SINGULAR PERTURBATION EXPANSION FOR RELATIVISTIC QUANTUM EVOLUTION IN THE SCHWINGER-DEWITT PROPER TIME FORMALISM

by

Jeffrey E. Diamond, B.Sc.

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ABSTRACT

The problem of relativistic quantum evolution in the Schwinger-Dewitt proper time formalism is examined for a system of particles which couple to arbitrary external electromagnetic and scalar fields via the Schrödinger equation. A large mass singular perturbation expansion is obtained for the propagator (the kernel of the proper time evolution operator) and is shown to be asymptotic in the sense that the error incurred in truncating the series is of the same order in the inverse mass as the first neglected term. This expansion is used to derive a similar expansion for the Green's functions associated with the Klein-Gordon and Dirac equations. Consistency with relativistic causality is examined by projecting the evolution operator onto states with a given mass (Hamiltonian) spectrum and considering the perturbation expansion for the kernel of the resulting operator.

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INTRODUCTION

This thesis is concerned with the problem of quantum evolution in proper time τ of a relativistic system of particles. The parameter τ is separate from the geometrical time $t \equiv x^0$ which plays the role of a physical observable in contrast to nonrelativistic quantum mechanics where t is the evolution parameter. The physical interpretation of this parameter τ is discussed in [AHP] and [HP]. The first chapter of the thesis outlines a classical Lagrangian and Hamiltonian formulation of the relativistic dynamics of a particle interacting with external electromagnetic fields. Here τ arises as a parameter describing the classical trajectory of the particle and is a monotonic function of the geometrical time t. The Hamiltonian which governs this motion and which, by the correspondence principle, governs the quantum evolution of the particle is given by

$$H(x,p) = (2M)^{-1}(p^{\mu}-A^{\mu}(x))(p_{\mu}-A_{\mu}(x)) \tag{0.1}$$

where $x \equiv (x^0,x^1,x^2,x^3)$ is the contravariant representation of the space-time vector x and $p \equiv (p_0,p_1,p_2,p_3)$ is the conjugate momentum. We use the standard convention

$$a_{\mu} \equiv g_{\mu\nu}a^{\nu} \tag{0.2}$$

where repeated indices are summed from 0 to 3 and the components of the metric tensor $g_{\mu\nu} = +1$ when $\mu = \nu = 0$, $g_{\mu\nu} = -1$ where $\mu = \nu \neq 0$ and $g_{\mu\nu} = 0$ otherwise. In chapters two through four the corresponding Hamiltonian operator is generalized to one which contains a matrix valued potential $V(x)$ as well as an added scalar potential $v(x)$ and the space time vector x is generalized to be a member of R^d instead of R^4. The metric tensor $g_{\mu\nu}$ is assumed to have the form

$$g_{\mu\nu} = \begin{cases} +1 & \mu = \nu = 0, m-1 \\ -1 & \mu = \nu = m, d-1 \\ 0 & \text{otherwise} \end{cases} \tag{0.3}$$

where m is some integer between 1 and $d-1$. This generalization of the Minkowski space-time ($m = 1$) makes it possible for the formulation to describe systems with more than one particle. Thus the relevant Hamiltonian is given by

$$H = (2M)^{-1}[(P-A(x))^{\mu}(P-A(x))_{\mu} + V(x)] + v(x) \tag{0.4}$$

and the quantum evolution of the system is given by the Schrodinger equation

$$i \frac{d}{d\tau} \psi(t) = H\psi(\tau) \tag{0.5}$$

where the wave function ψ is an element of $L^2(R^d)$, H is an operator on that Hilbert space and $d/d\tau$ represents the strong derivative in the corresponding norm. The second chapter uses the theory of linear differential equations in Banach spaces outlined in [KREIN] along with some easily satisfied assumptions (continuity, differentiability) on the form of the potentials A^{μ}, v, V to show the existence of unique solutions to (0.5) which satisfy the initial condition

$$\psi(0) = \psi_0 \in L^2(R^d) \;. \tag{0.6}$$

The linear map $\psi(0) \to \psi(\tau)$ defines a bounded linear operator $u(\tau)$ on the Hilbert space $L^2(R^d)$ and can sometimes be expressed as an integral operator

$$[U(\tau)\psi_0](x) = \int dy\, K(x,y;\tau)\psi_0(y) \;. \tag{0.7}$$

The kernel $K(x,y;\tau)$ of this operator is called the propagator of evolution and is the object of interest in this thesis. The propogator gives us a convenient form for the evolution operator $u(\tau)$ as well as providing some insight into the space-time geometry of the evolution process. For example, we will examine consistency with relativistic causality by considering the extent of the support of the propagator in the space like $[(x-y)^2 < 0]$ and time-like $[(x-y)^2 > 0]$ regions of R^d. The nonrelativistic form of this propogator (m = d) has recently been examined in [POM], [OPC] and [SAKSENA]. In [OPC] the Dyson series (see [DY1] and [DY2]) is used to construct the propogator for systems interacting with potentials which can be represented as Fourier transforms of complex bounded measures. In [SAKSENA] an approximate evolution operator is obtained by truncating a singular perturbation series expansion in the inverse mass parameter (M^{-1}) and is shown to be an asymptotic approximation in the $L^2(R^d)$ norm of the exact evolution $u(\tau)$ (i.e., the error is of the same order in M^{-1} (as $M \to \infty$) as the first neglected term of the series). In chapter three the method used in [SAKSENA] is applied to the relativistic case. In [SCHWINGER] the relativistic propagator is determined in the special case of electromagnetic fields which are constant and uniform (i.e., independent of their space time argument). The Green's function for the Dirac equation is then obtained from this propagator via a Fourier-Laplace transform. This analysis is summarized in chapter three. The result is then generalized to the case of non-uniform fields where the method of [SAKSENA] is applied in order to obtain an approximate propagator for the evolution. In chapter four the approximate propagator is used to obtain an expansion for the Green's functions for the Dirac and Klein-Gordon equations via a Fourier-Laplace transform. In the last section of chapter four the evolution operator is projected onto states with a given mass (Hamiltonian) spectrum and it is seen that a positive spectrum is associated with time-like correlation in the propagator while a negative spectrum is asspciated with space-like correlation. If the spectrum is strictly positive the value of the approximate propagator $K_N(x,y;\tau)$ is seen to decay exponentially in the space-like region ($(x-y)^2 < 0$) while if the spectrum is strictly negative it is seen to decay exponentially in the time-like region ($(x-y)^2 > 0$). This is an excellent example of the way in which the propagator gives insight into the space-time geometry of the evolution as well as the preservation of relativistic causality in the problem.

CHAPTER ONE

CLASSICAL RELATIVISTIC DYNAMICS

1.1 Non-covariant Formulation

In undertaking a discussion regarding the classical and quantum dynamics of a single relativistic particle we must begin with the classical equations of motion. This is somewhat unpedagogical since in the end we would like to say that the entire treatment began from postulating a particular form for the action, the rest being determined from there. In the beginning, however, the only guidance we have in postulating such an action is the experimentally verified equations of motion. These are specifically the Lorentz force which acts on a charged particle in the presence of an external electromagnetic field, and Poyntings Theorem, both given below.

a) Lorentz Force

$$\vec{F} = \vec{E} + \vec{v} \times \vec{B} \tag{1.1}$$

$\vec{F}(t)$ = Force on particle

$\vec{E}(\vec{x},t)$ = electric field

$\vec{B}(\vec{x},t)$ = magnetic field

$\vec{v}(t)$ = velocity of particle

c = speed of light = 1

$\vec{x} = (x^1, x^2, x^3)$ = position of particle

$t = x^0$ = time

b) Poynting's Theorem

$$\frac{d}{d\tau}\,(\varepsilon - \phi) = \vec{v}\cdot\vec{E} \qquad (1.2)$$

$\varepsilon(t)$ = energy of particle

$\phi(\vec{x},t)$ = electric potential $(\vec{E} = -\vec{\nabla}\phi)$

These can be combined into a covariant form.

$$M\,\frac{du^{\mu}}{d\tau} = F^{\mu\nu}u_{\nu} \qquad (1.3)$$

where

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \qquad \vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{a}$$

$$\tau \equiv (1-|\vec{v}|^2)^{\frac{1}{2}}t \qquad A^0 \equiv \phi(\vec{x},t)$$

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \qquad A^i \equiv a^i(\vec{x},t) \quad (i = 1,3)$$

$$\vec{a} = (a^1,a^2,a^3)(\vec{x},t) \qquad F^{\mu\nu}(\vec{x},t) \equiv \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}$$

In the absence of electromagnetic fields the equation of motion is given by

$$\frac{d}{d\tau}\,(M\vec{v}/(1-|\vec{v}|^2)^{\frac{1}{2}}) = 0\,. \qquad (1.4)$$

Suppose now we set $M = 1$ and attempt to postulate the Lagrangian for which the Euler-Lagrange equations reduce to exactly this form. Try

$$L = -(1-|\vec{v}|^2)^{\frac{1}{2}} \qquad (1.5)$$

$$\frac{\partial L}{\partial \vec{v}} = \vec{v}(1-|\vec{v}|^2)^{-\frac{1}{2}}$$

$$\frac{\partial L}{\partial \vec{x}} = 0 \text{ .}$$

Then the Euler-Lagrange equations are

$$\frac{d}{dt} \left(\vec{v}/(1 - |\vec{v}|^2)^{\frac{1}{2}}\right) = 0 \tag{1.6}$$

exactly as in (1.4). Now turn on the electromagnetic field by adding a velocity dependent term to the Lagrangian exactly as we do in nonrelativistic mechanics.

$$L = -(1 - |\vec{v}|^2)^{\frac{1}{2}} - \phi + \vec{a} \cdot \vec{v} \tag{1.7}$$

Then the Euler-Lagrange equations give us exactly the Lorentz Force.

$$\frac{d}{dt} \left(M\vec{v}/(1 - |\vec{v}|^2)^{\frac{1}{2}}\right) = \vec{E} + \vec{v} \times \vec{B}. \tag{1.8}$$

With this much success we now proceed to use this Lagrangian to develop an Hamiltonian formulation and from there to hopefully quantize the system from the Hamiltonian point of view. First we define the momenta.

$$\vec{p} \equiv \frac{\partial L}{\partial \vec{v}} = \vec{v}(1 - |\vec{v}|^2)^{\frac{1}{2}} + \vec{a} \text{ .} \tag{1.9}$$

Now define the Hamiltonian

$$H(\vec{p},\vec{x}) = \vec{p}\cdot\vec{v}(\vec{x},\vec{p}) - L(\vec{x},\vec{v}(\vec{x},\vec{p})) \text{ .} \tag{1.10}$$

Here it is understood that we must first invert (1.9) to find:

$$\vec{v}(\vec{x},\vec{p}) = (\vec{p}-\vec{a})(|\vec{p}-\vec{a}|^2 + 1)^{-\frac{1}{2}} . \tag{1.11}$$

Then we have:

$$H = \vec{p}\cdot(\vec{p}-\vec{a})/(1 + |\vec{p}-\vec{a}|^2)^{\frac{1}{2}} + [1 - |\vec{p}-\vec{a}|^2/(1 + |\vec{p}-\vec{a}|^2)]^{\frac{1}{2}}$$

$$+ \phi - \vec{a}\cdot(\vec{p}-\vec{a})/(1 + |\vec{p}-\vec{a}|^2)^{\frac{1}{2}}$$

$$= [1 + |\vec{p}-\vec{a}|^2]^{\frac{1}{2}} + \phi . \tag{1.12}$$

Then Hamilton's equations of motion become:

$$\frac{dp_i}{dt} = - \frac{\partial H}{\partial x_i} = \frac{\partial a_j}{\partial x^i} (p-a)^j (1 + |\vec{p}-\vec{a}|^2)^{-\frac{1}{2}} + \frac{\partial \phi}{\partial x^i} \tag{1.13}$$

$$\frac{dx_i}{dt} = (p-a)_i/(1 + |\vec{p}-\vec{a}|^2)^{\frac{1}{2}} . \tag{1.14}$$

Substitute $\partial a_i/\partial t = (\partial a_i/\partial x^j)(\partial x^j/\partial t)$, recall $t = x^0$ and multiply both sides of (1.13) by $dt/d\tau = dx^0/d\tau$.

$$\frac{d}{d\tau} (p-a)_i = \left[(p-a)^j \left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}\right)/(1 + |\vec{p}-\vec{a}|^2)^{\frac{1}{2}} + \frac{\partial \phi}{\partial x^i}\right]\frac{dt}{d\tau}$$

$$+ \frac{\partial \phi}{\partial x^i} \frac{dx^0}{d\tau}$$

$$= F_{ij} \frac{dx^i}{d\tau} + F_{i0} \frac{dx^0}{d\tau} = F_{i\nu} u^\nu . \tag{1.15}$$

(Note that summation convention used here is that roman letters are summed from 1 to 3 and greek letters from 0 to 3.) Clearly equation (1.15) is the last three components of equation (1.3), the Lorentz Force. Now with a classical

Hamiltonian formulation in place we can attempt to quantize the system by postulating an associated Schrödinger equation. If we take $\hbar = 1$ this becomes:

$$i \frac{\partial\psi}{\partial t} = H_{op}\psi = [((-i\vec{\nabla}-\vec{a})^2 + 1)^{\frac{1}{2}} + \phi]\psi(x,t) . \tag{1.16}$$

It is immediately apparent that there will be some difficulty in interpreting the square root in (1.16). H_{op} could be interpreted, in the abstract sense of an operator on the Hilbert space $L^2(R^3)$, via the spectral theorem. However, this will not yield a simple differential operator and thus can be difficult to work with in practice. We can perhaps get around this difficulty by squaring (1.16).

$$(i \frac{\partial}{\partial t} - \phi)^2\psi = [(-i\vec{\nabla}-\vec{a})^2 + 1]\psi . \tag{1.17}$$

In covariant notation this becomes:

$$[(i\partial-A)^{\mu}(i\partial-A)_{\mu} + 1]\psi = 0 . \tag{1.18}$$

This is known as the Klein-Gordon equation. Squaring equation (1.16) in this manner is somewhat ad-hoc and it would clearly be preferable to have a treatment of this problem which is manifestly Lorentz covariant at every stage. By beginning with a Lorentz invariant action we can arrive at the Klein-Gordon equation in a much more systematic way following the Dirac-Bergman method of quantization of constrained Hamiltonian systems (see [DIRAC]). In order to do this we will require some preliminary discussion of constrained dynamics, but first we examine a somewhat different covariant formulation of the problem.

1.2 Covariant Formulation #1

We introduce now a manifestly covariant approach to the single relativistic particle. We must begin with an expression for the action which is a Lorentz invariant

$$A = \int L \, d\tau \, . \tag{1.19}$$

The Lagrangian L must be a Lorentz invariant. The simplest Lorentz invariant which is a function of coordinates and generalized velocities is simply the Lorentz "Length" of the velocity.

$$L = \frac{1}{2} u^{\mu} u_{\mu} \, . \tag{1.20}$$

The Euler-Lagrange equations for this action are given by:

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial u^{\mu}} \right) = 0 = \frac{du_{\mu}}{d\tau} \, . \tag{1.21}$$

This agrees with equation (1.3) in the absence of electromagnetic fields ($F^{\mu\nu} = 0$). We can include interaction with an external electromagnetic field in a minimal fashion by adding a term to the Lagrangian

$$L \to \frac{1}{2} u^{\mu} u_{\mu} + u^{\mu} A_{\mu} \, . \tag{1.22}$$

The corresponding Euler-Lagrange equations are:

$$\frac{d}{d\tau}(u_\nu + A_\nu) = u^\mu(\partial_\nu A_\mu) . \tag{1.23}$$

Substitute $dA_\nu/d\tau = (\partial_\mu A_\nu)u^\mu$. Then we have:

$$\frac{du_\nu}{d\tau} = (\partial_\nu A_\mu - \partial_\mu A_\nu)u^\mu = F^{\nu\mu}u_\mu . \tag{1.24}$$

This agrees with (1.3). Now we proceed with the Hamiltonian formulation.

$$p_\nu = \frac{\partial L}{\partial u^\nu} = u_\nu + A_\nu$$

$$H = p^\nu u_\nu - L = p^\nu(p_\nu - A_\nu) - \frac{1}{2}(p^\mu - A^\mu)(p_\mu - A_\mu) - (p^\mu - A^\mu)A_\mu$$

$$= \frac{1}{2}(p-A)^\nu(p-A)_\nu . \tag{1.25}$$

Hamilton's equations become:

$$\frac{dp^\nu}{d\tau} = -\frac{\partial H}{\partial x_\nu} = (p-A)^\mu(\partial^\nu A_\mu) \tag{1.26}$$

$$\frac{dx^\nu}{d\tau} = \frac{\partial H}{\partial p_\nu} = (p-A)^\nu . \tag{1.27}$$

Substitute $A_\nu \equiv dA_\nu/d\tau = (\partial_\mu A_\nu)u^\mu = (\partial_\mu A_\nu)(p-A)^\mu$. Then (1.26) becomes:

$$\frac{d}{d\tau}(p-A)_\nu = (p-A)^\mu(\partial_\nu A_\mu - \partial_\mu A_\nu) = F_{\nu\mu}(p-A)^\mu \tag{1.28}$$

or

$$\frac{du_\nu}{d\tau} = F_{\nu\mu}u^\nu \quad (\text{since } u_\nu = (p-A)_\nu).$$

Note we have assumed that the vector potential A^μ has no explicit τ-dependence (i.e., $dA_\nu/d\tau = 0$). Consider the "0-th" component of equation (1.28).

$$\frac{d}{d\tau}(\varepsilon - \phi) = \vec{u}\cdot\vec{E} \qquad (\varepsilon = p^0 = \text{energy}) \tag{1.29}$$

but $d\phi/d\tau = \vec{\nabla}\phi\cdot d\vec{x}/d\tau + (\partial\phi/\partial t)(\partial t/\partial\tau)$. Substituting back into (1.29) we have

$$\frac{d\varepsilon}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} \qquad \text{or} \qquad \frac{\partial\varepsilon}{\partial t} = \frac{\partial\phi}{\partial t}\ .$$

Thus the energy ε is constant in time, or conserved, if the scalar potential ϕ is time independent. This agrees with the nonrelativistic result. If we separate the space and time components of (1.27) we get

$$\vec{u} \equiv \frac{d\vec{x}}{d\tau} = (\vec{p} - \vec{a})$$

$$\frac{dt}{d\tau} = \varepsilon - \phi \tag{1.30}$$

$$2H = \text{constant} = (E - \phi)^2 - |\vec{p}-\vec{a}|^2 = (E - \phi)^2 - |\vec{u}|^2 \tag{1.31}$$

$$\frac{dt}{d\tau} = \varepsilon - \phi = [2H + |\vec{u}|^2]^{\frac{1}{2}}\ . \tag{1.32}$$

Consider this equation in the instantaneous rest frame of the particle (i.e., x = u = 0). Then we can identify τ with the proper time of the particle if:

$$\frac{dt_{rest}}{d\tau} = (2H)^{\frac{1}{2}} = 1$$

$$\text{i.e. } H = \frac{1}{2} = Mc^2/2 \quad (\text{recall } M = c = 1)\ . \tag{1.33}$$

Then we have:

$$\frac{dt}{d\tau} = [1 + |\vec{u}/c|^2]^{\frac{1}{2}} \to 1 + \frac{1}{2}|\vec{u}/c|^2 + O(c^{-4}) \quad \text{as } c \to \infty$$

$$\frac{d\vec{x}}{dt} = [\frac{dt}{d\tau}\vec{E} + (\vec{u}/c) \times \vec{B}]\frac{d\tau}{dt}$$

$$= \vec{E} + \frac{d\vec{x}}{dt} \times (\vec{B}/c) \ . \tag{1.34}$$

We can now use this Hamiltonian formulation as a starting point for quantization of this system. We postulate a Schrödinger equation.

$$i\frac{d}{d\tau}\psi(x,\tau) = (\frac{1}{2})(i\partial - A)^{\mu}(i\partial - A)_{\mu}\psi(x,\tau) \ . \tag{1.35}$$

The corresponding Ehrenfest Theorem is given by the following.

$$\frac{d}{d\tau}\langle x^{\nu}\rangle = i\langle[H,x^{\nu}]\rangle = \langle(p-A)^{\nu}\rangle \tag{1.36}$$

$$\frac{d}{d\tau}\langle p^{\nu}\rangle = i\langle[H,p^{\nu}]\rangle$$

$$= (\frac{1}{2})\langle(p-A)^{\mu}\partial^{\nu}A_{\mu} + \partial^{\nu}A_{\mu}(p-A)^{\mu}\rangle$$

or

$$\frac{d}{d\tau}\langle u_{\nu}\rangle = \frac{1}{2}\langle F^{\mu\nu}u_{\nu} + u_{\nu}F^{\mu\nu}\rangle \ . \tag{1.37}$$

Clearly (1.37) is the quantum analog of equation (1.3). One difficulty with this formulation of the problem is that if we replace τ by some arbitrary function of τ, say $f(\tau)$, we obtain a different Schrödinger equation. τ is, however, meant to be some parameter describing evolution of the system and it should not make any difference to the physics exactly how we choose this parametrization. We now investigate a formulation of the problem which is explicitly reparametrization invariant.

1.3 Covariant Formulation #2

Suppose that we are given some arbitrary Lagrangian and we wish to put the corresponding action principle on a covariant footing. Begin with the following.

$$A = \int L(\vec{x},t,\frac{d\vec{x}}{dt})\,dt\,. \tag{1.38}$$

Now treat the time, t, not as a parameter but as a fourth generalized coordinate and suppose that the coordinates and velocities are functions of some parameter, λ, a monotone function of t. Denote differentiation with respect to λ by a "prime" symbol.

$$\frac{d}{d\lambda}\,f(\lambda) \equiv f'(\lambda)\,.$$

then

$$A = \int L(x_\nu,x_i'/x_0')\,x_0'\,d\lambda \qquad (x^0 = t)$$

or

$$A = \int \Lambda(x_\mu,x_\mu')\,d\lambda \tag{1.39}$$

where Λ is now homogeneous in the first degree with respect to the generalized velocities x_μ', i.e.,

$$\Lambda(x_\mu, ax'_\mu) = a\Lambda(x_\mu, x'_\mu) \; . \tag{1.40}$$

The action is parametrization invariant in the following sense.

$$A = \int L(x_\mu, x'_\mu/x'_0)\, x'_0 \, d\lambda$$

$$= \int L\left(x_\mu, \frac{dx_i}{df(\lambda)} \Big/ \frac{dx_0}{df(\lambda)}\right) \frac{dx_0}{df(\lambda)}\, df(\lambda) \; .$$

In other words, if we make the substitution $\lambda \to f(\lambda)$ (f is some arbitrary function) the action A does not change. Now since Λ is homogeneous of the first degree we have:

$$x'_\mu \frac{\partial \Lambda}{\partial x'_\mu} = \Lambda \; . \tag{1.41}$$

Differentiating with respect to x'_ν leads to:

$$x'_\mu \frac{\partial^2 \Lambda}{\partial x'_\mu \partial x'_\nu} = 0 \; . \tag{1.42}$$

This implies that the Jacobian determinant, J, must vanish.

$$J = \det\left[\frac{\partial^2 \Lambda}{\partial x'_\mu \partial x'_\nu}\right] = 0 \; .$$

This in turn means that the momenta ($p^\nu = \partial\Lambda/\partial x'_\nu$) are not all independent. There must be at least one relation between all the p^ν. Thus the particle is not free to roam about through all of phase space but is constrained to remain on some submanifold of the phase space. Thus in order to continue with this procedure we require some discussion of the dynamics of constrained system. We present this discussion before returning to the relata-vistic particle.

1.4 Constraint Dynamics

Consider a system with a Lagrangian defined in such a way that the momenta are not all independent but satisfy some constraint. For example (define $\dot{f}(\tau) \equiv df(\tau)/d\tau$ for any function $f(\tau)$ of the evolution parameter τ):

$$L = \frac{1}{3} \sum_i x_i^3 \rightarrow p_i = \dot{x}_i^2 .$$

Then clearly all the p_i must be positive so that we can "cover" only $(\frac{1}{2})^n$ of phase space (the positive sector) with the allowed momenta. It is important to realize that these constraints arise directly from the definition of the Lagrangian and not from any external assumptions. These are called "primary constraints". Suppose we have a number of such constraints labeled by the integers m, in the following form:

$$\phi_m = 0 .$$

We define our Hamiltonian:

$$H = p^\nu \dot{x}_\nu - L .$$

However, the Hamiltonian defined in this way is not uniquely determined since we can add any linear combination of the constraints and will only be adding zero. Thus the Hamiltonian H^* below is "as good as" the original Hamiltonian.

$$H^* = H + \lambda_m \phi^m \qquad \lambda_m \in R .$$

Consider now a variation of the Hamiltonian

$$\delta H = \dot{x}^\nu \delta p_\nu - \frac{\partial L}{\partial x_\nu} \delta x_\nu .$$

Now we cannot make arbitrary variations δp and δx since we must conform to the primary constraints. Using the method of Lagrange multipliers to minimize the action $A = \int p^\nu \dot{x}_\nu - H\, d\tau$ we arrive at the following:

$$\dot{x}^\nu = \frac{\partial H}{\partial p_\nu} + \lambda_m \frac{\partial \phi^m}{\partial p_\nu} \tag{1.43}$$

$$-\frac{\partial L}{\partial x_\nu} = \frac{\partial H}{\partial x_\nu} + \lambda_m \frac{\partial \phi^m}{\partial x_\nu}$$

or, substituting $p_\nu = \partial L/\partial x^\nu$:

$$p_\nu = -\frac{\partial H}{\partial x_\nu} - \lambda_m \frac{\partial \phi^m}{\partial x_\nu} \tag{1.44}$$

For any function $g(p,x)$ of phase space its evolution is given by:

$$\dot{g} = \frac{\partial g}{\partial x_\nu} \dot{x}_\nu + \frac{\partial g}{\partial p_\nu} \dot{p}_\nu$$

or substituting from (1.43) - (1.44) and making use of the Poisson bracket $[f,g] \equiv (\partial f/\partial x_\nu)(\partial g/\partial p^\nu) - (\partial f/\partial p_\nu)(\partial g/\partial x^\nu)$ we get:

$$\dot{g} = [g,H] + \lambda_m[g,\phi^m] . \tag{1.45}$$

Since λ_m is an arbitrary function of the evolution parameter τ but not a function of phase space:

$$\dot{g} = [g,H + \lambda_m\phi^m] = [g,H_T] \tag{1.46}$$

where

$$H_T = H + \lambda_m\phi^m .$$

It is important to realize at this stage that we must satisfy the constraints $\phi^m = 0$ but we must only apply these after we have calculated the Poisson brackets. Thus we define a specific notation for equations which are to be applied only at this later stage. They are called "weak equations" and are denoted by the symbol "$\approx$" as in $\phi^m \approx 0$. A weak equation is satisfied only on the surface of constraint.

Besides the constraints themselves which must be satisfied for all τ the τ-derivative of the constraints must also be satisfied in order to maintain consistency. Using (1.45) this gives us (1.47) below.

$$\dot{\phi}_j = [\phi_j,H] + \lambda_m[\phi_j,\phi^m] \approx 0 . \tag{1.47}$$

We assume that the equations (1.47) do not lead to any inconsistencies. This could happen, for example, if we had a Lagrangian of the form L = x. Then the Euler-Lagrange equations would immediately give us $1 = 0$. Thus we cannot have any arbitrary Lagrangian. There are three possibilities for each of the equations (1.47).

a) Some of these equations may be identically satisfied, i.e., they would reduce to $0 = 0$. These need not be considered further.

b) If one of the equations reduces to an equation independent of the λ's it is of the form:

$$\eta(x,p) \approx 0 . \tag{1.48}$$

These are called secondary constraints. These give rise again to further consistency conditions.

$$\dot{\eta} = [\eta,H] + \lambda_m[\eta,\phi^m] \approx 0 . \tag{1.49}$$

These in turn may be of type a or b (leading to further secondary constraints) or of type c. We must continue this process until we have "flushed out" all the secondary constraints. Though this appears complicated, in practice it usually only requires a few steps until all of the consistency conditions are determined.

c) If an equation does not fall into category a) or b) above, then it imposes a condition on the λ's.

The above introduction to constrained dynamics will now provide the tools necessary to proceed with the reparametrization invariant formulation of the relativistic particle problem.

1.5 Covariant Formulation #2 (continued)

Begin with the action considered in section 1.1 and use the method outlined in section 1.3 to express this in a covariant fashion.

$$A = \int - [1 - (\frac{d\vec{x}}{dt})^2]^{\frac{1}{2}} dt$$

$$= \int \Lambda\left(x^\mu, \frac{dx^\mu}{d\tau}\right) d\tau \qquad (1.50)$$

where

$$\Lambda = -[1 - \left|\frac{d\vec{x}}{dt}\right|^2]^{\frac{1}{2}} \frac{dt}{d\tau} = -[u^\mu u_\mu]^{\frac{1}{2}}$$

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \ .$$

Clearly the action is Lorentz invariant as well as reparametrization ($\tau \to f(\tau)$) invariant. The Euler-Lagrange equations are given by (1.51) below.

$$\frac{d}{d\tau} (u_\nu/(u_\mu u^\mu)^{\frac{1}{2}}) = 0 \ . \qquad (1.51)$$

The momenta are defined by:

$$p_\nu = \frac{\partial L}{\partial u^\nu} = u_\nu/(u_\mu u^\mu)^{\frac{1}{2}} \ . \qquad (1.52)$$

Now clearly the p_ν are of unit "length" ($p^\nu p_\nu = 1$) so that the allowed momenta are only those on the unit hyperbola defined by

$$\phi = p^\nu p_\nu - 1 = 0 \ . \qquad (1.53)$$

This is the primary constraint. Also it is the only primary constraint since any point p on this hyperbola can be expressed in the form (1.52) if we take $u_\nu = p_\nu$. Then $p^\nu p_\nu = 1$ implies $p^\nu = u^\nu/(u^\mu u_\mu)^{\frac{1}{2}}$.

Now since Λ is homogeneous in the first degree in the velocities u^ν we have:

$$u^\nu \frac{\partial L}{\partial u^\nu} = L \Rightarrow H = u^\nu \frac{\partial L}{\partial u^\nu} - L = 0$$

so that the Hamiltonian vanishes identically. The only consistency condition

$$\dot{\phi} = [\phi, \phi] = 0$$

is satisfied identically so that there are no secondary constraints and no conditions on the Lagrange multiplier λ. Thus our total Hamiltonian H_T is given by:

$$H_T = H + \lambda\phi = \lambda(\tau)(p^\nu p_\nu - 1) \qquad (1.54)$$

i.e., by the constraints alone. Hamilton's equations become:

$$\frac{dx^\nu}{d\theta} = p^\nu \qquad \frac{dp^\nu}{d\theta} = 0 \qquad (1.55)$$

where $\theta = \tau\lambda(\tau)$ is an arbitrary function of τ. This demonstrates explicitly the reparametrization invariance in the equations of motion.

If we now wish to quantize this system we follow the procedure outlined in [DIRAC]. We make the dynamical variables x^ν and p^ν into operators on a Hilbert space and replace the Poisson brackets of phase space functions with the commutator of self adjoint operators associated with these functions. Then we define the Schrödinger equation

$$i\,\frac{d\psi}{d\tau} = H_T \;=\; \lambda(\tau)\,(p^\nu p_\nu - 1)$$

or

$$i\,\frac{d\psi}{d\theta} = (p^\nu p_\nu - 1) \quad . \tag{1.56}$$

Note again the explicit reparametrization invariance inherent in the arbitrary function $\theta(\tau)$. We further impose a condition on the wave function.

$$\phi_j \psi = 0 \qquad\qquad (\phi_j = \text{constraints}).$$

In this case we have only one constraint so that the supplementary condition becomes

$$(p^\nu p_\nu - 1)\psi = 0 \; . \tag{1.57}$$

This is the Klein-Gordon equation. Now separate variables in (1.56) by setting $\psi(x,\theta) = T(\theta)\psi(x)$. Then we find

$$\psi(x,\theta) = e^{i\gamma\theta}\phi(x) \tag{1.58}$$

$$(p^\nu p_\nu - 1)\psi(x) = \gamma \; . \tag{1.59}$$

Now applying the condition (1.57) yields $\gamma = 0$. Substituting this back into (1.58) - (1.59) we have:

$$\psi(x,\theta) = \psi(x) \tag{1.60}$$

$$(p^{\nu}p_{\nu} - 1)\psi(x) = 0 \,. \tag{1.61}$$

Thus we have no evolution and the system is governed completely by the Klein-Gordon equation. By separating variables and applying the constraint (1.57) in this manner we have lost all of the evolution described by (1.56). There is perhaps something to be learned, however, by studying this evolution without immediately imposing the form $\psi(x,\theta) = T(\theta)\psi(x)$ and the constraint (1.57). The remainder of this thesis is devoted to the study of this evolution as well as its relationship to the Klein-Gordon equation.

CHAPTER TWO

SCHRÖDINGER EVOLUTION

2.1 The Schrödinger Equation

A Schrödinger equation describing world time evolution of a relativistic system can be written:

$$i \frac{d}{d\tau} \psi = H\psi . \tag{2.1}$$

We interpret (2.1) as an abstract equation describing evolution in the Hilbert space $\mathcal{H} = L^2(R^d, C^k)$ of square integrable functions defined on R^d. The inner product for this Hilbert space is given by:

$$(\phi, \psi) = \int dx \, \phi^\dagger(x) \psi(x) \tag{2.2}$$

where $\phi^\dagger$ represents the complex conjugate transpose of the column vector ϕ. We interpret the derivative on the left hand side of (2.1) as the strong derivative in the Hilbert space $\mathcal{H}$. The Hamiltonian H is interpreted as an operator defined on $\mathcal{H}$ in the following way. We first define the operator $\hat{H}$.

$$D(\hat{H}) = C_0^\infty(R^d, C^k)$$

$$\hat{H}\phi = \{(2M)^{-1}[(P^\mu - A^\mu)(P_\mu - A_\mu) + V] + v\}\phi$$

$$= \{(2M)^{-1} \sum_{\mu,\nu=0}^{d-1} g_{\mu\nu}[(P^\mu - A^\mu)(P^\nu - A^\nu) + V] + v\}\phi \tag{2.3}$$

where

i) $$g_{\mu\nu} = \begin{cases} +1 & \mu = \nu = 0, m-1 \\ -1 & \mu = \nu = m, d-1 \\ 0 & \mu \neq \nu \end{cases}$$

ii) $D(A^{\mu}) = D(P^{\mu}) = D(V) = D(v) = C_0^{\infty}(R^d, C^k)$

iii) $M \in R\backslash\{0\}$

iv) $(P^{\mu}\phi)(x) \equiv i \dfrac{\partial\phi}{\partial x_{\mu}}(x)$

v) $(A^{\mu}\phi)(x) \equiv A^{\mu}(x)\phi(x)$

where the function A^{μ}: $R^d \to R$ is continuously differentiable

vi) $(V\phi)x \equiv V(x)\phi(x)$

where V: $R^d \to C^{k\times k}$ is piecewise continuous and hermitian (i.e., $V^{\dagger}(x) = V(x)$ for every $x \in R^d$ where $V^{\dagger}(x)$ denotes the complex conjugate transpose of the $k\times k$ matrix $V(x)$)

vii) $(v\phi)(x) \equiv v(x)\phi(x)$

where v: $R^d \to R$ is piecewise continuous.

Since A^{μ}, V, v are piecewise continuous and A^{μ} continuously differentiable the operator H maps its domain C_0^{∞} into the Hilbert space $\mathcal{H} = L^2(R^d, C^k)$. This is because $\hat{H}$ does not disturb the compactness of support of the functions in C_0^{∞} so that $\hat{H}\phi$ has compact support if ϕ does. The following will show that $\hat{H}$ is also symmetric.

LEMMA 2.1: H *is symmetric.*

Proof:

a) The operators P^μ ($\mu = 0, d-1$) are symmetric. Let $f,g \in C_0^\infty$. Then:

$$(f,P^\mu g) = i \int dx\, f^\dagger(x)\, \partial^\mu g(x)$$
$$= i[\int d^4x\, \partial^\mu(f^\dagger g) - (\partial^\mu f^\dagger)g]$$
$$= \int d^4x(i\partial^\mu f)^\dagger g = (P^\mu f,g)\,. \qquad (2.4)$$

The first term in (2.4) vanishes as a result of the divergence theorem and the compactness of support of f and g.

b) Since the functions A^μ and v are real valued the associated multiplication operators are symmetric.

c) The operator V is symmetric: Since V(x) is hermitian for all $x \in R^d$:

$$(Vf,g) = \int dx\, f^\dagger(x)V^\dagger(x)g(x) = \int d^4x\, f^\dagger(x)V(x)g(x)$$
$$= (f,Vg)\,.$$

d) Since the operators P^μ and A^μ are all symmetric then $(P^\mu - A^\mu)$ and $(P^\mu - A^\mu)^2$ are also symmetric for each $\mu = 0, d-1$. Thus the operator $(P-A)^\mu(P-A)_\mu = \sum_{i=0}^{m-1} (P_i - A_i)^2 - \sum_{j=m}^{d-1} (P_j - A_j)^2$ is also symmetric.

$\hat{H}$ is a linear combination with real coefficients of the symmetric operators $[(P-A)^\mu(P-A)_\mu]$, V and v and is therefore symmetric. $\square$

Since $\hat{H}$ is symmetric it is closable. (See [SINHA; p.51].) Define the operator H which appears in (2.1) to be the closure of $\hat{H}$ as defined above. Since H is the closure of a symmetric operator it is both closed and symmetric. The following analysis will show that for such an operator, equation (2.1) has a unique solution for given initial data (ψ $(\tau = 0) = \psi_0 \in D(H)$).

2.2 Linear Differential Equations in Banach Space

Consider the following equation in a Banach space E.

$$\frac{d}{d\tau} \psi = A\psi \tag{2.5}$$

where ψ: $[0,T] \to E$ and A is a linear operator densely defined on E.

Definition 2.2: A solution of equation (2.5) on the segment [0,T] is a function satisfying:

i) $\psi(\tau) \in D(A)$ *for all* $\tau \in [0,T]$.

ii) There exists a strong derivative $\frac{d}{d\tau}\psi(\tau)$ *at each point* τ *of* $[0,T]$.

iii) Equation (2.5) is satisfied for all $\tau \in [0,T]$.

The Cauchy problem on [0,T] is the problem of finding a solution of (2.5) which satisfies the initial condition:

$$\psi(0) = \psi_0 \in D(A) . \tag{2.6}$$

Definition 2.3: *The Cauchy problem on* $[0,T]$ *is said to be uniformly correct if:*

i) For any $\psi_0 \in D(A)$ *it has a unique solution.*

ii) The solution depends continuously on the initial data in the sense that a sequence $\psi_n(0) \to 0$ $(\psi_n(0) \in D(A))$ *implies that* $\psi_n(\tau) \to 0$ *uniformly in* τ *on each finite interval* $[0,T]$.

iii) $\psi(\tau)$ *and* $d\psi/d\tau$ *are continous for* $\tau \in [0,T]$.

For a uniformly correct Cauchy problem the family of operators $u(\tau)$ defined by

$$\psi(\tau) = u(\tau)\psi_0 \tag{2.7}$$

is well defined on D(A) and linear. Since D(A) is dense in E, $u(\tau)$ may be extended by continuity to a bounded linear operator defined on all of E denoted by $u(\tau)$ also. $u(\tau)$ is called the evolution operator for the Cauchy problem (2.5).

THEOREM 2.3: *The evolution operator of a uniformly correct Cauchy problem has the following properties:*

i) $u(\tau): D(A) \to D(A)$.

ii) $u(\tau)$ *is uniformly bounded in* τ

$$||u(\tau)|| \leq M \qquad \tau \in [0,T]. \tag{2.8}$$

iii) $u(\tau)$ *is strongly continuous in* $[0,T]$.

iv) $u(\tau_1+\tau_2) = u(\tau_1)u(\tau_2) \qquad (0 \le \tau_1, \tau_2 < T)$ (2.9)

$u(0) = I$. (2.10)

v) On D(A), $u(\tau)$ *is strongly differentiable and*

$$\frac{d}{d\tau}u(\tau) = Au(\tau) . \tag{2.11}$$

Proof: See [Krein, p.195.]

In view of Theorem 2.3 it is useful to know under what conditions the Cauchy problem associated with (2.1) is uniformly correct. Here the Banach space E is the Hilbert space H and the following theorem and proposition give these conditions.

THEOREM 2.4: In order that the Cauchy problem for equation (2.5) with a closed operator A *be uniformly correct it is necessary and sufficient that the resolvent* $R(\lambda) = [A-\lambda]^{-1}$ $(\lambda \in C)$ *should satisfy the condition:*

$$||R^n(\lambda)|| \le \frac{M}{(\mathrm{Re}\,\lambda - w)^n} \qquad (\mathrm{Re}\,\lambda > w) \tag{2.12}$$

for some w *and* M *and for all integers* $n > 0$. *Here for the corresponding semigroup we have the inequality:*

$$||u(\tau)|| \le M e^{w\tau} . \tag{2.13}$$

Proof: See [Krein; p.51.]

<u>*Proposition 2.5:*</u> *If the operator H is closed and symmetric then the Cauchy problem for equation (2.1) is uniformly correct and the corresponding semigroup* $u(\tau)$ *satisfies the inequality* $\|u(\tau)\| \leq 1$.

<u>Proof:</u> a) Let $\phi \in D(H)$. Then

$$\|\phi\| \leq \frac{1}{|\mathrm{Re}\,\lambda|} \|[iH+\lambda]\phi\| \qquad \mathrm{Re}\ \lambda \neq 0 .$$

Let $\phi \in D(H)$. Then $\phi \in D(iH+\lambda) = D(H)$

$$\begin{aligned} \|\phi\| \, \|[iH+\lambda]\phi\| &\geq |(\phi,[iH+\lambda]\phi)| \\ &= |(\phi,[H-\lambda i]\phi)| \geq |\mathrm{Im}(\phi,[H-\lambda i]\phi)| \\ &= |\mathrm{Im}(\phi,-\lambda i\phi)| = |\mathrm{Re}\ \lambda| \, \|\phi\|^2 \end{aligned}$$

where we have used the following:

$$H \text{ symmetric} \Rightarrow (\phi,H\phi) \in R$$

a) follows immediately.

b) $[iH+\lambda]$ is invertible for $\mathrm{Re}\ \lambda \neq 0$.

Let $\phi \in D(H)$ and suppose that $[-iH-\lambda]\phi = 0$. Then from a):

$$\|\phi\| \leq \frac{1}{|\mathrm{Re}\ \lambda|} \|[iH+\lambda]\phi\| = 0 .$$

Thus $[iH+\lambda]\phi = 0 \Rightarrow \phi = 0$ so that $[+iH-\lambda]$ is invertible.

c) Define the resolvent $R(-iH,\lambda) = [-iH-\lambda]^{-1}$. Then $R(-iH,\lambda)$ is bounded and the following inequality holds.

$$\|R(-iH,\lambda)\| \leq |\mathrm{Re}\ \lambda|^{-1} \qquad \mathrm{Re}\ \lambda \neq 0 .$$

Let $u \in D(R(-iH,\lambda))$, $\mathrm{Re}\ \lambda \neq 0$. Then:

i) $u = [-iH-\lambda]\phi$ for some $\phi \in D(H)$

ii) $R(-iH,\lambda)u = \phi \in D(H)$

iii) $[-iH-\lambda]R(-iH,\lambda)u = [-iH-\lambda]\phi = u$.

Now using the result of a):

$$\|R(-iH,\lambda)u\| \leq \frac{1}{|\mathrm{Re}\ \lambda|} \|[iH + \lambda]R(-iH,\lambda)u\|$$

$$= \frac{1}{|\mathrm{Re}\ \lambda|} \|u\| .$$

c) follows immediately.

Since H is closed and $R(-iH,\lambda)$ satisfies the resolvent estimate (2.13) the above proposition is proved by virtue of Theorem 2.4. □

2.3 Spectral Decomposition of H

Suppose that the operator H is not only symmetric but also self-adjoint. Then by the spectral theorem (see [SINHA; p.197]) there exists a unique spectral family of projections $\{E_\lambda\}$ such that:

i) $E_\lambda E_\mu = E_{\min(\lambda,\mu)}$

ii) $\underset{\lambda \to -\infty}{\text{s-lim}}\ E_\lambda = 0$ and $\underset{\lambda \to +\infty}{\text{s-lim}}\ E_\lambda = I$

iii) $H = \int_{-\infty}^{+\infty} \lambda\ dE_\lambda$

i.e., $(f,Hg) = \int_{-\infty}^{+\infty} \lambda \, d(f,E_\lambda g)$ for $f \in H$, $g \in D(H)$.

Then we can define:

$$u(\tau) \equiv \exp(-iH\tau) = \int_{-\infty}^{+\infty} \exp(-i\lambda\tau) dE_\lambda$$

i.e., $(f,u(\tau)g) = \int_{-\infty}^{+\infty} \exp(-i\lambda\tau) d(f,E_\lambda g) \quad f \in H, \; g \in D(H)$.

By proposition 5.11 in [SINHA] $u(\tau)$ forms a strongly continuous one parameter unitary group whose infinitismal generator is the self adjoint operator H. Thus $u(\tau)$ satisfies the following.

i) strong continuity:

$$\text{s-}\lim_{\tau \to 0} (u(t+\tau) - u(t)) = 0 \text{ for every } t \in R$$

ii) unitarity:

$$u^*(\tau) = u^{-1}(\tau)$$

iii) group property:

$$u(t)u(s) = u(s)u(t) = u(t+s) \text{ for every } t,s \in R$$

$$u(0) = I$$

iv) generator:

$$\text{s-}\lim_{\tau \to 0} i(u(\tau) - I)/\tau = H \quad \text{on} \quad D(H)$$

v) commutation:

$$Hu(\tau) = u(\tau)H \qquad \text{on} \qquad D(H)$$

vi) Schrödinger equation:

$$i \frac{d}{d\tau} u(\tau) = H u(\tau) \qquad \text{on} \qquad D(H).$$

Since the Cauchy problem for equation (2.1) is uniformly correct it has a unique solution and thus $u(\tau)$ as defined here is identical to $u(\tau)$ as defined above in terms of the solution of (2.1) (section 2.2). □

2.4 Relativistic Covariance

The wave functions ψ which appear in (2.1) are assumed to transform under the Lorentz transformation as follows. (The term "Lorentz transformation" is used here in the generalized sense of a representation of the Desitter group $SO(n,m)$ $(n+m = d)$ of transformations of R^d which leave the product

$$x^\mu x_\mu \equiv \sum_{\mu=0}^{m-1} x_\mu^2 - \sum_{\mu=m}^{d-1} x_\mu^2$$

invariant for every $x \in R^d$.) If an observer in a reference frame 0 describes the state of a system by the function $\psi\colon R^d \to C^k$ (k is some positive integer; for example $k = 1$ for spin zero particles and $k = 2$ or 4 for spin half particles), then an observer in a reference frame 0' would describe the same state by the function $\psi': R^d \to C^k$ where ψ and ψ' are related by (2.14).

$$\psi' = S\psi \tag{2.14}$$

S is a unitary operator defined by

$$(S\psi)(x') = S(\Lambda)\psi(x) \ : \ x' = \Lambda x \tag{2.15}$$

where Λ represents the Lorentz transformation from 0 to 0' and $S(\Lambda)$ is a $k \times k$ unitary matrix representation of the Lorentz group SO(n,m).

$$S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1\Lambda_2) \tag{2.16}$$

The following establishes the relativistic covariance of equation (2.1).

<u>*Proposition 2.6:*</u> *Let* A^μ, V, v *be the fields and potentials in* (2.3) *as observed in the frame* 0 *and* $A^{\mu'}$, V', v' *be the same fields as observed from frame* 0'. *Assume that these fields transform in the following way.*

$$v'(x') = v(x) \tag{2.17}$$

$$A^{\mu'}(x) = \Lambda^\mu{}_\nu A^\nu(x) \tag{2.18}$$

$$V'(x') = S(\Lambda)V(x)S^{-1}(\Lambda) \tag{2.19}$$

Here $\Lambda^\mu{}_\nu$ *are the components of the* d-*dimensional matrix associated with the linear transformation* Λ *and satisfy*

$\Lambda_{\mu\nu}\Lambda^{\nu\rho} = g^{\rho}_{\mu}$. *Define the operator* H' *to be the closure of* $\hat{H}'$ *where* $\hat{H}'$ *is the operator defined in (2.3) with the function* A^{μ}, V, v *replaced by the functions* $A^{\mu'}$, V', v' *respectively. Then*

i) $H' = SHS^{-1}$. (2.20)

ii) *Equation (2.1) is covariant in the following sense.*

$$i \frac{d}{d\tau} \psi = H\psi \Rightarrow i \frac{d}{d\tau} \psi' = H'\psi' .$$

<u>Proof of i)</u>:

1a) $[(P-A')^{\mu}(P-A')_{\mu}S\psi](x')$

$$= \left(i \frac{\partial}{\partial x'_{\mu}} - A^{\mu'}(x')\right)\left(i \frac{\partial}{\partial x^{\mu'}} - A'_{\mu}(x')\right)\psi'(x')$$

$$= \Lambda^{\mu}{}_{\rho}\Lambda_{\mu\sigma}\left(i \frac{\partial}{\partial x_{\rho}} - A^{\rho}(x)\right)\left(i \frac{\partial}{\partial x_{\sigma}} - A^{\sigma}(x)\right)S(\Lambda)\psi(x)$$

$$= S(\Lambda)g_{\rho\sigma}[(P-A)^{\rho}(P-A)^{\sigma}\psi](x)$$

$$= [S(P-A)^{\mu}(P-A)_{\mu}\psi](x')$$

$\therefore \quad (P-A')^{\mu}(P-A')_{\mu}S = S(P-A)^{\mu}(P-A)_{\mu}$

1b) $[v'S\psi](x') = v'(x')S(\Lambda)\psi(x)$

$$= S(\Lambda)v(x)\psi(x) = S(\Lambda)[v\psi](x) = [Sv\psi](x')$$

$\therefore \quad v'S = Sv$

1c) $[V'S\psi](x') = V'(x')S(\Lambda)\psi(x)$

$$= S(\Lambda)V(x)S^{-1}(\Lambda)S(\Lambda)\psi(x)$$

$$= S(\Lambda)[V\psi](x) = [SV\psi](x')$$

$\therefore \quad V'S = SV$.

From 1a) - 1c) we find that

$$\hat{H}'S = ((2M)^{-1}[(P-A')^{\mu}(P-A')_{\mu} + V'] + v)S$$

$$= S((2M)^{-1}[(P-A)^{\mu}(P-A)_{\mu} + V] + v) = S\hat{H} .$$

Consider a sequence $\{\psi_n\} \in C_0^{\infty}$ which converges to the function ψ in L^2 norm. Since S is unitary it is bounded and continuous so that

$$\psi_n \to \psi \Rightarrow S\psi_n \to S\psi .$$

Therefore we have the following convergence in the norm topology on H.

$$SH\psi = S \lim_{n\to\infty} \hat{H}\psi_n = \lim_{n\to\infty} S\hat{H}\psi_n$$

$$= \lim_{n\to\infty} \hat{H}'S\psi_n = H'S\psi .$$

Therefore $H' = SHS^{-1}$.

Proof of ii):

$$i \frac{d}{d\tau} \psi' = i \lim_{h\to 0} [S\psi(\tau+h) - S\psi(\tau)]/h$$

$$= i\, S \lim_{h\to 0} [\psi(\tau+h) - \psi(\tau)]/h$$

$$= S\, i \frac{d}{d\tau} \psi = SH\psi = (SHS^{-1})(S\psi)$$

$$= H'\psi' . \qquad \square$$

Example 2.7: Consider the case where ψ is a spinor (i.e., the appropriate representation for spin half particles). Then $k = 4$ and $S(\Lambda)$ satisfies (2.21).

$$S(\Lambda)\sigma_{\rho\sigma}S^{-1}(\Lambda) = \Lambda_{\mu\rho}\Lambda_{\sigma\nu}\sigma^{\mu\nu} \tag{2.21}$$

$$\sigma^{\mu\nu} \equiv \frac{i}{4}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) \tag{2.22}$$

γ^{μ} ($\mu = 0,3$) are 4×4 matrices which satisfy the anticommutation relation (2.23) below. (We use the representation for γ^{μ} given by [Bjorken and Drell] (1964) for which the matrices $\sigma^{\mu\nu}$ are all hermitian.)

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\nu} = 2g^{\mu\nu} . \tag{2.23}$$

Consider the hermitian matrix potential $V(x) = \sigma^{\mu\nu}F_{\mu\nu}(x)$ where $F_{\mu\nu}(x)$ represents the electomagnetic field tensor.

$$F_{\mu\nu}(x) = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(x) . \tag{2.24}$$

The tensor $F_{\mu\nu}$ transforms under the action of the Lorentz group as a second rank tensor according to (2.25).

$$F'_{\mu\nu}(x') = \Lambda_{\mu\rho}\Lambda_{\nu\sigma}F^{\rho\sigma}(x) \tag{2.25}$$

Therefore, V(x) transforms according to (2.19) above.

$$\begin{aligned} V'(x') &= \sigma^{\mu\nu}F'_{\mu\nu}(x) = \Lambda_{\mu\rho}\Lambda_{\nu\sigma}\sigma^{\mu\nu}F^{\rho\sigma}(x) \\ &= [S(\Lambda)\sigma_{\rho\sigma}S^{-1}(\Lambda)]F^{\rho\sigma}(x) \\ &= S(\Lambda)[\sigma_{\rho\sigma}F^{\rho\sigma}(x)]\, S^{-1}(\Lambda) \\ &= S(\Lambda)V(x)S^{-1}(\Lambda)\ . \end{aligned} \tag{2.26}$$

2.5 Covariance of Expectation Values of Observables

With each observable quantity of a quantum mechanical system we associate a self-adjoint operator. If a measurement of this quantity is performed on a large number of identical systems then the average measurement obtained is given by the expectation value of the associated operator. The expectation value of a self adjoint operator B for a system in the state described by the function $\psi \in D(B)$ is defined as follows.

$$\langle B\rangle_{\psi} = (\psi, B\psi)\ . \tag{2.26}$$

Consider a set of observable quantities b^{μ} which transform as the components of a vector under the Lorentz transformation.

$$b'_\mu = \Lambda_{\mu\nu} b^\nu . \tag{2.27}$$

Let B_μ be the associated self-adjoint operators and define

$$B'_\mu = \Lambda_{\mu\nu}(SB^\nu S^{-1}) . \tag{2.28}$$

Then the associated expectation values are related covariantly according to

$$\begin{aligned} \langle B'_\mu \rangle_{\psi'} &= \Lambda_{\mu\nu}(S\psi, [SB^\nu S^{-1}]S\psi) \\ &= \Lambda_{\mu\nu}(\psi, B^\nu \psi) \\ &= \Lambda_{\mu\nu} \langle B^\nu \rangle \psi . \end{aligned} \tag{2.29}$$

Similarly if $B_{\mu\nu}$ are the operators associated with the components of a second rank tensor we define

$$B'_{\mu\nu} = \Lambda_{\mu\rho}\Lambda_{\mu\sigma}(SB^{\rho\sigma}S^{-1}) . \tag{2.30}$$

Then the analog of (2.29) becomes

$$\langle B'_{\mu\nu} \rangle_{\psi'} = \Lambda_{\mu\rho}\Lambda_{\mu\sigma}\langle B^{\rho\sigma}\rangle_\psi . \tag{2.31}$$

This is easily generalized to tensors of any rank. Suppose the set of quantities b_μ transforms as a tensor of rank r. (Here μ represents a sequence of r positive numbers $\mu_i \leq d$ $(i = 1,r)$.) Thus b'_μ is defined by

$$b'_\mu = \sum_{\substack{\nu_i \leq d \\ (i=1,r)}} T^\nu_\mu b_\nu \quad ; \quad T^\nu_\mu = \prod_{i=1}^{r} \Lambda_{\mu_i}^{\ \nu_i} . \tag{2.32}$$

Then if B_μ are the associated operators we define

$$B'_\mu = \sum_{\substack{\nu_i \le d \\ (i=1,r)}} T^\nu_\mu (S B_\nu S^{-1}) \; . \tag{2.33}$$

Then the expectation values transform covariantly.

$$\langle B'_\mu \rangle_{\psi'} = \sum_{\substack{\nu_i \le d \\ (i=1,r)}} T^\nu_\mu \langle B_\nu \rangle_\psi \; . \tag{2.34}$$

Note that Since S is unitary and the operators B_ν are self-adjoint the operators B'_μ are also self adjoint.

CHAPTER THREE

THE PROPAGATOR

3.1 Systems with Constant and Uniform Electromagnetic Fields

Definition 3.1: A one parameter family of functions $K(x,y;\tau)$: $R^d \times R^d \to C^{k\times k}$ *that are measurable and locally integrable on* $R^d \times R^d$ *is called the propagator for the evolution* $\{u(\tau);\ \tau \in R^+\}$ *if for all* $f \in C_0^\infty(R^d, C^k)$

$$[u(\tau)f](x) = \int dy\ K(x,y;\tau) f(y) \tag{3.1}$$

for almost all x. □

In most circumstances the propagator can be obtained as the fundamental solution of

$$i \frac{\partial}{\partial \tau} K(x,y;\tau) = H_{op} K(x,y;\tau) \tag{3.2}$$

$$\lim_{\tau \to 0} K(x,y;\tau) = \delta(x-y) \tag{3.3}$$

where H_{op} is the differential operator defined by

$$H_{op}K(x,y;\tau) \equiv \left\{(2M)^{-1}\left[\left(i \frac{\partial}{\partial x^\mu} - A_\mu(x)\right)\left(i \frac{\partial}{\partial x_\mu} - A^\mu(x)\right) + V(x)\right] + v(x)\right\} K(x,y;\tau) . \tag{3.4}$$

In [SCHWINGER], Schwinger has developed a method whereby for constant and uniform electromagnetic fields the solution $K(x,y;\tau)$ to (3.2) - (3.3) can be calculated exactly.

Consider the case where $v(x) = V(x) = 0$. On C_0^∞, $H = H_{op}$ (see (2.1)). Recall that $\hat{H}$ is symmetric and suppose that its closure H is self-adjoint. Then $u(\tau) = \exp(-iH\tau)$ is unitary and strongly continuously differentiable and satisfies

$$i \frac{d}{d\tau} u(\tau) = H u(\tau) \tag{3.5}$$

$$[H, u(\tau)] \equiv H u(\tau) - u(\tau) H = 0 . \tag{3.6}$$

Define the following operators:

$$x^\mu(\tau) \equiv u^\dagger(\tau) x^\mu u(\tau) \tag{3.7}$$

$$\Pi^\mu(\tau) \equiv u^\dagger(\tau) \Pi^\mu u(\tau) \tag{3.8}$$

where x^μ and Π^μ are defined by

$$D(\hat{x}^\mu) = D(\hat{\Pi}^\mu) = C_0^\infty$$

$$[\hat{x}^\mu \phi](x) = x^\mu \phi(x) \quad \text{for} \quad \phi \in C_0^\infty \tag{3.9}$$

$$[\hat{\Pi}^\mu \phi](x) = \left\{ i \frac{\partial}{\partial x_\mu} - A^\mu(x) \right\} \phi(x) \quad \text{for} \quad \phi \in C_0^\infty \tag{3.10}$$

and x^μ, Π^μ are the closures of the operators $\hat{x}^\mu$, $\hat{\Pi}^\mu$ respectively. The the Hiesenberg operators x^μ, Π^μ satisfy the Ehrenfest relations:

$$\frac{d}{d\tau} x^\mu(\tau) = i[H, x^\mu] = \frac{\Pi^\mu}{M} \tag{3.11}$$

$$\frac{d}{d\tau} \Pi^\mu(\tau) = i[H, \Pi^\mu] = \frac{F^{\mu\rho}}{M} \Pi_\rho + \frac{i}{M} \frac{\partial}{\partial x^\rho} F^{\mu\rho} \tag{3.12}$$

since $[\Pi_\mu, \Pi_\nu] = i\left[\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right] = iF_{\mu\nu}$.

If $F^{\mu\nu}$ is constant and uniform (i.e., $\partial F^{\mu\rho}/\partial x^\rho = 0$) then in matrix notation we can write:

$$\frac{d}{d\tau} x(\tau) = \frac{\Pi}{M}(\tau) \tag{3.13}$$

$$\frac{d}{d\tau} \Pi(\tau) = \frac{F}{M} \Pi(\tau) . \tag{3.14}$$

Since F is constant and uniform we can define the matrix

$$\exp(F\tau/M) \equiv \lim_{n\to\infty} \sum_{j=0}^{n} (F\tau/M)^j \tag{3.15}$$

which converges and has the property

$$\frac{\partial}{\partial\tau} \exp(F\tau/M) = -\frac{F}{M} \exp(F\tau/M) . \tag{3.16}$$

Also if we impose the condition $\det(F) \neq 0$ then the inverse of F exists

$$F^{-1\,\alpha\beta} = \mathfrak{f}^{\alpha\beta}/\det(F) \tag{3.17}$$

where $\mathfrak{f}^{\alpha\beta}$ is the dual electromagnetic field tensor

$$\mathfrak{f}^{\alpha\beta} \equiv \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} . \tag{3.18}$$

In four dimensions (d = 4) we have

$$F = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

$\vec{E} = (E_1, E_2, E_3)$ = electric field

$\vec{B} = (B_1, B_2, B_3)$ = magnetic field

$$\det(F) = -\frac{1}{3} F_{\alpha\beta} \delta^{\alpha\beta} = \vec{E}\cdot\vec{B} \tag{3.19}$$

Thus $\det(F) \neq 0$ is a Lorentz invariant statement ((3.19) indicates that $\det(F)$ is a Lorentz invariant) which requires that the electric and magnetic fields not be perpendicular. Now we can solve relations (3.13) - (3.14).

$$\Pi(\tau) = \exp(F\tau/M)\Pi(0) \tag{3.20}$$

$$x(\tau) - x(0) = F^{-1}[\exp(F\tau/M) - 1]\Pi(0) \tag{3.21}$$

We can now write $\Pi(\tau)$ as a function of $x(\tau)$ and $x(0)$.

$$\Pi(\tau) = \frac{1}{2} F \exp(-F\tau/2M)[\sinh(F\tau/2M)]^{-1}[x(\tau) - x(0)] \tag{3.22}$$

and so by the antisymmetry of F

$$\hat{H} = \frac{\Pi^2(0)}{2M} = \frac{\Pi^2(\tau)}{2M} = [x(\tau) - \dot{x}(0)]A[x(\tau) - x(0)] \tag{3.25}$$

$$= x(\tau)Ax(\tau) - 2x(\tau)Ax(0) + x(0)Ax(0)$$

$$- \frac{i}{2} \mathrm{tr}[F\coth(F\tau/2M)] \tag{3.26}$$

where $A \equiv (8M)^{-1}F^2[\sinh(F\tau/2M)]^{-2}$ and where we have used the commutator (3.27) below to arrive at (3.26) from (3.25).

$$[x_\mu(\tau), x_\nu(\tau)] = i[F^{-1}(\exp(-F\tau/M) - 1)]_{\mu\nu} . \tag{3.27}$$

Let $f, g \in C_0^\infty$. Since $u(\tau)$ is unitary and x is symmetric we have (denoting $u(\tau)$ simply by u):

$$\begin{aligned}(f, Hug) &= (f, uHg) \\ &= (f, u[x(\tau)Ax(\tau) - 2x(\tau)Ax(\tau) + x(0)Ax(0) \\ &\qquad - \tfrac{i}{2}\operatorname{tr}[F\coth(F\tau/2M)]]g) \\ &= (xAxf, ug) - 2(xAf, ug) + (f, uxAxg) \\ &\qquad - \tfrac{i}{2}\operatorname{tr}[F\coth(F\tau/2M)](f, g) .\end{aligned} \tag{3.28}$$

Now suppose that on C_0^∞ $u(\tau)$ is an integral operator with kernel $K(x, y; \tau) \in C^1(R^d \times R^d \times R)$ such that

$$[u(\tau)f](x) = \int dy\, K(x, y; \tau) f(y) \quad \text{for} \quad f \in C_0^\infty . \tag{3.29}$$

Then from (3.28) we have:

$$\begin{aligned}(f, Hug) = \int & dx dy f^\dagger(x) K(x, y; \tau)[(x-y)A(x-y) \\ & - \tfrac{1}{2}\operatorname{tr}(F\coth(F\tau/2M))]g(y) .\end{aligned} \tag{3.30}$$

Now define $\psi(x; \tau) = [u(\tau)g](x) = \int dy\, K(x, y; \tau) g(y)$. Then since $K(x, y; \tau) \in C^1(\tau)$ and $g \in C_0^\infty$:

$$\frac{\partial}{\partial\tau}\psi(x; \tau) = \frac{\partial}{\partial\tau}\int dy K(x, y; \tau) g(y) = \int dy \frac{\partial K}{\partial\tau}(x, y; \tau) g(y) . \tag{3.31}$$

This result follows from a theorem in analysis which states that (3.31) is valid for all $\tau \in R$ provided the following conditions are satisfied:

i) For almost all $y \in R^d$, $K(x,y;\tau)g(y)$ is $C^1(\tau)$.

ii) For every $\tau \in R$, $\frac{\partial}{\partial\tau} K(x,y;\tau)g(y) \in L^1(R^d_y)$.

iii) For every closed interval $T \in R$, there exists a function $g_T \in L^1(R^d_y)$ such that for all $(\tau,y) \in T \times R^d$

$$\left|\frac{\partial K}{\partial\tau}(x,y;\tau)g(y)\right| \leq g_T(y) .$$

Here, since $g(y)$ has compact support we can simply take

$$g_T(y) = \sup_{\tau\in T} \left|\frac{\partial K}{\partial\tau}(x,y;\tau)g(y)\right| \in L^1(R^d_y) .$$

Define $\psi_\delta(\tau) = \frac{1}{\delta}[u(\tau+\delta) - u(\tau)]g$, $\delta > 0$; then

$$\frac{d}{d\tau} ug = \operatorname*{s-lim}_{\delta \to 0} \psi_\delta .$$

From the strongly convergent sequence $\psi_\delta(\tau)$ we can extract a subsequence $\psi'_\delta(\tau)$ which converges pointwise almost everywhere in R^d [RUDIN; Theorem 3.12]. However, since $\psi(\tau) \in C^1(\tau)$ this pointwise limit is simply the partial derivative $\frac{\partial\psi}{\partial\tau}(\tau)$. Thus

$$\frac{d}{d\tau}\psi(\tau) = \frac{\partial}{\partial\tau}\psi(\tau) = \int dy \frac{\partial K}{\partial\tau}(x,y;\tau)g(y) . \tag{3.32}$$

Now combining equations (3.2), (3.30) and (3.32) we find

$$(f,(\frac{d}{d\tau} - H)u(g) = 0$$

$$= \int dxdy f^{\dagger}(x) \left\{ i \frac{\partial K}{\partial \tau} - [(x-y)A(x-y) - \frac{1}{2} tr(F\coth(F\tau/2M) \right\}]Kg(y) . \qquad (3.33)$$

In order to satisfy (3.33) it is sufficient that $K(x,y;\tau)$ satisfy

$$i \frac{\partial}{\partial \tau} K(x,y;\tau) = [(x-y)A(x-y) - \frac{1}{2} tr(F\coth(F\tau/2M))]K(x,y;\tau) . \qquad (3.34)$$

This may be integrated to yield

$$K(x,y;\tau) = c(x,y)2M\tau^{-d/2} \exp\{- \frac{1}{2} tr \ln [(F\tau/2M)^{-1} \sinh(F\tau/2M)] + \frac{i}{4} (x-y)[F\coth(F\tau/2M)](x-y)\} \qquad (3.35)$$

where $c(x,y)$ can be determined as follows. Let $f,g \in C_0^{\infty}$. Then since $K(x,y;\tau) \in C^k(R_x^d \times R_g^d \times R)$ we can use the analog of the above theorem regarding the interchange of integration with partial differentiation with respect to a parameter in the integrand to show

$$(f,\Pi ug) = (\Pi f,ug) = \int dx [(i\partial-A)_x f(x)]^{\dagger} [ug](x)$$

$$= \int dx\, f^{\dagger}(x)(i\partial-A)_x [ug](x)$$

$$= \int dx\, f^{\dagger}(x)(i\partial-A)_x \int dy\, K(x,y;\tau)g(y)$$

$$= \int dx\, f^{\dagger}(x) \int dy (i\partial-A)_x K(x,y;\tau)g(y)$$

$$= \iint dxdy\, f^{\dagger}(x)(i\partial-A)_x K(x,y;\tau)g(y) . \qquad (3.36)$$

The third step involves integration by parts where the surface term vanishes because of the compactness of support of f. The fifth step uses the theorem referred to above.

$$[u\Pi g](x) = \int dy\, K(x,y;\tau)(i\partial - A)_y g(y)$$

$$= \int dy [(-i\partial - A)_y K(x,y;\tau)] g(y) . \qquad (3.37)$$

The last step involves integration by parts where the surfaces term vanishes because of the compactness of support of g(y). From relation (3.22) we derive:

$$(f,\Pi(0)u(\tau)g) = (f,u(\tau)\Pi(\tau)g)$$

$$= \frac{1}{2} F \exp(-F\tau/2M)[\sinh(F\tau/2M)]^{-1}(f,u[x(\tau) - x(0)]g)$$

$$= \frac{1}{2} F \exp(-F\tau/2M)[\sinh(F\tau/2M)]^{-1}$$

$$\times \int dxdy\ f^{\dagger}(x)K(x,y;\tau)g(y)(x-y) \qquad (3.38)$$

$$(f,u(\tau)\Pi(0)g)$$

$$= \frac{1}{2} F \exp(F\tau/2M)[\sinh(F\tau/2M)]^{-1}(f,u(\tau)[x(\tau) - x(0)]g)$$

$$= \frac{1}{2} F \exp(F\tau/2M)[\sinh(F\tau/2M)]^{-1}$$

$$\times \int dxdy\ f^{\dagger}(x)K(x,y;\tau)g(y)(x-y) . \qquad (3.39)$$

Thus comparing equations (3.36) to (3.38) and (3.37) to (3.39) we have

$$0 = \int dxdy\ f^{\dagger}(x)\{(i\partial-A)_x$$

$$-\frac{1}{2}\ F\exp(-F\tau/2M)[\sinh(F\tau/2M)]^{-1}(x-y)]K(x,y;\tau)g(y) \tag{3.40}$$

$$0 = \int dxdy\ f^{\dagger}(x)\{(-i\partial - A)$$

$$-\frac{1}{2}\ F\exp(F\tau/2M)[\sinh(F\tau/2M)]^{-1}\}K(x,y;\tau)g(y) \tag{3.41}$$

which is satisfied if:

$$\{(i\partial-A)_x - \frac{1}{2}F\exp(-F\tau/2M)[\sinh(F\tau/2M)]^{-1}(x-y)\}K(x,y;\tau) = 0 \tag{3.42}$$

$$\{(-i\partial-A)_y - \frac{1}{2}F\exp(F\tau/2M)[\sinh(F\tau/2M)]^{-1}(x-y)\}K(x,y;\tau) = 0\ . \tag{3.43}$$

Now substituting (3.35) into (3.42) - (3.43) yields

$$[(i\partial-A)_x - \frac{1}{2}\ F(x-y)]c(x,y) = 0 \tag{3.44}$$

$$[(-i\partial-A)_y\ \ \frac{1}{2}\ F(x-y)]c(x,y) = 0\ . \tag{3.45}$$

This can be integrated to yield the following solution:

$$c(x,y) = c(x,x)\exp\left[-i\int_y^x d\xi^{\mu}[A(\xi) + \frac{1}{2}\ F(\xi-y)]_{\mu}\right]. \tag{3.46}$$

Since $A(\xi) + \frac{1}{2}\ F(\xi-y)$ has a vanishing curl the integral is independent of the path of integration and if we integrate along a straight path from y to x the second term does not contribute because of the antisymmetry of F $(F_{\mu\nu} = -F_{\nu\mu})$. Thus

$$c(x,y) = c(x,x)\exp\left|-i\int_0^1 d\xi\ A_\mu(w(\xi))(x-y)^\mu\right| \qquad (3.47)$$

where $w(\xi) \equiv y + \xi(x-y)$.

Consider the special case $A^\mu = 0$. Then $K(x,y;\tau)$ is given by

$$K(x,y;\tau)\Big|_{A^\mu=0} = c(x,x)\left(\frac{2M}{\tau}\right)^{d/2}\exp\left[\frac{iM}{2\tau}(x-y)^2\right] . \qquad (3.48)$$

Comparing this to the well known result for the free propogator:

$$K_0(x,y;\tau) = \left(\frac{M}{2\pi\tau}\right)^{d/2}\exp\left(-i\frac{\pi}{4}\,\mathrm{sgn}(g_{\mu\nu})\right)\exp\left[\frac{iM}{2\tau}(x-y)^2\right] \qquad (3.49)$$

where $\mathrm{sgn}(g_{\mu\nu})$ = (number of positive eigenvalues of $g_{\mu\nu}$)

− (number of negative eigenvalues of $g_{\mu\nu}$) .

We must have

$$c(x,x) = (4\pi)^{-d/2}\exp\left(-i\frac{\pi}{4}\,\mathrm{sgn}(g_{\mu\nu})\right) . \qquad (3.50)$$

Thus the candidate solution for equations (3.2) - (3.3) is given by

$$\begin{aligned} K(x,y;\tau) = {} & \left(\frac{M}{2\pi\tau}\right)^{d/2}\exp\left(-i\frac{\pi}{4}\,\mathrm{sgn}(g_{\mu\nu})\right) \\ & \times \exp\Bigg\{-i\int_0^1 d\xi\ A_\mu(w(\xi))(x-y)^\mu \\ & - \frac{1}{2}\,\mathrm{tr}\ln\left[(F\tau/2M)^{-1}\sinh(F\tau/2M)\right] \\ & + \frac{i}{4}(x-y)F\,\mathrm{cotanh}(F\tau/2M)(x-y)\Bigg\} . \end{aligned} \qquad (3.51)$$

One can now verify that $K(x,y;\tau)$ above satisfies (3.2) and also that it satisfies (3.3) in the following sense:

$$\text{for } \phi \in C_0^\infty, \quad \lim_{\tau\to 0^+} \int dy\, K(x,y;\tau)\phi(y) = \phi(x) . \tag{3.52}$$

The proof of (3.52) depends on the following stationary phase asymptotic formula.

THEOREM 3.2: *Let* $h: R^d \times R^d \to C$ *be a function satisfying*

i) there exists a compact set $\gamma \in R^d$ *whose interior contains* supp $h(z,\lambda)$ *for all* $\lambda \in R^+$.

ii) for every d-*component multi-index* β *of length* $|\beta| \le d$ *the partial derivatives* $\nabla^\beta h: R^d \times R^d \to C$ *exist and are continuous.*

For $0 \neq \lambda \in R^+$ *define the integral*

$$I(\lambda) = (\pi\lambda)^{d/2}\exp(-i\tfrac{\pi}{4}\,\mathrm{sgn}(g_{\mu\nu}))\int dy\, \exp\{\frac{iz^2}{\lambda}\}h(z,\lambda),$$

then $\lim_{\lambda\to 0} I(\lambda) = h(0,0)$.

Proof: See [POM, Lemma 4; Fed]. □

3.2 Singular Perturbation Expansion for Non Uniform Fields

Consider now the more general case when the electromagnetic field $F(x)$ is independent of the parameter τ but has some non trivial dependence on its space time argument x.

Take for the Hamiltonian $\hat{H}$ the most general form defined by (2.3) where $A^{\mu}(x)$, $V(x)$, $v(x) \neq 0$. As in the case of the uniform electromagnetic field the kernel (if it exists) of the evolution operator $u(\tau)$ for this more general system is often given by the solution of equationss (3.2) - (3.3). Taking a cue from the uniform field case we postulate that the solution of (3.2) - (3.3) can be expressed in the following form:

$$K(x,y;\tau) = K_0 \exp[-iJ]T \tag{3.53}$$

where:

i) K_0 is the free propagator

$$K_0(x,y;\tau) \equiv \left(\frac{M}{2\pi\tau}\right)^{d/2} \exp\left(-i\,\frac{\pi}{4}\,\mathrm{sgn}(g_{\mu\nu})\right)\exp\left[\frac{iM}{2\tau}\,(x-y)^2\right] \tag{3.54}$$

ii) J is a gauge carrying phase factor

$$J(x,y) \equiv \int_0^1 d\xi\,(x-y)^{\mu}A_{\mu}(w(\xi)) + v(w(\xi))\tau \tag{3.55}$$

$$(w(\xi) \equiv y + \xi(x-y))$$

iii) $T = T(x,y;\tau)$ is to be determined.

Then straightforward calculation shows that $K(x,y;\tau)$ satisfies the following relation:

$$\left(i\,\frac{\partial}{\partial\tau} - H_{op}\right)K(x,y;\tau) = K_0 \exp[-iJ](O_0 + M^{-1}O_1)T \tag{3.56}$$

where the operators 0_0 and 0_1 are defined by

$$0_0 \equiv i\left(\frac{\partial}{\partial\tau} + \frac{(x-y)^{\mu}}{\tau}\,\partial^x_{\mu}\right) \tag{3.57}$$

$$0_1 \equiv \frac{1}{2}\,\partial^2_x + i\bar{f}^{\mu}\partial^x_{\mu} - \frac{1}{2}\,(\bar{f}^2 - i\partial^{\mu}_x\bar{f}_{\mu}) + \frac{V}{2} \tag{3.58}$$

$$\bar{f}_{\nu} \equiv \int_0^1 d\xi \;\; (x-y)^{\mu}F_{\mu\nu} + (\partial_{\nu}V)\tau \;\; (w(\xi)) \tag{3.59}$$

where $\partial^x_{\mu} \equiv \partial/\partial x^{\mu}$, $\partial^2_x \equiv \partial^x_{\mu}\partial^{\mu}_x = \partial^2/\partial x^{\mu}\partial x_{\mu}$ and $\bar{f}^2 \equiv \bar{f}^{\mu}\bar{f}_{\mu}$. Thus if $T(x,y;\tau)$ satisfies

$$(0_0 + M^{-1}0_1)T = 0 \tag{3.60}$$

then $K(x,y;\tau)$ satisfies (3.2). Also if $T(x,y;\tau)$ satisfies

$$T(x,y;0) = 1 \tag{3.61}$$

then $K(x,y;\tau)$ satisfies (3.3). In general equation (3.60) is as difficult to solve exactly as is the original problem (3.2); however, perturbation techniques can be used to (at least formally) solve (3.60) - (3.61) in the limit of very large mass M. We suppose that T is given by

$$T = \sum_{n=0}^{\infty} M^{-n}\; T_n(x,y;\tau) \,. \tag{3.62}$$

Substituting (3.62) into (3.60) - (3.61) and comparing coefficients of similar powers of M^{-1} yields the following recursion relations

$$T_0 = 1 \tag{3.63}$$

$$O_0 T_n = -O_1 T_{n-1} \, . \tag{3.64}$$

Relations (3.63) - (3.64) can be solved using the method of characteristics. Consider the following partial differential equation.

$$O_0 f(x,y;\tau) = i\left(\frac{\partial}{\partial\tau} + \frac{(x-y)^\mu}{\tau}\,\partial^x_\mu\right) f \equiv g(x,y;\tau) \, . \tag{3.65}$$

The resulting characteristic equations are (in symmetric form)

$$\frac{d\tau}{i} = \frac{d(x-y)^\mu}{i(x-y)^\mu/\tau} = \frac{df}{g} \, . \tag{3.66}$$

Thus the base characteristics are given by (integrating the first of equations (3.66))

$$(x-y)^\mu = \alpha^\mu \tau \qquad\qquad \alpha^\mu = \text{arbitrary constants.} \tag{3.67}$$

Integrating the outside equation in (3.66) along the path given in (3.67) yields

$$f(y,x-y,\tau) = -i\int_0^\tau g\left(y,(x-y)\,\frac{\sigma}{\tau},\sigma\right)d\sigma + c\left(\frac{x-y}{\tau}\right) \tag{3.68}$$

where $c\colon R^d \to C$ is some arbitrary function. If we require that c is a meromorphic function which is finite as x tends to y and zero in the limit as τ tends to zero, then because of the form of its argument $((x-y)/\tau)$ c can only be identically zero. Then

$$f = 0_0^{-1} g \tag{3.69}$$

where the operator 0_0^{-1} is defined by

$$0_0^{-1} h(x,y;\tau) = -i\tau \int_0^1 d\xi \; h(w(\xi),y;\tau\xi) \, . \tag{3.70}$$

Thus the solution of (3.63) - (3.64) is given by

$$T_0 = 1 \qquad T_n = (-0_0^{-1} 0_1)^n T_0 \qquad (n > 0) \tag{3.71}$$

Note that the operator 0_1 defined in (3.58) depends only on the electromagnetic field $F_{\mu\nu}$ and not on the potentials A^μ. Thus since $F_{\mu\nu}$ is gauge invariant (i.e., invariant under the transformation $A^\mu(x) \to A^\mu(x) + \partial^\mu \lambda(x)$; $\lambda: R^d \to R$) all the coefficients T_n are gauge invariant. All of the gauge structure is contained in the factor J defined in (3.55). If we replace A^μ by $(A^\mu + \partial^\mu \lambda) \equiv A^{\mu'}$, then (3.55) becomes

$$\begin{aligned} J'(x,y) &\equiv \int_0^1 d\xi \, (x-y)^\mu A'_\mu(w(\xi)) + v(w(\xi))\tau \\ &= \int_0^1 d\xi \, (x-y)^\mu A_\mu(w(\xi)) + v(w(\xi)) \\ &\qquad + \int_0^1 d\xi \, (x-y)^\mu \partial_\mu \lambda(w(\xi)) \\ &= J + [\lambda(x) - \lambda(y)] \end{aligned} \tag{3.72}$$

so that we must replace $K(x,y;\tau)$ by

$$K'(x,y;\tau) = K_0 \exp[-iJ']T = \exp[i(\lambda(y) - \lambda(x)]K(x,y;\tau) \,. \tag{3.73}$$

Define H'_{op} to be H_{op} as defined in (3.4) with A^μ replaced by $A^{\mu'} = A^\mu + \partial^\mu\lambda$. Then straightforward calculation shows that

$$H'_{op}K'(x,y;\tau) = \exp(i\Delta\lambda)K(x,y;\tau), \quad \Delta\lambda \equiv \lambda(y) - \lambda(x) \,. \tag{3.74}$$

Thus if $K(x,y;\tau)$ satisfies $i\partial K/\partial\tau = H_{op}K$ then K' satisfies

$$i \frac{\partial K}{\partial \tau} = \exp(i\Delta\lambda) \frac{\partial K}{\partial \tau} = \exp(i\Delta\lambda)\hat{H}K = H'K' \,, \tag{3.75}$$

so that our Schrödinger equation is manifestly gauge covariant. Noting that the expression (3.51) for the exact propogator for the constant and uniform electromagnetic field is given in terms of a convergent power series in the parameter M^{-1}, we compare the first few terms up to order $[0(M^{-2})$ as $M \to \infty]$ of this series with the perturbation series obtained above. From the perturbation series for uniform field $F^{\mu\nu}$ and zero potential $(v(x) = V(x) = 0)$ we obtain:

$$T_0 = 1 \tag{3.76}$$

$$T_1 = \frac{i}{24} \tau (x-y)^\mu [F^2]_{\mu\nu} (x-y)^\nu \tag{3.78}$$

$$T_2 = \frac{1}{2} T_1^2 - \frac{\tau^2}{48} \operatorname{tr}(F^2) \,. \tag{3.79}$$

This agrees with the first three terms of the exact kernel obtained in the previous section. Though the two series begin the same, they do not necessarily coincide throughout. In fact, perturbation series obtained in this manner often diverge and serve as only asymptotic approximations. However, A. Saksena (see [SAKSENA]) has outlined a method where by truncating the perturbation series after a finite number of terms we can obtain an approximate kernel. The evolution operator generated by this kernel in turn generates a wave function which approximates the exact wave function in the $L^2(R^d)$ norm topology. An error bound is determined for this approximation and is shown to be of the same order in inverse mass (M^{-1} as $M \to \infty$) as the first neglected term. The following summarizes these results and applies them to the relativistic system we are considering here.

3.3 The Approximate Propagator

Consider the perturbation series obtained above. The approximate propagator is defined by truncating the series and multiplying by a smooth cut-off function $((\chi_d * \rho_\ell)(x-y))$ so that the approximate propagator has comapct support in $(x-y)$. This is justified by the presence of the oscillatory factor $K_0 = \exp(iM(x-y)^2/\tau)$ which appears in the series. When the mass M is large this factor oscillates rapidly except near $x = y$ so that the value of the wave function at the point x should receive only very small contribution from the initial data at the point y if $|x-y|$ is large.

Define the smooth cut-off function mollifier as follows:

$$\chi_D(x) \equiv \begin{cases} 1 & |x| < D \qquad D > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.79}$$

$$\rho_\ell(x) = \begin{cases} 0 & x > \ell \\ k_\ell \exp(-\ell^2/(\ell^2-|x|^2)) & \text{otherwise} \end{cases} \tag{3.80}$$

where $\ell > 0$ and $k_\ell \equiv [\ell^d \int_{x\leq 1} dx \exp(-(1-|x|^2)^{-1})]^{-1}$

$$(\chi_D * \rho_\ell)(x) \equiv \int dy\ \chi_D(x-y)\rho_\ell(y)dy\ . \tag{3.81}$$

The above function has the following properties:

1°: $(\chi_D * \rho_\ell)$ is infinitely continuously differentiable and has compact support

$$(\chi_D * \rho_\ell) \in C_0^\infty(R^d)\ .$$

2°: The support of $(\chi_D * \rho_\ell)$ lies within a spherical ball of radius $D+\ell$ and has the numerical value 1 inside a spherical ball of radius $D-\ell$

$$(\chi_D * \rho_\ell)(x) = \begin{cases} 1 & |x| < D-\ell \\ 0 & |x| > D+\ell \end{cases}$$

3°: The derivatives of $(\chi_D * \rho_\ell)$ are bounded and their support lies within a spherical shell of inner radius $D-\ell$

and outer radius $D+\ell$, i.e., for every multi-index $\beta \neq 0$

$$\nabla^{\beta}(\chi_D * \rho_\ell)\ (x) \neq 0 \quad \text{only if} \quad D-\ell \leq |x| \leq D+\ell$$

$$\|\nabla^{\beta}(\chi_D * \rho_\ell)\|_{L^1(R^d)} = C(\beta, D, \ell) < \infty$$

4°: For any integer $n > 0$ define

$$\chi_n^{+}(x) \equiv \sup_{|\beta| \leq n} |\nabla^{\gamma}(\chi_D * \rho_\ell)|$$

then $\chi_n^{+} \in L^1(R^d) \cap L^{\infty}(R^d)$.

Define the approximate propagator as follows.

$$K_N(x,y;\tau) \equiv (\chi_D * \rho_\ell)(x-y) W_N(x,y;\tau) \tag{3.82}$$

where

$$W_N(x,y;\tau) \equiv K_0 \exp(-iJ) \sum_{n=0}^{N} M^{-n}\, T_n \,. \tag{3.83}$$

Some straightforward calculation reveals that $K_M(Q)$ (here we define the shorthand notation $Q = (x,y;\tau)$) satisfies the following inhomogeneous partial differential equation.

$$(i\frac{d}{d\tau} - \hat{H}) K_N(Q)$$

$$= M^{-(N+1)} R_N^0(Q) + \sum_{i=1}^{3} R_N^i(Q) \equiv R_N(Q) \tag{3.84}$$

where

$$R_N^0(\Omega) \equiv K_0(\Omega)\exp(-iJ(\Omega))[0_1T_N(\Omega)](\chi_D * \rho_\ell)(x-y) \qquad (3.85)$$

$$R_N^1(\Omega) \equiv (2M)^{-1}[\partial_x^2(\chi_D * \rho_\ell)(x-y)]W_N(\Omega) \qquad (3.86)$$

$$R_N^2(\Omega) \equiv M^{-1}[\partial_x^\mu(\chi_D * \rho_\ell)(x-y)]\partial_\mu^x W_N(\Omega) \qquad (3.87)$$

$$R_N^3(\Omega) \equiv iM^{-1}[A^\mu(x)\partial_\mu^x(\chi_D * \rho_\ell)(x-y)]W_N(\Omega) \qquad (3.88)$$

With the functions $R_N^j(\Omega)$ $(j = 0,3)$ and $R_N(\Omega)$ associate the operators $\tilde{R}_N^j$ and $\tilde{R}_N$, which are integral operators with kernels $R_N^j(\Omega)$ and $R_N(\Omega)$ respectively.

$$(\tilde{R}_N^j\phi)(x;\tau) \equiv \int dy\, R_N^j(x,y;\tau)\phi(y) \qquad \phi \in L^2(R^d) \qquad (3.89)$$

$$(\tilde{R}_N\phi)(x;\tau) \equiv \int dy\, R_N(x,y;\tau)\phi(y) \qquad \phi \in L^2(R^d)\,. \qquad (3.90)$$

If the potentials A^μ, v, V fall into the class of potentials defined below, then we can find an upper bound on the $L^2(R^d)$ norm of the image under $\tilde{R}_N$ of certain sufficiently smooth wave functions $\phi \in L^2(R^d)$. This allows the establishment of an upper bound on the error associated with using the approximate propagator $K_N(x,y;\tau)$.

Definition 3.3: *For integer* $N > 0$ *we say that the potentials* A^μ, v, V *belong to the class* A(N) *if*

i) A^μ, v, V $\in C^N(R^d)$

ii) *for all multi-index* β *with* $|\beta| \leq N$

$$|\nabla^\beta A^\mu|(x) \le C_A(\lambda) \equiv k^{|\beta|}C_A < \infty \tag{3.91}$$

$$|\nabla^\beta v|(x) \le C_v(\beta) \equiv k^{|\beta|}C_v < \infty \tag{3.92}$$

$$|\nabla^\beta V|(x) \le C_V(\beta) \equiv k^{|\beta|}C_V < \infty . \tag{3.93}$$

Definition 3.4: *For* $\phi \in L^2(R^d)$ *define the norm*

$$|||\phi|||_n \equiv \sup_{|\beta|\le n} \|\nabla^\beta \phi\| . \tag{3.94}$$

Associated with this norm is a complete Sobolev space S_n *given by*

$$S_n = \{\phi \in L^2(R^d) : |||\phi|||_n < \infty\} .$$

For potentials in the class A(N) the coefficient functions $T_n(x,y;\tau)$ and their derivatives satisfy the following bounds.

LEMMA 3.5: *Let* $n \ge 1$ *and suppose* A^μ, v, V *are in the class* A(2N+n). *We define the following quantities*

i) $$|g| \equiv \sup_{x\in R} \left(\sum_{\mu=0}^{d-1} |g_{\mu\nu}x^\nu|^2\right)^{\frac{1}{2}} / |x| \le \left|\sum_{\mu,\nu=0}^{d-1} |g_{\mu\nu}|^2\right|^{\frac{1}{2}} \tag{3.95}$$

ii) $$\lambda_i = 2^{2i-3} + \frac{1}{2}\lambda_{i-1};\ \lambda_1 = 1 \tag{3.96}$$

iii) $$Z \equiv Z(|x-y|,\tau) = |g|[2C_A\sqrt{d}\ (k|x-y|+n) + \tau k C_V]$$

Then $|\nabla_x^\beta T_n|(x,y;\tau)$ *satisfies the bound*

$$|\nabla_x^\beta T_n| \le (\tfrac{d\tau}{2})^n k^{|\beta|} Z[2^{|\beta|}(Z + \lambda_n k + C_V)]^{2n-1}|g| . \qquad (3.97)$$

Proof. The proof for $g = I$ ($g^{\mu\nu} = 1$ for $\mu = \nu$ and $g^{\mu\nu} = 0$ otherwise) and $V = 0$ is given in [SAKSENA; Lemma 3.1]. The generalization to $g \neq I$ is achieved by using the inequality (3.99) in place of (3.98) below.

$$|x \cdot y| \equiv \left|\sum_{i=0}^{d-1} x^i t^i\right| \le |x|\,|y| \qquad (3.98)$$

$$x^\mu y_\mu = |g_{\mu\nu} x^\mu y^\nu| \le |g|\,|x|\,|y| . \qquad (3.99)$$

The proof for $V \neq 0$ follows analogously with the inductive proof in [SAKSENA] except for the added matrix structure (which is accounted for by reinterpretation of the norms involved; see the appendix on notation) and the addition of the term $V/2$ to the operator 0_1 (which is accounted for by the addition to (3.97) of the term involving C_V). □

LEMMA 3.6: Let δ be an arbitrary d-component multi-index and assume that $\phi \in S_{n+|\delta|+1}$. Further assume that A^μ, v, V belong to the potential class A(2N+n+|δ|+1). Then

$$\|\nabla^\delta \tilde{R}_N^i \phi\| \le \text{const.}\ M^{d/2-n-1} \tau^{n-d/2}$$
$$\times \|\chi^\dagger_{n+|\delta|+2}\|_1 \ |||\phi|||_{n+|\delta|+1} \qquad (3.100)$$

For i = 1,3, the constant above depends only on D, ℓ, and the constants $\tilde{C}_A$, $\tilde{c}_V$, $\tilde{C}_V$.

Proof. The proof is analogous to the proof in [SAKSENA; Lemma 3.3] with Lemma 3.5 replacing [SAKSENA; Lemma 3.1]. □

LEMMA 3.7: *Let* δ *be an arbitrary* d-*component multi-index and assume that* $\phi \in S_{d+|\delta|}$. *Further assume that the potentials* A^{μ}, v, V *belong the class* A(2N+d+|δ|). *Then*

$$||\nabla^{\delta} R_N^0 \phi|| \leq \text{ const. } M^0 \tau^N ||\overset{+}{\chi}_{d+|\delta|}||_1 |||\phi|||_{d+|\delta|} . \qquad (3.101)$$

Proof. The proof is analogous to that in [SAKSENA; Lemma 3.4] with Lemma 3.5 replacing [SAKSENA; Lemma 3.1]. Also the factor $\exp(-i\lambda(x^i-y^i)^2)$ replaces the factor $\exp(+i\lambda(x^i-y^i)^2)$ which appears in [SAKSENA] where $g_{ii} = -1$. This is accounted for by replacing [SAKSENA; equations (3.66) - (3.69)] by their complex conjugates. □

Define the approximate evolution operator $u_N(\tau)$ by

$$[u_N(\tau)\phi](x) = \int dy\, K_N(x,y;\tau)\phi(y) \quad , \quad \phi \in L^2(R^d) . \qquad (3.102)$$

For potentials in the class A(2N+2) the operators $u_N(\tau)$ are bounded and strongly continuously differentiable with respect to τ (see [SAKSENA; proposition 3.3]). This combined with the smoothness and compact support of K_N is used to show (see [SAKSENA; equation 3.62])

$$[[i \frac{d}{d\tau} - H]u_N(\tau)\phi](x) = \int dy\, [i \frac{\partial}{\partial\tau} - H_{op}]K_N(\Omega)\phi(y) \qquad (3.103)$$

where $d/d\tau$ represents the strong derivative and $\partial/\partial\tau$ the pointwise partial derivative. Thus $[u_N \phi]$ satisfies the following abstract inhomogeneous equation.

$$[i \frac{d}{d\tau} - H]u_N(\tau)\phi = \tilde{R}_N \phi . \tag{3.104}$$

Now the following theorem [KREIN; Theorem 3.1, 3.3] concerning the solution of such inhomogeneous equations can be used to compare to the solution of the corresponding homogeneous equation.

THEOREM 3.8: *If the Cauchy problem for the homogeneous equation*

$$\frac{d\psi}{d\tau} = A\psi \qquad \tau \in [0,T] \qquad \psi(0) = \phi \tag{3.105}$$

is uniformly correct and if $f(\tau)$ *has a continuous derivative then the equation*

$$\frac{d\psi}{d\tau} = A \quad + f(\tau) \tag{3.106}$$

has a unique solution given by

$$\psi(\tau) = u(\tau)\phi + \int_0^\tau u(\tau)f(\tau)d\tau .$$

Proof. See [KREIN; Theorem 3.1, 3.3]. □

Now applying the above theorem yields

$$||[u(\tau) - u_N(\tau)]\phi|| = \left|\left|\int_0^\tau u(\tau)\tilde{R}_N(\tau)\phi\, d\tau\right|\right|$$

$$\leq \int_0^\tau \|u(\tau)\| \, \|R_N(\tau)\phi\| \, d\tau \leq \int_0^\tau \|R_N(\tau)\phi\| \, d\tau \tag{3.107}$$

Thus an application of Lemmas 3.6 and 3.7 gives rise to the following proposition.

Proposition 3.9: Define $N_1 \equiv N_1(N,d) = \max(3N+d/2+1, 2N+d)$, $N_2 \equiv N_2(N,d) = \max(N+d/2+1, d)$. *Let* $\phi \in S_N$ *and assume the potentials* A^μ, v, V *belong to the class* $A(N_1)$. *Then*

$$\|[u(\tau) - u_N(\tau)]\phi\|$$

$$\leq C(D,\ell,C_A,C_v,C_V)\, M^{-(N+1)} \tau^{N+1} \|\chi^\dagger_{N_2+1}\|_1 \, |||\phi|||_{N_2} \, . \tag{3.108}$$

Proof. See [SAKSENA; Theorem 3.1]. □

Thus for sufficiently smooth potentials and initial data the error associated with the approximate evolution operator $u_N(\tau)$ is of the same order as the first neglected term of the perturbation expansion.

3.4 Relativistic Covariance

Define the functions χ'_D, ρ'_ℓ as follows:

$$\chi'_D(x') \equiv \chi_D(x) \qquad x' = \Lambda x \tag{3.109}$$

$$\rho'_\ell(x') \equiv \rho_\ell(x) \qquad x' = \Lambda x \, . \tag{3.110}$$

Define the approximate propagator $K'_N : R^d \times R^d \times R \to C^{k\times k}$ to be the function obtained from K_N defined in (3.82) -

(3.83) by replacing the functions A^{μ}, v, V (which define K_N through (3.53) - (3.59) and (3.71)) with the functions $A^{\mu'}$, v', V' defined in (2.17) - (2.19) and by replacing the functions χ_D and ρ_{ℓ} with χ'_D, ρ'_{ℓ} respectively. Then by an argument analogous to the proof of proposition 2.6 we could show that K_N and K'_N are related by

$$K'_N(x',y';\tau) = S(\Lambda)K_N(x,y;\tau)S^{-1}(\Lambda) . \tag{3.111}$$

If we define $u'_N(\tau)$ to be the integral operator with kernel K'_N (analogously to (3.102)), then $u'_N(\tau)$ and $u_N(\tau)$ are related by

$$u'_N(\tau) = S\, u_N(\tau)\, S^{-1} . \tag{3.112}$$

Define the operator $u'(\tau)$ as follows.

$$u'(\tau) \equiv S\, u(\tau)\, S^{-1} . \tag{3.113}$$

Then one can show that $u'(\tau)$ satisfies

$$i \frac{d}{d\tau} u'(\tau) = H'\, u'(\tau) \tag{3.114}$$

$$\lim_{\tau \to 0} u'(\tau) = I . \tag{3.115}$$

Thus $u'(\tau)$ is the unique evolution operator for the Cauchy problem associated with the Hamiltonian H'. Then since S is unitary:

$$\|[u'_N(\tau) - u'(\tau)]\phi'\| = \|[u_N(\tau) - u(\tau)]\phi\| \tag{3.116}$$

$$|||\phi'|||_N = |||\phi|||_N \tag{3.117}$$

where $\phi' \equiv S\phi$. Thus the inequality (3.108) in proposition 3.9 is covariant in the sense that it is equivalent to the inequality

$$\|[u'_N(\tau) - u'(\tau)]\phi'\|$$
$$\leq C(D,\ell,\tilde{C}_A,\tilde{C}_V,\tilde{C}_V)M^{-(N+1)}\tau^{N+1}\|\chi^+_{N_2+1}\|_1|||\phi'|||_{N_2}\,. \tag{3.118}$$

Note also that the function $(\chi'_D*\rho'_\ell)$ satisfies properties 1° and 4° of the function $(\chi_D*\rho_\ell)$ and also satisfies properties 2° and 3° if D and ℓ are replaced by D' and ℓ' where

$$D' \equiv D(|\Lambda^{-1}| + |\Lambda|)/2 \tag{3.119}$$

$$\ell' \equiv \ell(|\Lambda^{-1}| + |\Lambda|)/2\,. \tag{3.120}$$

This can be seen from the following argument. Suppose that $x \in R^d$ satisfies the inequality

$$C_1 < |x| < C_2\,.$$

Then $x' = \Lambda x$ satisfies the inequality

$$C'_1 < |x'| < C'_2 \tag{3.121}$$

where $C_1' = C_1/|\Lambda^{-1}|$ and $C_2' = C_2|\Lambda|$.

Proof:

$$|x'| = |\Lambda x| \leq |\Lambda||x| < |\Lambda|C_2 = C_2'$$

$$C_1 < |x| = |\Lambda^{-1}x'| \leq |\Lambda^{-1}||x'|$$

(3.121) follows immediately. $\square$

By a similar argument we could also show that if A^μ, v, V are in the class A(N), then the potentials $A^{\mu'}$, v', V' are also in the class A(N). Thus we could obtain (3.118) in exactly the same way as we obtained (3.108) if we replace χ_D, ρ_ℓ, A^μ, v, V with χ_D', ρ_ℓ', $A^{\mu'}$, v', V'. This demonstrates the relativistic covariance of the treatment in this chapter. Note that in making a Lorentz transformation the spherical symmetry of the function $(\chi_D * \rho_\ell)$ is lost since the function $(\chi_D' * \rho_\ell')$ will not have this symmetry in general. This is the only aspect of the above analysis which is not completely covariant. However, it is only the properties 1° - 4° of the function which are important in the analysis and since these properties are preserved we need not be concerned about the loss of this symmetry. In fact, rather than specifying the function $(\chi_D * \rho_\ell)$ exactly, as we have done, we could simply have defined it to be some arbitrary member of the class of functions which satisfy properties $1^{\circ} - 4^{\circ}$. Since we have an example of such a function, this class is not empty and since properties $1^{\circ} - 4^{\circ}$ are preserved under Lorentz transformation, our description would then be completely covariant.

CHAPTER FOUR

APPLICATIONS

4.1 Green's Functions for Scalar and Spinor Valued Fields

The Green's function $G_{sc}: R^4 \times R^4 \to C$ for a free scalar field is defined by the equation

$$(P_x^2 - M^2) G_{sc}(x,y) = \delta(x-y) \tag{4.1}$$

where $P_x^\mu \equiv i\partial/\partial x_\mu$ and $P_x^2 = P_x^\mu P^x_\mu$. Using the momentum representation we have

$$G_{sc}(x,y) = (2\Pi)^{-4} \int dk \exp(ik^\mu(x-y)_\mu) \tilde{G}_{sc}(k) \tag{4.2}$$

where $\tilde{G}_{sc}(k): R^4 \to C$ is given by

$$\tilde{G}_{sc}(k) = (k^2 - M^2)^{-1} \qquad (k^2 = k^\mu k_\mu) \tag{4.3}$$

so that $G_{sc}(x,y)$ can be expressed formally as

$$G_{sc}(x,y) = (2\Pi)^{-4} \int dk \exp(ik^\mu(x-y)_\mu)(k^2-M^2)^{-1} . \tag{4.4}$$

Expression (4.4), however, is not well-defined untill we specify the way in which we go around the poles at $k^2 = M^2$. This is related to the fact that the solution of (4.1) is not unique in that we can add to $G_{sc}(x,y)$ any solution of the homogeneous equation

$$(P_x^2 - M^2) D(x,y) = 0 \tag{4.5}$$

and still obtain a solution to (4.1). A particular solution to (4.1) can be obtained by specifying boundary conditions on $G_{sc}(x,y)$. Here we shall consider the causal Green's function which describes the casual relationship between the processes of creation and annihilation of particles at different space time points. The causal Green's function is given by (see [BOG])

$$G^{c}_{sc}(x,y) = \theta(x^{o}-y^{o})\, D^{-}(x-y) - \theta(y^{o}-x^{o})D^{+}(x-y) \qquad (4.6)$$

where

$$D^{-}(x-y) \equiv i(2\pi)^{3} \int dk\, \exp(ik^{\mu}(x-y)_{\mu})\,\delta(k^{2}-M^{2})\theta(-k^{o}) \qquad (4.7)$$

and

$$D^{+}(x-y) \equiv (2\pi)^{3} \int dk\, \exp(ik^{\mu}(x-y)_{\mu})\,\delta(k^{2}-M^{2})\,\theta(k^{o})\ . \qquad (4.8)$$

Here δ represents the Dirac delta function and θ the heaviside function. $D^{-}(x-y)$ and $D^{+}(x-y)$ are the negative and positive frequency parts (respectively) of the Pauli-Jordan function which is a solution to the homogeneous equation (4.5). Thus for $x^{o} > y^{o}$, $G^{c}_{sc}(x,y)$ is proportional to $D^{-}(x-y)$ which is the matrix element describing the creation of a scalar particle at the space-time point y and its subsequent annihilation at the point x. Conversely for $y^{o} > x^{o}$ $G^{c}_{sc}(x,y)$ is proportional to $D^{+}(x-y)$ which corresponds to creation at x and annihilation at y. The causal Green's function can also be specified by choosing the path of integration which is consistent with regarding the square of the mass M which appears in the denominator of its momentum representation as containing an infinitesimal negative imaginary part (see [BOG; p.143]).

$$G^{c}_{sc}(x,y) = \lim_{\epsilon\to 0^{+}} (2\pi)^{-4} \int dk \exp(ik^{\mu}(x-y)_{\mu})(k^{2}-(M^{2}-i\epsilon))^{-1}$$

$$= \lim_{z\in C\to(M^{2},0^{-})} \int dk \exp(ik^{\mu}(x-y)_{\mu})(k^{2}-z)^{-1} . \tag{4.9}$$

The Green's function $G_{sp}(x,y)$ for a spinor valued field ($R^{4} \to C^{4}$) satisfies

$$(\not{P}_{x}-M)G_{sp}(x,y) = \delta(x-y) \tag{4.10}$$

where $\not{P} \equiv i\gamma^{\mu}\partial^{x}_{\mu}$. (The matrices γ^{μ} are defined in example 2.7.) It can be derived from that of the scalar field by the following argument.

$$G_{sp}(x,y) = \langle x|(\not{P}-M)^{-1}|y\rangle$$

$$= \langle x|(\not{P}+M)(\not{P}+M)^{-1}(\not{P}-M)^{-1}|y\rangle$$

$$= \langle x|(\not{P}+M)(P^{2}-M^{2})^{-1}|y\rangle$$

$$= (\not{P}+M)G_{sc}(x,y) . \tag{4.11}$$

In the presence of an external electromagnetic field with potential A^{μ} we must make the replacement

$$P^{\mu} \rightarrow \Pi^{\mu} \equiv P^{\mu} - A^{\mu} \tag{4.12}$$

in accordance with the prescription for minimal interaction. Then the Green's functions for the scalar and spinor fields are given by

$$\begin{aligned} G_{sc}(x,y) &= \langle x|(\Pi^2 - M^2)^{-1}|y\rangle \\ &= (2M)^{-1}\langle x|(\frac{\Pi^2}{2M} - \frac{M}{2})^{-1}|y\rangle \end{aligned} \tag{4.13}$$

$$\begin{aligned} G_{sp}(x,y) &= (\not{\Pi}+M)\langle x|(\not{\Pi}+M)^{-1}(\not{\Pi}-M)^{-1}|y\rangle \\ &= (2M)^{-1}(\not{\Pi}+M)\langle x|(\frac{\Pi^2}{2M} - \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu} - \frac{M}{2})^{-1}|y\rangle\ . \end{aligned} \tag{4.14}$$

The matrices $\sigma_{\mu\nu}$ are given in example 2.7. Now define H_{sc} and H_{sp} as

$$H_{sc} \equiv \Pi^2/2M \tag{4.15}$$

$$H_{sp} \equiv \Pi^2/2M - \sigma_{\mu\nu}F^{\mu\nu}/2\ . \tag{4.16}$$

Substituting this into (4.13) - (4.14) yields

$$G_{sc}(x,y) = (2M)^{-1}\langle x|(H_{sc} - \frac{M}{2})^{-1}|y\rangle \tag{4.17}$$

$$G_{sp}(x,y) = (\not{\Pi}+M)(2M)^{-1}\langle x|(H_{sp} - \frac{M}{2})^{-1}|y\rangle\ . \tag{4.18}$$

Now just as in the case above, with no electromagnetic field we get the causal Green's functions $G^{c}_{sc}(x,y)$ and

$G^c_{sp}(x,y)$ by considering the mass M to be complex and then taking the limit as it approaches the real axis from below.

$$G^c_{sc}(x,y) = \lim_{z \to (\frac{M}{2}, 0^-)} (2M)^{-1} \langle x|(H_{sc}-z)^{-1}|y\rangle \tag{4.19}$$

$$G^c_{sp}(x,y) = \lim_{z \to (\frac{M}{2}, 0^-)} (2M)^{-1} \langle x|(H_{sp}-z)^{-1}|y\rangle \, . \tag{4.20}$$

Note that H_{sc} and H_{sp} are two special cases of the Hamiltonian H defined in (2.3). H_{sc} corresponds to the case $v = V = 0$ and H_{sp} corresponds to the case $v = 0$, $V = -\sigma_{\mu\nu}F^{\mu\nu}/2$ (see example 2.7). Thus we will have the causal Green's functions in (4.19) - (4.20) above if we can find a kernel for the resolvent

$$R(H,z) = (H-z)^{-1} \tag{4.21}$$

where H is as in (2.3). This resolvent can be related to the evolution operator u(z) associated with H as follows.

$$R(H,z) = -i \int_{-\infty}^{0} d\tau \; u(\tau)\exp(iz\tau) \qquad \text{Im } z < 0 \, . \tag{4.22}$$

To see how this arises formally consider the operator HR(H,z).

$$HR(H,z) = -i \; H \int_{-\infty}^{0} d\tau \; u(\tau)\exp(iz\tau)$$

$$= -i \int_{-\infty}^{0} d\tau \; Hu(\tau)\exp(iz\tau)$$

$$= \int_{-\infty}^{0} d\tau \, \frac{du(\tau)}{d\tau} \, \exp(iz\tau)$$

$$= \int_{-\infty}^{0} d\tau \{ \frac{d}{d\tau} [u(\tau)\exp(iz\tau)] - i\, z \exp(iz\tau) u(\tau) \}$$

$$= [u(\tau)\exp(iz\tau)]_{-\infty}^{0} + z\, R(H,z) \, . \qquad (4.23)$$

Since Im $z < 0$ and $u(\tau)$ is a bounded operator for all τ, the lower limit in the first term does not contribute and we have

$$(H-z)R(H,z) = u(0) = I \, . \qquad (4.24)$$

Since $u(\tau) = \exp(-iH\tau)$ is a function of H it commutes with H and thus so does $R(H,z)$ from (4.22). Thus we also have

$$R(H,z)(H-z) = I \qquad (4.25)$$

so that $R(H,z) = (H-z)^{-1}$. Thus the kernel of the resolvent is given by

$$\langle x|R(H,z)|y\rangle = -i \int_{-\infty}^{0} d\tau \, \exp(iz\tau)\langle x|u(\tau)|y\rangle$$

$$= -i \int_{-\infty}^{0} d\tau \, \exp(iz\tau) K(x,y;\tau) \, . \qquad (4.26)$$

Now consider the expansion for $K(x,y;\tau)$ obtained in Chapter Three.

$$K(x,y;\tau) = K_0 \exp(-J) \sum_{n=0}^{\infty} (0_0^{-1} 0_1)^n T_0 \, , \quad T_0 = 1 \, . \qquad (4.27)$$

From the form of 0_0^{-1} and 0_1 it can be seen that the coefficients $T_n(x,y;\tau)$ must be polynomials in τ of degree 3n. Then we can write

$$\sum_{n=0}^{N} M^{-n} T_n(x,y;\tau) = \sum_{m=0}^{3N} \tau^m P_m(x,y;M) \tag{4.28}$$

where

$$P_m(x,y;M) = \sum_{n=0}^{m} M^{-n} [(\frac{d}{d\tau})^m T_n(x,y;\tau)]_{\tau=0} . \tag{4.29}$$

Define the functions $\bar{v}$ and J_0 as follows.

$$\bar{v} \equiv \bar{v}(x,y) = \int_0^1 d\xi \; v(w(\xi)) \tag{4.30}$$

$$J_0 \equiv J_0(x,y) = \int_0^1 d\xi (x-y)^\mu A_\mu(w(\xi)) . \tag{4.31}$$

Recall $w(\xi) \equiv y + \xi(x-y)$. Then $J = J_0 + \bar{v}$ and

$$K(x,y;\tau) = K_0(x,y;\tau)\exp(-iJ_0(x,y))\exp(-i\bar{v}(x,y)) \times \sum_{m=0}^{\infty} \tau^m P_m(x,y;M) . \tag{4.32}$$

Now substituting (4.32) into (4.28) we arrive at an expansion for the Green's function

$$\langle x|R(H,z)|y\rangle = \frac{M^2}{4\pi^2} \exp(-iJ_0(x,y)) \sum_{m=0}^{\infty} P_m(x,y;M) \times \int_{-\infty}^{0} d\tau \; \tau^{m-2} \exp(i \frac{M}{2\tau} (x-y)^2 + i(z-\bar{v}(x,y))\tau)$$

$$= \exp(-iJ_0(x,y)) \sum_{m=0}^{\infty} P_m(x,y;M) \left(\frac{d}{dz}\right)^m G_0(x,y;z-\bar{v}(x,y)) \tag{4.33}$$

where $G_0(x,y;z)$ is the free Green's function

$$G_0(x,y;z) = \langle x|(H_0-z)^{-1}|y\rangle \ ; \qquad (H_0 = P^\mu P_\mu/2M)$$

$$= \frac{M^2}{4\pi^2} \int_0^\infty d\tau\ \tau^{-2} \exp(-i \frac{M}{2\tau} (x-y)^2 + iz\tau) \tag{4.34A}$$

$$= -\frac{M^2}{4\pi^2} \left(\frac{2M\lambda}{z}\right)^{\frac{1}{2}} K_1(-i\sqrt{2\lambda zM}) \tag{4.34B}$$

where $\lambda \equiv (x-y)^2$ and K_1 is the modified Bessel function. As we let $z \to (M/2,0^-)$, this gives us the familiar free causal Green's function for the scalar field.

$$G^c_{sc}(x,y) = (2M)^{-1} \lim_{z\to(\frac{M}{2},0^-)} G_0(x,y;z)$$

$$= -\frac{iM}{4\pi^2\sqrt{-\lambda}} \theta(-\lambda)K_1(M\sqrt{-\lambda})$$

$$- \frac{M}{8\pi\sqrt{\lambda}} \theta(\lambda)[J_1(M\sqrt{\lambda}) - iN(M\sqrt{\lambda})] - \delta(\lambda) . \tag{4.35}$$

The term $\delta(\lambda)/4\pi$ results from regularization (see [GEL; Chapter 3]) of the Green's function and has support on the surface of the light cone. The above derivatives of $G_0(x,y;z)$ are given by

$$\left(\frac{d}{dz}\right)^m G_0(x,y;z) = \frac{(-1)^{m-1}}{2\pi^2} \left(\frac{M\lambda}{2z}\right)^{-\frac{1}{2}} K_{m-1}(-i\sqrt{2\lambda Mz}) \tag{4.36}$$

and for large $s \in c$ (see [ARFKEN; p.517])

$$K_\nu(s) \xrightarrow[s\to\infty]{} \left(\frac{\pi}{2s}\right)^{\frac{1}{2}} \exp(-s)$$

so that for large positive z

$$K_{m-1}(-i\sqrt{\lambda z}) \xrightarrow[z\to\infty]{} \left(\frac{\pi i}{2}\right)^{\frac{1}{2}} (\lambda z)^{-\frac{1}{4}} \times \begin{cases} \exp(i\sqrt{\lambda z}) \; ; \; \lambda > 0 \\ \exp(-\sqrt{-\lambda z}) \; ; \; \lambda < 0 \end{cases}$$

so that each term of the Green's function expansion is oscillatory in the time like region $\lambda > 0$ while it decays exponentially in the space like region $\lambda < 0$ (outside the light cone) where the points x and y are such that they cannot be connected by a light signal. This is consistent with the concept of causality in special theory of relativity where we would expect no (or very little) space like correlations. Also note that if z is large and negative the Green's function decays exponentially in the time like region instead. The following section will examine this behaviour by projecting the evolution operator $u(\tau)$ onto states with a particular mass spectrum.

4.2 Causality and the Mass Spectrum

Assume that the operator H defined in Chapter two is self adjoint. Then by the spectral theorem we can define the following operator

$$u_\chi(\tau) = \int_{-\infty}^{+\infty} dE_k \; \chi(k)\exp(-ik\tau) \tag{4.37}$$

where $\{E_k\}$ is the spectral family of projections associated with H and $\chi(k): R \to R$ is a smooth bounded function which represents the mass spectrum we wish to project the evolution $u(\tau) = \exp(-iH\tau)$ onto. The support of this function will determine the causal behaviour of the evolution. Typically $\chi(k)$ will be cut off smoothly above some $k = \delta > 0$ and have some decay as $k \to \infty$. We require that the Fourier transform $\tilde{\chi}(\lambda)$ fall off faster than $\lambda^{-(N+2)}$ where N is the number of terms used in the asymptotic expansion of $u(\tau)$ (see Chapter three). This will be necessary in order to bound the error in the expansion of $u_\chi(\tau)$. For example, we could take $\chi(k)$ to be a Schwartz space function of rapid decay.

Proposition 4.1: Define the function $K_N^\chi: R^d \times R^d \times R \to C^{k\times k}$ *as follows*

$$K_N^\chi(x,y;\tau) = -\frac{2M}{\Pi} \exp(-iJ(x,y)) \sum_{m=0}^{3N} P_m(x,y;M)$$

$$\times \; (\chi_D * \rho_\ell)(x-y) \int_{-\infty}^{+\infty} dk \; \mathrm{Im}[G_0^c(x,y;k)]\left(i\frac{d}{dk}\right)^m [\chi(k)\exp(-ik\tau)] \tag{4.38}$$

where $(\chi_D * \rho_\ell)$, J, P_m, G_0 *are defined in Chapters three* *and four and* $G_0(x,y;k)$ *is given by*

$$G_0^c(x,y;k) \equiv \lim_{z\to(k,0^-)} G_0^c(x,y;z) \qquad k \in R. \tag{4.39}$$

Define the operator $\tilde{K}_N^\chi(\tau)$ *by*

$$\tilde{K}_N^\chi(\tau)\phi\ (x) = \int dy\, K_N(x,y;\tau)\phi(y) \qquad \phi \in L^2(R^d). \tag{4.40}$$

Assume the function $\chi(k)$ *has compact support and satisfies*

$$c_j \equiv \binom{N}{j} \int_{-\infty}^{+\infty} d\lambda\, \tilde{\chi}(\lambda)\, |\lambda|^{N-j} < \infty \tag{4.41}$$

where $\tilde{\chi}(\lambda)$ *is the fourier transform of* $\chi(k)$

$$\tilde{\chi}(\lambda) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} dk\, \exp(-ik\lambda)\, \chi(k). \tag{4.42}$$

Then the operator $u_\chi(\tau)$ *satisfies*

$$u_\chi(\tau) = \tilde{K}_N^\chi(\tau) + \epsilon_N(\tau) \tag{4.43}$$

where $\epsilon_N(\tau)$ *is a family of operators on* $L^2(R^d)$ *which satisfy*

$$\|\epsilon_N(\tau)\phi\| \le (\text{constant})M^{-(N+1)} |||\phi|||_{N_2} \sum_{j=0}^{N} c_j |\tau|^j \tag{4.44}$$

for $\phi \in S_{N_2}$ *and* A^μ, v, V *in the class* $A(N_1)$ *(see Proposition 3.9 for definition of* N_1 *and* N_2*).*

<u>Proof</u>: See Section 4.3. □

Now since $G_0(x,y;k)$ is causal (i.e., decays exponentially in the spece like region) for $k > 0$ and anticausal (i.e., decays exponentially in the time like region) for $k < 0$ we see that (up to the order of the error term $\epsilon(\tau)$) if $\chi(k)$ has support only where $k > 0$ the projected evolution $u_x(\tau)$ is causal. Also if $\chi(k)$ has support only where $k < 0$ then the evolution is anitcausal.

4.3 Proof of Proposition 4.1

LEMMA 4.2: *If the function* f *satisfies*

$$\int_0^\infty f(t)dt < \infty$$

then

$$\lim_{\epsilon\to 0^+} \int_0^\infty \exp(-\epsilon t)dt = \int_0^\infty f(t)dt.$$

Proof: See [TITCHMARCH, p.26]. □

LEMMA 4.3: *If the function* f *satisfies*

$$\int_0^\infty f(t)dt < \infty \quad \textit{and} \quad \int_0^\infty f(-t)dt < \infty$$

then

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} \exp(-\epsilon|t|)f(t)dt = \int_{-\infty}^{+\infty} f(t)dt\ .$$

Proof:

$$\int_{-\infty}^{+\infty} f(t)dt = \int_0^\infty [f(t) + f(-t)]dt$$

$$= \lim_{\epsilon\to 0^+} \int_0^\infty \exp(-\epsilon t)[f(t) + f(-t)]dt$$

$$= \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} f(t)\exp(-\epsilon|t|)\,dt . \qquad \square$$

LEMMA 4.4:

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda\, \exp(-\epsilon|\lambda|)\, f(\tau-\lambda) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda\, \exp(-\epsilon|\lambda|)\, f(\lambda) .$$

Proof:

$$\text{L.H.S.} = \lim_{\epsilon\to 0^+} \int_{0}^{\infty} d\lambda\, \exp(-\epsilon\lambda)\, f(\tau-\lambda) + \int_{-\infty}^{0} d\lambda\, \exp(\epsilon\lambda)\, f(\tau-\lambda)$$

$$= \lim_{\epsilon\to 0^+} \left| \int_{-\infty}^{0} d\sigma\, \exp(-\epsilon(\tau-\sigma))\, f(\sigma) + \int_{0}^{\infty} d\sigma\, \exp(-\epsilon(\sigma-\tau))\, f(\sigma) \right|$$

$$= \lim_{\epsilon\to 0^+} \left| \int_{-\infty}^{0} d\sigma\, \exp(\epsilon\sigma)\, f(\sigma) + \int_{0}^{\infty} d\sigma\, \exp(-\epsilon\sigma)\, f(\sigma) \right|$$

$$= \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\sigma\, \exp(-\epsilon|\sigma|)\, f(\sigma) . \qquad \square$$

LEMMA 4.5: Let $K_0(x,y;\tau)$ and $G_0^c(x,y;k)$ be as defined previously. Then

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\sigma\, \exp(-\epsilon|\sigma|)\exp(ik\sigma)\,\sigma^n K_0(x,y;\sigma)$$

$$= -4M\left(-i\frac{d}{dk}\right)^n \mathrm{Im}[G_0^c(x,y;k)] .$$

Proof:

$$\text{L.H.S.} = (-i\frac{d}{dk})^n \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\sigma \exp(-\epsilon|\sigma|)\exp(ik\sigma)K_0(x,y;\sigma)\ .$$

Let $\sigma = -\sigma'$ in $\int_0^\infty d\sigma\,[\ldots.]$ and use $-K_0^*(\sigma) = K_0(-\sigma)$. Then comparing to (4.34A) and the definition of $K_0(x,y;\tau)$ we get the desired result. □

PROPOSITION 4.1:

Proof:

$$\chi(k) = (2\Pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} d\lambda \exp(ik\lambda)\tilde{\chi}(\lambda)$$

$$= (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(ik\lambda)\exp(-\epsilon|\lambda|)\tilde{\chi}(\lambda)\ . \qquad (4.45)$$

Substitute (4.45) into (4.38).

$$u_\chi(\tau) = (2\Pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} dE_k \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\exp(ik(\tau-\lambda))\tilde{\chi}(\tau)\ . \qquad (4.46)$$

Interchanging the spectral integral with the integration over λ and with the limiting process $(\epsilon \to 0^+)$ gives

$$u_\chi(\tau) = (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\chi(\lambda) \int_{-\infty}^{+\infty} dE_k \exp(-ik(\tau-\lambda))$$

$$= (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\chi(\lambda)u(\tau-\lambda)\ . \qquad (4.47)$$

Substitute the asymptotic expansion for $u(\tau)$. Let $\phi \in L^2(R^d)$.

$$[u(\sigma)\phi](x) = \int dy\,\phi(y)\left\{K_0(\sigma) \sum_{n=0}^{3N} \sigma^n A_n(x,y;M)\right\}$$

$$+ [E_N(\sigma)\phi](x) \qquad (4.48)$$

where

$$E_N(\sigma) \equiv u(\sigma) - u_N(\sigma) \qquad (4.49)$$

$$A_n(x,y;M) \equiv \exp(-iJ(x,y))(\chi_D * \rho_\ell)(x-y)P_n(x,y;M) \,. \qquad (4.50)$$

Then we have

$$[u_\chi(\tau)\phi](x) = (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\tilde{\chi}(\lambda)$$

$$\times \left\{\int dy\ \phi(y)K_0(x,y;\tau-\lambda) \sum_{n=0}^{N} (\tau-\lambda)^n A_n(x,y)\right.$$

$$\left. + [E_N(\tau-\lambda)\phi](x)\right\}$$

$$= [\epsilon(\tau)\phi](x) + \int dy\,\sigma(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\times (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\tilde{\chi}(\lambda)(\tau-\lambda)^n$$

$$\left.\times K_0(x,y;\tau-\lambda)\right\} \qquad (5.41)$$

where

$$\epsilon_N(\tau) \equiv (2\Pi)^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda \exp(-\epsilon|\lambda|)\tilde{\chi}(\lambda)E_N(\tau-\lambda) \,. \qquad (4.52)$$

Now substitute $\tilde{\chi}(\lambda) = (2\Pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} dk \exp(ik\lambda)\chi(k)$.

$$[u_\chi(\tau)\phi](x) = [\epsilon_N(\tau)\phi](x) + \int dy\ \phi(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\times\ (2\Pi)^{-1} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda\, \exp(-\epsilon|\lambda|)$$

$$\left.\times \int_{-\infty}^{+\infty} dk\ \chi(k)\exp(-ik\lambda)(\tau-\lambda)^n K_0(x,y;\tau-\lambda)\right\}.$$

Interchange the integration over k with the integration over λ and the limiting process. Then

$$[u_\chi(\tau)\phi](x) = [\epsilon_N(\tau)\phi](x) + \int dy\ \phi(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\times\ (2\Pi)^{-1} \int_{-\infty}^{+\infty} dk\, \exp(-ik\tau)\,\chi(k)$$

$$\times \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda\ \exp(-\epsilon|\lambda|)\exp(ik(\tau-\lambda))$$

$$\left.\times\ (\tau-\lambda)^n K_0(x,y;\tau-\lambda)\right\}.$$

Now applying Lemma 4.4 we get

$$[u_\chi(\tau)\phi](x) = [\epsilon_N(\tau)\phi](x) + \int dy\ \phi(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\times\ (2\Pi)^{-1} \int_{-\infty}^{+\infty} dk\, \exp(-ik\tau)\,\chi(k)$$

$$\left.\times \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\lambda\ \exp(-\epsilon|\lambda|)\exp(ik\lambda)\,\lambda^n K_0(x,y;\lambda)\right\}.$$

Now applying Lemma 4.5 we get

$$[u_\chi(\tau)\phi](x) = [\epsilon_N(\tau)\phi](x) - \frac{2M}{\Pi}\int dy\ \phi(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\left.\times\int_{-\infty}^{+\infty} dk\ \exp(-ik\tau)\chi(k)(-i\ \frac{d}{dk})^n \mathrm{Im}[G_0^C(x,y;k)]\right\}.$$

Now since $\chi(k)$ has compact support we can integrate by parts n times (dropping surface terms) to get

$$[u_\chi(\tau)\phi](x) = [\epsilon_N(\tau)\phi](x) - \frac{2M}{\Pi}\int dy\ \phi(y)\left\{\sum_{n=0}^{3N} A_n(x,y)\right.$$

$$\left.\times\int dk\ \mathrm{Im}[G_0^C(x,y;k)](i\ \frac{d}{dk})^n[\exp(-ik\tau)\chi(k)]\right\}$$

so that by (4.39)

$$u_\chi(\tau) = \tilde{K}_N^\chi(\tau) + \epsilon_N(\tau)\ . \tag{4.53}$$

<u>Error Bound:</u>

$$\epsilon_N(\tau) \equiv (2\Pi)^{=\frac{1}{2}}\int_{-\infty}^{+\infty} d\lambda\ \tilde{\chi}(\lambda)E_N(\tau-\lambda)\ . \tag{4.54}$$

From Chapter three we have for $\phi \in S_{N_2}$

$$||E_N(\sigma)\phi|| \leq \text{const. } M^{-(N+1)}|\sigma|^{N=1}|||\phi|||_{N_2}\ . \tag{4.55}$$

Thus we have

$$||\epsilon_N(\tau)\phi|| \leq (2\Pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} d\lambda|\chi(\lambda)|\ ||E_N(\tau-\lambda)\phi||$$

$$\leq (\text{const.})M^{-(N+1)}|||\phi|||_{N_2}\int_{-\infty}^{+\infty} d\lambda\ \tilde{\chi}(\lambda)|\tau-\lambda|^N$$

$$\leq (\text{const.}) M^{-(N+1)} |||\phi|||_{N_2} \sum_{j=0}^{N} \binom{N}{j} |\tau|^j \int_{-\infty}^{+\infty} \tilde{\chi}(\lambda) |\lambda|^{N-j}$$

$$= (\text{const.}) M^{-(N+1)} |||\phi|||_{N_2} \sum_{j=0}^{N} c_j \, |\tau|^j < \infty . \qquad \square$$

CONCLUSIONS

Though the exact propagator for systems such as the one considered here is difficult to obtain, it provides good geometrical insight into the space-time structure of the evolution. For this reason there is considerable advantage in having an approximate propagator which, though it does not give the exact evolution, approximates it in a controlled manner and contains much of the information necessary to provide this insight. The method used here, and in [SAKSENA], provides a relatively straightforward way to determine this approximation and has the advantage that it need only consider the region near the diagonal $(x = y)$ of the propagator $K(x,y;\tau)$ because of rapid oscillation and cancellations far off the diagonal. The propagator provides a good check of the consistency of the evolution theory with the concept of relativistic causality in that it provides a clear and explicit description of correlations between space-time points. The question of causality is more difficult to ask at the level of wave functions, for example, since there is no obvious way to separate the space-time manifold into time-like and space-like regions as there is in the case of the propagator. The standard interpretation of the relativistic Hamiltonian used here is that it is related to the mass of the particles in question. The Klein-Gordon equation is the quantum anolog of the classical relation $p^2 = m^2$ and can be interpreted as fixing the mass of the particle at a particular value or on mass shell. In chapter four the system is allowed to have a range of masses with associated probabilities determined by the spectrum $\chi(k)$. The free propagator $K_0(x,y;\tau)$ does not exhibit causal behaviour in that it has substantial space-like correlation. This might be viewed as a failure of consistency of the proper time evolution theory; however, section 4.2 shows that the evolution is indeed causal for states which have strictly positive mass spectra. This provides some insight into the nature of the spectrum of the Hamiltonian as well as answering the question of consistency with relativistic causality. The use of the asymptotic expansion for the propogator allows us to make similar statements about interacting systems without having to solve the evolution problem exactly. This is of considerable advantage since exact solutions for interacting systems are elusive.

APPENDIX

NOTATION

1. Summation Convention

a) Where greek indices are repeated they are summed from 0 to d-1.

b) With each contravariant vector (e.g., a^μ) is associated a covariant vector (a_μ) denoted by the same symbol (a) with the upper index replaced by a lower index of the same greek letter. These are related by

$$a_\mu \equiv g_{\mu\nu} a^\nu$$

where $g_{\mu\nu}$ are the components of the metric tensor defined in c) below.

c)
$$g_{\mu\nu} = \begin{cases} +1 & \mu = \nu = 0, m-1 \\ -1 & \mu = \nu = m, d-1 \\ 0 & \text{otherwise} \end{cases}$$

where m and d are positive integers and $m < d$.

2. Norms

The symbol $|\cdot|$ denotes either the euclidean norm of a complex vector or matrix or the norm of a multi-index (see this appendix).

a) If $x \in C^d$, then $|x| \equiv \sum_{i=1}^{d} x_i^* x$

where $d \geq 1$ is an integer and "*" denotes complex conjugation.

b) If $A \in C^{k \times k}$, then $|A| \equiv \sup\limits_{x \in C^k} \dfrac{|Ax|}{|x|}$.

3. Multi-Index

The symbol ∇^β, where $\beta = (\beta^1, \beta^2, \ldots, \beta^n)$ and β^i are all positive integers ($i = 1,n$), denotes the partial derivative

$$\nabla^\beta \equiv \partial^n / [(\partial x^1)^{\beta^1} (\partial x^2)^{\beta^2} \ldots (\partial x^n)^{\beta^n}] .$$

β is called a multi-index and the norm $|\beta|$ is defined by

$$|\beta| \equiv \sum_{i=1}^{n} \beta^i .$$

4. Sets of Functions

a) C^n denotes the class of functions which are n times continuously differentiable.

b) $C_0^\infty(R^d, C^k)$ denotes the set of functions which map R^d into C^k and which are infinitely continuously differentiable and have compact support. The symbols C_0^∞ and $C_0^\infty(R^d)$ are sometimes used where the domain and/or range of the functions can be understood from the context in which they appear.

c) $L^2(R^d, C^k)$ denotes the Hilbert space of functions which map R^d into C^k and which are square integrable in the sense that

$$f: R^d \to C^k \in L^2(R^d, C^k)$$

if

$$\int dx\, f^+(x) f(x) < \infty$$

where $f^+(x)$ denotes the complex conjugate transpose. The symbols L^2 and $L^2(R^d)$ are also used where their meaning can be understood from the context in which they are used.

5. Dirac Notation

a) If f and g are elements of the Hilbert space $L^2(R^d, C^k)$ and A is an operator defined on that Hilbert space, then the symbol $\langle f|A|g\rangle$ denotes the inner product

$$\langle f|A|g\rangle \equiv (f, Ag)$$

where (,) is the inner product associated with the Hilbert space.

b) If the operator A can be expressed as an integral operator

$$A\phi\ (x) = \int dy\, A(x,y)\, \phi(y)$$

where $\phi \in L^2(R^d, C^k)$ and A(x,y) maps $R^d \times R^d$ into $C^{k\times k}$, then the kernel A(x,y) is sometimes denoted by

$$\langle x|A|y\rangle \equiv A(x,y)\ .$$

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