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Topics in Exact Coverings

by

Jeffrey L. Allston

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DOCTOR of PHILOSOPHY

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Abstract

In this thesis, three categories of problems in Exact Combinatorial Coverings are discussed. All the coverings are minimal, in the sense that they use the fewest blocks possible; they also have specific restrictions imposed on them. The first category of covering requires that all pairs of elements from a v-set be covered exactly once. To eliminate the trivial case of generating a covering by taking all the elements and placing them in a single long block, we restrict the length of the longest block to be less than or equal to v-1, where v is the number of elements involved. The second category of covering problem examined involves covering all triples exactly once; as with the coverings of pairs, the length of the longest block is restricted to be less than or equal to v-1. In the third category, all pairs must be covered exactly twice; again, the length of the longest block is restricted to be less than or equal to v-1. For these bicoverings, a computer algorithm is employed for many of the results. In the final chapter of the thesis, a discussion is given of the number of non-isomorphic solutions for the classical covering problem for N(2,4,9).

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His never failing faith in me over these many years has been a constant source of inspiration throughout the many phases of writing this thesis. It was this inspiration which finally enabled me to put on the final "push" and finish.

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Chapter I

1.1 Introduction

Combinatorics is basically the study of finite sets, and most of the problems encountered fall into three major categories. The first category is the discussion of existence: can a structure be created that satisfies certain specified conditions. The second category is more numerical: if a structure exists, how many sets of elements (or blocks) are required in order to create the structure. Finally, we have the third category, that of determining the number of non-isomorphic structures satisfying a given set of conditions.

In this thesis, we shall be studying structures that satisfy certain minimal conditions and that consequently must exist. Most of our discussion will be concerned with problems of the second category, namely, the determination of how many subsets are required to create the structures we are studying. However, in the final chapter, we will include a discussion of a problem from the third category and will discuss the number of non-isomorphic solutions to a particular covering problem.

1.2 General Combinatorial Background

Balanced Incomplete Block Designs (BIBDs) were introduced by Frank Yates in 1935, and have been widely studied since then, both for their practical applications and their mathematical properties. They are significant tools in statistical analysis, and have been employed in communication theory; they are related to many combinatorial designs such as error correcting codes and Latin squares.

Covering and packing designs are a generalization of Balanced Incomplete Block Designs; in these designs, we drop the requirement for exactness and permit repetition of pairs in a covering and omission of pairs in a packing. Consideration of these systems as designs was first made explicit by Stanton, Kalbfleisch, and Mullin in [51], although various earlier papers had discussed these structures without emphasizing their design properties (see, for example, [23], [52], [40]). Since that time, a great deal of further work has been done by Mills, Mullin, Rees, Stanton, Stinson, Vanstone, and others.

In this thesis, we shall basically be discussing exact coverings (which are also exact packings). In order to meet the condition of exactness for every possible situation, it will be necessary to permit more than one block size in our covering designs. This approach first appeared in a 1948 paper by Erdös and de Bruijn, but no further developments took place for some time. Woodall [67] obtained an important inequality in 1968, and Stanton and Kalbfleisch [53] obtained a different inequality (1972) that subsumes the Erdös - de Bruijn result. Since that time, there have been many developments by various authors; these will be indicated in appropriate places during our discussion.

The general structure of this thesis will now be described. In Chapter II, we shall give a brief outline of the structures to be discussed and we shall give definitions of the various structures. The early definitions will deal with the classical background of the subject, and then we will present the generalizations which we shall be discussing. Some of these definitions will be repeated in later chapters in order to make the presentation simpler to follow without excessive referencing to Chapter II.

In Chapter III, we discuss exact covering designs where every pair must occur exactly once. These designs are the analogues of Balanced Incomplete Block Designs, and are of interest in Graph Theory as well as Combinatorics (cf. the various papers by Rees). In Chapter IV, we extend the discussion to the case where every triple occurs exactly once; the resulting designs are the natural generalization of the Steiner Systems S(3,k,v) introduced by Witt [66].

In Chapter V, we introduce the concept of a bicovering in which every pair occurs exactly twice. Much of the discussion of this case is carried out by means of a computational study, and the algorithm employed is given in an Appendix, along with various numerical results.

Finally, in Chapter VI, we do a study of the number of non-isomorphic solutions of two particular covering problems.

1.3 References

Many of the results obtained in this thesis have already been published in various journals during the last several years. The basic step forward was the introduction of pair-coverings with restricted largest block length (Allston, Stanton, and Cowan [3]);

this work is described in section 3.2. The work of section 3.3 appeared in Allston and Stanton ([7] and [8]).

The discussion in section 4.2 dealing with exact coverings of triples with specified longest block length appeared in Allston and Stanton [6]. The discussion of the two important special cases involving 20 or 21 elements appeared in Allston, Stanton, and Cowan ([1] and [2]). The results of section 4.7 are due to Allston, Stanton, and Wirmani-Prasad [9]. The discussion of section 4.8 is taken from Allston, Stanton, and Rogers [10].

The material in Chapter V has not yet appeared in print. The body of Chapter VI appeared in Allston, Stanton, and Wallis [4], although we here present a modified version which incorporates a correction found by Bate and van Rees [13].

Chapter II

2.1 Definition of the D and N Functions

Suppose that we have a set V consisting of v elements, denoted by the integers 1,2,3,... ,v. Suppose also that we have a family F of k-subsets selected from these v elements (k is normally taken to be less than v, since the case k = v is trivial). Now consider all the t-subsets of V.

If F has the property that all t-subsets of V occur at least λ times in F, then we say that we have a covering, and that the family F is a λ-covering of all t-sets. The most interesting coverings are those which have a minimal property, that is, the cardinality of F, or $|F|$, is as small as possible; if we are dealing with such a minimal family, we denote this minimum cardinality by the symbol $N_\lambda(t,k,v)$. The case $\lambda = 1$ is particularly important, and we normally consider this case unless otherwise stated; in this particular case, we omit the subscript altogether and simply write $N(t,k,v)$. Some authors use the symbol $C(t,k,v)$ for the covering number.

On the other hand, if the family F is selected to have the property that no t-subset occurs more than λ times, then we say that we have a packing, and the family F is called a λ-packing of all t-sets. Again, it is natural to restrict attention to those particular packings that are maximal, that is, that have $|F|$ as large as possible. We use the symbol $D_\lambda(t,k,v)$ to denote this maximum cardinality. Again, we normally take $\lambda = 1$, unless otherwise stated, and we write $D(t,k,v)$ in this case.

It is trivial, but very important, to note the results for $t = 1$, that is, when we are covering or packing all single elements. For this case, we immediately have

$$D(1,k,v) = \left\lfloor \frac{v}{k} \right\rfloor \quad \text{and} \quad N(1,k,v) = \left\lceil \frac{v}{k} \right\rceil.$$

This relationship allows us to obtain a well known bound on the numbers N and D; we call this the Fisher Bound, since it is obtained by counting the elements appearing in the blocks of the covering family F in two ways, in exactly the same way that the usual relationship $bk = vr$ is derived for Balanced Incomplete Block Designs.

Write down all the blocks in a covering family F; then each block contains k elements, and the number of blocks is N(t,k,v). Consequently, the number of elements in the covering array is just kN(t,k,v). Now, look at the blocks containing a particular element x; since all t-sets containing x must occur, x must occur with every (t-1)-set formed from the v elements, and these (t-1)-sets must appear in the (k-1)-sets that occur with x. Hence, the frequency of x is at least N(t-1,k-1,v-1), and this is true for each of the v elements x. Hence, the number of elements in the array is at least vN(t-1,k-1,v-1). Putting these two facts together, we have

$$kN(t,k,v) \geq vN(t-1,k-1,v-1).$$

This recursive relation is best employed as it stands. If t = 2, we can derive the slightly weaker form of the covering bound as

$$N(2,k,v) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \right\rceil \right\rceil$$

An exactly similar discussion can be carried out for the packing bound; we again write down all the blocks in a packing family F. Since there are D(t,k,v) blocks of length k, the number of elements in the packing array is simply kD(t,k,v). Now look at the blocks containing a particular element x; x can not occur with a (t-1)-set more than once in the (k-1)-sets which appear with x. Consequently, x can not appear in more than D(t-1,k-1,v-1) blocks, and it may appear in fewer blocks. Since there are v possibilities for the element x, the number of elements in the array is at most equal to vD(t-1,k-1,v-1), and we obtain the recursive relationship

$$kD(t,k,v) \leq vD(t-1,k-1,v-1).$$

This equation yields the packing bound for t = 2 in the usual form as

$$D(2,k,v) \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \right\rfloor \right\rfloor.$$

We conclude this section by describing results that have been obtained for packings and coverings with small values of v and k.

The results for N(2,3,v) were given by Fort and Hedlund [23]; for a much simpler presentation, along with the results for D(2,3,v), see Stanton and Rogers [56], where the results are phrased in terms of the defect graph of a packing and the excess graph of a covering. The packing bound is achieved for all values of v, except when v is congruent to 5 (mod 6); in this latter case, the maximal packing design has a number of blocks one less than the bound. Similarly, one finds that the covering bound is always achievable.

The results for N(2,4,v) are summed up in two papers by Mills [29,30]. With the aid of a considerable amount of computation, it is shown that the bound is always achieved except for v = 7,9,10 (when one extra block is needed), and for v = 19 (when two extra blocks are needed). The results for D(2,4,v) are summarized in Brouwer [17]. The numbers 8,9,10,11,17,19, are exceptions; for all other values, the result is that the bound is met for v not congruent to 7 or 10 (mod 12), and the bound, less one, is met for v congruent to 7 or 10 (mod 12). For v = 9 and 17, the packing number is the bound, less one; for v = 8,10,11,17, the packing number is the bound, less two; for v = 19, the packing number is the bound, less 3.

It is worth noting that Brouwer's paper contains an especially useful result about the cases v congruent to 7 or 10 (mod 12). In this case, all pairs are exactly covered once by a single block of size 7 and a collection of blocks of size 4.

2.2 Exact Covering and Packing Systems

Supppose that we start with a variety set comprising v elements denoted by the integers 1,2,3, ... , v; let t be an integer less than v. We define an exact or perfect covering of t-sets to be a selection of subsets formed from the variety set such that each subset is proper, and such that each t-set occurs exactly λ times. For example, if v = 7, t = 2, and $\lambda = 3$, then the following family of subsets is a perfect covering.

1236, 2347, 3451, 4562, 5673, 6714, 7125, + 21 pairs 12,13,14, ... , 56,57,67.

Another perfect covering would be provided by taking the first 7 sets in the above family (the quadruples) and repeating them to give a covering in 14 sets.

Not a great deal of work has been done on coverings with $\lambda > 1$, and we shall only consider the case $\lambda = 1$ in this section. With $\lambda = 1$, there are two trivial solutions. We exclude the trivial case when all elements are placed in a single set {1,2,3, ... ,v}. However, there is also always the trivial (but important) solution where we take a covering family made up of all possible t-sets. However, in order to have interesting covering families to study, we need further restrictions. Three particular kinds of coverings have been studied in considerable detail (cf. the survey given in [45]).

(1) If we impose the restriction that all sets in the covering family have the same cardinality k, we are dealing with Steiner Systems S(t,k,v); see, for example, [66] as well as section 2.3. Steiner Systems are particularly symmetric and fascinating, but only a dozen or so are known with $t > 3$. For $t = 2$, the Steiner Systems S(2,k,v) are just the well known Balanced Incomplete Block Designs, about which there is a very large literature. However, even in the case of Balanced Incomplete Block Designs, it frequently happens that an appropriate design does not exist. For example, if we take $v = 11$, then there are 55 pairs from an 11-set; if we are going to cover these pairs by blocks of equal size k (with $k > 2$), then $k(k-1)/2$ must divide 55. But 55 is not divisible by 3, or 6, or 10, or 15, or 21, or 28. This illustration emphasizes the fact that, if we are going to have a perfect covering family, then some of the block sizes will generally have to be different from one another.

(2) We might restrict the covering family by permitting two distinct block sizes in the covering family. As an example, let $v = 12$ and $t = 3$. Then a perfect covering of triples can be displayed as follows; take $A = \{1,2,3,4,5,6\}$ and $B = A + 6$ as two disjoint sextuples. Define a 1-factorization of A (see [50], for example) as follows.

$F_1 = \{(1,2), (3,4), (5,6)\}$, $F_2 = \{(1,3),(2,5),(4,6)\}$, $F_3 = \{(1,4), (3,5), (2,6)\}$, $F_4 = \{(1,5), (2,4), (3,6)\}$, $F_5 = \{(1,6), (2,3), (4,5)\}$.

Also, define a 1-factorization of B by taking sets $G_i = F_i + 6$. Then it is easy to see that we can construct a perfect covering family by taking the 2 sets A and B, together with the 45 quadruples formed by taking every pair from F_i with every pair from G_i (this gives 9 quadruples for any fixed i, and i may take on the values 1,2,3,4,5).

(3) Another restriction on the covering family that leads to problems of extreme interest is to demand that the family have minimal cardinality. This leads to the introduction of the concept of a g-covering. A g-covering is a covering family such that no other covering family has smaller cardinality; we denote the minimal cardinality by $g(1,t;v)$. For example, it can be shown that the family just constructed in the previous paragraph is minimal for $v = 12$ and $t = 3$; thus $g(1,3;12) = 45$, as shown in [48]. An example of a non-minimal family for $v = 12$ and $t = 3$ would be provided by taking the set

$C = \{1,2,3,4,5,6,7,8,9,10,11\}$, and the 55 triples of the form $\{c,d,12\}$, where c and d are elements of C; this would be a perfect covering in 56 sets.

We shall discuss g-coverings of pairs ($t = 2$) and of triples ($t = 3$). Only a few results are known for $t > 3$, largely because progress depends heavily on a knowledge of Steiner Systems.

2.3 Steiner Systems and t-designs

A Steiner System is a particular exact covering in which every t-set occurs exactly once in a collection of k-sets taken from a v-set. Such a Steiner System is denoted by the symbol S(t,k,v), and the number of blocks in the system is easily found as

$$|S(t,k,v)| = \frac{v(v-1)(v-2)...(v-t+1)}{k(k-1)(k-2)...(k-t+1)}.$$

If we consider a Generalized Steiner System in which each t-set occurs λ times, we use the notation $S_\lambda(t,k,v)$; these Generalized Steiner Systems are often called t-designs, although the term is unfortunate because it does not emphasize their relationship with the classical Steiner Systems.

If $t = 2$, a Generalized Steiner System $S_\lambda(t,k,v)$ is just an ordinary BIBD with parameter set (v,b,r,k,λ); the usual Balanced Incomplete Block Design relations immediately show that $r = \lambda(v-1)/(k-1)$ and $b = \lambda v(v-1)/k(k-1)$.

There are many tables of BIBDs; the earliest was produced by Fisher and Yates in their Statistical Tables for Biological, Agricultural, and Medical Research [22].

2.4 g-Coverings

The minimal number of incomplete blocks made up of elements from a v-set in such a way that every μ-set occurs exactly λ times in the blocks selected is designated by $g(\lambda,\mu;v)$. Almost all results will deal with the particular case $\lambda = 1$. Erdös and de Bruijn showed [18] that $g(1,2;v) = v$ and that this minimum is always given by a near-pencil, that is, by one block of length $v - 1$ plus $v - 1$ pairs. Exceptionally, the minimum is also attained if $v = k^2-k+1$ and a geometry of k points per line exists; then the geometry covers all pairs by v lines comprising k points each.

Allston, Stanton, and Cowan, in [3], introduced the quantity $g^{(k)}(\lambda,\mu;v)$; this is the minimal number of blocks required to cover all μ-sets exactly λ times, given that the largest block in the covering has length k. It is clear that $g(\lambda,\mu;v)$ is just the minimum value of the quantity $g^{(k)}(\lambda,\mu;v)$ when we allow k to range over all possible values from μ to $v - 1$.

If we set $\lambda = 1$, various bounds have been obtained. For example, Woodall [67] showed that

$$g \geq 1 + (v-k)\binom{k}{\mu-1}\left(1 - \frac{v-k-1}{2(k-\mu+2)}\right) \tag{2.4.1}$$

where k is the size of a block in the covering set (henceforth, we shall always use k as the size of the largest block in the covering set). Stanton and Kalbfleisch [53] showed that

$$g \geq 1 + \frac{k-\mu+2}{v-\mu+1}\binom{k}{\mu-1}(v-k) \ . \tag{2.4.2}$$

And it is trivial, by a counting argument, to obtain the Combinatorial Bound

$$g \geq \frac{\binom{v}{\mu}}{\binom{k}{\mu}} \ . \tag{2.4.3}$$

A stronger bound was obtained by Stinson [61] and it will be introduced in Chapter III along with a later bound due to Rees [36] and Rees and Stinson [38].

Chapter III

3.1 Coverings of Pairs; the Numbers $g(1,2;v)$

We first mention that perfect pair coverings, that is, families that cover all pairs exactly once, also appear in the literature under the name of finite linear spaces or under the name of pairwise balanced designs. We prefer the first and third of these three terms. Pairwise balanced designs also subsume the well known Balanced Incomplete Block Designs (BIBDs); these are just pairwise balanced designs for which only a single block size is involved.

As we have already stated, one of the first results on perfect pair-coverings was given in 1948 by de Bruijn and Erdös [18]. They proved that the cardinality of the minimal covering family is $g(1,2;v) = v$. Furthermore, they showed that this minimum could always be achieved by taking one block C containing all elements of the v-set except v, and then taking all pairs $\{c,v\}$, where c ranges over the $v - 1$ elements of C (this configuration is often called a near-pencil). In the special case that $v = k^2 - k + 1$, and a finite geometry exists (that is, $k - 1$ is a prime power), then the finite geometry provides a second minimal covering in v sets, and those v sets all contain precisely k elements (this second type of minimum, of course, can only occur if v is an integer selected from the set $\{7, 13, 21, 31, 57, 73, 91, ...\}$; it is just a Balanced Incomplete Block Design with parameters (v,b,r,k,λ), where $v = b = k^2 - k + 1$, $r = k$, $\lambda = 1$).

Actually, it is not too illuminating to consider the number g. In [3], the quantity $g^{(k)}(1,2;v)$ was introduced; this is the minimum size of a covering family that contains at least one block of length k but no block of larger size. Henceforth, in this chapter, we shall normally write $g(v)$ for $g(1,2;v)$, and we shall write $g^{(k)}(v)$ for $g^{(k)}(1,2;v)$. We thus see that the quantity $g(v)$ is rather accidental; it is just the global minimum for the quantity $g^{(k)}(v)$. We shall now provide a summary of developments in the theory, phrasing the results in terms of $g^{(k)}(v)$, since results appear more simple and direct in this context.

We have already referred to the important step forward taken by Woodall [67] in 1968; Woodall established a general bound for any t. In the special case when $t = 2$, Woodall's result specializes to

$$g^{(k)}(v) \geq W = 1 + \frac{(v-k)(3k-v+1)}{2}$$

We call W the Woodall bound.

Stanton and Kalbfleisch [53] used a variance method to obtain a bound which, specialized to the case $t = 2$, becomes

$$g^{(k)}(v) \geq 1 + \frac{k^2(v-k)}{(v-1)}$$

We call this quantity SK. If we make a graph of the function SK, we obtain a particularly simple proof of the Erdös-de Bruijn result (see [48] or [49]). Of course, we should also mention the combinatorial bound C; for small values of k, it is extremely useful to note that

$$g^{(k)}(v) \geq C = \frac{v(v-1)}{k(k-1)}$$

The combinatorial bound obviously gives the exact result for $k = 2$.

There are many open questions for particular values of k and v. However, a great deal of progress has now been made. In Section 3.2, we will give the results that we obtained in [3], where it was shown that the Woodall Bound gave the exact value for $g^{(k)}(v)$ as long as $k \geq (v-1)/2$ and v does not have the form 4m+1 (the latter case was handled in [55]). This basically means that the combinatorial bound C holds (approximately) from k equal to 2 until k reaches a value in the neighbourhood of $\sqrt{v}$; the SK bound then holds, approximately, from k in the neighbourhood of $\sqrt{v}$ to k in the neighbourhood of v/2; then the bound W holds exactly when k exceeds v/2 (special consideration is needed at the transitional points where we change from one bound to another). It thus becomes of importance to see just how close $g^{(k)}(v)$ is to the bound SK.

In [43], it was pointed out that the bound SK is exact when all other blocks meet the block of length k and when they form a resolvable balanced incomplete block design. A very important extension of this result was given by Stinson in [61]; Stinson defined s to be the greatest integer in the quantity (v-1)/k, and showed that

$$g^{(k)}(v) \geq 1 + \frac{(v-k)(2sk-v+k+1)}{s(s+1)}$$

We call this bound S (the Stinson bound). It is an improvement over the SK bound, except when SK is exact. Furthermore, if S is integral, then S is exact if and only if all blocks meet the base block of length k and form a resolvable pairwise balanced design with block sizes s and s+1. It is possible to give a particularly straightforward account of the behaviour of the Stinson bound, and we now proceed to do this.

Suppose that we consider all the blocks of the covering with the exception of the base block of length k. We say that there are b_i blocks of length i (in a block of length i, there may be i points not on the base block, or there may be one point on the base block and i-1 points not on the base block). Then, we may write

(3.1.1) $$g^{(k)}(v) - 1 = g-1 \text{ (for short)} = b_2 + b_3 + b_4 + b_5 + \ldots$$

We now count all pairs not in the base block and find:

(3.1.2) $$\frac{v(v-1)}{2} - \frac{k(k-1)}{2} = b_2 + 3b_3 + 6b_4 + 10b_5 + \ldots$$

Now let the points in the base block be called j (j = 1,2,3,...,k), and let b_{ij} denote the number of blocks of length i through point j. Clearly,

$$\Sigma_i (i-1)b_{ij} = v-k, \text{ for all } i;$$

thus we have $\Sigma_{i,j} (i-1)b_{ij} = k(v-k)$. However, we must not forget the blocks that do not meet the base block; suppose that b_{i0} denotes the number of blocks of length i that do not meet the base block; then $\Sigma_i (i-1)b_{i0} = \varepsilon$, where ε is a non-negative integer. Adding all of these expressions together gives us our third equation:

(3.1.3) $$k(v-k) + \varepsilon = b_2 + 2b_3 + 3b_4 + 4b_5 + \ldots$$

We now multiply these equations by the quantities s(s+1)/2, 1, and -(s+1), respectively, and add the three equations. This has the effect of eliminating the terms in b_{s+1} and b_{s+2} to leave the result

(3.1.4) $$s(s+1)(g-1) + (v^2-v+k^2-k) = 2(s+1)k(v-k) + 2\varepsilon(s+1) + 2P,$$

where P is the non-negative integer

$$b_s + b_{s+3} + 3(b_{s-1}+b_{s+4}) + 6(b_{s-2}+b_{s+5}) + \ldots$$

Then we find

$$(3.1.5) \qquad g - 1 = \frac{(v-k)(2sk-v+k+1)}{s(s+1)} + \frac{2\varepsilon}{s} + \frac{2P}{s(s+1)}.$$

Equation (3.1.5) gives the Stinson Bound when we ignore the last two terms, which are certainly non-negative.

It is easy to see that the optimal value for s, in Equation (3.1.5), is the greatest integer in (v-1)/k; suppose that we assign that particular value to s.

Now consider s = 1 (that is, k lies between v/2 and v). The Stinson bound thus becomes the Woodall bound W and we have

$$(3.1.6) \qquad g - 1 = \frac{(v-k)(3k-v+1)}{2} + 2\varepsilon + b_4 + 3b_5 + 6b_6 + \ldots = (W-1) + 2\varepsilon + P.$$

Equation (3.1.6) immediately gives us the

THEOREM. *The Woodall Bound can only be achieved if all blocks meet the long block of length k (that is, ε=0) and if all the other blocks have lengths 2 and 3 (thus the other blocks fall into resolution classes with blocks of lengths 1 and 2 hanging on to the points of the base block).*

That the Woodall bound is actually achieved in this region will be shown by a straightforward construction in Section 3.2 (cf. [3]).

We now turn our attention to the case when s = 2 in Equation (3.1.5); this is when k lies bewteen v/3 and v/2. Equation (3.1.5) can then be written in the form

$$(3.1.7) \qquad g = 1 + \frac{(v-k)(5k-v+1)}{6} + \varepsilon + \frac{P}{3} = S + \varepsilon + \frac{(b_2+b_5)}{3} + b_6 + \ldots$$

Now the Stinson bound may not be integral, but we see that it can not be achieved (in the nearest-integer sense) unless ε = 0 (recall that ε is an integer). Thus, we have the

THEOREM. *When $s = 2$ (that is, k lies between $v/3$ and $v/2$), the Stinson Bound is only attained if all blocks meet the base block.*

However, we can go further; the numerator $(v-k)(5k-v+1)$ is an even integer, and so the quantity $P/3 = (b_2+b_5)/3$ can only assume the values 0/3, 1/3, or 2/3. Thus we have the

THEOREM. *If the Stinson Bound is met with $s = 2$, then all blocks must have lengths 3 and 4, except that there may be one or two exceptional blocks with lengths 2 or 5.*

It is relatively easy to specify when these rogue blocks appear. We let $k = 6t+a$, and let $v = 2k+6u+b$; then one can carry out the requisite algebra and find

(3.1.8) $$b_2 + b_5 = 3\psi\left(\frac{X}{6}\right) - \frac{X}{2},$$

where $X = (a+b)(3a+1-b)$ and ψ denotes the ceiling function. Of course, in any particular case, it is probably easier to carry out the specific elimination that led to Equation (3.1.5).

For example, let us consider $v = 24$ and $k = 8$; then $a = 2$, $b = 2$ and $b_2+b_5 = 2$. In general, we find that, for $X = 0, 2, 4 \pmod 6$, respectively, then $b_2+b_5 = 0, 2, 1$, respectively.

We should add that the procedure used in obtaining Equation (5) is equally useful for other values of s. If $s = 3$ (that is, k lies between $v/4$ and $v/3$), then we get

(3.1.9) $$g - 1 = \frac{(v-k)(7k-v+1)}{12} + \frac{2\varepsilon}{3} + \frac{P}{6}.$$

From this equation, we can deduce easily that $\varepsilon = 0$ if the Stinson bound is met; furthermore, one can get a quantitive limitation on the number of rogue blocks in this case. However, this result is only a special case of a much more general theorem. Let us return to Equation (3.1.5), with s having its optimal value. If the Stinson bound is to be met, it is clear that $2\varepsilon/s$ can not exceed unity; hence the maximum length of any block disjoint from the base block is $s/2$ when s is even and $(s+1)/2$ when s is odd. This shows that the disjoint blocks are relatively "short", in order to keep down the value of ε. On the other hand, let us look at the quantity P and let us suppose that there is a block of length (s+1)-z disjoint from the base block. It will contribute an amount (z+1)z/2 to P and an amount (s-z) to e. The total contribution from this one disjoint block will thus be

$$\frac{(z+1)z}{s(s+1)} + \frac{2(s-z)}{s} = \frac{z^2 - z(2s+1) + 2s(s+1)}{s(s+1)}.$$

This quadratic function starts at the value 2 when z = 0 and decreases to the value $(s^2+s+2)/(s^2+s)$, which is always greater than 1, for z = s-1, that is, for block length 2. We thus see that any block disjoint from the base block must contribute more than one unit to Equation (3.1.5), and thence we obtain the following result.

THEOREM. *If the Stinson Bound is to be met, in the nearest-integer sense, then all blocks must meet the base block of length k.*

Recently, a further strengthening of the Stinson bound has been achieved. Rolf Rees [36], in his doctoral dissertation, was able to obtain a bound R that is, in some cases, stronger than S; if the bound R is exact, then all blocks must meet the base block of length k and they must have block sizes equal to s, s+1, or s+2. The exact properties of the R bound are rather complicated, but are described in detail by Rees and Stinson [38]; we will simply note that, if τ is defined to be the residue of (v-k), modulo s, then

$$R = 1 + \frac{(v-k)(2k(s-1+\tau/s)-v+k+1) + 2k\tau(1-t/s)}{(s^2-s+2\tau)}.$$

Tables have now been produced giving the values of $g^{(k)}(v)$ for most small v and k; in particular, [7] and [8] give the results for all $v \leq 22$ except in the cases v = 17,18, and 19, with k = 4. The case k = 4 is discussed, for all other values of v, in [57]. The value for $g^{(4)}(17)$ is quoted in [58]; see also [58], [46], [59], [11]. A few initial results for the case k = 5 are given in [5]. Buskens, Rees, Stanton, and Stinson, have extended the census of $g^{(k)}(v)$ up to v = 31 (with a number of blanks). However, the next natural range of values, $32 \leq v \leq 57$, presents many opportunities for discovery of exotic designs.

Rees [36] has given constructions for the cases v = 2k+2, 2k+3, and 2k+4 (see also [20]). The case v = 2k+1 was already given in [3] and [55]. A special instance of the case v = 2k+7 appears in [20].

3.2 The Quantity $g^{(k)}(1,2;v)$

3.2.1 Introduction. If we specialize the Woodall, Stanton-Kalbfleisch, and Combinatorial Bounds to the case $\mu = 2$, we have the lower bounds in the form

$$W = 1 + \frac{v-k}{2}(3k-v+1) \tag{3.2.1}$$

$$SK = 1 + \frac{k^2(v-k)}{v-1} \tag{3.2.2}$$

$$C = \frac{v(v-1)}{k(k-1)} \ . \tag{3.2.3}$$

It was shown (see [48], or, for more detail, [49]) that the Stanton-Kalbfleisch bound (3.2.2) easily produces the Erdös-de Bruijn Theorem.

In this section, we employ the numbers

$$g^{(k)}(1,2;v) = g^{(k)}(v)$$

as the cardinality of the minimal family of sets that covers all pairs, given that the elements are from a v-set and that the size of the longest block in a covering family is k. When the argument v is obvious, we simply write $g^{(k)}$.

As an example, suppose that we take $v = 13$; then we can construct the following table. The values $g^{(k)}(1,2;13)$ for $k > 6$ will be justified later in the section. Note that

$$W = 1 + \frac{3}{2}(13-k)(k-4) \ ,$$

$$SK = 1 + \frac{k^2(13-k)}{12} \ ,$$

$$C = \frac{156}{k(k-1)} \ .$$

k	g(k)	W	SK	C
13	1	1	1	1
12	13	13	13	2
11	22	22	22	2
10	28	28	26	2
9	31	31	28	3
8	31	31	28	3
7	28	28	26	4
6	24	22	22	6
5	19	13	18	8
4	13	1	13	13
3	26	<0	9	26
2	78	<0	5	78

For k = 2, 3, 4, the values follow from using the set of all pairs; from using the triple system on 13 elements; and from using the projective geometry on 13 elements.

3.2.2 The Construction of a Covering System for Large k. Suppose that $k = v - 2\alpha$; then the Woodall bound is

$$W = 1 + \alpha(2v-6\alpha+1) \ .$$

Now take a complete 1-factorization of the 2α points not contained in the block of length k. Form triples by associating all pairs in any 1-factor with the same point in the block of length k (this can be done so long as the number of 1-factors, which is $2\alpha-1$, is not greater then $v-2\alpha$). Use all pairs not contained in the block of length k or in the triples. Then the total number of blocks is

$$1 + (2\alpha-1)\alpha + \binom{v}{2} - \binom{v-2\alpha}{2} - \alpha(2\alpha-1)3$$

$$= 1 + \alpha(2v-6\alpha+1) = W.$$

The condition $2\alpha-1 \leq v-2\alpha$ simplifies to

$$\alpha \leq (v+1)/4.$$

Thus $k \geq v-(v+1)/2 = (v-1)/2$, and we have

THEOREM 3.2.2.1. *If v-k is even, then the Woodall bound gives* $g^{(k)}$ *for* $k \geq (v\text{-}1)/2$.

For v-k odd, we need a different factorization of the pairs on the v-k = $2\alpha+1$ points. The easiest way to get a complete 1-factorization of an even number of points (take 8 as an example) is to place 1 at the centre of a circle formed by the other 7 points, as shown in Figure 3.2.1.

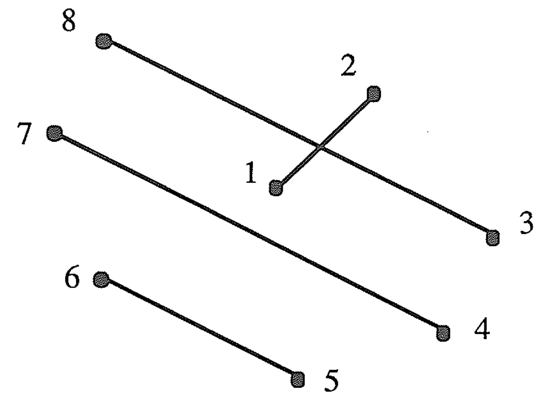


Figure 3.2.1: 1-Factorization of K_8

The first 1-factor is found by taking (1,2) and the three perpendicular chords, as shown. The other 1-factors are found by rotating this 1-factor about 1.

Similarly, if there is an odd number of points, say 7, we can place them on the circumference of a circle with centre 0.

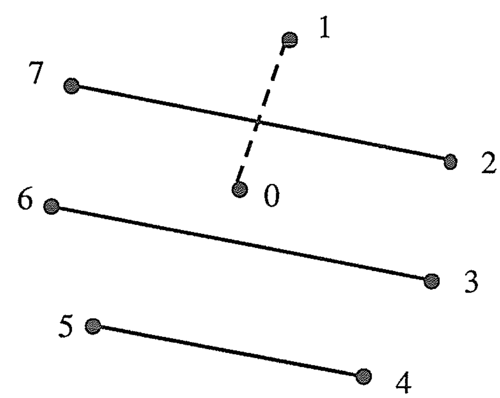


Figure 3.2.2

The first "1-factor" is found by taking "1", joining 0 to 1, and taking the chords perpendicular to (0,1). By rotation, we get 7 generalized 1-factors.

Now we make a construction analogous to that in Theorem 3.2.2.1. We form triples by taking all pairs in a 1-factor and adjoining the same point from the k-block. This is possible so long as

$$2\alpha+1 \le v-(2\alpha+1) \ ,$$

that is, so long as

$$\alpha \le (v-2)/4 \ ,$$

or

$$k \ge v-(v-2)/2-1 = v/2 \ .$$

The number of pairs needed to cover all pairs is now found as

$$\binom{v}{2} - \binom{v-2\alpha-1}{2} - 3\alpha(2\alpha+1)$$

$$= 2v\alpha-8\alpha^2-6\alpha+v-1.$$

So the total number of blocks is this number increased by $1+\alpha(2\alpha+1)$, that is,

$$2v\alpha-6\alpha^2+v-5\alpha \ .$$

But we easily calculate that

$$W = 1 + \frac{2\alpha+1}{2}(3v-6\alpha-3-v+1)$$

$$= 1 + (2\alpha+1)(v-3\alpha-1)$$

$$= 2\alpha v-6\alpha^2+v-5\alpha.$$

We thus have

THEOREM 3.2.2.2. *If v-k is odd, then the Woodall bound gives* $g^{(k)}$ *for* $k \geq v/2$.

These two theorems are easily merged into

THEOREM 3.2.2.3. *If* $v \equiv 1 \pmod 4$, *the Woodall bound holds for* $k > (v-1)/2$; *otherwise, the Woodall bound holds for all* $k \geq (v-1)/2$. *Thus, for k in these ranges, we have* $g^{(k)} = 1 + (v-k)(3k-v+1)/2$.

Actually, we can go a bit farther. If we use (2.1) and (2.2) from [49], it follows that, with t = 2, the Woodall bound W is only attainable if

$$\sum_{A(0)} \binom{k_i}{2} = 0 \ ,$$

where A(0) is the set of blocks disjoint to the block of length k, $\{k_i\}$ is the set of lengths for these blocks. Thus, we have the result that all other blocks meet the longest block. Furthermore, it is also required that

$$\sum_{A(1)} \binom{k_i-2}{2} = 0 \ ,$$

and this shows that the blocks meeting the longest block in 1 point (all others) have cardinalities 2 or 3. Thus we have

THEOREM 3.2.2.4. *The only configurations producing the bound W are those using pairs and triples, as described earlier in this section.*

3.2.3 The Values k = 2 and 3. It is trivial to note that

$$g^{(2)} = \binom{v}{2} \ .$$

Also, it is clear that $g^{(3)}$ is obtained by taking as many triples as possible; now this number (see, for example, [52]) is

$$D(2,3,v) = \left[\frac{v}{3}\left[\frac{v-1}{2}\right]\right] - \delta_{\alpha 5}$$

where [x] denotes the greatest integer not exceeding x, and α is the congruence class of v, modulo 5. It follows that we can state

THEOREM 3.2.3.1. *The value of* $g^{(2)}$ *is* $\binom{v}{2}$, *and the value of* $g^{(3)}$ *is*

$$\binom{v}{2} - 2\left[\frac{v}{3}\left[\frac{v-1}{2}\right]\right] + 2\delta_{\alpha 5} \ .$$

It is useful to record $g^{(3)}$ according to the form of v. We give two alternative forms.

v	$g^{(3)}$	$g^{(3)}$
6t	$6t^2+t$	$v(v+1)/6$
6t+1	$6t^2+t$	$v(v-1)/6$
6t+2	$6t^2+5t+1$	$v(v+1)/6$
6t+3	$6t^2+5t+1$	$v(v-1)/6$
6t+4	$6t^2+9t+4$	$(v^2+v+4)/6$
6t+5	$6t^2+9t+6$	$(v^2-v+16)/6$

Using the results proved so far, we can fill in the following table.

k \ v	2	3	4	5	6	7	8	9	10	11	12
2	1	3	6	10	15	21	28	36	45	55	66
3		1	4	6	7	7	12	12	19	21	26
4			1	5	8	10	11	12*	12*	13*	13*
5				1	6	10	13	15	16	16	18*
6					1	7	12	16	19	21	22
7						1	8	14	19	23	26
8							1	9	16	22	27
9								1	10	18	25
10									1	11	20
11										1	12
12											1

Table 3.2.1. $g^{(k)}(v)$ for $2 \leq v \leq 12$

The starred values do not follow from our theorems; rather we need some easy Lemmata.

LEMMA 3.2.3.2. *$g^{(4)}(9) = 12$.*

PROOF. The SK bound is 11; also it is clear that

1234	189	368	269	459	35
1567	258	478	379	27	46

provides a cover in 12 blocks.

If a pairs, b triples, c quadruples, provide a cover in 11 blocks, then

$$a + b + c = 11, \ \ a + 3b + 6c = 36 \ .$$

It follows that $2b + 5c = 25$, and we have one of 3 cases:

(1) $c = 5, b = 0, a = 6$;
(2) $c = 3, b = 5, a = 3$;
(3) $c = 1, b = 10, a = 0$;

Case (1) is impossible, since $D(2,4,9) = 3$. Case (2) is not possible since we get 1234, 1567, 4789; then we can not have 5 triples. Case (3) is impossible since the quadruple 1234 leaves 5 symbols to go with 1, and hence the use of triples only is impossible.

Indeed, Lemma 3.2.3.2 generalizes trivially to the result.

LEMMA 3.2.3.3. *If $v = t^2$, then $g^{(k)}(v) \geq t^2+t$ $(t\geq 3)$.*

PROOF. It is clear that

$$g^{(t^2-2)}(v) = 2t^2-4 > t^2+t$$

by using Theorem 3.2.2.3. Also, the counting bound shows that

$$g^{(t)}(v) \geq \frac{t^2(t^2-1)}{t(t-1)} = t^2+t.$$

Finally, the SK bound gives

$$g^{(t+1)}(v) \geq 1 + \frac{(t+1)^2(t^2-t-1)}{t^2-1} ,$$

that is,

$$g^{(t+1)}(v) \geq t^2+t - \frac{2}{t-1} .$$

This shows that $g^{(t+1)}(v) \geq t^2+t$ for $t > 3$, and the result for $t = 3$ follows from Lemma 3.2.3.2.

The general result then follows from the shape of the bounding curve

$$SK = 1 + \frac{k^2(v-k)}{v-1} = 1 + \frac{k^2(t^2-k)}{t^2-1}$$

between $k = t+1$ and $k = t^2-2$.

LEMMA 3.2.3.4. *$g^{(4)}(10) = 12$.*

PROOF. The SK bound is 12, and the cover

1234	258	26T	279
1567	369	378	468
189T	47T	459	35T

is trivially obtained.

LEMMA 3.2.3.5. *$g^{(4)}(11) = 13$.*

PROOF. Again, the SK bound is 13. Simply take an affine geometry on 9 points, adjoin T and E to two resolution classes, and add the pair {T,E}.

LEMMA 3.2.3.6. *$g^{(4)}(12) = 13$.*

PROOF. The SK bound of 13 is achieved by deleting a single point from the 13-point geometry.

For $g^{(5)}(12)$, matters are slightly more complicated; it is easy to get the SK bound of 17 and the construction

12345	T62	E63	V64
16789	T73	E74	V75
ITEV	T84	E85	V82
	T95	E92	V93

along with pairs 65, 72, 83, 94, shows that $g^{(5)}(12) \leq 19$.

Now blocks 12345, 6789T, imply at least $2 + 25 = 27$ blocks. Blocks 12345, 16789, imply at least $3 + 16 = 19$ blocks. So there can be only one block of length 5 if $g^{(5)}(12) < 19$. Let this block be B = {89TEV}. Then use a pairs, b triples, c quadruples, and we have

$$a + b + c = 16 + d \quad (d = 0 \text{ or } 1),$$

$$a + 3b + 6c = 56.$$

Then $2b + 5c = 40 - d$, and we have cases:

(1)	$d = 0$;	$c = 6$,	$b = 5$,	$a = 5$
(2)	$d = 0$;	$c = 4$,	$b = 10$,	$a = 2$
(3)	$d = 1$;	$c = 7$,	$b = 2$,	$a = 8$
(4)	$d = 1$;	$c = 5$,	$b = 7$,	$a = 5$
(5)	$d = 1$;	$c = 3$,	$b = 12$,	$a = 2$.

Now no quadruple is disjoint to B, or we would have at least $2 + 20 = 22$ blocks. If there is at most one quadruple through any point of B, then $c \leq 5$; also, if 2 quadruples pass through one point in B, we find that only 3 more are possible. This rules out Cases (1) and (3).

In Case (4), our 5 quadruples use up 5 triples from $A = \{1, 2, ..., 7\}$. So we can only get triples by using an element from B with a pair from A; since only $21 - 15 = 6$ pairs are available, we can not meet the requirement $b = 7$.

In Case (2), we only need 4 quadruples. This leaves 9 pairs free in A; but, even using all of them, we can not get 10 triples. Hence, we need only consider the case

$$a = 2, b = 12, c = 3, d = 1.$$

This can only occur if 3 triples from A are used for quadruples and the other 12 pairs from A are used to form triples. Then each point in B must occur with 3 or 1 points from A; hence the distribution of lines through the points of B is 3 (1 quadruple, 2 triples), 2 (1 pair, 3 triples). We may form the blocks:

89TEV, 8123,
845, 867,
9146, 925, 937.

If we now take T157, T24, T36, then we are forced to have E1 and V1. Triples E26, E35, E47 are available; so are V27, V34, V56. Thus we have achieved a construction and established

LEMMA 3.2.3.7. *$g^{(5)}(12) = 18$.*

3.2.4 The Case v = 13. It will be useful to give a slight strengthening of the SK bound before we complete the table in section 3.2.1.

From the derivation in [49], we see that the SK bound comes from using a positive variance and omitting the set A_0. Thus we have

LEMMA 3.2.4.1. *If the SK bound is an integer and if it gives the exact value of $g^{(k)}$, then all other blocks meet the block of length k and all of these other blocks have the same length t.*

Indeed, it follows at once that these other blocks form a BIBD with $1 + k(t-2)$ varieties, block size t-1, $\lambda = 1$, and this BIBD is resolvable into k resolution classes. It further follows that t-1 divides k-1.

There are 3 obvious cases in which the bound is exact. If t = 2, then v = k+1 and we have a near-pencil, If t = k, then $v = k^2-k+1$ and we have a projective geometry (in appropriate cases). If $t = 3$, and $v = 4m+3$, then $k = (v-1)/2 = 2m+1$, and we have one of the cases covered earlier.

However, if $v = 4m+1$, $k = (v-1)/2 = 2m$, we have $SK = 1+(v^2-a)/8$. This is an integer, but the $2m+1$ points not in the long blocks can not be partitioned into pairs to form triples. Thus Lemma 3.2.4.1 gives us

LEMMA 3.2.4.2. *If $v = 4m+1$, $k = 2m$, then the number of blocks strictly exceeds the bound $1 + (v^2-1)/8$.*

Now consider $g^{(5)}(13)$. The SK bound gives $g^{(5)}(13) \geq 18$. An easy construction

∞ 1234,	∞ 5678,	∞ 9TEV	
159	25T	35E	45V
16T	26E	36V	469
17E	27V	379	47T
18V	289	38T	48E

shows that $g^{(5)}(13) \leq 19$.

If $g^{(5)}(13) = 18$, we first note that any other block must meet the initial base block $B = \{\infty\ 1234\}$. For using the exact relation (2.5) from [49], we find that the number of blocks is at least

$$1 + \frac{200}{12 - \frac{1}{4}(a_0+3b_0+6c_0+10d_0)}$$

where there are a_0 blocks of length 2 disjoint from the base block, b_0 blocks of length 3, etc. (of course, it is clear $c_0 = d_0 = 0$). Even $a_0 = 1$, $b_0 = 0$, gives a bound of 19. So we find that all blocks meet the base block.

If there is a second block of length 5, we can immediately form at least $3 + 16 = 19$ blocks. So take a pairs, b triples, c quadruples, with

$$a + b + c = 17$$

$$a + 3b + 6c = 68.$$

Then $2b + 5c = 51$, whence we find:

(1) $b = 3,\ c = 9,\ a = 5$

(2) $b = 8,\ c = 7,\ a = 2$.

There can be at most 2 quadruples through any point on B. From this, we find there is no distribution of pairs and triples to points of B that works in Case (1) or Case (2). Hence we have

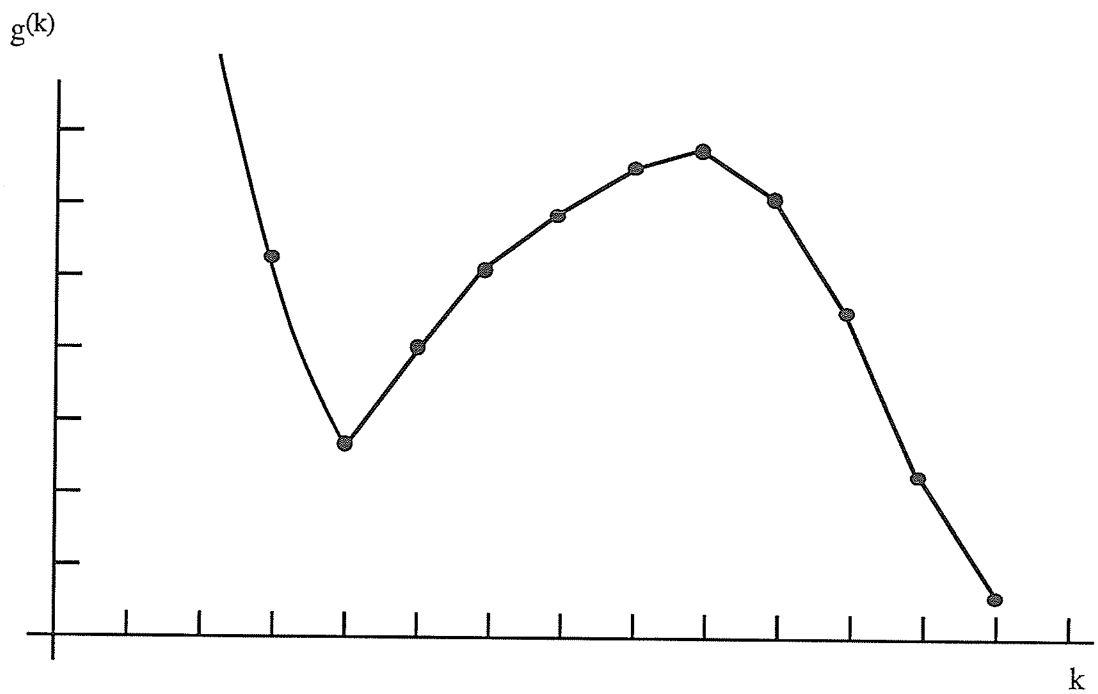
LEMMA 3.2.4.3. *$g^{(5)}(13) = 19$.*

We now move to the case $k = 6$ and use Lemma 3.2.4.2 to give the bound of 23. Actually $g^{(6)}(13) = 24$; this is a special case of the result

$$g^{(2a)}(4a+1) = 2a^2 + a + 1 + \lceil a/2 \rceil$$

which is established in Mullin, Stanton, Stinson [55]

3.2.5 Values of $g^{(k)}(v)$ which are near v. Let us return to section 3.2.1. and plot $g^{(k)}(v)$



Ignoring the case when $k = v$, which is trivial, we note that $g^{(k)}$ is usually much greater than v. Indeed, we can prove

THEOREM 3.2.5.1. *If $v = t^2+t+1+e$, where $0 \le e \le 2t+2$, and if $g^{(k)}(v) > v$, then (with a few small exceptions)*

$$g^{(k)}(v) \ge t^2+3t+1$$

unless $k = t+1$.

PROOF. From the shape of the graph of $g^{(k)}(v)$, it is clear that we need only consider the cases $k = v-2$, $k = t$, $k = t+1$, $k = t+2$.

For $k = v-2$, we have

$$g^{(k)} = 2v-4 = 2t^2+2t-2+2e = t^2+3t+1 + (t^2-t-3+2e).$$

Now $g^{(k)} > t^2+3t+1$ for $t \ge 3$. The only exception is for $t = 2$; there $g^{(5)}(7) = 10$.

For $k = t+1$, we use SK and have

$$g^{(k)}(v) \ge 1 + \frac{t^2+e}{t^2+t+e}(t+1)^2$$

$$= 1 + \frac{t^4+2t^3+(e+1)t^2+2et+e}{t^2+t+e}$$

$$= 1 + t^2 + t + \frac{e(t+1)}{t^2+t+e} .$$

For $k = t + 2$, we have

$$g^{(k)}(v) \ge 1 + \frac{(t+2)^2(t^2+e-1)}{t^2+t+e}$$

$$= 1 + t^2 + 3t + \frac{(t+4)(e-4)}{t^2+t+e} .$$

Clearly, the only exceptions that can occur are for e=0 (t = 2, 3, 4, 5), e = 1 (t=2, 3, 4), e=2 (t=2, 3).

For k = t, we use the C bound and have

$$g^{(k)}(v) \geq \frac{(t^2+t+1+e)(t^2+t+e)}{t(t-1)}$$

$$= t^2 + 3t + 2e + 5 + \frac{(4e+6)t+(e^2+e)}{t^2-t} \ .$$

Thus we have $g^{(k)}(v) \geq t^2+3t+1$ unless t and e have certain small values, so long as $k \neq t+1$.

Corollary. *if $v = t^2+t+1+e$, $k = t+1$, $g^{(k)}(v) < t^2+2t+1$, then the number of blocks of length $t+1$ is at least $(t+1)(t+2)/2$.*

PROOF. The worst case is when all other blocks have length t. With obvious meanings for x and y, we have

$$x+y = t^2+2t+1-a$$

$$(t+1)tx+(t-1)ty = (t^2+t+1+e)(t^2+t+e) \ .$$

$$\text{Then } 2tx = (t^2+t+1+e)(t^2+t+e) - (t^2-t)(t^2+2t+1-a)$$

$$= t^3+(3+2e+a)t^2 + (2+2e-a)t+e^2+e.$$

Thus $x \geq \frac{1}{2}(t^2+3t+2)$, even for a = e = 0.

3.3 A Census of Values of $g^{(k)}(v)$

3.3.1 Introduction. We now wish to extend the table of $g^{(k)}(v)$ as far as v = 21. In addition to the Combinatorial Bound, the Woodall Bound, and the Stanton-Kalbfleisch Bound, we shall use the Stinson bound

$$S = 1 + \frac{v-k}{s^2+s}(2sk-v+k+1), \text{ where } s = \lfloor (v-1)/k \rfloor.$$

We refer to [43] for a detailed description of the behaviour of $g^{(k)}(v)$ in various regions. Roughly, $g \geq C$ from $k = 2$ to about $\sqrt{v}$, equality holding for BIBDs; from $\sqrt{v}$ to $v/2$ (roughly), $g \geq SK$, with equality for certain resolvable designs (the amount by which g exceeds SK is frequently determined by S); from $v/2$ to v, $g = W$ (except for a few trivial exceptions), as described in Section 3.2.

k\v	13	14	15	16	17	18	19	20	21
2	78	91	105	120	136	153	171	190	210
3	26	35	35	46	48	57	57	70	70
4	13	20	20	20	31		35	39	39
5	19	19	20	20	20	21	21	21	21
6	24	25	25	26	27	28	28	29	29
7	28	29	29	32	33	34	35	36	36
8	31	34	36	37	39	40	43	43	45
9	31	36	40	43	45	46	46	50	51
10	28	35	41	46	50	53	55	56	59
11	22	31	39	46	52	57	61	64	66
12	13	24	34	43	51	58	64	69	73
13	1	14	26	37	47	56	64	71	77
14		1	15	28	40	51	61	70	78
15			1	16	30	43	55	66	76
16				1	17	32	46	59	71
17					1	18	34	49	63
18						1	19	36	52
19							1	20	38
20								1	21
21									1

Table 3.3.1 $g^{(k)}(v)$ for $13 \leq v \leq 21$

In the above table, we have collected a great many values for $g^{(k)}(v)$ that we have accumulated. The table indicates many patterns, and provides useful grist for more general results; also, the methods indicated in building up the table are instructive. We indicate how the results are obtained, giving sufficient detail to enable duplication. It has been simpler to work with k decreasing for purposes of the exposition.

3.3.2 The Larger k Values, $k \geq 9$. For $k > 10$ in the table, $g^{(k)}(v) = W$ in all cases. For $k = 10$, $g^{(k)}(v) = W$ except for $g^{(10)}(21)$; this latter value is a special case of $g^{(2m)}(4m+1)$, as determined in [55].

For $k = 9$, we find $g^{(9)}(21) \geq 51$, using the Stinson bound S. Indeed $S = 51$; hence, using [61], we know that $g^{(9)}(21) = 51$ if and only if all blocks have lengths $s+1$ and $s+2$ (here $s = 2$) and fall into 9 resolution classes. If there are a quadruples, b triples, then $a + b = 50$, $3a + b = 66$, whence $a = 8$, $b = 42$. It is then easy to apply a hill-climbing algorithm (by hand, in a case this small) to provide the design

1 2 3	1 4 7	1 5	1 6	1 8	1 t	1 e	1 v	1 9
4 5 6	2 5 e	2 4	2 v	2 7	2 8	2 t	2 9	2 6
7 8 9	3 9 t	3 e	3 7	3 v	3 4	3 8	3 6	3 5
t e v	6 8 v	6 9	4 8	4 e	5 7	4 9	4 t	4 v
		7 v	5 t	5 9	6 e	5 v	5 8	7 t
		8 t	9 e	6 t	9 v	6 7	7 e	8 e

Here, we use 1, 2, ..., 9, t, e, v to denote the 12 points not on the long block; the points of the long block are $A_1, A_2, \ldots, A_9$, and correspond to the preceding 8 resolutions. Thus, we have blocks

$$A_1 123,\ A_1 456,\ \ldots,\ A_2 147,\ A_2 25e,\ \ldots,\ A_3 15,\ A_3 24,\ \ldots, \text{etc.}$$

For $g^{(9)}(20)$, we use the S bound to give $g^{(9)}(20) \geq 49$. Suppose, if possible, that $g^{(9)}(20) = 49$. Then the largest other block has length 6 (a sextuple not meeting the long block produces a need for at least 2 + 9(6) blocks; a sextuple meeting the long block requires at least 2+2+8(5) = 44 blocks). Suppose that the numbers of sextuples, quintuples, quadruples, triples, and pairs are s, f, q, t, p, respectively; that the numbers A_i ($i = 1, \ldots, 9$) are s_i, f_i, q_i, t_i, p_i; that the numbers disjoint from the long block are $s_0, \ldots p_0$. Then we have the following relations.

$$s + f + q + t + p = 48$$

$$15s + 10f + 6q + 3t + p = 190 - 36 = 154$$

$$5s_i + 4f_i + 3q_i + 2t_i + p_i = 11 \quad (i > 0)$$

$$5s_0 + 4f_0 + 3q_0 + 2t_0 + p_0 \geq 0$$

Add the last set of equations to give

$$5s + 4f + 3q + 2t + p \geq 99.$$

We then deduce

$$14s + 9f + 5q + 2t = 106$$

$$4s + 3f + 2q + t \geq 51;$$

by subtraction,

$$6s + 3f + q \leq 4.$$

It follows that $s = 0$ (no sextuples).

If $f = q = 1$, then $t + p = 46$, $3t + p = 138$; hence $t = 46$, $p = 0$.

This is impossible, since there are 11 elements in 9 resolution classes and thus $q + p \geq 9$.

If $f = 1$, $q = 0$, the solution for t is non-integral, Thus $f = 0$.

In order to have t integral, q must be even. For $f = q = 0$, we find $t + p = 48$, $3t + p = 154$ (p negative); similarly, $q = 2$ gives $t + p = 46$, $3t + p = 142$, and this forces $t = 48$ (contradiction). Finally, $q = 4$ produces $t = 43$, $p = 1$; this is rejected, since $q + p$ must be 9 or more.

Thus, we have proved that $g^{(9)}(20) > 49$. We now show that the correct value is 50.

Using the previous notation, we obtain analogous equations as follows.

$$s + f + q + t + p = 49,$$

$$15s + 10f + 6q + 3t + p = 154,$$

$$5s + 4f + 3q + 2t + p \geq 99.$$

Thence, we deduce

$$6s + 3f + q \leq 5.$$

Again, $s = 0$. If $f = 1$, then $q = 0, 1, 2$. The only possibilities are $f = 1$, $q = 2$, $t = 43$, $p = 3$; $f = 1$, $q = 0$, $t = 48$, $p = 0$. Both are rejected on considering the value of $q + p$. We thus have $s = f = 0$, whence $q \leq 5$. Now $5q + 2t = 105$, whence q is odd. If $q = 1$, $t = 50$; if $q = 3$, $t = 45$, $p = 1$ (reject on basis of $q + p$). Hence the only possibility is $q = 5$, $t = 40$, $p = 4$. If this design is possible, each resolution must contain exactly one pair or one quadruple.

Let each resolution contain a pair and 5 triples (R of these) or a quadruple and 3 triples (S of these). Clearly, $R + S = 9$, $R = 5$, $S = 4$. It is then easy to use hill-climbing (again, a hand algorithm suffices) to produce the following 9 resolutions.

1 2 3	4 5 6	7 8 9	1 4 t	2 5 e	2	5	1	4
4 7	1 9	1 5	2 7	1 6	1 8	1 7	2 4	1 e
5 8	2 8	2 6	3 e	3 7	3 5	2 9	3 9	2 t
6 t	3 t	3 4	5 9	4 8	4 9	3 6	5 t	3 8
9 e	7 e	t e	6 8	9 t	7 t	4 e	6 7	5 7
					6 e	8 t	8 e	6 9

We have learned that Rees [37] has obtained a family of designs giving $g^{(2m+1)}(4m+4)$; this family will include the (9,20) case.

3.3.3 The Values of $g^{(8)}(v)$. We first find $g^{(8)}(20)$ since it is the simplest. The S bound is 43, and thus we know that, if this bound is obtained, a design with only triples and quadruples is required. If there are a quadruples and b triples, $a + b = 42$, $3a + b = 66$; thence $a = 12$, $b = 30$. Also, there must be R, S, T, of the quadruple-triple patterns (4,0), (2,3), and (0,6). Then $R + S + T = 8$, $4R + 2S = 12$, $3S + 6T = 30$; whence $(R,S,T) = (0,6,2)$ or $(1,4,3)$ or $(2,2,4)$ or $(3,0,5)$. Clearly, the last case is the most straightforward, and a hill-climbing algorithm (manual, of course) produces the following 8 resolutions

1 2 3	1 4 7*	1 5 9	1 6	1 8	1 t	1 e	1 v
4 5 6	2 5 e	2 4 v	2 9*	2 t	2 7	2 8	2 6
7 8 9	3 9 t	3 8 e	3 v	3 6*	3 5	3 4	3 7
t e v*	6 8 v	6 7 t	4 e	4 9	4 8	5 t	4 t
			5 7	5 v	6 e	6 9	5 8*
			8 t	7 e	9 v	7 v	9 e

Thus $g^{(8)}(20) = 43$. Furthermore, if we adjoin h to the starred blocks and three pairs (A_i,h) to the other 3 resolutions ($i = 3,6,7$), we at once have the corollary that $g^{(8)}(21) \leq 43 + 3 = 46$ (the S bound is 45). Note that the above array is corrected from that appearing in [7], where the spacing of the fourth row needs moving to the left and the addition of two pairs.

Let us now consider $g^{(8)}(19)$, where the S bound is 42. Clearly, we can get 43 blocks by deleting one point from the design just obtained; so we only need consider the possibility that $g^{(8)}(19) = 42$. As pointed out in [8], a design can actually be constructed on 42 elements.

For $g^{(8)}(18)$, we can delete 2 points from the design on 20 elements to give an upper bound of 42; the Stinson bound is 40. Proceed as above, if $g = 40$.

$$s + f + q + t + p = 39,$$

$$15s + 10f + 6q + 3t + p = 125,$$

$$5s + 4f + 3q + 2t + p \geq 80.$$

Thence

$$14s + 9f + 5q + 2t = 86,$$

$$4s + 3f + 2q + t \geq 41,$$

$$6s + 3f + q \leq 4.$$

The only possibilities are $f = q = 1$, $t = 36$, $p = 1$, and $q = 4$, $t = 33$, $p = 2$.

If $f = q = 1$, $t = 36$, $p = 1$, consider the quadruple, triple and singleton (using only the ten points not on the long block). There must be a point (of the ten) that lies on nine pairs (impossible, since there are only 8 resolutions). If $q = 4$, $t = 33$, $p = 2$, let α_i, β_i, γ_i ($i = 1, \ldots, 10$) be the frequency with which i appears in triples, pairs, singletons; then $\alpha_i + \beta_i + \gamma_i = 8$, $2\alpha_i + \beta_i = 9$. The only solutions are (3,3,2), (2,5,1), (1,7,0), and we assume there are R, S, T, of each pattern. Then $R+S+T = 10$, $3R+2S+T = 12$, $3R+5S+7T = 66$, $2T+S = 2$; the solutions are (1,0,9) and (0,2,8). The (1,0,9) solution would give a point, say 10, in two singletons; deletion of this point would give $g^{(8)}(17) \leq 38$; so this is impossible. But the (0,2,8) solution can be achieved by using blocks 9,5,6; 9; 9,7,8; 9,0; 9,1; 9,2; 9,3; 9,4; in the design given in example 3.1 of [55]. Hence $g^{(8)}(18) = 40$. The complete design is

0 1 2	9	0 3 4	9 0*
3 8	0	2 5	4 5
4 7	2 3	1 6	3 6
9 5 6	1 4	9 7 8	2 7
	5 8		1 8
	6 7		

9 1	9 2	9 3	9 4
5 0	6 0	7 0	8 0
2 8	1 3	2 4	1 7
3 7*	4 8*	1 5*	2 6*
4 6	5 7	6 8	3 5

We note that adjunction of t to the starred blocks, together with the addition of a singleton t to each of the first three resolutions, would produce a covering for $v = 19$, $k = 8$, in 43 blocks.

Since $g^{(8)}(v) = W$ for $v \leq 15$ and $g^{(8)}(17) = 39$, from [55], we need merely consider $g^{(8)}(21)$. We have already displayed a solution in 46 blocks, and so must now consider the possibility that $g^{(8)}(21) = 45$.

We need to let u be the number of septuples. Then

$$u + s + f + q + t + p = 44,$$

$$21u + 15s + 10f + 6q + 3t + p = 182,$$

$$6u + 5s + 4f + 3q + 2t + p \geq 104.$$

Proceeding as usual, we find

$$10u + 6s + 3f + q \leq 18.$$

Now $u \leq 1$; but if there is a 7-set, then it meets the 8-set and the other points and must lie on a second 7-set; hence $u = 0$. Then we can sieve the solutions and find there are no solutions except for $s = 0$; the (f,q,t,p) solutions are then (2,12,30,0), (1,15,27,1), and (0,8,24,2). The last is easily achieved by taking a different solution to the earlier $g^{(8)}(20) = 43$. Use the semi-cyclic solution displayed.

1 3 4	2 4 5	3 5 6	4 6 7	5 7 8	6 8 9	1 6	1 8
7 9 t	8 t e	9 e v	t v 1	e 1 2	v 2 3	4 e	6 e
2 6	3 7	4 8	5 9	6 t	7 e	2 9	4 9
5 e*	6 v*	7 1*	8 2*	9 3*	t 4*	7 v	2 7
8 v	9 1	t 2	e 3	v 4	1 5	5 t	5 v
						3 8	3 t

Now adjoin h to the six starred blocks and a singleton h to each of the last two resolutions. This achieves the bound and shows that $g^{(8)}(21) = 45$.

3.3.4 The Case k = 7. Here we have $g = W$ for $v = 13, 14, 15$. We need to discuss the cases $16 \leq v \leq 21$.

First, we calculate the bound S for $v = 21$ and find it to be 36 (integrally). So a solution ($s = 2$) may exist in triples and quadruples (numbers a and b, respectively). Then $a + b = 35$, $3a + 6b = 189$, whence $a = 7$, $b = 28$. Let x and y of each pass through any point on the long block; then $2x + 3y = 14$, and $(x,y) = (1,4)$ or $(4,2)$ or $(7,0)$. If there are R, S, T, of each pattern, then $R+S+T = 7$, $4R+2S = 28$, $R+4S+7T = 7$. Thus $R = 7$; each resolution contains 4 triples and a pair. Obviously, the only solution (cf. [43] for $v = 22$) is to take the 7 resolutions of a resolvable Steiner Triple System on 15 points, and then to delete one element. Thus $g^{(7)}(21) = 36$.

For $g^{(7)}(20)$, ths S bound is 36. Hence we need merely delete an element from the design just obtained.

For $g^{(7)}(19)$, the bound S is 35; this can be achieved by deleting 2 elements that form a pair in a resolution of the solution for $g^{(7)}(21)$. An alternative solution is cyclic and is provided by

1 5 9	1 3 4	2 4 5	3 5 6	4 6 7	5 7 8	6 8 9
2 6 t	7 9 t	8 t e	9 e v	t v 1	e 1 2	v 2 3
	2 8	3 9	4 t	5 e	6 v	7 1
3 7 e	6 e	7 v	8 1	9 2	t 3	e 4
4 8 v	5 v	6 1	7 2	8 3	9 4	t 5

For $g^{(7)}(18)$, the bound S is 34 and is integrally exact. So we must seek 7 resolutions ($s = 2$) into triples and pairs. We get $a + b = 33$, $3a + 6b = 132$; thus $a = 22$, $b = 11$. From $2x + 3y = 11$, we get $(x,y) = (1,3)$ or $(4,1)$. Taking R and S of each pattern, we find $R + S = 7$, $R + 4S = 22$, $3R + S = 11$. Thus $R = 2$, $S = 5$. From this information, a hill-climbing algorithm provides the following design.

1 2 3	1 4 7	1 5 9	2 6 t	4 8 t	6 7 e	3 8 e
4 5 6	2 5 e	2 8	1 e	1 6	1 8	1 t
7 8 9	3 9 t	3 6	3 7	2 7	2 9	2 4
t e	6 8	4 e	4 9	3 5	3 4	5 7
		7 t	5 8	9 e	5 t	6 9

Thus $g^{(7)}(18) = 34$.

For $g^{(7)}(17)$, the bound S is 33, and we can find a design in 33 blocks. Proceed as before, with the usual notation.

$$s + f + q + t + p = 32,$$

$$15s + 10f + 6q + 3t + p = 115,$$

$$5s + 4f + 3q + 2t + p \geq 70.$$

Then we find $6s + 3f + q \leq 7$.

Now $s = q = 1$ gives $t + p = 30$, $3t + p = 94$, $t = 32$ (impossible). And $s = 1$, $q = 0$, gives $t + p = 31$, $3t + p = 100$ (again impossible).

Hence $3f + q \leq 7$. It is easy to sieve possibilities and find that the only solutions occur for $(f,q,t,p) = (1,4,27,0)$ and $(0,7,24,1)$. The first solution is impossible (every point must lie on an even number of quadruples; hence abcd is a quadruple, we need axxx, bxxx, cxxx, dxxx, at least). In the second solution, we write $w + 2x + 3y = 10$ and suppose that there are R of pattern (1,0,3), S of pattern (1,3,1), T of pattern (0,5,0), U of pattern (0,2,2). Then $R+S+T+U = 7$, $R+S = 1$, $3S+5T+2U = 24$, $3R+S+2U = 7$. Then $(R,S,T,U) = (1,0,2,4)$ or $(0,1,3,3)$. Selecting the latter pattern, we easily construct a design as follows.

1 5	1 2	1 7	1 3 8	2 3 9	1 9 t	6 7 8
2 t	3 4	2 8	4 5 9	5 7 t	2 4 6	1 4
3 6	5 6	3 5	2 7	1 6	3 7	2 5
4 7	7 9	4 t	6 t	4 8	5 8	3 t
8 9	8 t	6 9				9

We can delete 9 from the last resolution to leave $g^{(7)}(16) \leq 32$. To prove 32 is the value, we must exclude the bound $S = 31$. The usual argument gives, in this latter case,

$$f + q + t + p = 30,$$

$$10f + 6q + 3t + p = 99,$$

$$4f + 3q + 2t + p \geq 63.$$

We deduce that $3f + q \leq 3$. The only solution is $f = 0$, $q = 3$, $t = 27$, $p = 0$, and this does not provide enough quadruples to cover each resolution; thus $g^{(7)}(16) = 32$. (Alternatively, we could note that $S = 31$ exactly; then we easily deduce $a = 27$, $b = 3$, since only quadruples and triples are allowed if the bound is exact).

3.3.5 The Case k = 6. If $k = 6$, the bound is 29; this can easily be achieved, using a method that has general application. Take the geometry on 31 points generated cyclically by (1,5,11,24,25,27). Delete points 5,11,24,25,27, to give 30 blocks covering 26 points; then delete 6,12,26,28, to give 29 blocks covering 22 points. Deleting one further point establishes that $g^{(6)}(21) = 29$.

Continuing the above process, we delete 8,14,30, to show that $g^{(6)}(19) \leq 28$. In this case, the bound is 27 (integrally). However, $s = 3$ and so there would have to be a quadruples, b quintuples, with $a + b = 26$, $6a + 10b = 156$. This would require $a = 26$, $b = 0$, and the resolutions can not be packed (13 is not a multiple of 3); thus $g^{(6)}(19) = 28$.

Another deletion shows that $g^{(6)}(18) \le 28$. Again, the bound is exactly 27 ($s = 2$); so there may be a solution with a triples, b quadruples, $a + b = 26$, $3a + 6b = 138$. Then $a = 6$, $b = 20$. Consider the 6 pairs and 20 triples in the resolutions; any single element occurs with 11 others and so must be in a pair. Thus the pairs involve all 12 elements and thus form a resolution. So the configuration involves just the 6 pairs and 20 triples of an STS on 13 points with one point deleted. It is well known that this system can not be resolved [26].

Finally, returning to the design on 19 elements, delete 17 and 23 to give $g^{(6)}(17) = 27$ (the bound here is 27). Delete point 29 to give $g^{(6)}(16) = 26$ (the bound is integrally 26 and our design naturally ends up with a 6-set, 1 2 4 9 12 19, and 10 quadruples, 15 triples).

There remains to determine whether $g^{(6)}(20)$ can equal 28. We set up the usual equations (there can be no other 6-set).

$$f + q + t + p = 27,$$

$$10f + 6q + 3t + p = 175,$$

$$4f + 3q + 2t + p \ge 84.$$

It follows that $3f + q \le 24$. Using a sieve on the solutions, we find there are none; hence $g^{(6)}(20) = 29$.

The three lowest values for $k = 6$ are now immediate. The value $g^{(6)}(13) = 24$ is given in both [55] and [3]. The value $g^{(6)}(15) = 25$ follows since the bound is 25, and we find there must be 18 quadruples and 6 triples. The resolutions are easily found as

123	456	789	147	258	369
49	18	16	26	19	15
57	29	24	38	34	27
68	37	35	59	67	48

Deletion of one element gives $g^{(6)}(14) \le 25$. However, the bound is 24, and so we must consider

$$f+q+t+p = 23,$$

$$10f+6q+3t+p = 76,$$

$$4f+3q+2t+p \ge 48.$$

A contradiction results at once. We get $3f+q \le 3$ with solution $f=0$, $q=3$, $t=19$, $p=1$; there are not enough quadruples and pairs to cover all points. Thus $g^{(6)}(14) = 25$.

3.3.6 The Case k = 5. This case is simple. For $g^{(5)}(13) = 19$, see [3]. For the values from 14 to 21, the bounds are 19, 20, 20, 20, 21, 21, 21, 21. All of these can be achieved by starting from the geometry on 21 points. Generate it by (0 1 4 14 16). First, delete points 0, 1, 4, 14, to show that the bound is met for 20, 19, 18, 17. Then delete 2, 5, 15, to give the results for 16, 15, 14, The last value $g^{(5)}(14) = 19$ is exactly integrally equal to the bound, a fact which mirrors the nature of the design (9 triples and 9 quadruples, besides the remaining block 16, 17, 20, 9, 11).

This case of a geometry can be discussed in general. For instance, consider $g^{(k)}(k_2\text{-}k+1\text{-}d)$, where $d \le k$. Then

$s = \left\lfloor \frac{k^2-k+1-d}{k} \right\rfloor = \left\lfloor k-1-\frac{d-1}{k} \right\rfloor = k-2$. The bound is

$$1 + \frac{k^2-2k+1-d}{(k-2)(k-1)}(2k(k-2)-(k^2-2k-d))$$

$$= 1 + \frac{(k^2-2k+1-d)}{k^2-3k+2}(k^2-2k+d)$$

$$= 1 + k^2-k-\frac{d(d-1)}{k(k-1)}.$$

We thus see that the bound verifies that the best results occur by deletion for d up to k. The discussion for $d > k$ is more complicated.

3.3.7 The Cases k < 5. For k = 2, the values are just the binomial coefficients $\binom{k}{2}$. For k = 3, they are just the pair-packing numbers, increased by the pair defect (see [56]). For k = 4, all values are known (see [57]) except the values for v = 17, 18, 19. We fill in $g^{(4)}(17) = 31$ from [63].

3.3.8 The Values for v = 22. We have most of the values between v = 22 and v = 32, which is the next natural limit. Since v = 22 is just past the bound for PG(2,4), the behaviour is particularly interesting and we include the values here

k	$g^{(k)}(22)$	k	$g^{(k)}22)$
2	231	11	67
3	85	12	76
4	42	13	82
5	25	14	85
6	29	15	85
7	36	16	82
8	46	17	76
9	53	18	67
10	59	19	55
		20	40
		21	22
		22	1

The values for k ≥ 11 are given by W; the values for 2, 3, 4, are given by C, [56], and [57]. For k = 10, the bound S is given as 59, and there is a solution in 10 resolutions of triples and pairs, namely,

1 2 3	1 4	1 5	1 6	1 7	1 8	1 9	1 t	1 e	1 v
4 5 6	2 5	2 6	2 4	2 8	2 9	2 7	2 e	2 v	2 t
7 8 9	3 6	3 4	3 5	3 9	3 7	3 8	3 v	3 t	3 e
t e v	7 t	7 e	7 v	4 t	4 e	4 v	4 7	4 8	4 9
	8 e	8 v	8 t	5 e	5 v	5 t	5 8	5 9	5 7
	9 v	9 t	9 e	6 v	6 t	6 e	6 9	6 7	6 8

For k = 9, S is likewise exactly 53 and we need 9 resolutions into 13 triples and 39 pairs. Hill-climbing produced the following solution.

123	14t	15h	27v	36t	3eh	47e
456	28e	26	16	18	17	1v
789	69v	34	39	2h	2t	29
te	35	9e	4h	49	4v	38
vh	7h	8v	5e	57	59	5t
		7t	8t	ev	68	6h

58h	9th
19	1e
24	25
37	3v
6e	48
tv	67

For $k = 8$, $S = 46$; the key here is to look at $g^{(8)}(23)$, where $S = 46$ exactly. A solution for $k = 8$, $v = 23$, in triples and pairs is given by the following 8 resolutions.

123	26f	3vf	68v	19v	2vh	18f	7tf
456	9ti	69e	14i	37e	59f	34t	249
789	47v	48h	57h	6th	36i	25e	8ei
tev	1wh	15t	ef	4f	8t	9h	3h
hfi	358	27i	2t	5i	4e	67	16
			39	28	17	vi	5v

This array can be derived from PG(2,7) by generating PG(2,7) cyclically from the initial block (1,6,7,9,19,38,42,49) and then retaining the block (53,1,2,4,33,37,44) and the 15 points 3, 7, 8, 9, 15, 22, 28, 30, 35, 43, 47, 49, 55, 56, 57. Deletion of a single point now produces a solution for $g^{(8)}(22)$. The value $g^{(7)}(22)$ comes from the resolutions of an STS on 15 elements (see [43]). For $g^{(6)}(22)$, the bound is exactly 29 and may be achieved by deleting 5 collinear points ABCDE from the 32-point geometry and then deleting 4 further points GHJK where GHJK are on a line through A. Finally, $g^{(5)}(22) = 30$ (delete 3 points from the BIBD on 25 points, as in [5]).

3.4 The Case $k = 4$ and $k = 5$

3.4.1 Values for $g^{(4)}(v)$. In this section, we record the values of $g^{(4)}(v)$ from [57]. Complete results were given there for all v with the exception of $v = 17$, 18, and 19. We note that the value of $g^{(4)}(19)$ has now been established as 35 in [46]. Also, $g^{(4)}(17) = 31$ (cf. [58], [59], [11]).

v	$g^{(4)}(v)$	exceptions
$12t - 1$	$12t^2 + t$	
$12t$	$12t^2 + t$	
$12t + 1$	$12t^2 + t$	
$12t + 2$	$12t^2 + 7t + 1$	
$12t + 3$	$12t^2 + 7t + 1$	
$12t + 4$	$12t^2 + 7t + 1$	
$12t + 5$	$12t^2 + 13t + 4$	$g^{(4)}(5) = 5$, $g^{(4)}(17) = 31$
$12t + 6$	$12t^2 + 13t + 4$	$g^{(4)}(6) = 8$, $g^{(4)}(18)$ not known
$12t + 7$	$12t^2 + 13t + 7$	$g^{(4)}(7) = 10$, $g^{(4)}(19) = 35$
$12t + 8$	$12t^2 + 19t + 8$	$g^{(4)}(8) = 11$
$12t + 9$	$12t^2 + 19t + 8$	$g^{(4)}(9) = 12$
$12t + 10$	$12t^2 + 19t + 11$	$g^{(4)}(10) = 12$

The values for $g^{(5)}(v)$ are largely undetermined. Allston and Stanton [5] have obtained partial results along the lines of [57], and we give these here.

3.4.2 A bound on $g^{(5)}(v)$. We let g_i ($i = 2,3,4,5$) represent the number of blocks of length i in an exact covering. Then the total number of blocks in the covering is

$$g = g_2 + g_3 + g_4 + g_5.$$

If Σ denotes a summation over all g blocks, then we have

$$\Sigma 1 = g,$$

$$\Sigma(k_i-4)(k_i-5) = 6g_2 + 2g_3$$

$$= \Sigma(k_i^2 - 9k_i + 20)$$

$$= \Sigma k_i(k_i-1) - 8\Sigma k_i + 20g$$

where k_i represent the g block lengths. Furthermore,

$$\Sigma k_i(k_i - 1) = v(v-1)$$

The frequency r_i of any element is at least $\lceil (v-1)/4 \rceil$; hence

$$\Sigma k_i = \Sigma r_i = v \lceil (v-1)/4 \rceil + \varepsilon,$$

where $\varepsilon \geq 0$. Combine these results, and we obtain

$$6g_2 + 2g_3 = v(v-1) - 8v \lceil (v-1)/4 \rceil - 8\varepsilon + 20g,$$

that is,

$$g = \frac{1}{20} \{6g_2 + 2g_3 + 8\varepsilon + v(8 \lceil (v-1)/4) \rceil - v + 1)\}.$$

Since g_2, g_3, and ε are non-negative, we have

THEOREM 3.4.1. *For an exact covering with at least one block of size 5, but no longer block, then*

$$g \geq \frac{v}{20} (8 \lceil (v-1)/4 \rceil - v + 1).$$

It is well known that Balanced Incomplete Block Designs with parameters

$$(20t + 1, 20t^2 + t, 5t, 5, 1)$$

and

$$(20t + 5, 20t^2 + 9t + 1, 5t + 1, 5, 1)$$

exist for all $t > 1$ (the second exists trivially for $t = 1$). The bound is, of course, exact for these designs.

3.4.3 The Cases of an Exact Bound. Suppose that we use $g_0(v)$ to denote the integer bound in Theorem 3.4.1. Then it is easy to calculate the following table of

$$g_0(v) = \lceil v(8\lceil (v-1)/4 \rceil - v + 1) / 20 \rceil.$$

v	$g_0(v)$
20t - 2, 20t - 1, 20t, 20t + 1	$20t^2 + t$
20t + 2, 20t + 3, 20t + 4, 20t + 5	$20t^2 + 9t + 1$
20t + 6, 20t + 7, 20t + 8, 20t + 9	$20t^2 + 17t + 4$
20t + 10, 20t + 11, 20t + 12, 20t + 13	$20t^2 + 25t + 8$
20t + 14, 20t + 15, 20t + 16, 20t + 17	$20t^2 + 33t + 14$

Now deletion of points (one, two, or three) from the BIBDs on 20t + 5 or on 20t + 1 points achieves the bound $g_0(v)$. Consequently, we have

THEOREM 3.4.2. *The quantity $g^{(5)}(v)$ is given by $g_0(v)$ for v congruent to -2, -1, 0, 1, 2, 3, 4, 5, modulo 20.*

We can also use $g_0(v)$ to throw light on the behaviour of D(2,5,v), which is the maximum number of quintuples from v elements with no repeated pair (the packing number).

Clearly, $D(2,5,v) = g^{(5)}(v) = g_0(v)$ for $v \equiv 1$ or 5, modulo 20. For $v \equiv 0$ or 4, modulo 20, we substitute in the exact relation

$$g = \frac{1}{20}\{6g_2 + 2g_3 + 8\varepsilon + v(8\lceil (v-1)/4 \rceil - v + 1)\},$$

and obtain $g_2 = g_3 = \varepsilon = 0$. This shows that the exact coverings contain only blocks of lengths 4 and 5 and that every element has the same frequency; it follows that the blocks of length 4 are all disjoint and thus we have established the well known result that the optimal packing for $v \equiv 0$ or 4 (modulo 20) occurs by taking BIBDs for $v \equiv 1$ or 5 (modulo 20) and deleting a single point. This process produces the value of $g^{(5)}(v)$ and D(2,5,v).

For $v \equiv -1$ or 3 (modulo 20), the situation is similar; for $v \equiv 20t + 3$, we have

$$20g = (20t^2 + 9t + 1)20$$

$$= 6g_2 + 2g_3 + 8\varepsilon + (20t + 3)(20t + 6).$$

Hence, $6g_2 + 2g_3 + 8\varepsilon = 2$. Thus $g_3 = 1$, and it follows that the optimal packing involves a single block of length 3, with all frequencies again equal to $5t + 1$. The configuration is derived from and extendible to the relevant BIBD.

For $v = 20t - 1$, we have

$$20g = (20t^2 + t)20$$

$$= 6g_2 + 2g_3 + 8e + (20t - 1)(20t + 2).$$

It follows again that $g_3 = 2$, and the same conclusion holds for $v = 20t + 3$.

For $v = 20t + 2$, we obtain

$$20(20t^2 + 9t + 1) = 6g_2 + 2g_3 + 8\varepsilon + (20t + 2)(20t + 7);$$

thus $6g_2 + 2g_3 + 8\varepsilon = 6$. Similarly, for $v = 20t - 2$,

$$20(20t^2 + t) = 6g_2 + 2g_3 + 8\varepsilon + (20t - 2)(20t + 3);$$

again, $6g_2 + 2g_3 + 8\varepsilon = 6$. In either case, we find $g_2 = 1$, $g_3 = 0$, or $g_2 = 0$, $g_3 = 3$. In both situations, $\varepsilon = 0$, and $r_i = \lceil (v-1)/4 \rceil$. Again, we find that we have an optimal packing derived by deleting three points from a BIBD (the two situations correspond to the cases when the three deleted points lie in one block, giving $g_2 = 1$, or lie in three blocks, giving $g_3 = 3$). We thus have

THEOREM 3.4.3. *The optimal configurations for exact coverings of $v = 20t + a$ points with blocks of sizes 2, 3, 4, 5 ($-2 \leq a \leq 5$) occur if and only if the design is a punctured BIBD with 0, 1, 2, or 3 points removed.*

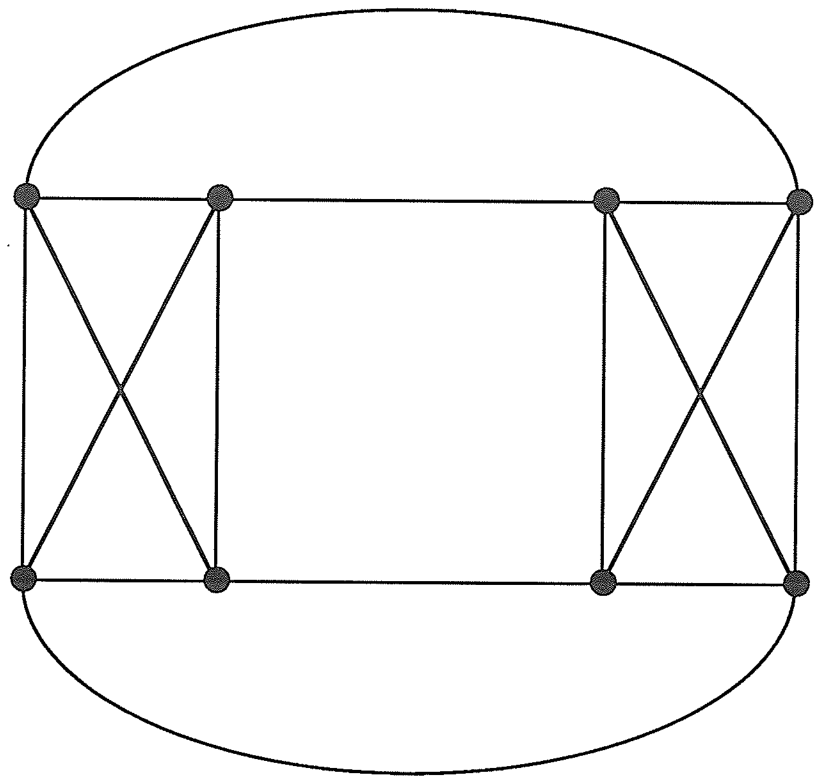


Figure 3.4.1.

3.4.4 An Improved Bound for $g^{(5)}$ in the Cases $v \equiv 9, 13, 17$ (modulo 20). Consider $v = 20t + 9$; then it is well known that

$$D(2,5,v) \leq \lfloor (20t + 9)(5t + 2)/5 \rfloor = 20t^2 + 17t + 3.$$

If this bound were achievable, the defect graph (consisting of all pairs not used) would contain

$$\binom{20t + 9}{2} - 10(20t^2 + 17t + 3) = 6$$

edges. Since every vertex in the defect graph has a valence divisible by 4, this is impossible. Consequently,

$$D(2, 5, 20t + 9) \leq 20t^2 + 17t + 2.$$

If $D(2, 5, 20t + 9) = 20t^2 + 17t + 2$, the sum of the valencies would be $2(16) = 32$. A vertex of valence 8 would require a minimum of 9 vertices in the graph; hence the graph contained 8 vertices of valence 4. Such a graph needs 16 K_2's to cover it; if K_3's are allowed, at least 5 K_3's and a K_2 would be needed (this is purely on count; actually such a configuration is not achievable); finally, if K_4's are allowed, one needs at least two K_4's and four K_2's (any vertex used in a K_4 must have an attached K_2). This last configuration is easily achieved (see Figure 3.4.1) and so we have

LEMMA 3.4.1. *If $D(2, 5, 20t + 9) = 20t^2 + 17t + 2$, and if the maximum number of K_5's is used, then $g^{(5)} \geq 20t^2 + 17t + 8$.*

However, there are two possibilities: either $D(2, 5, 20t + 9) < 20t^2 + 17t + 2$, or one does not use the maximal number of K_5's. Both these cases can be treated together.

If, for either reason, the number of K_5's used is $20t^2 + 17t + 1$, then the defect graph contains 26 edges and has a valence sum of 52. It follows that it has vertices of degrees 4 and 8, the distribution pattern being (13,0), (11,1), (9,2), (7,3), or (8,4). If only K_2's are used, we need 26 of them; if K_3's are used we need at least 9 blocks; if one K_4 is used, we need at least 8 blocks (again, we do not consider achievability of configurations). If three K_4's are used, 8 edges remain; this would require at least one edge left over at each vertex) that the configuration is not achievable. So at least 7 more blocks would be needed, and the total is again at least $20t^2 + 17t + 8$.

If the number of K_5's used is $20t^2 + 17t$, then the defect graph contains 36 edges; not more than 18 vertices, and the allowable valencies are 4, 8, 12. We want to show that 8 blocks or more are needed to cover it. Clearly, K_4's are required; indeed, if only 4 K_4's are used, then at least 4 K_3's are needed, and the total would be 8. Six K_4's is impossible, since it would require all vertices to have valence 12 (no edges left over), and this can not occur with 36 edges. If five K_4's could be used, there would be six lines left over and they would have to be in triangles; but that would mean no vertices of valence 4 (impossible). So again, we have $g^{(5)} \geq 20t^2 + 17t + 8$.

Finally, if $20t^2 + 17t - x$ K_5's are used, then the defect graph has $36 + 10x$ edges. For $x \geq 3$, even the use of K_4's for all blocks requires $6 + (2 + x)$ blocks or more. If $x = 2$ (56 edges), at least 11 blocks are needed. Thus we obtain

THEOREM 3.4.4. *For $v = 20t + 9$, $g^{(5)} \geq 20t^2 + 17t + 8 = g_0 + 4$.*

The discussion for $v = 20t + 17$, where the bound on $D(2,5,v)$ is $20t^2 + 33t + 13$, and the defect graph has 6, 16, 26, ... vertices, is exactly the same. Consequently, we can state

THEOREM 3.4.5. *For* $v = 20t + 17$, $g^{(5)} \geq 20t^2 + 13t + 18 = g_0 + 4$.

The situation is only slightly different for $v = 20t + 13$. The bound on $D(2, 5, 20t + 13)$ is $20t^2 + 25t + 7$; for this many quintuples, there would be a defect graph of 8 edges, but it can not be achieved. Hence, we must first consider the possibility of using $20t^2 + 25t + 6$ quintuples with a defect graph of 18 edges (all valencies 4) and 9 vertices. If all K_3's are used, one needs 6 (achievable; cf. Figure 3.4.2). If a K_4 is used, the remaining 12 edges can not be covered by 4 triangles. If two K_4's are used (the maximum allowable), again four K_2's (and more) are forced. Thus we have

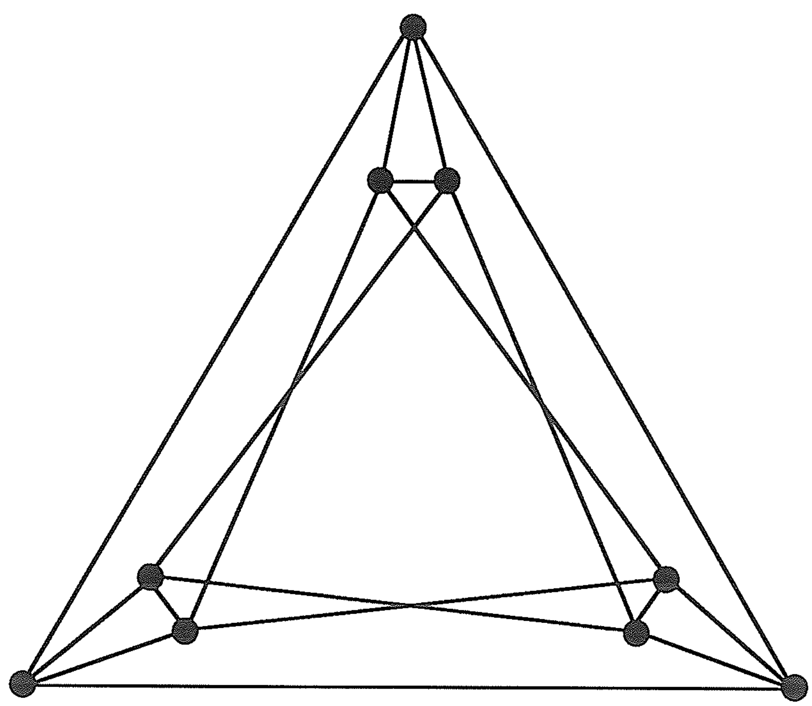


Figure 3.4.2

LEMMA 3.4.2. *If* $20t^2 + 25t + 6$ *quintuples are used, then* $g^{(5)} \geq 20t^2 + 25t + 12 = g_0 + 4$.

If the defect graph has 28 edges ($20t^2 + 25t + 5$ quintuples), then all vertices have valencies 4 and 8 (distribution patterns (14,0), (12,1), (10,2), (8,3), or (6,4)). Certainly, the covering can not be achieved with less than 10 blocks if one uses no K_4's. Three K_4's require 18 edges; the remaining 10 require at least 4 blocks. Four K_4's still need at least 4 more blocks. In any case, we find $g^{(5)} \geq g_0 + 4$.

The discussion for 38 edges is even easier; even 6 K_4's require 8 blocks and so, again, we need at least $g_0 + 4$. For more than 38 edges, there is no problem. Hence, we obtain

THEOREM 3.4.6. *For $v = 20t + 13$, $g^{(5)} \geq 20t^2 + 25t + 12 = g_0 + 4$.*

Chapter IV

4.1 Coverings of Triples; the Numbers g(1,3;v)

4.1.1 Introduction. In this chapter, we deal with the case $t = 3$, that is, we are looking at the covering of triples. We shall now write $g(v)$ to mean $g(1,3;v)$; similarly $g^{(k)}(v)$ will designate $g^{(k)}(1,3;v)$, that is, the minimal cardinality of a covering family that includes a block of length k but no block of larger size.

For $t = 3$, the Combinatorial Bound is given by

$$C = v(v-1)(v-2)/k(k-1)(k-2).$$

The SK Bound is given by the expression

$$SK = 1 + k(k-1)^2(v-k)/2(v-2),$$

and the Woodall Bound W is given by

$$W = 1 + k(v-k)(3k-v-1)/4.$$

Also, if s denotes the greatest integer in $(v-2)/(k-1)$, the Stinson Bound is given by

$$S = 1 + k(v-k)(2s(k-1)-v+k+1)/2s(s+1).$$

The over-all behaviour of $g^{(k)}(v)$ is described in [6], where a table of values is given for v up to 26; the C bound predominates for small k, and then the SK or S bound takes over. Finally, the W bound is exact (after being increased by a small perturbation factor) in the range between v/2 and v (the "long block" case).

A survey up to 1985 of progress in the $g^{(k)}(v)$ problem is found in [8]; the values of $g^{(k)}(v)$, for most of the values in the range $v \leq 26$, appear in [53], [2], [1], and [48]. These papers also introduced the hypothesis, established in some instances, that minimal families were to be found by puncturing Steiner systems, especially inversive planes. (A punctured Steiner System is merely one from which one or more points have been deleted.) This hypothesis was established in [34], where Hartman, Mullin, and Stinson proved that g(v) is basically a step function. If we have $v = q^2+1$, where q is a prime power, then the inversive planes $S(3, q+1, q^2+1)$ actually give the value of g;

furthermore, this value of $q(q^2+1)$ is also the minimum when $v = q^2+1-\alpha$, where α is small relative to q (for an exact formulation of the permissible size of α, we refer to [34]).

4.1.2 Known Results. It will be convenient to depart briefly from historical order; a very useful result can be found in [53], namely,

THEOREM 4.1.1. *Let x be an integer, $0 \leq x \leq k$. Let B be a block (set in the cover) containing k elements (in applications, B will almost always be the longest block), and let A(j) be the set consisting of all blocks that intersect block B in j elements. Then*

$$\sum_{j} \binom{j}{x} \sum_{A(j)} \binom{k_i-j}{t-x} = \lambda \binom{k}{x} \binom{v-k}{t-x} ,$$

where the k_i are the other block lengths.

PROOF. Count the occurrences of sets of t varieties of which exactly x are in B; there are

$$\binom{k}{x} \binom{v-k}{t-x}$$

of these sets, each occuring λ times.

However, a block of length k_i in A(j) contains j varieties from B, k_i-j not in B. So we select x varieties in B in $\binom{j}{x}$ ways and select the other t-x in $\binom{k_i-j}{t-x}$ ways. We get the result by multiplying these two numbers, summing over the blocks of A(j), and finally summing over j.

By taking $\lambda = 1$ in Theorem 4.1.1, one deduces that

$$\sum_{j} \sum_{A(j)} \binom{j}{x} \binom{k_i-j}{t-x} = \binom{k}{x} \binom{v-k}{t-x} .$$

Then no t-set is repeated, and A(j) is null for $j \geq t$. The basic equality, when written for x = t-1 and x = t-2 gives

$$\sum_j \sum_{A(j)} \binom{j}{t-1}\binom{k_i-j}{1} = \binom{k}{t-1}(v-k) ,$$

$$\sum_j \sum_{A(j)} \binom{j}{t-2}\binom{k_i-j}{2} = \binom{k}{t-2}\binom{v-k}{2} .$$

In the first equation, j must equal t-1; thus we have

$$\sum_{A(t-1)} \binom{k_i-t+1}{1} = \binom{k}{t-1}(v-k) . \tag{4.1.2.1}$$

In the second equation, j = t-1 or t-2, and subtract from the second; we obtain

$$\sum_{A(t-1)} (t-1)\binom{k_i-t+1}{2} + \sum_{A(t-2)} \binom{k_i-t+2}{2} = \binom{k}{t-2}\binom{v-k}{2} . \tag{4.1.2.2}$$

Multiply the first equation by t-1, and subtract from the second; we obtain

$$(t-1)\sum_{A(t-1)} \left\{\binom{k_i-t+1}{2}\binom{k_i-t+1}{1}\right\} + \sum_{A(t-2)} \binom{k_i-t+2}{2}$$

$$= \binom{k}{t-2}\binom{v-k}{2} - \binom{k}{t-1}(t-1)\binom{v-k}{1}$$

$$= (v-k)\binom{k}{t-1}\left(\frac{t-1}{k-t+2}\;\frac{v-k-1}{2} - (t-1)\right)$$

The second sum on the left-hand side is non-negative; thus

$$\sum_{A(t-1)} \left\{\binom{k_i-t+1}{2} - \binom{k_i-t+1}{-1}\right\} \le (v-k)\binom{k}{t-1}\left(\frac{v-k-1}{2(k-t+2)} - 1\right) .$$

The left hand side is

$$\sum_{A(t-1)} \frac{(k_i-t+1)(ki-t-2)}{2} = \sum_{A(t-1)} \frac{(k_i-t)(k_i-t-1)2}{2}$$

$$= \sum_{A(t-1)} \left\{ \binom{k_i-t}{2} - 1 \right\}$$

$$= \sum_{A(t-1)} \binom{k_i-t}{2} - a(t-1) \ ,$$

where $a(t-1) = |A(t-1)|$. This produces the result

$$a(t-1) \geq \sum_{A(t-1)} \binom{k_i-t}{2} + (v-k)\binom{k}{t-1}\left(1 - \frac{v-k-1}{2(k-t+2)}\right) .$$

But the number of blocks is certainly at least $1 + a(t-1)$, and we have

THEOREM 4.1.2. *(Woodall [67]). If there is a block of length k in a (1,t;v) cover, then the total number of blocks is at least*

$$1 + (v-k)\binom{k}{t-1}\left(1 - \frac{v-k-1}{2(k-t+2)}\right) .$$

However, this identity can be improved; for brevity, set $a = a(t-1)$ and $b = (v-k)\binom{k}{t-1}$ Then (4.1.2.1) and (4.1.2.2) can be rewritten as

(4.1.2.3) $$\sum k_i = (t-1)a+b \ ,$$

(4.1.2.4) $$\frac{t-1}{2}\sum \left(k_i^2 - k_i\,(2t-1)+t(t-1)\right) + \sum_{A(t-2)} \binom{k_i-t+2}{2} = \frac{v-k-1}{2}\,b\,\frac{t-1}{k-t+2} \ ,$$

where unspecified summations refer to the set $A(t-1)$.

If we denote the second summation in (4.1.2.4) by S, we have

$$\sum k_i^2 = \frac{v-k-1}{k-t+2}\,b - \frac{2S}{t-1} + (2t-1)\sum k_i - t(t-1)a.$$

Now, if $\overline{k}$ denotes the average value of the k_i in $A(t-1)$, then

$$\Sigma (k_i - \overline{k})^2 = \Sigma k_i^2 - a\overline{k}^2 \geq 0 \ .$$

Thus we have the usual variance inequality that $a\Sigma k_i^2 - (\Sigma k_i)^2 \geq 0$. This becomes

$$ab\,\frac{v\text{-}k\text{-}1}{k\text{-}t\text{+}2} - \frac{2aS}{t\text{-}1} + a^2(t\text{-}1)(2t\text{-}1) + ab(2t\text{-}1)$$

$$- a^2t(t\text{-}1)a^2(t\text{-}1)^2 - 2ab(t\text{-}1) - b^2 \geq 0 \, .$$

The coefficient of a^2 is $(t\text{-}1)(2t\text{-}1) - t(t\text{-}1) - (t\text{-}1)^2 = 0$; hence

$$ab\left\{\frac{v\text{-}t\text{+}1}{k\text{-}t\text{+}2} + 2t\text{-}1 - 2t\text{+}2\right\} - b^2 - \frac{2aS}{t\text{-}1} \geq 0 \, ,$$

(4.1.2.5)
$$ab\,\frac{v\text{-}t\text{+}1}{k\text{-}t\text{+}2} \geq b^2 + \frac{2aS}{t\text{-}1} \ .$$

In some cases, it is useful to retain the form (4.1.2.5); however, if we omit the term involving S, we immediately have

$$a \geq \frac{b(k\text{-}t\text{+}2)}{v\text{-}t\text{+}1} = \frac{k\text{-}t\text{+}2}{v\text{-}t\text{+}1}\binom{k}{t\text{-}1}(v\text{-}k) \ .$$

Since the number of blocks is again greater than or equal to 1+a, we have obtained

THEOREM 4.1.3. *(Stanton-Kalbfleisch [53]). Under the hypothesis of Theorem 4.1.2, the total number of blocks is at least*

$$1 + \frac{k\text{-}t\text{+}2}{v\text{-}t\text{+}1}\binom{k}{t\text{-}1}(v\text{-}k) \ .$$

Theorem 4.1.3 is especially useful for smaller values of k. In particular, it allows a direct proof of the Erdös - de Bruijn Theorem [18].

THEOREM 4.1.4. *(Erdös - de Bruijn). The covering number g(1,2;v) is equal to v.*

PROOF. By Theorem 4.1.3.

$$g(1,2;v) \geq 1 + \frac{k^2}{v-1}(v-k)$$

Consider the function

$$h(k,v) = 1 + \frac{k^2}{v-1}(v-k) - v$$

Then

$$\begin{aligned}(v-1)\ h(k,v) &= v-1+k^2v-k^3-v^2+v \\ &= -\ (v^2-(2+k^2)v+k^3+1) \\ &= -\ (v-(k+1))(v-(k^2-k+1))\end{aligned}$$

If $v = k + 1$ (that is, $k = v-1$), then $h(v-1,v) = 0$. This value can be achieved by taking a block

$$1\ 2\ \ldots\ (v-1)$$

and blocks (v,i) as i ranges from 1 to $v-1$.

If $v > k+1$, then $h(k,v)$ is positive so long as $v < k^2-k+1$ (that is, we get a larger family than we do for $v = k+1$). When $v = k^2-k+1$ (which occurs when there is a finite geometry with k points per line), we again achieve a situation where we have exactly v blocks (this time, all of equal size).

Finally, if $v > k^2-k+1$, we have the blocks all of size at most k; so v blocks can cover at most $v\binom{k}{2}$ pairs. But

$$v\binom{k}{2} = \frac{v(k)(k-1)}{2} = \frac{v(k^2-k)}{2} < \frac{v(v-1)}{2}\ ,$$

and so not all pairs are covered.

Theorem 4.1.4 has shown that $g(1,2;v) = v$ and has also shown there are either one or two minimal families in this case.

4.2 Exact Coverings of Triples (k large)

4.2.1 Introduction. In [3], we introduced $g^{(k)}(\lambda,t;v)$; this was the covering number under the restriction that there was a block of size k but no block of size greater than k. Clearly,

$$g(\lambda,t;v) = \min_{t \leq k \leq v-1} g^{(k)}(\lambda\, t;v)$$

It thus appears that the behaviour of $g^{(k)}(\lambda,t;v)$ is more fundamental than that of $g(\lambda,t;v)$.

Indeed, the value of g(1,3;v) is almost an accident; it depends on whether the minimum for small k is less than the functional value for k = v - 1.

4.2.2 The behaviour of g(1,3;v) for large k. For convenience, we list the four general bounds for g(1,3;v). These are as follows (in any case that a bound is non-integral, we must take the next integer above).

The Combinatorial Bound is

(4.2.2.1) $$C = \frac{v\,(v-1)\,(v-2)}{k\,(k-1)\,(k-2)}.$$

The Stanton-Kalbfleisch Bound (cf. [53] and [49]) is

(4.2.2.2) $$SK = 1 + \frac{k-1}{v-2}\binom{k}{2}(v-k).$$

The Woodall Bound (cf. [67] and [49]) is

(4.2.2.3) $$W = 1 + (v-k)\binom{k}{2}\left(1 - \frac{v-k-1}{2(k-1)}\right).$$

It is useful to write W in the form

(4.2.2.4) $$W = 1 + \frac{(v-k)k(3k-v-1)}{4}.$$

In addition, there is a bound due to D. R. Stinson which improves (4.2.2.3). For this bound, one needs to determine $s = \lfloor (v-2)/(k-1) \rfloor$. The bound then takes the form (cf. [61])

$$S = 1 + \frac{(v-k)}{s(s+1)} \binom{k}{2} \left(2s + 1 - \frac{v-2}{k-1}\right). \tag{4.2.2.5}$$

Just as in the case $t = 2$, the bound C predominates for small k.. Then the SK bound takes over, and finally the W bound predominates. We give a table for the case $v = 16$ (this is a value of v large enough to be typical).

Table 4.2.1: Lower bounds for $g^{(k)}(1,3;16)$

k	C	SK	W	S
3	560			
4	140			
5	56	27		28
6	28	55	16	56
7		82	64	85
8		113	113	113
9			159	
10			196	
11			221	
12			229	
13			216	
14			176	
15			106	
16			1	

It is easy to deduce from (4.2.2.3) and (4.2.2.4) that $W \geq SK$ so long as

$$v/2 \leq k \leq v-1$$

(the equality occurs if and only if $v/2 = k = v-1$). In this section, we show that, with the exception of small perturbations, $g^{(k)}(1,3;v)$ is equal to the bound W in this range; a more precise statement will be given later.

4.2.3 An improvement on the bound W. We first note that there are three trivial cases in which the bound W is exact.

(a) Clearly, if $k = v$, then $W = 1$ and the bound is exact (usually we exclude $k = v$ as a possibility).

(b) If $k = v - 1$, then

$$W = 1 + \frac{(v-1)(v-2)}{2}.$$

But, if $k = v - 1$, then we need this single long block plus all triples made up of the remaining element taken with every pair from the long block. So the value is

$$g^{(k-1)}(1,3;v) = 1 + \binom{v-1}{2} = W .$$

(c) If $k = v - 2$, and if v is even, then

$$W = 1 + \frac{2(v-2)(2v-7)}{4}.$$

We need to take the single long block and make quadruples consisting of the two elements not in this long block together with a set of disjoint pairs from the long block; we also need triples consisting of an element not from the long block together with all pairs from the long block not previously used. Thus we have

$$g^{(k-1)}(1,3;v) = 1 + \frac{v-2}{2} + 2\,\frac{(v-2)}{2}\,(v-4),$$

where we employ the well known fact that the elements of the long block have $(v - 3)$ 1-factors. Simplifying, we find that, in this case,

$$g^{(k-2)}(1,3;v) = 1 + \frac{v-2}{2}\,(2v-7) = W.$$

Henceforth, we exclude cases (a), (b), and (c). We now refer to [49] and quote the result

$$\sum_j \binom{j}{x} \sum_{A(j)} \binom{k_i - j}{t - x} = \lambda \binom{k}{x}\binom{v-k}{t-x}$$

proved there in Theorem 1 (the k_i are the various block lengths). By placing $\lambda = 1$, writing $x = t - 1$ and $x = t - 2$, and combining the equations, it was shown there that

$$(t-1) \sum_{A(t-1)} \frac{(k_i - t + 1)\,(k_i - t - 2)}{2} + \sum_{A(t-2)} \binom{k_i - t + 2}{2}$$

$$+ (t-1)(v-k) \binom{k}{t-1} \left(1 - \frac{v-k-1}{2(k-t+2)}\right) = 0.$$

Here $\Sigma_{A(n)}$ denotes the summation over all blocks which meet the longest block in n elements. This equation can be written as

$$(4.2.3.1) \qquad (t-1) \sum_{A(t-1)} \frac{(k_i - t + 1)\,(k_i - t - 2)}{2}$$

$$+ \sum_{A(t-2)} \binom{k_i - t + 2}{2 +} + (t-1)\,(W-1) = 0.$$

Now put $t = 3$ to give

$$(4.2.3.2) \qquad 2 \sum_{A(2)} \frac{(k_i - 2)\,(k_i - 5)}{2} + \sum_{A(1)} \binom{k_i - 1}{2} + 2(W-1) = 0.$$

The first term can be written as

$$(4.2.3.3) \qquad 2 \left\{ \sum_{A(2)} \binom{k_i - 3}{2} - \sum_{A(2)} 1 \right\} = 2 \sum_{A(2)} \binom{k_i - 3}{2} - 2\alpha_2$$

where we write α_i to denote the number of blocks which meet the long block in i elements.

Also, since W is a bound, we can write

(4.2.3.4) $$g^{(k)}(1,3;v) = 1 + \alpha_0 + \alpha_1 + \alpha_2 = W + \varepsilon,$$

where ε denotes the excess over the bound W. When we substitute (4.2.3.3) and (4.2.3.4) into (4.2.3.2), we obtain

(4.2.3.5) $$2\sum_{A(2)} \binom{k_i-3}{2} - 2\alpha_2 + \sum_{A(1)} \binom{k_i-1}{2} + 2(\alpha_0 + \alpha_1 + \alpha_2 - \varepsilon) = 0.$$

Divide by 2 and simplify to obtain

(4.2.3.6) $$\varepsilon = \alpha_0 + \alpha_1 + \sum_{A(2)} \binom{k_i-3}{2} + \frac{1}{2}\sum_{A(1)} \binom{k_i-1}{2}.$$

We might remark that analogous formulae hold for t = 2 and t = 4. For reference, we record these as

(4.2.3.7) $$\varepsilon = \alpha_0 + \sum_{A(0)} \binom{k_i}{2} + \sum_{A(1)} \binom{k_i-2}{2}$$

and

(4.2.3.8) $$\varepsilon = \alpha_0 + \alpha_1 + \alpha_2 + \sum_{A(3)} \binom{k_i-4}{2} + \frac{1}{3}\sum_{A(2)} \binom{k_i-2}{2}.$$

Now, suppose that there are 3 or more elements not in the long block; they must occur in a block, and it will meet the long block in 0, 1, or 2 elements. If it meets the long block in 0 elements, then $\alpha_0 > 0$; if it meets the long block in 1 element, then $\alpha_1 > 0$; if it meets the long block in 2 elements, then $k_i = 5$ and $\sum_{A(2)} \binom{k_i-3}{2} > 0$. In any case, we have $\varepsilon > 0$.

If there are 2 elements not in the long block and if k = v -2 is odd, then there must be a triple which meets the long block in exactly one element; again $\alpha_1 > 0$, and so $\varepsilon > 0$.

Our conclusion can be stated as

THEOREM 4.2.1. *For $g^{(k)}(1,3;v)$, we have*

$$\varepsilon = \alpha_0 + \alpha_1 + \sum_{A(2)} \binom{k_i-3}{2} + \frac{1}{2} \sum_{A(1)} \binom{k_i-1}{2}$$

Furthermore, If $k = v$ or $k = v - 1$ or $k = v - 2$ (v even) then

$$g^{(k)}(1,3;v) = W\ .$$

In all other cases, we have $\varepsilon > 0$ and $g^{(k)}(1,3;v) \geq W + 1$.

We shall see that this result can not be sharpened, since the bound W + 1 is attained in many cases.

4.2.4 The case of a long block of even length. We first recall the well known fact that a graph K_{2a} possesses (2a - 1) disjoint 1-factors (cf., for example, [50]). Thus the pairs from K_6 can be split into K_2's as follows.

12, 34, 56	13, 25, 46
16, 23, 45	15, 24, 36
14, 26, 35	

This splitting is called a 1-factorization. It is useful to consider 1-factorizations of K_{2a-1} as well. In this case, a 1-factor consists of K_2's and a single K_1; no K_1 can be repeated. Thus, K_{2a-1} has (2a - 1) 1-factors (again, cf. [50]); for example, the splitting for K_5 is simply

12, 34, 5	13, 25, 4	23, 45, 1
15, 24, 3	14, 35, 2	

These results on 1-factors will be useful in the next constructions.

First consider the case that k is even. The remaining points form a set of v - k elements. Suppose first that v - k is also even. Form a block of length v - k which is disjoint from the long block (clearly, $v - k \le v/2$ for this to be possible). We also take $v - k > 2$, by virtue of the result of Theorem 4.2.1 when $v - k = 2$.

Form quadruples by taking the Cartesian product of all 1-factors from the $(v - k)$ points with $(v - k - 1)$ 1-factors from the k points. The number of these is

$$\frac{k}{2}\,\frac{v-k}{2}\,(v-k-1),$$

Now form triples by taking the elements from the set of $(v - k)$ points with the remaining $(k - 1) - (v - k - 1)$ 1-factors from the k points. The number of these is

$$(v - k)\frac{k}{2}\,(2k - v).$$

All triples have now been accounted for, and the number of blocks is

$$2 + \frac{k}{4}(v-k)\,(v-k-1+4k-2v) = 2 + \frac{k(v-k)\,(3k-v-1)}{4} = W + 1.$$

Since, by Theorem 4.2.1, we can not do better, we obtain

THEOREM 4.2.2. *If $v/2 \le k \le v - 2$, and if k and $v - k$ are even, then*

$$g^{(k)}(1,3;v) = W + 1 = 2 + \frac{k(v-k)(3k-v-1)}{4}. \tag{4.2.4.1}$$

COROLLARY 4.2.2.1. *The bound $W + 1$ can only be achieved in the way indicated ($v - k$ elements in a single disjoint block).*

PROOF. This is immediate from (4.2.3.6), since α_1 must be zero (otherwise $\alpha_1 + \frac{1}{2}\sum_{A(1)}\binom{k_i-1}{2} > 1$, and α_2 must be zero (otherwise, since $v - k \ge 4$, $k_i \ge 6$ and we would have $\sum_{A(2)}\binom{k_i-3}{2} > 1$). Then $\alpha_0 = 1$, and we have our result.

We now consider the case that k is even and $v - k$ is odd, and we employ a similar construction. The number of quadruples formed by taking all 1-factors from the $(v - k)$ points with $(v - k)$ 1-factors from the k points is

$$\frac{k}{2}\,\frac{v\text{-}k\text{+}1}{2}\,(v\text{-}k\,).$$

The number of triples formed by taking elements from the (v - k) points with the remaining (k - 1) - (v - k) = (2k - v - 1) 1-factors is

$$\frac{k}{2}\,(v\text{-}k)\,(2k\text{-}v\text{-}1).$$

So the total number of blocks is

$$2+\frac{k}{4}\,(v\text{-}k)\,(v\text{-}k\text{+}1\text{+}2(2k\text{-}v\text{-}1)) = 2+\frac{k(v\text{-}k)(3k\text{-}v\text{-}1)}{4}$$

$$= W\;+1.$$

This gives us

THEOREM 4.2.3. *If $v\,/\,2 \le k \le v - 2$, and if k is even and v - k is odd, then*

(4.2.4.2) $\qquad g^{(k)}(1,3;v\,) = W + 1.$

COROLLARY 4.2.3.1. *The bound W + 1 can only be achieved by placing all v - k elements not in the long block in a single disjoint block.*

PROOF. This follows as for Corollary 4.2.2.1.

4.2.5 The case of a long block of odd length. The situation when k is odd is somewhat different in that, whereas $\varepsilon = 1$ for k even, we find that $\varepsilon > 1$ for k odd. This basically stems from the result of the following lemma.

LEMMA 4.2.5.1 *If AB represents any pair of points from the v - k points not in the long block and if k is odd, then there is at least one block containing AB that intersects the long block in a single point.*

PROOF: AB must occur with each element from the long block; the intersections of blocks containing AB with the long block can contain only 1 element or 2 elements; and intersections can not all contain 2 elements, since k is odd.

Now let us illustrate what happens in a couple of cases. Suppose that v - k = 3. If the pairs AB, AC, BC, all appear in separate blocks (triples), then they contribute $\varepsilon = 3(1.5) = 4.5$. On the other hand, if there is a single block ABC meeting the long block in a point, then $\varepsilon = 1 + 1.5 = 2.5$.

As a more complicated illustration, let v - k = 10 and suppose that the blocks ABCD, AEFG, AHKL, BEH, CFK, DGL, DEK, BFL, CGH, CEL, DFH, BGH, all meet the long block in single points. Their contribution to ε is

$$12 + \frac{3}{2}(6) + \frac{9}{2}(3) = \frac{69}{2},$$

as opposed to 45 + 45/2 = 135/2 if the pairs had all been in separate blocks. However, one block ABCDEFGHKL only contributes 1 + 45/2 = 47/2 to the excess. We are thus led to

LEMMA 4.2.5.2. *The minimal contribution to the excess from the fact that every pair of points not in the long block must occur in a block meeting the long block in a single point is* $1 + \frac{1}{2}\binom{v-k}{2}$.

Proof. As in the last example, let the v - k points be pair-covered by a set of blocks of length $m_1, m_2, \ldots, m_r$. Then

$$\sum \binom{m_i}{2} = \binom{v-k}{2}.$$

Each block of length m_i, extends to a block of length $m_i + 1$ by meeting the long block in a single point; so the total contribution to the excess is

$$r + \frac{1}{2}\sum \binom{m_i}{2} = r + \frac{1}{2}\binom{v-k}{2}.$$

On the other hand, if all v - k points are put in a block of length (v - k + 1), then the contribution to the excess is only

$$1 + \frac{1}{2}\binom{v-k}{2}.$$

Clearly, this is the best we can do. Also, we need $v - k + 1 \le k$, that is, $k \ge (v + 1)/2$. Lemma 4.2.5.2 immediately gives us

THEOREM 4.2.4. *If $(v + 1)/2 \le k \le v - 2$, and if k is odd, then*

$$g^{(k)}(1,3;v) \le W + 1 + \frac{1}{2}\binom{v-k}{2}.$$

COROLLARY 4.2.4.1. *Under these conditions,*

$$g^{(k)}(1,3;v) \le 2 + \frac{(v-k)\,(k-1)\,(3k-v+1)}{4}.$$

We shall now show that this bound is actually attained for k odd in the range $(v + 1) / 2 \le k \le v - 2$.

First, let k be odd and let $v - k$ be even. In addition to the two blocks of lengths k and $v - k + 1$, we require the following.

(a) $(k - 1)(v - k)$ triples containing the point A which lies on both blocks and also containing one point from each block.

(b) $\frac{1}{2}(v - k)\frac{1}{2}(k - 1)(v - k - 1)$ quadruples formed by taking the Cartesian product of all one-factors from the $(v - k)$ points (less A in the long block).

(c) $(v - k)\frac{1}{2}(k - 1)(2k - v - 1)$ triples formed by combining the $v - k$ points not in the long block with the remaining 1-factors of the $(k - 1)$ points (less A) in the long block.

The total number of these blocks, which cover all triples on the v points, is

$$2 + \frac{(v-k)\,(k-1)}{4}\,\{4 + (v-k-1) + (4k-2v-2)\}$$

$$= 2 + \frac{(v-k)(k-1)(3k-v+1)}{4}.$$

Since this is the bound in Corollary 4.2.4.1, we can do no better and thus have shown that the bound is attained.

The construction for k odd and v - k odd is similar, although the counts differ. We have two blocks intersecting in (A), together with the following blocks:

(a) (k - 1) (v - k) triples as before

(b) $\frac{1}{2}$(v - k + 1)$\frac{1}{2}$(k - 1)(v - k) blocks (some are quadruples and some are triples) formed by taking the Cartesian product of 1-factors.

(c) (v - k) $\binom{k-1}{2}$ (2k - v - 2) triples formed by taking single elements with 1-factors from the k - 1 points different from the A on the long block.

The total number of blocks then is given by

$$2 + \frac{(k-1)\,(v-k)}{4}\,\{4 + (v-k+1) + 2(2k-v-2)\}$$

$$= 2 + \frac{(v-k)\,(k-1)\,(3k-v+1)}{4}\ ,$$

as before. These two calculations establish

THEOREM 4.2.5. *If $(v + 1) / 2 \leq k \leq v - 2$, and if k is odd, then*

(4.2.5.1) $$g^{(k)}(1,3;v) = 2 + \frac{(v-k)(k-1)(3k-v+1)}{4}\ .$$

COROLLARY 4.2.5.1. *For the minimal configuration giving*

$$g^{(k)}(1,3;\ v) = 2 + \frac{(v-k\ )(k-1)(3k-v+1)}{4},$$

we must have two blocks of lengths k and v - k + 1 intersecting in a single point; the other blocks are triples or quadruples .

PROOF. Any other configuration would give (by Lemma 4.2.5.2) a contribution to the excess that would push the value over the stated lower bound.

4.3 An Upper Bound for g(1,3;v)

4.3.1 Introduction. We now derive an upper bound for g(1,3;v). The proof depends upon the following result.

LEMMA 4.3.1. *Let $v > u + 1$. Then*

$$g(1,u;v-1) \leq g(1,u;v)$$

PROOF. Begin with a minimal (1,u;v) design. Since $u<v-1$ and $\lambda=1$, no set of v-2 varieties occurs more than once. Hence the (1,u;v) design contains at most one block of length v-1. Now delete one variety from every block containing it. If the (1,u;v) design contains a block of length v-1, the variety deleted must belong to that block. There remain v-1 varieties arranged in g(1,u;v) blocks of length at most v-2, with every set of varieties occuring in one block. The theorem follows.

NOTE. We shall show that $g(1,3;7) = g(1,3;8)$, so that it is possible to have equality in Lemma 4.3.1.

THEOREM 4.3.1. *Let p be the smallest prime or prime power exceeding $\sqrt{v}$. Then*

$$g(1,3;v) \leq p(p^2+1)$$

PROOF. By hypothesis, $v \leq p^2+1$, and hence, by Lemma 4.3.1,

$$g(1,3;v) \leq g(1,3;p^2+1) .$$

But Witt [66] has proved the existence of a Steiner system $S(3,p+1,p^2+1)$. This is an arrangement of p^2+1 varieties in $p(p^2+1)$ blocks of length p+1, with each triple occuring exactly once. Therefore,

$$g(1,3;p^2+1) \leq p(p^2+1)$$

and the theorem is established.

Theorem 3 of Woodall [67] gives

$$g(1,3;v) \geq \frac{v(v-1)}{2(v-4)} \left\{ \sqrt{4v-7} - 3 \right\}$$

If $v = p^2+1$ where p is a prime or prime power, the ratio of the bounds in Theorem 4.3.1 and Woodall tends to one. (Their difference tends to infinity.)

Thus, if p is a prime or prime power,

$$\lim_{p\to\infty} \frac{g(1,3;p^2+1)}{p(p^2+1)} = 1 .$$

It would thus appear that the bound given in Theorem 4.3.1 is of the right order of magnitude.

It is of some interest to determine the values of v for which Theorem 4.3.1 gives a better bound than $1 + \binom{v-1}{2}$ (long block + pairs selected from the long block) (call this the LBP bound). By Bertrand's Theorem, if x is any real number greater than 1, then there exists a prime between x and 2x. Hence there exists a prime between $\sqrt{v}$ and $2\sqrt{v}$. The bound given by Theorem 4.3.1 is

$$p(p^2+1) < 2\sqrt{v}(4v+1)$$

This is easily shown to be less than $1+(v-1)(v-2)/2$ when $v>289$. The LBP Bound and the bound from Theorem 4.3.1 may be calculated and compared for each $v \le 289$, with the following result.

LEMMA 4.3.2. The only values of v for which the LBP Bound gives as good a bound as Theorem 4.3.1 are v = 4, 5, ..., 9, 11, 12, 13, and 27.

In the next section, we show that $g(1,3;v) = 1 + (v-1)(v-2)/2$ for v = 4, 5, 6, and 9, and that $g(1,3;v) < 1+(v-1)(v-2)/2$ for v = 7, 8, 12, and 13. Although taking a block of v-1 varieties together with the v-1 pairs involving the vth variety is minimal for all v, the similar (block of v-1 varieties, together with $\binom{v-1}{2}$ triples containing the vth element) construction is best for only a few small values of v. We summarize these results as follows:

THEOREM 4.3.2. *The (1,3;v) design described above is minimal for v = 4, 5, 6, and 9. It is not minimal for any other values of v.*

4.3.2 Results for Small v. We conclude this section with a detailed discussion of minimal (1,3;v) designs for small v.

v = 4. Only triples are allowed, and each triple must occur. Hence g(1,3;4) = 4. The unique minimal (1,3;4) design consists of all four triples.

v = 5. Blocks of lengths 3 and 4 are allowed. Since no triple is repeated, there is at most one block of four, and then there must be $\binom{5}{3} - \binom{4}{3} = 6$ triples. Hence g(1,3;5) = 7, and the unique minimal (1,3;5) design consists of a block of four and six triples.

v = 6. If a block of length 5 is used, there will be $\binom{6}{3} - \binom{5}{3} = 10$ triples, for a total of 11 blocks. If only triples are used, 20 blocks are required. The maximum number of blocks of four with no repeated triples is 3, and there must be $\binom{6}{3} - 3\binom{4}{3} = 8$ triples, for a total of 11 blocks. Thus g(1,3;6) = 11. There are two minimal (1,3;6) designs, one consisting of a block of five and ten triples, the other consisting of three blocks of four and eight triples.

v = 7. If a block of length six is used, there will be $\binom{7}{3} - \binom{6}{3} = 15$ triples, for a total of 16 blocks. If a block of length five is used, there must be at least 19 blocks by Woodall. It was shown in [52] that the maximum number of quadruples with no repeated triple is 7. Such a set of 7 quadruples is essentially unique, and may be obtained by deleting all blocks containing one variety in the unique Steiner quadruple system S(3,4,8). We must now add $\binom{7}{3} - 7\binom{4}{3} = 7$ triples, for a total of 14 blocks. Therefore, g(1,3;7) = 14, and the unique minimal (1,3;7) design consists of seven quadruples and seven triples.

v = 8. If blocks of lengths seven, six, or five are used, there must be at least 22 blocks by using Woodall's bound. There exists a unique Steiner quadruple system S(3,4;8) with 14 blocks containing each triple once. Hence g(1,3;8) = 14, and the unique minimal design is the Steiner system S(3,4,8).

v = 9. A block of length 8 and $\binom{8}{2} = 28$ triples give a (1,3;9) design with 29 blocks. We shall show that this design is minimal and unique. By Theorem 4 from [53], any other design with 29 or fewer blocks can have no block length greater than 5. The maximum number of quadruples with no repeated triple is 18 from [52]. If no blocks of length 5 are used, we require at least $\binom{9}{3} - 18\binom{4}{3} = 12$ triples, for a total of 30 blocks. Hence a minimal design not containing a block of 8 contains at least one block of 5. Since no triple is repeated, there are at most three blocks of 5.

Now let r, s, and t be the numbers of blocks of length 5, 4, and 3. respectively. Then the total number of blocks is

$$r + s + t = g$$

and the number of triples which they contain is

$$10r + 4s + t = \binom{9}{3} = 84.$$

Subtracting the first of these equations from the second gives

$$9r + 3s = 84 - g$$

and hence g is divisible by 3. But for k = 5, Woodall gives $g(1,3;9) \geq 26$. Hence a minimal design which contains a block of length 5 must have 27 blocks. This possibility was ruled out in [53].

v = 10. A Steiner quadruple system S(3,4,10) provides a (1,3;10) design with 30 blocks, so that $g(1,3;10) \leq 30$. Now Woodall implies that every block in a minimal design has length at most 5. If we denote the numbers of blocks 5, 4, and 3, by r, s, and t, respectively, we obtain as before

$$r + s + t = g$$

$$10r + 4s + t = \binom{10}{3} = 120$$

so that the number of blocks is divisible by 3. But Lemma 4.3.1 gives

$$g(1,3;10) \geq g(1,3;9) = 29$$

Hence any design with a block of length 5 contains at least 30 blocks, and $g(1,3;10) = 30$.

Furthermore, suppose that there exists a (1,3;v) design with 30 blocks which contain a block of length 5. Then it must also contain a triple (1,2,3), say. Delete this triple, and delete variety 1 from all blocks containing it. This gives a (1,3;9) design with 29 blocks consisting of blocks of lengths 5, 4, 3. This contradicts the above results for v=9. It follows that g(1,3;10) = 30, and that any minimal (1,3;10) design is a Steiner system S(3,4;10).

v = 11. The value of g(1,3;11) is the first difficult result for small v; it was determined in Allston, Stanton, and Wirmani-Prasad [9], and we give an account of the construction in Section 4.7.

v = 12. From the LBP Bound, we obtain g(1,3;12) ≤ 56. We improve this bound to give g(1,3;12) ≤ 47 by beginning with two disjoint blocks of length 6, say B = (1, 2, 3, 4, 5, 6) and $\overline{B}$ = (7, 8, 9, 10, 11, 12). Now we form 45 quadruples, each containing two varieties from B and two from $\overline{B}$; the result is a set of 47 blocks containing each triple once. Stanton and Dirksen [48] showed that this design is minimal.

v = 22. Witt proved the existence of a Steiner system S(3,6,22). It contains 77 blocks of six, with each triple occuring once. Hence g(1,3;22) ≤ 77. Also, by Theorem 4 from [53], any (1,3;22) design containing a block of length 7 or more must have more than 77 blocks. It follows that g(1,3;22) = 77, and that the only minimal (1,3;22) design is the Steiner System S(3,6;22).

4.4 The Cases v = 20 and v = 21

4.4.1 Introduction. In Section 4.3, we showed that $g(1,3;q^2+1) \le q(q^2+1)$ when q is a prime power; this result merely made use of the Steiner Systems (inversive geometries) $S(3,q+1,q^2+1)$. These inversive geometries are really the "smallest" Steiner Systems S(3,k,v) in a certain sense.

For example, suppose there exists an S(3,k,v); then it contains S(2,k-1,v-1), and this last system is a BIBD. Suppose, if possible, that $v < (k-1)^2+1$, (that is, that v is smaller than for an inversive geometry). Then

$$v = k^2-2k+2-\delta, \text{ where } \delta > 0.$$

The contained design

$$S(2,k-1,k^2-2k+1-\delta)$$

contains b blocks and has replication number r, where

$$b = \frac{(k^2-2k+1-\delta)(k^2-2k-\delta)}{(k-1)(k-2)}, \quad r = \frac{k^2-2k-\delta}{k-2} = k - \frac{\delta}{k-2}.$$

Since r is integral, we have $\delta = m(k-2)$, where m is an integer. But $r \geq k-1$ (Fisher's Inequality), and so

$$k-m \geq k-1.$$

It follows that $m = 1$, and the Steiner System $S(2,k-1,k^2-2k+1-\delta)$ is a symmetric BIBD with $\delta = k-2$. Thus we are are dealing with the symmetric BIBD given by $S(2,k-1,k^2-3k+3)$, and it is contained in the Steiner System

$$S(3,k,k^2-3k+4).$$

Since the number of blocks in this system is

$$\frac{(k^2-3k+4)(k^2-3k+3)(k^2-3k+2)}{k(k-1)(k-2)} = \frac{(k^2-3k+4)(k^2-3k+3)}{k},$$

it follows at once that k divides 12. Since $k > 3$, we see that $k = 4$, 6, or 12; thus the only Steiner Systems $S(3,k,v)$ for which v is smaller than k^2+1, its value for an inversive plane, are $S(3,4,8)$, $S(3,6,22)$, and $S(3,12,112)$; of course, the existence of the last system is hypothetical.

4.4.2 The Problem for v=21. Because of the almost unique character of $S(3,6,22)$, as pointed out in the introduction, it is of interest to determine $g(1,3;21)$. Clearly, since g is a non-decreasing function, $g(1,3;21) \leq 77$ (the value for $v = 22$).

First, we apply the Stanton-Kalbfleisch inequality for $k = 7$; the result produces at least 94 blocks, and so the minimal design for $v = 21$ can contain no block of length greater than 6. Furthermore, even if all blocks had length 6, we should need at least

$$\frac{\binom{21}{3}}{\binom{6}{3}} = 66.5$$

blocks. So we may assume $g(1,3;21) = 67 + m$ ($m \geq 0$). Clearly $m \leq 10$.

Suppose now that the minimal design includes a triples, b quadruples, c quintuples, d sextuples. Then we have

$$a + b + c + d = 67 + m$$

$$a + 4b + 10c + 20d = 1330.$$

Thus

$$19a + 16b + 10c = 10 + 20m$$

$$10a + 10b + 10c \leq 10 + 20m$$

$$a + b + c \leq 1 + 2m.$$

It follows that

$$d = 67 + m - a - b - c,$$

$$d \geq 66 - m.$$

Now d consists of sextuples on 21 elements; thus d can not exceed the packing number $D(3,6,21)$, where $D(t,k,v)$ is the maximum number of k-sets from a v-set with the property that no repeated t-set occurs.

Clearly, $D(1,4,19) = 4$. By standard packing theory, $D(2,5,20) \leq 20 / 5 \times 4 = 16$, and this bound can be achieved by using a Euclidean geometry on 16 points. Also, $D(3,6,21) \leq 21 / 6 \times 16 = 56$, and this bound can be achieved by taking the 56 blocks of an $S(3,6,22)$ that do not contain a specific symbol. Thus we have $d \leq 56$.

We now deduce that

$$66 - m \leq 56,$$

$$m \geq 10.$$

But m was at most 10; hence m = 10, and we have established that $g(1,3;21) = 77$.

4.4.3 The Design for v=21. The determination of m in the last subsection automatically gives a = b = 0, c = 21, d = 56. We thus have a minimal design of 21 quintuplets and 56 sextuplets. Suppose that we now denote the frequency of a pair ij by f(i,j); clearly, $f(i,j) \leq 19 / 3$, that is $f(i,j) \leq 6$.

Now let g_1 be the number of occurrences of ij in 5-blocks, and let g_2 be the number of occurrences of ij in 6-blocks. It follows that

$$g_1 + g_2 = f(i,j),$$

$$3g_1 + 4g_2 = 19.$$

Now $f(i,j) \geq 5$, and we find that the only solutions are:

Case 1: $g_1 = 1, \; g_2 = 4$ Case 2: $g_1 = 5, \; g_2 = 1.$

Let us now suppose there are U pairs (i,j) of the first type and V pairs (i,j) of the second type. Then

$$U + V = \binom{21}{2} = 210,$$

$$U + 5V = \text{number of pairs in the 5-blocks}$$

$$= 21 \binom{5}{2} = 210.$$

It follows that U = 210, that is, all pairs occur exactly once in the twenty-one 5-blocks. This establishes

THEOREM 4.4.1. *The twenty-one 5-blocks in the minimal design for g(1,3;21) form a unique configuration, namely, the 21-point projective geometry, PG(2,4).*

It is then trivial to adjoin a symbol ∞ to these 21 blocks and obtain a design on 22 varieties with 77 blocks of length 6, that is, an S(3,6,22). Our conclusion is

THEOREM 4.4.2. *Not only is g(1,3;21) = 77, but the 77 blocks of the minimal configuration are found by taking a Steiner System S(3,6,22) and deleting one symbol.*

4.4.4 The Case v = 20; Preliminaries. The result for g(1,3;20) is considerably more difficult and we shall now discuss it. We shall need various packing numbers D(t,k,v). It is trivial that D(1,4,18) = 4. Also, using the usual packing inequality,

$$D(2,5,19) \leq \frac{19}{5} D(1,4,18);\ \text{hence } D(2,5,19) \leq 15.$$

If any symbol has frequency 4, we can write blocks

1 2 3 4 5, 1 6 7 8 9, 1 10 11 12 13, 1 14 15 16 17.

Then 18 and 19, but no other symbols, can have frequency 4; hence there are at most 12 blocks in this case.

If no symbol has frequency 4, we have the packing number $\leq 3 \times 19 / 5$; this produces a bound of 11.

Since 12 blocks can be obtained from the 21-point geometry containing 2 specific symbols, we have D(2,5,19) = 12.

Finally, $D(3,6,20) \leq 20(12) / 6 = 40$.

But the bound of 40 can be attained by taking the Steiner System S(3,6,22) and deleting the blocks containing 2 specific symbols to leave 77 - 2(21) + 5 = 40 blocks. Hence, D(3,6,20) = 40.

Finally, we apply the Stanton-Kalbfleisch Inequality with v = 20, k = 7, to see that a (1,3;20) family must contain

$$1 + \binom{7}{2} 13 \frac{6}{18} = 92 \text{ blocks}$$

if there is a block of size 7 in the family. Hence g(1,3;20), must be attained for a family with largest block size not exceeding 6.

4.4.5 Blocks in a (1,3;20) Family. The Combinatorial Bound of $\frac{\binom{20}{3}}{\binom{6}{3}} = 57$ shows that we may set g(1,3;20) = 57 + e (e ≥ 0). Now let a, b, c, d, be the number of blocks of sizes 3, 4, 5, 6. We have

$$a + b + c + d = 57 + e$$

$$a + 4b + 10c + 20d = 1140,$$

Thus

$$19a + 16b + 10c = 20e$$

$$a + b + c \leq 2e,$$

$$d \geq 57 + e - 2e = 57 - e.$$

But d is as most 40, and so e is at least 17.

We now write d = 40 - f, e = 17 + h.. Our equations then become

$$a + b + c = 34 + f + h$$

$$a + 4b + 10c = 340 + 20f.$$

Then 9a + 6b = 10h - 10f, or 10h = 10f + 9a + 6b.

Now g(1,3;20) ≤ g(1,3;22) = 77; hence h ≤ 3. We tabulate all solutions.

1. h = 0, c = 34, d = 40
2. h = 1, c = 36, d = 39.

3. $h = 2, c = 38, d = 38.$
4. $h = 3, c = 40, d = 37.$
5. $h = 3, b = 5, c = 32, d = 40.$
6. $h = 3, a = b = 2, c = 33, d = 40.$

Now suppose that there is a symbol that appears only in the 5-blocks (r times) and in the 6-blocks (s times). Then it occurs with all 171 pairs on the other 19 symbols; hence

$$6r + 10s = 171, \text{ a contradiction.}$$

So no symbol can appear only in 5-blocks and 6-blocks. This rules out Cases 1-4, and establishes

THEOREM 4.4.3. *The number g(1,3;20) is equal to 77.*

Actually, we can go further; in Case 6, there are at most $3 + 3 + 4 + 4 = 14$ symbols in the 4 short blocks. This means that there exists a symbol appearing only in 5-blocks and 6-blocks; since we have seen that that is impossible, we have

THEOREM 4.4.4. *The minimal (1,3;20) family comprises five 4-blocks, thirty-two 5-blocks, forty 6-blocks.*

4.4.6 Structure of the Minimal Family. There are only 20 places in the five 4-blocks. Hence, all 20 symbols appear there (otherwise, some symbols would appear only in 5-blocks and 6-blocks).

Counting pairs as before, we have

$$3 + 6r + 10s = 171,$$

$$3r + 5s = 84.$$

Take the 4-blocks as

1 2 3 4, 5 6 7 8, 9 10 11 12, 13 14 15 16, 17 18 19 20.

Let us consider a pair like (1,2); it appears with 16 other symbols. Say it appears in x of the 5-blocks, y of the 6-blocks. Then $3x + 4y = 16$, and $x = 0$, $y = 4$; or $x = 4$,

$y = 1$. We suppose that P pairs have a 0-4 distribution and Q pairs have a 4-1 distribution.

Furthermore, if we look at a pair like (1,5), we may suppose it appears in x of the 5-blocks and y of the 6-blocks. Then $3x + 4y = 18$, and $x = 6$, $y = 0$ (S pairs) or $x = 2$, $y = 3$ (T pairs).

Now $P + Q = 30$, $S + T = 190$. Also, the number of pairs in the 5-blocks $= 320 = 4Q + 6S + 2T$; the number of pairs in the 6-blocks $= 600 = 4P + Q + 3T$.

Solving, we get $Q + S = 0$, $P = 30$, $T = 160$. This gives us

THEOREM 4.4.5. *All pairs occuring in the 4-blocks occur 0 times in the 5-blocks, 4 times in the 6-blocks. The pairs not occuring in the 4-blocks occur twice in the 5-blocks, thrice in the 6-blocks.*

Now we can find the r and s values; suppose we take a symbol such as 1. It occurs in pairs 12, 13, 14, appearing only in 6-blocks; these pairs have a total frequency of $3(4) = 12$, and they must all appear in distinct blocks. Thus we have $s = 12$, and consequently $r = 8$. This gives the expected result that each element appears with the same frequency.

By employing Theorem 4.4.5, we may take the first 5-block as 1 5 9 13 17; call this block B, and let B meet b_i of the other 5-blocks in i elements ($i = 0, 1, 2$). Then

$$b_0 + b_1 + b_2 = 31, \quad b_1 + 2b_2 = 35, \quad b_2 = 10.$$

Thus, we see that B (and, indeed, any 5-block) meets fifteen 5-blocks in 1 element, ten 5-blocks in 2 elements, and is disjoint from six 5-blocks. Now write the blocks in 2 sets, B_1 and B_2 (dots denote elements other than 1, 5, 9, 13, 17).

1 ...,	1 ...,	1 ...
5 ...,	5 ...,	5 ...
9 ...,	9 ...,	9 ...
13 ...,	13 ...,	13 ...
17 ...,	17 ...,	17 ...

1 5 9 13 17

1 5 ...,	1 9 ...,	1 13 ...,	1 17 ...
5 9 ...,	5 13 ...,	5 17 ...,	9 13 ...
9 17 ...,	13 17		

Six blocks disjoint from 1 5 9 13 17

The pairs that do not occur in the 4-blocks must occur in B_1 and B_2. We show that they must appear exactly once in B_1 and once in B_2.

First, we note that the set of 6-blocks contains 30 blocks of the form 1 5 ..., 1 5 ..., 1 5 ..., 1 9 ..., 1 9 ..., 1 9 ..., etc. (10 sets of pairs from 1, 5, 9, 13, 17); call these blocks B_3. There is also a set B_4 of 10 blocks disjoint from the set {1, 5, 9, 13, 17}.

Now we ask whether there can be a repeated pair in B_1. If it is a pair like 1a, then 1a can not appear in B_2. But all triples 1a5, 1a9, 1a13, 1a17, must occur. So they appear in B_3. This is a contradiction, since the frequency of 1a in B_3 is only 3.

If the pair ab is repeated in B_1, suppose it appears as 1ab and as 5ab. Then ab must occur 3 times in B_3. The only blocks in which it may occur have the form 9 13 ..., 9 17 ..., and 13 17 ..., and the occurrence of ab leads to a repeated triple.

We have thus established

THEOREM 4.4.6. *The set B_1 (and hence the set B_2 as well) contains a single occurrence of all pairs not appearing in the 4-blocks.*

4.4.7 Identification of the Design. We now can identify the design by adjunction. Let Y and Z be two new symbols; adjoin YZ to the 4-blocks, thus giving all triples containing YZ. Then adjoin Y to all blocks in B_1, Z to all blocks in B_2. The triples Y ... either occur in the augmented 4-blocks or in the augmented 5-blocks, in virtue of Theorems 4.4.5 and 4.4.6. Similarly, all triples containing Z occur. And the triples not containing YZ were given by the original design.

We have thus shown that the augmented design is a Steiner System $S(3,6,22)$ and have established

THEOREM 4.4.7. *The minimal set covering all triples on 20 elements is found by deleting two elements from a Steiner System $S(3,6,22)$ on 22 elements.*

Thus $g(1,3;v)$ is constant for $v = 20, 21, 22$; as v increases, this step-function behaviour becomes typical of the behaviour of g.

4.5 Results on Inversive Planes

In this section, we include the results by Mullin et al [34] showing that $g(1,3,q^2+1) = q^3 + q$ for all prime powers q, and that the minimal configurations are inversive planes of order q. They also show that for $q \geq 4$, $\alpha \geq 0$, the value of $g(1,3,q^2 - \alpha)$ is $q^3 + q$, when α is small relative to q.

THEOREM 4.5.1. *(Mullin, Hartman, and Stinson) [34]. For any prime power $q \geq 3$,*

$$g(1,3,q^2+1) = q^3+q$$

and the minimal configuration is an inversive plane.

THEOREM 4.5.2. *(Mullin, Hartman, and Stinson) [34]. Let q be a prime power such that $q \geq 4$ and let α be any positive integer satisfying the inequalities*

$$q \geq 3(\alpha^2\text{-}a)/2, \quad \alpha < q\text{-}5\text{+}12/(q+3)$$

Then $g(1,3,q^2+1-\alpha) = q^3+q$.

These results establish that $g(1,3;17) = 68$ and that $g(1,3;25) = g(1,3;26) = 130$ (one simply uses the inversive plane $S(3,5,17)$ and the inversive plane $S(3,6,26)$.

4.6 Table of $g^{(k)}(v)$ for small values of v

In this section, we make use of the results obtained to tabulate $g^{(k)}(1,3;v)$ for v up to 12. In forming Table 4.6.1, we have used the obvious fact that, for $k = 4$, we take $D(3,4,v)$ quadruples plus as many triples as are needed. Since the packing number $D(3,4,v)$ is known for all v (cf. [52], [14], [41]); in our table, the second row is merely

$$D(3,4,v) + \left\{ \binom{v}{3} - 4D(3,4,v) \right\}$$

Table 4.6.1. $g^{(k)}(1,3;v)$ for $3 \leq v \leq 12$.

k \ v	3	4	5	6	7	8	9	10	11	12
3	1	4	10	20	35	56	84	120	165	220
4		1	7	11	14	14	30	30	60	67
5			1	11	20	26	30	42*	45	*
6				1	16	28	38	44	47	47
7					1	22	41	56	68	77
8						1	29	53	74	90
9							1	37	87	98
10								1	46	86
11									1	56
12										1

This fact, with the results of the earlier sections, gives all the entries except two marked with asterisks.

LEMMA 4.6.1. *$g^{(5)}(1,3;10) = 42$.*

PROOF. Clearly 42 is a lower bound since we can take two disjoint blocks of length 5. Each has five disjoint near 1-factors, and the Cartesian product of the 1-factors contains 3(3) - 1 = 8 blocks (drop the block of length 2). Thus

$$g^{(5)}(1,3;10) \leq 2 + 5(8) = 42.$$

Now let the long block be 12345 and the other points be A, B, C, D, E. We must cover A, B, C, D, E by 10 blocks, 7 blocks, or 1 block (see the third column of the table). We have already dealt with one block ABCDE (it must be disjoint).

If the cover is 10 triples of the form ABC, they must meet the long block in 0, 1, or 2 elements. An intersection of 0 or 2 contributes 1 to the value of ε of equation 4.2.3.6, whereas an intersection of 1 contributes 2.5. However, Lemma 4.2.5.1 and the fact that a pair-covering of 5 elements contains at least 4 triples (such as ABC, ADE, BDE, CDE) guarantees that ε is at least 6(1) + 4(2.5) = 16. Hence, since W = 26, we can not obtain a value less that 42 in this way.

If the cover is ABCD, EAB, EAC, EAD, EBC, EBD, ECD, and if the pair-covering is made up of the six triples, then these contribute a minimum of 6(2.5) and ABCD contributes a minimum of 1. Again, we can not get a value less than 42.

Finally, let the cover be ABCD, EAB, EAC, EAD, EBC, EBD, ECD, and suppose the pair covering is ABCD, EAB, ECD. Then these blocks contribute to E an amount at least 4 + 2(2.5) + 4(1) = 13. However, Lemma 4.2.5.1 guarantees that AB and CD meet the long block in an odd number of unit intersections; hence there are two triples ABX and CDX at least, and they contribute another 2(1.5) = 3 units to ε. Hence, again, in this case, we can do no better that 42. This completes the demonstration of the lemma.

4.7 Discussion of g(1,3;11)

4.7.1 Introduction. The number $g^{(k)}(1,3;11)$, which we abbreviate in this section to $g^{(k)}$, is the minimal number of blocks needed to cover all 165 triples from an 11-set exactly, using a block of length k but no longer block; it has been given in [6] for all $k \neq 5$. Since $g = g(1,3;11)$ is the minimum value of $g^{(k)}$, and since $g^{(10)} = 46$, the determination of g requires finding whether or not $g^{(5)} < 46$.

If g_i (i = 3,4,5) represents the number of blocks of length i in a minimal $g^{(5)}$ cover, then

$$g_3 + g_4 + g_5 = g,$$

$$g_3 + 4g_4 + 10g_5 = 165.$$

As in [53], we deduce that $3(g_4 + 3g_5) = 165 - g$; consequently, g is a multiple of 3.

Our main tool will be two results from Section 4.4. We state these in a slightly different form for application to the present case.

THEOREM 4.7.1. *The number g can be written as*

$$g = W + \varepsilon,$$

where W is the Woodall bound

$$W = 1 + (v\text{-}k)\binom{k}{2}\left(1 - \frac{v\text{-}k\text{-}1}{2(k\text{-}1)}\right)$$

and $\varepsilon \geq 0$. *The value of* ε *is*

$$\varepsilon = \alpha_0 + \alpha_1 + + \Sigma_2 \binom{k_i - 3}{2} + \frac{1}{2}\Sigma_1 \binom{k_1 - 1}{2}.$$

Here we are using a base block 123...k, the k_i are the lengths of the other $g - 1$ blocks, α_i is the number of blocks meeting the base block in i points, Σ_i denotes summation over the α_i blocks meeting the base block in i points ($i = 0,1,2$).

We also make use of the following lemma from Section 4.4.

LEMMA 4.7.1. *If k is odd, then there exists at least one block containing the pair AB that meets the base block in a single point, for every choice of AB among the v - k elements not in the base block (more generally, the number of such blocks containing AB is odd).*

It will be convenient to denote the base block as 12345 and the other six points as A, B, C, D, E, F. We note that, for $k = 5$, $W = 23.5$.

4.7.2 Coverings of the points not in the base block. Suppose we define g_{ij} to be the number of blocks (other than the base block) that have length i and meet the base block in j points. If we let A denote any point not on the base block and 1 denote any point on the base block, then we can make the following table in which we record numbers of blocks and their types.

$$\begin{cases} g_{30}:\mathrm{AAAA} & g_{40}:\mathrm{AAAA} & g_{50}:\mathrm{AAAAA} \\ g_{31}:\mathrm{AA1} & g_{41}:\mathrm{AAA1} & g_{51}:\mathrm{AAAA1} \\ g_{32}:\mathrm{A11} & g_{42}:\mathrm{AA11} & g_{52}:\mathrm{AAA11} \end{cases} \qquad (4.7.2.1)$$

The row sums are just α_0, α_1, α_2, and the column sums are just g_3, g_4, and $g_5 - 1$. We can immediately write down the following relations:

$$\begin{cases} g_{30} + g_{41} + g_{52} + 4g_{40} + 4g_{51} + 10g_{50} = 20 \\ g_{31} + 3g_{41} + 6g_{51} + 2g_{42} + 6g_{52} = 75 \\ g_{32} + 2g_{42} + 3g_{52} = 60 \end{cases} \tag{4.7.2.1}$$

One may combine these equations to give the result

$$g = 43.5 + 1.5(g_{31} + g_{41}) - 3(g_{40} + 3g_{50}) \tag{4.7.2.2}$$

but it is more illuminating to use Theorem 4.7.1, which gives us

$$\varepsilon = g_{30} + g_{40} + g_{50} + 1.5g_{31} + 2.5g_{41} + 4g_{51} + g_{52} \tag{4.7.2.3}$$

We now split the discussion into 3 cases, according as the triples from A, B, C, D, E, F, are covered by using a quintuple, are covered using only triples, or are covered using quadruples and triples.

Case 1.

$$g_{50} = 1, \ \ g_{40} = g_{51} = 0.$$

$$\text{Then} \quad g_{30} + g_{41} + g_{52} = 10,$$

and we compute

$$\begin{aligned} \varepsilon &= g_{30} + 1 + 1.5g_{31} + 2.5g_{41} + g_{52} \\ &= 1 + (g_{30} + g_{41} + g_{52}) + 1.5(g_{31} + g_{41}) \\ &= 11 + 1.5(g_{31} + g_{41}) \end{aligned}$$

If the block of length 5 is ABCDE, then the ten "diagonal" blocks appearing in the $g_{30} + g_{41} + g_{52}$ blocks are FAB, FAC, FAD, FAE, FBC, FBD, FBE, FCD, FCE, FDE. Now we know that every one of the 15 pairs from A, ..., F, must occur at least once in the $g_{31} + g_{41}$ blocks. There are 10 pairs not involving F, and so, if x of them appear in the g_{41} blocks, then 10 - x must appear in the g_{31} blocks. Hence, $g_{31} + g_{41} \geq 10$, $\varepsilon \geq 26$, $g \geq 49.5$.

We have thus ruled out this case as a possibility for a minimum configuration.

Case 2. $g_{50} = g_{40} = g_{51} = 0$, that is, all triples from A,...,F, appear in the $g_{30} + g_{41} + g_{52} = 20$ diagonal blocks.

We have

$$\varepsilon = g_{30} + 1.5g_{31} + 2.5g_{41} + g_{52}$$

$$= 20 + 1.5(g_{31} + g_{41})$$

The 15 pairs from A, ..., F must be covered in the $g_{31} + g_{41}$ blocks. The most efficient covering is ABC, ADE, FBD, FCE, AF, BE, CD, and hence $g_{31} + g_{41} \geq 7$, $\varepsilon \geq 30.5$, $g \geq 54$. So we may reject this case.

Case 3. $g_{50} = 0$, $g_{40} + g_{51} \leq 3$ (the maximal number of 4-sets is given by ABCD, ABEF, CDEF, since the packing number D (3,4,6) = 3). In Case 3, we have

$$\varepsilon = g_{30} + g_{40} + 1.5g_{31} + 2.5g_{41} + 4g_{51} + g_{52}$$

$$= (g_{30} + g_{41} + g_{52}) + 1.5(g_{31} + g_{41}) + g_{40} + 4g_{51}.$$

Case 3A.1

$$g_{40} = 1,\ g_{52} = 0;\quad g_{30} + g_{41} + g_{52} = 16,$$

$$\varepsilon = 1 + 16 + 1.5(g_{31} + g_{41}).$$

As in the preceding case, $g_{31} + g_{41} \geq 7$; $\varepsilon \geq 27.5$, $g \geq 51$.

Case 3A.2

$$g_{40} = 0,\ g_{51} = 1;\quad g_{30} + g_{41} + g_{52} = 16,$$

$$\varepsilon = 4 + 16 + 1.5(g_{31} + g_{41}).$$

In this case, there is a block of the form ABCD1. However, the nine pairs AE, AF, BE, BF, CE, CF, DE, DF, EF must all occur in the $g_{31} + g_{41}$ blocks; the best we can do is AEF, BEF, CEF, DEF for a total of 4 (actually, one could strengthen this result to 5 by using the fact that any pair must occur an odd number of times). Hence $e \geq 26$, $g \geq 49.5$

Case 3B.1

$$g_{40} = 2, g_{51} = 0, \quad g_{30} + g_{41} + g_{52} = 12$$

$$\varepsilon = 2 + 12 + 1.5(g_{31} + g_{41}).$$

As before, $g_{31} + g_{41} \geq 7$, $\varepsilon \geq 24.5$, $g \geq 48$.

Case 3B.2

$$g_{40} = 1 = g_{51}, \quad g_{30} + g_{41} + g_{52} = 12,$$

$$\varepsilon = 5 + 12 + 1.5(g_{31} + g_{41}).$$

As in Case 3A.2, $g_{31} + g_{41} \geq 4$, $g \geq 46.5$,

Case 3B.3

$$g_{40} = 0, g_{51} = 2, \quad g_{30} + g_{41} + g_{52} = 12,$$

$$\varepsilon = 8 + 12 + 1.5(g_{31} + g_{41}).$$

We have blocks ABCD 1 and ABEF 2; hence the $g_{31} + g_{41}$ blocks must cover the pairs CE, CF, DE, DF, and this requires at least 2 blocks (again, we could strengthen this to 3). Hence $\varepsilon \geq 23$, $g \geq 46.5$

Case 3C.1

$$g_{40} = 3, g_{51} = 0, \quad g_{30} + g_{41} + g_{52} = 8,$$

$$\varepsilon = 3 + 8 + 1.5(g_{31} + g_{41}) \geq 11 + 1.5(7),$$

$g \geq 45.$

Case 3C.2

$$g_{40} = 2, g_{51} = 1, \quad g_{30} + g_{41} + g_{52} = 8,$$

$$\varepsilon = 2 + 4 + 8 + 1.5(g_{31} + g_{41}) \geq 20,$$

$$g \geq 43.5.$$

Case 3C.3

$$g_{40} = 1, g_{51} = 2, \quad g_{30} + g_{41} + g_{52} = 8,$$

$$\varepsilon = 1 + 8 + 8 + 1.5(g_{31} + g_{41}) \geq 20,$$

$$g \geq 43.5$$

Case 3C.4

$$g_{40} = 0, g_{51} = 3, \quad g_{30} + g_{41} + g_{52} = 8,$$

$$\varepsilon = 12 + 8 + 1.5(g_{31} + g_{41}).$$

In this case, we have blocks ABCD1, ABEF2, CDEF3; so $\varepsilon \geq 20$, $g \geq 43.5$.

Thus, we conclude that, if $g < 46$, then $g = 45$ (since g must be a multiple of 3). We thus have $g_4 + 3g_5 = 40$, $\varepsilon = 45 - 23.5 = 21.5$, and one of the four cases (3C) must hold. We summarize this result as

THEOREM 4.7.2. *If $g(1,3;11) < 46$, then $g(1,3,11) = 45$, and the six points not on the base block are covered by 3 quadruples and 8 triples.*

4.7.3 Investigation of the three-quadruple cases. In Case 3C.4, we have blocks ABCD1, ABEF2, CDEF3; also $g_{31} + g_{41} = 1$. But AB, DE, and EF must occur an odd number of times and so must all occur again in the $g_{31} + g_{41}$ blocks. This is an immediate contradiction.

Now consider Case 3C.3 with blocks CDEF, ABCD1, ABEF2. We also have $g_{31} + g_{41} = 3$. The pairs CE, CF, DE, DF, must occur in the $g_{31} + g_{41}$ blocks. Since they can not appear in a triple, $g_{31} = 4$ and we have a contradiction (note also that AB must appear again).

In Case 3C.2, we have blocks ABEF, CDEF, and ABCD1; here $g_{31} + g_{41} = 5$. The available triples are ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF, and the $g_{31} + g_{41}$ blocks must cover the 9 pairs AE, AF, BE, BF, CE, CF, DE, DF, EF. Clearly, we can cover only two pairs with any available triple; thus $g_{31} = 1$ and $g_{41} = 4$. Without loss of generality we may use triples ACE, ACF, BDE, BDF : the pair is EF . Now use Equations (4.7.2.1) to give $2g_{42} + 6g_{52} = 62$, $g_{32} + 2g_{42} + 3g_{52} = 60$; it easily follows that $g_{32} + g_{42} = 29$. Also, the $g_{30} + g_{52}$ blocks contain ADE, ADF, BCE, BCF .

We now look at occurrences of number pairs ij in the last row (the $g_{32} + g_{42} + g_{52}$ blocks). The pairs 23, 24, 25, 34, 35, 45, must appear in at least 3 blocks (with 3 pairs from A, ..., F, or with a triple, a pair, and a singleton); the pairs 12, 13, 14, 15, must appear in at least 4 blocks (with a triple and 3 singletons, or with 2 pairs and 2 singletons from A, ..., F). This gives a total of 4(4) + 6(3) = 34 blocks needed; however, even if $g_{52} = 4$, using all available triples ADE, ADF, BCE, BCF (with $g_{30} = 0$), we have only 33 blocks. So this case is rejected.

Finally, in Case 3C.1, we have blocks ABCD, ABEF, CDEF . Also, $g_{31} + g_{41} = 7$; hence $g_{31} = 3$, $g_{41} = 4$ (the optimal case). But the only triples available are ACE, ACF, ADE, ADF, BCE, BCF, BDE, BDF . Without loss of generality, we take triples ACE, ADF, BCF, BDE , in the g_{41} blocks; then, we must take pairs AB, CD, EF, in the g_{31} blocks. This leaves triples ACF, ADE, BDF, BCE, to appear in the $g_{30} + g_{52}$ blocks. As in Case 3C.2, we substitute in Equations (4.7.2.1) to obtain $g_{32} + g_{42} = 30$.

Now consider occurrences of the 15 letter pairs with the numbers. Every letter pair occurs with one single number (the g_{31} and g_{41} blocks) and two number pairs. Also, the blocks ACE, ADF, BCF, BDE, all have unit intersections with one another. Suppose, if possible, that ACE 1 and ADF 1 are blocks. Then our pair occurrences can be taken as follows:

AC	1	23	45	AD	1	-	-
AE	1	24	35	AF	1	-	-
CE	1	-	-	DE	1	-	-

No completion is possible since all 1-factors 23 45, 24 35, 25 34 with AD and AF lead to contradictions. Thus we may start the table for AB, AC, ... as follows.

AB	BC 3	CE 1
AC 1, 23, 45	BD 4	CF 3
AD 2	BE 4	DE 4
AE 1, 24, 35	BF 3	DF 2
AF 2	CD	EF

Now put in the 1-factors 14, 35; 15, 34; 13, 45; this can be done in only one way. Thus we are forced to AD 2, 14, 35; AF 2, 13, 45; DF 2, 15, 34. Similarly, all other pairings are forced as in the following table.

AB 5, 12, 34	BC 3, 14, 25	CE 1, 25, 34
AC 1, 23, 45	BD 4, 15, 23	CF 3, 45, 12
AD 2, 14, 35	BE 4, 25, 13	DE 4, 35, 12
AE 1, 24, 35	BF 3, 15, 24	DF 2, 15, 34
AF 2, 13, 45	CD 5, 24, 13	EF 5, 23, 14

We note that this pair table proves the existence of four blocks AAA 11, namely, ACF45, ADE35, BCE25. Our conclusion can be stated as

THEOREM 4.7.3. *There is a unique exact covering of all triples on 11 elements given by the following array.*

1 2 3 4 5 base block

$$\left.\begin{array}{l}\text{ACF45}\\ \text{ADE35}\\ \text{BDF15}\\ \text{BCE25}\end{array}\right\} g_{52} \qquad \left.\begin{array}{l}\text{AB5}\\ \text{CD5}\\ \text{EF5}\end{array}\right\} g_{31}$$

$$\left.\begin{array}{l} ABCD \\ ABEF \\ CDEF \end{array}\right\} g_{40}$$

$$\left.\begin{array}{l} ACE1 \\ ADF2 \\ BCF3 \\ BDE4 \end{array}\right\} g_{41} \qquad \left.\begin{array}{ll} A15 & D25 \\ A25 & D45 \\ B35 & E15 \\ B45 & E45 \\ C15 & F25 \\ C35 & F35 \end{array}\right\} g_{32}$$

$$\left.\begin{array}{lll} AB12 & BC14 & CE34 \\ AB34 & BD23 & CF12 \\ AC23 & BE13 & DE12 \\ AD14 & BF24 & DF34 \\ AE24 & CD24 & EF23 \\ AF13 & CD13 & EF14 \end{array}\right\} g_{42}$$

It is clear (delete element 5) that this array is an extension of a Steiner System S(3,4,10) on 30 blocks. We state this result as

THEOREM 4.7.4. *The unique exact covering of all triples on 11 elements is an extension of the Steiner System S(3,4,10).*

4.8 Computation of $g^{(6)}(1,3;13)$

4.8.1 Introduction. We use a notation similar to that used in Section 4.7. In this section, g_{ij} will denote the number of blocks of length i that contain j elements from the base block 123456. We may also immediately write

$$g^{(6)}(13) = W + \varepsilon$$

where the Woodall bound W is 43. We can also write down the four relations

$$g_{30} + g_{41} + g_{52} + 4(g_{40}+g_{51}+g_{62}) + 10(g_{50}+g_{61}) + 20g_{60} = 35,$$

$$g_{31} + 2g_{42} + 3g_{41} + 6g_{52} + 6g_{51} + 12g_{62} + 10g_{61} = 126,$$

$$g_{32} + 2g_{42} + 3g_{52} + 6g_{62} = 105,$$

$$\varepsilon = g_{30} + g_{41} + g_{52} + g_{40} + g_{50} + g_{60} + 4g_{51} + 6g_{61} + 3g_{62} + 1.5(g_{31}+g_{41})$$

4.8.2 The Cases of All Triples and of a Sextuple. We split the discussion into various cases, depending on how the points not in the base block are distributed.

Case 1. The triples from A, B, C, D, E, F, G (the seven points not in the base block) are all covered by triples. Then

$$g_{30} + g_{41} + g_{52} = 35, \ \varepsilon \geq 78, \ g \geq 78.$$

So this case does not provide a minimal design.

Case 2. There is a block ABCDEF, that is, $g_{60} = 1$. As a result,

$$g_{50} = g_{40} = g_{61} = g_{51} = g_{62} = 0. \text{ Then}$$

$$g_{30} + g_{41} + g_{52} = 15,$$

$$\varepsilon = 1 + 15 + 1.5(g_{31}+g_{41})$$

The least possible value for g will be 59, achieved with $g_{31} = g_{41} = 0$. Then

$$g_{42} + 3g_{52} = 63,$$

$$g_{32} + 2g_{42} + 3g_{52} = 105.$$

Thence

$$g_{32} + g_{42} = 42.$$

Now, the $g_{30} + g_{41} + g_{52}$ blocks must contain the 15 triples

GAB, GAC, GAD, GAE, GAF, GBC, GBD
GBF, GCD, GCE, GCF, GDE, GDF, GEF

Now, the g_{52} blocks GXXij have $f(A) \le 3$ for all A; thus $g_{52} \le 3(6)\div 2 = 9$. Hence $g_{42} \ge 63 - 27 = 36$.

It follows that the only distributions are

g_{32}	g_{42}	g_{52}
6	36	9
3	39	8
0	42	7

Now a number pair ij may occur with various letter patterns. However, all of these patterns involve a singleton (X) except for the pattern (XXX)(XX)(XX), of which there can be at most g_{52}. Consequently, the number of singletons is at least $15 - g_{52} \ge 6$. It follows that $g_{32} = 6$, $g_{42} = 36$, $g_{52} = 9$. We conclude that $f(A) = 3$ for all ij in the blocks GXXij; also $g_{30} = 6$.

This block distribution can be achieved uniquely. Up to isomorphism, we may take

GAB 12	GBE 35	GCF 15
GAC 34	GBF 46	GDE 14
GAD 56	GCE 26	GDF 23

The only possible g_{42} blocks are found by taking letter pairs with number pairs as follows.

AB: 36, 45	BF: 16, 24	CF: 24, 36
AC: 16, 25	BF: 13, 25	DE: 25, 36
AD: 13, 24	CE: 13, 45	DF: 16, 45

as well as

AE: 46, 15, 23	BD: 34, 26, 15
AF: 35, 26, 14	CD: 12, 35, 46
BC: 56, 14, 23	EF: 12, 34, 56

Finally, one needs the g_{30} blocks, GAE, GAF, GBC, GBD, GCD, GEF, and the g_{32} blocks G13, G16, G24, G25, G36, G45.

This configuration gives, on deletion of G, the familiar 47-block minimal system of 2 disjoint sextuples and 45 quadruples on 12 elements (cf. [48]).

4.8.3 Case of a Quintuple or a Quadruple. We now consider the case $g_{60} = 0$, $g_{50} + g_{61} = 1$ (that is, there is a block of length 5).

Case 3(a). $g_{60} = 0, g_{50} = 1, g_{61} = 0.$

Case 3(b). $g_{60} = 0, g_{50} = 0, g_{61} = 1.$

For both subcases, we have

$$\varepsilon = g_{30} + g_{41} + g_{52} + 1.5(g_{31}+g_{41}) + g_{40} + g_{50} + 4g_{51} + 6g_{61} + 3g_{62},$$

$$g_{51} + g_{40} + g_{62} \leq 2.$$

Hence, since

$$g_{30} + g_{41} + g_{52} + 4(g_{40}+g_{51}+g_{62}) = 25,$$

we have

$$g_{30} + g_{41} + g_{52} \geq 17.$$

Thus $\varepsilon \geq 17 + g_{50} + 4g_{61} \geq 18$.

Thus $g \geq 61$.

So Case 3 does not provide a minimum.

Case 4. $g_{60} = g_{50} = g_{61} = 0$, but we have

$$g_{40} + g_{51} + g_{62} > 0.$$

Now we know that D(3,4,7) = 7; hence

$$g_{40} + g_{51} + g_{62} = p \leq 7.$$

Furthermore,

$$g_{30} + g_{41} + g_{52} + 4(g_{40}+g_{51}+g_{62}) = 35.$$

Then

$$\varepsilon = g_{30} + g_{41} + g_{52} + 1.5(g_{31}+g_{41}) + g_{40} + 4g_{51} + 3g_{62}$$

$$\geq (g_{30}+g_{41}+g_{52}) + (g_{40}+g_{51}+g_{62})$$

$$= 35 - 4p + p = 35 - 3p.$$

But a minimum must have $g = 43 + \varepsilon \geq 78 - 3p$.

Hence, the minimum would be 57, and we would need

$$\begin{aligned}
&g_{30} + g_{41} + g_{52} = 7,\\
&g_{40} + g_{51} + g_{62} = 7,\\
&g_{31} + g_{41} = 0, \quad g_{51} = g_{62} = 0.
\end{aligned}$$

Then, we deduce

$$\begin{aligned}
&g_{40} = 7,\\
&g_{42} + 3g_{52} = 63,\\
&g_{32} + 2g_{42} + 3g_{52} = 105,\\
&g_{32} + g_{42} = 42.
\end{aligned}$$

But $g_{52} \leq 7$, $g_{42} \leq 42$, imply that $g_{52} = 7$, $g_{42} = 42$.

Hence, any pair ij must occur with one triple and two pairs (since $g_{32} = 0$). This requirement can not be met.

4.8.4 Conclusion. We summarize these results as a Theorem.

THEOREM 4.8.1. *The value of* $g^{(6)}(13)$ *is 59, and the minimal design contains 2 disjoint 6-blocks, 9 blocks of length 5, 36 blocks of length 4, and 12 blocks of length 3. The element not in the 2 blocks of length 6 occurs in all the blocks of length 5 and all the blocks of length 3. The design is an extension of the minimal* $g(1,3;12)$ *design formed by 2 disjoint 6-sets and the Cartesian products of their five 1-factors.*

Chapter V

5.1 Preliminaries

In this Chapter, we will be dealing with an algorithm to investigate some exact bicoverings, that is, exact coverings in which each pair must occur exactly twice. This problem is considerably more difficult because we do not have the usual inequalities (they are developed under the hypothesis that $\lambda = 1$). The main general result available is a remarkable theorem that has been proved by Ryser; this theorem answers a fundamental question about the behaviour of exact coverings when λ is equal to 2. We quote the theorem from [39].

THEOREM (Ryser). *Let $S = \{a_1, a_2, ..., a_m\}$ be an m-set and let $S_1, S_2, ..., S_n$ be n subsets of S. In this configuration we assume that each S_i and S_j with $i \neq j$ intersect in exactly λ elements of S. We also assume that $n > 1$, $\lambda \geq 1$, and that the number of elements in each S_i is greater than λ. Then the configuration has $m \geq n$ and if equality holds, the configuration satisfies one of the following two requirements:*

1) *Each of the replication numbers of the configuration equals a positive integer k and each S_i is a k-subset of S.*

2) *The configuration has exactly two distinct replication numbers r_1 and r_2, and these numbers satisfy*

$$r_1 + r_2 = n + 1.$$

For application, this form of the theorem is not convenient. If we think of the Ryser design as given in an incidence matrix form, we can obtain a dual theorem by interchanging the role of rows and columns in the matrix. For our purposes, the dual form of Ryser's Theorem is required.

DUAL OF RYSER'S THEOREM. *Let $a_1, ..., a_v$ be v elements and let $\{B_1, B_2, ..., B_b\}$ be b blocks. We assume that a_i and a_j $(i \neq j)$ occur together exactly λ times. We also assume that $v > 1$, $\lambda \geq 1$, and that each element occurs more than λ times. Then the design has $b \geq v$ and equality holds if and only if the design satisfies one of the following requirements.*

1) *Each of the blocks of the design contains k elements and each point occurs k times; thus we have a Balanced Incomplete Block Design with parameters* (v,k,λ).

2) *The design has exactly two block sizes* k_1 *and* k_2 *and these numbers satisfy the relation*

$$k_1 + k_2 = v + 1.$$

Ryser has also discussed Case 2 (he calls the designs in Case 2 by the name λ-designs) for $\lambda = 2$ and has established the following result.

RYSER'S THEOREM FOR $\lambda = 2$. *Let A be the incidence matrix of a* λ*-design with* $\lambda = 2$ *on n elements. Then, apart from row and column permutations, A is given by the following array.*

$$\begin{array}{ccccccc}
1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}$$

The net result of Ryser's work, in the dual form, is that an exact covering for $\lambda = 2$ has $b \geq v$ and that b can only be equal to v if we have

Case 1) A Balanced Incomplete Block Design with parameters

$$v = 1 + \frac{1}{2}k(k-1);\ k;\ 2.$$

Case 2) $v = 7$ and there are 3 blocks containing 5 elements and 4 blocks containing 3 elements.

5.2 A Computational Example

We shall be using a computer algorithm to obtain exact bicoverings for small v. To illustrate the algorithm, we shall discuss a small example in complete detail.

Let b_i be the number of blocks of length i in an exact covering. We illustrate our algorithm for v = 8. Since Ryser's Theorem guarantees b > 8, we must first try b = 9 and have

$$b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 9,$$

$$b_2 + 3b_3 + 6b_4 + 10b_5 + 15b_6 + 21b_7 = 2\binom{8}{2} = 56.$$

Then $2b_3 + 5b_4 + 9b_5 + 14b_6 + 20b_7 = 47.$

The possible value for b_i are stored as block vectors $(b_2, b_3, \ldots, b_7)$. A simple Diophantine sieving produces

b_2	b_3	b_4	b_5	b_6	b_7
5	1	1	0	0	2
4	2	0	1	1	1
2	4	1	0	1	1
5	0	0	3	0	1
3	2	1	2	0	1
1	4	2	1	0	1
2	1	5	0	0	1
5	0	1	0	3	0
4	0	2	1	2	0
1	5	0	1	2	0
2	2	3	0	2	0
2	3	0	3	1	0
3	0	3	2	1	0
0	5	1	2	1	0
1	2	4	1	1	0
3	1	0	5	0	0
1	3	1	4	0	0
2	0	4	3	0	0
0	2	5	2	0	0

For each specific block vector, we define distribution vectors $(a_2, a_3, a_4, a_5, a_6, a_7)$; a_j gives the number of times that any particular element appears in blocks of length j. Clearly, we have

$$a_2 + 2a_3 + 3a_4 + 4a_5 + 5a_6 + 6a_7 = 2(7) = 14.$$

It is trivial that $a_i \leq b_i$.

For example, the first block vector (5,1,1,0,0,2) has distribution vectors

$$\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 2 \\ 3 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 1 \end{array}$$

We now assign frequencies to these distribution vectors. Thus, there are f_1 elements with distribution (0,1,0,0,0,2), f_2 with distribution (2,0,0,0,0,2), f_3 with distribution (3,1,1,0,0,1), and f_4 with distribution (5,0,1,0,0,1). The following equations hold. We normally omit the first equation, since it is dependent on the others.

$$\begin{array}{rrrrrrrrl} f_1 & + f_2 & + & f_3 & + & f_4 & = & 8 \\ 2f_1 & +2f_2 & + & f_3 & + & f_4 & = & 2(7) = 14 \\ & & & f_3 & + & f_4 & = & 1(4) = 4 \\ f_1 & & + & f_3 & & & = & 1(3) = 3 \\ & 2f_2 & + & 3f_3 & + & 5f_4 & = & 5(2) = 10 \end{array}$$

In general, we reduce these equations to row echelon form.

In this case,

$$\begin{array}{rrrrrrl} f_1 & & & - & f_4 & = & 2.5 \\ & f_2 & & + & f_4 & = & 4.25 \\ & & f_3 & + & f_4 & = & 0.5 \\ & & & & 0 & = & 3.5 \end{array}$$

Clearly, this system of equations has no integral solution.

This discussion eliminates the first block vector. We tabulate the results for the remaining block vectors. (When we program the algorithm, we omit all those entries in the block vector that are zero.)

<u>Block Vector 2</u> (4,2,0,1,1,1) has distribution vectors

$$
\begin{array}{cccccc}
1 & 1 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & 0 & 1 & 1 \\
0 & 2 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 & 1 \\
4 & 2 & 0 & 0 & 0 & 1 \\
1 & 2 & 0 & 1 & 1 & 0 \\
3 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

The frequency equations are

$$
\begin{array}{lllllllllll}
f_1 & + f_2 & + f_3 & + f_4 & + f_5 & + f_6 & & & = & 7 \\
f_1 & + f_2 & & & & & + f_7 & + f_8 & = & 6 \\
 & & f_3 & + f_4 & + f_5 & & + f_7 & + f_8 & = & 5 \\
f_1 & & + 2f_3 & + f_4 & & + 2f_6 & + 2f_7 & + f_8 & = & 6 \\
f_1 & + 3f_2 & & + 2f_4 & + 4f_5 & + 4f_6 & + f_7 & + 3f_8 & = & 8
\end{array}
$$

In row echelon form

$$
\begin{array}{llllllllll}
f_1 & & & & & - f_6 & - 2f_7 & - f_8 & = & 0 \\
 & f_2 & & & & + f_6 & + 2f_7 & + f_8 & = & 5 \\
 & & f_3 & & & + f_6 & + f_7 & & = & 4 \\
 & & & f_4 & & & & & = & -2 \\
 & & & & f_5 & & & + f_8 & = & 1
\end{array}
$$

Since f_4 can not be less than zero, this case is ruled out.

<u>Block Vector 3</u> (2,4,1,0,1,1) has distribution vectors

$$
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 & 1 \\
0 & 4 & 0 & 0 & 0 & 1 \\
2 & 3 & 0 & 0 & 0 & 1 \\
0 & 3 & 1 & 0 & 1 & 0 \\
2 & 2 & 1 & 0 & 1 & 0 \\
1 & 4 & 0 & 0 & 1 & 0
\end{array}
$$

The frequency equations are

$$
f_1 + f_2 + f_3 + f_4 + f_5 = 7
$$

$$\begin{array}{lllllllllll}
f_1 & + f_2 & & & & + & f_6 & + f_7 & + f_8 & = & 6 \\
f_1 & & + f_3 & & & + & f_6 & + f_7 & & = & 4 \\
& f_2 & + 2f_3 & +4f_4 & + 3f_5 & + & 3f_6 & +2f_7 & +4f_8 & = & 12 \\
& f_2 & + f_3 & & + 2f_5 & & & +2f_7 & + f_8 & = & 4
\end{array}$$

In row echelon form

$$\begin{array}{llllllllll}
f_1 & & & & & - & f_6 & - f_7 & - f_8 & = 2 \\
& f_2 & & & & + & f_6 & + f_7 & + f_8 & = 3 \\
& & f_3 & & & + & f_6 & + f_7 & & = 1 \\
& & & f_4 & & & & - f_7 & & = 1 \\
& & & & f_5 & & & + f_7 & + f_8 & = 1
\end{array}$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
4	1	0	1	0	1	0	1
3	2	1	1	0	0	0	1
3	2	0	2	0	0	1	0
3	2	0	1	1	1	0	0
2	3	1	1	1	0	0	0

<u>Block vector 4</u> (5,0,0,3,0,1) has distribution vectors

0	0	0	2	0	1
4	0	0	1	0	1
2	0	0	3	0	0

The frequency equations are

$$\begin{array}{lllll}
f_1 & + f_2 & & = & 7 \\
2f_1 & + f_2 & + 3f_3 & = & 15 \\
& 4f_2 & + 2f_3 & = & 10
\end{array}$$

In row echelon form

$$\begin{array}{lllll} f_1 & & & = & 5 \\ & f_2 & & = & 2 \\ & & f_3 & = & 1 \end{array}$$

These equations have the solution

f_1	f_2	f_3
5	2	1

<u>Block Vector 5</u> (3,2,1,2,0,1) has distribution vectors

$$\begin{array}{cccccc}
0 & 0 & 0 & 2 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
0 & 2 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 2 & 0 & 0 \\
3 & 0 & 1 & 2 & 0 & 0 \\
2 & 2 & 0 & 2 & 0 & 0 \\
3 & 2 & 1 & 1 & 0 & 0
\end{array}$$

The frequency equations are

$$\begin{array}{l}
f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 7 \\
2f_1 + f_2 + f_3 + f_4 + 2f_7 + 2f_8 + 2f_9 + f_{10} = 10 \\
f_2 + f_5 + f_6 + f_7 + f_8 + f_{10} = 4 \\
2f_3 + f_4 + 2f_5 + f_6 + f_7 + 2f_9 + 2f_{10} = 6 \\
f_2 + 2f_4 + f_5 + 3f_6 + f_7 + 3f_8 + 2f_9 + 3f_{10} = 6
\end{array}$$

In row echelon form

$$\begin{array}{l}
f_1 - f_6 - f_7 - f_{10} = 1 \\
f_2 + f_6 + f_7 - f_9 = 3 \\
f_3 + f_7 - f_8 = 2 \\
f_4 + f_6 + f_8 + f_9 + f_{10} = 1 \\
f_5 + f_8 + f_9 + f_{10} = 1
\end{array}$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
4	0	0	0	1	1	2	0	0	0
3	1	1	0	1	1	1	0	0	0
2	2	2	0	1	1	0	0	0	0
3	1	0	1	1	0	2	0	0	0
2	2	1	1	1	0	1	0	0	0
1	3	2	1	1	0	0	0	0	0
4	0	0	0	0	0	3	1	0	0
3	1	1	0	0	0	2	1	0	0
3	2	0	0	0	0	2	0	1	0
4	1	0	0	0	0	2	0	0	1
2	2	2	0	0	0	1	1	0	0
2	3	1	0	0	0	1	0	1	0
3	2	1	0	0	0	1	0	0	1
1	3	3	0	0	0	0	1	0	0
1	4	2	0	0	0	0	0	1	0
2	3	2	0	0	0	0	0	0	1

<u>Block Vector 6</u> (1,4,2,1,0,1) has distribution vectors

1	0	1	1	0	1
0	2	0	1	0	1
0	1	2	0	0	1
1	2	1	0	0	1
0	4	0	0	0	1
0	2	2	1	0	0
1	3	1	1	0	0
0	4	2	0	0	0

The frequency equations are

$$
\begin{array}{lllllllll}
f_1 & + f_2 & + f_3 & + f_4 & + f_5 & & & & = 7 \\
f_1 & + f_2 & & & & + f_6 & + f_7 & & = 5 \\
f_1 & & + 2f_3 & + f_4 & & + 2f_6 & + f_7 & + 2f_8 & = 8 \\
 & 2f_2 & + f_3 & + 2f_4 & + 4f_5 & + 2f_6 & + 3f_7 & + 4f_8 & = 12 \\
f_1 & & & + f_4 & & & + f_7 & & = 2
\end{array}
$$

In row echelon form

$$
\begin{array}{llllll}
f_1 & & & - f_6 & & - f_8 = 1 \\
 & f_2 & & + f_6 & & = 3 \\
 & & f_3 & & - f_7 & = 2
\end{array}
$$

$$\begin{array}{lllllllllll}
f_4 & & + & f_6 & + & f_7 & + & f_8 & = & 1 \\
 & f_5 & & & + & f_7 & + & f_8 & = & 1
\end{array}$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
2	2	2	0	1	1	0	0
1	3	2	1	1	0	0	0
1	3	3	0	0	0	1	0
2	3	2	0	0	0	0	1

Block Vector 7 (2,1,5,0,0,1) has distribution vectors

0	1	2	0	0	1
2	0	2	0	0	1
0	1	4	0	0	0
2	0	4	0	0	0

The frequency equations are

$$\begin{array}{rrrrl}
f_1 & + f_2 & & & = 7 \\
2f_1 & +2f_2 & + 4f_3 & + 4f_4 & = 20 \\
f_1 & & + f_3 & & = 3 \\
 & 2f_2 & & + 2f_4 & = 4
\end{array}$$

In row echelon form

$$\begin{array}{rrrrl}
f_1 & & & - f_4 & = 5 \\
 & f_2 & & + f_4 & = 2 \\
 & & f_3 & + f_4 & =1.5 \\
 & & & 0 & =-3.5
\end{array}$$

Obviously, these equations have no solutions.

Block Vector 8 (5,0,1,0,3,0) has distribution vectors

1	0	1	0	2	0
4	0	0	0	2	0

The frequency equations are

$$\begin{array}{llcr} 2f_1 & +2f_2 & = & 18 \\ f_1 & & = & 4 \\ f_1 & +4f_2 & = & 10 \end{array}$$

In row echelon form

$$\begin{array}{llr} f_1 & & =8.67 \\ & f_2 & =0.33 \\ & 0 & =-4.67 \end{array}$$

Clearly, there are no solutions to these equations.

<u>Block Vector 9</u> (4,0,2,1,2,0) has distribution vectors

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 1 & 0 & 0 \end{array}$$

The frequency equations are

$$\begin{array}{lllllllcr} 2f_1 & +2f_2 & +2f_3 & + & f_4 & +f_5 & & = & 12 \\ f_1 & & & + & f_4 & & + f_6 & = & 5 \\ & f_2 & & + & f_4 & +2f_5 & + 2f_6 & = & 8 \\ & f_2 & +4f_3 & + & 2f_4 & +3f_5 & + 4f_6 & = & 8 \end{array}$$

In row echelon form

$$\begin{array}{llllllcr} f_1 & & & & -f_5 & - f_6 & = & 1 \\ & f_2 & & & +f_5 & & = & 4 \\ & & f_3 & & & & = & -1 \\ & & & f_4 & +f_5 & + 2f_6 & = & 4 \end{array}$$

There are no solutions to these equations ($f_3 < 0$).

Block Vector 10 (1,5,0,1,2,0) has distribution vectors

0	0	0	1	2	0
0	2	0	0	2	0
1	2	0	1	1	0
1	4	0	0	1	0
0	5	0	1	0	0

The frequency equations are

$$\begin{array}{llllllll} 2f_1 & +2f_2 & + & f_3 & + & f_4 & & = & 12 \\ f_1 & & + & f_3 & & & + f_5 & = & 5 \\ & 2f_2 & + & 2f_3 & + & 4f_4 & +5f_5 & = & 15 \\ & & & f_3 & + & f_4 & & = & 2 \end{array}$$

In row echelon form

$$\begin{array}{llllll} f_1 & & & & - f_5 & = 2 \\ & f_2 & & & + f_5 & = 3 \\ & & f_3 & & + f_5 & = 2 \\ & & & f_4 & & = 1 \end{array}$$

These equations have the following solutions

f_1	f_2	f_3	f_4	f_5
2	3	2	1	0
3	2	1	1	1
4	1	0	1	2

Block Vector 11 (2,2,3,0,2,0) has distribution vectors

1	0	1	0	2	0
0	2	0	0	2	0
2	1	0	0	2	0
0	0	3	0	1	0
1	1	2	0	1	0
2	2	1	0	1	0
1	2	3	0	0	0

The frequency equations are

$$
\begin{array}{lllllllll}
2f_1 & +2f_2 & +2f_3 & +f_4 & +f_5 & + & f_6 & & =12 \\
f_1 & & & +3f_4 & +2f_5 & + & f_6 & +3f_7 & =12 \\
 & 2f_2 & +f_3 & & f_5 & + & 2f_6 & +2f_7 & =6 \\
f_1 & & +2f_3 & & +f_5 & + & 2f_6 & +f_7 & =4
\end{array}
$$

In row echelon form

$$
\begin{array}{lllllllll}
f_1 & & & & -f_5 & - & 2f_6 & -3f_7 & =0 \\
 & f_2 & & & & & & & =2 \\
 & & f_3 & & f_5 & + & 2f_6 & +2f_7 & =2 \\
 & & & f_4 & +f_5 & + & f_6 & +2f_7 & =4
\end{array}
$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7
0	2	2	4	0	0	0
1	2	1	3	1	0	0
2	2	0	3	0	1	0
2	2	0	2	2	0	0
3	2	0	2	0	0	1

<u>Block Vector 12</u> (2,3,0,3,1,0) has distribution vectors

1	0	0	2	1	0
1	2	0	1	1	0
0	1	0	3	0	0
2	0	0	3	0	0
0	3	0	2	0	0
2	2	0	2	0	0

The frequency equations are

$$
\begin{array}{llllllll}
f_1 & +f_2 & & & & & & =6 \\
2f_1 & +f_2 & +3f_3 & +3f_4 & +2f_5 & + & 2f_6 & =15 \\
 & 2f_2 & +f_3 & & +3f_5 & + & 2f_6 & =9 \\
f_1 & +f_2 & & +2f_4 & & + & 2f_6 & =4
\end{array}
$$

In row echelon form

$$\begin{array}{lllllllll}
f_1 & & & & - f_5 & - & f_6 & = & 3 \\
 & f_2 & & & + f_5 & + & f_6 & = & 3 \\
 & & f_3 & & + f_5 & & & = & 3 \\
 & & & f_4 & & + & f_6 & = & -1
\end{array}$$

There are no solutions to these equations (neither f_4 nor f_6 is negative).

<u>Block Vector 13</u> (3,0,3,2,1,0) has distribution vectors

$$\begin{array}{cccccc}
1 & 0 & 0 & 2 & 1 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 3 & 0 & 1 & 0 \\
3 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 \\
3 & 0 & 1 & 2 & 0 & 0 \\
1 & 0 & 3 & 1 & 0 & 0
\end{array}$$

The frequency equations are

$$\begin{array}{lllllllllll}
f_1 & + f_2 & + f_3 & + f_4 & & & & & & = & 6 \\
2f_1 & + f_2 & & & +2f_5 & + & 2f_6 & + & f_7 & = & 10 \\
 & f_2 & + 3f_3 & + 2f_4 & +2f_5 & + & f_6 & + & 3f_7 & = & 12 \\
f_1 & +2f_2 & & + 3f_4 & & + & 3f_6 & + & f_7 & = & 6
\end{array}$$

In row echelon form

$$\begin{array}{lllllllllll}
f_1 & & & & - f_5 & - & f_6 & - & f_7 & = & 2 \\
 & f_2 & & & +2f_5 & + & 2f_6 & + & f_7 & = & 2 \\
 & & f_3 & & & - & f_6 & & & = & 2 \\
 & & & f_4 & & + & f_6 & + & f_7 & = & 2
\end{array}$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7
3	0	2	2	1	0	0
2	2	2	2	0	0	0
3	0	3	1	0	1	0

3	1	2	1	0	0	1
4	0	2	0	0	0	2

<u>Block Vector 14</u> (0,5,1,2,1,0) has distribution vectors

0	1	1	1	1	0
0	3	1	0	1	0
0	3	0	2	0	0
0	5	0	1	0	0

The frequency equations are

$$
\begin{array}{lllll}
f_1 & + f_2 & & & = 6 \\
f_1 & & + 2f_3 & + f_4 & = 10 \\
f_1 & + f_2 & & & = 4 \\
f_1 & + 3f_2 & + 3f_3 & + 5f_4 & = 15
\end{array}
$$

In row echelon form

$$
\begin{array}{lllll}
f_1 & & & - f_4 & = 5.14 \\
 & f_2 & & + f_4 & = 0.86 \\
 & & f_3 & + f_4 & = 2.43 \\
 & & & 0 & = -2
\end{array}
$$

Clearly, there are no solutions to these equations.

<u>Block Vector 15</u> (1,2,4,1,1,0) has distribution vectors

0	1	1	1	1	0
1	2	0	1	1	0
0	0	3	0	1	0
1	1	2	0	1	0
1	0	3	1	0	0
0	2	2	1	0	0
0	1	4	0	0	0
1	2	3	0	0	0

The frequency equations are

$$\begin{array}{lllllllll}
f_1 & + f_2 & + f_3 & + f_4 & & & & & = 6 \\
f_1 & + f_2 & & & + f_5 & + f_6 & & & = 5 \\
f_1 & & + 3f_3 & + 2f_4 & +3f_5 & + 2f_6 & + 4f_7 & +3f_8 & = 16 \\
f_1 & +2f_2 & & + f_4 & & + 2f_6 & + f_7 & +2f_8 & = 6 \\
 & f_2 & & + f_4 & + f_5 & & & + f_8 & = 2
\end{array}$$

In row echelon form

$$\begin{array}{lllllllll}
f_1 & & & & & - f_6 & - f_7 & - 2f_8 & = 2 \\
 & f_2 & & & & + f_6 & & + f_8 & = 1 \\
 & & f_3 & & & + f_6 & + f_7 & + f_8 & = 3 \\
 & & & f_4 & & & + f_7 & + f_8 & = 1 \\
 & & & & f_5 & & & & = 1
\end{array}$$

These equations have the following solutions:

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
4	0	1	0	1	1	1	0
3	0	2	1	1	1	0	0
3	1	2	0	1	0	1	0
4	0	2	0	1	0	0	1
2	1	3	1	1	0	0	0

<u>Block Vector 16</u> (3,1,0,5,0,0) has distribution vectors

0	1	0	3	0	0
2	0	0	3	0	0

The frequency equations are

$$\begin{array}{llll}
3f_1 & +3f_2 & = & 25 \\
f_1 & & = & 3 \\
 & 2f_2 & = & 6
\end{array}$$

In row echelon form

$$\begin{array}{lll}
f_1 & & =5.33 \\
 & f_2 & = 3 \\
 & 0 & =-2.33
\end{array}$$

Clearly, there are no solutions to these equations.

<u>Block Vector 17</u> (1,3,1,4,0,0) has distribution vectors

0	1	0	3	0	0
1	1	1	2	0	0
0	3	0	2	0	0
1	3	1	1	0	0

The frequency equations are

$$\begin{aligned} 3f_1 + 2f_2 + 2f_3 + f_4 &= 20 \\ f_2 + f_4 &= 4 \\ f_1 + f_2 + 3f_3 + 3f_4 &= 9 \\ f_2 + f_4 &= 2 \end{aligned}$$

In row echelon form

$$\begin{aligned} f_1 - f_4 &= 3.71 \\ f_2 + f_4 &= 4 \\ f_3 + f_4 &= 0.43 \\ 0 &= -2 \end{aligned}$$

Clearly, there are no solutions to these equations.

<u>Block Vector 18</u> (2,0,4,3,0,0) has distribution vectors

2	0	0	3	0	0
0	0	2	2	0	0
1	0	3	1	0	0
2	0	4	0	0	0

The frequency equations are

$$\begin{aligned} 3f_1 + 2f_2 + f_3 &= 15 \\ 2f_2 + 3f_3 + 4f_4 &= 16 \\ 2f_1 + f_3 + 2f_4 &= 4 \end{aligned}$$

In row echelon form

$$\begin{array}{llllll} f_1 & & & & = & 1 \\ & f_2 & & - f_4 & = & 5 \\ & & f_3 & + 2f_4 & = & 2 \end{array}$$

These equations have the following solutions

f_1	f_2	f_3	f_4
1	5	2	0
1	6	0	1

<u>Block Vector 19</u> (0,2,5,2,0,0) has distribution vectors

0	0	2	2	0	0
0	2	2	1	0	0
0	1	4	0	0	0

The frequency equations are

$$\begin{array}{lllll} 2f_1 & + f_2 & & = & 10 \\ 2f_1 & +2f_2 & + 4f_3 & = & 20 \\ & 2f_2 & + f_3 & = & 6 \end{array}$$

In row echelon form

$$\begin{array}{lllll} f_1 & & & = & 4 \\ & f_2 & & = & 2 \\ & & f_3 & = & 2 \end{array}$$

These equations have the solution

f_1	f_2	f_3
4	2	2

It is clear from this example that many of the block vectors are easily eliminated. However, in the cases where there are solutions for the frequency equations, we need to give further rules for elimination of solutions. The first such restriction comes from consideration of the pair table that we describe in the next section.

5.3 The Pair Count Table

Suppose, for a given block vector, that there are r distribution vectors (a_{i2}, ..., a_{is} where i = 1, 2, ..., r and s = v - 1). Suppose further that two elements of our design occur with distribution vectors i and j respectively. Clearly, it is possible to have i = j. Then the number of pairs formed by these two elements in the kth block is given by

$$\mathrm{Max}(0, a_{ik} + a_{jk} - b_k).$$

We now define

$$p_{ij} = \sum_k \mathrm{Max}(0, a_{ik} + a_{jk} - b_k).$$

These quantities p_{ij} are printed out in a pair count table. It is clear that, if p_{ij} is greater than 2, then the ith and jth distribution vectors can not occur together. As a special case, we note that $p_{ii} > 2$ implies that the ith distribution vector has frequency either 0 or 1.

As an illustration of the power of the pair count table, we look at block vector 5 from the example of the last section. There are 10 distribution vectors and so the pair count table is of size 10 by 10.

Block Vector 5 (3,2,1,2,0,1) has distribution vectors

0	0	0	2	0	1
1	0	1	1	0	1
0	2	0	1	0	1
2	1	0	1	0	1
1	2	1	0	0	1
3	1	1	0	0	1
1	1	1	2	0	0
3	0	1	2	0	0
2	2	0	2	0	0
3	2	1	1	0	0

The pair count table is obtained in the following form

3	2	2	2	1	1	2	2	2	1
2	2	1	1	2	3	2	3	1	2
2	1	3	2	3	2	2	1	3	2
2	1	2	2	2	3	1	3	3	3
1	2	3	2	4	4	2	2	2	4
1	3	2	3	4	5	2	4	3	5
2	2	2	1	2	2	3	4	3	4
2	3	1	3	2	4	4	6	4	5
2	1	3	3	2	3	3	4	5	5
1	2	2	3	4	5	4	5	5	6

The frequency equations had 16 solutions, which we repeat.

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
4	0	0	0	1	1	2	0	0	0
3	1	1	0	1	1	1	0	0	0
2	2	2	0	1	1	0	0	0	0
3	1	0	1	1	0	2	0	0	0
2	2	1	1	1	0	1	0	0	0
1	3	2	1	1	0	0	0	0	0
4	0	0	0	0	0	3	1	0	0
3	1	1	0	0	0	2	1	0	0
3	2	0	0	0	0	2	0	1	0
4	1	0	0	0	0	2	0	0	1
2	2	2	0	0	0	1	1	0	0
2	3	1	0	0	0	1	0	1	0
3	2	1	0	0	0	1	0	0	1
1	3	3	0	0	0	0	1	0	0
1	4	2	0	0	0	0	0	1	0
2	3	2	0	0	0	0	0	0	1

We now note that $p_{11} = 3$. Consequently, we can reject all solutions in which f_1 is greater than 1. This eliminates all solutions except solutions number 6, number 14, and number 15. Solution 6 is (1,3,2,1,1,0,0,0,0,0) and it is eliminated because p_{33} and p_{35} are both greater than 2. Solution 14 is (1,3,3,0,0,0,0,1,0,0), and it is eliminated because $p_{33} > 2$. Finally, solution 15 is (1,4,2,0,0,0,0,0,1,0), and it is eliminated because p_{33} and p_{39} are both greater than 2.

This illustration shows how effective the pair count table can be in quickly eliminating possible solutions.

5.4 Statistics on Some Program Runs

A program was written which uses the algorithms described in sections 5.2 and 5.3. We shall now discuss the output from several runs of this program which is listed in Appendix A.

Each computer listing is divided into cases. Within each case, the following information is always displayed:

- Case number;
- Block vector;
- Distribution vectors;
- Pair count table;
- System of equations generated from distribution vectors;
- System of equations in row echelon form;
- Column pointer vector.

The column pointer vector is used to keep track of the original order of the columns. This is required since the row echelon algorithm employed will interchange columns.

The remainder of the output for the case depends on the characteristics of the system of equations. The block vector can immediately be rejected if any of the following conditions on the row echelon form of the system of equations is met:

(A) { - Left hand side of equation equals 0 while the right hand side does not;
- LHS is positive - RHS is negative;
- LHS is all integral - RHS is not.

If the block vector is not rejected, the system of equations is converted from real numbers to integers (for increased processing speed). This new system is displayed along with upper bounds for each column.

The number of solutions is displayed, along with each solution. If the solution can be rejected by the pair count method (Condition B), this information is also displayed.

Finally, if any solutions are not rejected, a message is displayed indicating which solutions should be examined manually.

As a sample of output, a complete listing for the case $g(9) = 11$ is given in Appendix B.

5.4.1 Case of $g(8) = 9$.

Number of Block Vectors	19
Total Number of Cases remaining after (A)	10
Subcases Remaining after Pair Count Method (B)	2

5.4.2 Case of $g(9) = 10$.

Number of Block Vectors	47
Total Number of Cases remaining after (A)	33
Subcases Remaining after Pair Count Method (B)	25

5.4.3 Case of $g(9) = 11$.

Number of Block Vectors	54
Total Number of Cases remaining after (A)	38
Subcases Remaining after Pair Count Method (B)	84

5.4.4 Case of $g(10) = 11$.

Number of Block Vectors	110
Total Number of Cases remaining after (A)	77
Subcases Remaining after Pair Count Method (B)	123

5.4.5 Case of $g(11) = 11$.

Number of Block Vectors	203
Total Number of Cases remaining after (A)	122
Subcases Remaining after Pair Count Method (B)	107

5.5 The Multi-pair Criterion

A great many subcases are automatically rejected by the program, using either the row echelon criteria or the pair-count criterion. The remaining cases can usually be removed using the following "multi-pair criterion".

Suppose we have a block vector with entries $b_2, b_3, \ldots, b_r$ (many of these may be zero). A typical "case" consists of t frequencies $f_1, f_2, \ldots, f_t$, each f_i being associated with a distribution vector $a_{i2}, a_{i3}, \ldots, a_{ir}$. Pick any f_i (the larger f_i are best). Let F_i be the set of these f_i elements; we note that there are $f_i a_{is}$ elements in the blocks of length s. We write

$$f_i a_{is} = q_s b_s + r_s.$$

This means that we get the minimal number of F_i-pairs by using r_s blocks with q_s+1 elements and $b_s - r_s$ blocks with q_s elements. The total number of F_i-pairs is thus at least

$$\sum \left\{ r_s \binom{q_s+1}{2} + (b_s - r_s) \binom{q_s}{2} \right\},$$

where the summation is over s = 2 to s = r. This number can not exceed the total number of F_i-pairs. Hence

$$(*) \qquad \sum q_s \left\{ b_s (q_s - 1) + 2r_s \right\} \leq f_i (f_i - 1).$$

For example, in Case 75 (#14) of the g(10) = 11, we have

	$b_3 = 6$	$b_5 = 3$	$b_7 = 2$
$f_1 = 4$	1	1	2
$f_2 = 5$	2	2	1
$f_3 = 1$	4	1	1

Consider the four elements given by $f_1 = 4$. Then

$$f_1 a_3 = 4 = 0 \bullet 6 + 4$$
$$f_1 a_5 = 4 = 1 \bullet 3 + 1$$
$$f_1 a_7 = 8 = 4 \bullet 2 + 0$$

The summation is thus

$$0 + \frac{1}{2} \{ 3(0) + 2 \} + \frac{4}{2} \{ 2(3) + 0 \} = 13.$$

Since this summation should be ≤ 12, the case is rejected.

For later cases, one could put (*) into the program; however, one would not employ it unless the case survived both the various row-echelon criteria and the pair-count criterion. Future research will concentrate on this improvement to the algorithm.

5.6 Summary

The program considered the case g(8) = 9 and provided a solution given by the set of blocks

1 2
1 2
2 3 5 7
2 3 6 8
2 4 5 8
2 4 6 7
1 3 4 5 6
1 3 4 7 8
1 5 6 7 8

When element 8 is deleted, the result is a set of 7 blocks covering the pairs on 7 elements (the exceptional Ryser solution, not the Fano Gemoetry).

No solutions were produced when the case is g(9) = 10.

Single solutions were obtained in the cases g(9) = 11, g(10) = 11 and g(11) = 11. The last solution is merely the familiar BIBD (11, 11, 5, 5, 2) generated by the initial block (1, 2, 3, 5, 8). The solution for 10 elements is found by deleting 1 element to leave 5 blocks of length 4 and 6 blocks of length 5. The solution for g(9) = 11 can be obtained from the BIBD by deleting two elements to leave 2 blocks of lengh 3, 6 blocks of length 4 and 3 blocks of length 5.

Chapter VI

6.1 Introduction

We begin with some general remarks taken from Allston, Buskens, and Stanton [12]. Determination of the number of non-isomorphic Balanced Incomplete Block Designs with a given parameter set (v,b,r,k,λ) is a problem of considerable importance and even greater difficulty. Probably the case that has been most studied is that of a Steiner Triple System (15, 35, 7, 3, 1); cf. Fisher [21], Mathon and Rosa [28]. It is well known that there are eighty possible solutions for this set of design parameters, and the different designs are a fruitful source of examples and constructions. Since [21], there have been various papers devoted to similar problems in Balanced Incomplete Block Designs on a small number of varieties; cf., for example, [15], [16], [24], [54].

Stanton, Kalbfleisch, and Mullin [51] discussed the more general concept of a covering design; in such a design, every variety pair occurs at least once and we normally impose a minimality condition by demanding that the cardinality of the design be as small as possible. It is clear that, in a covering design, we may have to permit the repetition of a small number of pairs in order to ensure that all pairs do appear. The analogue of the BIBD identity $bk = rv$ is the inequality

$$k\, N(t,k,v) \geq v\, N(t-1, k-1, v-1),$$

where $N(t,k,v)$ is the minimum cardinality of a family of k-sets that cover every t-set from a given set of v elements ($t \leq k \leq v$). In this chapter, we shall only be concerned with pair coverings by sets of size 4, that is, the design will contain $N(2,4,v)$ blocks. It is well known (see, for example, Mills [30], [35]) that, provided v is not contained in the set $\{7, 9, 10, 19\}$, then

$$N(2,4,v) = \lceil v \lceil (v-1)/3 \rceil /4 \rceil.$$

For various general results on covering designs, we refer to [51].

Relatively little has been done in considering the number of non-isomorphic solutions for covering designs. The case of quadruples on 9 symbols (one of the four exceptional cases) was discussed in [4]; see also [13] for a small correction. Other results for small values are given in [35] and [42]. In this paper, we wish to consider the analogue of the case discussed in Fisher's 1940 paper; we have a variety set of 15 elements, but we wish to determine the number of designs that cover all pairs (minimally) by quadruples. Since N(2,4,15) = 19, we shall have a total of 19 quadruples in the design.

In a certain sense, 15 seems to be about the break point for manageable designs. If we look at the case of triples and take v less than 15, then the discussion is relatively simple. For example, if v is equal to 4, then N =3, and the unique design may be taken as 123, 124, 134, with 3 repeated pairs. If v = 5, then N = 4, and the unique design may be taken as 123, 145, 245, 345, with 2 repeated pairs. If v = 6, then N = 6, and the unique design may be taken as 123, 145, 126, 245, 346, 356, with 3 repeated pairs. It is well known that there are unique solutions for the case v = 7 (the Fano geometry generated cyclically from the block 124) and for v = 9 (the affine geometry found by deleting points 0, 1, 3, 9, from the projective geometry on 13 points generated cyclically from the block 0139). There are two solutions for v = 13 (both are given in Marshall Hall's book or in [28]). We have already noted that there are 80 solutions for v equal to 15. For v greater than 15, the number of solutions climbs astronomically; the number for v = 19 is not known, but Stinson and Seah [64] have shown that the number of triple sytems S(2,3,19) that satisfy the additional very powerful constraint that they contain both a subsystem S(2,3,7) and a subsystem S(2,3,9) is 13,529 (the number containing a subsystem S(2,3,9) is 244,457). The total number of systems on 19 points is well into the millions (cf. [62], where 2,395,687 are found, and where it is estimated that the total number is of the order of 10^9).

6.2 The N(2,4,9) Designs

6.2.1 Introduction. The superstructure of modern design theory can be said to stem from the pioneer work of Fisher and Yates in their Statistical Tables (1963) [22]. They were among the first to stress the value of Balanced Incomplete Block Designs, and Fisher, in a fundamental paper (1940) [21], discussed the number of non-isomorphic balanced incomplete block designs with parameter set (15,35,7,3,1). Stanton, Kalbfleisch, and Mullin (1970) [51] discussed the more general concept of covering designs in which every pair appears at least once and the cardinality of the design is as small as possible. We shall here be concerned with the number N(2,4,v), which is the cardinality of a minimal design in which a set of quadruples covers all pairs from {1, 2, ..., v}; this number is well known to be equal to

$$L(v) = \left\lceil \frac{v}{4} \left\lceil \frac{v-1}{3} \right\rceil \right\rceil$$

for all values of v except $v = 7, 9, 10, 19$; see, for example, Mills (1972) [29]. For these four exceptional values, one has:

$$\begin{aligned} N(7) &= 1 + L(7) &&= 5, \\ N(9) &= 1 + L(9) &&= 8, \\ N(10) &= 1 + L(10) &&= 9, \\ N(19) &= 2 + L(19) &&= 31. \end{aligned}$$

Various general results on covering designs were given by Stanton, Kalbfleisch, and Mullin (1970) [51]. In this section, we restrict our attention to the three 2-4-v designs for which $N(v) = 1 + L(v)$, that is, we consider the exceptional values 7, 9, 10.

We begin by retracing familiar ground and considering the possibility of the equation $N(2,4,9) = 7$. If we let r_i denote the frequency of element i, then $\Sigma r_i = 28$, and we deduce that $r_9 = 4$, $r_i = 3$ $(i < 9)$. Since every pair must occur, we see that there must be one repeated pair containing i $(i \neq 9)$ and four repeated pairs containing 9. Hence a covering design may be taken as containing repeated pairs 91, 92, 93, 94, 56, 78. The design consists of two sets of blocks: set A comprises four blocks containing 9, and set B comprises three blocks not containing 9. Also, A contains two each of 1,2,3,4, and one each of 5,6,7,8. In set B, we can not have a block 1234; since there are exactly three blocks, there must be a block of the form 125X. This forces set A to take the form 91XX, 91XX, 92XX, 92XX; there is then no place for symbol 5 in set A. Thus, we see that the equation $N(2,4,9) = 7$ is impossible; hence, a covering design on nine symbols must contain at least eight blocks.

6.2.2 Covering Designs with Eight Blocks and High Frequency. We now determine all solution sets for the equation $N(2,4,9) = 8$. Clearly, the equation $bk = 32$ implies that the frequencies of the nine elements obey the equation $\Sigma r_i = 32$. Thus, we have seven cases, the unspecified r_i being all equal to 3:

Case 1. $r_9 = 8$.
Case 2. $r_9 = 7, r_8 = 4$.
Case 3. $r_9 = 6, r_8 = 5$.
Case 4. $r_9 = 6, r_8 = r_7 = 4$.
Case 5. $r_9 = r_8 = 5, r_7 = 4$.
Case 6. $r_9 = 5, r_8 = r_7 = r_6 = 4$.
Case 7. $r_9 = r_8 = r_7 = r_6 = r_5 = 4$.

In this section, we consider the first five cases. Our discussion involves a series of lemmata.

LEMMA 6.2.1. *Case 1 is impossible.*

PROOF. Each block contains the symbol 9; the other 24 symbols generate $8(3) = 24$ pairs only. But we need the 28 pairs from the remaining 8 elements.

LEMMA 6.2.2. *Case 2 is impossible.*

PROOF. In this case, the 28 pairs on 1,2,...,8, must fit into one 4-block and seven 3-blocks. But such a configuration can accommodate only $6 + 7(3) = 27$ pairs.

LEMMA 6.2.3. *Case 3 is impossible.*

PROOF. Place 9 in the first six blocks; then 8 may occur in 0, 1, or 2 of the last two blocks. In these three cases, there remains space for 20, 19, or 18 pairs from the symbols 1,2,...,7. However, space for 21 pairs is needed.

LEMMA 6.2.4. *In case 4, a single solution exists.*

PROOF. If 9 occurs in $B_1,\ldots,B_6$, and 8 occurs in B_7 and B_8, then we have room for only 20 pairs from 1,...,7.

By the symmetry of 7 and 8, we may now assume three possibilities:

(A) 7 and 8 each occur once in B_7 and B_8,

(B) 7 occurs once in B_7, 8 occurs in neither B_7 nor B_8,

(ç) 7 and 8 do not occur in B_7 and B_8.

Case (A). If 7 and 8 both occur in B_7, the pair 78 can occur 0,1,2, or 3 times in the first six blocks. The first two possibilities provide space for only 13 and 14 pairs from 1,2,...,6 but 15 are required. The third possibility gives blocks 987X, 987X, 98XX, 97XX, 9XXX, 9XXX, 78XX, XXXX, and there is exactly space for the needed 15 pairs; hence no pairs from 1,2,...,6, can be repeated, and this is impossible. Finally, the last possibility leads to blocks 987X, 987X, 987X, 9XXX, 9XXX, 9XXX, 78XX, XXXX, and we have only five places in which the six symbols 1,2,...,6, can appear with 7; this is again impossible.

Hence, we must have 7 in B_7, 8 in B_8. In order to have space for 15 pairs from 1,...,6, we must have blocks

987X, 987X, 987X, 9XXX, 9XXX, 9XXX, 7XXX, 8XXX.

These immediately lead to

9871, 9872, 9873, 9XXX, 9XXX, 9XXX, 7456, 8456.

and the schema can not be completed.

Case (B). We start from

9XXX, 9XXX, 98XX, 98XX, 98XX, 98XX, 7XXX, XXXX.

Exactly one symbol c is repeated with 8, since there are only 8 spaces for 1,2,3,...,6,7. This repeated symbol c must occur with six others, and hence has frequency 4. Hence $c = 7$, and the skeleton takes the form

97XX, 9XXX, 9871, 9872, 9834, 9856, 7XXX, XXXX.

There are only 15 places for the pairs on 1,2,...,6, and so there are no repeats; thus we may assign the eighth block to be 1235. The pairs 14 and 16 must occur together, since $r_i = 3$. Hence B_2 is 9146 or B_7 is 7146. In either case, we need pairs 24, 26, 36, 45, and these will not fit into the skeleton.

Case (C). As in Case (B), we are forced to have 7 repeated with 8, and we have the skeleton

97XX, 97XX, 9871, 9872, 9834, 9856, XXXX, XXXX.

The symbols 3, 4, 5, 6, must occur in B_1 and B_2, with 7. Hence, B_7 and B_8 must contain both 1 and 2; since all pairs occur, no other pairs from 1,...,6 can be repeated. Thus, up to an isomorphism, we can give the solution

9735, 9746, 9871, 9872, 9834, 9856, 1236, 1245.

This can be described as 9871, 9872, and the combination of 97, 98, and 12 with the 1-factors of a 1-factorization of {3,4,5,6}.

LEMMA 6.2.5. *There is only one solution in Case 5.*

PROOF. There must be 2, 3, 4, or 5 pairs 98.

If 98 occurs twice, there is only space for 20 pairs on 1,...,7. If 98 occurs thrice, let the design be

9XXX, 9XXX, 8XXX, 98XX, 98XX, 98XX, 1234.

There is no room for repeats among 1,...,7. Hence B_1, B_2, B_3, and B_4 must contain two or more symbols from 5, 6, 7. This is not possible, since only three distinct pairs can be formed from 5, 6, 7.

If 98 occurs five times, we have a skeleton

98XX, 98XX, 98XX, 98XX, 98XX, XXXX, XXXX, XXXX,

and there are two repeats among the pairs on 1,...,7. If 7 occurs four times in $B_1,\dots,B_5$, we can not get the six pairs 71,...,76. If 7 occurs three times in $B_1,\dots,B_5$, we obtain blocks

9871, 9872, 9873, 98XX, 98XX, 7456, XXXX, XXXX;

but then 9 can not occur with 4, 5, 6.

If 7 occurs once with 98, we have space for only 13 pairs from 1,...,6. Hence we are forced to the skeleton

987X, 987X, 98XX, 98XX, 98XX, 7XXX, 7XXX, XXXX.

No repeats are permitted from 1,...,6; hence B_6, B_7, B_8, can not be completed.

There remains only the possibility of four pairs 98 in a skeleton

98XX, 98XX, 98XX, 98XX, 9XXX, 8XXX, XXXX, XXXX.

There is only one repeated pair from 1,...,7. If the last two blocks have only one element in common, we write them as abcd, aefg. Then, in order to complete B_5 and B_6, we are forced to use at least two repeated pairs. Hence, we must take the last two blocks in the form abcd, abef. Then pairs ga,...,gf, must be fitted into $B_1,...,B_6$; this requires the frequency of g to be 4, and hence g = 7. So we may take the skeleton

9871, 9872, 98XX, 98XX, 9734, 8756, XXXX, XXXX.

Completion is forced as

9871, 9872, 9836, 9845, 9734, 8756, 1235, 1246.

This solution can be written as 9871, 9872, and the combination of 98, 12, and {97,87} with the 1-factorization of {3,4,5,6}.

6.2.3 The Case of a Single Frequency of 5, Three Frequencies of 4. We shall establish

LEMMA 6.2.6. *Case 6 produces five solutions.*

PROOF. Suppose that pair 98 occurs only once. Then symbols 1,2,...,7, occur in 7 triples and a pair. If the pair is not repeated, we need 20 pairs fitted into 7 triples with a specific pair ab missing; hence, the 7 triples are

aXX, aXX, aXX, bXX, bXX, bXX, XXX.

The 10 pairs on the other 5 symbols can not be accommodated. Hence all 21 pairs occur in the 7 triples and form the unique Fano geometry. So we must fit the Fano geometry into the skeleton

9XXX, 9XXX, 9XXX, 9XXX, 9876, 8XXX, 8XXX, 8XXX.

The three lines with 8 can not concur in any point save 6 or 7, or that point would not occur with 9; for a similar reason, no 3 lines concurring in 1,...,5, can occur with 9.

We may then have

Case 6A. The three lines concurring in 6 occur with 8. If we use the standard cyclic representation of the Fano geometry, we obtain

9124, 9235, 9457, 9713, 9876, 8627, 8615, 8634.

Case 6B. The three lines occuring with 8 form a triangle with 6 omitted, 7 being a vertex:

9346, 9156, 9267, 9235, 9876, 8137, 8457, 8124.

Case 6C. The three lines occuring with 8 form a triangle with 6 omitted, 7 not being a vertex:

9457, 9672, 9346, 9561, 9876, 8137, 8124, 8235.

Now suppose that 98 occurs twice, and consider the skeleton

9XXX, 9XXX, 9XXX, 98XX, 98XX, 8XXX, 8XXX, XXXX.

If an element of frequency 3 occurs in B_4, then it must occur in B_8 and one further block, which can not be B_5, and it can occur in no repeats. Hence B_4, and similarly B_5, can not contain two elements of frequency 3.

If 7 lies in both B_4 and B_5, write these blocks as 987a, 987b. It may be that a and b both have frequency 3; then we write

9XXX, 9XXX, 9XXX, 987a, 987b, 8XXX, 8XXX, abcd,

and require triples aef, bef, 7ce, 7df; but then pairs cf and de can not be placed. So we are forced to consider B4 and B5 as 9876, 987b. If the last block contains 4 elements abcd of frequency three, then we need triples 7ae, 7cd; we also need triples 6be, 6XX, 6XX, and there is no room for the missing pairs ce, de, since 6 must occur with a, c, and d. Consequently, there must be two elements d and e, of frequency three, that do not lie in B8; this forces five triples dXX, dXX, deX, eXX, eXX.

The triple deb is required; hence, the last block contains 6 and we have, omitting 9 and 8, the skeleton

dXX, dXX, deb, eXX, eXX, 6, b, b6.

The missing symbols are a and c (three times), 6 and 7 (twice). So the last block must be b6ac, and we need pairs 7a, 7c; hence we obtain

d7a, d6c, deb, e7c, e6a, 6, b, b6ac,

and the complete solution is

8a6e, 8d6c, 9876, 987b, 9d7a, 9e7c, 9deb, b6ac,

or

8d7a, 8e7c, 9876, 987b, 9d6c, 9a6e, 9deb, b6ac.

In the first case, 96 occurs only once and we have a case already obtained. The second case can be written as

<u>Solution 6D</u>.

9156, 9346, 9245, 9876, 9872, 8147, 8357, 1236.

We must still consider the possible skeleton

9XXX, 9XXX, 9XXX, 9871, 9862, 8XXX, 8XXX, 12XX.

If the two elements of frequency three are missing from B_8, they must occur in triples 3XX, 3XX, 34X, 4XX, 4XX, and this is impossible since they must occur with 1 and 2; so the last block is 1234. Then we require the triples 156, 257, 345, 367, 467, and B_6 must be 8345 in order for 8 to meet 3, 4, and 5. There is only one solution, since 3 and 4 must occur with 9 and therefore blocks 9367 and 9467 are required. The solution is

Solution 6E.

9367, 9467, 9156, 9871, 9862, 8345, 8257, 1234,

We now summarize Solutions 6A, ..., 6E, according to the frequencies of the pairs 98, 97, 96; these are A(1,1,3), B(1,2,4), C(1,3,4), D(2,2,3), E(2,3,4). So the cases are non-isomorphic.

The case (x,4,4) is impossible; so we need merely consider the cases (3,3,3) and (3,3,4). The latter case leaves only five places for symbols 1,2,...,5, in the first five blocks, and so 8 can not occur with them all, since the frequency of 98 is 4. So we consider the skeleton

9XXX, 9XXX, 98XX, 98XX, 98XX, 8XXX, XXXX, XXXX.

If 987 occurs thrice, we have $B_3,\ldots,B_7$, as

987a, 987b, 987c, 8def, 7def.

Then def must occur with 9,a,b,c, and this impossible.

If we start from

9XXX, 976X, 986X, 987X, 9876.

we obtain

9876, 9861, 9762, 9873, 9XXX, 8245, 7145, 6345,

amd 49 is missing.

If we start from

96XX, 97XX, 986X, 987X, 9876,

then we obtain

9876, 9861, 9872, 9634, 975X, 8345, 7XXX, XXXX.

It is necessary that 3 occurs with 127 and 4 must occur with 127; this is impossible.

Finally, we may have

96XX, 976X, 986X, 987X, 987X.

The X symbols in $B_2,\ldots,B_5$, are all distinct; otherwise the repeated symbol could not occur with the other 4 symbols of frequency 3. So we have blocks

96XX, 9761, 9862, 9873, 9874, 725X, 815X.

It is necessary that 5 occur in B_1 and 5 must occur with 3 and 4. So we have

965X, 9761, 9862, 9873, 9874, 7235, 815X, XXXX,

and the missing symbols are 3, 4, 4, 2, 1, 6. The missing pairs are 64, 54, 63, 41, 42, 43, 12, 13. Hence we obtain

965X, 9761, 9862, 9873, 9874, 7235, 815X, 136X,

and completion is impossible.

We have thus completed the demonstration that Case 6 gives rise to six solutions.

6.2.4 The Case of All Frequencies 3 or 4. This is the most complex situation; we prove Lemma 6.2.7.

LEMMA 6.2.7. *There are 4 cases with $r_i = 4$, for $i > 4$, in which there is a repeated pair of elements with frequencies 3.*

PROOF. We have $r_i = 3$ $(i \leq 4)$. Suppose that a pair of these elements is repeated. We thus have blocks

12ab, 12cd, 1efg, 2efg.

Thus e, f, and g have frequency 4 and may be taken as 7, 8, 9.

If 789 occurs again, then, in order for all pairs to occur involving 7, 8, 9, we need

12ab, 12cd, 1789, 2789, 789X, 7XXX, 8XXX, 9XXX.

The last 4 blocks must be 789a, 7bcd, 8bcd, 9bcd, and we see that ac never occurs.

The last 4 blocks have the form

7XXX, 8XXX, 79XX, 89XX,

then we are forced to take

7bdX, 8acX, 79ac, 89bd.

Use of the permutation (ab)(cd)(78) allows us to complete the solution to

Solution 7A.

1234, 1256, 1789, 2789, 7456, 8356, 7935, 8946.

If the last 4 blocks have the form

79XX, 89XX, 78XX, XXXX,

we obtain blocks 79ac, 89bd; then we need 78ac and 78bd, an impossibility. The only other possibility is for the last 4 blocks to be

78XX, 78XX, 9XXX, 9XXX.

The possibility 78ab, 78cd, leaves out ac, ad, bc, bd and so can not be completed save as

1234, 1256, 1789, 2789, 7834, 7856, 9356, 9456.

This is solution 7B.

The other possibility is for the blocks to be

1234, 1256, 1789, 2789, 7835, 7846, 9XXX, 9XXX.

The last 2 blocks must be 936X, 945X, and we need no other pairs.

Solution 7C. 9364, 9453.

Solution 7D. 9364, 9456.

We now must consider the case when no pair from {1,2,3,4} is repeated.

LEMMA 6.2.8. *There is one solution to Case 7 with blocks*

12XX, 13XX, 14XX, 23XX, 24XX, 34XX, XXXX, XXXX.

PROOF. Write blocks

12XX, 13XX, 14XX, 23XX, 24XX, 34XX, 5678, 5679.

Place 5, 6, 7, with distinct 1-factors to give

126X, 135X, 147X, 237X, 245X, 346X, 5678, 5679.

Then 89 does not occur; so we place 5 and 6 with the same 1-factor

12XX, 1356, 14XX, 23XX, 2456, 34XX, 5678, 5679.

We may place 7 with 14 and 23 to give B_3 and B_4 as

147X, 237X.

We then may take B_1 as 1289, and are still missing 8, 8, 9, 9; so B6 must be 3489. There is no loss of generality in completing

Solution 7E.

1289, 1356, 1478, 2379, 2456, 3489, 5678, 5679.

We thus have to consider Case 7 when the blocks have the form

123X, 1XXX, 2XXX, 3XXX, 14XX, 24XX, 34XX, XXXX.

LEMMA 6.2.9. *There is one solution of the form*

1235, 1XXX, 2XXX, 3XXX, 14XX, 24XX, 34XX, 6789.

PROOF. If 5 is the repeated symbol with 4, we obtain

1235, 1XXX, 2XXX, 3XXX, 1456, 2457, 3489, 6789.

We then obtain

1235, 1789, 2689, 3567, 1456, 2457, 3489, 6789.

But then 58 and 59 do not occur.

If 5 is not the repeated symbol with 4, we obtain

1235, 1XXX, 2XXX, 3XXX, 1456, 2467, 3489, 6789,

or

1235, 1XXX, 2XXX, 3XXX, 1456, 2478, 3479, 6789.

The first case completes to

Solution 7F. 1789, 2895, 3675,

and the second can not be completed.

LEMMA 6.2.10. *There is one solution of the form*

1236, 1XXX, 2XXX, 3XXX, 145X, 245X, 34XX, 6789.

PROOF. B_5, B_6, and B_7 can be, up to isomorphism, taken as

1456, 2457, 3489, or as 1457, 2458, 3469.

The first case gives

1236, 1789, 289X, 357X, 1456, 2457, 3489, 6789.

We are missing 58 and 59; we also miss a 5 and a 6. So we obtain

<u>Solution 7G</u>.

1236, 1789, 2895, 3567, 1456, 2457, 3489, 6789.

The second case gives

1236, 189X, 279X, 3578, 1457, 3458, 3469, 6789.

We need a 5 and a 6 as well as pairs 56 and 59, and this requirement permits no solution.

LEMMA 6.2.11. *There are no solutions of the form*

1236, 1XXX, 2XXX, 3XXX, 146X, 246X, 34XX, 6789.

PROOF. The values of B_5, B_6, and B_7 may be taken in the form

1465, 2467, 3489, or 1467, 2468, 3459.

In the first case, we find B_2, B_3, B_4, as

1789, 2589, 357X.

and this does not permit r_5 to be 5.

In the second case, we find B_2, B_3, B_4, as

1589, 2579, 3578.

This is not a solution since 56 is absent.

LEMMA 6.2.12. *There are 3 solutions of the form*

1236, 1XXX, 2XXX, 3XXX, 147X, 247X, 34XX, 6789.

PROOF. We take B_5, B_6, and B_7 to be

(1) 1475, 2476, 3489, or (2) 1475, 2478, 3469, or
(3) 1476, 2478, 3459, or (4) 1478, 2479, 3456.

From (1) we obtain B_2, B_3, B_4, as 1895, 2589, 3576,

and get

Solution 7H.

1236, 1589, 2589, 3567, 1457, 2467, 3489, 6789.

From (2) we obtain B_2, B_3, B_4, as

1895, 2596, 3578,

and get

Solution 7I.

1236, 1589, 2569, 3578, 1457, 2478, 3469, 6789.

From (3), we obtain B_2, B_3, B_4, as

1589, 2596, 3578,

and get

Solution 7J.

1236, 1589, 2569, 3578, 1467, 2478, 3459, 6789.

Finally, from (4), we obtain B_2, B_3, B_4, as

159X, 258X, 3789.

Now 7 has already occured 4 times and has not appeared with 5; so this case can not be completed.

Lemma 6.2.12 completes the discussion of the final case, and we have

THEOREM 6.2 1. *Up to isomorphism, there are 10 solutions in Case 7.*

When we combine Theorem 6.2.1 with Lemmas 6.2.1 to 6.2.6, we obtain

THEOREM 6.2.2. *Up to isomorphism, there are seventeen designs satisfying the equation $N(2,4,9) = 8$.*

We summarize the relationship between the designs given in this section and those given by Bate and Van Rees in [13].

Design in this section	Design in [13]	Permutation from Case X to [13]
Case 4	Type 1	(1 4 8 3 6 9) (2 5 7)
Case 5	Type 2	(1 4 9 2 5 8) (3 7)
Case 6A	Type 3	(1 7) (2 5 9 4 8) (3 6)
Case 6B	Type 4	(1 8 4 7 3 6 2 5 9)
Case 6C	Type 5	(1 9) (2 5 8 4 6) (3 7)
Case 6D	Type 6	(1 8 3 7 2 5 9) (4 6)
Case 6E	Type 7	(1 8 4 6 2 7 3 5 9)
Case 7A	Type 13	(7 8 9)
Case 7B	Type 14	(3 5) (4 6)
Case 7C	Type 15	(1)
Case 7D	Type 16	(3 6)
Case 7E	Type 8	(3 4) (5 7)
Case 7G	Type 9	(2 3) (7 9)
Case 7H	Type 10	(1)
Case 7I	Type 12	(1)
Case 7J	Type 11	(1)
Case 7K	Type 17	(1)

6.2.5 The Cases of 7 and 10 Varieties. For completeness, we include the relatively easy discussion of the other two cases when $N(v) = 1 + L(v)$. If we have 7 varieties, then $\Sigma r_i = 20$. Now, r_7 can not be 5, since that would leave space for 15 pairs on 1, ..., 6, and no repeats would be allowed; but there is no triple system on 6 elements. If $r_7 = 4$, we consider the case when $r_6 = 4$. The skeleton containing only 3 pairs 76 gives insufficient space for the 10 pairs from 1,...,5; so we must take 4 pairs 76, and immediately obtain the design

7651, 7652, 7653, 7654, 1234,

by assigning the last block as 1234.

Now, suppose that $r_7 = 4$ and that no other frequency is 4, that is, the frequencies are $r_6 = r_5 = r_4 = r_3 = 3$, $r_2 = r_1 = 2$. Then 1 and 2 can occur only once each with 7, and we are constrained to the skeleton

71XX, 72XX, 7XXX, 7XXX, 12XX.

Let the last block be 1234; then the design is forced to be

7156, 7256, 7345, 7346, 1234.

Finally, let the frequencies be $r_1 = 2$, $r_i = 3$ $(i > 1)$. then the first two blocks may be written 1234, 1567. If no other block intersects either of these in three elements, we obtain the design

1234, 1567, 2356, 2457, 3467.

If there is a block with a triple intersection, we obtain the last three blocks as 2345, 67XX, 67XX, and the design is completeable to

1234, 1567, 2345, 2567, 3467.

Thus we have

THEOREM 6.2.3. *There are four non-isomorphic covering designs on seven elements.*

We now consider the case of ten varieties in a design with nine blocks. Suppose that the maximal frequency of a variety is x; then $x(3) + (9-x)6 \geq 36$, and we deduce

LEMMA 6.2.13. *The possible frequencies in a 2-4-10 covering design are 3,4,5,6.*

If there are two frequencies of 6, then the other eight symbols generate 28 pairs. But the two elements of frequency 6, even if they always occur together, leave space for at most 24 pairs. Thus we have

LEMMA 6.2.14. *The case $r_{10} = r_9 = 6$, $r_i = 3$ $(i < 9)$ is impossible.*

Now let $r_{10} = 6$, $r_9 = 5$, $r_8 = 4$, $r_i = 3$ $(i < 8)$. Even if 9 and 10 always occur together, we can accommodate at most 26 pairs from 1,...,8; hence, we obtain

LEMMA 6.2.15. *The case $r_{10} = 6$, $r_9 = 5$, $r_8 = 4$, $r_i = 3$ $(i < 8)$ is impossible.*

Finally, let $r_{10} = 6$, $r_9 = r_8 = r_7 = 4$, $r_i = 3$ $(i < 6)$. It is possible to have space for 28 pairs in a skeleton with (use T = 10)

T9XX (4 times), TXXX (twice), XXXX (thrice).

But then the four 8s must also appear in the first 6 blocks, and this is impossible since the symbols 1,2,3,...,8, complete B_1, B_2, B_3, B_4. Thus, we have

LEMMA 6.2.16. *Only frequencies 3,4,5 are possible in a 2-4-10 design.*

If we suppose there are a elements of frequency 5, b of frequency 4, c of frequency 3, then

$$a + b + c = 10,$$
$$5a + 4b + 3c = 36.$$

then $2a + b = 6$, and the only solutions (a,b,c) are (0,6,4), (1,4,5), (2,2,6), (3,0,7).

Take $r_1 = r_2 = 3$. Then we have blocks (use T = 10)

1234, 1567, 189T, 2XXX, 2XXX.

If B_4 and B_5 are 2567, 289T, then the frequencies of 5,...,T, are all forced to be at least 4. So we are dealing with the (0,6,4) case, and the remaining blocks are

3XXX, 3XXX, 4XXX, 4XXX.

Completion is impossible, and so we may take the first 5 blocks as

1234, 1567, 189T, 2568, 279T.

Again, $r_5 = r_6 = r_9 = r_T \geq 4$. Also, if $r_8 = 3$, we need the block 8347; then completion is uniquely possible to give the design

1234, 1567, 189T, 2568, 279T, 8347, 3456, 359T, 469T.

On the other hand, if $r_8 \neq 3$, we have r_7 and r_8 both ≥ 4. It follows that $a = 0$, $b = 6$, $c = 4$, and we have $r_3 = r_4 = 3$; again, the last four blocks must be

3XXX, 3XXX, 4XXX, 4XXX,

and we have to use symbols 5,6,7,8,9,T, twice. Also we need pairs 59, 5T, 69, 6T, 78. Since symbols 3 and 4 are equivalent, as are 5 and 6, and 9 and T, we may take the blocks as:

<u>Case 1</u>. 3596, 378T, 456T, 4789;

<u>Case 2</u>. 359X, 36XX, 469X, 45XX.

The second case then must be completed as

359T, 3678, 469T, 4578.

Cases 1 and 2 are isomorphic via the permutation (78)(59)(6T).

We thus have proved

THEOREM 6.2.4. *There are two 2-4-10 covering designs.*

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Appendix A

This appendix contains the FORTRAN source code for the algorithms described in Chapter 5. There are two parts to this appendix. The first part is the file COMMON.INC which contains the global variables used in the program. The second part is the actual FORTRAN program (BICOVER.FOR) itself.

COMMON.INC

```
      Implicit    None

      Integer*2   MAX_SIZE, LOGFILE
      Parameter   (MAX_SIZE = 50)
      Parameter   (LOGFILE = 10)
      Real*8            TOLERANCE
      Parameter   (TOLERANCE = 1.0d-6)

      Integer*2   V,                  ! Number of elements
     1            NUM_BLOCKS,         ! Number of blocks
     1            V_MINUS_1,          ! Oft used constant
     1            TWO_V_CHOOSE_2,           !  "    "        "
     1            TWO_V_MINUS_1,            !  "    "        "
     1            B(MAX_SIZE),              ! Block Vector
     1            A(MAX_SIZE,MAX_SIZE),     ! Distribution vectors
     1            PAIR_TABLE(2*MAX_SIZE,2*MAX_SIZE), ! Pair count
     1            NUM_CASES,          ! Case number of block vector
     1            START_CASE,         ! case number to start at
     1            B_PTR(MAX_SIZE),    ! Pointer and multiplier for B
     1            N_CHOOSE_2(2:MAX_SIZE), ! Used frequently
     1            TWO_N_CHOOSE_2(2:MAX_SIZE),
     1            ROWS_A, COLS_A,           ! # of rows and columns
     1            ROWS_EQN, COLS_EQN        ! # of rows and columns

      Real*8            EQN(MAX_SIZE,2*MAX_SIZE)
      Integer*2   COL_PTR(2*MAX_SIZE)

      Integer*2   IEQN(MAX_SIZE,2*MAX_SIZE),
     1            SOLUTIONS(20*MAX_SIZE, 2*MAX_SIZE),
     1            NUM_SOLUTIONS,
     1            UPPER(2*MAX_SIZE),
     1            CTR(2*MAX_SIZE),
     1            CTR_POS

      Common V, NUM_BLOCKS, V_MINUS_1, TWO_V_CHOOSE_2, TWO_V_MINUS_1,
     1      B, A,PAIR_TABLE, NUM_CASES,START_CASE, B_PTR, N_CHOOSE_2,
     1      TWO_N_CHOOSE_2, ROWS_A, COLS_A, ROWS_EQN, COLS_EQN,
     1      EQN, IEQN, SOLUTIONS, NUM_SOLUTIONS, UPPER,
     1      CTR, CTR_POS, COL_PTR
```

BICOVER.FOR

```
      Program BICOVER

!     This program implements the algorithms described in Chapter 5.

!     This program was designed to run on a DEC VAX computer.  With
!     minor modifications it can run on any computer with a FORTRAN
!     77 compiler.

      Include 'COMMON.INC'

      Integer*2   I                         ! A loop counter
      Character*23      TOD                 ! Time of day

      Accept *, V, NUM_BLOCKS, START_CASE ! Read in start up info

!     Calculate some useful constants

      N_CHOOSE_2(2) = 1
      TWO_N_CHOOSE_2(2) = 2
      Do I = 3, MAX_SIZE
          N_CHOOSE_2(I) = N_CHOOSE_2(I-1) + I - 1
          TWO_N_CHOOSE_2(I) = 2 * N_CHOOSE_2(I)
      End Do

      V_MINUS_1 = V - 1
      TWO_V_MINUS_1 = 2 * V_MINUS_1
      TWO_V_CHOOSE_2 = 2 * N_CHOOSE_2(V)

!     Open output log and dislpay run information

      Open (Unit=LOGFILE, File='G.OUT', Status='NEW',
      1     Carriagecontrol='LIST', Recordsize=511)

      Call LIB$DATE_TIME(TOD)       ! Get time of day

      Write (LOGFILE,10) V, NUM_BLOCKS, TOD
10    Format('Beginning G(',I2,') = ', I2, ' at ',A)

!     Do the actual processing

      Call BLOCK_VECTORS

      Call LIB$DATE_TIME(TOD)

      Write (LOGFILE,20) TOD
20    Format(//'Finished at ',A)

      Close (Unit=LOGFILE)

      End
```

```
      Subroutine BLOCK_VECTORS

!     This routine generates all possible block vectors, and, one at
!     a time, processes them

      Include 'COMMON.INC'

      Integer*2    PTR, BPTR, SUM, I
      Logical*1    KEEP_GOING

!     The B vector contains the block lengths.  Initially, we have a
!     block of length v - 1

      PTR = V_MINUS_1
      B(PTR) = TWO_V_CHOOSE_2 / N_CHOOSE_2(V_MINUS_1)

      KEEP_GOING = .true.
      Do While (KEEP_GOING)

!         Generate upper bounds based on the value of the current
!         position in the upper bound array (B)

          Do BPTR = PTR-1, 3, -1
            SUM = 0
            Do I = BPTR+1, V_MINUS_1
                SUM = SUM + B(I) * N_CHOOSE_2(I)
            End Do
            B(BPTR) = (TWO_V_CHOOSE_2 - SUM) / N_CHOOSE_2(BPTR)
          End Do

!         Assume the rest of the blocks are pairs

          SUM = 0
          Do I = 3, V_MINUS_1
            SUM = SUM + B(I)
          End Do
          B(2) = NUM_BLOCKS - SUM

!         If the number of blocks is correct, check the pair count

          If (B(2) .ge. 0) Then
            SUM = 0
            Do I = 2, V_MINUS_1
                SUM = SUM + B(I) * N_CHOOSE_2(I)
            End Do
            If (SUM .eq. TWO_V_CHOOSE_2) Then
                Call PROCESS_BLOCK
            End If
          End If

!         Recalculate the upper bounds

          B(2) = 0        ! Clear out number of pairs
          PTR = 3
          Do While (PTR .le. V_MINUS_1 .and. B(PTR) .eq. 0)
            PTR = PTR + 1
          End Do
```

```
    If (PTR .le. V_MINUS_1) Then
      B(PTR) = B(PTR) - 1
    Else
      KEEP_GOING = .false.
    End If
End Do

Return
End
```

```
      Options /Check=Bounds
      Subroutine PROCESS_BLOCK

!     This routine processes a block vector.  At this point, the "B"
!     vector has the correct number of blocks and the correct pair
!     count.  The distribution vectors ("A") will be generated.

      Include 'COMMON.INC'

      Integer*2    I, J, BOUND(MAX_SIZE), SIZE_BOUND, SUM
      Logical*1    KEEP_GOING
      Character*4  C_CASE

!     Display the block vector

      NUM_CASES = NUM_CASES + 1

      If (NUM_CASES .lt. START_CASE) Return

      Write(LOGFILE, 10) NUM_CASES, (I, I=2,V_MINUS_1),
     1      (B(I),I=2,V_MINUS_1)
10    Format(//50('-')/'Case',I5//<V-2>('B',I2,3X)/
     1      <6*(V-2)-3>('-')/
     1      <V-2>(I3,3X))

!     SYS$SETPRN is VAX specific and can be removed without any
!     adverse side effects.  SYS$SETPRN changes the process name
!     so I can monitor the progress of the execution of the program.

      Write (C_CASE,'(I4)') NUM_CASES
      Call SYS$SETPRN ('Case ' // C_CASE)

!     Generate upper bounds and process non-zero elements

      SIZE_BOUND = 0
      Do I = 2, V_MINUS_1
          If (B(I) .ne. 0) Then
            SIZE_BOUND = SIZE_BOUND + 1
            BOUND(SIZE_BOUND) = B(I)
            B_PTR(SIZE_BOUND) = I - 1
          End If
      End Do

!     Using the upper bounds just generated, generate all possible
!     distribution vectors.  The "A" matrix will contain the
!     distribution vectors.  Initially, A is empty.

      ROWS_A = 0
      COLS_A = SIZE_BOUND

      KEEP_GOING = .true.
      Do While (KEEP_GOING)

!         Sum over (I-1) * BOUND(I)

          SUM = 0
          Do I = 1, SIZE_BOUND
```

```
              SUM = SUM + BOUND(I) * B_PTR(I)
            End Do

!           If SUM is equal to 2(V-1), we have a distribution vector.
!           Copy it to the "A" matrix.

            If (SUM .eq. TWO_V_MINUS_1) Then
              ROWS_A = ROWS_A + 1
              Do I = 1, COLS_A
                  A(ROWS_A, I) = BOUND(I)
              End Do
            End If

!           We have to adjust the upper bounds. Find the first non-zero
!           upper bound and decrement it.  Then set all previous lower
!           bounds back to their original values.

            I = 1
            Do While (I .le. SIZE_BOUND .and. BOUND(I) .eq. 0)
              I = I + 1
            End Do
            If (I .le. SIZE_BOUND) Then
              BOUND(I) = BOUND(I) - 1
              Do J = I-1, 1, -1
                  BOUND(J) = B(B_PTR(J)+1)
              End Do
            Else
              KEEP_GOING = .false.
            End If
        End Do

!       Display the distribution vectors

        Write (LOGFILE, 20) (B_PTR(I)+1, I = 1, SIZE_BOUND),
        1      ((A(I,J),J=1,COLS_A), I=1,ROWS_A)
20      Format(//'Distribution Vectors'//
        1      <SIZE_BOUND>('A',I2,3X)/<6*SIZE_BOUND-3>('-')/
        1      (<COLS_A>(I3,3X)))

        If (ROWS_A .lt. 1) Then
            Write (LOGFILE, 30)
30          Format (/'No distribution vectors can be generated.')
        Else

!           For each block vector, generate the pair count, the
!           equations derived from the distribution vectors, put the
!           equations in row echelon form and attempt to solve them.

            Call GENERATE_PAIR_COUNT
            Call GENERATE_EQUATIONS
            Call ROW_ECHELON
            Call PROCESS_EQUATIONS
        End If

        Return
        End
```

```
      Subroutine GENERATE_PAIR_COUNT

!     This routine generates the pair count table.

      Include 'COMMON.INC'

      Integer*2   COMPRESSED_B(MAX_SIZE), I, J, K, SUM, COUNT

!     Remove the zero elements from the block vector

      I = 0
      Do J = 2, V_MINUS_1
          If (B(J) .ne. 0) Then
            I = I + 1
            COMPRESSED_B(I) = B(J)
          End If
      End Do

!     Generate pair table

!     PTij = SUM (Aik + Ajk - # blocks of length kl)

      Do I = 1, ROWS_A
          DO J = I, ROWS_A
            SUM = 0
            Do K = 1, COLS_A
                COUNT = A(I,K) + A(J,K) - COMPRESSED_B(K)
                If (COUNT .gt. 0) Then
                  SUM = SUM + COUNT
                End If
            End Do
            PAIR_TABLE(I,J) = SUM
            PAIR_TABLE(J,I) = SUM
          End Do
      End Do

      Write (LOGFILE, 10) ((PAIR_TABLE(I,J),J=1,ROWS_A), I=1,ROWS_A)
10    Format(//'Pair Table'//(<ROWS_A>I3))

      Return
      End
```

```
      Options /Check=Bounds
      Subroutine GENERATE_EQUATIONS

!     This routine generates the equations to be solved, from the
!     distribution vectors.

      Include 'COMMON.INC'

      Integer*2 I, J, COL_POS

      Do I = 1, COLS_A
          COL_POS = COLS_A + 1 - I
          Do J = 1, ROWS_A
            EQN(I,J) = A(J, COL_POS)
          End Do
          EQN(I, ROWS_A+1) = B(B_PTR(COL_POS)+1) * (B_PTR(COL_POS)+1)
      End Do

      ROWS_EQN = COLS_A
      COLS_EQN = ROWS_A + 1

      Write (LOGFILE, 10)
10    Format (//'Original System of Equations'/)

      Call PRINT_EQUATIONS

      Return
      End
```

```
      Subroutine PRINT_EQUATIONS

!     This routine prints out the equations

      Include 'COMMON.INC'

      Integer*2 I, J

      Write (LOGFILE, 10) ((EQN(I,J),J=1,COLS_EQN),I=1,ROWS_EQN)
10    Format (<COLS_EQN-1>(F6.2),' = ',F6.2)

      Return
      End
```

```
      Subroutine ROW_ECHELON

!     This routine places the equations in row echelon form.

      Include 'COMMON.INC'

      Integer*2 I, J, POSN, K, T_ROWS_EQN
      Real*8 MAX_ELEMENT, Dabs, T, MULT

!     Initialize pointer vectors

      Do I = 1, COLS_EQN-1
          COL_PTR(I) = I
      End Do

!     Work with the minimum number of rows possible

      If (ROWS_EQN .lt. COLS_EQN) Then
          T_ROWS_EQN = ROWS_EQN
      Else
          T_ROWS_EQN = COLS_EQN - 1
      End If

!     Process one row at a time

      Do I = 1, T_ROWS_EQN

!         Find pivot element

          POSN = I
          MAX_ELEMENT = Dabs(EQN(I,I))
          Do J = I+1, ROWS_EQN
            If (Dabs(EQN(J,I)) .gt. MAX_ELEMENT) Then
                POSN = J
                MAX_ELEMENT = Dabs(EQN(J,I))
            End If
          End Do

!         Get row containing pivotal element to the current row

          If (POSN .ne. I) Then
            Do J = I, COLS_EQN
                T = EQN(I,J)
                EQN(I,J) = EQN(POSN,J)
                EQN(POSN,J) = T
            End Do

          End If

!         If there is a zero as the pivotal element, try and find a
!         non-zero element on the current row

          If (MAX_ELEMENT .eq. 0.0d0) Then
            POSN = I
            Do J = I+1, COLS_EQN-1
                If (Dabs(EQN(I,J)) .gt. MAX_ELEMENT) Then
                  POSN = J
```

```
                MAX_ELEMENT = Dabs(EQN(I,J))
              End If
          End Do

!         If we found a non-zero element, switch the columns in
!         the equation and keep track of this switch in the
!         pointer vector

          If (POSN .ne. I) Then
              Do J = 1, ROWS_EQN
                T = EQN(J,I)
                EQN(J,I) = EQN(J,POSN)
                EQN(J,POSN) = T
              End Do

!             Switch column pointer vector

              J = COL_PTR(I)
              COL_PTR(I) = COL_PTR(POSN)
              COL_PTR(POSN) = J
          End If
        End If

!       Normalize the current row

        If (Dabs(EQN(I,I)) .le. TOLERANCE) EQN(I,I) = 0.0d0

        If (EQN(I,I) .ne. 1.0d0 .and. EQN(I,I) .ne. 0.0d0) Then
          Do J = COLS_EQN, I, -1
              EQN(I,J) = EQN(I,J) / EQN(I,I)
              If (Dabs(EQN(I,J)) .le. TOLERANCE) EQN(I,J) = 0.0d0
          End Do
        End If

!       Zero rows below the current row

        Do J = I+1, ROWS_EQN
          If (EQN(J,I) .ne. 0.0d0 .and. EQN(I,I) .ne. 0.0d0) Then
              MULT = EQN(J,I) / EQN(I,I)
              Do K = I, COLS_EQN
                EQN(J,K) = EQN(J,K) - MULT * EQN(I,K)
                If (Dabs(EQN(J,K)) .le. TOLERANCE) EQN(J,K) = 0.0d0
              End Do
          End If
        End Do

!       Zero rows above the current row

        Do J = I-1, 1, -1
          If (EQN(J,I) .ne. 0.0d0 .and. EQN(I,I) .ne. 0.0d0) Then
              MULT = EQN(J,I) / EQN(I,I)
              Do K = I, COLS_EQN
                EQN(J,K) = EQN(J,K) - MULT * EQN(I,K)
                If (Dabs(EQN(J,K)) .le. TOLERANCE) EQN(J,K) = 0.0d0
              End Do
          End If
        End Do
    End Do
```

```
      Write (LOGFILE, 10)
10    Format(//'Equations in Row Echelon Form'/)

      Call PRINT_EQUATIONS

      Write (LOGFILE, 20) (COL_PTR(I), I=1, COLS_EQN-1)
20    Format('Column pointer vector: ',<COLS_EQN-1>I3)

!     Clean up (remove any accumulated round off error) the equations

      Call CLEANUP_EQUATIONS

      Return
      End
```

```
      Subroutine PROCESS_EQUATIONS

!     This routine attempts to reject the system of equations based
!     on the following three criteria
!
!     LHS = 0                    RHS != 0
!     LHS all != 0               RHS < 0
!     LHS all integral           RHS not integral
!
!     If the equations can not be rejected, generate all possible
!     solutions

      Include 'COMMON.INC.'

      Logical*1    LHS_EQ_0_RHS_NE_0,
      1            LHS_GE_0_RHS_LT_0,
      1            LHS_INT_RHS_NOT_INT,
      1            FAILED

      FAILED = .true.
      If (.not. LHS_EQ_0_RHS_NE_0() ) Then
          If (.not. LHS_GE_0_RHS_LT_0() ) Then
            If (.not. LHS_INT_RHS_NOT_INT() ) Then
                Call INTEGERIZE_EQUATIONS
                Call SOLVE_EQUATIONS
                FAILED = .false.
            End If
          End If
      End If

      If (FAILED) Then
          Write (LOGFILE,10)
10        Format('*** Block vector rejected')
      End If

      Return
      End
```

```fortran
      Logical Function LHS_EQ_0_RHS_NE_0*1 ()

!     This function examines the equations in EQN for any rows that
!     have LHS coefficients equal to 0 and RHS not equal to 0.

      Include 'COMMON.INC'

      Logical*1   LHS_EQ_0
      Integer*2   I, J

!     Process equations starting with the last row since this row
!     will be the first fail this check.

      LHS_EQ_0_RHS_NE_0 = .false.
      I = ROWS_EQN

      Do While (I.ge.1 .and. .not.LHS_EQ_0_RHS_NE_0)
          LHS_EQ_0 = .true.
          J = 1
          Do While (J.lt.COLS_EQN .and. LHS_EQ_0)
            LHS_EQ_0 = EQN(I,J) .eq. 0.0d0
            J = J + 1
          End Do
          LHS_EQ_0_RHS_NE_0 = LHS_EQ_0.and.EQN(I,COLS_EQN) .ne. 0.0d0
          I = I - 1
      End Do

      If (LHS_EQ_0_RHS_NE_0) Then
          Write (LOGFILE, 10) I+1
10        Format(/'Equation',I3,' has LHS = 0 and RHS != 0')
      End If

      Return
      End
```

```
      Logical Function LHS_GE_0_RHS_LT_0*1 ()

!     This function examines the equations for a row whose LHS
!     coefficients are all >= 0 and whose RHS < 0.

      Include 'COMMON.INC'

      Logical*1   LHS_GE_0
      Integer*2   I, J

      LHS_GE_0_RHS_LT_0 = .false.
      I = ROWS_EQN

      Do While (I.ge.1 .and. .not.LHS_GE_0_RHS_LT_0)
          LHS_GE_0 = .true.
          J = 1
          Do While (J.lt.COLS_EQN .and. LHS_GE_0)
            LHS_GE_0 = EQN(I,J) .ge. 0.0d0
            J = J + 1
          End Do
          LHS_GE_0_RHS_LT_0 = LHS_GE_0.and.EQN(I,COLS_EQN) .lt. 0.0d0
          I = I - 1
      End Do

      If (LHS_GE_0_RHS_LT_0) Then
          Write (LOGFILE, 10) I + 1
10        Format(/'Equation',I3,' has LHS >= 0, RHS < 0')
      End If

      Return
      End
```

```fortran
      Logical Function LHS_INT_RHS_NOT_INT*1 ()

!     This routine examines the equations for a row whose LHS
!     coefficients are all integral but RHS is not.

      Include 'COMMON.INC'

      Logical*1   LHS_INT
      Integer*2   I, J
      Real*8            Dint

      LHS_INT_RHS_NOT_INT = .false.
      I = ROWS_EQN

      Do While (I.ge.1 .and. .not.LHS_INT_RHS_NOT_INT)
          LHS_INT = .true.
          J = 1
          Do While (J.lt.COLS_EQN .and. LHS_INT)
            LHS_INT = EQN(I,J) .eq. Dint(EQN(I,J))
            J = J + 1
          End Do
          LHS_INT_RHS_NOT_INT = LHS_INT .and.
     1      (EQN(I,COLS_EQN) .ne. Dint(EQN(I,COLS_EQN)))
          I = I - 1
      End Do

      If (LHS_INT_RHS_NOT_INT) Then
          Write(LOGFILE, 10) I + 1
10        Format(/'Equation',I3,' has LHS integral,RHS non integral')
      End If

      Return
      End
```

```
      Subroutine CLEANUP_EQUATIONS

!     This routine will get rid of any accumulated round off error

      Include 'COMMON.INC'

      Integer*2   I, J
      Integer*4   Jidint

      Real*8            CONST, ROUND_AMOUNT
      Parameter   (CONST = 10000.0d0)

      Do I = 1, ROWS_EQN
          Do J = 1, COLS_EQN
            ROUND_AMOUNT = 0.5d0/CONST
            If (EQN(I,J) .lt. 0.0d0) ROUND_AMOUNT = - ROUND_AMOUNT
            EQN(I,J) = Jidint((EQN(I,J)+ROUND_AMOUNT)* CONST) / CONST
          End Do
      End Do

      Return
      End
```

```
      Subroutine INTEGERIZE_EQUATIONS

!     This routine will copy the system of equations (currently in
!     double precision) to an equivalent system (in integer).  This
!     is done so that integer solutions to the system can be
!     obtained.  If the equations have non integral coefficients,
!     find the 'greatest common divisior' and divide through the
!     offending equation.
!
!     Also, this routine removes any equations that have been
!     reduced to all zeroes.

      Include 'COMMON.INC'

      Integer*2 I, J, NUM_DELETED
      Real*8 GCD_LIST(2*MAX_SIZE), GCD, T, Dabs
      Logical*1 ALL_ZERO

!     Process one equation at a time

      Do I = 1, ROWS_EQN

!         Set up parameter list for GCD routine

          Do J = 1, COLS_EQN
            GCD_LIST(J) = Dabs(EQN(I,J))
          End Do

          T = GCD (GCD_LIST, COLS_EQN)
          If (T .ne. 1.0d0) Then
            Do J = 1, COLS_EQN
                IEQN(I,J) = EQN(I,J) / T
            End Do
          Else
            Do J = 1, COLS_EQN
                IEQN(I,J) = EQN(I,J)
            End Do
          End If
      End Do

!     Remove any all zero rows

      NUM_DELETED = 0
      ALL_ZERO = .true.
      Do While (ALL_ZERO)
          J = 1
          Do While (J .le. COLS_EQN .and. ALL_ZERO)
            ALL_ZERO = IEQN(ROWS_EQN,J) .eq. 0
            J = J + 1
          End Do
          If (ALL_ZERO) Then
            ROWS_EQN = ROWS_EQN - 1
            NUM_DELETED = NUM_DELETED + 1
          End If
      End Do

      Write (LOGFILE, 10)
```

```
10      Format(/'Equations to be solved'//)

        Call PRINT_IEQUATIONS

        If (NUM_DELETED .ne. 0) Then
            Write (LOGFILE, 20) NUM_DELETED
20          Format(/'There were', I3,' equations deleted.')
        End If

        Return
        End
```

```
      Subroutine PRINT_IEQUATIONS

!     This routine prints out the integer form of the equations

      Include 'COMMON.INC'

      Integer*2 I, J

      Write (LOGFILE, 10) ((IEQN(I,J),J=1,COLS_EQN),I=1,ROWS_EQN)
10    Format (<COLS_EQN-1>(I4),' = ',I4)

      Return
      End
```

```
      Real Function GCD*8 (LIST, N)

!     This function will find the GCD of a LIST of N elements

      Implicit None

      Integer*2 N
      Real*8 LIST(N)

      Real*8 T
      Integer*2 START, I, J

      START = 1
      Do While (START .lt. N)

!         Sort LIST

          Do I = START, N-1
            Do J = I+1, N
                If (LIST(J) .lt. LIST(I)) Then
                  T = LIST(I)
                  LIST(I) = LIST(J)
                  LIST(J) = T
                End If
            End Do
          End Do

!         Find first non-zero entry

          Do While (LIST(START) .eq. 0.0D0 .and. START .le. N)
            START = START + 1
          End Do

!         If the row is all zero, exit with a GCD of 1

          If (START .gt. N) Then
            GCD = 1
            Return
          End If

!         Subtract smallest element from rest

          Do I = START+1, N
            LIST(I) = LIST(I) - LIST(START)
            If (LIST(I) .le. 1.0D-5) Then
                LIST(I) = 0.0D0
            End If
          End Do

      End Do

      GCD = LIST(N)

      Return
      End
```

```
      Subroutine SOLVE_EQUATIONS

!     This routine will generate all possible solutions to the
!     equations.

      Include 'COMMON.INC'

      Integer*2 I
      Logical *1 EQN_OK

!     Process trivial case first.  Same number of equations and
!     variables to solve for.  Also ensure that this solution obeys
!     the restrictions imposed by the pair count table.

      If (ROWS_EQN .eq. COLS_EQN-1) Then
          NUM_SOLUTIONS = 1
          Do I = 1, COLS_EQN-1
            SOLUTIONS(1,I) = IEQN(I,COLS_EQN) / IEQN(I,I)
          End Do
      Else
          NUM_SOLUTIONS = 0
          Call CALCULATE_UPPER_BOUNDS

!         Set up initial values for solutions

          Do I = 1, COLS_EQN-1
            CTR(I) = UPPER(I)
          End Do
          CTR_POS = COLS_EQN - 1

!         Keep trying until all counters have reached zero

          Do While (CTR_POS .ge. ROWS_EQN)
            If (EQN_OK(ROWS_EQN)) Then
                Call CHECK_REST_OF_SYSTEM
            End If

!           Decrement counter(s)

            If (CTR(CTR_POS) .ne. 0) Then
                CTR(CTR_POS) = CTR(CTR_POS) - 1
            Else
                Do While (CTR_POS.ge.ROWS_EQN.and. CTR(CTR_POS).eq.0)
                  CTR_POS = CTR_POS - 1
                End Do
                If (CTR_POS .ge. ROWS_EQN) Then
                  CTR(CTR_POS) = CTR(CTR_POS)  -  1
                  Do CTR_POS  = CTR_POS+1, COLS_EQN-1
                      CTR(CTR_POS) = UPPER(CTR_POS)
                  End Do
                  CTR_POS = COLS_EQN - 1
                End If
            End If
          End Do
      End If

      Call PRINT_SOLUTIONS
```

```
Return
End
```

```fortran
      Subroutine PRINT_SOLUTIONS

!     This routine prints out all possible solutions to the system
!     of equations.  As well, the possible solutions are checked with
!     the pair table count.

      Include 'COMMON.INC'

      Integer*2 I,J, K, NUM_GOOD, GOOD(MAX_SIZE)
      Logical*1 BAD

      Write (LOGFILE,10) NUM_SOLUTIONS
10    Format(//'The number of possible solutions is:',I3/)

      NUM_GOOD = 0
      Do I = 1, NUM_SOLUTIONS
          Write (LOGFILE,20) I, (SOLUTIONS(I,J),J=1,COLS_EQN-1)
20        Format (I3,') ',<COLS_EQN-1>(I4))

!         Check solution against pair count table

          BAD = .false.
          J = 1
          Do While (J.le.COLS_EQN-1 .and. .not. BAD)
            K = 1
            Do While (K.le.COLS_EQN-1 .and. .not. BAD)
                If (J. ne. K) Then
                    BAD = SOLUTIONS(I,J) .ge. 1 .and.
     1              SOLUTIONS(I,K) .ge. 1 .and.
     1              PAIR_TABLE(COL_PTR(J),COL_PTR(K)) .gt. 2
                Else
                    BAD = SOLUTIONS(I,J) .ge. 2 .and.
     1              PAIR_TABLE(COL_PTR(J),COL_PTR(J)) .gt. 2
                End If
                If (BAD) Then
                  Write (LOGFILE, 30) J, K,
     1              PAIR_TABLE (COL_PTR(J),COL_PTR(K))

30                Format ('Solution rejected.',I3,' and',I3,' occur',
     1            I3, ' times together')
                Else
                  K = K + 1
                End If
            End Do
            J = J + 1
          End Do

          If (.not.BAD) Then
            NUM_GOOD = NUM_GOOD + 1
            GOOD(NUM_GOOD) = I
          End If

      End Do

      If (NUM_GOOD .gt. 0) Then
          Write (LOGFILE, 40) (GOOD(I),I=1,NUM_GOOD)
40        Format(/'Check equations: ',<NUM_GOOD>I3)
```

```
End If

Return
End
```

```
      Subroutine CALCULATE_UPPER_BOUNDS

!     This routine attempts to calculate some upper bounds for
!     solving the Diophantine system of equations.

      Include 'COMMON.INC'

      Integer*2 I, J, T
      Logical*1 ALL_ZERO, ALL_POS

!     Assume the worst

      Do I = 1, COLS_EQN-1
          UPPER(I) = V
      End Do

!     Only element on row is on diagonal

      Do I = 1, ROWS_EQN
          ALL_ZERO = .true.
          J = I + 1
          Do While (J .lt. COLS_EQN .and. ALL_ZERO)
            ALL_ZERO = IEQN(I,J) .eq. 0
            J = J + 1
          End Do
          If (ALL_ZERO) Then
            UPPER(I) = IEQN(I,COLS_EQN)
          End If
      End Do

!     All elements are positive, then upper bound is RHS / element

      Do I = 1, ROWS_EQN
          If (UPPER(I) .eq. V) Then
            ALL_POS = .true.
            J = I + 1
            Do While (J .lt. COLS_EQN .and. ALL_POS)
                ALL_POS = IEQN(I,J) .ge. 0
                J = J + 1
            End Do
            If (ALL_POS) Then

!               Set bounds for the entire equation, if there is an
!               improvement

                Do J = I, COLS_EQN-1
                  If (IEQN(I,J) .ne. 0) Then
                      T = IEQN(I,COLS_EQN) / IEQN(I,J)
                      If (T .lt. UPPER(J)) Then
                        UPPER(J) = T
                      End If
                  End If
                End Do
            End If
          End If
      End Do
```

```
!     Use pair count table.  If any element on the pair count table
!     diagonal is > 2, set the corresponding upper bound to 1.

      Do I = 1, COLS_EQN-1
          If (PAIR_TABLE(COL_PTR(I),COL_PTR(I)) .gt. 2) Then
            UPPER(I) = 1
          End If
      End Do

      Write (LOGFILE, 10) (UPPER(I), I=1,COLS_EQN-1)
10    Format(/'Upper bounds: ',<COLS_EQN-1>(I3))

      Return
      End
```

```
      Logical Function EQN_OK*1 (ROW_NUM)

!     This function ensures that the possible solution being tried
!     passes the equation specified in ROW_NUM

      Include 'COMMON.INC'

      Integer*2 ROW_NUM

      Integer*2 I, SUM

      SUM = 0
      Do I = ROW_NUM, COLS_EQN-1
          SUM = SUM + IEQN(ROW_NUM,I) * CTR(I)
      End Do

      EQN_OK = SUM .eq. IEQN(ROW_NUM, COLS_EQN)

      Return
      End
```

```
      Subroutine CHECK_REST_OF_SYSTEM

!     This routines checks the rest of the system of equations for
!     validity based on the current counter values.

      Include 'COMMON.INC'

      Integer*2 I, J, SUM
      Logical*1 SYSTEM_OK, EQN_OK

      SYSTEM_OK = .true.
      I = ROWS_EQN - 1
      Do While (SYSTEM_OK .and. I .ge. 1)

!         Calculate what the value of the counter should be for
!         this equation

          SUM = IEQN(I, COLS_EQN)
          Do J = I+1, COLS_EQN-1
            SUM = SUM - IEQN(I,J) * CTR(J)
          End Do

          If (SUM .ge. 0 .and. SUM .le. UPPER(I)) Then
            If (IEQN(I,I) .eq. 1) Then
                CTR(I) = SUM
            Else
                CTR(I) = SUM / IEQN(I,I)
                SYSTEM_OK = EQN_OK(I)
            End If
            I = I - 1
          Else
            SYSTEM_OK = .false.
          End If
      End Do

!     If the system is ok, save it for later use

      If (SYSTEM_OK) Then
          NUM_SOLUTIONS = NUM_SOLUTIONS + 1
          Do I = 1, COLS_EQN-1
            SOLUTIONS(NUM_SOLUTIONS,I) = CTR(I)
          End Do
      End If

      Return
      End
```

Appendix B

Appendix B contains the output of a program run with g(9) = 11.

```
Beginning G( 9) = 11 at 26-JUN-1989 22:44:54.04

--------------------------------------------------
Case     1

B 2    B 3    B 4    B 5    B 6    B 7    B 8
---------------------------------------
  7      1      1      0      0      0      2

Distribution Vectors

A 2    A 3    A 4    A 8
---------------------
  0      1      0      2
  2      0      0      2
  4      1      1      1
  6      0      1      1
  7      1      0      1

Pair Table

  3  2  2  1  2
  2  2  1  2  3
  2  1  3  4  5
  1  2  4  6  6
  2  3  5  6  8

Original System of Equations

  2.00  2.00  1.00  1.00  1.00 =  16.00
  0.00  0.00  1.00  1.00  0.00 =   4.00
  1.00  0.00  1.00  0.00  1.00 =   3.00
  0.00  2.00  4.00  6.00  7.00 =  14.00

Equations in Row Echelon Form

  1.00  0.00  0.00  0.00 -1.00 =   1.00
  0.00  1.00  0.00  0.00  1.00 =   6.00
  0.00  0.00  1.00  0.00  1.00 =   4.00
  0.00  0.00  0.00  1.00  0.00 =  -2.00
Column pointer vector:   1  2  3  5  4

Equation  4 has LHS >= 0, RHS < 0
*** Block vector rejected

--------------------------------------------------
```

```
Case     2

B 2    B 3    B 4    B 5    B 6    B 7    B 8
-----------------------------------------------
  8      0      0      0      1      1      1


Distribution Vectors

A 2    A 6    A 7    A 8
--------------------------
  3      0      1      1
  4      1      0      1
  5      1      1      0


Pair Table

  2  1  1
  1  2  2
  1  2  4


Original System of Equations

  1.00  1.00  0.00 =   8.00
  1.00  0.00  1.00 =   7.00
  0.00  1.00  1.00 =   6.00
  3.00  4.00  5.00 =  16.00


Equations in Row Echelon Form

  1.00  0.00  0.00 =   8.50
  0.00  1.00  0.00 =  -0.50
  0.00  0.00  1.00 =  -1.50
  0.00  0.00  0.00 =   8.00
Column pointer vector:   1  2  3

Equation  4 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case     3

B 2    B 3    B 4    B 5    B 6    B 7    B 8
-----------------------------------------------
  7      0      1      1      0      1      1


Distribution Vectors

A 2    A 4    A 5    A 7    A 8
---------------------------------
  0      1      0      1      1
  3      0      0      1      1
  2      1      1      0      1
```

5	0	1	0	1
6	1	0	0	1
3	1	1	1	0
6	0	1	1	0
7	1	0	1	0

Pair Table

3	2	2	1	2	2	1	2
2	2	1	2	3	1	3	4
2	1	3	2	3	2	2	3
1	2	2	5	5	2	5	5
2	3	3	5	7	3	5	7
2	1	2	2	3	3	4	5
1	3	2	5	5	4	7	7
2	4	3	5	7	5	7	9

Original System of Equations

1.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00	=	8.00
1.00	1.00	0.00	0.00	0.00	1.00	1.00	1.00	=	7.00
0.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00	=	5.00
1.00	0.00	1.00	0.00	1.00	1.00	0.00	1.00	=	4.00
0.00	3.00	2.00	5.00	6.00	3.00	6.00	7.00	=	14.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	0.00	0.00	-1.00	-1.00	=	1.00
0.00	1.00	0.00	0.00	0.00	0.00	1.00	1.00	=	5.00
0.00	0.00	1.00	0.00	0.00	1.00	1.00	1.00	=	5.00
0.00	0.00	0.00	1.00	0.00	1.00	1.00	0.00	=	1.00
0.00	0.00	0.00	0.00	1.00	-1.00	-1.00	0.00	=	-3.00

Column pointer vector: 1 2 3 8 5 6 7 4

Equations to be solved

1	0	0	0	0	0	-1	-1	=	1
0	1	0	0	0	0	1	1	=	5
0	0	1	0	0	1	1	1	=	5
0	0	0	1	0	1	1	0	=	1
0	0	0	0	1	-1	-1	0	=	-3

Upper bounds: 1 5 1 1 1 1 1 1

The number of possible solutions is: 0

Case 4

B 2 B 3 B 4 B 5 B 6 B 7 B 8

---

5 2 2 0 0 1 1

Distribution Vectors

A 2	A 3	A 4	A 7	A 8
0	0	1	1	1
1	1	0	1	1
3	0	0	1	1
1	1	2	0	1
3	0	2	0	1
2	2	1	0	1
4	1	1	0	1
5	2	0	0	1
0	2	2	1	0
2	1	2	1	0
4	0	2	1	0
3	2	1	1	0
5	1	1	1	0

Pair Table

2	2	2	2	2	1	1	1	2	2	2	1	1
2	2	2	1	1	2	1	3	2	1	1	2	2
2	2	3	1	2	1	3	4	1	1	3	2	4
2	1	1	3	3	3	2	3	3	2	2	2	2
2	1	2	3	4	2	4	4	2	2	4	2	4
1	2	1	3	2	3	3	5	3	2	2	2	3
1	1	3	2	4	3	4	6	2	2	4	3	4
1	3	4	3	4	5	6	8	2	3	4	5	6
2	2	1	3	2	3	2	2	5	4	3	4	3
2	1	1	2	2	2	2	3	4	3	4	3	4
2	1	3	2	4	2	4	4	3	4	6	4	6
1	2	2	2	2	2	3	5	4	3	4	4	5
1	2	4	2	4	3	4	6	3	4	6	5	6

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
0.00  0.00 =   8.00
  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00  1.00  1.00  1.00
1.00  1.00 =   7.00
  1.00  0.00  0.00  2.00  2.00  1.00  1.00  0.00  2.00  2.00  2.00
1.00  1.00 =   8.00
  0.00  1.00  0.00  1.00  0.00  2.00  1.00  2.00  2.00  1.00  0.00
2.00  1.00 =   6.00
  0.00  1.00  3.00  1.00  3.00  2.00  4.00  5.00  0.00  2.00  4.00
3.00  5.00 =  10.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -2.00  0.00  0.00  0.00 -
1.00 -1.00 =   2.00
```

```
  0.00  1.00  0.00  0.00  0.00  1.00  0.00  1.00 -1.00 -1.00 -2.00
0.00 -1.00 =   2.00
  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  2.00
1.00  2.00 =   2.00
  0.00  0.00  0.00  1.00  0.00  1.00  1.00  1.00  1.00  0.00  0.00
0.00  0.00 =   2.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  1.00  1.00
1.00  1.00 =   1.00
Column pointer vector:   1  2  3  4  9  6  7  8  5 10 11 12 13

Equations to be solved


  1   0   0   0   0  -1  -1  -2   0   0   0  -1  -1 =    2
  0   1   0   0   0   1   0   1  -1  -1  -2   0  -1 =    2
  0   0   1   0   0   0   1   1   1   1   2   1   2 =    2
  0   0   0   1   0   1   1   1   1   0   0   0   0 =    2
  0   0   0   0   1   0   0   0   0   1   1   1   1 =    1

Upper bounds:   9  9  1  1  1  1  1  1  1  1  1  1  1


The number of possible solutions is: 25

  1)    4   1   1   0   1   1   1   0   0   0   0   0   0
Solution rejected.  3 and  7 occur  3 times together
  2)    5   0   1   0   1   1   0   1   0   0   0   0   0
Solution rejected.  3 and  8 occur  4 times together
  3)    3   2   1   0   1   1   0   0   1   0   0   0   0
Solution rejected.  5 and  6 occur  3 times together
  4)    5   1   0   0   1   0   1   1   0   0   0   0   0
Solution rejected.  2 and  8 occur  3 times together
  5)    3   3   0   0   1   0   1   0   1   0   0   0   0
Solution rejected.  7 and  9 occur  4 times together
  6)    3   2   1   1   1   0   1   0   0   0   0   0   0
Solution rejected.  3 and  7 occur  3 times together
  7)    4   2   0   0   1   0   0   1   1   0   0   0   0
Solution rejected.  2 and  8 occur  3 times together
  8)    4   1   1   1   1   0   0   1   0   0   0   0   0
Solution rejected.  2 and  8 occur  3 times together
  9)    2   3   1   1   1   0   0   0   1   0   0   0   0
Solution rejected.  4 and  5 occur  3 times together
 10)    4   2   0   0   0   1   1   0   0   1   0   0   0
Solution rejected.  6 and  7 occur  3 times together
 11)    5   1   0   0   0   1   1   0   0   0   0   1   0
Solution rejected.  6 and  7 occur  3 times together
 12)    5   1   0   0   0   1   0   1   0   1   0   0   0
Solution rejected.  2 and  8 occur  3 times together
 13)    6   0   0   0   0   1   0   1   0   0   0   1   0
Solution rejected.  6 and  8 occur  5 times together
 14)    3   3   0   0   0   1   0   0   1   1   0   0   0
 15)    4   2   0   0   0   1   0   0   1   0   0   1   0
 16)    3   2   1   1   0   1   0   0   0   1   0   0   0
Solution rejected.  4 and  6 occur  3 times together
 17)    3   3   0   1   0   1   0   0   0   0   1   0   0
Solution rejected.  4 and  6 occur  3 times together
 18)    4   1   1   1   0   1   0   0   0   0   0   1   0
Solution rejected.  4 and  6 occur  3 times together
```

```
19)    4   2   0   1   0   1   0   0   0   0   0   0   1
Solution rejected.  4 and  6 occur  3 times together
20)    3   3   0   1   0   0   1   0   0   1   0   0   0
21)    4   2   0   1   0   0   1   0   0   0   0   1   0
Solution rejected.  7 and 12 occur  3 times together
22)    4   2   0   1   0   0   0   1   0   1   0   0   0
Solution rejected.  2 and  8 occur  3 times together
23)    5   1   0   1   0   0   0   1   0   0   0   1   0
Solution rejected.  2 and  8 occur  3 times together
24)    2   4   0   1   0   0   0   0   1   1   0   0   0
Solution rejected.  4 and  9 occur  3 times together
25)    3   3   0   1   0   0   0   0   1   0   0   1   0
Solution rejected.  4 and  9 occur  3 times together
```

Check equations: 14 15 20

Case 5

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	7	0	0	0	1	1

Distribution Vectors

A 2	A 3	A 7	A 8
1	1	1	1
1	4	0	1
0	5	1	0
2	4	1	0
2	7	0	0

Pair Table

2	1	1	2	2
1	2	2	2	5
1	2	4	3	5
2	2	3	4	6
2	5	5	6	9

Original System of Equations

```
1.00  1.00  0.00  0.00  0.00 =   8.00
1.00  0.00  1.00  1.00  0.00 =   7.00
1.00  4.00  5.00  4.00  7.00 =  21.00
1.00  1.00  0.00  2.00  2.00 =   4.00
```

Equations in Row Echelon Form

```
1.00  0.00  0.00  0.00 -1.00 =   6.00
0.00  1.00  0.00  0.00  1.00 =   2.00
0.00  0.00  1.00  0.00  0.00 =   3.00
```

```
  0.00   0.00   0.00   1.00   1.00 =   -2.00
Column pointer vector:   1  2  3  4  5

Equation  4 has LHS >= 0, RHS < 0
*** Block vector rejected
```

Case 6

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	3	0	0	2	0	1

Distribution Vectors

A 2	A 3	A 6	A 8
0	2	1	1
2	1	1	1
4	0	1	1
3	3	0	1
5	2	0	1
0	3	2	0
2	2	2	0
4	1	2	0
5	3	1	0

Pair Table

2	1	1	3	2	3	2	1	2
1	1	2	2	3	2	1	2	3
1	2	4	3	5	1	2	4	4
3	2	3	5	6	3	2	3	6
2	3	5	6	7	2	3	4	7
3	2	1	3	2	5	4	3	4
2	1	2	2	3	4	3	3	5
1	2	4	3	4	3	3	5	6
2	3	4	6	7	4	5	6	8

Original System of Equations

```
  1.00   1.00   1.00   1.00   1.00   0.00   0.00   0.00   0.00 =    8.00
  1.00   1.00   1.00   0.00   0.00   2.00   2.00   2.00   1.00 =   12.00
  2.00   1.00   0.00   3.00   2.00   3.00   2.00   1.00   3.00 =    9.00
  0.00   2.00   4.00   3.00   5.00   0.00   2.00   4.00   5.00 =   10.00
```

Equations in Row Echelon Form

```
  1.00   0.00   0.00   0.00  -1.00   2.00   1.00   0.00  -1.00 =    4.00
  0.00   1.00   0.00   0.00   1.00  -1.00   0.00   1.00   2.00 =    7.00
  0.00   0.00   1.00   0.00   0.00   1.00   1.00   1.00   0.00 =    1.00
  0.00   0.00   0.00   1.00   1.00  -1.00  -1.00  -1.00   0.00 =   -3.00
Column pointer vector:   1  2  9  4  5  6  7  8  3
```

Equations to be solved

1	0	0	0	-1	2	1	0	-1	=	4
0	1	0	0	1	-1	0	1	2	=	7
0	0	1	0	0	1	1	1	0	=	1
0	0	0	1	1	-1	-1	-1	0	=	-3

Upper bounds: 9 9 1 1 1 1 1 1 1

The number of possible solutions is: 0

Case 7

B 2	B 3	B 4	B 5	B 6	B 7	B 8
6	1	0	2	1	0	1

Distribution Vectors

A 2	A 3	A 5	A 6	A 8
0	0	1	1	1
2	1	0	1	1
4	0	0	1	1
1	0	2	0	1
3	1	1	0	1
5	0	1	0	1
1	1	2	1	0
3	0	2	1	0
5	1	1	1	0
6	1	2	0	0

Pair Table

2	2	2	2	1	1	2	2	1	1
2	3	2	1	2	2	2	1	3	3
2	2	4	1	2	4	1	2	4	4
2	1	1	3	2	2	2	2	1	3
1	2	2	2	2	3	2	1	3	5
1	2	4	2	3	5	1	3	4	6
2	2	1	2	2	1	4	3	3	4
2	1	2	2	1	3	3	3	4	5
1	3	4	1	3	4	3	4	6	7
1	3	4	3	5	6	4	5	7	9

Original System of Equations

1.00	1.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	=	8.00
1.00	1.00	1.00	0.00	0.00	0.00	1.00	1.00	1.00	0.00	=	6.00

```
 1.00  0.00  0.00  2.00  1.00  1.00  2.00  2.00  1.00  2.00 =  10.00
 0.00  1.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00 =   3.00
 0.00  2.00  4.00  1.00  3.00  5.00  1.00  3.00  5.00  6.00 =  12.00
```

```
Equations in Row Echelon Form

 1.00  0.00  0.00  0.00  0.00 -1.00 -1.00  0.00 -1.00 -2.00 =   2.00
 0.00  1.00  0.00  0.00  0.00  0.00  1.00 -1.00  0.00  0.00 =   2.00
 0.00  0.00  1.00  0.00  0.00  1.00  0.00  1.00  1.00  1.00 =   1.00
 0.00  0.00  0.00  1.00  0.00  1.00  1.00  0.00  0.00  1.00 =   3.00
 0.00  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00 =   1.00
Column pointer vector:   1   2   3   4   7   6   5   8   9  10

Equations to be solved


  1   0   0   0   0  -1  -1   0  -1  -2 =    2
  0   1   0   0   0   0   1  -1   0   0 =    2
  0   0   1   0   0   1   0   1   1   1 =    1
  0   0   0   1   0   1   1   0   0   1 =    3
  0   0   0   0   1   0   0   1   1   1 =    1

Upper bounds:   9   1   1   1   1   1   3   1   1   1


The number of possible solutions is:  8

  1)     5   0   0   0   1   1   2   0   0   0
Solution rejected.   6 and  7 occur  3 times together
  2)     4   1   0   1   1   1   1   0   0   0
Solution rejected.   6 and  7 occur  3 times together
  3)     4   0   1   1   1   0   2   0   0   0
  4)     5   0   0   0   0   0   3   1   0   0
  5)     4   1   0   1   0   0   2   1   0   0
  6)     5   0   0   1   0   0   2   0   1   0
Solution rejected.   7 and  9 occur  3 times together
  7)     6   0   0   0   0   0   2   0   0   1
Solution rejected.   7 and 10 occur  5 times together
  8)     5   1   0   1   0   0   1   0   0   1
Solution rejected.   2 and 10 occur  3 times together

Check equations:   3   4   5


------------------------------------------------
Case    8

B 2   B 3   B 4   B 5   B 6   B 7   B 8
----------------------------------------
 4     3     1     1     1     0     1


Distribution Vectors

A 2   A 3   A 4   A 5   A 6   A 8
---------------------------------
 0     0     0     1     1     1
```

1	0	1	0	1	1
0	2	0	0	1	1
2	1	0	0	1	1
4	0	0	0	1	1
0	1	1	1	0	1
2	0	1	1	0	1
1	2	0	1	0	1
3	1	0	1	0	1
0	3	1	0	0	1
2	2	1	0	0	1
4	1	1	0	0	1
3	3	0	0	0	1
0	2	1	1	1	0
2	1	1	1	1	0
4	0	1	1	1	0
1	3	0	1	1	0
3	2	0	1	1	0
2	3	1	0	1	0
4	2	1	0	1	0
3	3	1	1	0	0

Pair Table

3	2	2	2	2	2	2	2	2	1	1	1	1	2	2	2	2	2	1	1	1
2	3	2	2	3	2	2	1	1	2	2	3	1	2	2	3	1	1	2	3	1
2	2	3	2	2	1	1	2	1	3	2	1	3	2	1	1	3	2	3	2	2
2	2	2	2	4	1	1	1	2	2	1	3	3	1	1	3	2	2	2	3	2
2	3	2	4	6	1	3	2	4	1	3	5	4	1	3	5	2	4	3	5	3
2	2	1	1	1	3	3	2	2	3	2	2	2	2	2	2	2	1	2	1	3
2	2	1	1	3	3	3	2	3	2	2	4	2	2	2	4	1	2	1	3	3
2	1	2	1	2	2	2	3	2	3	2	2	3	2	1	2	3	2	2	2	3
2	1	1	2	4	2	3	2	4	2	2	4	4	1	2	4	2	3	2	3	4
1	2	3	2	1	3	2	3	2	5	4	3	4	3	2	1	3	2	4	3	4
1	2	2	1	3	2	2	2	2	4	3	4	4	2	1	3	2	2	3	4	4
1	3	1	3	5	2	4	2	4	3	4	6	5	1	3	5	2	3	4	5	5
1	1	3	3	4	2	2	3	4	4	4	5	6	2	2	3	3	4	4	5	5
2	2	2	1	1	2	2	2	1	3	2	1	2	4	3	3	4	3	4	3	4
2	2	1	1	3	2	2	1	2	2	1	3	2	3	3	5	3	3	3	4	4
2	3	1	3	5	2	4	2	4	1	3	5	3	3	5	7	3	5	4	6	5
2	1	3	2	2	2	1	3	2	3	2	2	3	4	3	3	5	4	4	4	4
2	1	2	2	4	1	2	2	3	2	2	3	4	3	3	5	4	5	4	5	5
1	2	3	2	3	2	1	2	2	4	3	4	4	4	3	4	4	4	5	6	5
1	3	2	3	5	1	3	2	3	3	4	5	5	3	4	6	4	5	6	7	6
1	1	2	2	3	3	3	3	4	4	4	5	5	4	4	5	4	5	5	6	7

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00
1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 =   8.00
  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00
0.00  0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00 =   6.00
  1.00  0.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  0.00  0.00
0.00  0.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  1.00 =   5.00
  0.00  1.00  0.00  0.00  0.00  1.00  1.00  0.00  0.00  1.00  1.00
1.00  0.00  1.00  1.00  1.00  0.00  0.00  1.00  1.00  1.00 =   4.00
```

```
   0.00  0.00  2.00  1.00  0.00  1.00  0.00  2.00  1.00  3.00  2.00
 1.00  3.00  2.00  1.00  0.00  3.00  2.00  3.00  2.00  3.00 =   9.00
   0.00  1.00  0.00  2.00  4.00  0.00  2.00  1.00  3.00  0.00  2.00
 4.00  3.00  0.00  2.00  4.00  1.00  3.00  2.00  4.00  3.00 =   8.00
```

Equations in Row Echelon Form

```
   1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -
 1.00 -1.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 =   1.00
   0.00  1.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00  0.00  0.00
 0.00 -1.00  0.00  0.00  0.00 -1.00 -1.00  0.00  0.00 -1.00 =   0.00
   0.00  0.00  1.00  0.00  0.00  0.00 -1.00  0.00 -1.00  1.00  0.00 -
 1.00  0.00 -1.00 -1.00 -2.00  0.00 -1.00  0.00 -1.00 -1.00 =   0.00
   0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  2.00  0.00  1.00
 2.00  2.00  2.00  1.00  2.00  1.00  2.00  1.00  2.00  2.00 =   4.00
   0.00  0.00  0.00  0.00  1.00  0.00  1.00  1.00  1.00  1.00  1.00
 1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  1.00 =   3.00
   0.00  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00
 0.00  0.00  0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00 =   1.00
Column pointer vector:    1  2  3  4  6 14  7  8  9 10 11 12 13  5 15
16 17 18 19 20 21
```

Equations to be solved

```
    1    0    0    0    0    0    0    0    0   -1   -1   -1   -1    0    0    0    0
 0   -1   -1   -1  =      1
    0    1    0    0    0    0    0   -1   -1    0    0    0   -1    0    0    0   -1
-1    0    0   -1  =      0
    0    0    1    0    0    0   -1    0   -1    1    0   -1    0   -1   -1   -2    0
-1    0   -1   -1  =      0
    0    0    0    1    0    0    1    1    2    0    1    2    2    2    1    2    1
 2    1    2    2  =      4
    0    0    0    0    1    0    1    1    1    1    1    1    1    0    0    0    0
 0    0    0    1  =      3
    0    0    0    0    0    1    0    0    0    0    0    0    0    0    1    1    1
 1    1    1    1  =      1

Upper bounds:    1  1  1  4  1  1  1  1  1  1  1  1  1  1  1  1  1  1
1  1  1


The number of possible solutions is:   1

  1)     1    1    1    2    1    1    1    1    0    0    0    0    0    0    0    0
0   0    0    0    0
Solution rejected.  5 and  7 occur  3 times together


------------------------------------------------
Case     9

B 2   B 3   B 4   B 5   B 6   B 7   B 8
--------------------------------------
  5     0     4     0     1     0     1
```

Distribution Vectors

A 2	A 4	A 6	A 8
1	1	1	1
4	0	1	1
0	3	0	1
3	2	0	1
2	3	1	0
5	2	1	0
4	4	0	0

Pair Table

2	2	1	1	1	2	1
2	5	1	3	2	5	3
1	1	3	2	2	1	3
1	3	2	2	1	3	4
1	2	2	1	3	4	4
2	5	1	3	4	6	6
1	3	3	4	4	6	7

Original System of Equations

```
1.00  1.00  1.00  1.00  0.00  0.00  0.00 =   8.00
1.00  1.00  0.00  0.00  1.00  1.00  0.00 =   6.00
1.00  0.00  3.00  2.00  3.00  2.00  4.00 =  16.00
1.00  4.00  0.00  3.00  2.00  5.00  4.00 =  10.00
```

Equations in Row Echelon Form

```
1.00  0.00  0.00  0.00  2.00  1.00 -1.00 =   6.00
0.00  1.00  0.00  0.00 -1.00  0.00  1.00 =   0.00
0.00  0.00  1.00  0.00 -1.00 -1.00  1.00 =   2.00
0.00  0.00  0.00  1.00  1.00  1.00  0.00 =   1.00
Column pointer vector:   1  2  3  7  5  6  4
```

Equations to be solved

```
1   0   0   0   2   1  -1 =   6
0   1   0   0  -1   0   1 =   0
0   0   1   0  -1  -1   1 =   2
0   0   0   1   1   1   0 =   1
```

Upper bounds: 9 1 1 1 1 1 9

The number of possible solutions is: 0

Case   10

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	5	2	0	1	0	1

Distribution Vectors

A 2	A 3	A 4	A 6	A 8
1	0	1	1	1
0	2	0	1	1
2	1	0	1	1
1	1	2	0	1
0	3	1	0	1
2	2	1	0	1
1	4	0	0	1
1	2	2	1	0
0	4	1	1	0
2	3	1	1	0
1	5	0	1	0
0	5	2	0	0
2	4	2	0	0

Pair Table

2	2	3	2	1	2	1	2	1	2	1	1	2
2	2	2	1	1	1	2	1	2	1	3	2	1
3	2	4	2	1	3	2	2	1	3	3	1	2
2	1	2	3	2	3	1	2	1	2	1	3	3
1	1	1	2	2	1	3	1	2	1	3	4	3
2	1	3	3	1	3	3	2	1	2	3	3	4
1	2	2	1	3	3	4	1	3	3	4	4	4
2	1	2	2	1	2	1	3	3	3	3	4	4
1	2	1	1	2	1	3	3	4	3	5	5	4
2	1	3	2	1	2	3	3	3	4	5	4	5
1	3	3	1	3	3	4	3	5	5	6	5	5
1	2	1	3	4	3	4	4	5	4	5	7	6
2	1	2	3	3	4	4	4	4	5	5	6	7

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00
0.00  0.00 =   8.00
  1.00  1.00  1.00  0.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00
0.00  0.00 =   6.00
  1.00  0.00  0.00  2.00  1.00  1.00  0.00  2.00  1.00  1.00  0.00
2.00  2.00 =   8.00
  0.00  2.00  1.00  1.00  3.00  2.00  4.00  2.00  4.00  3.00  5.00
5.00  4.00 =  15.00
  1.00  0.00  2.00  1.00  0.00  2.00  1.00  1.00  0.00  2.00  1.00
0.00  2.00 =   4.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -2.00  2.00  1.00  1.00  0.00 -
1.00  0.00 =  2.00
  0.00  1.00  0.00  0.00  0.00  0.00  1.00 -1.00  0.00 -1.00  0.00
1.00 -1.00 =  4.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00  0.00  0.00  1.00  1.00
0.00  1.00 =  0.00
  0.00  0.00  0.00  1.00  0.00  1.00  1.00 -1.00 -1.00 -1.00 -1.00
1.00  0.00 =  2.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00  1.00
0.00  1.00 =  1.00
Column pointer vector:   1  2  3  4 12  6  7  8  9 10 11  5 13
```

Equations to be solved

1	0	0	0	0	-1	-2	2	1	1	0	-1	0	=	2
0	1	0	0	0	0	1	-1	0	-1	0	1	-1	=	4
0	0	1	0	0	1	1	0	0	1	1	0	1	=	0
0	0	0	1	0	1	1	-1	-1	-1	-1	1	0	=	2
0	0	0	0	1	0	0	1	1	1	1	0	1	=	1

Upper bounds: 9 9 1 1 1 1 1 1 1 1 1 9 1

The number of possible solutions is: 6

```
  1)    4   2   0   0   1   0   0   0   0   0   0   2   0
Solution rejected.  5 and 12 occur  4 times together
  2)    3   3   0   1   1   0   0   0   0   0   0   1   0
Solution rejected.  4 and  5 occur  3 times together
  3)    3   2   0   0   0   0   0   1   0   0   0   3   0
  4)    2   3   0   1   0   0   0   1   0   0   0   2   0
  5)    4   1   0   0   0   0   0   0   1   0   0   3   0
  6)    3   2   0   1   0   0   0   0   1   0   0   2   0
```

Check equations: 3 4 5 6

Case 11

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	1	1	3	0	0	1

Distribution Vectors

A 2	A 3	A 4	A 5	A 8
1	0	0	2	1
0	1	1	1	1
2	0	1	1	1
3	1	0	1	1
5	0	0	1	1
4	1	1	0	1
1	0	1	3	0
2	1	0	3	0

4	0	0	3	0
3	1	1	2	0
5	0	1	2	0

Pair Table

2	1	1	1	2	1	2	2	2	1	2
1	3	2	2	1	3	2	2	1	2	1
1	2	2	1	3	3	2	1	2	1	3
1	2	1	3	4	4	1	2	3	2	3
2	1	3	4	6	5	2	3	5	3	5
1	3	3	4	5	6	1	2	3	4	5
2	2	2	1	2	1	4	3	3	3	4
2	2	1	2	3	2	3	4	4	3	4
2	1	2	3	5	3	3	4	6	4	6
1	2	1	2	3	4	3	3	4	4	5
2	1	3	3	5	5	4	4	6	5	7

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00 =
8.00
  2.00  1.00  1.00  1.00  1.00  0.00  3.00  3.00  3.00  2.00  2.00 =
15.00
  0.00  1.00  1.00  0.00  0.00  1.00  1.00  0.00  0.00  1.00  1.00 =
4.00
  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00 =
3.00
  1.00  0.00  2.00  3.00  5.00  4.00  1.00  2.00  4.00  3.00  5.00 =
10.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00  0.00  0.00  0.00 -1.00 -1.00 =
4.00
  0.00  1.00  0.00  0.00  0.00  0.00 -1.00  0.00 -1.00  0.00 -1.00 =
2.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00 -1.00  0.00  0.00  1.00 =
1.00
  0.00  0.00  0.00  1.00  0.00  1.00  1.00  1.00  1.00  1.00  1.00 =
1.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00  1.00 =
1.00
Column pointer vector:   1  2  3  4  7  6  5  8  9 10 11
```

Equations to be solved

```
  1   0   0   0   0  -1   0   0   0  -1  -1 =   4
  0   1   0   0   0   0  -1   0  -1   0  -1 =   2
  0   0   1   0   0   1   1  -1   0   0   1 =   1
  0   0   0   1   0   1   1   1   1   1   1 =   1
  0   0   0   0   1   0   0   1   1   1   1 =   1
```

Upper bounds: 9 1 9 1 1 1 1 1 1 1 1

The number of possible solutions is: 0

Case 12

B 2	B 3	B 4	B 5	B 6	B 7	B 8
3	3	2	2	0	0	1

Distribution Vectors

A 2	A 3	A 4	A 5	A 8
1	0	0	2	1
0	1	1	1	1
2	0	1	1	1
1	2	0	1	1
3	1	0	1	1
1	1	2	0	1
3	0	2	0	1
0	3	1	0	1
2	2	1	0	1
3	3	0	0	1
0	1	2	2	0
2	0	2	2	0
1	2	1	2	0
3	1	1	2	0
2	3	0	2	0
0	3	2	1	0
2	2	2	1	0
3	3	1	1	0

Pair Table

3	2	2	2	3	1	2	1	1	2	2	2	2	3	2	1	1	2
2	1	1	1	1	2	2	2	1	2	2	2	1	1	2	2	1	1
2	1	2	1	3	2	4	1	2	3	2	3	1	3	2	1	2	2
2	1	1	2	2	1	2	3	2	4	1	1	2	2	3	2	1	3
3	1	3	2	4	2	4	2	3	5	1	3	2	4	4	1	2	4
1	2	2	1	2	3	4	3	2	3	2	2	1	2	1	3	2	3
2	2	4	2	4	4	6	2	4	4	2	4	2	4	2	2	4	4
1	2	1	3	2	3	2	4	3	4	2	1	2	1	3	4	3	3
1	1	2	2	3	2	4	3	3	5	1	2	1	2	3	3	3	4
2	2	3	4	5	3	4	4	5	7	1	2	3	4	5	3	4	6
2	2	2	1	1	2	2	2	1	1	4	4	3	3	3	4	3	3
2	2	3	1	3	2	4	1	2	2	4	5	3	5	3	3	4	4
2	1	1	2	2	1	2	2	1	3	3	3	3	3	4	4	3	4
3	1	3	2	4	2	4	1	2	4	3	5	3	5	5	3	4	5
2	2	2	3	4	1	2	3	3	5	3	3	4	5	6	4	4	6
1	2	1	2	1	3	2	4	3	3	4	3	4	3	4	5	4	4
1	1	2	1	2	2	4	3	3	4	3	4	3	4	4	4	4	5
2	1	2	3	4	3	4	3	4	6	3	4	4	5	6	4	5	6

```
Original System of Equations

  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00
0.00  0.00  0.00  0.00  0.00  0.00  0.00 =   8.00
  2.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00  2.00
2.00  2.00  2.00  2.00  1.00  1.00  1.00 =  10.00
  0.00  1.00  1.00  0.00  0.00  2.00  2.00  1.00  1.00  0.00  2.00
2.00  1.00  1.00  0.00  2.00  2.00  1.00 =   8.00
  0.00  1.00  0.00  2.00  1.00  1.00  0.00  3.00  2.00  3.00  1.00
0.00  2.00  1.00  3.00  3.00  2.00  3.00 =   9.00
  1.00  0.00  2.00  1.00  3.00  1.00  3.00  0.00  2.00  3.00  0.00
2.00  1.00  3.00  2.00  0.00  2.00  3.00 =   6.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 -1.00 -1.00  0.00
0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 =   0.00
  0.00  1.00  0.00  0.00  0.00  1.00  0.00  1.00  0.00 -1.00 -1.00 -
1.00 -1.00 -2.00 -2.00  0.00 -1.00 -2.00 =   4.00
  0.00  0.00  1.00  0.00  0.00  1.00  2.00  0.00  1.00  1.00  1.00
1.00  0.00  1.00  0.00  0.00  1.00  1.00 =   2.00
  0.00  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  2.00  1.00
0.00  1.00  1.00  2.00  1.00  1.00  2.00 =   2.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00
1.00  1.00  1.00  1.00  1.00  1.00  1.00 =   1.00
Column pointer vector:   1  2  3  4 11  6  7  8  9 10  5 12 13 14 15
16 17 18

Equations to be solved


   1    0    0    0    0   -1   -1   -1   -1   -1    0    0    0    0    0   -1   -1
-1 =    0
   0    1    0    0    0    1    0    1    0   -1   -1   -1   -1   -2   -2    0   -1
-2 =    4
   0    0    1    0    0    1    2    0    1    1    1    1    0    1    0    0    1
1 =    2
   0    0    0    1    0    0    0    1    1    2    1    0    1    1    2    1    1
2 =    2
   0    0    0    0    1    0    0    0    0    0    1    1    1    1    1    1    1
1 =    1

Upper bounds:    1  9  2  2  1  1  1  1  1  1  1  1  1  1  1  1  1  1


The number of possible solutions is: 37

  1)     1    4    0    1    1    1    0    0    0    0    1    0    0    0    0    0
0    0
Solution rejected.  1 and 11 occur  3 times together
  2)     1    3    1    2    1    1    0    0    0    0    0    0    0    0    0    0
0    0
  3)     1    4    0    2    1    0    1    0    0    0    0    0    0    0    0    0
0    0
```

```
  4)    1   4   1   0   1   0   0   1   0   0   1   0   0   0   0   0
0   0
Solution rejected.  1 and 11 occur  3 times together
  5)    1   3   2   1   1   0   0   1   0   0   0   0   0   0   0   0
0   0
Solution rejected.  4 and  8 occur  3 times together
  6)    1   5   0   0   1   0   0   0   1   0   1   0   0   0   0   0
0   0
Solution rejected.  1 and 11 occur  3 times together
  7)    1   4   1   1   1   0   0   0   1   0   0   0   0   0   0   0
0   0
  8)    1   5   1   0   1   0   0   0   0   1   0   0   0   0   0   0
0   0
Solution rejected.  3 and 10 occur  3 times together
  9)    0   5   1   1   1   0   0   0   0   0   1   0   0   0   0   0
0   0
Solution rejected.  3 and 11 occur  3 times together
 10)    0   4   2   2   1   0   0   0   0   0   0   0   0   0   0   0
0   0
 11)    1   5   0   0   0   1   0   0   0   0   1   0   1   0   0   0
0   0
Solution rejected.  1 and 11 occur  3 times together
 12)    1   4   0   2   0   1   0   0   0   0   0   1   0   0   0   0
0   0
 13)    1   4   1   1   0   1   0   0   0   0   0   0   1   0   0   0
0   0
 14)    1   5   0   1   0   1   0   0   0   0   0   0   0   1   0   0
0   0
Solution rejected.  1 and 14 occur  3 times together
 15)    1   5   1   0   0   1   0   0   0   0   0   0   0   0   1   0
0   0
 16)    1   5   0   1   0   0   1   0   0   0   0   0   1   0   0   0
0   0
 17)    1   6   0   0   0   0   1   0   0   0   0   0   0   0   1   0
0   0
 18)    1   5   0   0   0   0   0   1   0   0   1   1   0   0   0   0
0   0
Solution rejected.  1 and 11 occur  3 times together
 19)    1   4   1   1   0   0   0   1   0   0   0   1   0   0   0   0
0   0
Solution rejected.  3 and 12 occur  3 times together
 20)    1   4   2   0   0   0   0   1   0   0   0   0   1   0   0   0
0   0
 21)    1   5   1   0   0   0   0   1   0   0   0   0   0   1   0   0
0   0
Solution rejected.  1 and 14 occur  3 times together
 22)    1   5   0   1   0   0   0   0   1   0   0   1   0   0   0   0
0   0
 23)    1   5   1   0   0   0   0   0   1   0   0   0   1   0   0   0
0   0
 24)    1   6   0   0   0   0   0   0   1   0   0   0   0   1   0   0
0   0
Solution rejected.  1 and 14 occur  3 times together
 25)    1   6   0   0   0   0   0   0   0   1   0   1   0   0   0   0
0   0
 26)    0   6   0   1   0   0   0   0   0   0   1   1   0   0   0   0
0   0
Solution rejected. 11 and 12 occur  3 times together
```

```
 27)    0   6   1   0   0   0   0   0   0   0   1   0   1   0   0   0
0   0
Solution rejected.  3 and 11 occur  3 times together
 28)    0   7   0   0   0   0   0   0   0   1   0   0   1   0   0
0   0
Solution rejected. 11 and 14 occur  4 times together
 29)    1   5   1   0   0   0   0   0   0   0   1   0   0   0   0   1
0   0
Solution rejected.  1 and 11 occur  3 times together
 30)    1   6   0   0   0   0   0   0   0   0   1   0   0   0   0   0
1   0
Solution rejected.  1 and 11 occur  3 times together
 31)    0   5   1   2   0   0   0   0   0   0   0   1   0   0   0   0
0   0
Solution rejected.  3 and 12 occur  3 times together
 32)    0   5   2   1   0   0   0   0   0   0   0   0   1   0   0   0
0   0
 33)    0   6   1   1   0   0   0   0   0   0   0   0   0   1   0   0
0   0
Solution rejected.  3 and 14 occur  3 times together
 34)    0   6   2   0   0   0   0   0   0   0   0   0   0   0   1   0
0   0
 35)    1   4   2   1   0   0   0   0   0   0   0   0   0   0   0   1
0   0
 36)    1   5   1   1   0   0   0   0   0   0   0   0   0   0   0   0
1   0
 37)    1   6   1   0   0   0   0   0   0   0   0   0   0   0   0   0
0   1

Check equations:   2  3  7 10 12 13 15 16 17 20 22 23 25 32 34 35 36
37
```

Case 13

B 2	B 3	B 4	B 5	B 6	B 7	B 8
0	8	0	2	0	0	1

Distribution Vectors

A 3	A 5	A 8
4	2	0
6	1	0
8	0	0

Pair Table

2	3	4
3	4	6
4	6	8

Original System of Equations

```
  0.00  0.00  0.00 =   8.00
  2.00  1.00  0.00 =  10.00
  4.00  6.00  8.00 =  24.00
```

Equations in Row Echelon Form

```
  1.00  0.00 -1.00 =   4.50
  0.00  1.00  2.00 =   1.00
  0.00  0.00  0.00 =   8.00
Column pointer vector:   1  2  3
```

Equation 3 has LHS = 0 and RHS != 0
*** Block vector rejected

Case   14

B 2	B 3	B 4	B 5	B 6	B 7	B 8
4	0	5	1	0	0	1

Distribution Vectors

A 2	A 4	A 5	A 8
2	1	1	1
0	3	0	1
3	2	0	1
0	4	1	0
3	3	1	0
1	5	0	0
4	4	0	0

Pair Table

2	1	2	1	2	1	2
1	2	1	2	1	3	2
2	1	3	1	2	2	4
1	2	1	4	3	4	3
2	1	2	3	4	3	5
1	3	2	4	3	5	5
2	2	4	3	5	5	7

Original System of Equations

```
  1.00  1.00  1.00  0.00  0.00  0.00  0.00 =   8.00
  1.00  0.00  0.00  1.00  1.00  0.00  0.00 =   5.00
  1.00  3.00  2.00  4.00  3.00  5.00  4.00 =  20.00
  2.00  0.00  3.00  0.00  3.00  1.00  4.00 =   8.00
```

Equations in Row Echelon Form

```
  1.00   0.00   0.00   0.00   0.00 -1.00 -1.00 =    4.00
  0.00   1.00   0.00   0.00 -1.00   0.00 -1.00 =    4.00
  0.00   0.00   1.00   0.00   1.00   1.00   2.00 =    0.00
  0.00   0.00   0.00   1.00   1.00   1.00   1.00 =    1.00
Column pointer vector:    1  2  3  4  5  6  7
```

Equations to be solved

```
   1    0    0    0    0   -1   -1  =     4
   0    1    0    0   -1    0   -1  =     4
   0    0    1    0    1    1    2  =     0
   0    0    0    1    1    1    1  =     1
```

Upper bounds: 9 9 1 1 1 1 1

The number of possible solutions is: 1

1)   4   4   0   1   0   0   0

Check equations: 1

Case   15

B 2	B 3	B 4	B 5	B 6	B 7	B 8
1	5	3	1	0	0	1

Distribution Vectors

A 2	A 3	A 4	A 5	A 8
0	1	1	1	1
1	2	0	1	1
0	0	3	0	1
1	1	2	0	1
0	3	1	0	1
1	4	0	0	1
1	1	3	1	0
0	3	2	1	0
1	4	1	1	0
1	3	3	0	0
0	5	2	0	0

Pair Table

2	2	2	1	1	1	2	1	1	1	1
2	3	1	2	1	3	2	1	3	1	2
2	1	4	3	2	1	3	2	1	3	2
1	2	3	3	1	2	3	1	1	3	2
1	1	2	1	2	3	1	1	2	2	3
1	3	1	2	3	5	1	2	4	3	4

```
    2   2   3   3   1   1   5   3   3   4   3
    1   1   2   1   1   2   3   3   3   3   4
    1   3   1   1   2   4   3   3   5   4   4
    1   1   3   3   2   3   4   3   4   5   5
    1   2   2   2   3   4   3   4   4   5   6


Original System of Equations

   1.00   1.00   1.00   1.00   1.00   1.00   0.00   0.00   0.00   0.00   0.00 =
8.00
   1.00   1.00   0.00   0.00   0.00   0.00   1.00   1.00   1.00   0.00   0.00 =
5.00
   1.00   0.00   3.00   2.00   1.00   0.00   3.00   2.00   1.00   3.00   2.00 =
12.00
   1.00   2.00   0.00   1.00   3.00   4.00   1.00   3.00   4.00   3.00   5.00 =
15.00
   0.00   1.00   0.00   1.00   0.00   1.00   1.00   0.00   1.00   1.00   0.00 =
2.00


Equations in Row Echelon Form

   1.00   0.00   0.00   0.00   0.00  -1.00   0.00   1.00   0.00  -1.00  -1.00 =
3.00
   0.00   1.00   0.00   0.00   0.00   1.00   1.00   0.00   1.00   1.00   1.00 =
2.00
   0.00   0.00   1.00   0.00   0.00   0.00   1.00   0.00   0.00   1.00   1.00 =
2.00
   0.00   0.00   0.00   1.00   0.00   0.00   1.00   1.00   1.00   1.00   0.00 =
1.00
   0.00   0.00   0.00   0.00   1.00   1.00  -2.00  -1.00  -1.00  -1.00   0.00 =
1.00
Column pointer vector:   1   2   3  11   5   6   7   8   9  10   4

Equations to be solved


    1    0    0    0    0   -1    0    1    0   -1   -1 =    3
    0    1    0    0    0    1    1    0    1    1    1 =    2
    0    0    1    0    0    0    1    0    0    1    1 =    2
    0    0    0    1    0    0    1    1    1    1    0 =    1
    0    0    0    0    1    1   -2   -1   -1   -1    0 =    1

Upper bounds:    9   1   1   1   9   1   1   1   1   1   1


The number of possible solutions is: 11

  1)     4    0    0    0    3    0    1    0    0    0    1
Solution rejected.   7 and 11 occur   3 times together
  2)     3    1    1    0    3    0    1    0    0    0    0
Solution rejected.   3 and  7 occur   3 times together
  3)     4    0    1    0    2    1    1    0    0    0    0
Solution rejected.   3 and  7 occur   3 times together
  4)     3    1    1    0    2    0    0    1    0    0    1
Solution rejected.   3 and 11 occur   3 times together
  5)     4    0    1    0    2    0    0    0    1    0    1
```

```
Solution rejected.  3 and 11 occur  3 times together
  6)    5   0   0   0   2   0   0   0   0   1   1
Solution rejected. 10 and 11 occur  3 times together
  7)    4   1   1   0   2   0   0   0   0   1   0
Solution rejected.  3 and 10 occur  3 times together
  8)    4   0   1   0   1   1   0   1   0   0   1
Solution rejected.  3 and 11 occur  3 times together
  9)    5   0   1   0   1   1   0   0   0   1   0
Solution rejected.  3 and 10 occur  3 times together
 10)    4   1   1   1   1   0   0   0   0   0   1
Solution rejected.  3 and 11 occur  3 times together
 11)    5   0   1   1   0   1   0   0   0   0   1
Solution rejected.  3 and 11 occur  3 times together


--------------------------------------------------
Case   16

B 2    B 3    B 4    B 5    B 6    B 7    B 8
---------------------------------------------
  2      2      6      0      0      0      1


Distribution Vectors

A 2    A 3    A 4    A 8
-----------------------
  0      0      3      1
  1      1      2      1
  2      2      1      1
  1      0      5      0
  0      2      4      0
  2      1      4      0


Pair Table

  1   1   1   2   1   1
  1   1   3   1   1   1
  1   3   5   1   2   3
  2   1   1   4   3   4
  1   1   2   3   4   3
  1   1   3   4   3   4


Original System of Equations

  1.00  1.00  1.00  0.00  0.00  0.00 =   8.00
  3.00  2.00  1.00  5.00  4.00  4.00 =  24.00
  0.00  1.00  2.00  0.00  2.00  1.00 =   6.00
  0.00  1.00  2.00  1.00  0.00  2.00 =   4.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00 -1.00 -1.00 =   4.00
  0.00  1.00  0.00  0.00  2.00  1.00 =   4.00
  0.00  0.00  1.00  0.00  0.00  0.00 =   1.00
```

```
 0.00  0.00  0.00  1.00  0.00  1.00 =   0.00
Column pointer vector:   1  2  5  4  3  6

Equations to be solved


   1    0    0    0   -1   -1  =     4
   0    1    0    0    2    1  =     4
   0    0    1    0    0    0  =     1
   0    0    0    1    0    1  =     0

Upper bounds:    9   4   1   1   1   1


The number of possible solutions is:  2

  1)     5    2    1    0    1    0
Solution rejected.  2 and  5 occur  3 times together
  2)     4    4    1    0    0    0

Check equations:   2
```

Case   17

B 2	B 3	B 4	B 5	B 6	B 7	B 8
6	1	1	0	1	2	0

Distribution Vectors

A 2	A 3	A 4	A 6	A 7
1	0	1	0	2
2	1	0	0	2
4	0	0	0	2
0	1	1	1	1
2	0	1	1	1
3	1	0	1	1
5	0	0	1	1
5	1	1	0	1
6	1	1	1	0

Pair Table

3	2	2	2	2	1	1	2	2
2	3	2	2	1	2	2	3	3
2	2	4	1	1	2	4	4	4
2	2	1	3	2	2	1	2	3
2	1	1	2	2	1	2	2	4
1	2	2	2	1	2	3	3	5
1	2	4	1	2	3	5	4	6
2	3	4	2	2	3	4	6	7
2	3	4	3	4	5	6	7	9

Original System of Equations

2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	0.00	=	14.00
0.00	0.00	0.00	1.00	1.00	1.00	1.00	0.00	1.00	=	6.00
1.00	0.00	0.00	1.00	1.00	0.00	0.00	1.00	1.00	=	4.00
0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	1.00	=	3.00
1.00	2.00	4.00	0.00	2.00	3.00	5.00	5.00	6.00	=	12.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	0.00	-1.00	-1.00	0.00	-1.00	=	0.00
0.00	1.00	0.00	0.00	0.00	0.00	-1.00	-1.00	-1.00	=	-1.00
0.00	0.00	1.00	0.00	0.00	1.00	2.00	1.00	1.00	=	6.00
0.00	0.00	0.00	1.00	0.00	1.00	1.00	1.00	1.00	=	6.00
0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	1.00	=	-2.00

Column pointer vector: 1 2 3 4 8 6 7 5 9

Equation 5 has LHS >= 0, RHS < 0
*** Block vector rejected

Case 18

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	1	2	1	0	2	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 7
0	0	0	1	2
1	0	1	0	2
2	1	0	0	2
4	0	0	0	2
0	0	2	1	1
1	1	1	1	1
3	0	1	1	1
4	1	0	1	1
2	1	2	0	1
4	0	2	0	1
5	1	1	0	1
4	1	2	1	0

Pair Table

3	2	2	2	2	2	2	2	1	1	1	1
2	2	2	2	2	1	1	1	2	2	2	1
2	2	3	3	1	2	1	3	2	2	4	2
2	2	3	5	1	1	3	4	2	4	5	3
2	2	1	1	3	2	2	1	2	2	1	3
2	1	2	1	2	2	1	2	2	1	2	3
2	1	1	3	2	1	2	3	1	3	3	4

```
2  1  3  4  1  2  3  5  2  3  5  5
1  2  2  2  2  2  1  2  3  3  4  4
1  2  2  4  2  1  3  3  3  5  5  5
1  2  4  5  1  2  3  5  4  5  6  6
1  1  2  3  3  3  4  5  4  5  6  7
```

Original System of Equations

```
  2.00  2.00  2.00  2.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00
0.00 =  14.00
  1.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
1.00 =   5.00
  0.00  1.00  0.00  0.00  2.00  1.00  1.00  0.00  2.00  2.00  1.00
2.00 =   8.00
  0.00  0.00  1.00  0.00  0.00  1.00  0.00  1.00  1.00  0.00  1.00
1.00 =   3.00
  0.00  1.00  2.00  4.00  0.00  1.00  3.00  4.00  2.00  4.00  5.00
4.00 =  10.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 -
1.00 =   1.00
  0.00  1.00  0.00  0.00  0.00 -1.00 -1.00 -2.00  0.00  0.00 -1.00 -
2.00 =   0.00
  0.00  0.00  1.00  0.00  0.00  1.00  0.00  1.00  1.00  0.00  1.00
1.00 =   3.00
  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  0.00  1.00  1.00
1.00 =   1.00
  0.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00
2.00 =   4.00
Column pointer vector:   1  2  3  4  5  6  7  8  9 10 11 12
```

Equations to be solved

```
  1   0   0   0   0   0   0   0  -1  -1  -1  -1 =     1
  0   1   0   0   0  -1  -1  -2   0   0  -1  -2 =     0
  0   0   1   0   0   1   0   1   1   0   1   1 =     3
  0   0   0   1   0   0   1   1   0   1   1   1 =     1
  0   0   0   0   1   1   1   1   1   1   1   2 =     4
```

```
Upper bounds:   1  9  1  1  1  3  1  1  1  1  1  1
```

```
The number of possible solutions is:   4

  1)    1   3   0   1   1   3   0   0   0   0   0   0
  2)    1   3   1   0   1   2   1   0   0   0   0   0
  3)    1   4   0   0   1   2   0   1   0   0   0   0
  4)    1   4   0   0   0   3   1   0   0   0   0   0

Check equations:   1  2  3  4
```

Case 19

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	6	0	1	0	2	0

Distribution Vectors

A 2	A 3	A 5	A 7
0	0	1	2
0	2	0	2
2	1	0	2
0	3	1	1
2	2	1	1
0	5	0	1
2	4	0	1
0	6	1	0
2	5	1	0

Pair Table

3	2	2	2	2	1	1	1	1
2	2	2	1	1	2	1	2	1
2	2	4	1	3	1	3	1	2
2	1	1	1	1	2	1	4	3
2	1	3	1	3	1	2	3	4
1	2	1	2	1	4	3	5	4
1	1	3	1	2	3	4	4	5
1	2	1	4	3	5	4	7	6
1	1	2	3	4	4	5	6	7

Original System of Equations

2.00	2.00	2.00	1.00	1.00	1.00	1.00	0.00	0.00	=	14.00
1.00	0.00	0.00	1.00	1.00	0.00	0.00	1.00	1.00	=	5.00
0.00	2.00	1.00	3.00	2.00	5.00	4.00	6.00	5.00	=	18.00
0.00	0.00	2.00	0.00	2.00	0.00	2.00	0.00	2.00	=	4.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	0.00	-1.00	-1.00	-1.00	-1.00	=	1.00
0.00	1.00	0.00	0.00	-1.00	1.00	0.00	0.00	-1.00	=	2.00
0.00	0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	=	2.00
0.00	0.00	0.00	1.00	1.00	1.00	1.00	2.00	2.00	=	4.00

Column pointer vector: 1 2 3 4 5 6 7 8 9

Equations to be solved

1	0	0	0	0	-1	-1	-1	-1	=	1
0	1	0	0	-1	1	0	0	-1	=	2
0	0	1	0	1	0	1	0	1	=	2
0	0	0	1	1	1	1	2	2	=	4

Upper bounds: 1 9 1 4 1 1 1 1 1

The number of possible solutions is: 1

1) 1 3 1 3 1 0 0 0 0
Solution rejected. 3 and 5 occur 3 times together

Case 20

B 2	B 3	B 4	B 5	B 6	B 7	B 8
3	3	3	0	0	2	0

Distribution Vectors

A 2	A 3	A 4	A 7
1	0	1	2
0	2	0	2
2	1	0	2
1	0	3	1
0	2	2	1
2	1	2	1
1	3	1	1
3	2	1	1
1	3	3	0
3	2	3	0

Pair Table

2	2	2	2	1	1	1	2	1	2
2	3	2	1	2	1	3	2	2	1
2	2	3	1	1	2	2	3	1	2
2	1	1	3	2	2	1	2	3	4
1	2	1	2	2	1	2	1	4	3
1	1	2	2	1	2	1	2	3	4
1	3	2	1	2	1	3	3	4	4
2	2	3	2	1	2	3	4	4	5
1	2	1	3	4	3	4	4	6	6
2	1	2	4	3	4	4	5	6	7

Original System of Equations

2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	0.00	0.00	=	14.00
1.00	0.00	0.00	3.00	2.00	2.00	1.00	1.00	3.00	3.00	=	12.00
0.00	2.00	1.00	0.00	2.00	1.00	3.00	2.00	3.00	2.00	=	9.00
1.00	0.00	2.00	1.00	0.00	2.00	1.00	3.00	1.00	3.00	=	6.00

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00 -1.00 -1.00 -2.00 -2.00 -3.00 -3.00 =   0.00
  0.00  1.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00 =   4.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00  2.00  1.00  2.00 =   1.00
  0.00  0.00  0.00  1.00  1.00  1.00  1.00  1.00  2.00  2.00 =   4.00
Column pointer vector:   1  2  3  4  5  6  7  8  9 10

Equations to be solved


   1   0   0   0  -1  -1  -2  -2  -3  -3 =   0
   0   1   0   0   1   0   1   0   1   0 =   4
   0   0   1   0   0   1   1   2   1   2 =   1
   0   0   0   1   1   1   1   1   2   2 =   4

Upper bounds:   9  1  1  1  4  1  1  1  1  1


The number of possible solutions is:  6

  1)    3   1   1   1   3   0   0   0   0   0
  2)    4   1   0   1   2   0   1   0   0   0
Solution rejected.  2 and  7 occur  3 times together
  3)    4   0   1   0   4   0   0   0   0   0
  4)    4   1   0   0   3   1   0   0   0   0
  5)    5   0   0   0   3   0   1   0   0   0
  6)    5   1   0   0   2   0   0   0   1   0
Solution rejected.  5 and  9 occur  4 times together

Check equations:   1  3  4  5


------------------------------------------------
Case   21

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  0     8     1     0     0     2     0


Distribution Vectors

A 3   A 4   A 7
---------------
  2     0     2
  5     0     1
  8     0     0


Pair Table

  2  1  2
  1  2  5
  2  5  8


Original System of Equations

  2.00  1.00  0.00 =  14.00
```

```
  0.00  0.00  0.00 =   4.00
  2.00  5.00  8.00 =  24.00
```

Equations in Row Echelon Form

```
  1.00  0.00 -1.00 =   5.75
  0.00  1.00  2.00 =   2.50
  0.00  0.00  0.00 =   4.00
Column pointer vector:   1  2  3

Equation  3 has LHS = 0 and RHS != 0
*** Block vector rejected
```

Case 22

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	2	0	1	2	1	0

Distribution Vectors

A 2	A 3	A 5	A 6	A 7
0	0	0	2	1
1	0	1	1	1
1	2	0	1	1
3	1	0	1	1
5	0	0	1	1
2	2	1	0	1
4	1	1	0	1
0	1	1	2	0
2	0	1	2	0
2	2	0	2	0
4	1	0	2	0
3	2	1	1	0
5	1	1	1	0

Pair Table

3	2	2	2	2	1	1	2	2	2	2	1	1
2	2	1	1	2	2	2	2	2	1	1	1	2
2	1	3	2	2	3	2	2	1	3	2	2	2
2	1	2	2	4	2	3	1	1	2	3	2	3
2	2	2	4	6	3	5	1	3	3	5	3	5
1	2	3	2	3	4	4	2	1	2	2	3	4
1	2	2	3	5	4	5	1	2	2	3	4	5
2	2	2	1	1	2	1	3	3	3	2	3	2
2	2	1	1	3	1	2	3	3	2	3	2	4
2	1	3	2	3	2	2	3	2	4	4	3	4
2	1	2	3	5	2	3	2	3	4	5	4	5
1	1	2	2	3	3	4	3	2	3	4	4	5
1	2	2	3	5	4	5	2	4	4	5	5	6

```
Original System of Equations

  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00
0.00  0.00 =   7.00
  2.00  1.00  1.00  1.00  1.00  0.00  0.00  2.00  2.00  2.00  2.00
1.00  1.00 =  12.00
  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  0.00  0.00
1.00  1.00 =   5.00
  0.00  0.00  2.00  1.00  0.00  2.00  1.00  1.00  0.00  2.00  1.00
2.00  1.00 =   6.00
  0.00  1.00  1.00  3.00  5.00  2.00  4.00  0.00  2.00  2.00  4.00
3.00  5.00 =  10.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00  0.00  0.00  0.00  0.00 -
1.00 -1.00 =   1.00
  0.00  1.00  0.00  0.00  0.00  1.00  1.00  0.00  0.00 -1.00 -1.00
0.00  0.00 =   3.00
  0.00  0.00  1.00  0.00  0.00  1.00  0.00 -1.00 -1.00  0.00 -1.00
0.00 -1.00 =   1.00
  0.00  0.00  0.00  1.00  0.00  0.00  1.00  2.00  1.00  1.00  2.00
1.00  2.00 =   2.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00
1.00  1.00 =   2.00
Column pointer vector:   1  2  3  4  8  6  7  5  9 10 11 12 13

Equations to be solved


   1   0   0   0   0  -1  -1   0   0   0   0  -1  -1 =    1
   0   1   0   0   0   1   1   0   0  -1  -1   0   0 =    3
   0   0   1   0   0   1   0  -1  -1   0  -1   0  -1 =    1
   0   0   0   1   0   0   1   2   1   1   2   1   2 =    2
   0   0   0   0   1   0   0   0   1   1   1   1   1 =    2

Upper bounds:   1  9  1  2  1  1  1  1  1  1  1  1  1


The number of possible solutions is:  1

  1)    1   4   1   1   1   0   0   0   0   1   0   0   0
Solution rejected.  3 and 10 occur  3 times together


--------------------------------------------------
Case   23

B 2   B 3   B 4   B 5   B 6   B 7   B 8
-------------------------------------------
  3     4     1     0     2     1     0


Distribution Vectors

A 2   A 3   A 4   A 6   A 7
```

0	0	0	2	1
0	1	1	1	1
2	0	1	1	1
1	2	0	1	1
3	1	0	1	1
1	3	1	0	1
3	2	1	0	1
2	4	0	0	1
1	1	1	2	0
3	0	1	2	0
0	3	0	2	0
2	2	0	2	0
0	4	1	1	0
2	3	1	1	0
3	4	0	1	0

Pair Table

3	2	2	2	2	1	1	1	2	2	2	2	1	1	1
2	2	2	1	1	2	2	2	2	2	1	1	2	1	1
2	2	3	1	3	2	4	2	2	4	1	2	1	2	2
2	1	1	1	2	2	2	3	1	2	2	1	2	1	3
2	1	3	2	4	2	4	4	2	4	1	3	1	2	4
1	2	2	2	2	4	4	4	1	2	2	1	4	3	4
1	2	4	2	4	4	5	5	2	4	1	2	3	4	5
1	2	2	3	4	4	5	6	1	2	3	3	4	4	6
2	2	2	1	2	1	2	1	3	4	2	2	3	2	3
2	2	4	2	4	2	4	2	4	6	2	4	2	4	4
2	1	1	2	1	2	1	3	2	2	4	3	4	3	4
2	1	2	1	3	1	2	3	2	4	3	3	3	3	5
1	2	1	2	1	4	3	4	3	2	4	3	5	4	4
1	1	2	1	2	3	4	4	2	4	3	3	4	4	5
1	1	2	3	4	4	5	6	3	4	4	5	4	5	7

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
0.00  0.00  0.00  0.00 =   7.00
  2.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  2.00  2.00  2.00
2.00  1.00  1.00  1.00 =  12.00
  0.00  1.00  1.00  0.00  0.00  1.00  1.00  0.00  1.00  1.00  0.00
0.00  1.00  1.00  0.00 =   4.00
  0.00  1.00  0.00  2.00  1.00  3.00  2.00  4.00  1.00  0.00  3.00
2.00  4.00  3.00  4.00 =  12.00
  0.00  0.00  2.00  1.00  3.00  1.00  3.00  2.00  1.00  3.00  0.00
2.00  0.00  2.00  3.00 =   6.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00  0.00  0.00  0.00
0.00 -1.00 -1.00 -1.00 =   1.00
  0.00  1.00  0.00  0.00  0.00  1.00  0.00  0.00 -1.00 -1.00  0.00 -
1.00  1.00  0.00 -1.00 =   2.00
```

```
  0.00   0.00   1.00   0.00   0.00   0.00   1.00   0.00   1.00   1.00 -1.00
0.00 -1.00   0.00   0.00 =    0.00
  0.00   0.00   0.00   1.00   0.00   1.00   1.00   2.00   1.00   0.00   1.00
1.00   1.00   1.00   2.00 =    4.00
  0.00   0.00   0.00   0.00   1.00   0.00   0.00   0.00   0.00   1.00   1.00
1.00   1.00   1.00   1.00 =    2.00
Column pointer vector:    1  2  3  4  9  6  7  8  5 10 11 12 13 14 15

Equations to be solved


   1    0    0    0    0   -1   -1   -1    0    0    0    0   -1   -1   -1 =     1
   0    1    0    0    0    1    0    0   -1   -1    0   -1    1    0   -1 =     2
   0    0    1    0    0    0    1    0    1    1   -1    0   -1    0    0 =     0
   0    0    0    1    0    1    1    2    1    0    1    1    1    1    2 =     4
   0    0    0    0    1    0    0    0    0    1    1    1    1    1    1 =     2

Upper bounds:    1  9  1  4  1  1  1  1  1  1  1  1  1  1  1


The number of possible solutions is:   6

  1)    1    3    0    2    1    0    0    0    1    0    1    0    0    0    0
  2)    1    2    1    3    1    0    0    0    0    0    1    0    0    0    0
  3)    1    3    0    3    1    0    0    0    0    0    0    1    0    0    0
  4)    1    4    0    1    0    0    0    0    1    0    1    1    0    0    0
Solution rejected.  9 and 12 occur  3 times together
  5)    1    3    0    3    0    0    0    0    0    1    1    0    0    0    0
  6)    1    3    1    2    0    0    0    0    0    0    1    1    0    0    0
Solution rejected. 11 and 12 occur  3 times together

Check equations:   1  2  3  5


--------------------------------------------------
Case   24

B 2   B 3   B 4   B 5   B 6   B 7   B 8
----------------------------------------
  6     0     0     3     1     1     0


Distribution Vectors

A 2   A 5   A 6   A 7
---------------------
  1     1     1     1
  5     0     1     1
  2     2     0     1
  6     1     0     1
  3     2     1     0
  4     3     0     0


Pair Table

  2  2  1  2  1  1
  2  6  2  6  3  3
```

1	2	2	3	1	2
2	6	3	7	3	5
1	3	1	3	2	3
1	3	2	5	3	5

Original System of Equations

```
  1.00  1.00  1.00  1.00  0.00  0.00 =   7.00
  1.00  1.00  0.00  0.00  1.00  0.00 =   6.00
  1.00  0.00  2.00  1.00  2.00  3.00 =  15.00
  1.00  5.00  2.00  6.00  3.00  4.00 =  12.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00 -1.00 -1.00 =   5.00
  0.00  1.00  0.00  0.00  1.00  0.00 =  -1.00
  0.00  0.00  1.00  0.00  1.00  1.00 =   3.00
  0.00  0.00  0.00  1.00  0.00  1.00 =   2.00
Column pointer vector:    1  2  3  5  4  6
```

Equation 2 has LHS >= 0, RHS < 0
*** Block vector rejected

Case   25

B 2	B 3	B 4	B 5	B 6	B 7	B 8
4	2	1	2	1	1	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 6	A 7
1	0	0	1	1	1
0	1	1	0	1	1
2	0	1	0	1	1
1	2	0	0	1	1
3	1	0	0	1	1
0	1	0	2	0	1
2	0	0	2	0	1
1	1	1	1	0	1
3	0	1	1	0	1
2	2	0	1	0	1
4	1	0	1	0	1
3	2	1	0	0	1
0	0	1	2	1	0
1	1	0	2	1	0
3	0	0	2	1	0
0	2	1	1	1	0
2	1	1	1	1	0
4	0	1	1	1	0
3	2	0	1	1	0
4	2	1	0	1	0

1	2	1	2	0	0
3	1	1	2	0	0
4	2	0	2	0	0

Pair Table

2	2	2	2	2	2	2	1	1	1	2	1	2	2	2	1	1	2	1	2	1	1	2
2	3	3	3	2	1	1	2	2	2	1	3	2	1	1	3	2	2	2	3	2	1	1
2	3	3	2	3	1	1	2	3	1	3	3	2	1	2	2	2	4	2	4	1	2	2
2	3	2	4	3	2	1	2	1	3	3	3	1	2	1	3	2	2	3	4	2	1	3
2	2	3	3	4	1	2	1	3	3	4	4	1	1	3	2	2	4	4	5	1	2	4
2	1	1	2	1	3	3	2	2	3	2	2	2	2	2	2	1	1	2	1	3	2	3
2	1	1	1	2	3	3	2	3	2	4	2	2	2	3	1	1	3	2	2	2	3	4
1	2	2	2	1	2	2	2	2	2	2	3	2	1	1	2	1	2	1	3	3	2	3
1	2	3	1	3	2	3	2	4	2	4	4	2	1	3	1	2	4	2	4	2	4	4
1	2	1	3	3	3	2	2	2	3	4	4	1	2	2	2	1	2	3	4	3	3	5
2	1	3	3	4	2	4	2	4	4	5	5	1	2	4	1	2	4	4	5	3	4	6
1	3	3	3	4	2	2	3	4	4	5	6	1	1	2	3	3	4	4	6	3	4	5
2	2	2	1	1	2	2	2	2	1	1	1	4	3	3	3	3	3	2	2	3	3	2
2	1	1	2	1	2	2	1	1	2	2	1	3	3	3	3	2	3	3	3	3	2	4
2	1	2	1	3	2	3	1	3	2	4	2	3	3	5	2	3	5	4	4	2	4	5
1	3	2	3	2	2	1	2	1	2	1	3	3	3	2	4	3	2	3	4	4	3	3
1	2	2	2	2	1	1	1	2	1	2	3	3	2	3	3	2	4	3	5	3	3	4
2	2	4	2	4	1	3	2	4	2	4	4	3	3	5	2	4	6	4	6	3	5	5
1	2	2	3	4	2	2	1	2	3	4	4	2	3	4	3	3	4	5	6	3	4	6
2	3	4	4	5	1	2	3	4	4	5	6	2	3	4	4	5	6	6	8	4	5	6
1	2	1	2	1	3	2	3	2	3	3	3	3	3	2	4	3	3	3	4	5	4	5
1	1	2	1	2	2	3	2	4	3	4	4	3	2	4	3	3	5	4	5	4	5	6
2	1	2	3	4	3	4	3	4	5	6	5	2	4	5	3	4	5	6	6	5	6	8

Original System of Equations

1.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00
1.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00
0.00 = 7.00
1.00 1.00 1.00 1.00 1.00 0.00 0.00 0.00 0.00 0.00 0.00
0.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00 1.00 0.00 0.00
0.00 = 6.00
1.00 0.00 0.00 0.00 0.00 2.00 2.00 1.00 1.00 1.00 1.00
0.00 2.00 2.00 2.00 1.00 1.00 1.00 1.00 0.00 2.00 2.00
2.00 = 10.00
0.00 1.00 1.00 0.00 0.00 0.00 0.00 1.00 1.00 0.00 0.00
1.00 1.00 0.00 0.00 1.00 1.00 1.00 0.00 1.00 1.00 1.00
0.00 = 4.00
0.00 1.00 0.00 2.00 1.00 1.00 0.00 1.00 0.00 2.00 1.00
2.00 0.00 1.00 0.00 2.00 1.00 0.00 2.00 2.00 2.00 1.00
2.00 = 6.00
1.00 0.00 2.00 1.00 3.00 0.00 2.00 1.00 3.00 2.00 4.00
3.00 0.00 1.00 3.00 0.00 2.00 4.00 3.00 4.00 1.00 3.00
4.00 = 8.00

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 -1.00 -
2.00  2.00  2.00  2.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
0.00 =   4.00
  0.00  1.00  0.00  0.00  0.00  0.00 -1.00  0.00 -1.00 -1.00 -2.00 -
1.00  1.00  0.00 -1.00  1.00  0.00 -1.00 -1.00 -1.00 -1.00 -1.00 -
2.00 =   1.00
  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  2.00  1.00  2.00
2.00 -1.00 -1.00  0.00 -1.00  0.00  1.00  0.00  1.00  1.00  1.00
1.00 =   1.00
  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00  1.00  1.00
1.00 -1.00  0.00  0.00  0.00  0.00  0.00  1.00  1.00  1.00  0.00
1.00 =   0.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00
0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  1.00
1.00 =   2.00
  0.00  0.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  1.00  1.00
1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00  0.00  0.00
0.00 =   1.00
Column pointer vector:   1  2  3  4 21  6  7  8  9 10 11 12 13 14 15
16 17 18 19 20  5 22 23

Equations to be solved


   1   0   0   0   0   0   0  -1  -1  -1  -1  -2   2   2   2   1   1
1   1   0   0   0   0 =    4
   0   1   0   0   0   0  -1   0  -1  -1  -2  -1   1   0  -1   1   0
-1  -1  -1  -1  -1  -2 =    1
   0   0   1   0   0   0   1   1   2   1   2   2  -1  -1   0  -1   0
1   0   1   1   1   1 =    1
   0   0   0   1   0   0   0   0   0   1   1   1  -1   0   0   0   0
0   1   1   1   0   1 =    0
   0   0   0   0   1   0   0   0   0   0   0   0   1   1   1   1   1
1   1   1   0   1   1 =    2
   0   0   0   0   0   1   1   1   1   1   1   1  -1  -1  -1  -1  -1
-1  -1  -1   0   0   0 =    1

Upper bounds:    9  1  1  1  1  1  1  9  1  1  1  1  1  1  1  1  2  1
1  1  1  1  1


The number of possible solutions is: 56

  1)    1   1   1   1   0   1   1   1   0   0   0   0   1   1   0   0
0   0   0   0   0   0   0
Solution rejected.  2 and  3 occur  3 times together
  2)    2   1   0   0   0   1   1   1   0   0   0   0   1   0   0   1
0   0   0   0   1   0   0
Solution rejected.  2 and 16 occur  3 times together
  3)    2   0   1   1   0   1   1   1   0   0   0   0   1   0   0   1
0   0   0   0   0   0   0
Solution rejected.  4 and 16 occur  3 times together
  4)    2   1   0   1   0   1   1   1   0   0   0   0   1   0   0   0
1   0   0   0   0   0   0
Solution rejected.  2 and  4 occur  3 times together
  5)    2   1   1   0   0   1   1   1   0   0   0   0   0   1   0   1
0   0   0   0   0   0   0
Solution rejected.  2 and  3 occur  3 times together
```

```
  6)     3    1    0    0    0    1    1    1    0    0    0    0    0    0    0    1
1    0    0    0    0    0    0
Solution rejected.  2 and 16 occur  3 times together
  7)     2    1    0    1    0    1    1    0    1    0    0    0    1    0    0    1
0    0    0    0    0    0    0
Solution rejected.  2 and  4 occur  3 times together
  8)     2    1    1    0    0    1    1    0    0    1    0    0    1    0    0    1
0    0    0    0    0    0    0
Solution rejected.  2 and  3 occur  3 times together
  9)     3    1    0    0    0    1    1    0    0    0    0    1    1    0    0    1
0    0    0    0    0    0    0
Solution rejected.  2 and 12 occur  3 times together
 10)     2    1    1    1    1    1    1    0    0    0    0    0    1    0    0    0
0    0    0    0    0    0    0
Solution rejected.  2 and  3 occur  3 times together
 11)     3    1    1    0    1    1    1    0    0    0    0    0    0    0    0    1
0    0    0    0    0    0    0
Solution rejected.  2 and  3 occur  3 times together
 12)     2    1    0    0    0    1    0    2    0    0    0    0    1    1    0    0
0    0    0    0    1    0    0
Solution rejected. 13 and 14 occur  3 times together
 13)     2    0    1    1    0    1    0    2    0    0    0    0    1    1    0    0
0    0    0    0    0    0    0
Solution rejected. 13 and 14 occur  3 times together
 14)     2    1    0    1    0    1    0    2    0    0    0    0    1    0    1    0
0    0    0    0    0    0    0
Solution rejected.  2 and  4 occur  3 times together
 15)     3    0    0    0    0    1    0    2    0    0    0    0    1    0    0    1
0    0    0    0    1    0    0
Solution rejected. 13 and 16 occur  3 times together
 16)     3    0    0    1    0    1    0    2    0    0    0    0    1    0    0    0
1    0    0    0    0    0    0
Solution rejected. 13 and 17 occur  3 times together
 17)     3    1    0    0    0    1    0    2    0    0    0    0    1    0    0    0
0    0    1    0    0    0    0
 18)     3    0    1    0    0    1    0    2    0    0    0    0    0    1    0    1
0    0    0    0    0    0    0
Solution rejected. 14 and 16 occur  3 times together
 19)     3    1    0    0    0    1    0    2    0    0    0    0    0    1    0    0
1    0    0    0    0    0    0
 20)     3    1    0    0    0    1    0    2    0    0    0    0    0    0    1    1
0    0    0    0    0    0    0
Solution rejected.  2 and 16 occur  3 times together
 21)     4    0    0    0    0    1    0    2    0    0    0    0    0    0    0    1
1    0    0    0    0    0    0
Solution rejected. 16 and 17 occur  3 times together
 22)     2    1    0    1    0    1    0    1    1    0    0    0    1    1    0    0
0    0    0    0    0    0    0
Solution rejected.  2 and  4 occur  3 times together
 23)     3    0    0    1    0    1    0    1    1    0    0    0    1    0    0    1
0    0    0    0    0    0    0
Solution rejected.  4 and 16 occur  3 times together
 24)     3    1    0    0    0    1    0    1    1    0    0    0    0    1    0    1
0    0    0    0    0    0    0
Solution rejected.  2 and 16 occur  3 times together
 25)     2    1    1    0    0    1    0    1    0    1    0    0    1    1    0    0
0    0    0    0    0    0    0
Solution rejected.  2 and  3 occur  3 times together
```

26) 3 0 1 0 0 1 0 1 0 1 0 0 1 0 0 1
0 0 0 0 0 0 0
Solution rejected. 6 and 10 occur 3 times together
27) 3 1 0 0 0 1 0 1 0 1 0 0 1 0 0 0
1 0 0 0 0 0 0
Solution rejected. 6 and 10 occur 3 times together
28) 3 1 0 0 0 1 0 1 0 0 1 0 1 0 0 1
0 0 0 0 0 0 0
Solution rejected. 2 and 16 occur 3 times together
29) 3 1 0 0 0 1 0 1 0 0 0 1 1 1 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 12 occur 3 times together
30) 4 0 0 0 0 1 0 1 0 0 0 1 1 0 0 1
0 0 0 0 0 0 0
Solution rejected. 8 and 12 occur 3 times together
31) 3 1 0 0 1 1 0 1 0 0 0 0 1 0 0 0
0 0 0 0 1 0 0
Solution rejected. 5 and 6 occur 3 times together
32) 3 1 0 1 0 1 0 1 0 0 0 0 1 0 0 0
0 0 0 0 0 1 0
Solution rejected. 2 and 4 occur 3 times together
33) 3 0 1 1 1 1 0 1 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0
Solution rejected. 5 and 6 occur 3 times together
34) 3 1 1 0 1 1 0 1 0 0 0 0 0 1 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 3 occur 3 times together
35) 4 1 0 0 0 1 0 1 0 0 0 0 0 0 0 1
0 0 0 0 0 1 0
Solution rejected. 2 and 16 occur 3 times together
36) 4 0 1 0 1 1 0 1 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0
Solution rejected. 5 and 6 occur 3 times together
37) 4 1 0 0 1 1 0 1 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0
Solution rejected. 5 and 6 occur 3 times together
38) 3 1 0 0 0 1 0 0 1 1 0 0 1 0 0 1
0 0 0 0 0 0 0
Solution rejected. 2 and 16 occur 3 times together
39) 3 1 0 1 1 1 0 0 1 0 0 0 1 0 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 4 occur 3 times together
40) 4 1 0 0 1 1 0 0 1 0 0 0 0 0 0 1
0 0 0 0 0 0 0
Solution rejected. 2 and 16 occur 3 times together
41) 3 1 1 0 1 1 0 0 0 1 0 0 1 0 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 3 occur 3 times together
42) 4 1 0 0 1 1 0 0 0 0 0 1 1 0 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 12 occur 3 times together
43) 2 1 0 1 0 0 1 2 0 0 0 0 1 1 0 0
0 0 0 0 0 0 0
Solution rejected. 2 and 4 occur 3 times together
44) 3 0 0 1 0 0 1 2 0 0 0 0 1 0 0 1
0 0 0 0 0 0 0
Solution rejected. 4 and 16 occur 3 times together

```
 45)    3   1   0   0   0   0   1   2   0   0   0   0   0   1   0   1
0   0   0   0   0   0   0
Solution rejected.  2 and 16 occur  3 times together
 46)    3   1   0   0   0   0   1   1   0   1   0   0   1   0   0   1
0   0   0   0   0   0   0
Solution rejected.  2 and 16 occur  3 times together
 47)    3   1   0   1   1   0   1   1   0   0   0   0   1   0   0   0
0   0   0   0   0   0   0
Solution rejected.  2 and  4 occur  3 times together
 48)    4   1   0   0   1   0   1   1   0   0   0   0   0   0   0   1
0   0   0   0   0   0   0
Solution rejected.  2 and 16 occur  3 times together
 49)    3   0   0   1   0   0   0   3   0   0   0   0   1   1   0   0
0   0   0   0   0   0   0
Solution rejected. 13 and 14 occur  3 times together
 50)    4   0   0   0   0   0   0   3   0   0   0   0   0   1   0   1
0   0   0   0   0   0   0
Solution rejected. 14 and 16 occur  3 times together
 51)    3   1   0   0   0   0   0   2   0   1   0   0   1   1   0   0
0   0   0   0   0   0   0
Solution rejected. 13 and 14 occur  3 times together
 52)    4   0   0   0   0   0   0   2   0   1   0   0   1   0   0   1
0   0   0   0   0   0   0
Solution rejected. 13 and 16 occur  3 times together
 53)    4   0   0   1   1   0   0   2   0   0   0   0   1   0   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and  8 occur  3 times together
 54)    4   1   0   0   1   0   0   2   0   0   0   0   0   1   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and  8 occur  3 times together
 55)    5   0   0   0   1   0   0   2   0   0   0   0   0   0   0   1
0   0   0   0   0   0   0
Solution rejected.  5 and  8 occur  3 times together
 56)    4   1   0   0   1   0   0   1   0   1   0   0   1   0   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and  8 occur  3 times together

Check equations:  17 19


--------------------------------------------------
Case   26

B 2   B 3   B 4   B 5   B 6   B 7   B 8
-------------------------------------
  2     4     2     1     1     1     0


Distribution Vectors

A 2   A 3   A 4   A 5   A 6   A 7
-------------------------------
  1     0     0     1     1     1
  0     1     1     0     1     1
  2     0     1     0     1     1
  1     2     0     0     1     1
  0     0     2     1     0     1
  1     1     1     1     0     1
```

0	3	0	1	0	1
2	2	0	1	0	1
0	2	2	0	0	1
2	1	2	0	0	1
1	3	1	0	0	1
2	4	0	0	0	1
1	0	2	1	1	0
0	2	1	1	1	0
2	1	1	1	1	0
1	3	0	1	1	0
1	2	2	0	1	0
0	4	1	0	1	0
2	3	1	0	1	0
0	3	2	1	0	0
2	2	2	1	0	0
1	4	1	1	0	0
2	4	2	0	0	0

Pair Table

3	2	3	2	2	2	2	3	1	2	1	2	2	2	3	2	1	1	2	1	2	1	1
2	2	2	2	2	1	1	1	2	2	1	2	2	1	1	1	2	2	1	1	1	1	2
3	2	4	3	2	2	1	3	2	4	2	3	3	1	3	2	3	1	3	1	3	1	3
2	2	3	2	1	1	2	2	1	2	2	4	1	1	2	2	1	3	3	1	1	2	3
2	2	2	1	4	3	2	2	3	3	2	1	3	2	2	1	2	1	1	3	3	2	2
2	1	2	1	3	2	2	3	2	3	1	3	2	1	2	1	1	1	1	2	3	2	3
2	1	1	2	2	2	4	3	2	1	3	4	1	2	1	3	1	3	2	3	2	4	3
3	1	3	2	2	3	3	4	1	3	3	5	2	1	3	3	1	2	3	2	3	4	4
1	2	2	1	3	2	2	1	3	3	3	3	2	1	1	1	2	3	2	3	2	3	4
2	2	4	2	3	3	1	3	3	5	3	4	3	1	3	1	3	2	3	2	4	3	5
1	1	2	2	2	1	3	3	3	3	3	5	1	1	1	2	2	3	3	3	3	3	5
2	2	3	4	1	3	4	5	3	4	5	7	1	2	3	4	3	4	5	3	4	5	6
2	2	3	1	3	2	1	2	2	3	1	1	4	3	4	2	3	2	3	3	4	2	3
2	1	1	1	2	1	2	1	1	1	1	2	3	2	2	3	2	3	2	3	2	3	3
3	1	3	2	2	2	1	3	1	3	1	3	4	2	4	3	3	2	3	2	4	3	4
2	1	2	2	1	1	3	3	1	1	2	4	2	3	3	4	2	4	4	3	3	4	4
1	2	3	1	2	1	1	1	2	3	2	3	3	2	3	2	3	4	4	3	3	3	5
1	2	1	3	1	1	3	2	3	2	3	4	2	3	2	4	4	5	4	4	3	4	5
2	1	3	3	1	1	2	3	2	3	3	5	3	2	3	4	4	4	5	3	4	4	6
1	1	1	1	3	2	3	2	3	2	3	3	3	3	2	3	3	4	3	5	4	5	5
2	1	3	1	3	3	2	3	2	4	3	4	4	2	4	3	3	3	4	4	5	5	6
1	1	1	2	2	2	4	4	3	3	3	5	2	3	3	4	3	4	4	5	5	5	6
1	2	3	3	2	3	3	4	4	5	5	6	3	3	4	4	5	5	6	5	6	6	8

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00
1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
0.00 =    7.00
  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
0.00 =    6.00
  1.00  0.00  0.00  0.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00
0.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00
0.00 =    5.00
```

```
  0.00  1.00  1.00  0.00  2.00  1.00  0.00  0.00  2.00  2.00  1.00
0.00  2.00  1.00  1.00  0.00  2.00  1.00  1.00  2.00  2.00  1.00
2.00 =   8.00
  0.00  1.00  0.00  2.00  0.00  1.00  3.00  2.00  2.00  1.00  3.00
4.00  0.00  2.00  1.00  3.00  2.00  4.00  3.00  3.00  2.00  4.00
4.00 =  12.00
  1.00  0.00  2.00  1.00  0.00  1.00  0.00  2.00  0.00  2.00  1.00
2.00  1.00  0.00  2.00  1.00  1.00  0.00  2.00  0.00  2.00  1.00
2.00 =   4.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 -
1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00 -
1.00 =   2.00
  0.00  1.00  0.00  0.00  0.00  0.00 -1.00 -2.00  0.00 -1.00 -1.00 -
2.00  1.00  1.00  0.00  0.00  1.00  1.00  0.00 -1.00 -1.00 -1.00 -
1.00 =   2.00
  0.00  0.00  1.00  0.00  0.00  0.00 -1.00  0.00  0.00  1.00  0.00
0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00
1.00 =   0.00
  0.00  0.00  0.00  1.00  0.00  0.00  2.00  2.00  1.00  1.00  2.00
3.00 -2.00 -1.00 -1.00  0.00 -1.00  0.00  0.00  1.00  0.00  1.00
1.00 =   2.00
  0.00  0.00  0.00  0.00  1.00  0.00  1.00  1.00  1.00  1.00  1.00
1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00  1.00  0.00  0.00
0.00 =   1.00
  0.00  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00
0.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  1.00  1.00
1.00 =   2.00
Column pointer vector:   1  2  3  4  5 20  7  8  9 10 11 12 13 14 15
16 17 18 19  6 21 22 23

Equations to be solved


   1   0   0   0   0   0   0   0  -1  -1  -1  -1   1   1   1   1   0
0   0   0   0   0  -1 =    2
   0   1   0   0   0   0  -1  -2   0  -1  -1  -2   1   1   0   0   1
1   0  -1  -1  -1  -1 =    2
   0   0   1   0   0   0  -1   0   0   1   0   0   1   0   1   0   1
0   1   0   1   0   1 =    0
   0   0   0   1   0   0   2   2   1   1   2   3  -2  -1  -1   0  -1
0   0   1   0   1   1 =    2
   0   0   0   0   1   0   1   1   1   1   1   1  -1  -1  -1  -1  -1
-1  -1   1   0   0   0 =    1
   0   0   0   0   0   1   0   0   0   0   0   0   1   1   1   1   1
1   1   0   1   1   1 =    2

Upper bounds:   1  9  1  9  1  1  1  1  1  1  1  1  1  2  1  1  1  1
1  9  1  1  1


The number of possible solutions is: 94

  1)    1   4   0   0   0   1   1   1   0   0   0   0   1   0   0   0
0   0   0   0   0   0   0
```

```
Solution rejected.  1 and  8 occur  3 times together
  2)     1    3    0    1    0    1    1    0    0    0    0    0    1    0    0    0
0    0    0    1    0    0    0
Solution rejected.  6 and  7 occur  3 times together
  3)     1    2    0    2    1    1    1    0    0    0    0    0    1    0    0    0
0    0    0    0    0    0    0
Solution rejected.  5 and  6 occur  3 times together
  4)     1    3    1    0    0    1    1    0    0    0    0    0    0    1    0    0
0    0    0    1    0    0    0
Solution rejected.  1 and  3 occur  3 times together
  5)     1    2    1    1    1    1    1    0    0    0    0    0    0    1    0    0
0    0    0    0    0    0    0
Solution rejected.  1 and  3 occur  3 times together
  6)     1    4    0    0    0    1    1    0    0    0    0    0    0    0    1    0
0    0    0    1    0    0    0
Solution rejected.  1 and 15 occur  3 times together
  7)     1    3    0    1    1    1    1    0    0    0    0    0    0    0    1    0
0    0    0    0    0    0    0
Solution rejected.  1 and 15 occur  3 times together
  8)     1    3    1    0    1    1    1    0    0    0    0    0    0    0    0    1
0    0    0    0    0    0    0
Solution rejected.  1 and  3 occur  3 times together
  9)     1    4    0    0    0    1    0    1    0    0    0    0    0    1    0    0
0    0    0    1    0    0    0
Solution rejected.  1 and  8 occur  3 times together
 10)     1    3    0    1    1    1    0    1    0    0    0    0    0    1    0    0
0    0    0    0    0    0    0
Solution rejected.  1 and  8 occur  3 times together
 11)     1    4    0    0    1    1    0    1    0    0    0    0    0    0    0    1
0    0    0    0    0    0    0
Solution rejected.  1 and  8 occur  3 times together
 12)     1    3    0    1    0    1    0    0    0    0    0    0    0    1    0    0
0    0    0    2    0    0    0
Solution rejected.  6 and 14 occur  3 times together
 13)     1    2    0    2    1    1    0    0    0    0    0    0    0    1    0    0
0    0    0    1    0    0    0
Solution rejected.  5 and  6 occur  3 times together
 14)     1    4    0    0    0    1    0    0    0    0    0    0    0    0    0    1
0    0    0    2    0    0    0
Solution rejected.  6 and 16 occur  3 times together
 15)     1    3    0    1    1    1    0    0    0    0    0    0    0    0    0    1
0    0    0    1    0    0    0
Solution rejected.  5 and  6 occur  3 times together
 16)     1    3    0    0    0    0    1    1    1    0    0    0    1    1    0    0
0    0    0    0    0    0    0
Solution rejected.  1 and  8 occur  3 times together
 17)     0    4    0    0    0    0    1    1    0    0    0    0    1    1    0    0
0    0    0    1    0    0    0
Solution rejected.  7 and  8 occur  3 times together
 18)     0    3    0    1    1    0    1    1    0    0    0    0    1    1    0    0
0    0    0    0    0    0    0
Solution rejected.  5 and 13 occur  3 times together
 19)     0    4    0    0    1    0    1    1    0    0    0    0    1    0    0    1
0    0    0    0    0    0    0
Solution rejected.  5 and 13 occur  3 times together
 20)     1    3    0    0    1    0    1    1    0    0    0    0    1    0    0    0
0    1    0    0    0    0    0
Solution rejected.  1 and  8 occur  3 times together
```

```
 21)    0   3   1   0   1   0   1   1   0   0   0   0   0   2   0   0
0   0   0   0   0   0   0
Solution rejected.  3 and  8 occur  3 times together
 22)    0   4   0   0   1   0   1   1   0   0   0   0   0   1   1   0
0   0   0   0   0   0   0
Solution rejected.  7 and  8 occur  3 times together
 23)    1   3   0   0   1   0   1   1   0   0   0   0   0   1   0   0
1   0   0   0   0   0   0
Solution rejected.  1 and  8 occur  3 times together
 24)    1   2   0   1   0   0   1   0   1   0   0   0   1   1   0   0
0   0   0   1   0   0   0
Solution rejected. 13 and 14 occur  3 times together
 25)    1   1   0   2   1   0   1   0   1   0   0   0   1   1   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and  9 occur  3 times together
 26)    1   3   0   0   0   0   1   0   1   0   0   0   1   0   0   1
0   0   0   1   0   0   0
Solution rejected.  7 and 16 occur  3 times together
 27)    1   2   0   1   1   0   1   0   1   0   0   0   1   0   0   1
0   0   0   0   0   0   0
Solution rejected.  5 and  9 occur  3 times together
 28)    1   2   1   0   0   0   1   0   1   0   0   0   0   2   0   0
0   0   0   1   0   0   0
Solution rejected.  1 and  3 occur  3 times together
 29)    1   1   1   1   1   0   1   0   1   0   0   0   0   2   0   0
0   0   0   0   0   0   0
Solution rejected.  1 and  3 occur  3 times together
 30)    1   3   0   0   0   0   1   0   1   0   0   0   0   1   1   0
0   0   0   1   0   0   0
Solution rejected.  1 and 15 occur  3 times together
 31)    1   2   0   1   1   0   1   0   1   0   0   0   0   1   1   0
0   0   0   0   0   0   0
Solution rejected.  1 and 15 occur  3 times together
 32)    1   2   1   0   1   0   1   0   1   0   0   0   0   1   0   1
0   0   0   0   0   0   0
Solution rejected.  1 and  3 occur  3 times together
 33)    1   3   0   0   1   0   1   0   1   0   0   0   0   0   1   1
0   0   0   0   0   0   0
Solution rejected.  1 and 15 occur  3 times together
 34)    1   3   0   0   0   0   1   0   0   1   0   0   0   2   0   0
0   0   0   1   0   0   0
Solution rejected. 10 and 20 occur  3 times together
 35)    1   2   0   1   1   0   1   0   0   1   0   0   0   2   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and 10 occur  3 times together
 36)    1   3   0   0   1   0   1   0   0   1   0   0   0   1   0   1
0   0   0   0   0   0   0
Solution rejected.  5 and 10 occur  3 times together
 37)    1   3   0   0   0   0   1   0   0   0   1   0   1   1   0   0
0   0   0   1   0   0   0
Solution rejected.  7 and 11 occur  3 times together
 38)    1   2   0   1   1   0   1   0   0   0   1   0   1   1   0   0
0   0   0   0   0   0   0
Solution rejected.  5 and 13 occur  3 times together
 39)    1   3   0   0   1   0   1   0   0   0   1   0   1   0   0   1
0   0   0   0   0   0   0
Solution rejected.  5 and 13 occur  3 times together
```

40) 1 2 1 0 1 0 1 0 0 0 1 0 0 2 0 0
0 0 0 0 0 0 0
Solution rejected. 1 and 3 occur 3 times together
41) 1 3 0 0 1 0 1 0 0 0 1 0 0 1 1 0
0 0 0 0 0 0 0
Solution rejected. 1 and 15 occur 3 times together
42) 1 3 0 0 1 0 1 0 0 0 0 1 1 1 0 0
0 0 0 0 0 0 0
Solution rejected. 5 and 13 occur 3 times together
43) 0 3 0 1 0 0 1 0 0 0 0 0 1 1 0 0
0 0 0 2 0 0 0
Solution rejected. 13 and 14 occur 3 times together
44) 0 2 0 2 1 0 1 0 0 0 0 0 1 1 0 0
0 0 0 1 0 0 0
Solution rejected. 5 and 13 occur 3 times together
45) 0 4 0 0 0 0 1 0 0 0 0 0 1 0 0 1
0 0 0 2 0 0 0
Solution rejected. 7 and 16 occur 3 times together
46) 0 3 0 1 1 0 1 0 0 0 0 0 1 0 0 1
0 0 0 1 0 0 0
Solution rejected. 5 and 13 occur 3 times together
47) 1 3 0 0 0 0 1 0 0 0 0 0 1 0 0 0
0 1 0 2 0 0 0
Solution rejected. 7 and 18 occur 3 times together
48) 1 2 0 1 1 0 1 0 0 0 0 0 1 0 0 0
0 1 0 1 0 0 0
Solution rejected. 4 and 18 occur 3 times together
49) 1 4 0 0 0 0 1 0 0 0 0 0 1 0 0 0
0 0 0 1 0 1 0
Solution rejected. 7 and 22 occur 4 times together
50) 1 3 0 1 1 0 1 0 0 0 0 0 1 0 0 0
0 0 0 0 0 1 0
Solution rejected. 5 and 13 occur 3 times together
51) 0 3 1 0 0 0 1 0 0 0 0 0 0 2 0 0
0 0 0 2 0 0 0
52) 0 2 1 1 1 0 1 0 0 0 0 0 0 2 0 0
0 0 0 1 0 0 0
Solution rejected. 3 and 4 occur 3 times together
53) 0 4 0 0 0 0 1 0 0 0 0 0 0 1 1 0
0 0 0 2 0 0 0
54) 0 3 0 1 1 0 1 0 0 0 0 0 0 1 1 0
0 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
55) 0 3 1 0 1 0 1 0 0 0 0 0 0 1 0 1
0 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
56) 1 3 0 0 0 0 1 0 0 0 0 0 0 1 0 0
1 0 0 2 0 0 0
57) 1 2 0 1 1 0 1 0 0 0 0 0 0 1 0 0
1 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
58) 1 2 1 0 1 0 1 0 0 0 0 0 1 0 0
0 1 0 1 0 0 0
Solution rejected. 1 and 3 occur 3 times together
59) 1 3 0 0 1 0 1 0 0 0 0 0 0 1 0 0
0 0 1 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together

60) 1 4 0 0 0 0 1 0 0 0 0 0 0 1 0 0
0 0 0 1 1 0 0
Solution rejected. 20 and 21 occur 3 times together
61) 1 3 0 1 1 0 1 0 0 0 0 0 0 1 0 0
0 0 0 0 1 0 0
Solution rejected. 5 and 21 occur 3 times together
62) 1 3 1 0 1 0 1 0 0 0 0 0 0 1 0 0
0 0 0 0 0 1 0
Solution rejected. 1 and 3 occur 3 times together
63) 0 4 0 0 1 0 1 0 0 0 0 0 0 0 1 1
0 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
64) 1 3 0 0 1 0 1 0 0 0 0 0 0 0 1 0
0 1 0 1 0 0 0
Solution rejected. 1 and 15 occur 3 times together
65) 1 4 0 0 1 0 1 0 0 0 0 0 0 0 1 0
0 0 0 0 0 1 0
Solution rejected. 1 and 15 occur 3 times together
66) 1 3 0 0 1 0 1 0 0 0 0 0 0 0 0 1
1 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
67) 1 4 0 0 1 0 1 0 0 0 0 0 0 0 0 1
0 0 0 0 1 0 0
Solution rejected. 5 and 21 occur 3 times together
68) 1 3 0 0 0 0 0 1 1 0 0 0 0 2 0 0
0 0 0 1 0 0 0
Solution rejected. 1 and 8 occur 3 times together
69) 1 2 0 1 1 0 0 1 1 0 0 0 0 2 0 0
0 0 0 0 0 0 0
Solution rejected. 1 and 8 occur 3 times together
70) 1 3 0 0 1 0 0 1 1 0 0 0 0 1 0 1
0 0 0 0 0 0 0
Solution rejected. 1 and 8 occur 3 times together
71) 1 3 0 0 1 0 0 1 0 0 1 0 0 2 0 0
0 0 0 0 0 0 0
Solution rejected. 1 and 8 occur 3 times together
72) 0 4 0 0 0 0 0 1 0 0 0 0 0 2 0 0
0 0 0 2 0 0 0
Solution rejected. 8 and 20 occur 3 times together
73) 0 3 0 1 1 0 0 1 0 0 0 0 0 2 0 0
0 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
74) 0 4 0 0 1 0 0 1 0 0 0 0 0 1 0 1
0 0 0 1 0 0 0
Solution rejected. 5 and 20 occur 3 times together
75) 1 3 0 0 1 0 0 1 0 0 0 0 0 1 0 0
0 1 0 1 0 0 0
Solution rejected. 1 and 8 occur 3 times together
76) 1 4 0 0 1 0 0 1 0 0 0 0 0 1 0 0
0 0 0 0 0 1 0
Solution rejected. 1 and 8 occur 3 times together
77) 1 2 0 1 0 0 0 0 1 0 0 0 0 2 0 0
0 0 0 2 0 0 0
78) 1 1 0 2 1 0 0 0 1 0 0 0 0 2 0 0
0 0 0 1 0 0 0
Solution rejected. 5 and 9 occur 3 times together
79) 1 3 0 0 0 0 0 0 1 0 0 0 0 1 0 1
0 0 0 2 0 0 0

```
Solution rejected. 14 and 16 occur  3 times together
 80)    1   2   0   1   1   0   0   0   1   0   0   0   0   1   0   1
0   0   0   1   0   0   0
Solution rejected.  5 and  9 occur  3 times together
 81)    1   3   0   0   0   0   0   0   0   0   1   0   0   2   0   0
0   0   0   2   0   0   0
 82)    1   2   0   1   1   0   0   0   0   0   1   0   0   2   0   0
0   0   0   1   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 83)    1   3   0   0   1   0   0   0   0   0   1   0   0   1   0   1
0   0   0   1   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 84)    1   3   0   0   1   0   0   0   0   0   0   1   0   2   0   0
0   0   0   1   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 85)    0   3   0   1   0   0   0   0   0   0   0   0   0   2   0   0
0   0   0   3   0   0   0
 86)    0   2   0   2   1   0   0   0   0   0   0   0   0   2   0   0
0   0   0   2   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 87)    0   4   0   0   0   0   0   0   0   0   0   0   0   1   0   1
0   0   0   3   0   0   0
Solution rejected. 14 and 16 occur  3 times together
 88)    0   3   0   1   1   0   0   0   0   0   0   0   0   1   0   1
0   0   0   2   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 89)    1   3   0   0   0   0   0   0   0   0   0   0   0   1   0   0
0   1   0   3   0   0   0
Solution rejected. 14 and 18 occur  3 times together
 90)    1   2   0   1   1   0   0   0   0   0   0   0   0   1   0   0
0   1   0   2   0   0   0
Solution rejected.  4 and 18 occur  3 times together
 91)    1   4   0   0   0   0   0   0   0   0   0   0   0   1   0   0
0   0   0   2   0   1   0
Solution rejected. 14 and 22 occur  3 times together
 92)    1   3   0   1   1   0   0   0   0   0   0   0   0   1   0   0
0   0   0   1   0   1   0
Solution rejected.  5 and 20 occur  3 times together
 93)    1   3   0   0   1   0   0   0   0   0   0   0   0   0   0   1
0   1   0   2   0   0   0
Solution rejected.  5 and 20 occur  3 times together
 94)    1   4   0   0   1   0   0   0   0   0   0   0   0   0   0   1
0   0   0   1   0   1   0
Solution rejected.  5 and 20 occur  3 times together

Check equations:  51 53 56 77 81 85


--------------------------------------------------
Case   27

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  3     1     5     0     1     1     0


Distribution Vectors
```

A 2	A 3	A 4	A 6	A 7
0	1	1	1	1
2	0	1	1	1
3	1	0	1	1
1	0	3	0	1
2	1	2	0	1
0	1	3	1	0
2	0	3	1	0
3	1	2	1	0
1	0	5	0	0
2	1	4	0	0

Pair Table

3	2	3	1	2	2	1	2	1	1
2	3	4	1	2	1	2	3	1	1
3	4	6	2	4	2	3	5	1	3
1	1	2	2	1	1	1	1	3	2
2	2	4	1	3	1	1	3	2	3
2	1	2	1	1	3	2	2	3	3
1	2	3	1	1	2	3	3	3	3
2	3	5	1	3	2	3	5	3	4
1	1	1	3	2	3	3	3	5	4
1	1	3	2	3	3	3	4	4	5

Original System of Equations

1.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	=	7.00
1.00	1.00	1.00	0.00	0.00	1.00	1.00	1.00	0.00	0.00	=	6.00
1.00	1.00	0.00	3.00	2.00	3.00	3.00	2.00	5.00	4.00	=	20.00
1.00	0.00	1.00	0.00	1.00	1.00	0.00	1.00	0.00	1.00	=	3.00
0.00	2.00	3.00	1.00	2.00	0.00	2.00	3.00	1.00	2.00	=	6.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	=	4.00
0.00	1.00	0.00	0.00	0.00	0.00	1.00	0.00	-1.00	-1.00	=	3.00
0.00	0.00	1.00	0.00	0.00	0.00	0.00	1.00	1.00	1.00	=	-1.00
0.00	0.00	0.00	1.00	0.00	-1.00	-1.00	-1.00	1.00	0.00	=	1.00
0.00	0.00	0.00	0.00	1.00	1.00	1.00	1.00	0.00	1.00	=	2.00

Column pointer vector: 1 2 3 4 9 6 7 8 5 10

Equation 3 has LHS >= 0, RHS < 0
*** Block vector rejected

Case 28

B 2	B 3	B 4	B 5	B 6	B 7	B 8
0	6	3	0	1	1	0

Distribution Vectors

A 3	A 4	A 6	A 7
1	1	1	1
2	2	0	1
5	0	0	1
1	3	1	0
4	1	1	0
5	2	0	0

Pair Table

2	1	1	2	1	0
1	2	2	2	0	2
1	2	5	0	3	4
2	2	0	4	2	2
1	0	3	2	3	3
0	2	4	2	3	5

Original System of Equations

```
  1.00  1.00  1.00  0.00  0.00  0.00 =   7.00
  1.00  0.00  0.00  1.00  1.00  0.00 =   6.00
  1.00  2.00  0.00  3.00  1.00  2.00 =  12.00
  1.00  2.00  5.00  1.00  4.00  5.00 =  18.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 =   4.00
  0.00  1.00  0.00  0.00 -1.00  0.00 =   1.00
  0.00  0.00  1.00  0.00  1.00  1.00 =   2.00
  0.00  0.00  0.00  1.00  1.00  1.00 =   2.00
Column pointer vector:    1   2   3   4   5   6
```

Equations to be solved

```
   1    0    0    0    0   -1 =    4
   0    1    0    0   -1    0 =    1
   0    0    1    0    1    1 =    2
   0    0    0    1    1    1 =    2
```

Upper bounds: 9   9   1   1   1   1

The number of possible solutions is: 3

```
  1)    4   2   1   1   1   0
Solution rejected.  3 and  5 occur  3 times together
  2)    5   1   1   1   0   1
Solution rejected.  3 and  6 occur  4 times together
  3)    5   2   0   0   1   1
Solution rejected.  5 and  6 occur  3 times together
```

Case 29

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	0	1	4	0	1	0

Distribution Vectors

A 2	A 4	A 5	A 7
2	0	2	1
3	1	1	1
0	0	4	0
1	1	3	0
4	0	3	0
5	1	2	0

Pair Table

1	1	2	1	2	2
1	3	1	1	2	4
2	1	4	3	3	2
1	1	3	3	2	3
2	2	3	2	5	5
2	4	2	3	5	6

Original System of Equations

```
  1.00  1.00  0.00  0.00  0.00  0.00 =   7.00
  2.00  1.00  4.00  3.00  3.00  2.00 =  20.00
  0.00  1.00  0.00  1.00  0.00  1.00 =   4.00
  2.00  3.00  0.00  1.00  4.00  5.00 =  10.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00 -1.00 -1.00 =   3.00
  0.00  1.00  0.00  0.00  1.00  1.00 =   4.00
  0.00  0.00  1.00  0.00  1.00  0.00 =   4.00
  0.00  0.00  0.00  1.00  0.00  1.00 =  -2.00
Column pointer vector:   1  2  3  5  4  6
```

Equation 4 has LHS >= 0, RHS < 0
*** Block vector rejected

Case 30

B 2	B 3	B 4	B 5	B 6	B 7	B 8
3	2	2	3	0	1	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 7
0	1	0	2	1
2	0	0	2	1
0	0	2	1	1
1	1	1	1	1
3	0	1	1	1
2	2	0	1	1
0	2	2	0	1
2	1	2	0	1
3	2	1	0	1
1	0	1	3	0
0	2	0	3	0
2	1	0	3	0
0	1	2	2	0
2	0	2	2	0
1	2	1	2	0
3	1	1	2	0
2	2	2	1	0

Pair Table

2	2	1	1	1	2	2	1	2	2	3	2	1	1	2	1	1
2	3	1	1	3	2	1	2	3	2	2	3	1	2	1	3	1
1	1	3	2	2	1	3	3	2	2	1	1	2	2	1	1	2
1	1	2	1	2	2	3	2	3	1	2	1	1	1	1	1	2
1	3	2	2	4	3	2	4	4	2	1	3	1	3	1	3	3
2	2	1	2	3	4	3	3	5	1	3	3	1	1	2	3	3
2	1	3	3	2	3	5	4	4	1	2	1	3	2	3	2	4
1	2	3	2	4	3	4	4	5	1	1	1	2	3	2	3	4
2	3	2	3	4	5	4	5	6	1	2	3	2	3	3	4	5
2	2	2	1	2	1	1	1	1	3	3	3	3	3	2	3	2
3	2	1	2	1	3	2	1	2	3	5	4	3	2	4	3	3
2	3	1	1	3	3	1	1	3	3	4	4	2	3	3	4	3
1	1	2	1	1	1	3	2	2	3	3	2	3	3	3	2	3
1	2	2	1	3	1	2	3	3	3	2	3	3	4	2	4	3
2	1	1	1	1	2	3	2	3	2	4	3	3	2	3	3	3
1	3	1	1	3	3	2	3	4	3	3	4	2	4	3	4	4
1	1	2	2	3	3	4	4	5	2	3	3	3	3	3	4	5

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00
0.00  0.00  0.00  0.00  0.00  0.00 =   7.00
  2.00  2.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  3.00  3.00
3.00  2.00  2.00  2.00  2.00  1.00 =  15.00
  0.00  0.00  2.00  1.00  1.00  0.00  2.00  2.00  1.00  1.00  0.00
0.00  2.00  2.00  1.00  1.00  2.00 =   8.00
  1.00  0.00  0.00  1.00  0.00  2.00  2.00  1.00  2.00  0.00  2.00
1.00  1.00  0.00  2.00  1.00  2.00 =   6.00
  0.00  2.00  0.00  1.00  3.00  2.00  0.00  2.00  3.00  1.00  0.00
2.00  0.00  2.00  1.00  3.00  2.00 =   6.00
```

```
Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00  1.00
0.00  0.00 -1.00  0.00 -1.00 -1.00 =   2.00
  0.00  1.00  0.00  0.00  0.00  0.00 -1.00  0.00  0.00  1.00 -1.00
0.00 -1.00  0.00 -1.00  0.00 -1.00 =   0.00
  0.00  0.00  1.00  0.00  0.00 -1.00  0.00  0.00 -1.00  0.00 -1.00 -
1.00  0.00  0.00 -1.00 -1.00 -1.00 =   1.00
  0.00  0.00  0.00  1.00  0.00  2.00  2.00  2.00  3.00  1.00  1.00
1.00  1.00  1.00  2.00  2.00  3.00 =   4.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  0.00  0.00  1.00
1.00  1.00  1.00  1.00  1.00  1.00 =   2.00
Column pointer vector:   1  2  3  4 10  6  7  8  9  5 11 12 13 14 15
16 17

Equations to be solved


   1   0   0   0   0   0   0  -1  -1  -1   1   0   0  -1   0  -1  -1
=    2
   0   1   0   0   0   0  -1   0   0   1  -1   0  -1   0  -1   0  -1
=    0
   0   0   1   0   0  -1   0   0  -1   0  -1  -1   0   0  -1  -1  -1
=    1
   0   0   0   1   0   2   2   2   3   1   1   1   1   1   2   2   3
=    4
   0   0   0   0   1   0   0   0   0   0   1   1   1   1   1   1   1
=    2

Upper bounds:    9  1  1  4  1  1  1  1  1  1  1  1  1  1  1  1  1


The number of possible solutions is: 12

  1)    3   1   1   0   1   0   1   0   0   1   0   0   1   0   0   0
0
Solution rejected.  2 and 10 occur  3 times together
  2)    4   0   1   0   1   0   1   0   0   1   0   0   0   1   0   0
0
Solution rejected.  3 and  7 occur  3 times together
  3)    3   1   1   1   1   0   1   0   0   0   0   0   0   1   0   0
0
Solution rejected.  3 and  7 occur  3 times together
  4)    4   0   1   0   1   0   0   1   0   1   0   0   1   0   0   0
0
Solution rejected.  3 and  8 occur  3 times together
  5)    3   1   1   1   1   0   0   1   0   0   0   0   1   0   0   0
0
Solution rejected.  3 and  8 occur  3 times together
  6)    4   0   1   1   1   0   0   1   0   0   0   0   0   1   0   0
0
Solution rejected.  3 and  8 occur  3 times together
  7)    3   0   1   2   1   0   0   0   0   1   0   0   1   0   0   0
0
Solution rejected.  5 and 13 occur  3 times together
  8)    2   1   1   3   1   0   0   0   0   0   0   0   1   0   0   0
0
```

```
Solution rejected.  5 and 13 occur  3 times together
  9)     3   0   1   3   1   0   0   0   0   0   0   0   0   1   0   0
0
Solution rejected.  5 and 14 occur  3 times together
 10)     4   1   1   0   0   0   0   1   0   0   0   0   1   1   0   0
0
Solution rejected.  3 and  8 occur  3 times together
 11)     4   0   1   1   0   0   0   0   0   1   0   0   1   1   0   0
0
Solution rejected. 10 and 14 occur  3 times together
 12)     3   1   1   2   0   0   0   0   0   0   0   0   1   1   0   0
0
Solution rejected. 13 and 14 occur  3 times together


--------------------------------------------------
Case   31

B 2    B 3    B 4    B 5    B 6    B 7    B 8
---------------------------------------
  0      7      0      3      0      1      0


Distribution Vectors

A 3    A 5    A 7
---------------
  1      2      1
  3      1      1
  5      0      1
  2      3      0
  4      2      0
  6      1      0


Pair Table

  2  1  1  2  1  0
  1  1  2  1  0  2
  1  2  4  0  2  4
  2  1  0  3  2  2
  1  0  2  2  2  3
  0  2  4  2  3  5


Original System of Equations

  1.00  1.00  1.00  0.00  0.00  0.00 =   7.00
  2.00  1.00  0.00  3.00  2.00  1.00 =  15.00
  1.00  3.00  5.00  2.00  4.00  6.00 =  21.00


Equations in Row Echelon Form

  1.00  0.00  0.00 -1.00 -1.00 -2.00 =   2.00
  0.00  1.00  0.00  2.00  1.00  2.00 =   5.00
  0.00  0.00  1.00  0.00  1.00  1.00 =   2.00
Column pointer vector:   1  2  4  3  5  6
```

Equations to be solved

1	0	0	-1	-1	-2	=	2
0	1	0	2	1	2	=	5
0	0	1	0	1	1	=	2

Upper bounds: 9 5 1 1 2 1

The number of possible solutions is: 8

```
  1)    4   2   1   1   1   0
  2)    5   1   1   1   0   1
Solution rejected.  4 and  6 occur  4 times together
  3)    3   4   1   0   1   0
  4)    4   3   1   0   0   1
  5)    5   1   0   1   2   0
  6)    6   0   0   1   1   1
Solution rejected.  4 and  6 occur  4 times together
  7)    4   3   0   0   2   0
  8)    5   2   0   0   1   1
Solution rejected.  5 and  6 occur  3 times together
```

Check equations: 1 3 4 5 7

Case 32

B 2	B 3	B 4	B 5	B 6	B 7	B 8
1	4	3	2	0	1	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 7
0	1	0	2	1
0	0	2	1	1
1	1	1	1	1
0	3	0	1	1
1	0	3	0	1
0	2	2	0	1
1	3	1	0	1
0	1	2	2	0
1	2	1	2	0
0	4	0	2	0
1	1	3	1	0
0	3	2	1	0
1	4	1	1	0
1	3	3	0	0

Pair Table

```
 3  2  2  2  1  1  1  2  2  3  1  1  2  0
 2  2  1  1  3  2  1  2  1  1  2  1  0  2
 2  1  2  1  3  1  2  1  2  2  2  0  2  2
 2  1  1  3  1  2  3  1  2  4  0  2  3  2
 1  3  3  1  5  3  3  2  2  0  4  2  2  4
 1  2  1  2  3  2  2  1  0  2  2  2  2  3
 1  1  2  3  3  2  4  0  2  3  2  2  4  4
 2  2  1  1  2  1  0  3  2  3  3  2  2  2
 2  1  2  2  2  0  2  2  3  4  3  2  4  3
 3  1  2  4  0  2  3  3  4  6  2  4  5  3
 1  2  2  0  4  2  2  3  3  2  4  2  3  4
 1  1  0  2  2  2  2  2  2  4  2  3  3  4
 2  0  2  3  2  2  4  2  4  5  3  3  5  5
 0  2  2  2  4  3  4  2  3  3  4  4  5  6
```

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00
0.00  0.00  0.00 =   7.00
  2.00  1.00  1.00  1.00  0.00  0.00  0.00  2.00  2.00  2.00  1.00
1.00  1.00  0.00 =  10.00
  0.00  2.00  1.00  0.00  3.00  2.00  1.00  2.00  1.00  0.00  3.00
2.00  1.00  3.00 =  12.00
  1.00  0.00  1.00  3.00  0.00  2.00  3.00  1.00  2.00  4.00  1.00
3.00  4.00  3.00 =  12.00
  0.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00
0.00  1.00  1.00 =   2.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -1.00  0.00  0.00 -1.00 -
1.00 -1.00 -2.00 =  -1.00
  0.00  1.00  0.00  0.00  0.00  1.00  0.00  1.00 -1.00 -1.00  0.00
0.00 -1.00  0.00 =   3.00
  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00  0.00  1.00
0.00  1.00  1.00 =   2.00
  0.00  0.00  0.00  1.00  0.00  1.00  1.00  0.00  0.00  1.00  0.00
1.00  1.00  1.00 =   3.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00
1.00  1.00  1.00 =   2.00
Column pointer vector:   1  2  3  4  8  6  7  5  9 10 11 12 13 14
```

Equations to be solved

```
  1   0   0   0   0  -1  -1  -1   0   0  -1  -1  -1  -2 =   -1
  0   1   0   0   0   1   0   1  -1  -1   0   0  -1   0 =    3
  0   0   1   0   0   0   1   1   1   0   1   0   1   1 =    2
  0   0   0   1   0   1   1   0   0   1   0   1   1   1 =    3
  0   0   0   0   1   0   0   0   1   1   1   1   1   1 =    2
```

Upper bounds: 1 9 2 1 1 3 1 1 1 1 1 1 1 1

The number of possible solutions is: 32

```
  1)    1   2   1   1   1   2   0   0   1   0   0   0   0   0
  2)    1   2   2   0   1   2   0   0   0   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
  3)    1   3   0   1   1   1   1   0   1   0   0   0   0   0
Solution rejected.  4 and  7 occur  3 times together
  4)    1   3   1   0   1   1   1   0   0   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
  5)    1   2   1   1   1   1   0   1   0   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
  6)    0   3   2   1   1   1   0   0   0   1   0   0   0   0
Solution rejected.  4 and 10 occur  4 times together
  7)    1   2   2   1   1   1   0   0   0   0   0   1   0   0
  8)    1   3   1   1   1   1   0   0   0   0   0   0   1   0
Solution rejected.  4 and 13 occur  3 times together
  9)    1   3   0   1   1   0   1   1   0   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 10)    0   4   1   1   1   0   1   0   0   1   0   0   0   0
Solution rejected.  4 and  7 occur  3 times together
 11)    1   3   1   1   1   0   1   0   0   0   0   1   0   0
Solution rejected.  4 and  7 occur  3 times together
 12)    1   4   0   1   1   0   1   0   0   0   0   0   1   0
Solution rejected.  4 and  7 occur  3 times together
 13)    1   3   1   0   0   2   0   0   1   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 14)    1   4   0   0   0   1   1   0   1   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 15)    1   3   0   1   0   1   0   1   1   1   0   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 16)    0   4   1   1   0   1   0   0   1   1   0   0   0   0
Solution rejected.  4 and 10 occur  4 times together
 17)    1   3   1   1   0   1   0   0   1   0   0   1   0   0
 18)    1   4   0   1   0   1   0   0   1   0   0   0   1   0
Solution rejected.  4 and 13 occur  3 times together
 19)    1   3   1   1   0   1   0   0   0   1   1   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 20)    1   3   2   0   0   1   0   0   0   1   0   1   0   0
Solution rejected.  1 and 10 occur  3 times together
 21)    1   4   1   0   0   1   0   0   0   1   0   0   1   0
Solution rejected.  1 and 10 occur  3 times together
 22)    0   5   0   1   0   0   1   0   1   1   0   0   0   0
Solution rejected.  4 and  7 occur  3 times together
 23)    1   4   0   1   0   0   1   0   1   0   0   1   0   0
Solution rejected.  4 and  7 occur  3 times together
 24)    1   4   0   1   0   0   1   0   0   1   1   0   0   0
Solution rejected.  1 and 10 occur  3 times together
 25)    1   4   1   0   0   0   1   0   0   1   0   1   0   0
Solution rejected.  1 and 10 occur  3 times together
 26)    1   5   0   0   0   0   1   0   0   1   0   0   1   0
Solution rejected.  1 and 10 occur  3 times together
 27)    1   3   1   1   0   0   0   1   0   1   0   1   0   0
Solution rejected.  1 and 10 occur  3 times together
 28)    1   4   0   1   0   0   0   1   0   1   0   0   1   0
Solution rejected.  1 and 10 occur  3 times together
 29)    0   4   2   1   0   0   0   0   0   1   0   1   0   0
Solution rejected.  4 and 10 occur  4 times together
 30)    0   5   1   1   0   0   0   0   0   1   0   0   1   0
Solution rejected.  4 and 10 occur  4 times together
 31)    1   4   1   1   0   0   0   0   0   1   0   0   0   1
```

Solution rejected. 1 and 10 occur 3 times together
32) 1 4 1 1 0 0 0 0 0 0 0 1 1 0
Solution rejected. 4 and 13 occur 3 times together

Check equations: 1 7 17

Case 33

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	1	6	1	0	1	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 7
0	0	2	1	1
1	1	1	1	1
1	0	3	0	1
2	1	2	0	1
0	0	4	1	0
1	1	3	1	0
1	0	5	0	0
2	1	4	0	0

Pair Table

2	2	1	1	1	1	1	0
2	3	1	3	1	2	0	2
1	1	1	2	1	0	2	2
1	3	2	4	0	2	2	3
1	1	1	0	3	2	3	2
1	2	0	2	2	2	2	3
1	0	2	2	3	2	4	4
0	2	2	3	2	3	4	5

Original System of Equations

1.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	=	7.00
1.00	1.00	0.00	0.00	1.00	1.00	0.00	0.00	=	5.00
2.00	1.00	3.00	2.00	4.00	3.00	5.00	4.00	=	24.00
0.00	1.00	0.00	1.00	0.00	1.00	0.00	1.00	=	3.00
0.00	1.00	1.00	2.00	0.00	1.00	1.00	2.00	=	4.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	1.00	0.00	-1.00	-1.00	=	2.00
0.00	1.00	0.00	0.00	0.00	1.00	1.00	1.00	=	3.00
0.00	0.00	1.00	0.00	-1.00	-1.00	1.00	0.00	=	2.00
0.00	0.00	0.00	1.00	1.00	1.00	0.00	1.00	=	2.20
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	=	-3.20

Column pointer vector: 1 2 3 7 5 6 4 8

```
Equation  5 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case   34

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  0     3     7     0     0     1     0


Distribution Vectors

A 3   A 4   A 7
---------------
  2     2     1
  2     4     0


Pair Table

  2  1
  1  2


Original System of Equations

  1.00  0.00 =   7.00
  2.00  4.00 =  28.00
  2.00  2.00 =   9.00


Equations in Row Echelon Form

  1.00  0.00 =   7.00
  0.00  1.00 =   3.50
  0.00  0.00 = -12.00
Column pointer vector:   1  2

Equation  3 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case   35

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  6     0     1     0     4     0     0


Distribution Vectors

A 2   A 4   A 6
---------------
  1     0     3
```

3	1	2
6	0	2

Pair Table

2	1	2
1	1	3
2	3	6

Original System of Equations

```
3.00  2.00  2.00 =  24.00
0.00  1.00  0.00 =   4.00
1.00  3.00  6.00 =  12.00
```

Equations in Row Echelon Form

```
1.00  0.00  0.00 =   6.00
0.00  1.00  0.00 =   4.00
0.00  0.00  1.00 =  -1.00
```

Column pointer vector: 1 2 3

Equation 3 has LHS >= 0, RHS < 0
*** Block vector rejected

Case 36

B 2	B 3	B 4	B 5	B 6	B 7	B 8
5	0	2	1	3	0	0

Distribution Vectors

A 2	A 4	A 5	A 6
1	0	0	3
2	0	1	2
0	2	0	2
3	1	0	2
1	2	1	1
4	1	1	1
5	2	0	1

Pair Table

3	2	2	2	1	1	2
2	2	1	1	1	2	2
2	1	3	2	2	1	2
2	1	2	2	1	2	4
1	1	2	1	3	2	3
1	2	1	2	2	4	5

```
   2  2  2  4  3  5  7


Original System of Equations

   3.00  2.00  2.00  2.00  1.00  1.00  1.00 =  18.00
   0.00  1.00  0.00  0.00  1.00  1.00  0.00 =   5.00
   0.00  0.00  2.00  1.00  2.00  1.00  2.00 =   8.00
   1.00  2.00  0.00  3.00  1.00  4.00  5.00 =  10.00


Equations in Row Echelon Form

   1.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 =   0.00
   0.00  1.00  0.00  0.00  1.00  1.00  0.00 =   5.00
   0.00  0.00  1.00  0.00  1.00  0.00  0.00 =   4.00
   0.00  0.00  0.00  1.00  0.00  1.00  2.00 =   0.00
Column pointer vector:   1  2  3  4  5  6  7

Equations to be solved


    1    0    0    0   -1   -1   -1  =     0
    0    1    0    0    1    1    0  =     5
    0    0    1    0    1    0    0  =     4
    0    0    0    1    0    1    2  =     0

Upper bounds:   1  5  1  0  1  1  1


The number of possible solutions is:  0



--------------------------------------------------
Case   37

B 2    B 3    B 4    B 5    B 6    B 7    B 8
----------------------------------------------
  2      5      0      1      3      0      0


Distribution Vectors

A 2    A 3    A 5    A 6
------------------------
  1      0      0      3
  0      1      1      2
  2      0      1      2
  0      3      0      2
  2      2      0      2
  1      3      1      1
  1      5      0      1
  2      5      1      0


Pair Table
```

3	2	3	2	3	1	1	1
2	2	2	1	1	1	1	2
3	2	4	1	3	2	1	3
2	1	1	2	1	1	3	3
3	1	3	1	3	1	3	4
1	1	2	1	1	2	3	5
1	1	1	3	3	3	5	6
1	2	3	3	4	5	6	8

Original System of Equations

```
 3.00  2.00  2.00  2.00  2.00  1.00  1.00  0.00 =  18.00
 0.00  1.00  1.00  0.00  0.00  1.00  0.00  1.00 =   5.00
 0.00  1.00  0.00  3.00  2.00  3.00  5.00  5.00 =  15.00
 1.00  0.00  2.00  0.00  2.00  1.00  1.00  2.00 =   4.00
```

Equations in Row Echelon Form

```
 1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -2.00 =   0.00
 0.00  1.00  0.00  0.00 -1.00  0.00 -1.00 -1.00 =   3.00
 0.00  0.00  1.00  0.00  1.00  1.00  1.00  2.00 =   2.00
 0.00  0.00  0.00  1.00  1.00  1.00  2.00  2.00 =   4.00
Column pointer vector:   1  2  3  4  5  6  7  8
```

Equations to be solved

```
  1   0   0   0   0  -1  -1  -2 =   0
  0   1   0   0  -1   0  -1  -1 =   3
  0   0   1   0   1   1   1   2 =   2
  0   0   0   1   1   1   2   2 =   4

Upper bounds:   1  9  1  4  1  2  1  1
```

The number of possible solutions is: 5

```
  1)    0   4   1   3   1   0   0   0
Solution rejected.  3 and  5 occur  3 times together
  2)    1   3   1   3   0   1   0   0
Solution rejected.  1 and  3 occur  3 times together
  3)    1   4   0   2   1   1   0   0
Solution rejected.  1 and  5 occur  3 times together
  4)    1   4   1   2   0   0   1   0
Solution rejected.  1 and  3 occur  3 times together
  5)    1   5   0   1   1   0   1   0
Solution rejected.  1 and  5 occur  3 times together
```

Case 38

B 2	B 3	B 4	B 5	B 6	B 7	B 8
3	2	3	0	3	0	0

Distribution Vectors

A 2	A 3	A 4	A 6
1	0	0	3
0	0	2	2
1	1	1	2
3	0	1	2
2	2	0	2
0	1	3	1
2	0	3	1
1	2	2	1
3	1	2	1
3	2	3	0

Pair Table

3	2	2	3	2	1	1	1	2	1
2	2	1	1	1	2	2	1	1	2
2	1	1	2	2	1	1	1	1	3
3	1	2	4	3	1	3	1	3	4
2	1	2	3	4	1	1	2	3	4
1	2	1	1	1	3	3	3	2	4
1	2	1	3	1	3	4	2	4	5
1	1	1	1	2	3	2	3	3	5
2	1	1	3	3	2	4	3	4	6
1	2	3	4	4	4	5	5	6	8

Original System of Equations

3.00	2.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	0.00	=	18.00
0.00	2.00	1.00	1.00	0.00	3.00	3.00	2.00	2.00	3.00	=	12.00
0.00	0.00	1.00	0.00	2.00	1.00	0.00	2.00	1.00	2.00	=	6.00
1.00	0.00	1.00	3.00	2.00	0.00	2.00	1.00	3.00	3.00	=	6.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	0.00	-1.00	-1.00	-1.00	-1.00	-2.00	=	0.00
0.00	1.00	0.00	0.00	-1.00	1.00	1.00	0.00	0.00	0.00	=	3.00
0.00	0.00	1.00	0.00	2.00	1.00	0.00	2.00	1.00	2.00	=	6.00
0.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	1.00	1.00	=	0.00

Column pointer vector: 1 2 3 4 5 6 7 8 9 10

Equations to be solved

1	0	0	0	0	-1	-1	-1	-1	-2	=	0
0	1	0	0	-1	1	1	0	0	0	=	3
0	0	1	0	2	1	0	2	1	2	=	6
0	0	0	1	0	0	1	0	1	1	=	0

Upper bounds: 1 9 6 1 1 1 1 1 1 1

```
The number of possible solutions is:   6

   1)     1    3    3    0    1    1    0    0    0    0
   2)     1    4    2    0    1    0    0    1    0    0
   3)     0    4    4    0    1    0    0    0    0    0
   4)     1    2    5    0    0    1    0    0    0    0
   5)     1    3    4    0    0    0    0    1    0    0
   6)     0    3    6    0    0    0    0    0    0    0

Check equations:    1   2   3   4   5   6


--------------------------------------------------
Case   39

B 2    B 3    B 4    B 5    B 6    B 7    B 8
----------------------------------------
  0      7      1      0      3      0      0


Distribution Vectors

A 3    A 4    A 6
---------------
  3      0      2
  4      1      1


Pair Table

  1  0
  0  2


Original System of Equations

  2.00  1.00 =  18.00
  0.00  1.00 =   4.00
  3.00  4.00 =  21.00


Equations in Row Echelon Form

  1.00  0.00 =  10.20
  0.00  1.00 =  -2.40
  0.00  0.00 =   6.40
Column pointer vector:   1  2

Equation  3 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case   40

B 2    B 3    B 4    B 5    B 6    B 7    B 8
----------------------------------------
  3      3      0      3      2      0      0
```

Distribution Vectors

A 2	A 3	A 5	A 6
0	1	1	2
2	0	1	2
0	3	0	2
2	2	0	2
1	1	2	1
3	0	2	1
1	3	1	1
3	2	1	1
0	2	3	0
2	1	3	0
2	3	2	0

Pair Table

2	2	3	2	1	1	2	1	1	1	1
2	3	2	3	1	3	1	3	1	2	1
3	2	5	4	2	1	4	3	2	1	3
2	3	4	4	1	3	3	4	1	1	3
1	1	2	1	1	2	1	1	2	2	2
1	3	1	3	2	4	1	3	2	4	3
2	1	4	3	1	1	3	3	3	2	3
1	3	3	4	1	3	3	4	2	3	4
1	1	2	1	2	2	3	2	4	3	4
1	2	1	1	2	4	2	3	3	4	4
1	1	3	3	2	3	3	4	4	4	5

Original System of Equations

```
  2.00  2.00  2.00  2.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00 =
12.00
  1.00  1.00  0.00  0.00  2.00  2.00  1.00  1.00  3.00  3.00  2.00 =
15.00
  1.00  0.00  3.00  2.00  1.00  0.00  3.00  2.00  2.00  1.00  3.00 =
9.00
  0.00  2.00  0.00  2.00  1.00  3.00  1.00  3.00  0.00  2.00  2.00 =
6.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -2.00 -1.00 -1.00 -2.00 =
3.00
  0.00  1.00  0.00  0.00  0.50  1.50  0.50  1.50  1.00  1.00  1.00 =
3.00
  0.00  0.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00  0.00  1.00 =
0.00
  0.00  0.00  0.00  1.00  0.50  0.50  0.50  0.50  0.00  1.00  1.00 =
3.00
Column pointer vector:   1  2  3  9  5  6  7  8  4 10 11
```

Equations to be solved

1	0	0	0	0	-1	-1	-2	-1	-1	-2	=	3
0	2	0	0	1	3	1	3	2	2	2	=	6
0	0	1	0	0	0	1	1	1	0	1	=	0
0	0	0	2	1	1	1	1	0	2	2	=	6

Upper bounds: 9 1 1 1 6 1 1 1 1 1 1

The number of possible solutions is: 4

```
  1)    4   0   0   1   3   1   0   0   0   0   0
  2)    5   0   0   1   1   1   0   0   0   1   0
Solution rejected.  4 and 10 occur  3 times together
  3)    3   0   0   0   6   0   0   0   0   0   0
  4)    4   0   0   0   4   0   0   0   0   1   0
```

Check equations: 1 3 4

Case 41

B 2	B 3	B 4	B 5	B 6	B 7	B 8
4	0	3	2	2	0	0

Distribution Vectors

A 2	A 4	A 5	A 6
2	0	1	2
0	2	0	2
3	1	0	2
0	1	2	1
3	0	2	1
1	2	1	1
4	1	1	1
2	3	0	1
2	2	2	0
3	3	1	0

Pair Table

2	2	3	2	3	1	3	1	1	1
2	3	2	1	1	2	1	3	1	2
3	2	4	1	3	1	4	3	1	3
2	1	1	2	2	1	1	1	2	2
3	1	3	2	4	1	4	1	3	3
1	2	1	1	1	1	1	2	2	2
3	1	4	1	4	1	4	3	3	4
1	3	3	1	1	2	3	3	2	4
1	1	1	2	3	2	3	2	3	4
1	2	3	2	3	2	4	4	4	5

```
Original System of Equations

  2.00  2.00  2.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00 =  12.00
  1.00  0.00  0.00  2.00  2.00  1.00  1.00  0.00  2.00  1.00 =  10.00
  0.00  2.00  1.00  1.00  0.00  2.00  1.00  3.00  2.00  3.00 =  12.00
  2.00  0.00  3.00  0.00  3.00  1.00  4.00  2.00  2.00  3.00 =   8.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -2.00 -2.00 -3.00 =  -2.00
  0.00  1.00  0.00  0.00 -1.00  0.00 -1.00  0.00 -1.00 -1.00 =   1.00
  0.00  0.00  1.00  0.00  1.00  1.00  2.00  2.00  2.00  3.00 =   4.00
  0.00  0.00  0.00  1.00  1.00  1.00  1.00  1.00  2.00  2.00 =   6.00
Column pointer vector:   1  2  3  4  5  6  7  8  9 10

Equations to be solved


   1   0   0   0   0  -1  -1  -2  -2  -3 =   -2
   0   1   0   0  -1   0  -1   0  -1  -1 =    1
   0   0   1   0   1   1   2   2   2   3 =    4
   0   0   0   1   1   1   1   1   2   2 =    6

Upper bounds:   9  1  1  6  1  4  1  1  1  1


The number of possible solutions is:  4

  1)    1   1   1   4   0   1   0   1   0   0
Solution rejected.  1 and  3 occur  3 times together
  2)    1   1   1   3   0   3   0   0   0   0
Solution rejected.  1 and  3 occur  3 times together
  3)    2   1   0   3   0   2   0   1   0   0
Solution rejected.  2 and  8 occur  3 times together
  4)    2   1   0   2   0   4   0   0   0   0

Check equations:   4


--------------------------------------------------
Case   42

B 2   B 3   B 4   B 5   B 6   B 7   B 8
-----------------------------------------
  1     5     1     2     2     0     0


Distribution Vectors

A 2   A 3   A 4   A 5   A 6
---------------------------
  0     1     0     1     2
  1     1     1     0     2
  0     3     0     0     2
  0     0     1     2     1
```

1	1	0	2	1
0	2	1	1	1
1	3	0	1	1
0	4	1	0	1
1	5	0	0	1
1	2	1	2	0
0	4	0	2	0
1	4	1	1	0

Pair Table

2	2	2	2	2	1	1	1	2	1	1	0
2	4	2	2	2	2	2	2	3	2	0	2
2	2	3	1	1	1	2	3	4	0	2	2
2	2	1	3	2	2	1	1	0	3	2	2
2	2	1	2	3	1	2	0	2	3	2	2
1	2	1	2	1	1	0	2	2	2	2	2
1	2	2	1	2	0	2	2	4	2	3	3
1	2	3	1	0	2	2	4	4	2	3	4
2	3	4	0	2	2	4	4	6	3	4	5
1	2	0	3	3	2	2	2	3	4	3	4
1	0	2	2	2	2	3	3	4	3	5	4
0	2	2	2	2	2	3	4	5	4	4	5

Original System of Equations

```
  2.00  2.00  2.00  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00
0.00 =  12.00
  1.00  0.00  0.00  2.00  2.00  1.00  1.00  0.00  0.00  2.00  2.00
1.00 =  10.00
  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00
1.00 =   4.00
  1.00  1.00  3.00  0.00  1.00  2.00  3.00  4.00  5.00  2.00  4.00
4.00 =  15.00
  0.00  1.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00  0.00
1.00 =   2.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00 -2.00 -2.00 -2.00 -2.00 -
3.00 =  -2.00
  0.00  1.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00 -1.00
0.00 =   0.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00  2.00  2.00  1.00  2.00
2.00 =   5.00
  0.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00
1.00 =   4.00
  0.00  0.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00  1.00
1.00 =   2.00
Column pointer vector:   1  2  3  4  5  6  7  8  9 10 11 12
```

Equations to be solved

```
 1   0   0   0   0  -1  -1  -2  -2  -2  -2  -3 =   -2
```

```
   0    1    0    0    0    0    0    0    0    0   -1    0 =     0
   0    0    1    0    0    1    1    2    2    1    2    2 =     5
   0    0    0    1    0    1    0    1    0    1    1    1 =     4
   0    0    0    0    1    0    1    0    1    1    1    1 =     2

Upper bounds:    9  1  1  1  1  4  2  1  1  1  1  1


The number of possible solutions is: 27

  1)     3    0    0    0    1    4    1    0    0    0    0    0
  2)     2    0    1    1    1    3    1    0    0    0    0    0
  3)     3    0    0    1    1    3    0    0    1    0    0    0
  4)     3    0    1    0    1    3    0    0    0    1    0    0
Solution rejected.  5 and 10 occur  3 times together
  5)     3    1    0    0    1    3    0    0    0    0    1    0
  6)     4    0    0    0    1    3    0    0    0    0    0    1
  7)     3    0    0    1    1    2    1    1    0    0    0    0
  8)     4    0    0    0    1    2    0    1    0    1    0    0
Solution rejected.  5 and 10 occur  3 times together
  9)     2    1    1    1    1    2    0    0    0    0    1    0
 10)     3    0    1    1    1    2    0    0    0    0    0    1
 11)     3    0    1    1    1    1    0    1    0    1    0    0
Solution rejected.  3 and  8 occur  3 times together
 12)     3    1    0    1    1    1    0    1    0    0    1    0
Solution rejected.  8 and 11 occur  3 times together
 13)     4    0    0    1    1    1    0    1    0    0    0    1
Solution rejected.  8 and 12 occur  4 times together
 14)     3    0    0    1    0    3    2    0    0    0    0    0
 15)     4    0    0    0    0    3    1    0    0    1    0    0
 16)     3    0    1    1    0    2    1    0    0    1    0    0
Solution rejected.  4 and 10 occur  3 times together
 17)     3    1    0    1    0    2    1    0    0    0    1    0
Solution rejected.  7 and 11 occur  3 times together
 18)     4    0    0    1    0    2    1    0    0    0    0    1
Solution rejected.  7 and 12 occur  3 times together
 19)     4    0    0    1    0    2    0    0    1    1    0    0
Solution rejected.  4 and 10 occur  3 times together
 20)     4    1    0    0    0    2    0    0    0    1    1    0
Solution rejected. 10 and 11 occur  3 times together
 21)     5    0    0    0    0    2    0    0    0    1    0    1
Solution rejected. 10 and 12 occur  4 times together
 22)     4    0    0    1    0    1    1    1    0    1    0    0
Solution rejected.  4 and 10 occur  3 times together
 23)     3    1    1    1    0    1    0    0    0    1    1    0
Solution rejected.  4 and 10 occur  3 times together
 24)     4    0    1    1    0    1    0    0    0    1    0    1
Solution rejected.  4 and 10 occur  3 times together
 25)     4    1    0    1    0    1    0    0    0    0    1    1
Solution rejected. 11 and 12 occur  4 times together
 26)     4    1    0    1    0    0    0    1    0    1    1    0
Solution rejected.  4 and 10 occur  3 times together
 27)     5    0    0    1    0    0    0    1    0    1    0    1
Solution rejected.  4 and 10 occur  3 times together

Check equations:    1  2  3  5  6  7  9 10 14 15
```

Case 43

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	2	4	1	2	0	0

Distribution Vectors

A 2	A 3	A 4	A 5	A 6
0	1	0	1	2
2	0	0	1	2
0	0	2	0	2
1	1	1	0	2
2	2	0	0	2
1	0	2	1	1
0	2	1	1	1
2	1	1	1	1
0	1	3	0	1
2	0	3	0	1
1	2	2	0	1
0	0	4	1	0
1	1	3	1	0
2	2	2	1	0
0	2	4	0	0
2	1	4	0	0

Pair Table

3	3	2	2	3	2	3	2	1	1	2	1	1	2	1	0
3	5	2	3	4	3	2	4	1	3	2	1	2	3	0	2
2	2	2	2	2	1	1	1	2	2	1	2	1	0	2	2
2	3	2	2	4	1	2	2	1	2	2	1	0	2	2	2
3	4	2	4	6	2	3	4	2	3	4	0	2	4	2	3
2	3	1	1	2	1	1	2	1	2	0	3	2	2	2	3
3	2	1	2	3	1	3	2	1	0	2	2	2	3	3	2
2	4	1	2	4	2	2	3	0	2	2	2	2	4	2	3
1	1	2	1	2	1	1	0	2	2	2	3	2	2	4	3
1	3	2	2	3	2	0	2	2	4	2	3	3	3	3	5
2	2	1	2	4	0	2	2	2	2	2	2	2	3	4	4
1	1	2	1	0	3	2	2	3	3	2	5	4	3	4	4
1	2	1	0	2	2	2	2	2	3	2	4	3	4	4	4
2	3	0	2	4	2	3	4	2	3	3	3	4	5	4	5
1	0	2	2	2	2	3	2	4	3	4	4	4	4	6	5
0	2	2	2	3	3	2	3	3	5	4	4	4	5	5	6

Original System of Equations

```
  2.00  2.00  2.00  2.00  2.00  1.00  1.00  1.00  1.00  1.00  1.00
0.00  0.00  0.00  0.00  0.00 =  12.00
  1.00  1.00  0.00  0.00  0.00  1.00  1.00  1.00  0.00  0.00  0.00
1.00  1.00  1.00  0.00  0.00 =   5.00
  0.00  0.00  2.00  1.00  0.00  2.00  1.00  1.00  3.00  3.00  2.00
4.00  3.00  2.00  4.00  4.00 =  16.00
```

```
  1.00  0.00  0.00  1.00  2.00  0.00  2.00  1.00  1.00  0.00  2.00
0.00  1.00  2.00  2.00  1.00 =   6.00
  0.00  2.00  0.00  1.00  2.00  1.00  0.00  2.00  0.00  2.00  1.00
0.00  1.00  2.00  0.00  2.00 =   4.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00  0.00  1.00  0.00  0.00 -1.00  0.00
0.00  0.00  0.00  0.00 -1.00 =   2.00
  0.00  1.00  0.00  0.00  0.00  0.50 -0.50  0.50 -0.50  0.50 -0.50
0.00  0.00  0.00 -1.00  0.00 =   0.00
  0.00  0.00  1.00  0.00  0.00  0.00 -1.00 -1.00  0.00  0.00 -1.00 -
1.00 -1.00 -2.00 -1.00 -1.00 =   0.00
  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00  1.00  2.00
2.00  1.00  2.00  2.00  2.00 =   4.00
  0.00  0.00  0.00  0.00  1.00  0.50  0.50  0.50  0.50  0.50  0.50
0.00  1.00  1.00  1.00  1.00 =   3.00
Column pointer vector:   1  2  3  4 12  6  7  8  9 10 11  5 13 14 15
16

Equations to be solved


   1   0   0   0   0   0   1   0   0  -1   0   0   0   0   0  -1 =
2
   0   2   0   0   0   1  -1   1  -1   1  -1   0   0   0  -2   0 =
0
   0   0   1   0   0   0  -1  -1   0   0  -1  -1  -1  -2  -1  -1 =
0
   0   0   0   1   0   0   1   1   1   1   2   2   1   2   2   2 =
4
   0   0   0   0   2   1   1   1   1   1   1   0   2   2   2   2 =
6

Upper bounds:   1  1  9  4  1  6  1  1  4  1  2  1  1  1  1  1


The number of possible solutions is: 20

  1)    1   0   2   0   1   2   1   0   1   0   0   1   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  2)    1   0   1   2   1   2   1   0   1   0   0   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  3)    1   0   2   1   1   2   1   0   0   0   1   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  4)    1   0   2   1   1   1   1   1   1   0   0   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  5)    1   0   3   0   1   1   1   1   0   0   1   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  6)    1   0   3   0   1   1   1   0   0   0   0   1   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  7)    1   0   2   2   1   1   1   0   0   0   0   0   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
  8)    1   0   3   1   1   1   1   0   0   0   0   0   0   1   0   0
Solution rejected.  1 and  7 occur  3 times together
  9)    1   0   3   1   1   0   1   1   0   0   0   0   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
```

```
 10)    1   0   4   0   1   0   1   1   0   0   0   0   0   1   0   0
Solution rejected.  1 and  7 occur  3 times together
 11)    1   0   1   1   0   3   1   0   2   0   0   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 12)    1   0   2   0   0   3   1   0   1   0   1   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 13)    1   0   2   1   0   3   1   0   0   0   0   0   0   0   1   0
Solution rejected.  1 and  7 occur  3 times together
 14)    1   0   2   0   0   2   1   1   2   0   0   0   0   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 15)    1   0   3   0   0   2   1   1   0   0   0   0   0   0   1   0
Solution rejected.  1 and  7 occur  3 times together
 16)    1   0   2   1   0   2   1   0   1   0   0   0   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 17)    1   0   3   0   0   2   1   0   1   0   0   0   0   1   0   0
Solution rejected.  1 and  7 occur  3 times together
 18)    1   0   3   0   0   2   1   0   0   0   1   0   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 19)    1   0   3   0   0   1   1   1   1   0   0   0   1   0   0   0
Solution rejected.  1 and  7 occur  3 times together
 20)    1   0   4   0   0   1   1   0   0   0   0   0   1   1   0   0
Solution rejected.  1 and  7 occur  3 times together


--------------------------------------------------
Case   44

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  0     4     5     0     2     0     0


Distribution Vectors

A 3   A 4   A 6
---------------
  0     2     2
  3     0     2
  1     3     1
  4     1     1
  2     4     0


Pair Table

  2  2  1  1  1
  2  4  1  4  1
  1  1  1  1  2
  1  4  1  4  2
  1  1  2  2  3


Original System of Equations

  2.00  2.00  1.00  1.00  0.00 =  12.00
  2.00  0.00  3.00  1.00  4.00 =  20.00
  0.00  3.00  1.00  4.00  2.00 =  12.00
```

```
Equations in Row Echelon Form

   1.00   0.00   0.00  -1.00  -1.00 =    1.00
   0.00   1.00   0.00   1.00   0.00 =    2.00
   0.00   0.00   1.00   1.00   2.00 =    6.00
Column pointer vector:    1   2   3   4   5

Equations to be solved


    1    0    0   -1   -1 =     1
    0    1    0    1    0 =     2
    0    0    1    1    2 =     6

Upper bounds:    9   1   6   1   1


The number of possible solutions is:   2

   1)     2    1    5    1    0
Solution rejected.  2 and   4 occur   4 times together
   2)     3    1    3    1    1
Solution rejected.  2 and   4 occur   4 times together


--------------------------------------------------
Case    45

B 2    B 3    B 4    B 5    B 6    B 7    B 8
---------------------------------------
   4      1      0      5      1      0      0


Distribution Vectors

A 2    A 3    A 5    A 6
---------------------
   1      1      2      1
   3      0      2      1
   0      0      4      0
   2      1      3      0
   4      0      3      0


Pair Table

   2   1   1   1   1
   1   3   1   1   3
   1   1   3   2   2
   1   1   2   2   3
   1   3   2   3   5


Original System of Equations

   1.00   1.00   0.00   0.00   0.00 =    6.00
   2.00   2.00   4.00   3.00   3.00 =   25.00
```

```
  1.00  0.00  0.00  1.00  0.00 =   3.00
  1.00  3.00  0.00  2.00  4.00 =   8.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00 -1.00 =   4.00
  0.00  1.00  0.00  0.00  1.00 =   2.00
  0.00  0.00  1.00  0.00  0.00 =   4.00
  0.00  0.00  0.00  1.00  1.00 =  -1.00
Column pointer vector:   1  2  3  4  5

Equation  4 has LHS >= 0, RHS < 0
*** Block vector rejected


--------------------------------------------------
Case   46

B 2   B 3   B 4   B 5   B 6   B 7   B 8
-----------------------------------------
  2     3     1     4     1     0     0


Distribution Vectors

A 2   A 3   A 4   A 5   A 6
---------------------------
  0     0     1     2     1
  1     1     0     2     1
  0     2     1     1     1
  2     1     1     1     1
  1     3     0     1     1
  2     3     1     0     1
  0     0     0     4     0
  1     0     1     3     0
  0     2     0     3     0
  2     1     0     3     0
  1     2     1     2     0
  2     3     0     2     0


Pair Table

  2  1  2  2  1  2  2  2  1  1  1  0
  1  1  1  2  2  3  2  1  1  2  0  2
  2  1  3  2  3  4  1  1  1  0  2  2
  2  2  2  4  3  5  1  2  0  2  2  3
  1  2  3  3  4  5  1  0  2  2  2  4
  2  3  4  5  5  7  0  2  2  3  4  5
  2  2  1  1  1  0  4  3  3  3  2  2
  2  1  1  2  0  2  3  3  2  3  2  2
  1  1  1  0  2  2  3  2  3  2  2  3
  1  2  0  2  2  3  3  3  2  4  2  4
  1  0  2  2  2  4  2  2  2  2  2  3
  0  2  2  3  4  5  2  2  3  4  3  5
```

```
Original System of Equations

  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00
0.00 =   6.00
  2.00  2.00  1.00  1.00  1.00  0.00  4.00  3.00  3.00  3.00  2.00
2.00 =  20.00
  1.00  0.00  1.00  1.00  0.00  1.00  0.00  1.00  0.00  0.00  1.00
0.00 =   4.00
  0.00  1.00  2.00  1.00  3.00  3.00  0.00  0.00  2.00  1.00  2.00
3.00 =   9.00
  0.00  1.00  0.00  2.00  1.00  2.00  0.00  1.00  0.00  2.00  1.00
2.00 =   4.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00  0.00 -1.00 -1.00  0.00 -1.00 -1.00 -1.00 -
2.00 =   0.00
  0.00  1.00  0.00  0.00  0.00  0.00  1.00 -1.00  0.00  0.00 -1.00
0.00 =   2.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00  0.00  1.00  0.00  1.00
1.00 =   3.00
  0.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00
1.00 =   1.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00  1.00
1.00 =   3.00
Column pointer vector:   1  2  3  4  7  6  5  8  9 10 11 12

Equations to be solved


   1   0   0   0   0  -1  -1   0  -1  -1  -1  -2 =    0
   0   1   0   0   0   0   1  -1   0   0  -1   0 =    2
   0   0   1   0   0   1   1   0   1   0   1   1 =    3
   0   0   0   1   0   1   0   1   0   1   1   1 =    1
   0   0   0   0   1   0   0   1   1   1   1   1 =    3

Upper bounds:   9  9  1  1  1  1  1  1  1  1  1  1


The number of possible solutions is:  6

  1)    2   2   1   0   1   0   1   1   1   0   0   0
Solution rejected.  3 and  7 occur  3 times together
  2)    3   1   1   0   1   0   1   0   1   1   0   0
Solution rejected.  3 and  7 occur  3 times together
  3)    3   2   0   0   1   0   1   0   1   0   1   0
Solution rejected.  5 and  9 occur  3 times together
  4)    4   1   0   0   1   0   1   0   1   0   0   1
Solution rejected.  5 and  9 occur  3 times together
  5)    2   3   1   0   1   0   0   0   1   0   1   0
Solution rejected.  5 and  9 occur  3 times together
  6)    3   2   1   0   1   0   0   0   1   0   0   1
Solution rejected.  5 and  9 occur  3 times together


--------------------------------------------------
Case   47
```

B 2	B 3	B 4	B 5	B 6	B 7	B 8
3	0	4	3	1	0	0

Distribution Vectors

A 2	A 4	A 5	A 6
0	1	2	1
3	0	2	1
1	2	1	1
2	3	0	1
1	1	3	0
2	2	2	0
0	4	1	0
3	3	1	0

Pair Table

2	2	1	1	2	1	1	0
2	5	2	3	3	3	0	3
1	2	1	2	1	0	2	2
1	3	2	4	0	2	3	4
2	3	1	0	3	2	2	2
1	3	0	2	2	2	2	3
1	0	2	3	2	2	4	3
0	3	2	4	2	3	3	5

Original System of Equations

1.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	=	6.00
2.00	2.00	1.00	0.00	3.00	2.00	1.00	1.00	=	15.00
1.00	0.00	2.00	3.00	1.00	2.00	4.00	3.00	=	16.00
0.00	3.00	1.00	2.00	1.00	2.00	0.00	3.00	=	6.00

Equations in Row Echelon Form

1.00	0.00	0.00	0.00	-1.00	-1.00	-1.00	-2.00	=	1.00
0.00	1.00	0.00	0.00	0.00	0.00	-1.00	0.00	=	-1.00
0.00	0.00	1.00	0.00	2.00	1.00	2.00	2.00	=	6.00
0.00	0.00	0.00	1.00	0.00	1.00	1.00	1.00	=	3.00

Column pointer vector: 1 2 3 5 4 6 7 8

Equations to be solved

1	0	0	0	-1	-1	-1	-2	=	1
0	1	0	0	0	0	-1	0	=	-1
0	0	1	0	2	1	2	2	=	6
0	0	0	1	0	1	1	1	=	3

Upper bounds: 9 1 6 1 1 3 1 1

```
The number of possible solutions is:  7

  1)    4   0   1   1   1   1   1   0
Solution rejected.  5 and  7 occur  3 times together
  2)    5   0   0   1   1   0   1   1
Solution rejected.  5 and  7 occur  3 times together
  3)    3   0   3   1   0   1   1   0
  4)    4   0   2   1   0   0   1   1
Solution rejected.  7 and  8 occur  3 times together
  5)    5   0   0   0   1   2   1   0
Solution rejected.  5 and  7 occur  3 times together
  6)    4   0   2   0   0   2   1   0
  7)    5   0   1   0   0   1   1   1
Solution rejected.  6 and  8 occur  3 times together

Check equations:   3  6


--------------------------------------------------
Case   48

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  0     5     2     3     1     0     0


Distribution Vectors

A 3   A 4   A 5   A 6
---------------------
  0     1     2     1
  2     1     1     1
  4     1     0     1
  2     0     3     0
  1     2     2     0
  4     0     2     0
  3     2     1     0
  5     2     0     0


Pair Table

  2  1  1  2  2  1  1  1
  1  1  2  1  1  1  1  3
  1  2  4  1  1  3  3  5
  2  1  1  3  2  3  1  2
  2  1  1  2  3  1  2  3
  1  1  3  3  1  4  2  4
  1  1  3  1  2  2  3  5
  1  3  5  2  3  4  5  7


Original System of Equations

  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00 =   6.00
  2.00  1.00  0.00  3.00  2.00  2.00  1.00  0.00 =  15.00
  1.00  1.00  1.00  0.00  2.00  0.00  2.00  2.00 =   8.00
```

```
  0.00  2.00  4.00  2.00  1.00  4.00  3.00  5.00 =  15.00


Equations in Row Echelon Form

  1.00  0.00  0.00  0.00 -1.00 -1.00 -1.00 -2.00 =   1.00
  0.00  1.00  0.00  0.00  2.00  1.00  1.00  2.00 =   5.00
  0.00  0.00  1.00  0.00  0.00  1.00  0.00  0.00 =   2.00
  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00 =   1.00
Column pointer vector:   1  2  4  5  3  6  7  8

Equations to be solved


   1    0    0    0   -1   -1   -1   -2 =    1
   0    1    0    0    2    1    1    2 =    5
   0    0    1    0    0    1    0    0 =    2
   0    0    0    1    0    0    1    1 =    1

Upper bounds:    9  5  1  1  1  1  1  1


The number of possible solutions is:  6

  1)    3   2   1   1   1   1   0   0
Solution rejected.  3 and  6 occur  3 times together
  2)    2   4   1   1   0   1   0   0
Solution rejected.  3 and  6 occur  3 times together
  3)    4   1   1   0   1   1   1   0
Solution rejected.  3 and  6 occur  3 times together
  4)    5   0   1   0   1   1   0   1
Solution rejected.  3 and  6 occur  3 times together
  5)    3   3   1   0   0   1   1   0
Solution rejected.  3 and  6 occur  3 times together
  6)    4   2   1   0   0   1   0   1
Solution rejected.  2 and  8 occur  3 times together


--------------------------------------------------
Case   49

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  1     2     5     2     1     0     0


Distribution Vectors

A 2   A 3   A 4   A 5   A 6
---------------------------
  0     0     1     2     1
  1     1     0     2     1
  1     0     2     1     1
  0     2     1     1     1
  0     1     3     0     1
  1     2     2     0     1
  0     1     2     2     0
  1     2     1     2     0
```

```
   0      0      4      1      0
   1      1      3      1      0
   1      0      5      0      0
   0      2      4      0      0
```

Pair Table

```
   3   3   2   2   1   1   2   2   1   1   1   0
   3   4   3   3   1   3   2   4   1   2   1   1
   2   3   2   1   1   2   1   2   1   1   3   1
   2   3   1   3   2   3   2   3   0   1   1   2
   1   1   1   2   2   2   0   1   2   1   3   3
   1   3   2   3   2   4   1   3   1   2   3   3
   2   2   1   2   0   1   2   3   2   1   2   2
   2   4   2   3   1   3   3   5   1   3   2   2
   1   1   1   0   2   1   2   1   3   2   4   3
   1   2   1   1   1   2   1   3   2   2   4   3
   1   1   3   1   3   3   2   2   4   4   6   4
   0   1   1   2   3   3   2   2   3   3   4   5
```

Original System of Equations

```
  1.00  1.00  1.00  1.00  1.00  1.00  0.00  0.00  0.00  0.00  0.00
0.00 =   6.00
  2.00  2.00  1.00  1.00  0.00  0.00  2.00  2.00  1.00  1.00  0.00
0.00 =  10.00
  1.00  0.00  2.00  1.00  3.00  2.00  2.00  1.00  4.00  3.00  5.00
4.00 =  20.00
  0.00  1.00  0.00  2.00  1.00  2.00  1.00  2.00  0.00  1.00  0.00
2.00 =   6.00
  0.00  1.00  1.00  0.00  0.00  1.00  0.00  1.00  0.00  1.00  1.00
0.00 =   2.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00  0.00 -1.00  0.00 -1.00  0.00 -1.00 -1.00 -
1.00 =   1.00
  0.00  1.00  0.00  0.00  0.00  0.00 -1.00  1.00 -1.00  0.00 -1.00 -
1.00 =  -3.00
  0.00  0.00  1.00  0.00  0.00  1.00  1.00  0.00  1.00  1.00  2.00
1.00 =   5.00
  0.00  0.00  0.00  1.00  0.00  1.00  1.00  0.00  0.00  0.00  0.00
1.00 =   3.00
  0.00  0.00  0.00  0.00  1.00  0.00  0.00  1.00  1.00  1.00  1.00
1.00 =   3.00
Column pointer vector:   1  2  3  4  7  6  5  8  9 10 11 12
```

Equations to be solved

```
    1    0    0    0    0   -1    0   -1    0   -1   -1   -1 =     1
    0    1    0    0    0    0   -1    1   -1    0   -1   -1 =    -3
    0    0    1    0    0    1    1    0    1    1    2    1 =     5
    0    0    0    1    0    1    1    0    0    0    0    1 =     3
    0    0    0    0    1    0    0    1    1    1    1    1 =     3
```

```
Upper bounds:   1  1  5  1  3  1  3  1  1  3  1  1


The number of possible solutions is:  3

  1)    1    0    2    0    3    0    3    0    0    0    0    0
  2)    1    1    1    0    2    0    3    0    1    0    0    0
Solution rejected.  1 and  2 occur  3 times together
  3)    1    0    2    1    2    0    2    0    1    0    0    0

Check equations:   1  3


--------------------------------------------------
Case   50

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  0     1     9     0     1     0     0


Distribution Vectors

A 3   A 4   A 6
---------------
  1     3     1


Pair Table

  2


Original System of Equations

  1.00 =   6.00
  3.00 =  36.00
  1.00 =   3.00


Equations in Row Echelon Form

  1.00 =  12.00
  0.00 =  -6.00
  0.00 =  -9.00
Column pointer vector:   1

Equation  3 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case   51

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  3     1     1     6     0     0     0
```

```
Distribution Vectors

A 2   A 3   A 4   A 5
---------------------
  0     0     0     4
  1     0     1     3
  2     1     0     3
  3     1     1     2


Pair Table

  2  1  1  0
  1  1  0  2
  1  0  2  3
  0  2  3  5


Original System of Equations

  4.00  3.00  3.00  2.00 =  30.00
  0.00  1.00  0.00  1.00 =   4.00
  0.00  0.00  1.00  1.00 =   3.00
  0.00  1.00  2.00  3.00 =   6.00


Equations in Row Echelon Form

  1.00  0.00  0.00 -1.00 =   3.75
  0.00  1.00  0.00  1.00 =   4.00
  0.00  0.00  1.00  1.00 =   1.00
  0.00  0.00  0.00  0.00 =   2.00
Column pointer vector:   1  2  3  4

Equation  4 has LHS = 0 and RHS != 0
*** Block vector rejected


--------------------------------------------------
Case   52

B 2   B 3   B 4   B 5   B 6   B 7   B 8
---------------------------------------
  1     3     2     5     0     0     0


Distribution Vectors

A 2   A 3   A 4   A 5
---------------------
  0     0     0     4
  1     0     1     3
  0     2     0     3
  0     1     2     2
  1     2     1     2
  0     3     2     1
```

Pair Table

```
  3  2  2  1  1  0
  2  2  1  1  1  1
  2  1  2  0  1  2
  1  1  0  2  1  3
  1  1  1  1  2  3
  0  1  2  3  3  5
```

Original System of Equations

```
  4.00  3.00  3.00  2.00  2.00  1.00 =  25.00
  0.00  1.00  0.00  2.00  1.00  2.00 =   8.00
  0.00  0.00  2.00  1.00  2.00  3.00 =   9.00
  0.00  1.00  0.00  0.00  1.00  0.00 =   2.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00  0.00 -1.00 -1.00 =   1.00
  0.00  1.00  0.00  0.00  1.00  0.00 =   2.00
  0.00  0.00  1.00  0.00  1.00  1.00 =   3.00
  0.00  0.00  0.00  1.00  0.00  1.00 =   3.00
Column pointer vector:   1  2  3  4  5  6
```

Equations to be solved

```
   1    0    0    0   -1   -1  =     1
   0    1    0    0    1    0  =     2
   0    0    1    0    1    1  =     3
   0    0    0    1    0    1  =     3
```

Upper bounds: 1 2 3 3 2 1

The number of possible solutions is: 1

```
  1)    1    2    3    3    0    0
```

Check equations: 1

Case   53

B 2	B 3	B 4	B 5	B 6	B 7	B 8
2	0	5	4	0	0	0

Distribution Vectors

```
A 2   A 4   A 5
---------------
```

0	0	4
1	1	3
2	2	2
0	4	1
1	5	0

Pair Table

4	3	2	1	0
3	2	2	0	1
2	2	2	1	3
1	0	1	3	4
0	1	3	4	5

Original System of Equations

```
  4.00  3.00  2.00  1.00  0.00 =  20.00
  0.00  1.00  2.00  4.00  5.00 =  20.00
  0.00  1.00  2.00  0.00  1.00 =   4.00
```

Equations in Row Echelon Form

```
  1.00  0.00  0.00 -1.00 -1.00 =   1.00
  0.00  1.00  0.00  2.00  1.00 =   4.00
  0.00  0.00  1.00  0.00  1.00 =   4.00
Column pointer vector:   1  2  4  3  5
```

Equations to be solved

```
   1   0   0  -1  -1 =    1
   0   1   0   2   1 =    4
   0   0   1   0   1 =    4
```

Upper bounds: 1 4 1 2 1

The number of possible solutions is: 0

Case 54

B 2	B 3	B 4	B 5	B 6	B 7	B 8
0	2	6	3	0	0	0

Distribution Vectors

A 3	A 4	A 5
2	0	3
1	2	2

```
  0      4      1
  2      4      0


Pair Table

  5  3  1  2
  3  1  0  1
  1  0  2  2
  2  1  2  4


Original System of Equations

  3.00  2.00  1.00  0.00 =  15.00
  0.00  2.00  4.00  4.00 =  24.00
  2.00  1.00  0.00  2.00 =   6.00


Equations in Row Echelon Form

  1.00  0.00  0.00 -1.00 =  -3.00
  0.00  1.00  0.00  2.00 =  12.00
  0.00  0.00  1.00  0.00 =   0.00
Column pointer vector:   1  2  4  3

Equations to be solved


   1   0   0  -1 =   -3
   0   1   0   2 =   12
   0   0   1   0 =    0

Upper bounds:   1  9  1  6


The number of possible solutions is:  2

  1)    1   4   0   4
Solution rejected.  1 and  2 occur  3 times together
  2)    0   6   0   3

Check equations:   2


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```