DEN)
10          SSC(I,J)=DCONJG(SS(I,J))
       CALL FREQRE(ORDER,SS,ST,X2)
       CALL FREQRE(ORDER,SSC,STC,X2)
       T1=(ST(M,M)+STC(M,M))/2.0D0
       MT1=CDABS(T1)
       MT2=-20.0D0*DLOG10(MT1+1.0D-15)
C
       SUMD=D(N)
       SUMP=P(N)
       DO 11 J=1,NN
            K=N-J
            SUMP=X2*SUMP+P(K)
            SUMD=X2*SUMD+D(K)
11     CONTINUE
       T2=SUMP/SUMD
       MT3=CDABS(T2)
       MT4=-20.0D0*DLOG10(MT3+1.0D-15)
       ATTNG=MT4
       ATTNSS=MT2
       RETURN
       END

       SUBROUTINE PRMULT(CZ,ORDER,POLY1)
C
```

```
C     A SUBROUTINE TO MULTIPLY OUT A PRODUCT FORM POLYNOMIAL OF THE
C     FORM
C                POLY1 = (Z + CZ1)(Z + CZ2)...(Z + CZN)
C
C     ORDER - NUMBER OF ZEROS (DEGREE OF POLY IS (N+1))
C     CZ - COMPLEX VECTOR OF ZEROS
C     POLY1 - COMPLEX VECTOR OF COEFFICIENTS OF THE POLYNOMIAL WITH
C             ZERO ORDER TERM IN POLY1(1), AND MOST SIGNIFICANT
C             DEGREE COEFFICIENT IS IN POLY1(N+1)
C
      COMPLEX*16 CZ(40),POLY1(40),POLY2(40),TEMP(40)
      COMPLEX*16 ONE,ZERO
      INTEGER ORDER
      ONE=DCMPLX(1.0D0,0.0D0)
      ZERO=DCMPLX(0.0D0,0.0D0)
      DO 1 I=1,(ORDER+1)
         POLY1(I)=ZERO
         POLY2(I)=ZERO
1        TEMP(I)=ZERO
      POLY1(1)=CZ(1)
      POLY1(2)=ONE
      POLY2(2)=ONE
      DO 2 I=2,ORDER
         POLY2(1)=CZ(I)
         CALL PYMULT(POLY1,I,POLY2,2,TEMP,NUM)
         DO 3 J=1,NUM
3           POLY1(J)=TEMP(J)
2     CONTINUE
      RETURN
      END

      SUBROUTINE OUTFUN(ORDER,NORDER,BETA,CS,CSN,CZ,CZN,AZ,P,Q,D)
C
C   A SUBROUTINE TO WRITE OUT THE LAPLACE AND Z-DOMAIN POLES AND
C   ZEROS ALONG WITH THE COMPLEX ALLPASS FUNCTION
C
C     ORDER - ORDER OF THE FILTER (NUMBER OF POLES)
C     NORDER - NUMBER OF ZEROS
C     BETA - UNIMODULAR CONSTANT MULTIPLYING ALLPASS
C     CS,CSN - COMPLEX VECTOR OF LAPLACE DOMAIN POLES AND ZEROS
C     CZ,CZN - COMPLEX VECTOR OF Z-DOMAIN POLES AND ZEROS
C     AZ - COMPLEX VECTOR OF POLES CHOSEN FOR ALLPASS FUNCTION
C     P - COMPLEX VECTOR OF COEFFICIENTS OF NUMBERATOR OF
C             Z-DOMAIN TRANSFER FUNCTION G WITH ZEROETH ORDER TERM IN
C             P(1)
C     Q - COMPLEX VECTOR OF COEFFICIENTS OF NUMBERATOR OF
C             Z-DOMAIN SPECTRAL COMPLEMENT OF THE TRANSFER FUNCTION H
C             WITH ZEROETH ORDER TERM IN Q(1)
C     D - COMPLEX VECTOR OF COEFFICIENTS OF DENOMINATOR OF
C             Z-DOMAIN TRANSFER FUNCTION G AND SPECTRAL COMPLEMENT OF
C             THE TRANSFER FUNCTION H WITH ZEROETH ORDER TERM IN Q(1)
C
```

```
      COMPLEX*16  BETA,CS(40),CZ(40),AZ(40),P(40)
      COMPLEX*16  Q(40),D(40),CSN(40),CZN(40)
      REAL*8  PI,REBETA,IMBETA,PHASE
      INTEGER  ORDER,ND2,M
      DATA  PI/3.141592653589793D0/
      REBETA=DREAL(BETA)
      IMBETA=DIMAG(BETA)
      PHASE=180.0D0*DATAN2(IMBETA,REBETA)/PI
C
      ND2=ORDER/2
      M=ORDER+1
      DO  34  I=1,ORDER
         WRITE  (5,60)  I,CS(I)
34    CONTINUE
      WRITE  (5,*)
      DO  35  I=1,ORDER
         WRITE  (5,61)  I,CZ(I)
35    CONTINUE
      WRITE  (5,*)
      DO  38  I=1,NORDER
         WRITE  (5,55)  I,CSN(I)
38    CONTINUE
      WRITE  (5,*)
      DO  39  I=1,NORDER
         WRITE  (5,56)  I,CZN(I)
39    CONTINUE
      WRITE  (5,*)
      DO  37  I=1,ND2
         WRITE  (5,63)  I,AZ(I)
37    CONTINUE
      WRITE  (5,*)
      DO  24  I=1,M
         WRITE  (5,50)  I,P(I)
24    CONTINUE
      WRITE  (5,*)
      DO  25  I=1,M
         WRITE  (5,51)  I,Q(I)
25    CONTINUE
      WRITE  (5,*)
      DO  26  I=1,M
         WRITE  (5,52)  I,D(I)
26    CONTINUE
      WRITE  (5,*)
      WRITE  (5,54)  BETA,CDABS(BETA),PHASE
50    FORMAT('  ','I=',I2,2X,'P     =',2F16.8)
51    FORMAT('  ','I=',I2,2X,'Q     =',2F16.8)
52    FORMAT('  ','I=',I2,2X,'D     =',2F16.8)
60    FORMAT('  ','I=',I2,2X,'CS    =',2F16.8)
61    FORMAT('  ','I=',I2,2X,'CZ    =',2F16.8)
63    FORMAT('  ','I=',I2,2X,'AZ    =',2F16.8)
54    FORMAT('  ','BETA=  ',2F8.6,'  MAG  BETA=  ',F8.6,'  PHASE=  ',F12.4)
55    FORMAT('  ','I=',I2,2X,'CSN  =',2F16.8)
```

```
56    FORMAT(' ','I=',I2,2X,'CZN =',2F16.8)
      RETURN
      END

      SUBROUTINE ZROOT(CS,ORDER,CZ)
C
C     A SUBROUTINE TO APPLY THE BILINEAR TRANSFORM TO THE S-PLANE
C     ZEROS OR POLES MAPPING THEM TO THE Z-DOMAIN
C
C          CS - COMPLEX VECTOR OF POLES IN LAPLACE DOMAIN
C                 ORDER - ORDER OF THE FILTER.
C                 CZ - COMPLEX VECTOR OF THE Z-DOMAIN POLES
C
      COMPLEX*16 CS(40),CZ(40)
      INTEGER MIN,ORDER
      M=ORDER/2
      DO 1 I=1,ORDER
      CZ(I)=((1.0D0,0.0D0)+CS(I))/((1.0D0,0.0D0)-1.0D0*CS(I))
1     CONTINUE
      RETURN
      END

      SUBROUTINE PYMULT(X,NUMX,Y,NUMY,Z,NUMZ)
C
C     A SUBROUTINE TO MULTIPLY TWO COMPLEX POLYNOMIALS Z=X*Y
C
C     X,Y,Z - COMPLEX VECTORS CONTAINING THE POLYNOMIALS
C                    WITH ZEROETH ORDER COEFFICIENTS IN X(1), Y(1),
C                    AND Z(1)
C                NUMX,NUMY,NUMZ - LENGTH OF EACH COMPLEX POLYNOMIAL
C
      COMPLEX*16 X(40),Y(40),Z(40)
      INTEGER NUMX,NUMY,NUMZ
      IF((NUMX.EQ.0).OR.(NUMY.EQ.0)) THEN
         NUMZ=0
      ELSE
         NUMZ=NUMX+NUMY-1
         DO 1 I=1,NUMZ
         Z(I)=(0.0D0,0.0D0)
         J=1
2        K=1+I-J
         IF(K.LE.0) GO TO 1
         IF(J.GT.NUMX) GO TO 1
         IF(K.GT.NUMY) GO TO 3
         Z(I)=Z(I)+X(J)*Y(K)
3        J=J+1
         GO TO 2
1        CONTINUE
      END IF
      RETURN
      END
```

```
      SUBROUTINE OPTROT(TYPE,FP,FS,AP,AS,ORDER,NORDER,CS,CSN,CZ,
     * CZN,FAIL,FIXORD,ORDERF,TRUEAS,CHKRIP)
C
C     A SUBROUTINE TO FIND THE ORDER AND THE Z-DOMAIN
C     POLES AND ZEROS FOR A LOWPASS COMPLEX DIGITAL FILTER THAT
C     SPLITS THE DESIGN MARGIN BETWEEN THE PASSBAND AND THE STOPBAND.
C     THE ORDER IS FIRST FOUND THAT MEETS THE FREQ. SPECS, THEN THE
C     PASSBAND RIPPLE IS DECREASED UNTIL THE DESIGN MARGIN CRITERIA
C     IS SATISFIED (THIS ALLOWS THE QUANTIZED FILTER TO MEET THE
C     FREQ. SPECS)
C
C        TYPE - TYPE OF FILTER (BUTTERWORTH, CHEBYSHEV,
C                        INVCHEBYSHEV,ELLIPTIC)
C                FP - IS THE UPPER LIMIT OF THE PASSBAND (IN
C                     RADIANS)
C                FS - IS THE LOWER LIMIT OF THE STOPBAND (IN
C                     RADIANS)
C                AP - IS THE MAXIMUM RIPPLE IN THE PASSBAND (IN DB)
C                AS - IS THE MIN ATTEN IN THE STOPBAND (IN DB)
C                ORDER - IS THE ORDER OF THE DIGITAL FILTER (NUMBER
C                        OF POLES)
C        NORDER - NUMBER OF ZEROS IN THE LAPLACE DOMAIN
C                CS - COMPLEX VECTOR OF THE LAPLACE-DOMAIN POLES
C                CSN - COMPLEX VECTOR OF THE LAPLACE-DOMAIN ZEROS
C                CZ - COMPLEX VECTOR OF THE Z-DOMAIN POLES
C                CZN - COMPLEX VECTOR OF THE Z-DOMAIN ZEROS
C                FAIL - LOGICAL VARIABLE THAT IS TRUE IF THE SPECS
C                       CANNOT BE MET
C        FIXORD - LOGICAL VARIABLE THAT FIXES THE ORDER OF
C                        THE DIGITAL FILTER AT ORDERF IF TRUE
C        ORDERF - FIXED ORDER OF THE FILTER IF FIXORD IS
C                        TRUE
C        CHKRIP - LOGICAL VARIABLE, IF TRUE THE PASSBAND
C                        RIPPLE IS DECREASED FOR THE DESIGN MARGIN
C        TRUEAS - THE ACTUAL ATTENUATION AT THE STOPBAND
C                        FREQ. IN DB.
C               NTYPE - THE TYPE OF FILTER
C                     1 = BUTTERWORTH
C                     2 = CHEBYSHEV
C                     3 = INVERSE CHEBYSHEV
C                     4 = ELLIPTIC
C
      INTEGER ORDER,NTYPE,NORDER,ORDERF
      CHARACTER TYPE*12
      REAL*8 FP,FS,AP,AS,WC,WA
      REAL*8 AP1,AS1,RS,DELTA,MARP,MARS,RZ,DF,FP1
      REAL*8 FS1,TRUEAS
      COMPLEX*16 CS(40),CZ(40),CSN(40),CZN(40)
      LOGICAL FAIL,FAIL1,FIXORD,CHKRIP
      DATA MAXORD/20/
      IF(TYPE.EQ.'ELLIPTIC    ') THEN
       NTYPE=4
```

```
      ELSE IF(TYPE.EQ.'INVCHEBYSHEV') THEN
       NTYPE=3
      ELSE IF(TYPE.EQ.'CHEBYSHEV    ') THEN
       NTYPE=2
      ELSE IF(TYPE.EQ.'BUTTERWORTH ') THEN
       NTYPE=1
      END IF
      FAIL=.TRUE.
      DF=FS-1.0D0*FP
      FP1=FP+0.05D0*DF
      FS1=FS-0.05D0*DF
      WC=DTAN(FP1/2.0D0)
      WA=DTAN(FS1/2.0D0)
       AP1=AP
      IF(FIXORD) THEN
       IF(ORDERF.GT.MAXORD) THEN
        WRITE (5,*) 'ORDER FIXED AT ',ORDERF,' IS TOO LARGE'
        GO TO 5
       ELSE
        IF(NTYPE.EQ.1) THEN
         CALL BTWHRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,
     *      FIXORD,ORDERF)
        ELSE IF(NTYPE.EQ.2) THEN
         CALL CHEBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,
     *      FIXORD,ORDERF)
        ELSE IF(NTYPE.EQ.3) THEN
         CALL INCBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,
     *      FIXORD,ORDERF)
        ELSE IF(NTYPE.EQ.4) THEN
         CALL ELTCRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,
     *      FIXORD,ORDERF)
        ELSE
         WRITE (5,*) '1 FILTER TYPE UNRECONIZABLE   TYPE=',TYPE
         FAIL=.TRUE.
         GO TO 5
        END IF
        FAIL=FAIL1
        GO TO 6
       ENDIF
      ELSE
      ORDER=2
2     AP1=AP
      IF(NTYPE.EQ.1) THEN
       CALL BTWHRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.2) THEN
       CALL CHEBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.3) THEN
       CALL INCBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.4) THEN
```

```
      CALL ELTCRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE
       WRITE (5,*) '1 FILTER TYPE UNRECONIZABLE   TYPE=',TYPE
       FAIL=.TRUE.
       GO TO 5
      END IF
      IF(.NOT.FAIL1) GO TO 1
3     ORDER=ORDER+2
      IF(ORDER.GT.MAXORD) GO TO 5
      GO TO 2
1     CONTINUE
      IF(CHKRIP) THEN
      DELTA=(1.0D0-1.0D0*AS/AS1)/10.0D0
      IF(10.0*DELTA.GT.AP1) DELTA=AP1/10.0D0
7     MARP=AP1/AP
      MARS=AS/AS1
      IF(MARP.LT.MARS) GO TO 4
      AP1=AP1-1.0D0*DELTA
      IF(NTYPE.EQ.1) THEN
       CALL BTWHRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.2) THEN
       CALL CHEBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.3) THEN
       CALL INCBRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE IF(NTYPE.EQ.4) THEN
       CALL ELTCRT(WC,WA,AP1,AS,CS,CSN,ORDER,NORDER,FAIL1,AS1,FIXORD,
     *      ORDERF)
      ELSE
        WRITE (5,*) '2 FILTER TYPE UNRECONIZABLE   TYPE=',TYPE
        FAIL=.TRUE.
        GO TO 5
      END IF
      IF(FAIL1) GO TO 3
      GO TO 7
4     FAIL=.FALSE.
      AP=AP1
      ELSE
       FAIL=FAIL1
       AP=AP1
      ENDIF
      ENDIF
6     TRUEAS=AS1
      CALL ZROOT(CS,ORDER,CZ)
      IF(NORDER.GT.0) CALL ZROOT(CSN,NORDER,CZN)
5     RETURN
      END

      SUBROUTINE INITIA(FIXORD,ORDERF,FIXDEN,IDENF,GENFRE,FILEARR,
```

```
     * MAXDNM,FIXFAL,FAILDN,CHKRIP,LOUTSP)
C
C  SUBROUTINE TO READ IN PARAMETERS THAT INITIAL THE PROGRAM FROM
C  FILE CP.INI
C
C     FIXORD - (LOGICAL) TO FIX THE ORDER OF THE FILTER
C                   REGARDLESS OF THE STOPBAND ATTENUATION
C     ORDERF - (INTEGER) THE ORDER OF THE FILTER WHEN FIXED
C     FIXDEN - (LOGICAL) TO GIVE A LOWER BOUND OF THE DENOMINATOR
C        OF THE QUANTIZED SCATTERING MATRIX
C     IDENF  - (INTEGER) THE DENOMINATOR OF THE SCATTERING MATRIX
C     GENFRE - (LOGICAL) TO GENERATE THE ACTUAL - INFINITE
C                   PRESICION FREQUENCY RESPONSE OF THE SCATTERING
C                   MATRIX
C     FILEARR- ARRAY OF NAMES OF FILES SAVED WITH FREQ SPECT.
C     MAXDNM - (INTEGER) THE MAXIMUM ALLOWED DENOMINATOR
C     FIXFAL - (LOGICAL) WHETHER OF NOT TO USE FAILDN AS
C                   DENOMINATOR WHEN FREQ SPECT FAILS UP TO MAXDNM
C     FAILDN - DENOMINATOR USED WITH FIXFAL
C     CHKRIP - (LOGICAL) WHETHER TO CKECK THE PASSBAND RIPPLE
C        AS WELL AS THE ORDER IS SUBROUTINE OPTRT
C     LOUTSP - (LOGICAL) FLAG WHETHER TO SAVE (G - S*T G S)
C        MATRIX IN FILE CPSP.DAT
C
      INTEGER ORDERF,IDENF,MAXDNM,FAILDN
      LOGICAL FIXORD,FIXDEN,GENFRE,FIXFAL,CHKRIP,LOUTSP
      CHARACTER FILEARR(4)*10
      OPEN(UNIT=4,FILE='CP.INI')
      FILEARR(1)='SPECT1.DAT'
      FILEARR(2)='SPECT2.DAT'
      FILEARR(3)='SPECT3.DAT'
      FILEARR(4)='SPECT4.DAT'
      READ(4,4) FIXDEN,IDENF
      READ(4,5) FIXORD,ORDERF
      READ(4,6) GENFRE
      READ(4,7) MAXDNM
      READ(4,6) FIXFAL
      READ(4,7) FAILDN
      READ(4,6) CHKRIP
      READ(4,6) LOUTSP
4     FORMAT(L1,2X,I7)
5     FORMAT(L1,7X,I2)
6     FORMAT(L1)
7     FORMAT(I10)
      CLOSE(UNIT=4)
      RETURN
      END

      SUBROUTINE SPECS(TYPE,FP,FS,AP,AS)
C
C  A SUBROUTINE TO READ IN THE DESIGN SPECS FOR THE DIGITAL LOWPASS
C  FILTER FROM THE TEXT  FILE CP.SPC
```

```
C
C     TYPE - TYPE OF THE FILTER (BUTTERWORTH, CHEBYSHEV,
C                INVCHEBYSHEV,ELLIPTIC)
C     FP - PASSBAND FREQ. IN RADIANS
C     FS - STOPBAND FREQ. IN RADIANS
C     AP - PASSBAND RIPPLE OR PASSBAND ATTENUATION IN DB.
C     AS - STOPBAND ATTENUATION IN DB.
C
      CHARACTER*12 TYPE
      REAL*8 FP,FS,AP,AS
      OPEN(UNIT=4,FILE='CP.SPC')
      READ(4,1) TYPE
      READ(4,2) FP,FS
      READ(4,3) AP,AS
1     FORMAT(A12)
2     FORMAT(F17.12,2X,F17.12)
3     FORMAT(F17.12,2X,F17.12)
      CLOSE(UNIT=4)
      RETURN
      END

      SUBROUTINE ISSOUT(ORDER,IDEN,ISSR,ISSI)
C
C   SUBROUTINE TO WRITE OUT THE QUANTIZED SCATTERING MATRIX TO THE
C   LIST DEVICE
C
C     ORDER - THE ORDER OF THE DIGITAL FILTER
C     IDEN - INTEGER DENOMINATOR OF THE SCATTERING MATRIX
C     ISSR - INTEGER MATRIX OF THE REAL PART OF THE QUANTIZED
C                SCATTERING MATRIX
C     ISSI - INTEGER MATRIX OF THE IMAGINARY PART OF THE
C                QUANTIZED SCATTERING MATRIX
C
      INTEGER ORDER,IDEN,ISSR(11,11),ISSI(11,11),M
      WRITE (5,*)
      WRITE (5,*) 'QUANTIZED SCATTERING MATRIX'
      WRITE (5,*) 'IDEN=',IDEN
      M=ORDER/2+1
      DO 1 I=1,M
         WRITE (5,*) 'COLUMN ',I
         DO 1 J=1,M
      WRITE (5,2) J,ISSR(J,I),ISSI(J,I)
1     CONTINUE
2     FORMAT(' ',I2,3X,I8,'+j',I8)
      RETURN
      END

      SUBROUTINE PRNISS(ORDER,IDEN,ISSR,ISSI)
C
C   SUBROUTINE TO WRITE THE QUANTIZED SCATTERING MATRIX INTO TEXT
C   FILE ISS.DAT
C
```

```
C       ORDER - THE ORDER OF THE DIGITAL FILTER
C       IDEN - INTEGER DENOMINATOR OF THE SCATTERING MATRIX
C       ISSR - INTEGER MATRIX OF THE REAL PART OF THE QUANTIZED
C                 SCATTERING MATRIX
C       ISSI - INTEGER MATRIX OF THE IMAGINARY PART OF THE
C                 QUANTIZED SCATTERING MATRIX
C
        INTEGER ORDER,IDEN,ISSR(11,11),ISSI(11,11),M
        OPEN(UNIT=8,FILE='ISS.DAT',STATUS='NEW')
        WRITE (8,*) 'QUANTIZED SCATTERING MATRIX'
        WRITE (8,*) 'IDEN=',IDEN
        M=ORDER/2+1
        DO 1 I=1,M
           WRITE (8,*) 'COLUMN ',I
           DO 1 J=1,M
        WRITE (8,2) J,ISSR(J,I),ISSI(J,I)
1       CONTINUE
2       FORMAT(' ',I2,3X,I8,'+j',I8)
        CLOSE(UNIT=8)
        RETURN
        END

        SUBROUTINE OUTA(ORDER,PA,DA)
C
C     SUBROUTINE TO WRITE OUT THE COMPLEX ALLPASS TO THE LIST DEVICE
C
C       ORDER - THE ORDER OF THE DIGITAL FILTER
C       PA - COMPLEX VECTOR OF COEFFICIENTS OF ALLPASS NUMERATOR
C       DA - COMPLEX VECTOR OF COEFFICIENTS OF ALLPASS DENOMINATOR
C
        COMPLEX*16 PA(40),DA(40)
        INTEGER ORDER,MM
        MM=ORDER/2+1
        DO 5 I=1,MM
           WRITE (5,7) I,PA(I),DA(I)
5       CONTINUE
7       FORMAT(' ','I=',I2,' PA=',2F8.5,' DA=',2F8.5)
        RETURN
        END

        SUBROUTINE OUTALZ(ORDER,Z,ALPHA)
C
C     SUBROUTINE TO WRITE OUT THE COMPLEX REFERENCE PORT IMPEDANCES AN
C     THE COMPLEX TWO-PORT PARAMETERS CALLED ALPHA, ALONG WITH THE LA
C     PARAMETER WHICH IS A UNIMODULAR COMPLEX MULTIPLIER
C
C       ORDER - ORDER OF THE FILER
C       Z - COMPLEX VECTOR OF THE PORT IMPEDANCES
C       ALPHA - COMPLEX VECTOR OF THE COMPLEX TWO-PORT PARAMETERS
C
        INTEGER ORDER,MM,M
        COMPLEX*16 Z(40),ALPHA(40)
```

```
      REAL*8  PI,PHASE
      DATA  PI/3.141592653589793D0/
      M=ORDER/2
      MM=M+1
      WRITE  (5,*)
      WRITE  (5,*)  'PORT  IMPEDANCES'
      DO  1  I=1,MM
         J=I-1
         WRITE  (5,3)  J,Z(I)
1     CONTINUE
      DO  2  I=1,M
         WRITE  (5,4)  I,ALPHA(I)
2     CONTINUE
      PHASE=DATAN2(DIMAG(ALPHA(MM)),DREAL(ALPHA(MM)))*180.0D0/PI
      WRITE  (5,8)  ALPHA(MM),CDABS(ALPHA(MM)),PHASE
8     FORMAT('  ','EPSIL=  ',2F8.6,'  MAG  EPSIL=  ',F8.6,'  PHASE=',F12.4)
3     FORMAT('  ','PORT=',I2,'  Z=',2F10.6)
4     FORMAT('  ','ADAPTOR=',I2,'  ALPHA=',2F10.6)
      RETURN
      END

      SUBROUTINE  GENOMG(FP,FS,M,OMG)
C
C     A  SUBROUTINE  TO  GENERATE  AN  ARRAY  OF  FREQUENCY  VALUES
C     FOR  TESTING  THE  FREQUENCY  RESPONSE  OF  A  TRANSFER  FUNCTION.
C
C              FP  -  THE  UPPER  LIMIT  OF  THE  FILTER  PASS  BAND  (IN
C                     RADIANS).
C              FS  -  THE  LOWER  LIMIT  OF  THE  FILTER  STOP  BAND  (IN
C                     RADIANS).
C              M  -  THE  TOTAL  NUMBER  OF  FREQUENCY  VALUES  TO  BE
C                    GENERATED  (M  IS  A  POWER  OF  2).
C              OMG  -  THE  VECTOR  CONTAINING  THE  FREQUENCY  VALUES.
C
      INTEGER  M,L,J,J1
      REAL*8  OMG(512),FP,FS,PI
      DATA  PI/3.141592653589793D0/
      OMG(1)=FS
      OMG(2)=FP
      L=3
      J=2
1     J1=J-1
      DO  2  K=1,J1,2
      OMG(L)=FS+DBLE(K)*(PI-FS)/DBLE(J)
      L=L+1
2     CONTINUE
      DO  3  K=1,J1,2
      OMG(L)=DBLE(J-K)*FP/DBLE(J)
      L=L+1
3     CONTINUE
      J=J*2
      IF(L-1.LT.M)  GO  TO  1
```

```
      RETURN
      END

      SUBROUTINE  SSQNT(ORDER,SS,IDEN,ISSR,ISSI)
C
C     A SUBROUTINE TO QUANTIZE THE ELEMENTS OF THE STATE VARIABLE
C     MATRICIES FROM REAL VALUES TO RATIONAL VALUES USING
C     MAGNITUDE TRUNCATION.
C     ORDER - ORDER OF THE FILTER
C     SS - INFINITE PRECISION SCALED COMPLEX SCATTERING MATRIX
C     IDEN - DENOMINATOR OF QUANTIZED SCATTERING MATRIX
C     ISSR - MATRIX OF THE REAL PART OF THE QUANTIZED SCATTERING
C               MATRIX
C     ISSI - MATRIX OF THE IMAG PART OF THE QUANTIZED SCATTERING
C               MATRIX
C
      INTEGER  ORDER,ISSR(11,11),ISSI(11,11),IDEN,M
      COMPLEX*16  SS(11,11)
      REAL*8  DEN
      M=ORDER+1
      DEN=DBLE(IDEN)
      DO 1 I=1,M
           DO 1 J=1,M
                ISSR(I,J)=IDNINT(DREAL(SS(I,J))*DEN)
        ISSI(I,J)=IDNINT(DIMAG(SS(I,J))*DEN)
1     CONTINUE
      RETURN
      END

      SUBROUTINE  FINDG(ORDER,Z,G)
C
C  A SUBROUTINE TO GENERATE THE REAL CONDUCTANCE MATRIX FROM THE
C  COMPLEX REFERENCE IMPEDANCE MATRIX OF THE PORTS
C
      COMPLEX*16  Z(40)
      REAL*8  G(40)
      INTEGER  ORDER,M
      M=ORDER/2+1
      DO 1 I=1,M
          G(I)=1.0D0/DREAL(Z(I))
          WRITE (5,*) 'I=',I,' G=',G(I)
1     CONTINUE
      RETURN
      END

      SUBROUTINE  LOSSYT(ORDER,G,ISSR,ISSI,IDEN,LOSSY,LOUTSP)
C
C  A SUBROUTINE TO TEST WHETHER A WAVE DIGITAL NETWORK IS
C  PEUDOPASSIVE BY CHECKING IF (G - S*T G S) IS POSITIVE DEFINITE
C  WITH THE COMPLEX DEFINITION OF POSITIVE DEFINITENESS AFTER BEING
C  UPPER TRIANGULARIZED
C
```

```
C       ORDER - ORDER OF THE DIGITAL FILTER
C       G - REAL CONDUCTANCE MATRIX OF THE REFERENCE PORTS
C       IDEN - DENOMINATOR OF QUANTIZED SCATTERING MATRIX
C       ISSR - MATRIX OF THE REAL PART OF THE QUANTIZED SCATTERING
C                  MATRIX
C       ISSI - MATRIX OF THE IMAG PART OF THE QUANTIZED SCATTERING
C                  MATRIX
C       LOSSY - LOGICAL VARIABLE THAT IS TRUE IF PSEUDOPASSIVE
C       LOUTSP - LOGICAL VARIABLE THAT IF TRUE WILL SAVE (G - S*T G
C                  S) IN A TEXT FILE CPSP.DAT
C
        INTEGER ISSR(11,11),ISSI(11,11),IDEN,ORDER
        COMPLEX*16 S(11,11),SP(11,11),SSP(11,11),SUM
        REAL*8 DEN,G(40),X,Y
        LOGICAL LOSSY,LOUTSP
        LOSSY=.FALSE.
        DEN=DBLE(IDEN)
        M=ORDER/2
        MM=M+1
        DO 1 I=1,MM
           DO 1 J=1,MM
              X=DBLE(ISSR(I,J))/DEN
          Y=DBLE(ISSI(I,J))/DEN
1             S(I,J)=DCMPLX(X,Y)
        DO 2 I=1,MM
           DO 2 J=1,MM
              SP(I,J)=DCONJG(S(J,I))*DCMPLX(G(J),0.0D0)
2       CONTINUE
        DO 5 I=1,MM
           DO 5 J=1,MM
              SUM=DCMPLX(0.0D0,0.0D0)
              DO 3 K=1,MM
3          SUM=SUM+SP(I,K)*S(K,J)
           SSP(I,J)=SUM
5       CONTINUE
        DO 4 I=1,MM
           DO 4 J=1,MM
              IF(I.EQ.J) THEN
           SP(I,J)=DCMPLX(G(I),0.0D0)-1.0D0*SSP(I,J)
        ELSE
           SP(I,J)=-1.0D0*SSP(I,J)
        ENDIF
4       CONTINUE
        IF(LOUTSP) CALL OUTSP(ORDER,SP)
        CALL POSDEF(MM,SP,LOSSY)
        RETURN
        END

        SUBROUTINE OUTSP(ORDER,SP)
C
C  A SUBROUTINE TO SAVE THE COMPLEX MATRIX (G - S*T G S) IN FILE
C  CPSP.DAT
```

```
C
      INTEGER ORDER,M
      COMPLEX*16 SP(11,11)
      OPEN(UNIT=9,FILE='CPSP.DAT',STATUS='NEW')
      M=ORDER/2+1
      WRITE (9,*) ' THE (G - S*T G S)  MATRIX FOR TESTING
     * PSEUDOLOSSY'
      WRITE (9,*)
      DO 1 I=1,M
         WRITE (9,*) 'COLUMN=',I
         DO 2 J=1,M
            WRITE (9,3) J,SP(J,I)
2        CONTINUE
1     CONTINUE
3     FORMAT(' ',5X,'I=',I4,' SP=',2F15.10)
      CLOSE(UNIT=9)
      RETURN
      END

      SUBROUTINE DCRISS(ORDER,ISSR,ISSI)
C
C     A SUBROUTINE TO DECREMENT THE MAGNITUDE OF THE ELEMENTS OF
C     THE REAL AND IMAGINARY PARTS OF THE QUANTIZED SCATTERING MATRI:
C     BY ONE.
C     ORDER - THE ORDER OF THE FILTER
C     ISSR - MATRIX OF THE REAL PART OF THE QUANTIZED SCATTERING
C                MATRIX
C     ISSI - MATRIX OF THE IMAG PART OF THE QUANTIZED SCATTERING
C                MATRIX
C
      INTEGER ORDER,ISSR(11,11),ISSI(11,11),M
      M=ORDER/2+1
      DO 1 I=1,M
         DO 1 J=1,M
            CALL DCRVAR(ISSR(I,J))
      CALL DCRVAR(ISSI(I,J))
1     CONTINUE
      RETURN
      END

      SUBROUTINE DCRVAR(M)
      INTEGER M
C
C     A SUBROUTINE TO DECREMENT THE MAGNITUDE OF AN INTEGER BY
C     ONE.
C
      IF(M.EQ.0) GO TO 1
      IF(M.GT.0) THEN
         M=M-1
      ELSE
         M=M+1
      END IF
```

```
1     RETURN
      END

      SUBROUTINE POSDEF(M,A,POS)
C
C A SUBROUTINE TO TEST WHETHER A HERMITIAN MATRIX IS POSITIVE
C DEFINITE USING THE COMPLEX DEFINITION OF POSITIVE DEFINITENESS
C
C                 M - THE ORDER OF MATRIX A.
C                 A - THE HERMITIAN MATRIX.
C                 POS - LOGICAL VARIBLE THAT IS TRUE IF THE MATRIX A
C                       IS POSITIVE DEFINITE.
C
      INTEGER M
      COMPLEX*16 A(11,11),X,S(11,11)
      LOGICAL POS
      POS=.FALSE.
      DO 1 I=1,M
      DO 2 J=1,M
      S(I,J)=A(I,J)
2     CONTINUE
1     CONTINUE
      N=M
      NN=N-1
      DO 7 J=1,NN
        I1=J+1
        IF(CDABS(S(J,J)).LT.1.0D-5) GO TO 3
        DO 5 I=I1,N
          IF(CDABS(S(I,J)).LT.1.0D-10) GO TO 5
          X=-1.0D0*S(I,J)/S(J,J)
          J1=J+1
          DO 6 J2=J1,N
            IF(CDABS(S(J,J2)).LT.1.0D-10) GO TO 6
            S(I,J2)=S(I,J2)+X*S(J,J2)
6         CONTINUE
5       CONTINUE
7     CONTINUE
      DO 4 J=1,M
         IF(CDABS(S(J,J)).LT.1.0D-5) THEN
           WRITE (5,*) '## S<1.0D-5   J=',J,' S=',S(J,J)
           GO TO 3
         ENDIF
4     CONTINUE
      POS=.TRUE.
3     RETURN
      END

      SUBROUTINE FINDA(AZ,ORDER,BETA,PA,DA)
C
C     THIS SUBROUTINE FINDS THE ACTUAL ALLPASS NUMERATOR PA AND
C     DENOMINATORDA POLYNOMIALS. THE ARRAY AZ CONTAINS THE Z DOMAIN
C     POLES AND ORDER IS TWICE THE ORDER OF THE ALLPASS
```

```
C
      COMPLEX*16  AZ(40),PA(40),DA(40),POLY(40),ONE,PROD,BETA
      INTEGER  ORDER,M,MM
      M=ORDER/2
      MM=M+1
      ONE=DCMPLX(1.0D0,0.0D0)
      DO  1  I=1,M
1        POLY(I)=ONE/AZ(I)
      CALL  PRMULT(POLY,M,PA)
      PROD=ONE
      DO  2  I=1,M
2        PROD=PROD*AZ(I)
      DO  3  I=1,MM
3        PA(I)=PA(I)*PROD*BETA
      DO  4  I=1,M
4        POLY(I)=DCONJG(AZ(I))
      CALL  PRMULT(POLY,M,DA)
      RETURN
      END

      SUBROUTINE  TESTG(PA,DA,P,D,ORDER,PASSED)
C
C  SUBROUTINE TO TEST IF THE ALLPASS DECOMPOSITION WAS SUCCESSFUL AND
C  THE COEFFICIENTS OF THE TWO ALLPASS FUNCTIONS ADDED TOGETHER
C  EQUALS THE COEFFICIENTS OF THE TRANSFER FUNCTION
C      PA - COEFFICIENTS OF THE NUMERATOR OF ALLPASS
C      DA - COEFFICIENTS OF THE DENOMINATOR OF ALLPASS
C      P - COEFFICIENTS OF THE NUMERATOR OF TRANSFER FUNCTION
C      D - COEFFICIENTS OF THE DENOMINATOR OF TRANSFER FUNCTION
C      PASSED - LOGICAL VARIABLE THAT IS TRUE IF THE ALLPASS
C                    FUNCTION IS CORRECT
C
      LOGICAL  PASSED
      COMPLEX*16  PA(40),DA(40),P(40),D(40),POLY(40)
      COMPLEX*16  MULT1(40),MULT2(40),MULT3(40),SUMP(40)
      REAL*8  DELTA
      INTEGER  ORDER,M,MM,NUM
      M=ORDER/2
      MM=M+1
      PASSED=.TRUE.
      DO  1  I=1,MM
1        POLY(I)=DCONJG(DA(I))
      CALL  PYMULT(POLY,MM,PA,MM,MULT1,NUM)
      DO  2  I=1,MM
2        POLY(I)=DCONJG(PA(I))
      CALL  PYMULT(POLY,MM,DA,MM,MULT2,NUM)
      DO  3  I=1,NUM
3        SUMP(I)=(MULT1(I)+MULT2(I))*0.5D0
      DO  4  I=1,MM
4        POLY(I)=DCONJG(DA(I))
      CALL  PYMULT(DA,MM,POLY,MM,MULT3,NUM)
      DO  5  I=1,NUM
```

```
      DELTA=DABS(DREAL(P(I))-1.0D0*DREAL(SUMP(I)))/DABS(DREAL(P(I)))
      IF(DELTA.GT.9.0D-2) THEN
      WRITE (5,*) 'RE P NOT MATCH I=',I,' P=',P(I),' SUMP=',SUMP(I)
      PASSED=.FALSE.
      ENDIF
      IF(DABS(DIMAG(P(I))-1.0D0*DIMAG(SUMP(I))).GT.5.0D-5) THEN
      WRITE (5,*) 'IM P NOT MATCH I=',I,' P=',P(I),' SUMP=',SUMP(I)
      PASSED=.FALSE.
      ENDIF
      DELTA=DABS(DREAL(D(I))-1.0D0*DREAL(MULT3(I)))/
     * DABS(DREAL(D(I)))
      IF(DELTA.GT.9.0D-2) THEN
      WRITE (5,*) 'RE D NOT MATCH I=',I,' D=',D(I),' MULT3=',MULT3(I)
      PASSED=.FALSE.
      ENDIF
      IF(DABS(DIMAG(D(I))-1.0D0*DIMAG(MULT3(I))).GT.5.0D-5) THEN
      WRITE (5,*) 'IM D NOT MATCH I=',I,' D=',D(I),' MULT3=',MULT3(I)
      PASSED=.FALSE.
      ENDIF
 5    CONTINUE
      RETURN
      END

      SUBROUTINE ALPHAZ(DA,PA,BETA,ORDER,Z1,ALPHA,Z)
C
C    THIS SUBROUTINE DETERMINES ALL OF THE PARAMETERS ASSOCIATED WITH
C     THE TWO PORT COMPLEX ADAPTORS (ALPHA) AND THE FINAL UNIMODULAI
C     MULTIPLIER EPSILON STORED IN ALPHA(ORDER/2+1)
C    IT ALSO DETERMINES THE PORT IMPEDANCES GIVEN THE INPUT PORT
C     IMPEDANCE
C     ORDER - ORDER OF THE DIGITAL FILTER
C     PA - COEFFICIENTS OF THE NUMERATOR OF ALLPASS
C     DA - COEFFICIENTS OF THE DENOMINATOR OF ALLPASS
C     BETA - UNIMODULAR CONSTANT MULTIPLYING ALLPASS
C     Z1 - IMPEDANCE OF THE INPUT-OUTPUT PORT
C     ALPHA - COMPLEX VECTOR OF PARAMETERS OF COMPLEX TWO-PORT
C                ADAPTORS
C     Z - IMPEDANCE VECTOR OF PORTS
C
      COMPLEX*16 ONE,ANUM,ADEN,POLY(40),POLYD(40),POLYP(40)
      COMPLEX*16 DA(40),PA(40),BETA,ALPHA(40),X1,X2,X3,X4,Z1,Z(40)
      COMPLEX*16 SCAL
      REAL*8 R,X,ZR,ZX,DEN
      INTEGER ORDER,M,MM,MSC
      ONE=DCMPLX(1.0D0,0.0D0)
      M=ORDER/2
      MM=M+1
      MSC=MM
      DO 1 I=1,MM
         POLYP(I)=PA(I)
 1       POLYD(I)=DA(I)
      DO 2 I=1,M
```

```
      ANUM=(DCONJG(POLYD(1))/POLYD(MSC))-1.0D0*DCONJG(BETA)
      ADEN=(DCONJG(POLYD(MSC))/POLYD(1))-1.0D0*DCONJG(BETA)
      ALPHA(I)=ANUM/ADEN
      X1=BETA*(DCONJG(ALPHA(I))-ONE)
      X2=DCONJG(ALPHA(I))*(ALPHA(I)-ONE)
      X3=ALPHA(I)-ONE
      X4=BETA*ALPHA(I)*(DCONJG(ALPHA(I))-ONE)
      DO 3 J=1,(MSC-1)
          POLYP(J)=X1*DCONJG(POLYD(MSC+1-J)) + X2*POLYD(J)
          POLY(J)=X3*POLYD(J+1) + X4*DCONJG(POLYD(MSC-J))
3     CONTINUE
      DO 4 J=1,(MSC-1)
4         POLYD(J)=POLY(J)
      MSC=MSC-1
2   CONTINUE
    ALPHA(MM)=POLYP(1)/POLYD(1)
C
    Z(1)=Z1
    SCAL=ONE
    DO 5 I=2,MM
      DEN=1.0D0-CDABS(ALPHA(I-1))**2
      R=DREAL(Z(I-1))
      X=DIMAG(Z(I-1))
      ZR=((DREAL(ALPHA(I-1))-1.0D0)**2+DIMAG(ALPHA(I-1))**2)*R/DEN
      IF(ZR.LT.0.0D0) THEN
        WRITE (*,*) 'REAL Z <0  I=',I,' ZR=',ZR
        WRITE (5,*) 'REAL Z <0  I=',I,' ZR=',ZR
      ENDIF
      ZX=2.0D0*DIMAG(ALPHA(I-1))*R/DEN - X
      Z(I)=DCMPLX(ZR,ZX)
      SCAL=SCAL*(CDABS(ONE - ALPHA(I-1))**2)/DEN
      Z(I)=Z(I)/SCAL
5   CONTINUE
    DO 6 I=1,M
      IF(CDABS(ALPHA(I)).GT.1.0D0) THEN
        WRITE (5,*) 'ALPHA >1  I=',I,' ALPHA=',ALPHA(I)
      ENDIF
6   CONTINUE
    RETURN
    END

    SUBROUTINE SSCATT(ORDER,ALPHA,SS)
    COMPLEX*16 ALPHA(40),SS(11,11),X1,X3,X4,ONE
    REAL*8 E1,E2,X2,X5
    LOGICAL EVEN
    INTEGER ORDER,M,MM,NUM
C
C   THIS SUBROUTINE CALCULATES THE SCALED SCATTERING MATRIX OF THE
C   COMPLEX ALLPASS USING THE SUBROUTINES ALOWES,ALOWOS,AHIGHS,AIS
C   ORDER - ORDER OF THE DIGITAL FILTER
C   ALPHA - VECTOR OF THE PARAMETERS OF THE COMPLEX TWO-PORT
C             ADAPTORS
```

```
C        SS - INFINITE PRECISION SCALED SCATTERING MATRIX
C
         ONE=DCMPLX(1.0D0,0.0D0)
         M=ORDER/2
         MM=M+1
         DO 1 I=1,MM
             DO 1 J=1,MM
1                SS(I,J)=DCMPLX(0.0D0,0.0D0)
         X1=ALPHA(1)-ONE
         SS(1,1)=-DCONJG(ALPHA(1))*X1/DCONJG(X1)
         X2=DSQRT(1.0D0-CDABS(ALPHA(1))**2)
         SS(1,2)=-X2*X1/CDABS(X1)
         IF(M.EQ.1) THEN
             CALL ALOWOS(M,ALPHA,SS)
         ELSE IF(M.EQ.2) THEN
             CALL ALOWES(1,ALPHA,SS)
         ELSE
             E1=DBLE(INT(M/2.0))
             E2=DBLE(M/2.0)
             X3=ALPHA(2)-ONE
             X4=X3/DCONJG(X3)
             X5=DSQRT(1.0D0-CDABS(ALPHA(2))**2)
             SS(2,1)=DCONJG(ALPHA(2))*X2*X4*X1/CDABS(X1)
             SS(3,1)=X2*X5*X1*X3/(CDABS(X1)*CDABS(X3))
             CALL AHIGHS(ALPHA,SS)
             EVEN=.FALSE.
             IF(DABS(E1-E2).LT.1.0D-8) EVEN=.TRUE.
             I=3
11           CONTINUE
                 IF(EVEN) THEN
             NUM=I+1
         ELSE
             NUM=I
         ENDIF
         IF(NUM.EQ.M) THEN
             IF(EVEN) THEN
                 CALL ALOWES(I,ALPHA,SS)
             ELSE
                 CALL ALOWOS(I,ALPHA,SS)
             ENDIF
             GO TO 10
         ELSE
             CALL AIS(I,ALPHA,SS)
             I=I+2
             GO TO 11
         ENDIF
         ENDIF
10       CONTINUE
         RETURN
         END

         SUBROUTINE ALOWES(I,ALPHA,SS)
```

```
      COMPLEX*16  ALPHA(40),SS(11,11),X1,X2,X3,X4,X5,ONE,EPSIL
      REAL*8  X6,X7
      INTEGER  I
C
C     THIS SUBROUTINE CALCULATES THE BOTTOM TWO ROWS OF THE SCALED
C     SCATTERING MATRIX FOR EVEN ORDERED ALLPASS
C     NOTE THAT I REFERS TO THE Ith ROW OF THE A MATRIX OR THE I+1th
C     ROW OF THE SCATTERING MATRIX
C
      ONE=DCMPLX(1.0D0,0.0D0)
      X1=ALPHA(I+1)-ONE
      X2=X1/DCONJG(X1)
      X3=ALPHA(I)-ONE
      X4=X3/CDABS(X3)
      X5=-X1/CDABS(X1)
      X6=DSQRT(1.0D0-CDABS(ALPHA(I))**2)
      X7=DSQRT(1.0D0-CDABS(ALPHA(I+1))**2)
      EPSIL=ALPHA(I+2)
      SS(I+1,I)=DCONJG(ALPHA(I+1))*X6*X2*X4
      SS(I+1,I+1)=-ALPHA(I)*DCONJG(ALPHA(I+1))*X2
      SS(I+1,I+2)=X7*EPSIL*X5
      SS(I+2,I)=-X6*X7*X4*X5
      SS(I+2,I+1)=ALPHA(I)*X7*X5
      SS(I+2,I+2)=ALPHA(I+1)*EPSIL
      RETURN
      END

      SUBROUTINE  ALOWOS(I,ALPHA,SS)
      COMPLEX*16  ALPHA(40),SS(11,11),ONE,X1,X2
      REAL*8  X3
      INTEGER  I
C
C     THIS SUBROUTINE CALCULATES THE BOTTOM ROW OF THE SCALED
C     SCATTERING MATRIX FOR ODD ORDERED ALLPASS
C     NOTE THAT I REFERS TO THE Ith ROW OF THE A MATRIX OR THE I+1th
C     ROW OF THE SCATTERING MATRIX
C
      ONE=DCMPLX(1.0D0,0.0D0)
      X1=ONE-ALPHA(I)
      X2=X1/CDABS(X1)
      X3=DSQRT(1.0D0-CDABS(ALPHA(I))**2)
      SS(I+1,I)=X3*X2*ALPHA(I+1)
      SS(I+1,I+1)=ALPHA(I)*ALPHA(I+1)
      RETURN
      END

      SUBROUTINE  AHIGHS(ALPHA,SS)
      COMPLEX*16  ALPHA(40),SS(11,11),ONE,X3,X4,X5,X6,X7,X8
      REAL*8  X1,X2
C
C     THIS SUBROUTINE CALCULATES THE FIRST TWO ROWS OF THE A SCALED
C     STATE VARIABLE MATRIX
```

```
C
       ONE=DCMPLX(1.0D0,0.0D0)
       X1=DSQRT(1.0D0-CDABS(ALPHA(2))**2)
       X2=DSQRT(1.0D0-CDABS(ALPHA(3))**2)
       X3=ALPHA(2)-ONE
       X4=X3/DCONJG(X3)
       X5=ALPHA(3)-ONE
       X6=X5/DCONJG(X5)
       X7=X3/CDABS(X3)
       X8=X5/CDABS(X5)
       SS(2,2)=-ALPHA(1)*DCONJG(ALPHA(2))*X4
       SS(2,3)=DCONJG(ALPHA(3))*X1*X6*X7
       SS(2,4)=X1*X2*X7*X8
       SS(3,2)=-ALPHA(1)*X1*X7
       SS(3,3)=-ALPHA(2)*DCONJG(ALPHA(3))*X6
       SS(3,4)=-ALPHA(2)*X2*X8
       RETURN
       END

       SUBROUTINE  AIS(I,ALPHA,SS)
       COMPLEX*16  ALPHA(40),SS(11,11),ONE
       COMPLEX*16  X4,X5,X6,X7,X8,X9,X10,X11
       REAL*8  X1,X2,X3
       INTEGER  I
C
C      THIS SUBROUTINE CALCULATES THE Ith AND I+1th ROWS OF THE A
C      STATE VARIABLE SCALED MATRIX OR THE I+1th AND I+2th ROWS OF THE
C      SCALED SCATTERING MATRIX
C      NOTE THAT I REFERS TO THE Ith ROW OF THE A MATRIX OR THE I+1th
C      ROW OF THE SCATTERING MATRIX
C
       ONE=DCMPLX(1.0D0,0.0D0)
       X1=DSQRT(1.0D0-CDABS(ALPHA(I))**2)
       X2=DSQRT(1.0D0-CDABS(ALPHA(I+1))**2)
       X3=DSQRT(1.0D0-CDABS(ALPHA(I+2))**2)
       X4=ONE-ALPHA(I)
       X5=X4/CDABS(X4)
       X6=ALPHA(I+1)-ONE
       X7=X6/DCONJG(X6)
       X8=X6/CDABS(X6)
       X9=ALPHA(I+2)-ONE
       X10=X9/DCONJG(X9)
       X11=X9/CDABS(X9)
       SS(I+1,I)=-DCONJG(ALPHA(I+1))*X1*X7*X5
       SS(I+1,I+1)=-ALPHA(I)*DCONJG(ALPHA(I+1))*X7
       SS(I+1,I+2)=DCONJG(ALPHA(I+2))*X2*X10*X8
       SS(I+1,I+3)=X2*X3*X8*X11
       SS(I+2,I)=-X1*X2*X5*X8
       SS(I+2,I+1)=-ALPHA(I)*X2*X8
       SS(I+2,I+2)=-ALPHA(I+1)*DCONJG(ALPHA(I+2))*X10
       SS(I+2,I+3)=-ALPHA(I+1)*X3*X11
       RETURN
```

```
      END

      SUBROUTINE SSOUT(ORDER,SS)
C
C   SUBROUTINE TO WRITE THE INFINITE PRECISION SCALED COMPLEX
C   SCATTERING MATRIX
C
      INTEGER ORDER,M,MM
      COMPLEX*16 SS(11,11)
      M=ORDER/2
      MM=M+1
      WRITE (5,*)
      WRITE (5,*) 'SCALED SCATTERING MATRIX'
      DO 1 I=1,MM
         WRITE (5,*) 'COLUMN ',I
         DO 2 J=1,MM
            WRITE (5,3) J,SS(J,I)
2        CONTINUE
1     CONTINUE
3     FORMAT(' ',5X,I2,5X,2F8.4)
      RETURN
      END

      SUBROUTINE TEST(ORDER,ISSR,ISSI,IDEN,NUMF,OMG,FCENT,AP,AS,PASS,
     *  I,BANDF)
C
C     A SUBROUTINE TO TEST IF A LOW PASS STATE VARIABLE DESIGN
C     PASSES THE FREQUENCY DOMAIN SPECIFICATIONS.
C
C     PARAMETERS  ORDER - THE ORDER OF THE DF.
C                 NUMF - THE NUMBER OF POINTS TO BE TESTED ON THE
C                        FREQUENCY AXIS.
C                 OMG - AN ARRAY CONTAINING THE FREQUENCY
C                       VALUES TO BE USED (IN RADIANS).
C                 FCENT - THE MIDWAY POINT BETWEEN THE PASSBAND
C                         AND STOPBAND LIMITS.
C                 AP - THE MAX RIPPLE IN THE PASSBAND (IN DB).
C                 AS - THE MIN ATTEN IN THE STOPBAND (IN DB).
C                 PASS - A LOGICAL VARIABLE THAT IS TRUE IF
C                        THE DESIGN PASSES.
C
      INTEGER ORDER,ISSR(11,11),ISSI(11,11),ISSIC(11,11),I
      INTEGER IDEN,M,NUMF,BANDF
      COMPLEX*16 S(11,11),SC(11,11)
      LOGICAL PASS
      REAL*8 OMG(512),AP,AS,FCENT,OMEGA
      REAL*8 ATTNDB(512),MINAS,MINAP,MAXAP,X
      PASS=.FALSE.
      M=ORDER/2+1
      BANDF=0
      OMEGA=OMG(1)
      DO 5 I=1,M
```

```
         DO  5  J=1,M
5          ISSIC(I,J)=-ISSI(I,J)
       CALL  TRISFR(ORDER,ISSR,ISSI,IDEN,OMEGA,S)
       CALL  TRISFR(ORDER,ISSR,ISSIC,IDEN,OMEGA,SC)
       MINAS=-20.0D0*DLOG10(CDABS(S(M,M)+SC(M,M))/2.0D0+1.0D-15)
       OMEGA=OMG(2)
       CALL  TRISFR(ORDER,ISSR,ISSI,IDEN,OMEGA,S)
       CALL  TRISFR(ORDER,ISSR,ISSIC,IDEN,OMEGA,SC)
       MINAP=-20.0D0*DLOG10(CDABS(S(M,M)+SC(M,M))/2.0D0+1.0D-15)
       MAXAP=MINAP
       DO  1  I=3,NUMF
          OMEGA=OMG(I)
          CALL  TRISFR(ORDER,ISSR,ISSI,IDEN,OMEGA,S)
          CALL  TRISFR(ORDER,ISSR,ISSIC,IDEN,OMEGA,SC)
          ATTNDB(I)=-20.0D0*DLOG10(CDABS(S(M,M)+SC(M,M))/2.0D0+
     *                 1.0D-15)
          IF(ATTNDB(I).LT.MINAP)  MINAP=ATTNDB(I)
1      CONTINUE
       DO  50  I=3,NUMF
          OMEGA=OMG(I)
          IF(OMEGA.GT.FCENT)  THEN
             IF(ATTNDB(I).LT.MINAS)  MINAS=ATTNDB(I)
        X=MINAS-1.0D0*MINAP
             IF(X.LT.AS)  THEN
          BANDF=-1
          GO  TO  2
       ENDIF
          ELSE
             IF(ATTNDB(I).GT.MAXAP)  MAXAP=ATTNDB(I)
        X=MAXAP-1.0D0*MINAP
             IF(X.GT.AP)  THEN
          BANDF=1
          GO  TO  2
       ENDIF
          ENDIF
50     CONTINUE
       PASS=.TRUE.
2      RETURN
       END

       SUBROUTINE  TRISFR(ORDER,ISSR,ISSI,IDEN,OMEGA,S)
C
C      TRISFR  -  TRIANGLE  SCATTERING  MATRIX  FREQUENCY  RESPONSE
C
C      A  SUBROUTINE  TO  TRIANGULARIZE  THE  MATRIX  S  WHERE  S  IS  A  SPECIAL
C        MATRIX  BUILT  FOR  TAKING  THE  RESPONSE  AT  A  PARTICULAR
C        FREQUENCY.
C
C                    S:  IS  THE  RESULTING  UPPER  TRIANGULAR  MATRIX.
C
       INTEGER  ORDER,ISSR(11,11),ISSI(11,11),IDEN,M,N
       REAL*8  OMEGA,DEN,RES(11,11),IMS(11,11)
```

```
      COMPLEX*16  S(11,11),X1,X2,X
      M=ORDER/2
      N=M+1
      X1=DCMPLX(0.0D0,OMEGA)
      X2=CDEXP(X1)
      DEN=DBLE(IDEN)
      CALL  REALSS(ORDER,ISSR,ISSI,IDEN,RES,IMS)
      S(N,N)=DCMPLX(RES(1,1),IMS(1,1))
      DO  1  I=1,M
          S(N,I)=DCMPLX(RES(1,I+1),IMS(1,I+1))
          S(I,N)=DCMPLX(-1.0D0*RES(I+1,1),-1.0D0*IMS(I+1,1))
          DO  2  J=1,M
              X=DCMPLX(-1.0D0*RES(I+1,J+1),-1.0D0*IMS(I+1,J+1))
              IF(I.EQ.J)  THEN
                  S(I,J)=X+X2
      ELSE
                  S(I,J)=X
      ENDIF
2     CONTINUE
1     CONTINUE
      CALL  GENTRI(N,S)
      RETURN
      END

      SUBROUTINE  REALSS(ORDER,ISSR,ISSI,IDEN,RES,IMS)
C
C  SUBROUTINE  TO  MAKE  THE  QUANTIZED  SCATTERING  MATRIX  ELEMENTS  REAL
C
      INTEGER  ISSR(11,11),ISSI(11,11),IDEN,N,ORDER
      REAL*8  RES(11,11),IMS(11,11),DEN
      N=ORDER/2+1
      DEN=DBLE(IDEN)
      DO  1  I=1,N
          DO  1  J=1,N
              RES(I,J)=DBLE(ISSR(I,J))/DEN
1     IMS(I,J)=DBLE(ISSI(I,J))/DEN
      RETURN
      END

      SUBROUTINE  GENTRI(N,S)
C
C     A  SUBROUTINE  TO  UPPER  TRIANGULARIZE  A  COMPLEX  MATRIX.
C
C     PARAMETERS   N:  IS  THE  ORDER  OF  MATRIX  S.
C                  S:  IS  A  COMPLEX  MATRIX.
C
      COMPLEX*16  S(11,11),X
      INTEGER  N,NN,I1,J,J1,I,J2
      NN=N-1
      DO  7  J=1,NN
      I1=J+1
      DO  5  I=I1,N
```

```
      IF(CDABS(S(I,J)).LT.1.0D-10) GO TO 5
      X=-1.0D0*S(I,J)/S(J,J)
      J1=J
      DO 6 J2=J1,N
      IF(CDABS(S(J,J2)).LT.1.0D-10) GO TO 6
      S(I,J2)=S(I,J2)+X*S(J,J2)
6     CONTINUE
5     CONTINUE
7     CONTINUE
      RETURN
      END

      SUBROUTINE BTWHRT(WC,WA,AP,AS1,CS,CSN,ORDER,NORDER,FAIL1,
     *   TRUEAS,FIXORD,ORDERF)
C
C     A SUBROUTINE TO GENERATE THE ROOTS OF A LOW PASS
C     BUTTERWORTH FILTER IN THE ANALOG DOMAIN.
C
C     PARAMETERS  WC: IS THE UPPER LIMIT OF THE PASSBAND (IN
C                       RAD/SEC).
C                 WA: IS THE LOWER LIMIT OF THE STOPBAND (IN
C                       RAD/SEC).
C                 AP: IS THE MAX RIPPLE IN THE PASSBAND (IN DB).
C                 AS1: IS THE MIN ATTEN IN THE STOPBAND (IN DB).
C                 CS: IS AN ARRAY CONTAINING THE COMPLEX ROOTS.
C                 RS: IS THE REAL ROOT.
C                 ORDER: IS THE ORDER OF THE FILTER (MUST BE
C                          ODD).
C                 FAIL1: IS TRUE IF THE FILTER SPECS CANNOT BE
C                          MET.
C                 TRUEAS: IS THE ACTUAL MIN STOPBAND ATTEN (IN
C                           DB).
C
      REAL*8 EPSL,GC,WC,WA,AP,AS1,AS,RS,PHI,PI,TRUEAS
      INTEGER ORDER,NORDER,ORDERF
      LOGICAL FAIL1,FIXORD
      COMPLEX*16 CS(40),CSN(40)
      DATA PI/3.141592653589793D0/
      NORDER=0
      GC=10.0D0**(-0.05D0*AP)
      IF(FIXORD) ORDER=ORDERF
      EPSL=DSQRT(1.0D0-1.0D0*GC**2)/(GC*WC**ORDER)
      AS=10.0*DLOG10(1.0D0+(EPSL**2)*(WA**(2*ORDER)))
      TRUEAS=AS
      IF(FIXORD) AS=1000.0D0
      IF(AS.LT.AS1) THEN
         FAIL1=.TRUE.
      ELSE
         FAIL1=.FALSE.
         RS=-1.0D0*EPSL**(DBLE(-1.0D0)/DBLE(ORDER))
         DO 1 I=1,ORDER
         PHI=PI*DBLE(2*I-1)/DBLE(2*ORDER)
```

```
      CS(I)=DCMPLX(RS*DSIN(PHI),-1.0D0*RS*DCOS(PHI))
1     CONTINUE
      END IF
      RETURN
      END

      SUBROUTINE CHEBRT(WC,WA,AP,AS1,CS,CSN,ORDER,NORDER,FAIL1,
     *  TRUEAS,FIXORD,ORDERF)
C
C     A SUBROUTINE TO GENERATE THE S-DOMAIN ROOTS FOR A
C     CHEBYSHEV LOW PASS FILTER GIVEN THE FREQUENCY DOMAIN
C     SPECIFICATIONS.
C
C     PARAMETERS  WC: IS THE UPPER LIMIT OF THE PASSBAND (IN
C                     RAD/SEC).
C                WA: IS THE LOWER LIMIT OF THE STOPBAND (IN
C                     RAD/SEC).
C                AP: IS THE MAX RIPPLE IN THE PASSBAND (IN DB).
C                AS1: IS THE MIN ATTEN IN THE STOPBAND (IN DB).
C                CS: IS AN ARRAY CONTAINING THE COMPLEX ROOTS.
C                ORDER: IS THE ORDER OF THE FILTER (MUST BE
C                        EVEN).
C                FAIL1: IS A LOGICAL VARIABLE THAT IS TRUE IF
C                        SPECS CANNOT BE MET.
C                TRUEAS: IS THE TRUE STOPBAND MIN ATTEN (IN DB).
C
      INTEGER ORDER,ORDERF
      LOGICAL FAIL1,FIXORD
      COMPLEX*16 CS(40),CSN(40)
      REAL*8 WC,WA,AP,AS1,AS,RS,PI,CN,EPSL,GC,PHI
      REAL*8 V,X,XX,WR,TRUEAS
      DATA PI/3.141592653589793D0/
      NORDER=0
      GC=10.0D0**(-0.05D0*AP)
      IF(FIXORD) ORDER=ORDERF
      EPSL=DSQRT(1.0D0-1.0D0*GC**2)/GC
      WR=WA/WC
      CALL ARCOSH(WR,XX)
      CN=DCOSH(DBLE(ORDER)*XX)
      AS=10.0D0*DLOG10(1.0D0+(EPSL*CN)**2)
      TRUEAS=AS
      IF(FIXORD) AS=1000.0D0
      IF(AS.LT.AS1) THEN
        FAIL1=.TRUE.
      ELSE
        FAIL1=.FALSE.
        WR=1.0D0/EPSL
        CALL ARSINH(WR,XX)
        V=XX/DBLE(ORDER)
        RS=-1.0D0*WC*DSINH(V)
        X=WC*DCOSH(V)
        DO 1 K=1,ORDER
```

```
         PHI=PI*DBLE(2*K-1)/DBLE(2*ORDER)
         CS(K)=DCMPLX(RS*DSIN(PHI),X*DCOS(PHI))
1        CONTINUE
      END IF
      RETURN
      END

      SUBROUTINE ARCOSH(X,Y)
C
C     A SUBROUTINE TO CALCULATE Y=INVCOSH(X).
C
      REAL*8 X,Y,Z
      Z=X+DSQRT(X**2-1.0D0)
      Y=DLOG(Z)
      RETURN
      END

      SUBROUTINE ARSINH(X,Y)
C
C     A SUBROUTINE TO CALCULATE Y=INVSINH(X).
C
      REAL*8 X,Y,Z
      Z=X+DSQRT(X**2+1.0)
      Y=DLOG(Z)
      RETURN
      END

      SUBROUTINE INCBRT(WC,WA,AP,AS1,CS,CSN,ORDER,NORDER,FAIL1,
     *   TRUEAS,FIXORD,ORDERF)
C
C     A SUBROUTINE TO GENERATE THE S-DOMAIN ROOTS FOR AN INVERSE
C     CHEBYSHEV LOW PASS FILTER GIVEN THE FREQUENCY DOMAIN
C     SPECIFICATIONS.
C
C     PARAMETERS  WC: IS THE UPPER LIMIT OF THE PASSBAND (IN
C                     RAD/SEC).
C                 WA: IS THE LOWER LIMIT OF THE STOPBAND (IN
C                     RAD/SEC).
C                 AP: IS THE MAX RIPPLE IN THE PASSBAND (IN DB).
C                 AS1: IS THE MIN ATTEN IN THE STOPBAND (IN DB).
C                 CS: IS AN ARRAY CONTAINING THE COMPLEX ROOTS.
C                 ORDER: IS THE ORDER OF THE FILTER (MUST BE
C                        ODD).
C                 FAIL1: IS A LOGICAL VARIABLE THAT IS TRUE IF
C                        THE SPECS CANNOT BE MET.
C                 TRUEAS: IS THE TRUE STOPBAND MIN ATTEN (IN DB).
C
      INTEGER ORDER,ORDERF,NORDER,I,II
      LOGICAL FAIL1,FIXORD
      COMPLEX*16 CS(40),CSN(40)
      REAL*8 WC,WA,AP,AS1,AS,RS,PI,CN,EPSL,GC,PHI,
     *RR,XX,V,WR,TRUEAS
```

```
      DATA PI/3.141592653589793D0/
      GC=10.0D0**(-0.05D0*AP)
      IF(FIXORD) ORDER=ORDERF
      NORDER=ORDER
      WR=WA/WC
      CALL ARCOSH(WR,XX)
      CN=DCOSH(DBLE(ORDER)*XX)
      EPSL=GC/(DSQRT(1.0D0-1.0D0*GC**2)*CN)
      AS=-10.0D0*DLOG10(EPSL**2/(1.0D0+EPSL**2))
      TRUEAS=AS
      IF(FIXORD) AS=1000.0D0
      IF(AS.LT.AS1) THEN
         FAIL1=.TRUE.
      ELSE
         FAIL1=.FALSE.
         WR=1.0/EPSL
         CALL ARSINH(WR,XX)
         V=XX/DBLE(ORDER)
         RS=-1.0D0*WA/DSINH(V)
         DO 1 K=1,ORDER
         PHI=PI*DBLE(2*K-1)/DBLE(2*ORDER)
         RR=WA/((DSIN(PHI)*DSINH(V))**2+(DCOS(PHI)*DCOSH(V))**2)
         CS(K)=DCMPLX(-1.0D0*RR*DSIN(PHI)*DSINH(V),RR*DCOS(PHI)*
     *DCOSH(V))
         CSN(K)=DCMPLX(0.0D0,WC/DCOS(PHI))
1        CONTINUE
      END IF
      RETURN
      END

      SUBROUTINE ELTCRT(WC,WA,AP,AS1,CS,CSN,ORDER,NORDER,FAIL1,
     *   TRUEAS,FIXORD,ORDERF)
C
C     A SUBROUTINE TO USE THE LOW PASS SPECIFICATIONS (IN
C     S-PLANE) TO
C     GENERATE THE REQUIRED ORDER AND ROOT LOCATIONS
C     IN THE S-PLANE FOR AN ELLIPTICAL LP FILTER.
C
C     PARAMETERS    WC: IS THE UPPER LIMIT OF THE PASSBAND (IN
C                       RADIANS).
C                  WA: IS THE LOWER LIMIT OF THE STOPBAND (IN
C                       RADIANS).
C                  AP: IS THE MAX ATTEN IN THE PASSBAND (IN DB).
C                  AS1: IS THE MIN ATTEN IN THE STOPBAND (IN DB).
C                  CS: IS AN ARRAY CONTAINING THE COMPLEX ROOTS.
C                  ORDER: IS THE ORDER OF THE FILTER (MUST BE
C                          ODD).
C                  FAIL1: IS A LOGICAL VARIABLE THAT IS TRUE
C                          IF THE SPECS CANNOT BE MET.
C                  TRUEAS: IS THE ACTUAL MIN STOPBAND ATTEN (IN
C                           DB).
C
```

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 J0,J1,J2,J3,J4,PHI,
     *K,RS,PMARG,SMARG,TRUEAS,PI
      INTEGER ORDER,ORDERF,NORDER,I,II,KCOUNT,N,M
      COMPLEX*16 S,S0,S1,S2,S3,S4,S5,F5,CS(40),CSN(40)
      LOGICAL FAIL1,FIXORD
      F1(A)=A**2+DSQRT(A**4-1.0D0)
      F2(A,Y)=(Y+1.0D0/Y)/A/2.0D0
      F5(A,S)=(S-1.0D0/S)/DCMPLX(2.0D0*A,0.0D0)
      IF(FIXORD) ORDER=ORDERF
      NORDER=ORDER
      PI=3.141592653589793D0
      K=DSQRT(10.0D0**(AP/10.0D0)-1.0D0)
      A0=DSQRT(WA/WC)
      A1=F1(A0)
      A2=F1(A1)
      A3=F1(A2)
      A4=F1(A3)
      N=ORDER
40    J4=DBLE(N)*DLOG10(2.0D0*A4)-1.0D0*DLOG10(2.0D0)
      CALL FF3(J4,J3)
      CALL FF3(J3,J2)
      CALL FF3(J2,J1)
      CALL FF3(J1,J0)
      AS=10.0D0*DLOG10(1.0D0+K*K*10.0D0**(4.0D0*J0))
      TRUEAS=AS
      IF(FIXORD) AS=1000.0D0
      IF(AS.GT.AS1) GO TO 45
      FAIL1=.TRUE.
      GO TO 10
45    R00=-1.0D0*J0-1.0D0*DLOG10(K)
      CALL FF4(J0,R00,R10)
      CALL FF4(J1,R10,R20)
      CALL FF4(J2,R20,R30)
      CALL FF4(J3,R30,R40)
      AN=N
      X=10.0D0**(J4-1.0D0*R40)
      R50=(X+DSQRT(X**2+1.0D0))**(1.0D0/AN)
      DO 20 I=1,N
      AR=PI/N*(I-(N+1)/2-0.5D0)
      M=N/2
C THIS HANDLES THE CASE OF ODD ORDER
      IF((M*2-N).NE.0) AR=PI/N*(I-(N+1)/2)
      C1=R50*DCOS(AR)
      C2=R50*DSIN(AR)
      S5=DCMPLX(C1,C2)
      S4=F5(A4,S5)
      S3=F5(A3,S4)
      S2=F5(A2,S3)
      S1=F5(A1,S2)
      S0=F5(A0,S1)
      CS(I)=S0*DCMPLX(A0*WC,0.0D0)
```

```
20    CONTINUE
      DO 30 I=1,NORDER
        II=I/2
        IF((II*2-I).EQ.0) THEN
          Y4=A4/DCOS(PI*(I-1)/N/2.0D0)
          Y3=F2(A3,Y4)
          Y2=F2(A2,Y3)
          Y1=F2(A1,Y2)
          Y0=F2(A0,Y1)
          W0=Y0*DSQRT(WC*WA)
          CSN(I-1)=DCMPLX(0.0D0,W0)
          CSN(I)=DCMPLX(0.0D0,-1.0D0*W0)
        ENDIF
30    CONTINUE
      FAIL1=.FALSE.
10    RETURN
      END

      SUBROUTINE FF3(J1,J2)
      REAL*8 J1,J2,X1,X2
      IF(J1.GT.8.0) THEN
          J2=0.5D0*J1-0.5D0*DLOG10(2.0D0)
      ELSE
          X1=10.0D0**J1
          X2=DSQRT((X1+1.0/X1)/2.0D0)
          J2=DLOG10(X2)
      END IF
      RETURN
      END

      SUBROUTINE FF4(J,R1,R2)
      REAL*8 J,R1,R2,X1,X2
      IF((J+R1).GT.8.0) THEN
          R2=R1+J+DLOG10(2.0D0)
      ELSE
          X1=10.0D0**(J+R1)
          X2=X1+DSQRT(X1**2+1.0D0)
          R2=DLOG10(X2)
      END IF
      RETURN
      END

      SUBROUTINE L2NORM(ORDER,SS,NORM)
C
C     A SUBROUTINE TO GENERATE THE L2NORM OF THE STATE VARIABLE
C     FILTER OUTPUT.
C
C     PARAMETERS  ORDER: IS THE ORDER OF THE FILTER.
C                 SS: IS THE STATE VARIABLE MATRIX.
C                 NORM: IS THE L2 NORM OF THE OUTPUT.
C
      COMPLEX*16 SS(11,11),X1(11),X2(11)
```

```
      REAL*8  NORM,PSTOR
      INTEGER  ORDER
      M=ORDER/2+1
      X1(1)=DCMPLX(1.0D0,0.0D0)
      DO 3  I=2,M
      X1(I)=DCMPLX(0.0D0,0.0D0)
3     CONTINUE
      NORM=0.0D0
5     CALL  MULTMV(M,SS,X1,X2)
      NORM=NORM+CDABS(X2(1))**2
      X1(1)=DCMPLX(0.0D0,0.0D0)
      PSTOR=0.0D0
      DO 4  I=2,M
      X1(I)=X2(I)
      PSTOR=PSTOR+CDABS(X2(I))**2
4     CONTINUE
      IF(PSTOR.GT.3.0D-3)  GO  TO  5
      RETURN
      END

      SUBROUTINE  MULTMV(N,A,X,Y)
C
C     A SUBROUTINE TO MULTIPLY A MATRIX TIMES A VECTOR (Y=AX).
C
C     PARAMETERS   N: IS THE ORDER OF MATRIX A.
C                 A: IS AN NXN COMPLEX MATRIX.
C                 X,Y: ARE NX1 COMPLEX VECTORS.
C
      COMPLEX*16  A(11,11),X(11),Y(11)
      DO 1  I=1,N
      Y(I)=0.0D0
      DO 2  J=1,N
      Y(I)=Y(I)+A(I,J)*X(J)
2     CONTINUE
1     CONTINUE
      RETURN
      END

      SUBROUTINE  GENWKD(ORDER,ISSR,ISSI,IDEN,KDIAG,WDIAG,AGAIN)
C
C     A SUBROUTINE TO CALCULATE 'K' AND 'W' MATRICIES DIAGONAL
C     ELEMENTS FOR A QUANTIZED DISCRETE TIME STATE VARIABLE
C     SYSTEM.
C
C     PARAMETERS   ORDER: IS THE ORDER OF THE SYSTEM.
C                 ISSR,ISSI: ARE THE STATE VARIABLE MATRICIES.
C                 IDEN: IS THE DENOMINATOR OF THE STATE
C                       VARIABLE MATRICIES.
C                 KDIAG: ARE THE DIAGONAL ELEMENTS OF MATRIX K.
C                 WDIAG: ARE THE DIAGONAL ELEMENTS OF MATRIX W.
C                 AGAIN: IS THE NOISE AMPLITUDE BOUND.
C
```

```
      REAL*8  WDIAG(11),KDIAG(11),PSTOR,AGAIN,DEN
      COMPLEX*16  SS(11,11),Y(11)
      INTEGER  ORDER,ISSR(11,11),ISSI(11,11),IDEN
      DEN=DBLE(IDEN)
      M=ORDER/2+1
      MM=ORDER/2
      DO  1  I=1,M
         DO  1  J=1,M
1           SS(I,J)=DCMPLX(DBLE(ISSR(I,J))/DEN,DBLE(ISSI(I,J))/DEN)
      AGAIN=0.0D0
      DO  2  I=1,MM
         KDIAG(I)=CDABS(SS(I+1,1))**2
         WDIAG(I)=CDABS(SS(1,I+1))**2
         AGAIN=AGAIN+CDABS(SS(1,I+1))
2     CONTINUE
10    CONTINUE
      DO  4  I=1,MM
      Y(I)=0.0D0
      DO  4  J=1,MM
         Y(I)=Y(I)+SS(I+1,J+1)*SS(J+1,1)
4     CONTINUE
      PSTOR=0.0D0
      DO  3  I=1,MM
         SS(I+1,1)=Y(I)
         PSTOR=PSTOR+CDABS(Y(I))**2
         KDIAG(I)=KDIAG(I)+CDABS(Y(I))**2
3     CONTINUE
      IF(PSTOR.GT.1.0D-3)  GO  TO  10
20    CONTINUE
      DO  7  I=1,MM
      Y(I)=0.0D0
      DO  7  J=1,MM
         Y(I)=Y(I)+SS(J+1,I+1)*SS(1,J+1)
7     CONTINUE
      PSTOR=0.0D0
      DO  5  I=1,MM
      SS(1,I+1)=Y(I)
      PSTOR=PSTOR+CDABS(Y(I))**2
      WDIAG(I)=WDIAG(I)+CDABS(Y(I))**2
      AGAIN=AGAIN+CDABS(Y(I))
5     CONTINUE
      IF(PSTOR.GT.1.0D-3)  GO  TO  20
      RETURN
      END

      SUBROUTINE  FINDGH(TYPE,FP,FS,AP,CS,CSN,CZ,CZN,ORDER,NORDER,
     *   BETA,AZ,P,Q,D)
C
C     A  SUBROUTINE  TO  FIND  G(Z)  AND  H(Z)  AND  BETA
C             G(Z)  =  GGAIN  P(Z)             H(Z)  =  Q(Z)
C                             D(Z)                      D(Z)
C     WHERE  THE  COEFFICIENT  OF  THE  HIGHEST  POWER  OF  Z  IN  D(Z)  IS
```

```
C       UNITY AND THE ORDERS OF P(Z), Q(Z), AND D(Z) IS ORDER+1
C       AZ IS THE Z POLES OF ALLPASS OF THE FORM
C                A = (Z + AZ1)(Z + AZ2)...(Z + AZN)
C
C       TYPE - THE TYPE OF THE DIGITAL FILTER
C       ORDER - ORDER OF THE DIGITAL FILTER
C       NORDER - ORDER OF THE NUMERATOR OF THE TRANSFER FUNCTION
C       FP - PASSBAND FREQ
C       FS - STOPBAND FREQ
C       AP - PASSBAND RIPPLE IN DB
C       AS - STOPBAND ATTENUATION IN DB
C       CS - LAPLACE POLES
C       CSN - LAPLACE ZEROS
C       CZ - Z-DOMAIN POLES
C       CZN - Z-DOMAIN ZEROS
C       BETA - UNIMODULAR CONSTANT MULTIPLYING ALLPASS
C       AZ - Z-DOMAIN POLES CHOSEN FOR THE COMPLEX ALLPASS
C       P - COEFFICIENTS OF THE TRANSFER FUNCTION NUMBERATOR
C       Q - COEFFICIENTS OF THE SPECTRAL COMPLEMENT OF THE TRANSFER
C           FUNCTION NUMBERATOR
C       D - COEFFICIENTS OF THE TRANSFER FUNCTION DENOMINATOR
C
        COMPLEX*16 CZZ(40),CZ(40),D(40),P(40),Q(40),GGAIN,ONE,CSN(40)
        COMPLEX*16 CS(40),TEMP(40),DD(40),DDD(40),PP(40),QQ(40),DEN
        COMPLEX*16 SUM,AZ(40),BETA,SUMP,SUMQ,SUMD,PROD,G1,H1,CZN(40)
        COMPLEX*16 BETAN,CHN(40),CHNZ(40),GZGAIN,HZGAIN,PRODD
        COMPLEX*16 PRODP,PRODQ
        INTEGER ORDER,ND2,M,MM,NUM,NORDER
        REAL*8 GC,AP,FS,FP,MAXGM,GCN,DF,FP1,FS1,WC,WA
        LOGICAL EVEN
        CHARACTER*12 TYPE
        ONE=DCMPLX(1.0D0,0.0D0)
        DF=FS-1.0D0*FP
        FP1=FP+0.05D0*DF
        FS1=FS-0.05D0*DF
        WC=DTAN(FP1/2.0D0)
        WA=DTAN(FS1/2.0D0)
        ND2=ORDER/2
        EVEN=.FALSE.
        DO 14 I=1,6
          IF((4*(I-1)).EQ.ORDER) EVEN=.TRUE.
14      CONTINUE
        IF((TYPE.EQ.'BUTTERWORTH ').OR.(TYPE.EQ.'CHEBYSHEV   ')) THEN
          IF(EVEN) THEN
             DO 13 I=1,ND2
                AZ(I)=-1.0D0*CZ(2*I-1)
13           CONTINUE
          ELSE
             DO 15 I=1,ND2
                AZ(I)=-1.0D0*CZ(2*I)
15           CONTINUE
          ENDIF
```

```
      ELSE
         IF(.NOT.EVEN) THEN
            DO 19 I=1,ND2
               AZ(I)=-1.0D0*CZ(2*I-1)
19          CONTINUE
         ELSE
            DO 20 I=1,ND2
               AZ(I)=-1.0D0*CZ(2*I)
20          CONTINUE
         ENDIF
      ENDIF
      DO 1 I=1,ORDER
1        CZZ(I)=-1.0D0*CZ(I)
      CALL PRMULT(CZZ,ORDER,D)
      IF(NORDER.EQ.0) THEN
         DO 3 I=1,ORDER
3           TEMP(I)=ONE
      ELSE
         DO 16 I=1,NORDER
            TEMP(I)=-1.0D0*CZN(I)
16       CONTINUE
         NUM=ORDER-NORDER
         IF(NUM.GT.0) THEN
            MM=NORDER+1
      DO 17 I=MM,ORDER
17       TEMP(I)=ONE
         ENDIF
      ENDIF
      DO 40 I=1,ORDER
40       CHN(I)=WC*WA/CSN(I)
      CALL ZROOT(CHN,ORDER,CHNZ)
      PRODD=ONE
      PRODP=ONE
      PRODQ=ONE
      DO 21 I=1,ORDER
         PRODD=PRODD*(ONE+CZZ(I))
         PRODP=PRODP*(ONE+CZN(I))
         PRODQ=PRODQ*(ONE+CHNZ(I))
21    CONTINUE
      CALL PRMULT(TEMP,ORDER,P)
      M=ORDER+1
      SUMD=DCMPLX(0.0D0,0.0D0)
      SUMP=DCMPLX(0.0D0,0.0D0)
      DO 12 I=1,M
         SUMD=SUMD+D(I)
12       SUMP=SUMP+P(I)
      IF(TYPE.EQ.'BUTTERWORTH  ') THEN
         GC=1.0D0
      ELSEIF (TYPE.EQ.'CHEBYSHEV    ') THEN
         GC=10.0D0**(-0.05D0*AP)
      ELSEIF (TYPE.EQ.'INVCHEBYSHEV') THEN
         GC=1.0D0
```

```
      ELSEIF (TYPE.EQ.'ELLIPTIC      ') THEN
        GC=10.0D0**(-0.05D0*AP)
      ELSE
        WRITE (5,*) 'FILTER TYPE UNRECOGNIZABLE IN FINDGH'
        GO TO 50
      ENDIF
C
      GGAIN=GC*SUMD/SUMP
      GZGAIN=GC*PRODD/PRODP
      HZGAIN=(1.0D0-GC)*PRODD/PRODQ
      DO 11 I=1,M
         P(I)=GGAIN*P(I)
11    CONTINUE
      CALL PYMULT(P,M,P,M,PP,MM)
      DO 4 I=1,M
         DDD(I)=D(M-I+1)
4     CONTINUE
      CALL PYMULT(D,M,DDD,M,DD,MM)
      DO 5 I=1,MM
         QQ(I)=DD(I)-1.0D0*PP(I)
5     CONTINUE
      Q(1)=CDSQRT(QQ(1))
      DEN=DCMPLX(2.0D0,0.0D0)*Q(1)
      Q(2)=QQ(2)/DEN
      IF(ND2.GE.2) THEN
         J=2
7        J=J+1
         SUM=DCMPLX(0.0D0,0.0D0)
         DO 6 I=2,J
6        SUM=SUM+Q(I)*Q(J+1-I)
         Q(J)=(QQ(J)-1.0D0*SUM)/DEN
         IF(J.LE.(ND2+1)) GO TO 7
         DO 8 I=(ND2+2),M
8           Q(I)=Q(M+1-I)
      ELSE
         Q(3)=Q(1)
      ENDIF
C
      SUMQ=DCMPLX(0.0D0,0.0D0)
      DO 9 I=1,M
9        SUMQ=SUMQ+Q(I)
      G1=GC
      H1=DSQRT(1.0D0-GC**2)
      PROD=ONE
      DO 10 I=1,ND2
10       PROD=PROD*(ONE+AZ(I))/(ONE+DCONJG(AZ(I)))
      IF(DABS((CDABS(PROD)-1.0D0)).GT.1.0D-8) THEN
         WRITE (5,*) '1 MAG PROD NOT EQUAL TO 1 - PROD= ',PROD
      ENDIF
      BETA=DCMPLX(DREAL(G1),DREAL(H1))/PROD
      IF(DABS((CDABS(BETA)-1.0D0)).GT.1.0D-8) THEN
         WRITE (5,*) 'MAG BETA NOT EQUAL TO 1 - BETA= ',BETA
```

```
      ENDIF
50    RETURN
      END
```

REFERENCES

[1] A. Fettweis, "Digital filter structures related to classical filter networks," *Arch. Elek. Übertragung*, vol. 25, pp. 79-89, Feb. 1971.

[2] A. Fettweis, "Some principles of designing digital filters imitating classical filter structures," *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 314-316, Mar. 1971.

[3] A. Fettweis, "Pseudopassivity, sensitivity, and stability of wave digital filters," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 668-673, Nov. 1972.

[4] A. Sedlmeyer and A. Fettweis, "Digital filters with true ladder configuration," *Int. J. Circuit Theory Appl.*, vol. 1, pp. 5-10, Mar. 1973.

[5] A. Fettweis and K. Meerkötter, "Suppression of parasitic oscillations in wave digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 239-246, Mar. 1975.

[6] A. Fettweis, "Wave digital filters: theory and practise," *Proc. IEEE*, vol.74, pp. 270-327, Feb. 1986.

[7] K. Meerkötter, "Incremental pseudopassivity of wave digital filters," in *Proc. 1st European Signal Processing Conf.* (EUSIP-CO-80), (Lausanne, Switzerland, Sept. 1980), pp. 27-32

[8] A. Fettweis and K. Meerkötter, "On parasitic oscillations in wave digital filters under looped conditions," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp.475-481, Sept. 1977.

[9] A. Fettweis, "On the connection between muliplier work length limitation and roundoff noise in digital filters," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 486-491, Sept. 1972.

[10] A. Fettweis, "Roundoff noise and attenuation sensitivity in digital filters with fixed-point arithmetic," *IEEE Trans. Circuit Theroy*, vol. CT-20, pp. 174-175, Mar. 1973.

[11] A. Fettweis, "On the evaluation of roundoff noise in digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 896, Nov. 1975.

[12] A. Fettweis and K. Meerkötter, "On Adaptors for wave digital filters," *IEEE Trans. Acoust., Speech, Singal Processing*, vol. ASSP-23, pp. 515-525, Dec. 1975.

[13] G. O. Martens and K. Meerkötter, "On n-port adaptors for wave digital filters with applications to bridged-T filters," in *Proc. 1976 IEEE Int. Symp. on Circuits and Systems* (Munich, W. Germany, Apr. 1976), pp. 514-517.

[14] A. Fettweis, "Canonic realization of ladder wave digital filters," *Int. J. Circuit Theory Appl.*, vol. 5, no. 4, pp. 321-332, Dec. 1975.

[15] A. Fettweis, "On sensitivity and roundoff noise in wave digital filters," *IEEE Trans. Acoust., Speech, Singal Processing*, vol. ASSP-22, no. 5, pp. 383-384, Oct. 1974.

[16] L. B. Jackson, "On the interaction of roundoff noise and the dynamic range in digital filters," *Bell Syst. Tech. J.*, vol. 49, pp. 159-184, Feb. 1970.

[17] C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters." *IEEE Trans. Circuits and Syst.*, vol. CAS-23, pp. 551-562, Sept. 1976.

[18] G. Verkroost and H. J. Butterweck, "Suppression of parasitic oscillations in wave digital filters and related structures by means of controlled rounding," *Arch. Elek. Übertragung*, vol. 30, pp. 181-186, May 1976.

[19] A. Fettweis, H. Levin, and A. Sedlmeyer, "Wave digital lattice filters," *Int. J. Circuit Theory Appl.*, vol. 2, pp. 203-211, June 1974.

[20] W. Wegener, "On the design of wave digital lattice filters with short coefficient word length and optimal dynamic range," *IEEE Trans. Circuits and Syst.*, vol. CAS-2, pp. 1091-1098, Dec. 1978.

[21] G. O. Martens and M. R. Jarmasz, "Design of digital filters with a diagonal Lyapunov function based on elliptic reference filters," *IEEE Int. Symp. on Circuits and Systems* (Rome, Italy, May 1982), vol. 2, pp. 527-530.

[22] G. O. Martens and H. H. Lê, "Wave digital adaptors for reciprocal second-order

sections," *IEEE Trans. Circuits and Syst.*, vol. CAS-25, no. 12, pp. 1077-1083, Dec. 1978.

[23] G. O. Martens, "Voltage scattering matrix analysis of n-ports with application to wave digital filters," in *Proc. 1980 Int. Conf. on Circuits and Computers* (Port Chester, NY, USA, Oct 1980), pp. 1180-1183.

[24] L. Gazsi, "Explicit formulas for lattice wave digital filters," *IEEE Trans. Circuits and Syst.*, vol. CAS-32, pp. 68-88, Jan. 1985.

[25] P. R. Moon and G. O. Martens, "A digital filter structure requiring only m-bit delays, shifters, inverters, and m-bit adders plus simple logic circuitry," *IEEE Trans. Circuits and Syst.*, vol. CAS-27, no. 10, pp. 901-908, Oct. 1980.

[26] A. Fettweis, "Principles of complex wave digital filters," *Int. J. Circuit Theory Appl.*, vol. 9, pp. 119-134, Apr. 1981.

[27] K. Meerkötter, "Complex passive networks and wave digital filters," in *Proc. Eur. Conf. on Circuit Theory and Design* (Warsaw, Poland, Sept. 1980), vol. 2, pp.24-35.

[28] K. Meerkötter, "Antimetric wave digital filters derived from complex reference filters," in *Proc. Eur. Conf. on Circuit Theory and Design* (Stuttgart, W. Germany, Sept. 1983).

[29] P. P. Regalia, S. K. Mitra, and P. P. Vaidyanathan, "The digital allpass filter: a versatile signal processing building block," *Proc. IEEE*, vol. 76, no. 1, pp. 19-37 Jan. 1988.

[30] P. P. Vaidyanathan, P. P. Regalia, and S. K. Mitra, "Design of doubly-complemenary IIR digital filters using a single complex allpass filter, with multirate applications," *IEEE Trans. Circuits and Syst.*, vol. CAS-34, no. 4, pp. 378-389, Apr. 1987.

[31] P. P. Regalia and S. K. Mitra, "Low sensitivity active filter realization using a complex allpass filter," *IEEE Trans. Circuits and Syst.*, vol. CAS-34, no. 4, pp. 390-399, Apr. 1987.

[32] P. I. Richards, "Resistor-transmission-line circuits," *Proc. IRE*, vol. 36, no. 2, pp. 217-220, Feb. 1948.

[33] H. K. Kwan, "Amplitude scaling of arbitrary linear digital networks," *IEEE Trans. Acoust., Speech, Singal Processing*, vol. ASSP-32, pp. 1240-1242, Dec. 1984.

[34] A. N. Willson, "Some effects of quantization and adder overflow on the forced response of digital filters," *Bell System Technical Journal*, vol. 51, pp. 863-887, Apr. 1972.

[35] M. Büttner, "Elimination of limit cycles in digital filters with very low increase in the quantization noise," *IEEE Trans. Circuits and Syst.*, vol. CAS-24, pp. 300-304, June 1977.

[36] T. A. C. M. Claasen, W. F. G. Mecklenbräuker, and J. B. H. Peek, "Effects of quantization and overflow in recursive digital fiters," *IEEE Trans. ASSP*, vol. ASSP-24, pp. 517-529, Dec. 1976.

[37] T. A. C. M. Claasen, W. F. G. Mecklenbräuker, and J. B. H. Peek, "Quantization noise analysis for fixed-point digital filters using magnitude truncation for quantization," *IEEE Trans. Circuits and Syst.*, vol. CAS-22, pp. 887-895, Nov. 1975.

[38] T. A. C. M. Claasen, W. F. G. Mecklenbräuker, and J. B. H. Peek, "On the stability of the forced response of digital filters with overflow nonlinearities," *IEEE Trans. Circuits and Syst.*, vol. CAS-22, no. 8, pp. 692-696, Aug. 1975.

[39] A. B. Sripad and D. L. Snyder, "A necessary and sufficient condition for quantization errors to be uniform and white," *IEEE Trans. Acoust., Speech, Singal Processing*, vol. ASSP-25, pp. 442-448, Oct. 1977.

[40] H. J. Butterweck, "On the quantization noise contribution in digital filters which are uncorrelated with the output signal," *IEEE Trans. Circuits and Syst.*, vol. CAS-26, pp. 901-910, Nov. 1979.

[41] C. Barnes and A. Fam, "Minimum norm recursive digital filters that are free of overflow limit cycles," *IEEE Trans. Circuits and Syst.*, vol. CAS-24, no. 10, pp. 569-574, Oct. 1977.

[42] K. Erichson and A. Michel, "Stability analysis of fixed-point digital filters using computer generated Lyapunov functions - part II: wave digital filters and lattice digital filters," *IEEE Trans. Circuits and Syst.*, vol. CAS-32, no. 2, pp. 132-

142, Feb. 1985.

[43] C. Webb, " Design of Wave-State-Variable Lattice Filters", M. Sc. Thesis, University of Manitoba, Winnipeg, Manitoba, Canada, 1988.

[44] V. Belevitch, *Classical Network Theory*, San Francisco, CA: Holden-Day, 1968.

[45] A. Antoniou, *Digital Filters: Analysis and Design*, New York, NY: McGraw-Hill, 1979.

[46] Z. Nehari, *Conformal Mapping*, New York, NY: McGraw-Hill, 1952.

[47] D. Humphery, *The Analysis, Design, and Synthesis of Electrical Filters*, Englewood Cliffs, NJ: Prentice-Hall, 1970.

[48] C. Chen, *One-Dimensional Digital Signal Processing*, New York, NY: Marcel Dekker, 1979.

[49] E. Kuh and R. Rohrer, *Theory of Linear Active Networks*, London, EN: Holden-Day, 1969.

[50] R. Newcomb, *Linear Multiport Synthesis*, New York, NY: McGraw-Hill, 1966.

[51] G. Temes and S. K. Mitra, *Modern Filter Theory and Design*, London, EN: John Wiley and Sons, 1973.

[52] P. Saeks, *Generalized Networks*, New York, NY: Holt, Rinehart and Winston, 1972.

[53] M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*, New York, NY: John Wiley and Sons, 1965.

[54] R. A. Roberts and C. T. Mullis, *Digital Signal Processing*, Reading, Massachusetts: Addison-Wesley, 1987.

[55] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*, Englewood Cliffs, NJ: Prentice-Hall, 1975.

[56] P. R. Moon, "On the Reduction of Finite-Wordlength Effects in State-Variable Digital Filters," Ph. D. Thesis, University of Manitoba, Winnipeg, Manitoba,

Canada, 1983.

[57] E. D. DeLuca and G. O. Martens, "A new coefficient quantization scheme to suppress limit-cyles in state-variable digital filters", *Proc 6th European Conf on Circuit Theroy and Design*(ECCTD'83), pp. 157-158, Stuttgart, W. Germany, Sept. 1983.

[58] H. J. Orchard, "Loss sensitivities in singly and doubly terminated filters", *IEEE Trans. Circuits and Syst.*, vol. CAS-26, no. 5, pp. 293-297, May 1979.

[59] R. F. Baum, "Design of unsymmetrical band-pass filters", *Proc. IRE*, vol. CT-4, no. 2, pp. 33-40, June 1957.

[60] A. Fettweis, "Passivity and losslessness in digital filtering", *Archiv Für Elektronik Und Übertragungstechnik*, vol. 42, no. 1, pp. 1-8, Jan./Feb. 1988.

[61] R. V. Churchill, *Complex Variables and Applications*, New York, NY: McGraw-Hill, 1960.