THE UNIVERSITY OF MANITOBA

CONTRIBUTIONS TO STATISTICAL QUALITY CONTROL

by

KARUNARATHNAGE PIYASENA HAPUARACHCHI

A THESIS

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CONTENTS

ABSTRACT

The use of acceptance sampling plans in the context of acceptance quality control is described in chapter 1. Construction of commonly used acceptance sampling plans and control charts is briefly discussed in chapter 2.

In construction of acceptance sampling plans or control charts, the quality measurements are assumed to be normally distributed. We have examined the effect of departure from normality on $\bar{X}$ and R-charts using Tukey's λ-family of distributions. It is found that there is a serious effect on control charts even if the departure from normality is moderate. The results are given in chapter 3.

In chapter 4, we have constructed sampling plans based on the exponential distribution using maximum likelihood and minimum variance unbiased estimators of μ (location parameter) and σ (scale parameter). Acceptance sampling plans for μ and σ are also given. Constants required to construct control charts for μ and σ and for the range for this distribution are also derived.

Chapter 5 is devoted to examine the effect of serial correlation on acceptance sampling plans. This can be done by comparing OC curves, sample size and producer's risk, α, with that of the independence case assuming the process standard deviation, σ, is known. For large n, sampling plans can be constructed using the central limit theorem. When σ is unknown and for small n, a simulation study reveals that there is still no satisfactory method of obtaining sampling plans.

When constructing sampling plans for material of the bulk form, a multistage sampling procedure is often used. The variance of the sample mean is generally estimated by estimating the variance components from the analysis of variance table. This procedure may lead one or more estimates of variance components to be negative. To avoid this situation, Bayesian method of estimating variance components using Lindley's (1980) technique is proposed. These estimates can then be used to obtain an appropriate sampling plan for bulk material. This procedure is discussed in chapter 6.

In chapter 7, prediction and prediction intervals for p are derived using the predictive distribution approach. The construction of Bayesian control charts for p is also discussed. The robustness of posterior distribution of p is examined. It appears that the Bayes intervals are quite robust to the departures from the prior distribution.

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CHAPTER ONE

1.1 Introduction

In the opening remarks of their book, Sinha and Willborn (1985) state,

" Everybody is for good quality and against poor quality. No matter what is made, sold, serviced or provided, every citizen in each society depends on the quality of such business transactions. Numerous kinds and sizes of organizations, companies, and institutions exist for the specific provision for, or solution to, human needs. All have quality objectives in common. However, many do not achieve their objectives of quality at all times and often fall below the desired expectations of their customers."

According to the above statement the term "quality" applies not only to the quality of manufactured products, but also to services provided by various other organizations. They also point out that the "quality" is generally below the desired expectations of the consumers. Given this perception, it is the responsibility and obligation of those who provide products and services to maintain and improve their existing quality standards.

In order to improve the existing "quality" one has to understand the meaning of "quality". As defined by Sinha and Willborn (1985), "Quality is the totality of all attributes and characteristics of a product or service as specified, required, and expected." This definition applies to goods that are manufactured by a production process or services provided by organizations such as hospitals, schools, restaurants, and the like.

In order for an enterprise to be successful the quality of its products and services must be maintained and improved continuously. This can be achieved by the improvement of the existing process being used. As reviewed by Deming (1982), process improvement leads to the following advantages:

(1) Consistency of the final product,

(2) Reduces re-work, mistakes, rejection, waste of manpower, machine time, and materials,

(3) Increases output of final product,

(4) Enhances the competitive position in the market place,

(5) Creates more jobs,

(6) Employees are happy at work,

(7) Better employer-employee relationship with improved morale.

In other words, low quality means high cost and loss of competitive market position. Several examples are presented in Deming (1982) to illustrate how to achieve improved productivity by proper techniques of control of quality. In order to achieve the above results of improved quality, Deming (1982) presents the following 14 points for management:

"1a. Innovate. Allocate resources to fulfill the long-range needs of the company and of the customer. The next quarterly dividend is not as important as existence of the company 5, 10, or 20 years from now. One requirement for innovation is faith that there will be a future.

b. Put resources into plans for product and service for the future, taking into account:

Possible materials, adaptability, probable cost.

Method of production; possible changes in equipment.

New skills required, and in what number?

Training and retraining of personnel.

Training of supervisors.

Cost of production.

Performance in the hands of the user.

Satisfaction of the user.

c. Put resources into education (points 12 and 13).

2. Learn the new philosophy. We can no longer accept defective material, material unsuited to the job, defective workmanship, defective product, equipment out of order.

3. Eliminate dependence on mass inspection for quality. Instead, depend on vendors that use process control through statistical techniques. The purchaser is entitled to the control charts for critical characteristic of purchased material, as evidence of quality, uniformity, and cost.

4. Reduce the number of suppliers for the same item. You will be lucky to find for any item one vendor that can

furnish evidence of repeatable, dependable quality, and that knows what his costs will be. Price has no meaning without evidence of quality. Demand and expect suppliers to use statistical process control, and to furnish evidence thereof.

5. Use statistical techniques to identify the 2 sources of waste: faults of the system, or common causes (85%), and local faults (15%): strive constantly to reduce waste. (Dr. Joseph M. Juran said this years ago.)

6. Institute better training on the job with the help of statistical methods.

7. Provide supervision with the statistical methods; encourage use of these methods to identify which defects should be investigated for solution. The aim of supervision should be to help people to do a better job.

8. Drive out fear throughout the organization. The economic loss resulting from fear to ask questions or report trouble is appalling.

9. Help reduce waste by putting together as a team the people that work on design, research, sales, and production.

10. Eliminate use of goals and slogans posted for the work-force in an attempt to increase productivity. Zero defects is an example. Such slogans, in the absence of quality control, will be interpreted correctly by the work-force as management's hope for a lazy way out, and as an indication that the management has abandoned the job, acknowledging their total inadequacy.

11. Examine closely the impact of work standards in production. Work standards are exacting a heavy toll on the economy. There is a better way.

12. Institute elementary statistical training on a broad scale. Thousands of people must learn simple but powerful statistical methods.

13. Institute a vigorous program for retraining people in new skills, to keep up with changes in materials, methods, design of product, and machinery.

14. Make maximum use of statistical knowledge and talent in your company"

(Sinha and Willborn, 1985, pp. 74-75).

The above 14 points apply everywhere to small as well as large organizations, which includes both service organizations and production industries. One of the essential features of these 14 points is the responsibility of the top management to carry them out.

In the discussion of the 14 points, Deming strongly recommends the use of simple but powerful statistical techniques for process control purposes. A discussion of statistical methods used for quality control will be presented later.

1.2 Statistical Quality Control

Pioneering work on the use of statistical quality control techniques was started in the 1920's in the United States, with the development of acceptance sampling plans by

Dodge and Romig (1929), and the use of control charts by Shewhart (1926) for the purpose of process control. Shewhart advocated that quality and work must be measured step-by-step in a process in the constant quest for improvement of quality and the use of control charts was a major element of this task.

Acceptance sampling is an important branch in modern statistical quality control. It has been used to ensure that products produced in a process are consistent with the requirements placed on the process. Detailed descriptions of acceptance sampling for measurements and attributes will be given later. The use of acceptance sampling has been widely accepted and used in many and varied branches of industry. The following examples illustrate the application of sampling inspection in various situations (Wetherill, 1977).

Example 1. Large batches of electronic components have been purchased by a computer manufacturer and it is known that each batch contains an unknown number of defective items. It was decided that if a batch contains p_0 or more proportion of defective items, the batch is considered to be unacceptable. Thus, a sample of n items is drawn from each batch and if the number of defective items is greater than some value c the batch is rejected; otherwise the batch is accepted.

Example 2. Samples of milk were taken and films of milk were prepared on slides. The amount of bacteria on each film is counted using a microscope and an estimate of the density of bacteria is then computed. This figure is then used to determine the grade of milk.

Example 3. A sample of 5 items is drawn every hour from the production, and a quality characteristic (length, strength etc.) is measured for each item in the sample. Assuming the quality characteristic is normally distributed with some mean (process mean) and standard deviation (process standard deviation) and using the sample results it is possible to determine if the process is functioning in accordance with the desired characteristic (in control). If the process is not in control, some corrective action is required. The sample results can also be used to determine if the batches (a batch in this case is the items produced in a period of one hour) should be accepted or rejected.

Examples 1 and 2 illustrate the application of sampling inspection for acceptance or rejection of immediate lots while example 3, describes a situation referred to as process control. In this latter case, interest is more on controlling the process; however, as noted, the sample information could be used for acceptance/rejection of the lot examined.

All of the above examples have one feature in common, i.e., a small number of items is drawn from the batches regardless of whether the purpose is acceptance/rejection of product or control of the process. The sample information may then be used effectively for decision making purposes.

In contrast to this sampling procedure it might be that consideration would be given to inspecting all of the products from a manufacturing process. The following reasons however can be given for not undertaking a one hundred percent inspection.

(1) Absolutely accurate information about a batch or a process is never required. Hence one hundred percent inspection is a waste of available resources such as money, labour, time, etc. An estimate of the proportion defective in sampling by attributes or an estimate of the process mean in variable sampling is thus sufficient for batch (or process) sentencing. However, if the purpose is to sort all defective items from the entire lot, then 100% inspection is necessary.

(2) It is known that, the standard deviation of an estimate (usually the sample proportion or sample mean) decreases as the sample size increases (e.g., the standard error of estimate for the mean is proportional to the reciprocal of the square root of the sample size). Thus, for example, in order to reduce the standard error by one half it is necessary to increase the sample size by a factor of four. Therefore, after some point it is impractical or not cost effective to increase the sample size to achieve absolute accuracy.

(3) In situations where sampling is destructive or prohibitively costly one hundred percent inspection is ruled

out and a sample must be taken. One example of this is testing missiles where testing is clearly destructive and costly. Another example of destructive testing is the testing of photographic film. In addition, some laboratory analyses are known to be very expensive.

(4) Research suggests that contrary to one's expectations one hundred percent inspection does not provide accurate information. Hill (1962) indicates that it is reasonable to assume that the probability of correctly identifying a defective to be 0.9 because of inspection error. Thus, a sample may be preferable to 100% inspection.

1.3 The Role of Acceptance Sampling:

As indicated by Dodge (1969) and Schilling (1982), there is a significant distinction between acceptance sampling plans and acceptance quality control. As has been already mentioned, acceptance sampling plans are specific procedures which, when applied, decide acceptance or rejection of the lot/process inspected. On the other hand, acceptance quality control can be compared to process quality control. Schilling (1982) points out that "..., acceptance quality control exploits various acceptance sampling plans as tactical elements in overall strategies designed to achieve desired ends. Such strategies utilize elements of system engineering, industrial psychology, and, of course, statistics and probability theory, together with other diverse disciplines, to bring pressure to bear, maintain and improve the quality of the submitted product". Thus one should utilize acceptance sampling plans to "achieve desired ends" which is the overall maintenance and improvement of the quality of products and services. A question then arises as to how one can effectively use supplementary information from acceptance sampling plans to facilitate process control. Schilling (1985) has provided an excellent discussion to demonstrate when and how to use acceptance sampling plans to achieve the full benefit of statistical quality control. He also points out that sampling procedures must continually be made to match existing conditions, and inspection should be discontinued when the process indicates that it is in statistical control. In such situations, sampling plans may then be supplemented by check inspections, audits or other similar methods. However, sampling inspection must be reinstituted whenever the evidence shows that the process is out of control.

As has been mentioned earlier one of the purposes of acceptance sampling is to distinguish between good and bad lots. However many authors have noted that "we cannot inspect quality into a lot" under perfect state of statistical control (Barnand (1954), Case and Keats (1982), David (1985) and Schilling (1985)). In other words, the act of sampling inspection is neutral; little or no quality improvement is gained by using sampling plans when a process proportion nonconforming is constant. This was first demonstrated by Mood (1943) in the following theorem which states that,

" If X has the binomial distribution $P(X) = \binom{N}{X} p^{X} (1-p)^{N-X}$, then x and X-x are independently distributed ".

Here X is the number of defective items in a lot of size N and x is the number of defective items in a random sample of size n. Mood (1943) further states that,

" Thus, sampling of lots drawn from a binomial population will provide no basis whatsoever for inferences concerning the remainder of the lot."

Mood's theorem thus implies that if the fraction defective is constant, the number of defective items occurring from one lot to the other is independently binomially distributed. It also indicates that the sample data (usually the number of defective items in the sample) provide no information whatsoever about the quality of the remainder of the lot (unsampled portion of the lot). Suppose a producer submits lots from a process with a constant proportion defective. According to Mood's theorem, simple acceptance or rejection will not change the quality of incoming lots. The proportion defective that the consumer will receive remains the same as that of the original process. That is, there is no quality improvement achieved by sampling inspection if the underlying process distribution is known to be binomial.

Implication of using the binomial distribution as a possible model has been examined by Barnard (1954) and David (1985). Case and Keats (1982) examined the relationship between the number of defective items in the sample and number of defective items in the remainder of the lot for several probability distributions including the binomial distribution and suggest that the use of the binomial assumption renders sampling useless and inappropriate.

However, if the fraction defective fluctuates from lot to lot the quality of the outgoing product can be improved by imposition of a suitable acceptance sampling plan. This has been illustrated by Schilling (1982, pp. 2-3) using a simple example. Case and Keats (1982) also indicate that pure binomial distribution is rarely realized in practice and the use of most of the widely used sampling plans is justified.

We noted earlier that Mood's theorem applies only to processes producing constant proportion defective. Although the sample data from such a process provides no information about the quality of the unsampled portion of the lot, one could utilize the sample information as a means of obtaining an estimate of the quality of the entire lot. In fact, the commonly used estimator x/n of the proportion defective, p, possesses good mathematical and statistical properties (x/n is the uniformly minimum variance unbiased estimator of p). This estimator could then be used for acceptance/rejection of lots as follows. Let p_0 be the acceptable proportion defective. i.e., any lot containing p_0 or less proportion defective is accepted. If the estimate, $\hat{p} = x/n \leq p_0$, then the lot is accepted. If $\hat{p} > p_0$, the lot is then rejected.

This estimator can also be used for the more important purpose which is to maintain and improve the process. This is achieved by plotting p against the sample number and obtaining a well known p-chart. If a point on this chart lies outside the control limits, the process is said to be "out of control" and some corrective action is required. One should also realise that while the act of inspection is neutral to process quality in the presence of a binomial prior, it is also true that product cannot be improved by not inspecting a given lot or the process. In spite of Mood's theorem, sampling inspection thus becomes essential for batch sentencing as well as process improvement. Schilling (1985) notes that, acceptance sampling when incorporated with acceptance control, has the following advantages:

(1) Protection for the consumer and the producer,

(2) Information of quality history which could be used for research and development,

(3) Feedback of information of data on quality for process control purposes,

(4) Encourages the producer to maintain and improve the process by the economic, psychological, and political leverage on the producer.

He further provides the following applications as examples of situations where acceptance sampling can be used:

(1) Incoming and final inspection,

(2) Inspection of product moving from one stage of a production process to another,

(3) Inspection of products while they are being produced (in-process inspection),

(4) In setting up of machines,

(5) Process monitoring and adjustment,

(6) Field surveillance.

In summary then, in many practical situations the purpose of acceptance sampling is to act as a monitor of process (or quality); rather than to determine the quality.

The above mentioned advantages associated with statistical process control cannot be achieved in a relatively short period of time. Also acceptance control cannot be applied to processes where few items are manufactured (i.e., batch production). In such cases some form of acceptance sampling or 100% inspection is necessary to guarantee the quality of the outgoing product. As noted earlier, if the inspection is costly or destructive, sampling inspection must be used. If, however, a process involves continuous mass production of a product, acceptance control, together with process control can be scientifically applied. This is done by adjusting the acceptance sampling procedures to match the existing manufacturing environment and state of control of the process (Schilling (1985)). Selection of an appropriate sampling procedure depending, on extent and nature of quality history and production environment has been discussed by Schilling (1985).

1.4 Statistical Quality Control and Normality Assumption:

The use of acceptance sampling in acceptance control has been discussed previously regardless of Mood's theorem. The use of control charts is another statistical technique widely applied by industry for the purpose of process control. The constants (or tables) for constructing acceptance sampling plans and control charts are available in any standard text book on statistical quality control such as Duncan (1986) and Burr(1976) for both types of quality characteristics i.e. attributes and variables. In attribute sampling the sampling unit (usually the unit of product) is classified into two classes; defective or non-defective. When the quality characteristic involved is a measurement on a continuous scale (such as length, diameter, strength etc.), it is referred to as variable sampling.

When dealing with variable sampling (whether the purpose could be acceptance sampling or construction of control charts) two assumptions are generally made about the measurements of quality characteristic. These assumptions are:

(1) the observations are normally distributed, and

(2) the observations are independent.

The most commonly used sampling plans by variables (e.g., MIL-STD-414) as well as control charts ($\bar{X}$ and R - Charts) have been developed based on the above assumptions. Much of the available statistical literature on acceptance sampling by variables deals with the normal distribution, (Lieberman and Resnikoff, 1955; Owen, 1964,1966,1969). Today the existing variable sampling plans and $\bar{X}$ and R -Chart constants are used widely by quality control practitioners without questioning the validity of the assumption of normality. This could be due to the fact that existing tables are simple and easy to use and relatively few computations are needed. Obviously if the assumption of normality is not valid, erroneous decisions are made and the consequences may be costly. There is, therefore, a need to pay close attention to this normality assumption.

The variables sampling plans generally deal with the fraction nonconforming based on the sample mean and standard deviation. Duncan (1986, p. 256) has discussed an interesting example to show that the effect of non-normality becomes more serious when the fraction nonconforming is very small. For example, assuming the quality variable to be normally distributed, probability that the sample mean lies below one sided 3-s.d. limit is 0.00135. However, for a distribution having skewness and kurtosis given by 1.00 and 1.5, respectively, the same probability is 0.01000, which is considerably larger than that for a normal distribution. Duncan further suggests that practitioners should be cautious when they are dealing with fraction nonconforming below 0.01 because the effect of non-normality on acceptance sampling plans becomes serious.

Another practical situation where non-normality occurs has been discussed by Grant and Leavenworth (1980, p. 527). They consider the case where a producer has applied one hundred percent inspection by attributes to a lot, and subsequently the same lot is subjected to a variables acceptance sampling by a consumer. In this situation, the use of a sampling plan based on the assumption of normality may lead to the rejection of a lot when in fact the actual lot distribution may not contain any defective items. They also point out that with today's tighter standards in terms of smaller acceptable quality levels, the assumption of normality may not be appropriate.

Srivastava (1961) is one of the pioneers who examined the effect on acceptance sampling plans of departures from normality using Edgeworth series distribution. Das and Mitra (1964) have used the Gram-Charlier series, but the normal approximation was applied in determining the effect on sampling plans. Montgomery (1985) has also examined the robustness of sampling plans by variables. Schneider (1985) used the truncated normal distribution for this purpose. All these authors, from their studies, have concluded that the acceptance sampling plans are sensitive to departures from normality and the sampling plans based on the assumption of normality should not be used if the normality assumption cannot be justified. Owen (1969) has discussed acceptance sampling plans by variables with special emphasis on departures from normality. He has suggested a simple modification to normal sampling plans if the underlying distribution deviates from normality.

American National Standard ANSI/ASQC Z 1.9 (1980) points out that,

" This standard assumes the underlying distribution of individual measurement to be normal in shape. Failure of this assumption to be valid will affect the OC curve and probabilities based on the curves. In particular it will affect the estimate of percent nonconforming calculated from the mean and the standard deviation of the distribution. The assumption should be verified prior to use of the standard. A variety of statistical tests and graphical techniques are available for this purpose. A person knowledgeable in statistics should be consulted who can advise whether the distribution appears suitable for sampling by variables."

ANSI/ASQC Z 1.9 - 1980 is the modified version of MIL - STD - 414 to achieve correspondence with the standard ISO/ DIS 3951 and matching with MIL - STD - 105D. As has been indicated, tables and procedures contained in this standard are constructed for normally distributed variables. Although the authors of this standard recognize the importance of the departure from the assumption, they do not offer any alternative when the normality assumption is called into question. Duncan (1986, p. 301) has, therefore, suggested that a test for normality be made part of this standard.

Duncan also points out that the assumption of normality should be verified before applying these plans to a given process or a lot. Several statistical techniques such as chi-square goodness of fit or Kolmogorov-Smirnov tests are available to aid in this process. Graphical techniques are also very useful in identifying the appropriate underlying probability distribution. One such widely used technique is probability plotting. Hahn and Shapiro (1967) have discussed the general theory of constructing probability plots for any distribution. Probability plots for commonly occurring distributions (e.g., normal, gamma and Weibull) are also discussed in their book. Quality control practitioners should, therefore, use these statistical and graphical techniques to examine the assumption of normality.

Consider the following example taken from Duncan (1986, p. 42). The data consist of 145 observations drawn at random from a production line and the variable of interest is overall height of fragmentation of bomb bases. Duncan used this data set to construct control chart limits (see Duncan (1986), pp. 480 -483) for $\bar{X}$ and R -charts assuming the underlying distribution of heights to be normal. The same data set has been subjected to a test of the validity of the assumption of normality (see Duncan (1986), pp. 635- 637) using chi- square goodness of fit test and normal probability plots. Chi-square goodness of fit suggests the data do not conform to a normal distribution while probability plot indicates the data belong to a mixture distribution. Therefore the tabulated normal distribution control chart constants should not be used in this case or the construction of acceptance sampling plans based on the assumption of normality may not be appropriate. An attempt to identify the appropriate underlying probability distribution and to determine constants applicable to this new distribution would be indicated.

1.5 Outline of the Dissertation:

Most of the material presented in this dissertation deals with modification of control charts and acceptance sampling plans. Chapter 2 is devoted to a review of sampling plans that are widely accepted and commonly used. These include both types of sampling plans; by attributes and by variables.

Because the assumption of normality is crucial to acceptance sampling and control charts by variables, an attempt is made in this dissertation to investigate this assumption. A robustness study was developed to examine the effect of non-normality on control chart limit constants for $\bar{X}$ and R - Charts. For this study Tukey's λ - family of distributions was used and the results are reported in Chapter 3. Description of control charts by variables will also be given in this chapter.

As has been noted previously, until recently no sufficient attention has been given to non-normal distributions as potential competitors to the normal distribution in statistical quality control. The two parameter exponential distribution is presented as a suitable alternative for quality control purposes and one and two sided sampling plans to control the percent defective are derived in Chapter 4. Sampling plans for process parameters (location ,μ, and scale ,σ) are also given in this chapter as well as construction of control charts for μ and σ.

As was discussed earlier, the second assumption regarding acceptance sampling is the assumption of independence. Relatively few authors have discussed the modification of control chart constants in the presence of data correlation (see Vasilopoulos and Stamboulis (1978)). However, no attempt has been made to investigate the effect of serial correlation on acceptance sampling plans. In Chapter 5, the observations are assumed to follow a stationary autoregressive process of order p. The corresponding sampling plans by variables to control the proportion of defective falling outside the specification limits are derived assuming the process standard deviation (σ) is known. For large samples using the central limit theorem, sampling plans can be obtained. But at present there is no satisfactory method to obtain sampling plans for small samples. In this chapter, a report is given on a simulation study to examine properties of sampling plans based on small samples in the presence of serial correlation. Several examples to illustrate the application of sampling plans to serially correlated data are presented in this chapter.

The sampling plans presented in standard texts on quality control relate to processes producing units of products which are readily identifiable, such as an electric bulb. These discrete items can be used as sampling units in a given acceptance sampling procedure. However, in practice another type of material can be given where the sampling units are not easily identifiable. Examples of such materials are truck loads of cement, wheat or fertilizer, piles of coal, shipments of wool etc. Sampling from these types of material is known as bulk sampling because the product consists of large quantity of material. Discussion on bulk sampling and problems associated with sampling and estimation can be found in Bicking (1967). Schilling (1982) in his book has provided an excellent chapter on bulk sampling.

In chapter 6 a special problem associated with bulk sampling will be examined. Consider the situation where n_1 primary sampling units are selected from the bulk material (e.g., n_1 bales of wool from a shipment of wool) and n_2 secondary sampling units are selected from each n_1 primary unit. Let X_{ij} be the measurement of some quality characteristic from the jth secondary unit of the ith primary unit. A linear model of the form

$$X_{ij} = \mu + \alpha_i + e_{ij}$$

$i=1,2,\ldots,n_1$; $j=1,2,\ldots,n_2$ can be used to represent X_{ij} where μ is the overall mean (usually the lot mean or the process average), α_i is the random variable representing the variability between primary units and e_{ij} is the random error component. We assume that $e_{ij} \sim N(0, \sigma_1^2)$, $\alpha_i \sim N(0, \sigma_2^2)$ and α_i and e_{ij} are independent. In constructing an appropriate sampling plan for mean, μ, the variance of the sample mean, is usually required. It can be shown that

$$V(\bar{X}) = \sigma_2^2/n_1 + \sigma_1^2/n_1n_2$$

and quality control practitioners usually use the traditional moment estimators of σ_1^2 and σ_2^2 obtained from the analysis of variance table. One of the problems associated with this procedure is the fact that one of the estimators of variance components may be negative if the mean square error is larger than the mean square due to primary units. Such negative estimators of variance components are not acceptable and statisticians usually set them to be zero. In this chapter, we give Bayes estimators of the variance components of the above model and these estimators are always positive. Bayes estimators of σ_1^2 and σ_2^2 can then be used to set up acceptance sampling plans for bulk material.

In chapter 7, prediction and prediction intervals for proportion defective (p) using the predictive distribution approach are discussed. These prediction intervals can be used in a similar manner for process control purposes as discussed by Hahn (1969, 1970) and Whitmore (1986). The effect of the prior distribution on the posterior distribution of p is also discussed in this chapter. Direction for future research in the area of statistical quality control will also be presented in this chapter.

CHAPTER TWO

ACCEPTANCE SAMPLING PLANS AND CONTROL CHARTS

In this chapter acceptance sampling plans and control charts by attributes and by variables that are widely used by industry are briefly discussed. A brief description of definitions of terms involved in statistical quality control will also be given.

Before proceeding to a description of acceptance sampling plans and control charts, it is useful to make some remarks about specification limits, control limits and tolerance limits. In the following discussion, let X be the quality measurement of a product (e.g., tensile strength).

Usually, specification limits refer to individual items and they are determined by quality control engineers by examining the capability of a given process. An item is considered acceptable only if X lies within specified limits of the quality variable. For example, consider the tensile strength of an item should be at least L, then if X is less than L, the item is considered unacceptable. L is called a lower specification limit. Similarly, an upper specification limit can be defined.

The control limits are entirely different to specification limits. Control limits pertain to means not to individual items. Generally, lower and upper control limits are constructed using sample information. A given process is said to be in control if the sample mean of a current sample lies inside the control limits although the quality measurement of an individual item may be outside the specification limits.

The tolerance limits can be explained using the following example. Suppose it is required to find two limits, a and b, such that

$$P[(X \leq a) \leq p/2] \text{ and } P[(X \geq b) \leq p/2] = \alpha.$$

Then (a,b) is said to be a two sided $100(1-\alpha)\%$ tolerance limit for X. Similarly, one sided tolerance limits can be defined.

2.1 Definitions of Terms:

The following definitions shall be used throughout this dissertation, and are taken from the American Society for Quality Control Standard (1962). These definitions are used in sampling systems such as MIL-STD-105D and MIL-STD-414 and in text books like Hald (1982) and Schilling (1982).

(i) Acceptable Quality Level (AQL):

The maximum percent defective (or the maximum number of defects per hundred units) that, for purposes of acceptance sampling, can be considered satisfactory as a process average.

(ii) Lot Tolerance Percent Defective (LTPD):

Lot tolerance percent defective - expressed in percent defective, the poorest quality in an individual lot that should be accepted.

(iii) Average Outgoing Quality (AOQ):

The expected average quality of an outgoing product including all accepted lots after the latter has been inspected 100 percent and the defective units replaced by good units.

(iv) Average Outgoing Quality Limit (AOQL):

The average outgoing quality limit is defined as the maximum of average outgoing quality i.e., AOQL = max AOQ.

2.2 Notation:

The following notation is used in this dissertation.

N : the number of items in the lot (or lot size),

n : the sample size (the number of items drawn from the lot into the sample),

X : the number of defective units in the lot,

x : the number of defective units in the sample,

c : acceptance number,

p : proportion defective in the lot ($p = X/N$),

p_1 : acceptable quality level,

p_2 : lot tolerance percent defective,

$P(p)$: probability of acceptance of a lot given that proportion defective in the lot is p,

α : producer's risk: the probability that the lot is rejected when the process is operating at AQL i.e., $P(p_1)$,

β : consumer's risk: the probability that the lot is accepted when the process is operating at LTPD i.e., $P(p_2)$.

2.3 Description of Acceptance Sampling Plans:

The main purpose of an acceptance sampling plan is to decide the acceptance/rejection of a given "lot". For this reason, a concept of a lot which is useful in statistical quality control has been suggested by Simon (1944). He defines " a lot is an aggregation of articles which are essentially alike". Thus, before applying quality control techniques, the practitioners should be able to identify an ideal lot for inspection. Grant and Leavenworth (1980, pp. 502-504) have discussed the use of control chart technique to identify such lots which are known as "grand lots".

An acceptance sampling plan is usually represented by its operating characteristic (OC) curve. This can also be used as an analytical tool of the sampling plan. The OC curve of a given sampling plan is obtained by plotting probabilities of lot acceptance for various values of process parameter (e.g., process proportion defective for attribute sampling or process mean for variable sampling).

Two types of OC curves are recognized in the literature.

(i) Type A curves relate to sampling from individual lots and are obtained by plotting the probability that the lot is accepted against the proportion defective in the lot.

(ii) Type B curves relate to sampling from a continuous process and are obtained by plotting the proportion of lots accepted against the process proportion defective.

The appropriate probability distribution of the lot quality must be used to obtain each type of OC curve.

2.3.1 Sampling Plans by Attributes:

(i) Single Sampling Plans:

Single sampling plans can be considered to be basic to all sampling plans just as simple random sampling is in Survey Sampling. The efficiencies of other sampling plans are determined by comparing the sample size required to achieve the same producer's and consumer's risks with that of the single sampling plan.

The parameters of a plan are acceptable quality level (p_1), lot tolerance percent defective (p_2), producer's risk (α), consumer's risk (β) and the plan is given by sample size (n) and acceptability constant (c). The values of p_1, p_2, α, and β are generally specified and the plan is then derived. The operation of the plan is as follows: A sample of items of size n is taken from the lot (applied to both type A and type B sampling) and the number defective, x, in the sample (or the number of defects per 100 items) is counted. The number defective is then compared with the acceptability constant, c. If $x > c$, the lot is rejected and if $x \leq c$ the lot is accepted.

To obtain a sampling plan we can use tables such as MIL-STD-105D (1963) and Dodge and Romig (1959). Graphical techniques may also be used to derive a single sampling plan. Such graphical procedures include binomial nomograph by Larson (1966), f-binomial nomograph by Landy (1971) and Thorndyke chart by Thorndyke (1926). Dodge and Romig (1959), Duncan (1986) and Hald (1982) are among the best sources for theoretical treatment of single sampling plans.

(ii) Double and Multiple Sampling Plans:

A double sampling plan is specified by five parameters n_1, n_2, c_1, c_2 and c_3. This plan works as follows: A sample of size n_1 is taken and if the first sample contains c_1 or less defective items (denoted by x_1) the lot is accepted. If it contains c_1 or more defective items the lot is rejected. If x_1 is greater than c_1 but less than c_2, the second sample of size n_1 is taken and the number of defective units in this sample is obtained (denoted by x_2). If the combined number of defective items ($x_1 + x_2$) is less than or equal to c_3 the lot is accepted; otherwise the lot is rejected (i.e.; $x_1 + x_2 > c_2$).

The principal advantage of a double sampling plan over a single sampling plan with the same producer's and consumer's risks is the reduction in the total inspection. This is because average sample number (ASN) (i.e., the expected number of items to be sampled) for a double sampling plan is always less than the ASN for a single sampling plan. This may lead to a considerable saving of available resources such as labour, money, and time, etc.

A multiple sampling plan is a natural extension of a double sampling plan in which further additional samples are taken to achieve even more discrimination in the disposition of a lot. A decision is reached at the k-th sample and the rejection number at the k-th stage is set to be $c_k + 1$.

A multiple sampling plan is much more flexible and offers even more reduction in sample size over single and double sampling plans which leads to increased economy. However, efficiency of this sort may be costly because multiple plans are often difficult to administer usually with extra workload.

As in the case of single sampling plans, double and multiple sampling plans, up to seven steps are found in the MIL-STD-105D tables and matched double sampling plans (the OC curves of double plans coincide with single plans) are tabulated in Dodge and Romig (1959). Several other sources from which to obtain double and multiple plans are Duncan (1986) and Schilling and Johnson (1979).

(iii) Sequential Sampling Plans:

Application of Wald's (1947) Sequential Probability Ratio Test (SPRT) to quality control situations leads to the development of sequential sampling plans. At each stage of sampling, a sample of one unit is taken and based on the sample results, a decision to accept, reject or take further samples is taken. Sampling is terminated when a decision is reached to accept or to reject the lot or process sampled. As such, the sample size is not specified; in fact it is a random variable and often very small. The sample size can only be determined after the test is terminated. It can be shown that a sequential sampling procedure has the optimum property that the average sample number is as low as possible Wald (1947). Thus, sequential sampling plans are more suitable in cases where it is absolutely necessary to keep the sample size as small as possible (e.g., destructive and costly testing). Single, double and multiple sampling plans are special cases of sequential plans.

A sequential sampling plan is generally implemented using a chart such as given in Figure 2.1 where

n = number of items drawn

x_k = number of defective items found in sample k

Σx_k = total number of defective items in n units

y_2 = $sk + h_2$ is the reject limit at sample k

y_1 = $sk + h_1$ is the accept limit at sample k.

The cumulative number defective is plotted on this chart against the number of samples taken and the decision to accept or to reject is taken when the plot crosses the accept or reject lines. This is a graphical procedure equivalent of the process:

reject the lot (or process) if $\sum x_k \geq y_2$

accept the lot (or process) if $\sum x_k \leq y_1$

continue sampling if $y_1 < \sum x_k < y_2$.

Formulas for the construction of sequential plans for given values of p_1, p_2, α, and β have been derived by Wald (1947) and the Statistical Research Group (1945). These are:

$$h_1 = \frac{\log[(1-\alpha)/\beta]}{\log[\{p_2(1-p_1)\}/\{p_1(1-p_2)\}]}$$

$$h_2 = \frac{\log[(1-\beta)/\alpha]}{\log[\{p_2(1-p_1)\}/\{p_1(1-p_2)\}]}$$

$$s = \frac{\log[(1-p_2)/(1-p_1)]}{\log[\{p_2(1-p_1)\}/\{p_1(1-p_2)\}]} .$$

Sequential sampling plans can be obtained from Statistical Research Group (1945) tables. Also a table of plans for α=.05 and β=.10 is given in Schilling (1982). An excellent discussion on this subject is given in Burr (1976). For theoretical development and applications see Wald (1947) and Wetherill (1975).

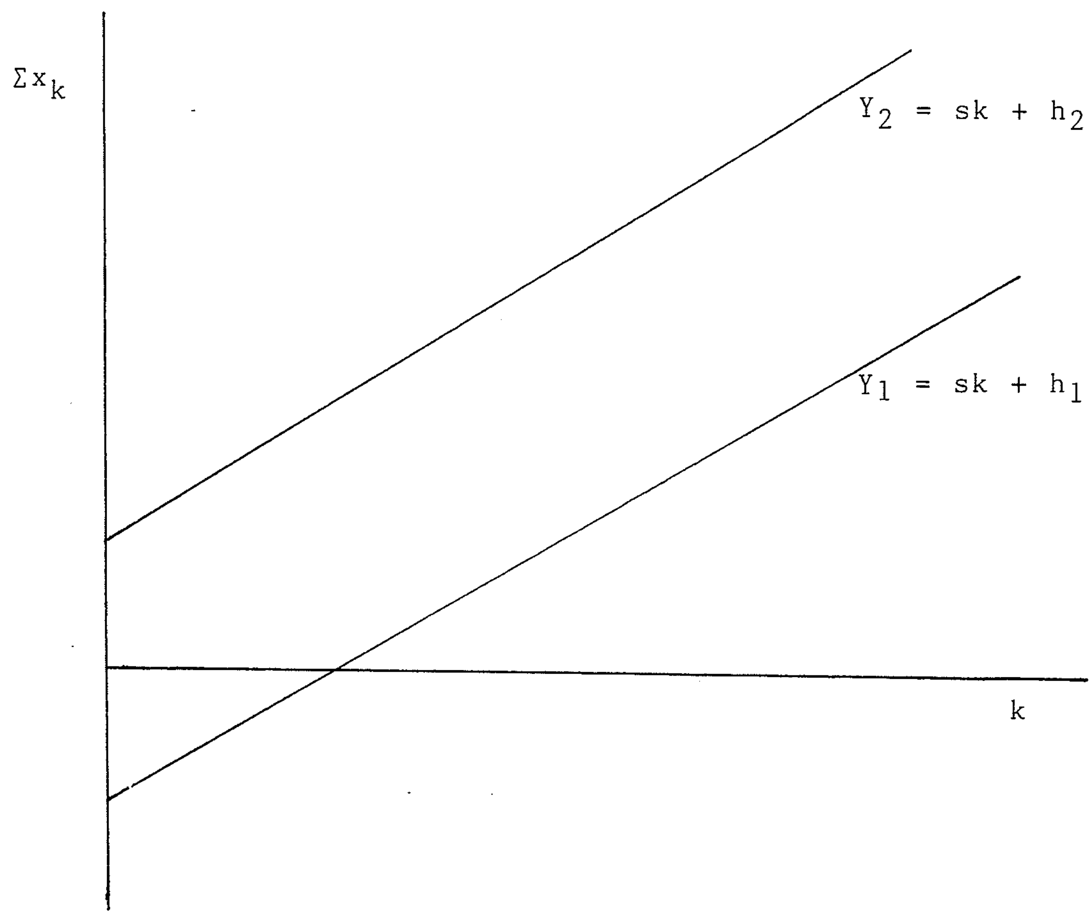


Figure 2.1. Sequential Test Criterion.

Efficiencies of single, double, multiple and sequential sampling plans are usually evaluated by comparing various measures of these plans. A commonly used measure is the average sample number (ASN). The following table taken from Schilling (1982, pp.154) compares sample sizes of double, multiple, and sequential plans against a single plan with n=50, c=2, α=.05, and β=.10. Note that the smaller size is an indication of greater efficiency.

Plan	ASN
Single	50
Double	43
Multiple	35
Sequential	33.5

Obviously the sequential plan has obtained the highest efficiency in terms of the ASN relative to a single and double plans, but a large gain is not noted from a multiple to a sequential plan. Other measures of these plans which can be used for comparison purposes include Average Outgoing Quality (AOQ), Average Outgoing Quality Limit (AOQL), and Average Total Inspection (ATI); and these are discussed in Schilling (1982).

Sequential plans can also be constructed when the number of defects per unit (or defects per 100 units) is involved. In this case, the Poisson distribution is used as the underlying model and using SPRT the following parameters are derived:

$$h_1 = \frac{\log[(1-\alpha)/\beta]}{\log\mu_2 - \log\mu_1}$$

$$h_2 = \frac{\log[(1-\beta)/\alpha]}{\log\mu_2 - \log\mu_1} \quad \text{and}$$

$$s = \frac{\mu_2 - \mu_1}{\log\mu_2 - \log\mu_1}$$

where μ_1 is producer's quality level (PQL) μ_2 is consumer's quality level (CQL) and $\mu_2 > \mu_1$ Schilling (1982, p.161).

2.3.2 Sampling Plans by Variables:

When the measurement of the quality characteristic involved is a variable such as length, time, weight, etc., it can easily be converted to an attribute. This is accomplished by choosing specification limits for the characteristic and examining whether the quality measurement lies within the specification limits or not. If it lies within the limits, the item is classified as good; otherwise it is defective. When deciding upon acceptance sampling plans we, therefore, have a choice to make; to use attribute or variable sampling.

In general, measurements provide more information about items or a process involved than simple classification of units into defective or non-defective. This leads to a considerable saving in sample size compared to attribute sampling to obtain the same discrimination.This is illustrated using the following table taken from Schilling (1982, p.224) by comparing average sample number for various plans matched to single plan by attributes n=50, c=2:

Plan	ASN
Single attribute	50
Double attribute	43
Multiple attribute	35
Sequential attribute	33.5
Variable (σ unknown)	27
Variable (σ known)	12

It must be emphasized efficiency of this sort can only be achieved if the underlying probability distribution of individual measurements is known and is stable. Thus, in many practical applications the measurements are assumed to be independently, normally distributed. The sampling plans by variables presented in this section are derived by using the normality assumption.

As pointed out earlier, the principal advantage of using variable plans is the greater reduction in sample size. For other advantages and disadvantages of sampling plans by variables, see Schilling (1982).

(i) Sampling by Variables for Proportion Nonconforming:

In this section the sampling plans to control the proportion of items lying outside specification limits are presented. The following notation is used.

k : acceptability constant

U : upper specification limit

L : lower specification limit

μ : process average (mean)

σ : process standard deviation

$\bar{X} = \sum X_i/n$: sample mean.

Definitions of p_1, p_2, α, and β remain as before.

It should be noted that X_i 's are the measurements of the quality characteristic and they are assumed to be independently, normally distributed random variables. A practitioner who uses these plans should be aware of this limitation of assumption of normality. In this regard, Duncan (1986, p.312) has given a "special warning" to practitioners.

Duncan (1986, p.548) also suggests that the decision of a practitioner to use the "normal analysis" depends on his personal inclinations, and this makes it impossible to obtain the theoretical OC curve of a given sampling plan. The assumption of normality should, therefore, be given serious consideration by the practitioners.

Two separate cases can be identified; σ known and σ unknown. Although σ known case is rarely realized in many practical situations, both cases are discussed for completeness. In either case, a sampling plan is generally specified by n and k where k is the acceptability constant.

<u>Case I:</u> <u>One Sided Sampling Plans (σ known)</u>

The rule is to accept a lot if $\bar{X} - k\sigma \geq L$ or $\bar{X} + k\sigma \leq U$ and the sampling plan (n and k) is derived in order to satisfy (for lower specification limit)

$$P(\bar{X} - k\sigma \geq L \mid p_1) = 1-\alpha,$$

and

$$P(\bar{X} - k\sigma \geq L \mid p_2) = \beta.$$

Using these probability statements, it can be shown that

$$P(Z \leq \sqrt{n}(k-z_1)) = \alpha \tag{2.1}$$

and

$$P(Z \geq \sqrt{n}(k-z_2)) = \beta \tag{2.2}$$

where

$$Z = \sqrt{n}(\bar{X}-\mu)/\sigma \sim N(0,1);$$

and z_1 and z_2 are such that

$$\int_{z_1}^{\infty}[\exp(-z^2/2)]/\sqrt{2\pi} = p_1 \text{ and } \int_{z_2}^{\infty}[\exp(-z^2/2)]/\sqrt{2\pi} = p_2.$$

From (2.1) and (2.2), we obtain

$$\sqrt{n}(k-z_1) = -z_\alpha \tag{2.3}$$

and

$$\sqrt{n}(k-z_2) = z_\beta. \tag{2.4}$$

Solving (2.3) and (2.4) for n, we get

$$n = [(z_\alpha - z_\beta)/(z_1 - z_2)]^2. \tag{2.5}$$

k is then given by

$$k = z_1 - z_\alpha/\sqrt{n} \tag{2.6}$$

or

$$k = z_2 + z_\beta/\sqrt{n}. \tag{2.7}$$

Note that two values are possible for k depending on whether equation (2.3) or (2.4) is used. In practice, it is common to take the average of two values derived above.

For an upper specification limit using the above procedure it can be easily shown that the same formulas (2.5), (2.6) and (2.7), are obtained. Two sided sampling plans by variables can be constructed in a similar manner.

Tables of sampling plans by variables are available for normally distributed measurements. Perhaps, the best source is the MIL-STD-414. Theory of MIL-STD-414 is given in Lieberman and Resnikoff (1955). Description of MIL-STD-414 can be found in the literature on quality control such as Duncan (1986), Burr (1976) and Schilling (1982).

Case II: σ-unknown

To illustrate the construction of sampling plans when σ is unknown, consider the acceptance rule $\bar{X} + ks \le U$ for an upper specification limit where s is the sample standard deviation. The acceptance rule can be expressed as

$$\{\sqrt{n}(U-\mu)/\sigma - \sqrt{n}(\bar{X}-\mu)/\sigma\}/(s/\sigma) > \sqrt{n}k. \qquad (2.8)$$

The expression on the left hand side of (2.8) is distributed as a non-central t-distribution with f= n-1 degrees of freedom and non-centrality parameter

$$\delta = \sqrt{n}(U-\mu)/\sigma = \sqrt{n}K_{1-p} = \sqrt{(f+1)}K_{1-p},$$

where K_{1-p} is the upper p-percent point in the standard normal distribution.

The probability of acceptance is then

$$\begin{aligned} P(t > \sqrt{n}k) &= 1 - P(t \le \sqrt{n}k) \\ &= 1 - P(t/\sqrt{n-1} < \sqrt{n/(n-1)}k) \\ &= 1 - P(n-1, \sqrt{n}K_{1-p}, \sqrt{n/(n-1)}k), \end{aligned}$$

where $P(n-1, \sqrt{n}K_{1-p}, \sqrt{n/(n-1)}k)$ is the probability that the t value is less than or equal to $\sqrt{n/(n-1)}k$ with (n-1) degrees of freedom and non-centrality parameter $\sqrt{n}K_{1-p}$. This probability integral has been tabulated by Resnikoff and Lieberman (1957). Using these tables we can construct the sampling plans as follows:

For (p_1, α),

$$P(t \ge \sqrt{n}k/p_1) = 1-\alpha$$

or

$$P(n-1, \sqrt{n}K_{1-p_1}, \sqrt{n/(n-1)}k) = \alpha. \qquad (2.9)$$

Similarly for (p_2, β),

$$P(n-1, \sqrt{n}K_{1-p_2}, \sqrt{n/(n-1)}k) = 1-\beta. \tag{2.10}$$

The solution for n and k can be obtained by trial and error method using equations (2.9) and (2.10).

However, Duncan (1986, p. 269) has given an approximate procedure for finding n and k for large samples under the assumption that $\bar{X} + ks$ is approximately normally distributed with mean $\mu + k\sigma$ and variance approximately equal to $[1/n + k^2/2n]\sigma^2$. This method yields

$$k = (z_2 z_\alpha + z_1 z_\beta)/(z_\alpha + z_\beta)$$

and

$$n = (1+k^2/2)[(z_\alpha + z_\beta)/(z_1 - z_2)]^2.$$

It is noted that in this case, because the process standard deviation is unknown, one must take $(1 + k^2/2)$ times as many samples as the case when σ is known. It is also noted that the formula for k remains as before. Several other procedures for choosing n and k are discussed in Resnikoff and Lieberman (1957).

Once again MIL-STD-414 provides tables for the construction of sampling plans when σ is unknown. Two types of procedures are given in these tables; one is the Variability Unknown - Standard Deviation Method, where s is used to estimate σ and other the is the Variability Unknown - Range Method, where $\bar{R}/d_2$ is used to estimate σ. Here $\bar{R}$ is the mean of the ranges and d_2 is the expectation of the relative range R/σ.

(c) Sampling Plans by Variables for Process Parameter:

The sampling plans discussed in the previous section are concerned with controlling the proportion of items lying outside the specification limits. In this section, sampling plans for average quality of material or variability in the quality of the material will be given. Plans of this type are widely used in the sampling of bulk material that comes in bags, boxes, drums, bales etc.

(i) Sampling Plans for Process Average(μ):

Let μ_1 and μ_2 be the acceptable quality level (AQL) and rejectable quality level (RQL) respectively. When a lower specification limit is given $\mu_1 > \mu_2$ and for a upper limit $\mu_2 > \mu_1$. Consider finding a sampling plan for the case where the lower limit is specified. The rule is to accept the lot if $\bar{X} \geq k$, and the problem is to find n and k to satisfy

$$P(\text{accept the lot}|\mu_1) = 1-\alpha$$

and

$$P(\text{accept the lot}|\mu_2) = \beta.$$

In other words,

$$P(\bar{X} \geq k|\mu_1) = 1-\alpha \qquad (2.11)$$

and

$$P(\bar{X} \geq k|\mu_2) = \beta. \qquad (2.12)$$

Assume the process standard deviation is known to be σ. Then from (2.11) and (2.12) we get,

$$(k - \mu_1)/(\sigma/\sqrt{n}) = -z_\alpha$$

and

$$(k - \mu_2)/(\sigma/\sqrt{n}) = z_\beta.$$

We then obtain

$$n = [\sigma(z_\alpha + z_\beta)/(\mu_2 - \mu_1)]^2 \tag{2.13}$$

and k is found either from

$$k = \mu_1 - z_\alpha \sigma/\sqrt{n} \quad \text{or}$$

$$k = \mu_2 + z_\beta \sigma/\sqrt{n}. \tag{2.14}$$

For upper specification limit (2.13) still gives n, but k is given by

$$k = \mu_1 + z_\alpha \sigma/\sqrt{n} \quad \text{or}$$

$$k = \mu_2 - z_\beta \sigma/\sqrt{n}.$$

(ii) Sampling Plans for Process Standard Deviation

In this case let σ_1 be the acceptable process standard deviation and σ_2 be the rejectable process standard deviation ($\sigma_2 > \sigma_1$). Assume X 's are independently, normally distributed and s is the sample standard deviation.

We would then use the criteria

accept the lot if $s^2 \le k$,

and

reject the lot if $s^2 > k$.

As such, the following probability equations can be formed:

$$P(s^2 \le k|\sigma_1) = 1-\alpha$$

and

$$P(s^2 \le k|\sigma_2) = \beta.$$

But it is known that, $(n-1)s^2/\sigma^2$ follows a chi-square distribution with (n-1) degrees of freedom. Using this property of s^2, it can be shown that,

$$\chi^2_{n-1,1-\alpha} = (n-1)k/\sigma_1^2 \tag{2.15}$$

$$\chi^2_{n-1,\beta} = (n-1)k/\sigma_2^2. \tag{2.16}$$

Solving (2.15) and (2.16) for n, n cannot be found directly. Thus, by dividing (2.15) by (2.16), we get

$$\sigma_2/\sigma_1 = \sqrt{\chi^2_{n-1,1-\alpha} / \chi^2_{n-1,\beta}} \tag{2.17}$$

Burr (1974) has tabulated the ratio given in (2.17) for several values of n, α and β. As a result, n can be obtained from these tables and k can then be found using either (2.15) or (2.16).

(iii) Sequential Sampling Plans by Variables

As has been indicated earlier the assumption of normality is crucial in the case of sampling plans by variables. Thus, the sequential sampling plans given in this section are also based on this assumption. It should be emphasized that the superiority of these plans over attribute and other variable sampling plans rests entirely on the validity of the normality assumption.

As given in section 2.3.1(iii), the accept/reject criterion is:

Reject the lot (or process) if $\sum X_i \geq Y_2$,

Accept the lot (or process) if $\sum X_i \leq Y_1$,

Continue sampling if $Y_1 < \sum X_i < Y_2$

where $Y_1 = sk + h_1$ and $Y_2 = sk + h_2$.

Case I: Sequential Plans for Process Average

When the measurements are assumed to be independently, normally distributed it can be easily shown using SPRT that

$$s = (\mu_1 + \mu_2)/2,$$
$$h_1 = [\log\{(1-\alpha)/\beta\}/(\mu_2-\mu_1)]\sigma^2,$$
$$h_2 = [\log\{(1-\beta)/\alpha\}/(\mu_2-\mu_1)]\sigma^2.$$

Here α = producer's risk, β = consumer's risk, μ_1 = AQL, μ_2 = RQL and σ^2 is the process variance assumed to be known.

Case II: Sequential Plans for Process Variance

Once again using the theory of SPRT the constants required for this plan are:

$$s = [\log(\sigma_2^2/\sigma_1^2)]/[(1/\sigma_1^2) - (1/\sigma_2^2)],$$

$$h_1 = [2\log(1-\alpha)/\beta)]/[(1/\sigma_1^2) - (1/\sigma_2^2)],$$

$$h_2 = [2\log[(1-\beta)/\alpha)]/[(1/\sigma_1^2) - (1/\sigma_2^2)].$$

2.4 Other Sampling Plans

The acceptance sampling plans discussed in section 2.3 of this chapter are the ones that are commonly used by the quality control practitioners. Apart from these plans, other special plans and procedures have been developed for various sampling situations for both attributes and variables. These include no-calc plans, lot plot plans, narrow limit gauging, mixed attribute-variable plans and Phillips standard sampling system. A detailed discussion of these plans are given in Schilling (1982).

Most of the sampling plans presented in this chapter are concerned with the individual lots that are readily identifiable. When a continuous production process is involved individual lots cannot be easily determined, rather they must be artificially created. For example, items produced in a period of one hour can be considered as a lot. Several sampling plans have been developed to deal with this situation. They are known as continuous sampling plans (CSP). An excellent discussion of various continuous sampling plans are given in Wetherill (1977) and Schilling (1982).

When dealing with lot by lot inspection the sampling plans discussed so far utilize the information obtained from the sample of the immediate lot inspected for the purpose of acceptance/rejection. However, there are situations where for economic considerations the sample size should be kept at a minimum while a reasonable protection against passing unacceptable lots should be maintained. In order to achieve this criterion, special sampling plans and procedures have been developed by incorporating the information gathered from samples of previous lots inspected. These plans are known as cumulative sampling plans and include skip-lot sampling plans, chain sampling plans, deferred sentencing schemes, demerit rating plans and cumulative results criterion plan (CRC). MIL-STD-105D and MIL-STD-414 can also be considered belonging to this category because the results of the previous lots examined are used to switch from normal to tightened, tightened to normal or normal to reduced inspection. Schilling (1982) has provided a summary of these plans.

Another type of special sampling plans known as compliance sampling deals with situations such as compliance testing of standards set by government (e.g.; testing of safety products and drugs), validation of suppliers inspection, inspection of extremely tight standards etc. In these types of plans an acceptance number c=0 is commonly used. These plans include lot sensitive sampling plans(LSP), tightened-normal-tightened scheme (TNT), quick switching system (QSS) and simplified grand lot procedure. Once again the reader is referred to Schilling (1982) for a discussion of compliance sampling plans.

If a process is in statistical control, it is optimum to use 100% inspection or not to inspect at all (Deming, 1982). Deming, therefore, suggests that acceptance sampling should not be used as a tool in quality assurance. However, there are situations where proportion nonconforming in incoming lots fluctuates markedly from lot to lot indicating that the process is not in statistical control. Zero defects sampling plans which are found to be useful in this case have been proposed by Hahn (1986). This plan works as follows: A random sample of size n is taken from a lot and if there is no defective items in the sample, the lot is accepted. Otherwise, the lot is rejected. Hahn (1986) has also proposed a method to estimate the proportion nonconforming in the accepted lot based on a zero defects sampling plan.

When there is a need to use 100% inspection to ensure that the items conform to the standard level of quality, a large number of inspections have to be performed. If the inspections are lengthy or expensive, a considerable amount of time or cost may be involved (e.g., destructive testing). To avoid this difficulty, a procedure known as group sampling can be used. This procedure tests a group of items simultaneously rather than testing individual items. A complete discussion of this procedure is given by Mundel (1984). Robust group testing has been discussed in Hwang (1984). This procedure can also be applied to sequential sampling (Schilling, 1982, p. 155).

In contrast to the preceding sampling plans and schemes, Dodge and Romig (1959) has developed sampling plans to be applied to series of lots with rectification. The rectification scheme used by them is as follows: One hundred percent inspection of rejected lots is carried out and any defective items are replaced with good ones. In the accepted lots, only the defective items in the sample are replaced with good items. As such, the total inspection and the outgoing quality of the product become random variables. Dodge and Romig (1959) sampling plans are then derived by minimizing the average total inspection with respect to a fixed value of the level of quality designated by average outgoing quality limit (AOQL) or lot tolerance percent defective (LTPD). These plans can, therefore, be considered as optimum sampling plans. The theory of these sampling plans together with the tables of plans and their operating characteristic curves are given in Dodge and Romig (1959).

Dodge and Romig (1959) sampling inspection tables have been developed to control AOQL values ranging from 1% to 10%. This is because, in the past, these AOQL values were acceptable for many customers. However, with today's tighter standards, manufacturers have been asked to provide quality levels in the range of .05% to .005% and Japanese customers demand an AOQL of .001% (Cross (1984)). These percentages are translated to getting 500 parts per million to 10 parts per million (ppm). It should be noted that such tight standards are difficult to verify and every lot must be almost perfect. Cross (1984) has developed parts per million AOQL plans to deal with this situation. The acceptance number for all the plans developed by Cross is zero. A table of ppm AOQL plans for 500, 250, 100, and 50 ppm, their OC curves and AOQL curves are also given in Cross (1984).

Another type of optimum sampling plan is known as Bayesian sampling plans. The essential feature of the Bayesian scheme is that the lot quality is assumed to follow a non-degenerate distribution which is known as the prior distribution. These plans are then derived by minimizing the average cost with respect to the prior distribution of the process. Various prior distributions and cost functions that are widely used in quality control work can be found in Hald (1982) and Wetherill (1977). Hald (1982) is perhaps the best source for a detailed discussion of the theory and application of Bayesian sampling plans by attributes.

2.5 Control Charts

Control charts form an important component in statistical quality control and are widely used by practitioners in various branches in industry. The principal application of control charts is in the area of process control. Other purposes of control charts are discussed in Schilling and Nelson (1976).

As in the case of acceptance sampling plans control charts can be constructed when the quality characteristic is an attribute or a variable. A description of control charts by variables will be given in chapter 3. This section is devoted to a brief discussion of control charts by attributes.

(i) Control Charts for Proportion Defective (p-Chart)

Let p' be the process fraction defective and assume that p' is known. The fraction defective in the sample is denoted by p and is given by x/n, where x is the number of defective items in a random sample of size n. Assuming x to follow a binomial distribution with parameters n and p', it is well known that $E(p) = p'$ and $Var(p) = p'(1-p')/n$. The 3 s.d. control limits for p' can then be set up as

$$\text{lower control limit} = p' - 3\sqrt{p'(1-p')/n}$$

and

$$\text{upper control limit} = p' + 3\sqrt{p'(1-p')/n}.$$

In the situation where p' is unknown, p' is commonly estimated by $\bar{p} = [\Sigma\, x_i/n_i]/k$ where x_i is the number of defective items in the sample i, n_i is the sample size of the sample i, and k is the number of such samples. When the n_i's are all equal, the above estimate becomes $\bar{p} = \Sigma\, p_i/k$ where $p_i = x_i/n$ is the fraction defective in the sample i. The variance of $\bar{p}$ can then be estimated by $\bar{p}(1-\bar{p})/n$. Thus 3-s.d. control limits are

$$\text{lower control limit} = \bar{p} - 3\sqrt{\bar{p}(1-\bar{p})/n}$$

and

$$\text{upper control limit} = \bar{p} + 3\sqrt{\bar{p}(1-\bar{p})/n}.$$

The 2-s.d. control limits are similar.

(ii) Control Charts for Number of Defects (c-Charts)

In many cases of industrial applications the variable of interest is the number of defects per unit of product. The nature of the unit depends on the characteristic of the product. For example, if the product is oil cloth, and is inspected for blemishes, the unit may be 100 square meters. If transistor radios are inspected for number of defects the natural unit of product is the radio itself, Duncan (1986).

Let c' be the average number of defects per unit of product and c' is assumed to be known. If c is the number of defects in a random sample of a single unit, then using the Poisson distribution as an appropriate model, $E(c) = c'$ and $Var(c) = c'$. Thus when c' is given the 3 s.d. control limits for c are

$$\text{lower control limit} = c' - 3\sqrt{c'}$$

and

$$\text{upper control limit} = c' + 3\sqrt{c'}.$$

When the value of c' is not given the estimator $\bar{c} = \sum c_i/k$ of c' is commonly used where c_i is the number of defects in the sample i and k is the number of samples. The 3-s.d. control limits are then obtained by substituting $\bar{c}$ for c'.

$$\text{Lower control limit} = \bar{c} - 3\sqrt{\bar{c}}$$

and

$$\text{upper control limit} = \bar{c} + 3\sqrt{\bar{c}}.$$

The above discussion of c-charts is concerned with the number of defects in a unit of product when a sample consists of a single item. Another type of commonly used control charts by attributes closely related to c-charts is the u-chart. These charts are used to control the average number of defects per unit of product when a group of items is sampled at a time. The following notation is commonly used in constructing u-charts.

n = the number of units in the sample i,

c = the total number of defects in the sample i,

$u = c/n$ = the average number of defects per unit in the sample i

k = the number of samples of size n

Burr (1976).

Let u' be the process average number of defects per unit. u' is then estimated by $\bar{u}$ the average of u's of the k samples. The variance of $\bar{u}$ can be shown to be u'/n. Hence the 3 s.d. control limits are:

$$\text{lower control limit} = \bar{u} - 3\sqrt{\bar{u}/n}$$

and

$$\text{upper control limit} = \bar{u} + 3\sqrt{\bar{u}/n}.$$

The above discussion is a brief presentation of control charts by attributes. An excellent discussion of construction and maintenance of control charts is given in Burr (1976) and Duncan (1986).

CHAPTER THREE

ROBUSTNESS OF $\bar{X}$ AND R-CHARTS

3.1 Introduction

Since the introduction of control charts by Shewhart, they have been widely used by industry for various purposes such as (Schilling and Nelson (1976)) studies of process capability; measurement capability studies; presentation of results of designed experiments; acceptance sampling for process parameters; process control.

When the process parameters (i.e., the process mean μ and the process standard deviation σ) are not given, it is common to use $\bar{\bar{X}} = \sum_{i=1}^{k} \bar{X}_i/k$ to estimate the process average μ, where $\bar{X}_i$ is the mean of the sample i based on a sample of n units and k is the number of such samples taken from the process. The process variability is estimated using the mean of the ranges $\bar{R} = \sum_{i=1}^{k} R_i/k$, where R_i is the range of sample i (Nelson (1975)). The control limits of the $\bar{X}$ and R charts are then determined by assuming the underlying probability distribution of the quality characteristic to be normal.

Burr (1967), Schilling and Nelson (1976), Balakrishnan and Kocherlakota (1983) have examined the effects of non-normality on $\bar{X}$ and R-charts. These authors suggest from their studies, that the ordinary normal curve control limit factors can be used provided the population does not depart markedly from normality. An excellent early summary on variable acceptance sampling with emphasis on non-normality has been given by Owen(1969).

In this chapter, non-normality on the 3-standard deviation (3-s.d.) and 2-standard deviation (2-s.d.) control limits of the $\bar{X}$ and R control charts is investigated. In order to carry out an empirical investigation it is necessary to generate data from a non-normal probability distribution, and to determine the performance of the normality based control limits applied to these data. The non-normal probability distribution used should have a characteristic or characteristics which differ in some particular manner from the normal probability distribution. The performance of the normality based control limits is assessed by determining how often the limits are exceeded under varying degrees of non-normality as compared with the known frequency under normality.

A possible family of probability distributions which offers attractive non-normality characteristics is the Tukey's λ-family, and it is this family of distributions which was used to determine empirically the performance of the control limits. This family was chosen because it consists of a broad range of symmetric probability distributions one of which is the normal distribution, but which differ according to the size of the tail areas. It is felt that the control limits may well be sensitive to the magnitude of the tails of the underlying probability distributions.

3.2 Tukey's λ-family

The Tukey's λ-family of probability distributions is a large family of probability distributions which have a common property of being symmetric but differ according to the magnitude of the tail areas. Each of the distributions can be explicitly represented by

$$y = F^{-1}(p) = [p^{\lambda} - (1-p)^{\lambda}]/\lambda, \quad 0 < p < 1, \tag{3.1}$$

where the F^{-1} is the inverse function of the distribution function $p = F(y)$. By varying the shape parameter λ one can obtain a wide spectrum of symmetric distributions from very heavy tailed to thin tailed. The well-known distributions such as the uniform, normal, logistic and Cauchy distributions, up to a scale parameter, are either members of the family (uniform ($\lambda = 1$ and 2), logistic ($\lambda = 0$)) or can be closely approximated by a member of the family, Cauchy ($\lambda \approx -1$)). In particular, the normal distribution with 0 mean and standard deviation .6874 can be closely approximated by $\lambda = .14$. Another advantage of using this family is that the simulated random observations x can be easily obtained through simulating the uniform random observations p. The Tukey's λ-family has been widely used recently in robustness studies and simulation (Chan and Rhodin (1980)).

The fact that Tukey's λ-family of distributions range from thin tailed to heavy tailed, can be seen from the following Table 3.1 of "tail lengths" as measured by τ and ρ

where

$$\tau = \frac{F^{-1}(.9975)}{F^{-1}(.975)} \quad \text{and} \quad \rho = \frac{F^{-1}(.95)}{F^{-1}(.75)}$$

(Chan and Rhodin, (1980)).

TABLE 3.1

Variances and Tail Lengths of the λ-Family of Distributions

(λ)	Variance	τ	ρ
1.00 (uniform)	0.3333	1.0231	1.2667
0.30	1.3688	1.2595	2.2427
0.20	1.7804	1.3503	2.3654
Normal	2.1163	1.4322	2.4387
0.14	2.1103	1.4192	2.4494
0.10	2.3780	1.4732	2.5101
-0.10	4.1805	1.8493	2.8795
-0.20	7.4851	2.1303	3.1130
-0.30	13.3547	2.4960	3.3859

Filliben (1969) did an extensive study of this family.

3.3 The Control Limits

Assuming the underlying probability distribution to be normal, the construction of control charts ($\bar{X}$ and R-charts) have been discussed in standard literature on statistical quality control. The constants required to obtain 3-s.d. control chart limits (A_1, A_2, d_2, d_3 for $\bar{X}$-charts and D_1, D_2, D_3, D_4 for R-charts) when underlying probability distribution is normal for the two situations where the process standard deviation is known and unknown are available. The control charts construction procedure is discussed fully in Burr (1976, pp. 86-87) and a table of normal control chart constants is also given in Burr (1976, p. 486). It is, therefore, the common practice to assume that the measurements are normally distributed and construct $\bar{X}$ -charts and R-charts using these tabulated constants.

3.4 Assumption of Normality

If the underlying distribution of the quality measurements is normal the probabilities that the sample mean falling outside 3-s.d. and 2-s.d. control limits for $\bar{X}$-chart are 0.0027 and 0.0456, respectively. The corresponding probabilities for the range from a normal distribution are given in Table 3.3. For any other underlying distribution these probabilities could be more or less, depending on the lengths of the tails of the distribution. Hence, the question arises as to what extent the non-normality affects the $\bar{X}$- and R-charts.

3.5 Control Limit Factors for the λ-family of Distributions

In order to compute the control limit factors for Tukey's λ-family we need the expectation and the standard deviation of the range. These can be obtained from the moments of order statistics $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$. For the λ-family the k-th moment about 0 of the r-th order statistic of a sample of size n is (Joiner and Rosenblatt (1971))

$$\mu_r^k = r\cdot\binom{n}{r}\left[\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,\beta\{(k-j)\lambda+r,\ j\lambda+n-r+1\}\right]/\lambda^k, \qquad (3.2)$$

where β is the beta-function. In particular, the expectation of the range $R = X_{(n)}-X_{(1)}$ is

$$E(R) = 2n[1-\lambda\beta(n,\lambda)]/\{\lambda(n+\lambda)\}. \qquad (3.3)$$

The product moment of the smallest and largest order statistics is:

$$\begin{aligned} E(X_{(1)}X_{(n)}) = {} & n(n-1)[2\beta(\lambda+1,n-1)/(2\lambda+n) \\ & + \{(-1)^n-1\}\beta(\lambda+n,\lambda+1)\beta(\lambda+1,n-1) \\ & - \sum_{j=0}^{n-2}\binom{n-2}{j}(-1)^j\{\beta(\lambda+1,j+1)/(n+\lambda-j-1)\}]/\lambda^2. \end{aligned} \qquad (3.4)$$

The variance of the λ-distribution is

$$\sigma^2 = 2[1-\lambda\beta(\lambda,\lambda)/2]/\{\lambda^2(2\lambda+1)\}. \qquad (3.5)$$

We can then use (3.2) to (3.5) to obtain the expectation and the standard deviation of the range, and compute the control limit constants d_2, d_3, D_1, D_2, D_3 and D_4 for the λ-family of distributions.

3.6 Comparison of the Probabilities Outside the Control Limits and Control Limit Constants Between Normal and λ-family of Distributions

As has been noted, the purpose of this study is to investigate the effect of non-normality on $\bar{X}$- and R-charts. This will be examined by looking at

(1) The probability that the sample mean $\bar{X}$ from the λ-distribution lies outside 3-s.d. and 2-s.d. control limits,

(2) The probability that the range from the λ-distribution lies outside 3-s.d. and 2-s.d. control limits, and

(3) The comparison of the normality constants with the control limit constants A_2, d_2, d_3, D_1, D_2, D_3 and D_4 for the λ-family of distributions.

We generated 10,000 random samples of sizes n = 2, 5, 8, 10 and 15 for each shape parameter λ = 1.00, 0.30, 0.20, 0.14, 0.10, -0.10, -0.20 and -0.30. The APL random number generator was used to generate the random samples. The selection of these λ-values was done so as to include distributions with thin to thick tails (see Table 3.1).

In order to check the accuracy of the simulation, the simulated probabilities that $|\bar{X}-\mu| > 3\sigma/\sqrt{n}$ and $> 2\sigma/\sqrt{n}$ reported for $\lambda = 1$ (i.e., the uniform distribution U(0,a) were compared with the theoretical probabilities, which can be obtained from the distribution function of the sample total

$$F(t) = \frac{1}{1.2\ \ldots .n}\,[(t/a)^n - (n/1)(t/a-1)^n$$
$$+(n(n-1))(t/a-2)^n/(1.2) - \ldots]$$

(Uspensky (1965)). We note from the following theoretical and simulated probabilities that they are quite close.

Sample size(n)	Theoretical $P(\lvert\bar{X}-\mu\rvert \geq \frac{3\sigma}{\sqrt{n}})$	Simulated $P(\lvert\bar{X}-\mu\rvert \geq \frac{3\sigma}{\sqrt{n}})$	Theoretical $P(\lvert\bar{X}-\mu\rvert \geq \frac{2\sigma}{\sqrt{n}})$	Simulated $P(\lvert\bar{X}-\mu\rvert \geq \frac{2\sigma}{\sqrt{n}})$
2	0.0000	0.0000	0.0336	0.0331
5	0.0009	0.0009	0.0430	0.0429
8	0.0016	0.0016	0.0440	0.0432
10	0.0019	0.0019	0.0445	0.0448
15	0.0021	0.0020	0.0447	0.0448

(1) $\bar{X}$-Charts

The probabilities that $\bar{x}$ is outside 3-s.d. and 2-s.d. limits using simulations for various n and λ combinations are given in Table 3.2. Examination of Table 3.2 for the 3-s.d. probabilities leads to the following observations:

(i) For $\lambda = 1$ (i.e., the uniform distribution) the probabilities are much less than 0.0027. Schilling and Nelson (1976) have concluded that the samples of size 25 or more are required for the sample mean from a uniform distribution to approach the normal distribution. Our results also support their finding.

(ii) For $\lambda = 0.30$, 0.20, 0.14 and 0.10, the results are encouraging. For these distributions the majority of the probabilities are close to 0.0027, except for distributions with $\lambda = 0.30$, 0.20 and n = 2 and 5. This suggests that the normal approximation for the sample mean could be used for these distributions even with small samples. This is because these distributions are not thick tailed.

(iii) We note that for distributions with shape parameters λ = -0.10, -0.20 and -0.30, the 3-s.d. tail probabilities are considerably larger than 0.0027. Therefore, these distributions may require larger samples (usually greater than 15) before the normal approximation may be used with acceptable accuracy.

(iv) For a fixed n, it is noted that the probability tends to increase as λ decreases.

Interesting results are found in Table 3.2 for the 2-s.d. probabilities. Almost all the probabilities for all combinations of λ and n are close to 0.0456 which is the probability that the sample mean from a normal distribution falls outside 2-s.d. limits. However, this is not the case for the uniform distribution (i.e.,λ = 1 and n = 2, 5 and 8) and λ = 0.30 and n = 2 and 5. This finding indicates that we can apply the normal approximation with small samples with reasonable accuracy to construct 2-s.d. control limits for $\bar{X}$-charts even if departure from normality is considerable.

(2) R-Charts

The probabilities of the range lying outside 3-s.d. and 2-s.d. control limits for Tukey's λ-distributions considered in this study are given in Table 3.3. The accuracy of the simulation results is checked through comparing the theoretical probabilities with the simulated probabilities for the uniform distribution (i.e., $\lambda = 1$).

Comparison of the probabilities in Table 3.3 with the theoretical values for the normal distribution when 3-s.d. is used as limit reveals that the effect is not sizeable for small departures from normality. For instance, for $\lambda = 0.10$ and $n = 5$ the probability that the range lying outside 3-s.d. limit is 0.0055 while for the normal distribution the same probability is 0.0058. This finding is generally true for small departures from normal distribution. However, the same conclusion does not hold for considerable deviations from normality.

The probabilities given in Table 3.3 when 2-s.d. limit is used are very close to those for the normal distribution for all the combinations of n and except for $\lambda = 1$ and $\lambda = -0.30$ This is because for $\lambda = 1$, the distribution is very thin tailed and for $\lambda = -0.30$ the distribution becomes heavy tailed. This finding indicates that non-normality has no severe effects even if the magnitude of the departure is moderate, if one wishes to use 2-s.d. control charts for the range.

(3) Control Limit Constants

The control limit constants A_2, d_2, d_3, D_1, D_2, D_3 and D_4 are given in Tables 3.4 to 3.7 respectively for several λ-distributions and for the normal distribution.

Comparison of these constants with those for normal distribution indicates that if the departure from normality is considerable, the use of normal distribution control limit constants may not be appropriate. This is particularly true for heavy tailed distributions such as the distribution with $\lambda = -0.30$ and very thin tailed distributions such as the distribution with $\lambda = 1.00$, i.e., the uniform distribution.

3.7 Numerical Example

Davis (1952) analyzed 417 observations on lifetimes of 110 - volt internally frosted incandescent lamps, taken from 42 weekly quality control forced - life test samples. A Chi-square goodness of fit test for normality was performed by Davis and concluded that normal distribution fits the data adequately. The analysis on kurtosis suggests that the data are more peaked at the mean and flatter in the tails. A subsequent analysis of Davis' data using a Chi-square goodness of fit of test for various members belonging to the λ-family (λ = -.2, -.1, 0, .1, .14, .2, .25) was carried out by Chan and Rhodin (1980). The Chi-square values and the kurtosis for the above distributions are given on Table 7 (p.234) of their paper. Examination of this table indicates that, although Davis' data can be fitted to normal distribution adequately, the λ-distribution with shape parameter -.1 is an even better fit.

In this chapter, Davis' data are reanalyzed using probability plots for the λ-distribution. The smallest 5 and the largest 5 observations in the data set can be considered as outliers and hence are excluded. One of the practices to obtain probability plots for any distribution is to plot the ordered values obtained from a random sample of size n against the expected values of standardized ordered observations. The values $F^{-1}\{(i-c)/(n-2c+1)\}$, $i = 1,2,\ldots,n$ is frequently used as an approximation to the expected value of the i-th order statistic $Y_{(i)}$ from the standardized distribution for which $E(Y_{(i)})$ cannot be given explicitly. Also several values of c are proposed in the literature, but $c = 1/2$ is found to be acceptable to a wide variety of probability distributions and sample sizes (Hahn and Shapiro (1967) 292-294). For the λ-distribution $E(Y_{(i)})$ can be obtained directly from the definition given in (3.1) and is given by

$$E(Y_{(i)}) = [\{(i-.5)/n\}^{\lambda} - \{(n+.5-i)/n\}^{\lambda}]/\lambda.$$

The probability plots for the λ-distribution are then obtained by plotting $X_{(i)}$ against $E(Y_{(i)})$. For Davis' data the plots for the distribution with $\lambda = -.1$ and the normal distribution are given in Figures 3.1 and 3.2, respectively. It can be seen from the plots that both the normal and $\lambda = -.1$ distributions fit the data quite well, although the $\lambda = -.1$ distribution seems to be slightly better. However, analysis of the data by Chan and Rhodin (1982) indicated that the observations are more peaked at the mean (which could not be easily shown through a probability plot) and flatter in the tails, and hence $\lambda = -.1$ would be a more appropriate distribution.

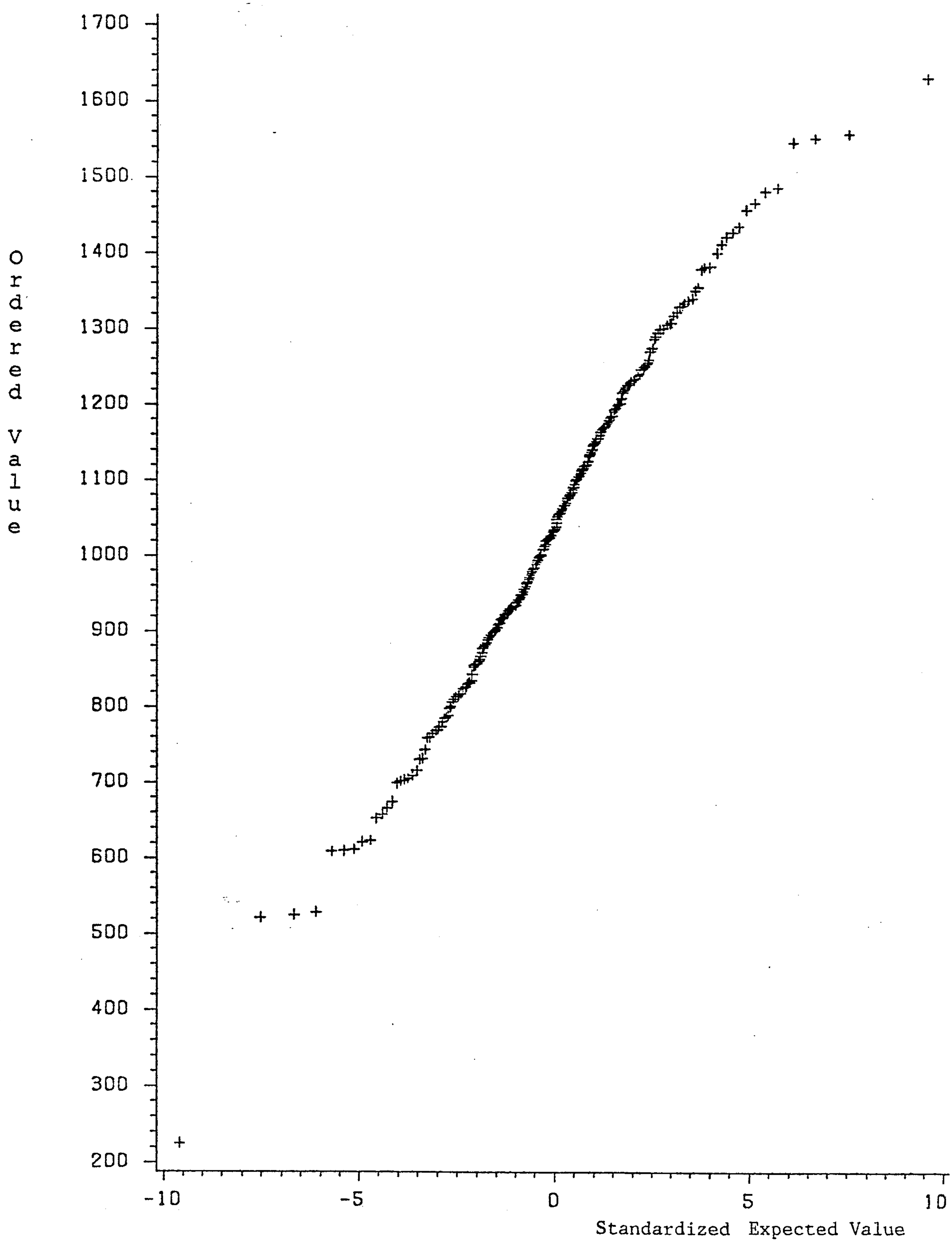


Figure 3.1. Probabilty Plot (λ=-.1) for Davis' Data

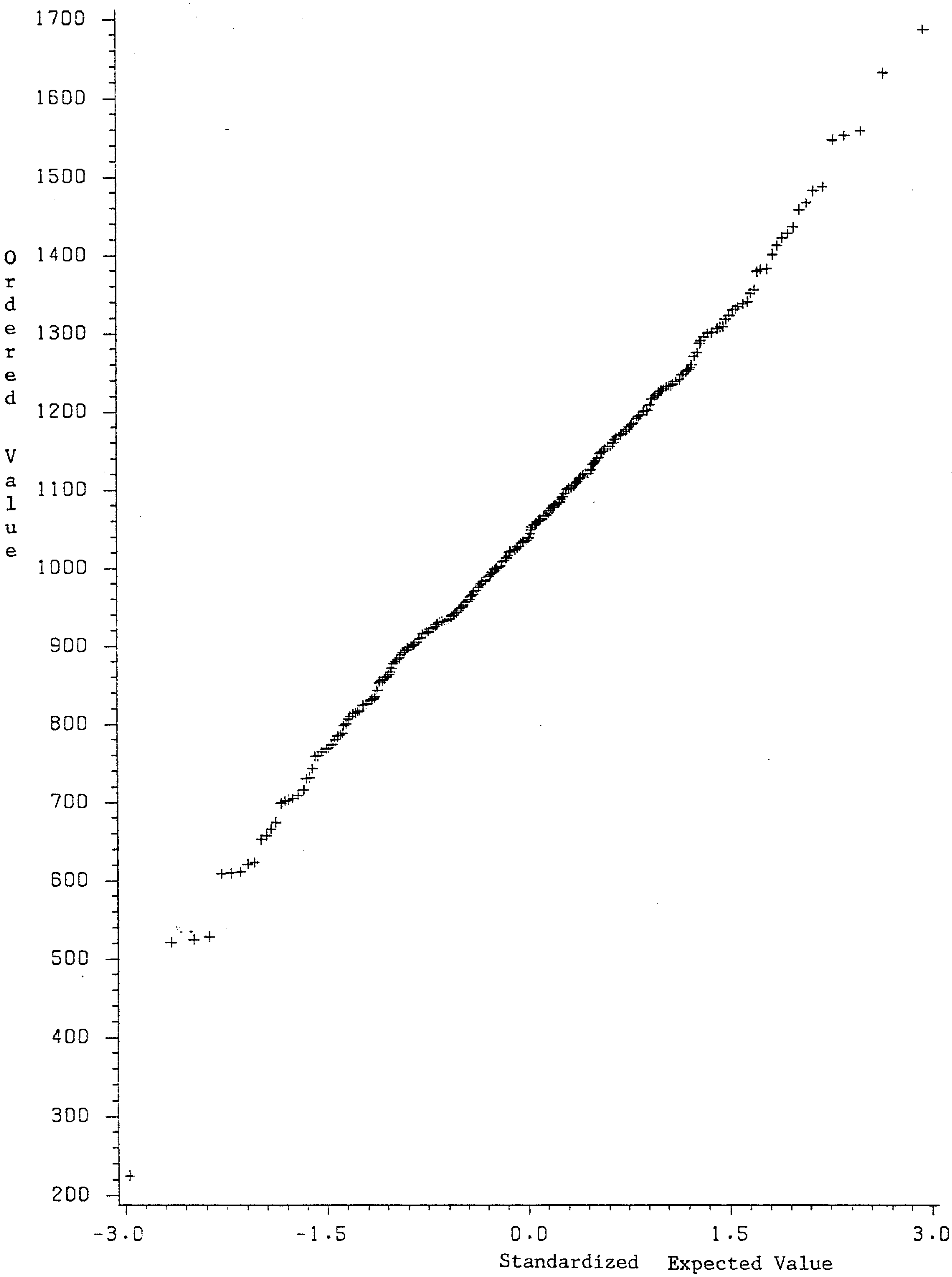


Figure 3.2 Normal Probability Plot for Davis' Data

Thus, in this situation, the use of λ-distribution with $\lambda = -.1$ distribution to construct $\bar{X}$ and R-charts is more appropriate than that of the normal distribution. We can, therefore, apply the constants A_2, D_3 and D_4 in Tables 3.3 and 3.7 for $\lambda = -.1$ and construct $\bar{X}$ and R-charts as follows.

Suppose that five samples of size n = 5 are selected randomly from Davis' data set to construct the control charts. The sample mean and the range for the samples are given in the following table. The grand mean is $\bar{\bar{X}} = 980.88$ and $\bar{R} = 515.4$.

Sample	Observations					$\bar{X}_i$	R_i
1	1226	1181	709	732	905	905.6	517
2	1195	910	705	1055	1303	1033.6	490
3	1022	1002	775	1438	984	1044.2	663
4	824	1255	909	1086	1160	1046.8	431
5	1078	658	732	769	1134	874.2	476

(i) <u>$\bar{X}$-Chart:</u>

We have $\bar{\bar{X}} = 980.88$ and $\bar{R} = 515.4$. From Table 3.4 for $n = 5$ and $\lambda = -.1$, $A_2 = .597$

Hence, lower control limit = 980.88 - .597(515.4) = 673.19

and

upper control limit = 980.88 + .597(515.4) = 1288.57.

For the normal distribution, A_2 = .577 and a similar computation gives lower control limit = 683.21 and upper control limit = 1278.27.

(ii) R-Charts:

The lower and upper control limits for R-Charts are $D_3\bar{R}$ and $D_4\bar{R}$. For the distribution with $\lambda = -.1$, n=5, we have from Table 3.7 that $D_3 = 0.000$ and $D_4 = 2.478$. Thus, the two limits are:

lower control limit = 0.000(515.4) = 0.00

and

upper control limit = 2.478(515.4) = 1277.16.

For the normal distribution, the lower control limit = 0.00 and the upper control limit = 1090.07.

We note in this case that the use of λ-distribution with $\lambda = -.1$ would yield wider limits than the ones obtained using the normal distribution.

3.8 Conclusions

The following conclusions can be made from the above discussion on $\bar{X}$ and R charts as well as the constants necessary for $\bar{X}$ and R Charts.

(1) Moderate sample sizes (generally less than 15) may be sufficient for constructing $\bar{X}$-charts for distributions with $\lambda \geq 0.10$. However, when the shape parameter λ is negative, 3-s.d. tail probabilities are much greater than

0.0027 for all sample sizes considered in this study. Thus, for those heavy tailed distributions control charts based on the assumption of normality may not be appropriate. If one wishes to construct 3-s.d. control limit constants, it is advisable to use larger samples, preferably greater than 15. This finding is in agreement with both Burr (1964) and Schilling and Nelson (1976).

(2) The same conclusion is reached when the control limit factors from λ-family are compared with those for normal distribution. It is found that for thick tailed distributions (i.e., when λ is negative) the control limit constants are not close to normal distribution constants. This suggests that degree of non-normality should be taken into consideration when constructing $\bar{X}$ and R-charts.

(3) We reach similar conclusions by comparison of the probabilities of the distribution of the ranges in Table III. That is, for the distributions that are close to the normal distribution, the tabulated constants may be used with small samples, but for those distributions with negative λ, constants appropriate to the distribution should be used; with fairly large samples normal distribution constants may be used.

(4) We also observe that sample size and shapes of distributions have little influence on the probabilities when 2-s.d. limit is used. This provides a justification for using normal control limits in constructing 2-s.d. control limits for a wide range of distributions belonging to Tukey's λ-family.

From this study, we note that if a marked departure from normality is observed, there could be a serious effect on the performance of control limit constants derived from the normality assumption. It may well be that, given a sufficient amount of data, an investigation of the form of the underlying distribution prior to the construction of control charts would be warranted. One might consider the use of probability plots developed in this chapter, or a goodness of fit test such as the Kolmogorov-Smirnov test, to attempt to identify the process distribution. Should there be evidence suggesting that the process distribution is a particular member of the Tukey's λ-family the use of Tables 3.4 to 3.7 would provide control limit constants appropriate for that member of the family.

TABLE 3.2

Probabilities That the Sample Mean Lies Outside

Control Limits for the Tukey's λ-family

λ/n	2	5	8	10	15
	3-Standard Deviation Limit				
1.00	.0000	.0009	.0016	.0019	.0020
0.30	.0007	.0019	.0021	.0023	.0023
0.20	.0017	.0024	.0026	.0025	.0026
Normal	.0027	.0027	.0027	.0027	.0027
0.14	.0024	.0026	.0025	.0027	.0027
0.10	.0034	.0029	.0027	.0028	.0028
-0.10	.0098	.0069	.0053	.0049	.0041
-0.20	.0130	.0091	.0083	.0075	.0066
-0.30	.0136	.0122	.0109	.0107	.0092
	2-Standard Deviation Limit				
1.00	.0331	.0429	.0432	.0448	.0448
0.30	.0403	.0438	.0452	.0455	.0436
0.20	.0455	.0444	.0460	.0462	.0463
Normal	.0456	.0456	.0456	.0456	.0456
0.14	.0456	.0466	.0445	.0445	.0451
0.10	.0472	.0469	.0460	.0447	.0453
-0.10	.0506	.0487	.0484	.0486	.0465
-0.20	.0492	.0487	.0486	.0475	.0479
-0.30	.0425	.0450	.0443	.0455	.0439

TABLE 3.3

Probabilities That The Range Lies Outside Control Limits for the Tukey's λ-family

λ\n	2	5	8	10	15
			3-Standard Deviation Limit		
1.00	.0000	.0013	.0063	.0082	.0107
(Theoret.)	.0000	.0013	.0064	.0082	.0105
0.30	.0061	.0009	.0007	.0011	.0014
0.20	.0077	.0032	.0019	.0021	.0024
Normal	.0092	.0058	.0043	.0044	.0045
0.14	.0090	.0040	.0040	.0038	.0039
0.10	.0099	.0055	.0049	.0045	.0049
-0.10	.0154	.0129	.0129	.0117	.0132
-0.20	.0167	.0156	.0153	.0142	.0149
-0.30	.0153	.0155	.0152	.0148	.0148
			2-Standard Deviation Limit		
1.00	.0381	.0348	.0419	.0436	.0449
(Theoret.)	.0000	.0348	.0421	.0437	.0452
0.30	.0432	.0397	.0381	.0436	.0431
0.20	.0449	.0419	.0417	.0428	.0449
Normal	.0464	.0405	.0329	.0433	.0441
0.14	.0446	.0399	.0424	.0436	.0434
0.10	.0456	.0409	.0422	.0430	.0441
-0.10	.0457	.0411	.0410	.0411	.0421
-0.20	.0425	.0413	.0397	.0394	.0398
-0.30	.0355	.0358	.0348	.0346	.0352

TABLE 3.4

Values of d_2 and $A_2 = 3/(\sqrt{n}d_2)$

$\lambda \backslash n$	2	5	8	10	15
		Values of d_2			
1.00	1.155	2.309	2.694	2.834	3.031
0.30	1.143	2.335	2.816	3.017	3.346
0.20	1.136	2.332	2.838	3.056	3.424
Normal	1.128	2.326	2.847	3.078	3.472
0.14	1.129	2.327	2.849	3.079	3.473
0.10	1.123	2.321	2.854	3.093	3.506
-0.10	1.069	2.249	2.838	3.121	3.645
-0.20	1.015	2.157	2.765	3.068	3.648
-0.30	0.919	1.977	2.581	2.894	3.512
-0.40	0.733	1.598	2.129	2.414	2.997
		Values of A_2			
1.00	1.837	0.581	0.394	0.335	0.256
0.30	1.856	0.575	0.377	0.314	0.231
0.20	1.867	0.575	0.374	0.310	0.226
Normal	1.881	0.577	0.373	0.308	0.223
0.14	1.878	0.577	0.372	0.308	0.223
0.10	1.888	0.578	0.372	0.307	0.221
-0.10	1.984	0.597	0.374	0.304	0.213
-0.20	2.090	0.622	0.384	0.309	0.212
-0.30	2.308	0.679	0.411	0.328	0.221
-0.40	2.894	0.834	0.498	0.393	0.258

TABLE 3.5

Values of d_3

$\lambda \backslash n$	2	5	8	10	15
1.00	0.817	0.617	0.455	0.386	0.278
0.30	0.832	0.769	0.679	0.637	0.562
0.20	0.843	0.823	0.759	0.726	0.668
Normal	0.853	0.864	0.820	0.797	0.755
0.14	0.852	0.863	0.817	0.792	0.748
0.10	0.859	0.893	0.861	0.843	0.809
-0.10	0.925	1.108	1.172	1.201	1.255
-0.20	0.984	1.271	1.409	1.477	1.608
-0.30	1.074	1.493	1.732	1.857	2.104
-0.40	1.209	1.800	2.182	2.389	2.816

TABLE 3.6

Values of $D_1 = d_2 - 3d_3$ and $D_2 = d_2 + 3d_3$

λ\n	5	8	10	15
		D_1 Values		
1.00	0.458	1.329	1.676	2.197
0.30	0.028	0.779	1.106	1.667
0.20	0.000	0.561	0.878	1.420
Normal	0.000	0.387	0.687	1.207
0.10	0.000	0.271	0.564	1.079
-0.10	0.000	0.000	0.000	0.000
-0.20	0.000	0.000	0.000	0.000
-0.30	0.000	0.000	0.000	0.000
-0.40	0.000	0.000	0.000	0.000
		D_2 Values		
1.00	4.160	4.059	3.992	3.865
0.30	4.642	4.853	4.928	5.032
0.20	4.801	5.115	5.234	5.428
Normal	4.912	5.307	5.469	5.737
0.10	5.000	5.437	5.622	5.933
-0.10	5.573	6.354	6.724	7.410
-0.20	5.970	6.992	7.499	8.472
-0.30	6.456	7.777	8.465	9.824
-0.40	6.998	8.675	9.581	11.445

TABLE 3.7

Values of $D_3 = 1 - 3d_3/d_2$ and $D_4 = 1 + 3d_3/d_2$

$\lambda \backslash n$	5	8	10	15
		D_3 Values		
1.00	0.198	0.493	0.592	0.725
0.30	0.021	0.277	0.367	0.496
0.20	0.000	0.198	0.287	0.414
Normal	0.000	0.136	0.223	0.348
0.10	0.000	0.095	0.182	0.308
-0.10	0.000	0.000	0.000	0.000
-0.20	0.000	0.000	0.000	0.000
-0.30	0.000	0.000	0.000	0.000
-0.40	0.000	0.000	0.000	0.000
		D_4 Values		
1.00	1.802	1.507	1.409	1.275
0.30	1.988	1.723	1.633	1.504
0.20	2.059	1.802	1.713	1.585
Normal	2.115	1.864	1.777	1.652
0.10	2.154	1.905	1.818	1.692
-0.10	2.478	2.239	2.155	2.033
-0.20	2.768	2.529	2.444	2.322
-0.30	3.266	3.013	2.925	2.797
-0.40	4.379	4.075	3.969	3.819

CHAPTER FOUR

SAMPLING PLANS AND CONTROL CHARTS FOR THE TWO PARAMETER EXPONENTIAL DISTRIBUTION

4.1 Introduction

Until fairly recently the normal distribution has been widely used in the construction of acceptance sampling by variables. Based on the assumption that the quality measurements are normally distributed, the acceptance sampling plans are given in standard texts such as Duncan (1986), Burr (1976), and Schilling (1982), and Owen (1964,1966). One and two-sided sampling plans based on Weibull distribution are given in Honso, Ohta and Kase (1981). Kocherlakota and Balakrishnan (1983a) used the modified maximum likelihood estimators to construct sampling plans for mixtures of normal distributions. An excellent summary of problems associated with non-normality in acceptance sampling plans is given by Owen (1969). He has proposed a simple modification to sampling plans if the underlying probability distribution is non-normal.

4.2 The Two Parameter Exponential Distribution

4.2.1 Description of the Exponential Distribution

The quality of a product is closely related to its life time. If a product can be used for a longer period of time without breakdowns and repairs, it is considered to be of superior quality. Thus, in this situation, quality characteristic of a product is its "life time". Among several probability distributions, that have been considered for life testing and reliability studies, the exponential distribution plays an important role. The probability density function of this distribution is given by

$$f(x;\mu,\sigma) = 1/\sigma \exp-\{(x-\mu)/\sigma\}, \; x\geq\mu, \; \mu,\sigma>0 \qquad (4.1)$$

where X is the random variable representing the quality characteristic and μ is the location parameter , σ is the scale parameter with $\mu+\sigma$ being the mean life of the distribution. It is common practice for a manufacturer to guarantee that a product (e. g., television) would perform for a specified period of time without any breakdowns and this time period is known as the guaranteed time. For the two parameter exponential distribution μ is, therefore, called the guaranteed time.

One and two-sided sampling plans based on this distribution have been constructed by Kocherlakota and Balakrishnan (1983b) using the maximum likelihood estimators of μ and σ. Their sampling plans have been derived by using one point on the operating characteristic (OC) curve, namely fixing the acceptable quality level, p_1, and the producer's risk, α. There is one practical limitation to their sampling plans; that is, although the producer is protected against rejecting lots having quality p_1 or better, the consumer is not given any assurance regarding the quality that is received.

In this chapter, sampling plans to accommodate two points on the OC curve based on the exponential distribution are presented. The two points are the acceptable quality level, p_1, and the lot tolerance percent defective, p_2. The corresponding producer's and consumer's risks are given by α and β. In addition, sampling plans based on minimum variance unbiased estimators (MVUE) of μ and σ are also provided. It should be noted that MVUE's are also the best linear unbiased estimators (BLUE) of μ and σ. Sampling plans to give assurance about the parameters (μ and σ) of the distribution given in (4.1) are also considered. Finally control charts for μ and σ based on the maximum likelihood estimators of μ and σ are derived in this chapter. Several numerical examples are presented to illustrate the applications of the sampling plans and the control charts.

4.2.2 Parameter Estimation

The maximum likelihood estimators of μ and σ are

$$\hat{\mu} = X_{(1)},$$

$$\hat{\sigma} = \sum_{i=1}^{n}(X_i-\mu)/n \qquad \text{(when } \mu \text{ is known)}$$

and

$$\hat{\sigma} = \sum_{i=2}^{n}\{(n-r+1)(X_{(i)}-X_{(i-1)})\}/n$$

$$= \bar{X}-X_{(1)} \qquad \text{(when } \mu \text{ is unknown)}$$

where $X_{(1)}, X_{(2)},\ldots,X_{(n)}$ is the set of order statistics of the random sample $X_1, X_2,\ldots,X_n$. It is well known that $X_{(1)}$ is statistically independent of $\hat{\sigma}$ and

$$2n\{(X_{(1)}-\mu)\}/\sigma \sim \chi^2_2 \quad \text{and} \quad 2n\hat{\sigma}/\sigma \sim \chi^2_{2n-2}.$$

4.3 One Sided Sampling Plans Based on ML Estimators

For convenience the notation given in Kocherlakota and Balakrishnan (1983b) shall be used in this chapter. Let K_p and K_{1-p} be the lower and upper p-percentage points in the exponential distribution given in (4.1). This leads to

$$K_p = (L-\mu)/\sigma \text{ and } K_{1-p} = (U-\mu)/\sigma$$

where L and U are the lower and upper specification limits such that $P(X<L) = p$ and $P(X>U) = p$. It can be shown that $K_p = -\ln(1-p)$ and $K_{1-p} = -\ln p$. The following separate cases will be considered in deriving the sampling plans depending whether the parameters μ and σ are known or unknown.

4.3.1 μ unknown and σ known

It is required to find n and k such that

$$P(\text{accept the lot}\,|\,p_1) = 1-\alpha \tag{4.2}$$

and

$$P(\text{accept the lot}\,|\,p_2) = \beta \tag{4.3}$$

where n is the necessary sample size and k is the acceptability constant. The acceptance rule in this case is accept the lot if $\hat{\mu} + k\sigma \geq L$ or $\hat{\mu} + k\sigma \leq U$. Using (4.2) and (4.3) and lower specification limit one can obtain

$$P\{2n(\hat{\mu}-\mu)/\sigma < 2n(K_{p_1}-k)\} = \alpha$$

and

$$P\{2n(\hat{\mu}-\mu)/\sigma < 2n(K_{p_2}-k)\} = 1 - \beta.$$

Now using the distributional properties given in the preceding section we get

$$n(K_{p_1} - k) = -\ln(1-\alpha) \qquad (4.4)$$

and

$$n(K_{p_2} - k) = -\ln\beta. \qquad (4.5)$$

Solving (4.4) and (4.5) for n we get

$$n = \ln\{(1-\alpha)/\beta\}/(K_{p_2} - K_{p_1}).$$

Substituting $K_{p_1} = -\ln(1-p_1)$ and $K_{p_2} = -\ln\ (1-p_2)$ gives

$$n = \ln\{(1-\alpha)/\beta\}/\ln\ \{(1-p_1)/(1-p_2)\}. \qquad (4.6)$$

Substituting n, given in (4.6), into (4.5) and simplifying we obtain

$$k = [\frac{\ln\{(1-p_1)/(1-p_2)\}}{\ln\{(1-\alpha)/\beta\}}]\ln(1-\alpha) - \ln(1-p_1). \qquad (4.7)$$

Similarly using equation (4.5)

$$k = [\frac{\ln\{(1-p_1)/(1-p_2)\}}{\ln\{(1-\alpha)/\beta\}}]\ln(\beta) - \ln(1-p_2). \qquad (4.8)$$

Because we have two separate values for k, the usual practice, by quality control practitioners, is to take the average of the two values obtained from (4.7) and (4.8).

For the upper specification limit, using the same procedure the following equations are obtained.

$$-n(K_{1-p_1} - k) = \ln\alpha \qquad (4.9)$$

and

$$-n(K_{1-p_2} - k) = \ln(1-\beta). \qquad (4.10)$$

Solving (4.9) and (4.10) for n yields

$$n = \ln\{(1-\beta)/\alpha\}/\ln(p_1/p_2).$$

The possible values of k are

$$k = \left[\frac{\ln(p_2-p_1)}{\ln\{(1-\beta)/\alpha\}}\right]\ln\alpha - \ln p_1 \tag{4.11}$$

and

$$k = \left[\frac{\ln(p_2/p_1)}{\ln\{(1-\beta)/\alpha\}}\right]\ln(1-\beta) - \ln p_2. \tag{4.12}$$

Once again the average of the values obtained from (4.11) and (4.12) is used.

4.3.2 <u>μ known and σ unknown</u>

In this case, the batch is accepted if $\mu + k\hat{\sigma} \geq L$ or $\mu + k\hat{\sigma} \leq U$. The corresponding probability statements for the lower specification limit are:

$$P\{(\hat{\sigma}/\sigma)<(K_{p_1}/k)\}=\alpha$$

and

$$P\{(\hat{\sigma}/\sigma)<(K_{p_2}/k)\}=1-\beta.$$

It is well known that $\hat{\sigma}/\sigma$ has a gamma distribution with parameters n and n and the proof of this theorem is given in Sinha and Kale (1980). Thus, the following equations are obtained using the above probability restrictions. For p_1 and α

$$K_{p_1}/k = \Gamma_{n,n;\alpha} \quad \text{or}$$

$$-\{\ln(1-p_1)\}/k = \Gamma_{n,n;\alpha} \tag{4.13}$$

and for p_2 and β we get

$$-\{\ln(1-p_2)\}/k = \Gamma_{n,n;1-\beta} \tag{4.14}$$

where $\Gamma_{n,n;\alpha}$ and $\Gamma_{n,n;1-\beta}$ are the lower α and $1-\beta$ percent points of the gamma distribution with parameters n and n, respectively. From (4.13) and (4.14) n can be found as the solution of the equation

$$\frac{\ln(1-p_1)}{\ln(1-p_2)} = \frac{\Gamma_{n,n;\alpha}}{\Gamma_{n,n;1-\beta}} . \tag{4.15}$$

Equation (4.15) can be solved iteratively for n using tables of the incomplete gamma function. Once n is obtained from (4.15), k may be found either from (4.13) or (4.14). As before we take the average of the two values of k.

The ratio $\Gamma_{n,n;\alpha}/\Gamma_{n,n;1-\beta}$ is tabulated for various combinations of α and β for n ranging from 2 to 100. The tabulated results are given in Table I in the Appendix. One can then use this table to obtain the required sample size for specified values of α, β, p_1 and p_2 as follows. The ratio $\ln(1-p_1)/\ln(1-p_2)$ is first computed. Choose n from Table I which gives the value of this ratio closest to $\Gamma_{n,n;\alpha}$ / $\Gamma_{n,n;1-\beta}$ for the specified values of α and β. For example, let p_1 =.032, p_2=.05, α=.05, and β=.10, then $\ln(1-p_1)/\ln(1-p_2)=\ln(.968)/\ln(.95)$ =.634. The ratio $\Gamma_{n,n;.05}/\Gamma_{n,n;.90}$ which is closest to .634 is obtained when n=42. Hence the necessary sample size is n = 42. An example for this situation can be given by considering the situation where the guaranteed time of a product (e.g., television) is specified by the manufacturer. Thus, in this case, μ is known but σ is unknown.

4.3.3 μ and σ both unknown

In this case, the rule is to accept if $\hat{\mu} + k\hat{\sigma} \geq L$ or $\hat{\mu} + k\hat{\sigma} \leq U$. For lower specification limit and (p_1,α) we have

$$P\{2n(\hat{\mu}-\mu)/\sigma + 2nk\hat{\sigma}/\sigma \geq 2nK_{p_1})\} = 1-\alpha$$

and for (p_2,β)

$$P\{2n(\hat{\mu}-\mu)/\sigma + 2nk\hat{\sigma}/\sigma \geq 2nK_{p_2})\} = \beta.$$

Using the distributional properties of $\hat{\mu}$ and $\hat{\sigma}$ these probability statements can be expressed as

$$P(Y_1 + kY_2 < 2nK_{p_1}) = \alpha \qquad (4.16)$$

and

$$P(Y_1 + kY_2 < 2nK_{p_2}) = 1-\beta \qquad (4.17)$$

where $Y_1 \sim \chi^2_2$ and $Y_2 \sim \chi^2_{2n-2}$.

From equation (4.16) we obtain

$$\alpha = \int_0^{2nK_{p_1}/k} f(y_2)[1 - \exp 1/2(2nK_{p_1}-ky_2)]dy_2.$$

Upon integrating and simplifying, the above expression yields

$$\alpha = P(\chi^2_{2n-2} \leq 2nK_{p_1}/k) - [\{\exp(-nK_{p_1})\}/(1-k)^{n-1}]$$
$$.[P\{\chi^2_{2n-2} \leq 2nK_{p_1}(1-k)/k\}] \qquad (4.18)$$

Similarly using expression (4.17) the corresponding equation for (p_2,β) is

$$1-\beta = P(\chi^2_{2n-2} \leq 2nK_{p_2}/k) - [\{\exp(-nK_{p_2})\}/(1-k)^{n-1}]$$
$$.[P\{\chi^2_{2n-2} \leq 2nK_{p_2}(1-k)/k\}]. \qquad (4.19)$$

Equations (4.18) and (4.19) may be solved iteratively to obtain solutions for n and k. For the upper specification limit the two probability statements are

$$P(Y_1 + kY_2 \le 2nK_{1-p_1}) = 1 - \alpha$$

and

$$P(Y_1 + kY_2 \le 2nK_{1-p_2}) = \beta.$$

The following equations involving n and k are then obtained

$$1-\alpha=P(\chi^2_{2n-2}\le 2nK_{1-p_1}/k) - [\{\exp(-nK_{1-p_1})\}/(1-k)^{n-1}] \cdot[P\{\chi^2_{2n-2} \le 2nK_{1-p_1}(1-k)/k\}] \qquad (4.20)$$

$$\beta = P(\chi^2_{2n-2} \le 2nK_{1-p_2}/k) - [\{\exp(-nK_{1-p_2})\}/(1-k)^{n-1}] \cdot[P\{\chi^2_{2n-2} \le 2nK_{1-p_2}(1-k)/k\}]. \qquad (4.21)$$

4.4 Two Sided Sampling Plans

The following definitions are required in this section. For producer's risk, α, let $p_1' = P(X < L)$ and $p_2' = P(X>U)$ and $p_1 = p_1' + p_2'$ where p_1 is the acceptable quality level. Similarly for consumer's risk, β, $p_2 = p_1'' + p_2''$ where p_2 is the LTPD and p_1'' and p_2'' are defined appropriately.

4.4.1 μ.unknown and σ known

The rule is to accept a batch if $L \le \hat{\mu} + k\sigma \le U$. For (p_1,α) the probability statement becomes

$$P\{2n(K_{p_1'}-k) \le Y \le 2n(K_{1-p_2'}-k)\}=1-\alpha$$

where $Y\sim\chi^2_2$. Integrating the above expression yields

$$\exp\{-n(K_{p_1'}-k)\} - \exp\{-n(K_{1-p_2'}-k)\}=1-\alpha \quad \text{or}$$

$$\exp(nk)\{\exp -nK_{p_1'} - \exp -nK_{1-p_2'}\}=1-\alpha. \tag{4.22}$$

For p_2 and β, a similar procedure gives

$$P\{2n(K_{p_1''}-k) \leq Y \leq 2n(K_{1-p_2''}-k)\}=\beta \quad \text{or}$$

$$\exp(nk)\{\exp -nK_{p_1''} - \exp -nK_{1-p_2''}\}=\beta. \tag{4.23}$$

Solving (4.22) and (4.23), n is obtained as the solution of the equation

$$\frac{\exp(-nK_{p_1'}) - \exp(-nK_{1-p_2'})}{\exp(-nK_{p_1''}) - \exp(-nK_{1-p_2''})} = \frac{1-\alpha}{\beta} \quad \text{or}$$

$$\frac{(1-p_1')^n - (p_2')^n}{(1-p_1'')^n - (p_2'')^n} = \frac{1-\alpha}{\beta} \tag{4.24}$$

where $K_{p_1'} = -\ln(1-p_1')$, $K_{p_1''} = -\ln(1-p_1'')$, $K_{1-p_2'} = -\ln p_2'$ and

$K_{1-p_2''} = -\ln p_2''$.

The solution for n can be found from (4.24) by an iterative technique or using the Table II in the Appendix, which includes the tabulation of the ratio

$$\frac{(1-p_1')^n - (p_2')^n}{(1-p_1'')^n - (p_2'')^n}$$

for various combinations of p_1 and p_2 and n ranging from 2 to 100. For convenience $p_1' = p_2'$ and $p_1'' = p_2''$ are used in this tabulation. This table can be used in a similar manner as described in section 4.3.2. Once n has been obtained k can be found either from

$$k = [\ln\{(1-\alpha)/\{\exp(n\ln(1-p_1')) - \exp(n\ln p_2')\}\}]/n$$

or

$$k = [\ln\{\beta/\{\exp(n\ln(1-p_1'')) - \exp(n\ln p_2'')\}\}]/n$$

By further simplifying the above formulas one can obtain

$$k = [\ln\{(1-\alpha)/\{(1-p')^n - (p')^n\}\}]/n$$

and

$$k = [\ln\{\beta/\{(1-p_1'')^n - (p_2'')^n\}\}]/n.$$

4.4.2 <u>μ known and σ unknown</u>

In this case the lot is accepted if $L \le \mu + k\hat{\sigma} \le U$ such that the probability statements for (p_1,α) and (p_2,β) are satisfied. It can be shown that for (p_1,α)

$$P[(K_{p_1'}/k) \le Y \le (K_{1-p_2'}/k)] = 1-\alpha$$

where $Y \sim \Gamma_{n,n}$ and which yields

$$\Gamma[n,n;(K_{1-p_2'}/k)] - \Gamma[n,n;(K_{p_1'}/k)] = 1-\alpha \qquad (4.25)$$

Similarly for p_2 and β

$$\Gamma[n,n;(K_{1-p_2''}/k)] - \Gamma[n,n;(K_{p_1''}/k)] = \beta. \qquad (4.26)$$

n and k may be found iteratively from (4.25) and (4.26).

4.4.3 <u>μ and σ both unknown</u>

As in the previous cases the lot is accepted if $L \le \hat{\mu} + k\hat{\sigma} \le U$ and n and k are chosen to satisfy the following probability statements. For p_1 and α

$$P(L \le \hat{\mu} + k\hat{\sigma} \le U) = 1-\alpha$$

and for p_2

$$P(L \le \hat{\mu} + k\hat{\sigma} \le U) = \beta.$$

But $2n(\hat{\mu}-\mu)/\sigma \sim \chi_2^2$ and $2n\hat{\sigma}/\sigma \sim \chi_{2n-2}^2$.

Hence the probability restrictions become

$$P(2nK_{p_1'} \le Y_1 + kY_2 \le 2nK_{1-p_2'}) = 1-\alpha \tag{4.27}$$

and

$$P(2nK_{p_1''} \le Y_1 + kY_2 \le 2nK_{1-p_2''}) = \beta. \tag{4.28}$$

The expression (4.27) after integrating becomes

$$1-\alpha = P[2nK_{p_1'}/k \le Y_2 \le 2nK_{1-p_2'}/k]$$

$$+ \exp(-nK_{p_1'}) \int_0^{(2nK_{p_1}/k)} [\exp\{-(1-k)y_2/2\}\, y_2^{n-2}]/\{2^{n-1}(n-1)\}dy_2$$

$$-\exp(-nK_{1-p_2'}) \int_0^{(2nK_{1-p_2'}/k)} [\exp\{-(1-k)y_2/2\} y_2^{n-2}]/\{2^{n-1}(n-1)\}dy_2.$$

Similarly the probability statement (4.28) can be expressed as

$$\beta = P[2nK_{p_1''}/k \le Y_2 \le 2nK_{1-p''}/k]$$

$$+ \exp(-nK_{p_1'}) \int_0^{(2nK_{p_1''}/k)} [\exp\{-(1-k)y_2/2\} y_2^{n-2}]/\{2^{n-1}(n-1)\}dy_2$$

$$-\exp(-nK_{1-p_2''}) \int_0^{(2nK_{1-p_2''}/k)} [\exp\{-(1-k)y_2/2\} y_2^{n-2}]/\{2^{n-1}(n-1)\}dy_2.$$

In order to solve the above equations the following cases are recognized.

<u>Case I: k<0</u>

Using the relationship (4.27) we get

$$\int_0^{\infty} \int_{d_1-ky_2}^{d_2-ky_2} f(y_1,y_2)\,dy_1 dy_2$$

where $d_1=2nK_{p_1'}$, $d_2=2nK_{1-p_2'}$ and $f(y_1,y_2)$ is the joint distribution of y_1 and y_2. Upon integrating the above expression one can obtain

$$\int_0^{\infty}[\exp\{-(d_1-ky_2)/2\} - \exp\{-(d_2-ky_2)/2\}]f(y_2)dy_2.$$

Now, substituting for $f(y_2)$ and integrating over y_2 yields

$$1-\alpha = [\exp(-d_1/2) - \exp(-d_2/2)]/(1-k)^{-(n-1)}.$$

Similarly from (4.28) we get

$$\beta = [\exp(-d_1'/2) - \exp(-d_2'/2)]/(1-k)^{-(n-1)}$$

where $d_1'=2nK_{p_1''}$, and $d_2'=2nK_{1-p_2''}$.

But $\exp(-d_1/2) = \exp(-nK_{p_1'}) = \exp(n\ln(1-p_1') = (1-p_1')^n$.

Similarly $\exp(-d_2/2) = (p_2')^n$ $\exp(-d_1'/2) = (1-p_1'')^n$

and $\exp(-d_2'/2) = (1-p_2'')^n$.

Hence $1-\alpha = [(1-p_1')^n - (p_2')^n]/(1-k)^{-(n-1)}$ (4.29)

and

$$\beta = [(1-p_1'')^n - (p_2'')^n]/(1-k)^{-(n-1)}. \quad (4.30)$$

Using (4.29) and (4.30), n can be determined as the solution of the equation

$$(1-\alpha)/\beta = \frac{(1-p_1')^n - (p_2')^n}{(1-p_1'')^n - (p_2'')^n}. \quad (4.31)$$

Once again Table II in the Appendix can be used to obtain the necessary sample size from equation (4.31).

The two solutions for k are

$$k = 1 - [\{(1-p_1')^n - (p_2')^n\}/(1-\alpha)]^{1/(n-1)}$$

and

$$k = 1 - [\{(1-p_1'')^n - (p_2'')^n\}/\beta]^{1/(n-1)}.$$

Case II $0<k<1$

From the basic relationship (4.27) it can be shown that

$$1-\alpha=P(d_1/k\le Y_2\le d_2/k)$$

$$+\exp(-d_1/2)\int_0^{d_1/k}[\{\exp-(1-k)y_2/2\}/\{2^{n-1}(n-1\}]y_2dy_2$$

$$-\exp(-d_2/2)\int_0^{d_2/k}[\{\exp-(1-k)y_2/2\}/\{2^{n-1}(n-1)\}]y_2dy_2.$$

This can further be simplified to obtain

$$1-\alpha = P(d_1/k \le Y_2 \le d_2/k) + [\exp(-d_1/2)P\{Y_2\le(1-k)d_1/k\}$$

$$-\exp(-d_2/2)P\{Y_2\le(1-k)d_2/k\}]/(1-k)^{-(n-1)}. \quad (4.32)$$

For (p_2,β) a similar procedure gives

$$\beta = P(d_1' /k \le Y_2\le d_2' /k) + [\exp(-d_1'/2)P\{Y_2\le(1-k)d_1'/k\}$$

$$-\exp(-d_2'/2)P\{Y_2\le(1-k)d_2'/k\}]/(1-k)^{-(n-1)}. \quad (4.33)$$

We can then obtain n and k by solving the equations (4.32) and (4.33).

The operational procedures of the sampling plans given in the preceding sections are illustrated in the figures 4.1 and 4.2. The following examples illustrate the practical application of these plans.

Example 1.

Suppose a sampling plan is required such that $p_1=.01$, $p_2=.08$, $\alpha=.05$, and $\beta=.10$. Using formula (4.6) for n when the lower specification limit is specified, the required sample size is given by

$$n = [\ln\{(1-\alpha)/\beta\}]/[\ln\{(1-p_1)/(1-p_2)\}]$$

$$= 2.25129/.0733 = 31.$$

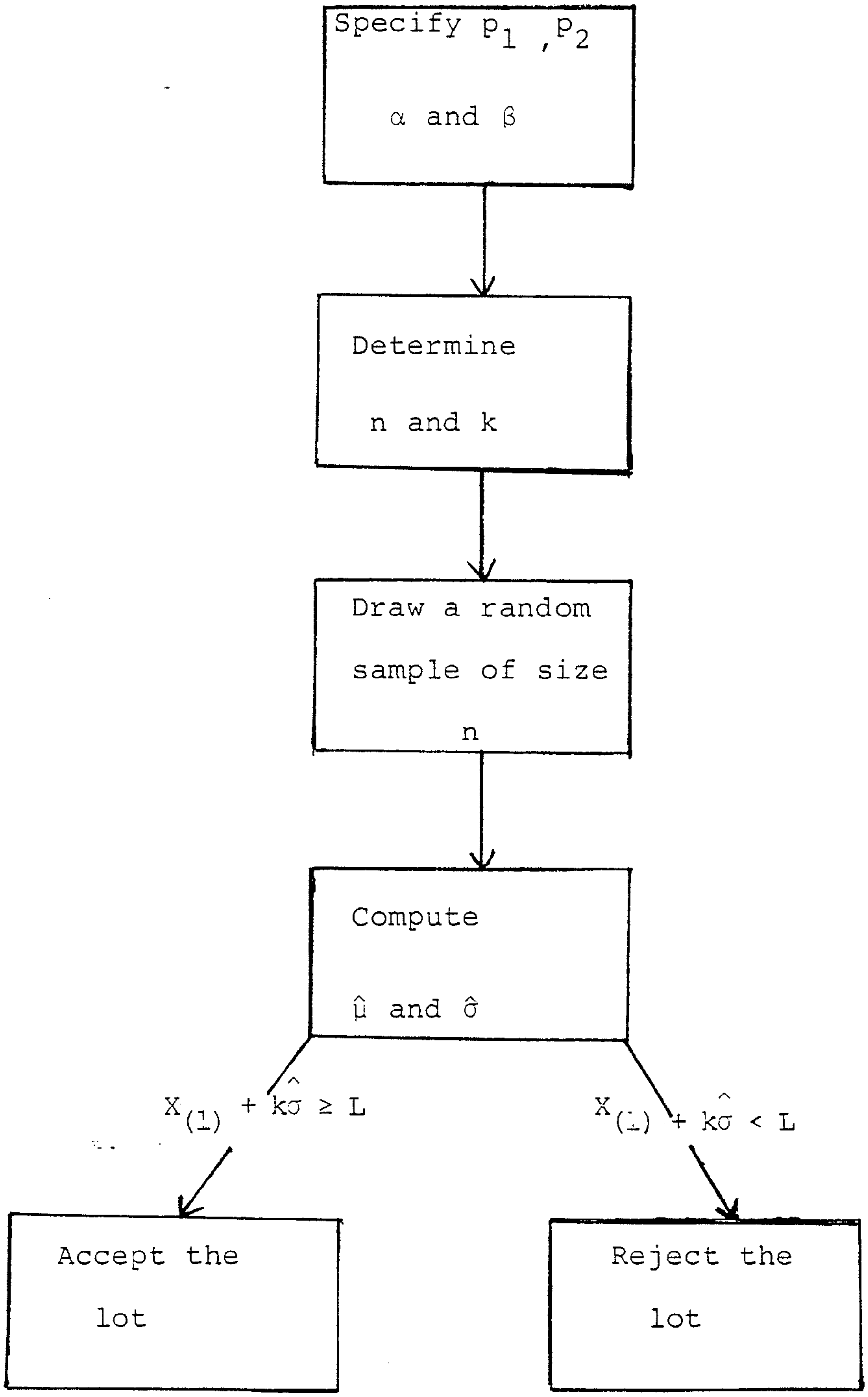


Figure 4.1. The operational procedure of one-sided sampling plans for the exponential distribution.

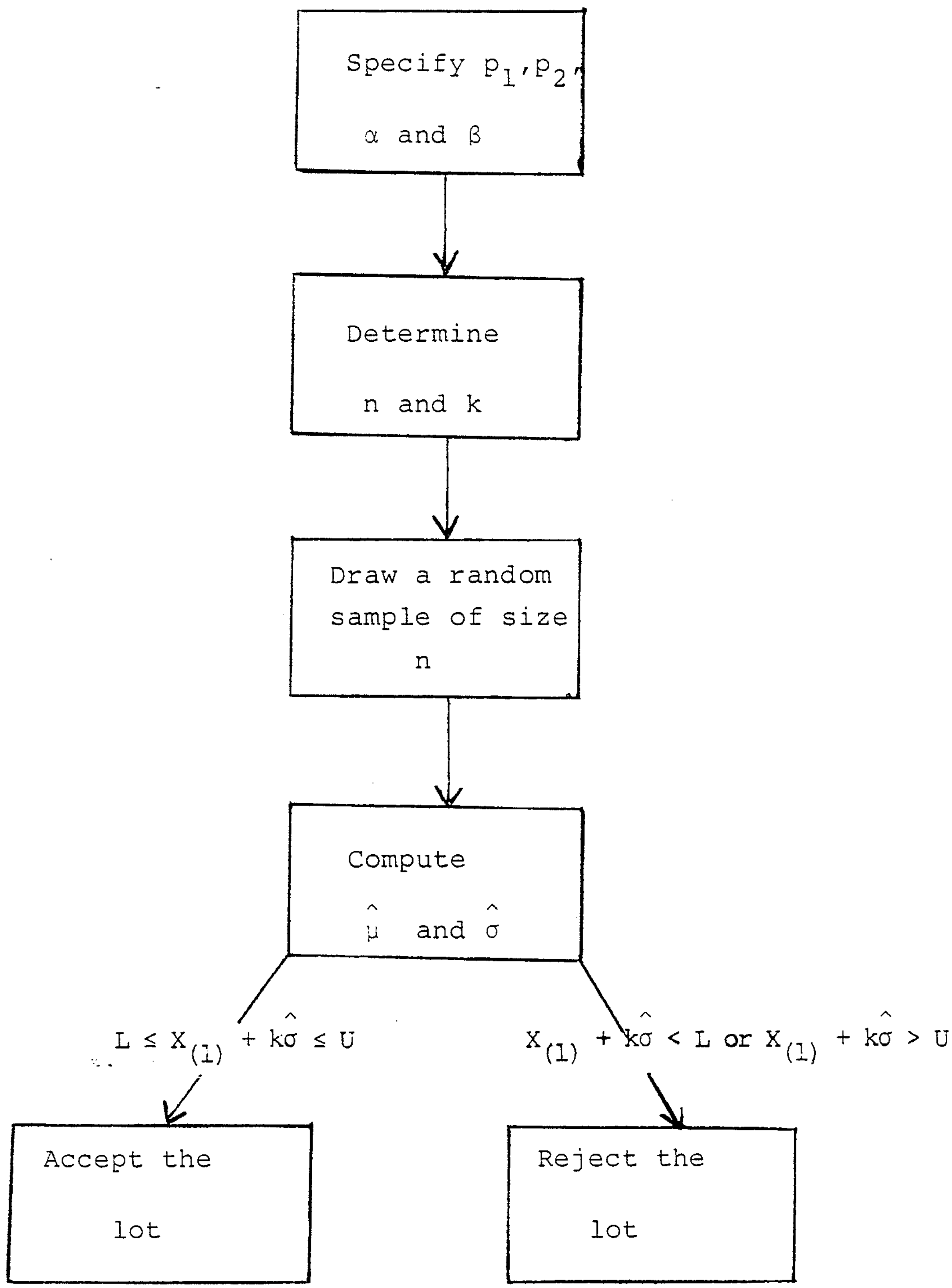


Figure 4.2. The operational procedure of two-sided sampling plans for the exponential distribution.

The two values of k are

$$k = \ln(1-\alpha)/n - \ln(1-p_1)$$

$$= \ln(.95)/31 - \ln(.99) = .0083$$

and

$$k = \ln\beta/n - \ln(1-p_2)$$

$$= \ln(.10)/31 - \ln(.92) = .0066.$$

The average value of k is $(.0083 + .0066)/2 = .0079$. Hence the sampling plan is n=30 and k=.0079. That is, a random sample of size 30 is taken from the production process and if $X_{(1)} + k\sigma \geq L$, the lot is considered acceptable. Otherwise the lot is rejected.

Sinha and Kale (1980) have analyzed 30 observations taken from the two parameter exponential distribution. The data set is given on page 34 of their book. The maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ are

$$\hat{\mu} = X_{(1)} = 20 \text{ hours and}$$

$$\hat{\sigma} = \bar{X} - X_{(1)} = 598 - 20 = 578, \text{ respectively.}$$

Now suppose that σ is known to be 578 and L is 30 hours. The lot is then rejected since $20+(.0079)(578) = 24.57 < 30$.

Example 2.

Bain (1978, p. 178) has reported 40 observations drawn from the exponential distribution with parameters μ=10 and σ=100. In this case a sampling plan is obtained assuming μ is known to be 10. Let $p_1 = .001$, $p_2 = .0016$, $\alpha = .05$ and $\beta = .10$. The maximum likelihood estimate of σ is $\hat{\sigma} = \bar{X} - \mu = 103.121 - 10 = 93.121$. The ratio $\ln(1-p_1)/\ln(1-p_2) = \ln(.999)/\ln(.9984) = .625$. Now using Table I in the appendix it can be seen that the ratio $\Gamma_{n,n;.05}/\Gamma_{n,n;.90}$ which is closest to .625 is obtained when n = 40. The necessary sample size is, therefore, 40. The acceptability constant is computed as $k = -\ln(1-p_1)/\Gamma_{40,40;.05} = .001$. Thus, the sampling plan is given by n=40 and k= .001. Now assume that the lower limit is specified to be 9.5 hours. The lot is accepted if $\mu + k\hat{\sigma} \geq L$. Since $\mu + k\hat{\sigma} = 10 + .001(93.121) = 10.093 > 9.5$, the lot is considered acceptable.

4.5. Sampling Plans Based on Minimum Variance Unbiased Estimators (MVUE)

The minimum variance unbiased estimators $\tilde{\mu}$ and $\tilde{\sigma}$ of μ and σ for the two parameter exponential distribution given in (4.1) are

$$\tilde{\mu} = X_{(1)} - \sigma/n \quad (\sigma \text{ known})$$

$$\tilde{\mu}^* = X_{(1)} - \tilde{\sigma}^*/n \quad (\sigma \text{ unknown})$$

$$\tilde{\sigma} = \sum_{i=1}^{n} (X_i - \mu)/n = \bar{X} - \mu, \quad (\mu \text{ known}) \text{ and}$$

$$\tilde{\sigma}^* = \sum_{i=1}^{n-1} \{(n-1)(X_{(i+1)} - X_{(i)})\}/(n-1) \quad (\mu \text{ unknown}).$$

It can be shown that

$$2n\{X_{(1)} - \mu\}/\sigma \sim \chi^2_2 \text{ and } 2(n-1)\tilde{\sigma}^*/\sigma \sim \chi^2_{2n-2}.$$

When μ is known $\tilde{\sigma}/\sigma$ has a gamma distribution with parameters n and n.

4.5.1. μ unknown and σ known

The decision rule is to accept a lot if $\tilde{\mu} + k^*\sigma \geq L$ or $\tilde{\mu} + k^*\sigma \leq U$ with the same probability restrictions given in the previous section. Using the procedure described in the preceding section it can be easily shown that the necessary sample size is

$$n = [\ln\{(1-\alpha)/\beta\}]/[\ln\{(1-p_1)/(1-p_2)\}].$$

Note that the sample size remains the same as in the case of the sampling plan based on the maximum likelihood estimators. It can be shown that the acceptability constant, k^*, can be expressed as $k^* = 1/n + k$, where k is the acceptability constant associated with the sampling plan based on the maximum likelihood estimator.

4.5.2 <u>μ known and σ unknown</u>

In this case MVUE, $\tilde{\sigma}$, of σ is also the maximum likelihood estimator. Thus, the sampling plan corresponding to MVUE coincides with the sampling plan based on the MLE given in section 4.3.2.

4.5.3 <u>μ and σ both unknown</u>

The lot is accepted if $\tilde{\mu}^* + k^* \tilde{\sigma}^* \geq L$ or $\tilde{\mu}^* + k^* \tilde{\sigma}^* \leq U$. For lower specification limit it can be easily shown that

$$P(Y_1 + aY_2 < 2nK_{p1}) = \alpha$$

where $a = (nk^* - 1)/(n - 1)$. Using the result (4.18) one can obtain

$$\alpha = P(Y_2 \leq 2nK_{p_1}/a) - [\{\exp(-nK_{p_1})\}/(1-a)^{n-1}$$

$$.P\{Y_2 \leq 2nK_{p_1}(1-a)/a\}]$$

and for (p_2, β)

$$1-\beta = P(Y_2 \le 2nK_{p_2}/a) - [\{\exp(-nK_{p_2})\}/(1-a)^{n-1}$$

$$.P\{Y_2 \le 2nK_{p_2}(1-a)/a\}].$$

The same procedure that can be used to solve the equations (4.18) and (4.19) may be used to obtain the solutions for n and a. Once n and a are found, k* could be obtained by $a = (nk^* - 1)/(n - 1)$.

For an upper specification limit the corresponding equations are

$$1-\alpha = P(Y_2 \le 2nK_{1-p_1}/a) - [\{\exp(-nK_{1-p_1})\}/(1-a)^{n-1}]$$

$$.[P\{Y_2 \le 2nK_{1-p_1}(1-a)/a\}]$$

and

$$\beta = P(Y_2 \le 2nK_{1-p_2}/a) - [\{\exp(-nK_{1-p_2})\}/(1-a)^{n-1}]$$

$$.[P\{Y_2 \le 2nK_{1-p_2}(1-a)/a\}].$$

4.6 Two Sided Sampling Plans

4.6.1 μ unknown and σ known

Applying the procedure used in section 4.4.1, the required sample size can be obtained from the relationship

$$\frac{(1-p_1')^n - (p_2')^n}{(1-p_1'')^n - (p_2'')^n} = (1-\alpha)/\beta.$$

Once again this relationship is the same as that of section 4.4.1 and Table II in the Appendix may be used to solve the above equation for n. The constants k* and k are related as $k^* = 1/n + k$ where k is the acceptability constant derived in section 4.4.1.

4.6.2 μ known and σ unknown

Once again the UMVU estimator, $\tilde{\sigma}$, of σ coincides with that of the maximum likelihood estimator. The sampling plan presented in section 4.4.2, therefore, applies to this situation.

4.6.3 μ and σ both unknown

In this case using the results in section 4.4.3 the following relationships can be established.

Case I: $a < 0$

The sample size is given by

$$\frac{(1-p_1')^n - (p_2')^n}{(1-p_1'')^n - (p_2'')^n} = (1-\alpha)/\beta.$$

Using Table II in the Appendix, solution for n can be obtained. The constants a and n are related in the following manner.

$$a = 1 - [\{(1-p_1')^n - (p_2')^n\}/(1-\alpha)]^{1/(n-1)}$$

and

$$a = 1 - [\{(1-p_1'')^n - (p_2'')^n\}/\beta]^{1/(n-1)}$$

where $a = (nk^* - 1)/(n - 1)$.

Case II: $0 < a < 1$

In this case n and a are obtained as the solutions of the following equations.

$$1-\alpha = P(d_1/a \le Y_2 \le d_2/a) + [\exp(-d_1/2)P\{Y_2 \le d_1(1-a)/a\} - \exp(-d_2/2)P\{Y_2 \le d_2(1-a)/a\}]/(1-a)^{n-1}$$

and

$$\beta = P(d_1'/a \le Y_2 \le d_2'/a) + [\exp(-d_1'/2)P\{Y_2 \le d_1'(1-a)/a\} - \exp(-d_2'/2)P\{Y_2 \le d_2'(1-a)/a\}]/(1-a)^{n-1}$$

where $d_1 = 2nK_{p_1'}$, $d_2 = 2nK_{p_2'}$, $d_1' = 2nK_{p_1''}$, and $d_2' = 2nK_{p_2''}$.

4.7 Acceptance sampling plans to give assurance regarding population parameters

In the previous section sampling plans to control the percentage of output lying outside upper and lower specification limits were derived. For normally distributed random variables, acceptance sampling plans to give assurance regarding population mean when the process standard deviation is known or unknown are given in Duncan (1986). In this section acceptance sampling plans concerned with the guaranteed time of the product when sampling from an exponential distribution will be developed.

4.7.1 Derivation of sampling plans for μ with specified α, β, μ_1 and μ_2.

(i) σ known

The rule is to accept a batch if $\hat{\mu} > k_1$ such that $P(\hat{\mu} > k_1 | \mu_1) = 1-\alpha$ and $P(\hat{\mu} > k_1 | \mu_2) = \beta$ are satisfied. Here k_1 is defined to be the acceptance limit and α and β are producer's and consumer's risks, respectively. Schilling (1982) defines μ_1 as the acceptable process level (APL) and μ_2 as the rejectable process level (RPL). The maximum likelihood estimators of μ and σ are used in the derivation of the appropriate sampling plans.

Using the fact that $2n(\hat{\mu}-\mu)/\sigma$ is a chi-square random variable with 2 degrees of freedom, the above probability statements can be expressed as

$$P(Y_1 \geq 2n(k_1-\mu_1)/\sigma) = \alpha$$

and

$$P(Y_1 \geq 2n(k_1-\mu_2)/\sigma) = 1-\beta$$

where $Y_1 \sim \chi^2_2$. We can then obtain

$$-n(k_1-\mu_1)/\sigma = \ln(1-\alpha) \tag{4.34}$$

and

$$-n(k_1-\mu_2)/\sigma = \ln\beta. \tag{4.35}$$

Solving (4.34) and (4.35) for n and k_1 we get

$$n = \sigma\ln\{\beta/(1-\alpha)\}/(\mu_2-\mu_1)$$

and

$$k_1 = \mu_1 - [(\mu_2-\mu_1)\ln(1-\alpha)]/\ln(\beta/(1-\alpha)) \text{ or}$$
$$k_1 = \mu_2 - [(\mu_2-\mu_1)\ln\beta]/\ln(\beta/1-\alpha)).$$

The required sampling plan is then given by n and k_1. That is, a random sample of n items is drawn from the process under consideration and k_1 is computed. If $X_{(1)} \geq k_1$ the lot (or process) is considered acceptable. Otherwise the lot is rejected.

(ii) <u>σ unknown</u>

In this case for (μ_1,α), it can be shown that

$$P[(Y_1/2)/(Y_2/2n-2) > \{(2n-2)/2\}(k_1-\mu_1)/\hat{\sigma}] = 1-\alpha$$

where $Y_1 \sim \chi^2_2$, $Y_2 \sim \chi^2_{2n-2}$ and Y_1 and Y_2 are independent. Hence the ratio $(Y_1/2)/(Y_2/2n-2)$ has an F distribution with 2 and 2n-2 degrees of freedom. The above probability statement then becomes

$$P[F_{2,2n-2} < \{(2n-2)/2\}(k_1-\mu_1)] = \alpha$$

which gives

$$(n-1)(k_1-\mu_1)/\hat{\sigma} = F_{2,2n-2;\alpha}. \tag{4.36}$$

Similarly for (β,μ_2),

$$(n-1)(k_1-\mu_2)/\hat{\sigma} = F_{2,2-2n;1-\beta}. \tag{4.37}$$

Equations (4.36) and (4.37) can be solved iteratively to obtain the solutions of n and k_1.

4.7.2 Derivation of sampling plans to give assurance regarding the scale parameter σ with specified α, β, σ_1 and σ_2

(i) μ known

In this case the lot is accepted if $\hat{\sigma} < k_1$ such that $P(\hat{\sigma} < k_1|\sigma_1) = 1-\alpha$ and $P(\hat{\sigma} < k_1|\sigma_2) = \beta$ are satisfied. Here σ_1 and σ_2 are APL and RPL, respectively and $\hat{\sigma}$ is the maximum likelihood estimator of σ. Using the distributional properties of $\hat{\sigma}$ it can be shown that the equations required to find n and k_1 are

$$k_1/\sigma_1 = \Gamma_{n,n;1-\alpha} \tag{4.38}$$

and

$$k_1/\sigma_2 = \Gamma_{n,n;\beta}. \tag{4.39}$$

The solution of n is then given by solving the equations (4.38) and (4.39) as

$$\sigma_2/\sigma_1 = \Gamma_{n,n;1-\alpha}/\Gamma_{n,n;\beta} \tag{4.40}$$

which can be easily solved using table I in the Appendix in a similar manner described in section 4.3.2. Once n is obtained from (4.40) k_1 may be found either from (4.38) or (4.39).

(ii) μ unknown

It can then be shown that the two equations involving n and k_1 are

$$2nk_1/\sigma_1 = \chi^2_{2n-2;1-\alpha} \tag{4.41}$$

$$2nk_1/\sigma_2 = \chi^2_{2n-2;\beta}. \tag{4.42}$$

Solving (4.41) and (4.42) we get

$$\sigma_2/\sigma_1 = \chi^2_{2n-2;1-\alpha} / \chi^2_{2n-2;\beta}.$$

The ratio $\chi^2_{2n-2;1-\alpha}/ \chi^2_{2n-2;\beta}$ is tabulated for various combinations of α, β and n ranging from 2 to 100. The tabulated results are given in Table III in the Appendix. This table can be used to obtain the solution of n in a similar manner described in the preceding section.

4.7.3 Sampling plans for σ with two acceptance limits

In this section we consider deriving the sampling plans with two acceptance limits. These plans are similar to two sided plans discussed in section 4.4 of this chapter. The following two cases may be identified for this purpose.

(i) μ known

A given process or lot is accepted if $k_1 < \hat{\sigma} < k_2$ where k_1 and k_2 are the two acceptance limits. The probability restrictions become

$$P(k_1 < \hat{\sigma} < k_2 | \sigma_1) = 1-\alpha$$

and

$$P(k_1 < \hat{\sigma} < k_2 | \sigma_2) = \beta.$$

Using the properties of $\hat{\sigma}$ the following equations can be obtained.

$$k_1/\sigma_1 = \Gamma_{n,n;\alpha/2} \tag{4.42}$$

$$k_2/\sigma_1 = \Gamma_{n,n;1-\alpha/2} \tag{4.43}$$

$$k_1/\sigma_2 = \Gamma_{n,n;(1-\beta)/2} \tag{4.44}$$

and

$$k_2/\sigma_2 = \Gamma_{n,n;(1+\beta)/2} \tag{4.45}$$

Solving (4.42) and (4.44) we get

$$\sigma_2/\sigma_1 = \Gamma_{n,n;\alpha/2}/\Gamma_{n,n;(1-\beta)/2} \tag{4.46}$$

or using (4.43) and (4.45)

$$\sigma_2/\sigma_1 = \Gamma_{n,n;1-\alpha/2}/\Gamma_{n,n;(1+\beta)/2} \tag{4.47}$$

is obtained. (4.46) or (4.47) can be solved iteratively to find n and then k_1 may be found from (4.42) or (4.44) and the solution of k_2 can be found from (4.43) or (4.45). Note that in this case the upper and lower tail probabilities are taken to be the same. For example, for producer's risk ,α, both tail probabilities are set to be $\alpha/2$.

(ii) <u>μ unknown</u>

As indicated earlier MLE ,$\hat{\sigma}$, of σ is $\bar{X} - X_{(1)}$ and applying the same procedure as in the preceding section the following equations are obtained.

$$2nk_1/\sigma_1 = \chi^2_{2n-2;\alpha/2} \tag{4.48}$$

$$2nk_2/\sigma_1 = \chi^2_{2n-2;1-\alpha/2} \tag{4.49}$$

$$2nk_1/\sigma_2 = \chi^2_{2n-2;(1-\beta)/2} \tag{4.50}$$

and

$$2nk_2/\sigma_2 = \chi^2_{2n-2;(1+\beta)/2}. \tag{4.51}$$

In order to find the solutions of n, k_1 and k_2 the same procedure explained in the preceding section can be applied to equations (4.48)-(4.51).

4.8 Control charts for the two parameter exponential distribution

In this section the control charts for the location (μ) and scale (σ) parameters of the distribution considered in (4.1) are derived. The maximum likelihood estimators of μ and σ are used to derive the control chart constants. A numerical example to illustrate the application of the procedures is also given. The following separate cases are identified and discussed.

4.8.1 Control charts for the location parameter

In order to construct control charts for the location parameter, the variance of the estimator , $X_{(1)}$, of μ is required. Using the distributional properties of $X_{(1)}$, it can be shown that the variance of $X_{(1)}$ is σ^2/n^2. The standard deviation of $X_{(1)}$ is, therefore, σ/n. In this section 3 standard deviation (3 s.d.) control chart limits are derived. The 2 s.d. control chart limits are similar.

(i) μ and σ are known

The 3 s.d. control chart limits for μ is given by

$$\mu \pm 3\sigma/n.$$

Let $A_1 = 3/n$. Then the lower and upper control limits are $\mu - A_1\sigma$ and $\mu + A_1\sigma$, respectively.

(ii) μ known and σ unknown

The lower and upper control limits are

$$\mu \pm 3\hat{\sigma}/n$$

$$= \mu \pm A_1\hat{\sigma}$$

where $\hat{\sigma} = \bar{X} - \mu$ is the maximum likelihood estimate of σ.

(iii) <u>μ unknown and σ known</u>

The two limits are given by

$$X_{(1)} \pm A_1\sigma.$$

(iv) <u>μ and σ both unknown</u>

Using both estimates the following limits are obtained.

$$X_{(1)} \pm A_1\hat{\sigma}.$$

4.8.2 <u>Control charts for the scale parameter</u>

The following estimators of σ are used to construct 3 s.d. control chart limits for the scale parameter.

(a) $\hat{\sigma} = \sum_{i=1}^{n}(X_i - \mu)/n$; μ known and

(b) $\tilde{\sigma} = \bar{X} - X_{(1)}$; μ is unknown.

As mentioned earlier $\hat{\sigma}/\sigma$ has a gamma distribution with parameters n and n. The variance of $\hat{\sigma}$ is, therefore, $n^3\sigma^2$ and the standard deviation of $\hat{\sigma}$ is $n^{3/2}\sigma$. Similarly $2n\tilde{\sigma}/\sigma$ has a chi-square distribution with 2n-2 degrees of freedom. The variance of $\tilde{\sigma}$ is then given by $(n-1)\sigma^2/n^2$, and the standard deviation of $\tilde{\sigma}$ is $\sqrt{(n-1)}\sigma/n$. Once again the 3 s.d. control chart limits are given for the following cases. The 2 s.d. control chart limits are similar.

(i) <u>μ known and σ known</u>

The lower and upper control chart limits are given by

$$\sigma \pm 3n^{3/2}\sigma$$

$$= (1 \pm 3n^{3/2})\sigma.$$

Let $A_2 = 1 - 3n^{3/2}$ and $A_3 = 1 + 3n^{3/2}$. Hence the two limits are $A_2\sigma$ and $A_3\sigma$.

(ii) <u>μ known and σ unknown</u>

$$\hat{\sigma} \pm 3n^{3/2}\hat{\sigma}$$

$$= (1 \pm 3n^{3/2})\hat{\sigma}.$$

Thus, the 3 s.d. control limits become $A_2\hat{\sigma}$ and $A_3\hat{\sigma}$.

(iii) <u>μ unknown and σ known</u>

$$\sigma \pm 3\sqrt{\{(n-1)/n\}}\sigma$$

$$= (1 \pm 3\sqrt{\{(n-1)/n\}}\sigma.$$

Let $A_4 = 1 - 3\sqrt{(n-1)}/n$ and $A_5 = 1 + 3\sqrt{(n-1)}/n$. The control limits then become $A_4\sigma$ and $A_5\sigma$.

(iv) <u>μ and σ both unknown</u>

$$\tilde{\sigma} \pm 3\sqrt{\{(n-1)/n\}}\tilde{\sigma}$$

$$= \{1 \pm 3\sqrt{(n-1)}/n)\}\tilde{\sigma}.$$

The control limits are $A_4\tilde{\sigma}$ and $A_5\tilde{\sigma}$.

4.8.3 <u>Control Charts for the Range</u>

By using the same procedure described in Chapter 3 for constructing R-charts for normally distributed observations, the range charts for the exponential distribution can be obtained. For this purpose the expectation and the standard deviation of the relative range, $W = R/\sigma$, are required. These can be obtained using the probability density function of the relative range,

$$f(r) = (n-1)e^{-r}(1-e^{-r})^{n-2}. \qquad 0<r<\infty$$

Let the expectation of the relative range be d_2. Then d_2 is given by

$$d_2 = (n-1)\int_0^{\infty} re^{-r}(1-e^{-r})^{n-2}\,dr.$$

A closed form solution for d_2 is not available, but expanding $(1-e^{-r})^{n-2}$ using a binomial expansion and integrating over r, it can be shown that

$$d_2 = (n-1)\left[\sum_{k=0}^{n-2}\binom{n-2}{k}(-1)^k/(1+k)^2\right].$$

Similarly $E(W^2) = 2(n-1)\left[\sum_{k=0}^{n-2}\binom{n-2}{k}(-1)^k/(1+k)^3\right].$

Using these expressions, expectation and the standard deviation of the relative range can be obtained. An unbiased estimate of the scale parameter can then be obtained by $\hat{\sigma} = \bar{R}/d_2$, where $\bar{R} = \sum_{i=1}^{k} R_i/k$ and R_i is the range of the sample i of size n, and k is the number of such samples. One can then construct the following range charts based on the exponential distribution.

(i) <u>σ known</u>

The 3 s. d. control chart limits are given by

$$E(R) \pm 3\sigma_R$$
$$= d_2\sigma \pm 3d_3\sigma = (d_2 \pm 3d_3)\sigma$$

where d_3 is the standard deviation of the relative range. Let $D_1 = d_2 - 3d_3$ and $D_2 = d_2 + 3d_3$. Then the lower and upper control limits are $D_1\sigma$ and $D_2\sigma$.

(ii) <u>σ unknown</u>

In this case using the estimate, $\bar{R}/d_2$, of σ the following limits can be given.

$$\bar{R} \pm 3d_3(\bar{R}/d_2)$$
$$= (1 \pm 3d_3/d_2)\bar{R}.$$

Let $D_3 = 1 - 3d_3/d_2$ and $D_4 = 1 + 3d_3/d_2$. Thus, the lower and upper control limits are $D_3\bar{R}$ and $D_4\bar{R}$, respectively.

The control chart limit constants A_1, A_2, A_3, A_4, and A_5 are tabulated for the exponential distribution. These values are given in Table 4.1 for various sample sizes. The constants d_2, d_3, D_1, D_2, D_3 and D_4 are tabulated in Table 4.2. One can then construct the control charts using these tabulated values when the underlying probability distribution of the quality characteristic is found to be exponential.

Numerical Examples:

(i) Consider Sinha and Kale's (1980, p.34) data set. The maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ are 20 and 578, respectively as given in section 4.4 of this chapter. The following control charts can then be presented.

(i) For the location parameter 3 s. d. control chart limits are $X_{(1)} \pm A_1\hat{\sigma}$. From Table 4.1 the value of A_1 is 0.1. Hence

lower control limit = 20 - 0.1(578) = 0.00 and

upper control limit = 20 + 0.1(578) = 77.8.

(ii) The control chart limits for the scale parameter are $A_4\hat{\sigma}$ nd $A_5\hat{\sigma}$. From Table 4.1, A_4 = 0.461 and A_5 = 1.538 Thus,

lower control limit = 0.461(578) = 266.46

upper control limit = 1.538(578) = 888.96.

In order to construct range charts for Sinha and Kale's (1980, p.35) data, the following six subsamples each of size 5 were randomly obtained from their data.

Sample	Observations					R_i
1	20	214	445	697	1016	996
2	27	232	472	798	1033	1006
3	52	238	503	805	1086	1034
4	61	371	526	909	1192	1131
5	110	393	581	976	1322	1212
6	122	426	627	1001	1681	1559

In this case n = 5 and $\bar{R}$ = 1156.33. From Table 4.2 the required constants are D_3 = 0.000 and D_4 = 6.234. Hence for the range chart

lower control limit = 0.000(1156.33) = 0.000

and

upper control limit = 6.234(1156.33) = 7208.56.

(ii) Grubbs (1971) has given mileages at which nineteen military carriers failed. For his data, $X_{(1)}$=162, $\bar{X}$=997.21, $\tilde{\sigma}$=835.21 and n=19. The 3-s.d. control limits for μ is then given by $X_{(1)} \pm A_1\tilde{\sigma}$. From Table 4.1, A_1=0.159 and

lower control limit = 162 - (0.159)(835.21) =29.21

and

upper control limit = 162 + (0.159)(835.21) =294.79.

The control charts for σ is given by $A_4\tilde{\sigma}$ and $A_5\tilde{\sigma}$. From Table 4.1, A_4=0.330, A_5=1.669 and the control limits for σ is given by

lower control limit = (.330)(835.21) = 275.62

and

upper control limit = (1.669)(835.21) = 1393.97.

Table 4.1. Control Chart Constants for the Exponential Distribution

n	A_1	A_2	A_3	A_4	A_5
2	1.500	0.000	9.485	0.000	2.500
3	1.000	0.000	16.588	0.000	2.414
4	0.750	0.000	25.000	0.000	2.999
5	0.600	0.000	34.541	0.000	2.200
6	0.500	0.000	45.091	0.000	2.118
7	0.429	0.000	56.561	0.000	2.049
8	0.375	0.000	68.882	0.008	1.992
9	0.333	0.000	82.000	0.057	1.943
10	0.300	0.000	95.868	0.100	1.900
11	0.273	0.000	110.449	0.138	1.862
12	0.250	0.000	125.708	0.171	1.829
13	0.231	0.000	141.616	0.201	1.799
14	0.214	0.000	158.150	0.227	1.773
15	0.200	0.000	175.284	0.252	1.748
16	0.187	0.000	193.000	0.274	1.726
17	0.176	0.000	211.278	0.294	1.706
18	0.167	0.000	230.103	0.313	1.687
19	0.159	0.000	249.457	0.330	1.669
20	0.150	0.000	269.328	0.346	1.653
21	0.143	0.000	289.702	0.361	1.639
22	0.136	0.000	310.567	0.375	1.625
23	0.130	0.000	331.912	0.388	1.612
24	0.125	0.000	353.727	0.401	1.599
25	0.120	0.000	376.000	0.412	1.587
26	0.115	0.000	398.728	0.423	1.576
27	0.111	0.000	445.486	0.433	1.566
28	0.107	0.000	469.509	0.443	1.556
29	0.103	0.000	469.509	0.453	1.547
30	0.100	0.000	493.950	0.461	1.538

Table 4.2. Control Chart Constants for the Exponential Distribution

n	d_2	d_3	D_1	D_2	D_3	D_4
2	0.000	1.000	0.000	4.000	0.000	4.000
3	1.500	1.118	0.000	4.854	0.000	5.025
4	1.833	1.167	0.000	5.334	0.000	5.712
5	2.083	1.194	0.000	5.665	0.000	6.234
6	2.283	1.210	0.000	5.913	0.000	6.660
7	2.449	1.223	0.000	6.118	0.000	7.007
8	2.593	1.229	0.000	6.280	0.000	7.329
9	2.718	1.236	0.000	6.426	0.000	7.597
10	2.829	1.241	0.000	6.552	0.000	7.839
11	2.929	1.245	0.000	6.664	0.000	8.058
12	3.019	1.250	0.000	6.769	0.000	8.246
13	3.103	1.252	0.000	6.859	0.000	8.435
14	3.180	1.254	0.000	6.942	0.000	8.608
15	3.252	1.256	0.000	7.014	0.000	8.779
16	3.318	1.258	0.000	7.092	0.000	8.913
17	3.381	1.259	0.000	7.155	0.000	9.063
18	3.439	1.261	0.000	7.222	0.000	9.182
19	3.495	1.262	0.000	7.278	0.000	9.308
20	3.548	1.262	0.000	7.333	0.000	9.434
21	3.598	1.263	0.000	7.387	0.000	9.546
22	3.645	1.265	0.000	7.440	0.000	9.644
23	3.691	1.265	0.000	7.486	0.000	9.753
24	3.734	1.267	0.000	7.535	0.000	9.841
25	3.776	1.266	0.000	7.574	0.000	9.948
26	3.816	1.267	0.015	7.617	0.000	10.036
27	3.855	1.266	0.057	7.652	0.000	10.036
28	3.892	1.267	0.091	7.693	0.000	10.215
29	3.927	1.269	0.129	7.725	0.000	10.284
30	3.962	1.268	0.158	7.766	0.000	10.374

CHAPTER FIVE

ACCEPTANCE SAMPLING PLANS BY VARIABLES IN THE PRESENCE OF SERIAL CORRELATION

5.1. Introduction

In the construction of quality control charts ($\bar{X}$, R, and s charts) and acceptance sampling plans by variables two assumptions are generally made. These assumptions are:

(1) the measurements of quality characteristic are normally distributed, and

(2) the measurements are independent.

Under these conditions, constants required to construct control charts and acceptance sampling plans are available in standard literature such as Duncan (1986) and Schilling (1982).

Examination of the effect of non-normality (assumption 1) on $\bar{X}$ and R-charts has been discussed in Chapter 3 using Tukey's λ-family of distributions. It has been observed in Chapter 3 that the use of control chart constants based on the normal distribution may not be appropriate for thick tailed distributions belonging to Tukey's λ-family. Thus, the application of the normal approximation may give rise to undesirable results if used without examining the form of the underlying probability distribution of the quality measurements.

Srivastava (1961), Das and Mitra (1964) and Schneider (1985) have examined the robustness of acceptance sampling by variables to departures from normality. These authors have concluded from their studies that variable acceptance sampling plans have little robustness. However, the assumption of independence (assumption 2) has not been studied in any depth and is usually taken for granted. Because the measurements of quality variables are often serially generated there is reason to doubt that the assumption of independence is valid.

Assumption of independence associated with control charts was first examined by Stamboulis (1971). An autoregressive process of order 1 (AR(1)) was used to describe the process and a simple modification of the control limit constants for $\bar{X}$ and s-charts was proposed. Stamboulis' result was extended to an ARMA(p,q) process by Vasilopoulos (1974). His result has been used to investigate the effect of serial correlation on control chart constants using an AR(2) (i.e., ARMA(2,0)) process by Vasilopoulos and Stamboulis (1978). They have also provided the modified control chart constants when the measurements follow an AR(2) process. Johnson and Bagshaw (1974) have examined the effect of serial correlation on cusum charts. Cusum charts refer to control charts based on cumulative total of deviations from a reference value which is usually taken to be the process average. The applications of time series for process control purposes are considered by Pandit and Wu (1983).

Most of the work above is concerned with the examination of the effect of serial correlation on constants of $\bar{X}$ and s-charts. There has been little or no research undertaken to study the implications of dependence of the data on acceptance sampling plans. In this Chapter the effect of serial correlation on acceptance sampling plans will be investigated. The measurements are assumed to follow an autoregressive process of order p.

5.2 Preliminaries

Let X_t be the t-th measurement of the quality characteristic under study, where X_t is modeled as

$$X_t = Y_t + \mu \quad \text{where}$$

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \ldots + \theta_p Y_{t-p} + e_t \qquad (5.1)$$

where μ is a constant (usually the process mean) and e_t's are assumed to be independently, normally distributed with zero mean and constant variance, σ_e^2. Y_t is then said to follow an autoregressive process of order p. If θ_i, i=1,2,...,p are all zero, we then have the usual model assumed in quality control where X_t's are independently normally distributed with mean μ and variance σ_e^2.

Box and Jenkins (1976) have provided several examples from industry which are shown to be closely approximated by AR(1) or AR(2) processes. They show that such autoregressive processes are mathematically tractable and useful. There are in fact many practical situations where the data follow or at least can be closely approximated by an AR process.

A class of time series known as stationary time series plays an important role in quality control applications. A stationary time series refers to a process in a particular state of equilibrium. In other words, the process is in statistical control. Thus, such a process has a constant mean, μ, which defines the level about which the series fluctuates, and a constant variance, σ^2, which measures the spread of the series about its mean. This class of time series also has the property that the covariance between X_t and X_s (i.e., $Cov(X_t, X_s)$), is a function only of the distance (s-t). This type of stationarity is also known as stationarity in the wide sense, covariance stationarity, weak stationarity or simply stationarity.

The process given in (5.1) is stationary if all the roots of the characteristic equation,

$$m^p + \theta_1 m^{p-1} + \dots + \theta_p = 0,$$

are less than one in absolute value. For example, consider an AR(1) process, $Y_t = \theta_1 Y_{t-1} + e_t$. This process is stationary if $|\theta_1|<1$. For an AR(2), $Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + e_t$, the characteristic equation is

$$m^2 + \theta_1 m + \theta_2 = 0. \tag{5.2}$$

This process is stationary if both roots of (5.2) are less than one in absolute value. This condition is the same as the coefficients, θ_1 and θ_2 lie in a triangular region:

$$\theta_1 + \theta_2 <-1,$$
$$\theta_1 - \theta_2 < 1,$$
$$-2 < \theta_1 < 2$$

(see Vasilopoulos and Stamboulis (1978) and Fuller (1976)).

Now consider a sample $X_1, X_2, \ldots, X_n$ of size n is taken from the process given in (5.1). The sample mean $\bar{X} = \sum_{t=1}^{n} X_t/n$ and the variance of the sample mean is given by

$$\sigma_{\bar{X}}^2 = \frac{1}{n} \sum_{h=-(n-1)}^{n-1} \left(1 - \frac{h}{n}\right) \gamma(h)$$

where $\gamma(h) = \mathrm{Cov}(X_t, X_{t+h})$, $h = \pm 0, \pm 1, \ldots$ is the autocovariance function of the process (Anderson,1971). $\sigma_{\bar{X}}^2$ can also be written as

$$\sigma_{\bar{X}}^2 = \frac{1}{n}\{\sigma^2 + 2\sum_{h=1}^{n-1}(1 - \frac{h}{n})\gamma(h)\} \tag{5.3}$$

where σ^2 is the variance of X_t.

$\sigma_{\bar{X}}^2$ can be estimated by estimating σ^2 through the autocovariance function, $\gamma(h)$. A commonly used estimator of $\gamma(h)$ is

$$\hat{\gamma}(h) = \sum_{t=1}^{n-h} (X_t - \mu)(X_{t+h} - \mu)/n \qquad \text{if } \mu \text{ is known}$$

and

$$\hat{\gamma}^*(h) = \sum_{t=1}^{n-h} (X_t - \bar{X})(X_{t+h} - \bar{X})/n \qquad \text{if } \mu \text{ is unknown.}$$

Hence an estimator of σ^2 is given by

$$\hat{\sigma}^2 = \sum_{t=1}^{n}(X_t - \mu)^2/n \qquad \text{if } \mu \text{ is known}$$

and

$$\hat{\sigma}^{*2} = \sum_{t=1}^{n}(X_t - \bar{X})^2/n \qquad \text{if } \mu \text{ is unknown.}$$

By substituting $\hat{\sigma}^2$ and $\hat{\gamma}(h)$ or $\hat{\sigma}^{*2}$ and $\hat{\gamma}^*(h)$ in (5.3) an estimator of $\sigma_{\bar{X}}^2$ may be obtained.

It is well known that

$$\lim_{n\to\infty} n\sigma_{\bar{X}}^2 = \lim_{n\to\infty} \gamma(0) \sum_{h=-(n-1)}^{n-1} \left(1 - \frac{h}{n}\right)\rho(h)$$

$$= \sum_{-\infty}^{\infty} \rho(h),$$

where $\gamma(0)=\sigma^2$ and $\rho(h)$ is the autocorrelation function of the process under consideration (Fuller, 1976, pp.232).

If the spectral density, $f(w)$, of the process is continuous $\sum_{-\infty}^{\infty} \rho(h) = 2\pi f(0)$, where $f(0)$ is the value of the spectral density evaluated at $w=0$. Thus, the variance of the sample mean for large n can be approximated by $2\pi f(0)/n$. The spectral density of an AR(p) process evaluated at $w=0$ is

$$f(0) = [\prod_{j=1}^{p}(1 - 2m_j + m_j^2)^{-1}]\sigma_e^2/2\pi$$

where the m_j's are the roots of the characteristic equation and σ_e^2 is the variance of the error term. Upon simplification of the above expression and using the large sample result, variance of the sample mean of an AR(p) process can be approximated by

$$\sigma_{\bar{x}}^2 = \{\frac{1}{\prod_{j=1}^{p}(1-m_j)^2}\}\sigma_e^2/n. \tag{5.4}$$

In particular, for an AR(1) process, $m_1=\theta_1$ and $f(0) = [\frac{1}{2\pi(1-\theta_1)^2}]\sigma_e^2$. But $\sigma^2= \sigma_e^2/(1-\theta_1^2)$ and therefore

$$\therefore \sigma_{\bar{x}}^2=[(1-\theta_1^2)\sigma^2]/[(1-\theta_1)^2 n]$$
$$= [(1+\theta_1)/(1-\theta_1)]\sigma^2/n.$$

Similarly the variance of an AR(2) process is given by

$$\sigma^2= (1+m_1m_2)\sigma_e^2/[(1-m_1m_2)(1-m_1^2)(1-m_2^2)]$$

and

$$f(0) = \sigma_e^2/[2\pi(1-m_1)^2(1-m_2)^2]$$

where m_1 and m_2 are the roots of the characteristic equation.

So $\sigma_{\bar{X}}^2 = [(1-m_1m_2)(1+m_1)(1+m_2)\sigma^2]/[n(1+m_1m_2)(1-m_1)(1-m_2)]$.

The variance formula for sample mean given in (5.3) and the large sample results stated above in (5.4) are derived by assuming that X_t can be expressed as an infinite linear combination of the error terms. However, the variance of the sample mean can also be derived when X_t is expressed as a finite linear combination of the error terms. This can be easily demonstrated using an AR(1) process as follows.

Suppose $Y_t = \theta_1 Y_{t-1} + e_t$ where the e_t's are assumed to be independently identically normally distributed with zero mean and variance σ_e^2. Without loss of generality assume that $\mu=0$. When $Y_0 = 0$, Y_t can be expressed as

$$Y_t = \sum_{i=1}^{t} \theta_1^{t-i} e_i$$

and

$$\sum_{t=1}^{n} Y_t = (1+\theta_1+\theta_1^2+\ldots+\theta_1^{n-1})e_1 + (1+\theta_1+\theta_1^2+\ldots+\theta_1^{n-2})e_2 + \ldots + (1+\theta_1)e_{n-1} + e_n.$$

It can now be seen that $\sum_{t=1}^{n} Y_t$ is a linear combination of the e_t's. Thus, the expectation of $\sum_{t=1}^{n} Y_t$ is 0 and the variance of $\sum_{t=1}^{n} Y_t$ is given by

$$\sum_{i=1}^{n} [(1-\theta_1^i)/(1-\theta_1)]^2 \sigma_e^2.$$

Furthermore, since the e_t's are i.i.d. normal random variables it can be seen that the sample mean, $\bar{X}$, follows a normal distribution with zero mean and variance given by

$$\sum_{i=1}^{n} [(1-\theta_1^i)/(1-\theta_1)]^2 \sigma_e^2/n^2. \qquad (5.5)$$

In a similar manner, normality of the sample mean from an autoregressive process of any order, p, can be established.

In summary, we have

(i) the exact variance of $\bar{X}$:

$$\sigma_{\bar{X}}^2 = \sigma^2 a_1/n, \text{ where } a_1 = 1+2\sum_{h=1}^{n-1}(1-h/n)\rho(h),$$

(ii) a large sample approximation to the variance of $\bar{X}$:

$$\sigma_{\bar{X}}^2 = \sigma_e^2 a_2/n, \text{ where } a_2 = 1/[\prod_{j=1}^{p}(1-m_j)^2],$$

(iii) assuming $X_0 = 0$, variance of $\bar{X}$:

$$\sigma_{\bar{X}}^2 = \sigma_e^2 a_3/n \text{ where } a_3 = \sum_{i=1}^{n}[(1-\theta_1^i)/(1-\theta_1)]^2/n.$$

5.3 <u>Derivation of One Sided Sampling Plans</u>

The following standard notation will be used in this chapter.

p_1 = Acceptable quality level (AOQ),

p_2 = Lot tolerance percent defective (LTPD),

α = Producer's risk,

β = Consumer's risk,

U = Upper specification limit,

L = Lower specification limit.

As has been noted in section 5.2, the sample mean, $\bar{X}$, from an AR(p) process is approximately, normally distributed with mean, μ and variance given by (5.3) or (5.4) or $\bar{X}$ is exactly, normally distributed if X_0=0. Under this condition, for an AR(1) process, the sample mean is normally distributed with mean μ and the variance of the sample mean is given by (5.5). Methods for constructing acceptance sampling plans by variables when the quality measurements are assumed to be normally distributed are well documented in the standard literature on statistical quality control such as Duncan (1986) and Schilling (1982). Hence these procedures can be applied to obtain the acceptance sampling plans by variables in the presence of serial correlation. For this purpose several cases are identified.

5.3.1 Case I

(a) σ known

For one sided sampling plans the rule is to accept the batch if $\bar{X} - k\sigma \geq L$ or $\bar{X} + k\sigma \leq U$ where k is the acceptability constant. Using the normal distribution theory, the values of n and k can be found for given values of p_1, p_2, α, and β by solving the following equations:

$$\sqrt{n/a_3}(k-z_1) = -z_\alpha \tag{5.6}$$

and

$$\sqrt{n/a_3}(k-z_2) = z_\beta \tag{5.7}$$

Where z_1 and z_2 are standard normal scores defined in section 2.3.2 (Case I). Note that in this case the variance formula (5.5) is used because the sample mean is normally distributed. Thus, for an AR(1) process, solving (5.6) and (5.7) for k we get

$$k = (z_1 z_\beta + z_2 z_\alpha)/(z_\alpha + z_\beta). \tag{5.8}$$

However, two values for n are obtained depending whether equation (5.6) or (5.7) is used. A closed form solution for n is not possible and a trial and error method may be used to solve the equations (5.6) or (5.7) for

n. Equations (5.6), (5.7), and (5.8) are valid for both lower and upper specification limits.

The usual procedure described in standard text books on quality control is, first, find n using equations (5.6) and (5.7), and then use the solution of n to obtain k either from (5.6) or (5.7). In this case, because a_3 is a function of n, obtaining a solution for n is much more difficult. In order to avoid this difficulty a sampling plan may be obtained by using one point on the operating characteristic (OC) curve. In many situations this is done by fixing the producer's risk, α, at the acceptable quality level, p_1. We will then have one probability statement involving p_1 and α which is given by

$$P(\bar{X} - k\sigma' \geq L \mid p_1) = 1-\alpha$$

or

$$P(Z \leq \sqrt{n/a_3}(k-z_1)) = \alpha.$$

The solution for k for a given sample size, n, is then given by $k = z_1 - z_\alpha\sqrt{a_3/n}$.

<u>(b) σ known and large n</u>

In this case, any of the variance formulas (5.3), (5.4), or (5.5) can be used. But a closed form solution for n is not possible if the formula (5.3) and (5.5) are used in the derivation of the sampling plan. To avoid this difficulty, the variance formula (5.4) is used in this section.

The procedure discussed in the preceding section can then be applied to this case to obtain

$$\sqrt{n/a_2}(k-z_1) = -z_\alpha \tag{5.9}$$

and

$$\sqrt{n/a_2}(k-z_2) = z_\beta. \tag{5.10}$$

Then, the solution for n is given by

$$n = [(z_\alpha + z_\beta)/(z_2 - z_1)]^2 a_2. \tag{5.11}$$

Substituting n from (5.11) in (5.9) or (5.10), two values of k are then obtained. They are

$$k = z_1 - z_\alpha[(z_2 - z_1)/(z_\alpha + z_\beta)] \tag{5.12}$$

and

$$k = z_2 + z_\beta[(z_2 - z_1)/(z_\alpha + z_\beta)]. \tag{5.13}$$

In practice k is taken to be the average of the two values obtained from (5.12) and (5.13). As before these equations hold true for both lower and upper specification limits.

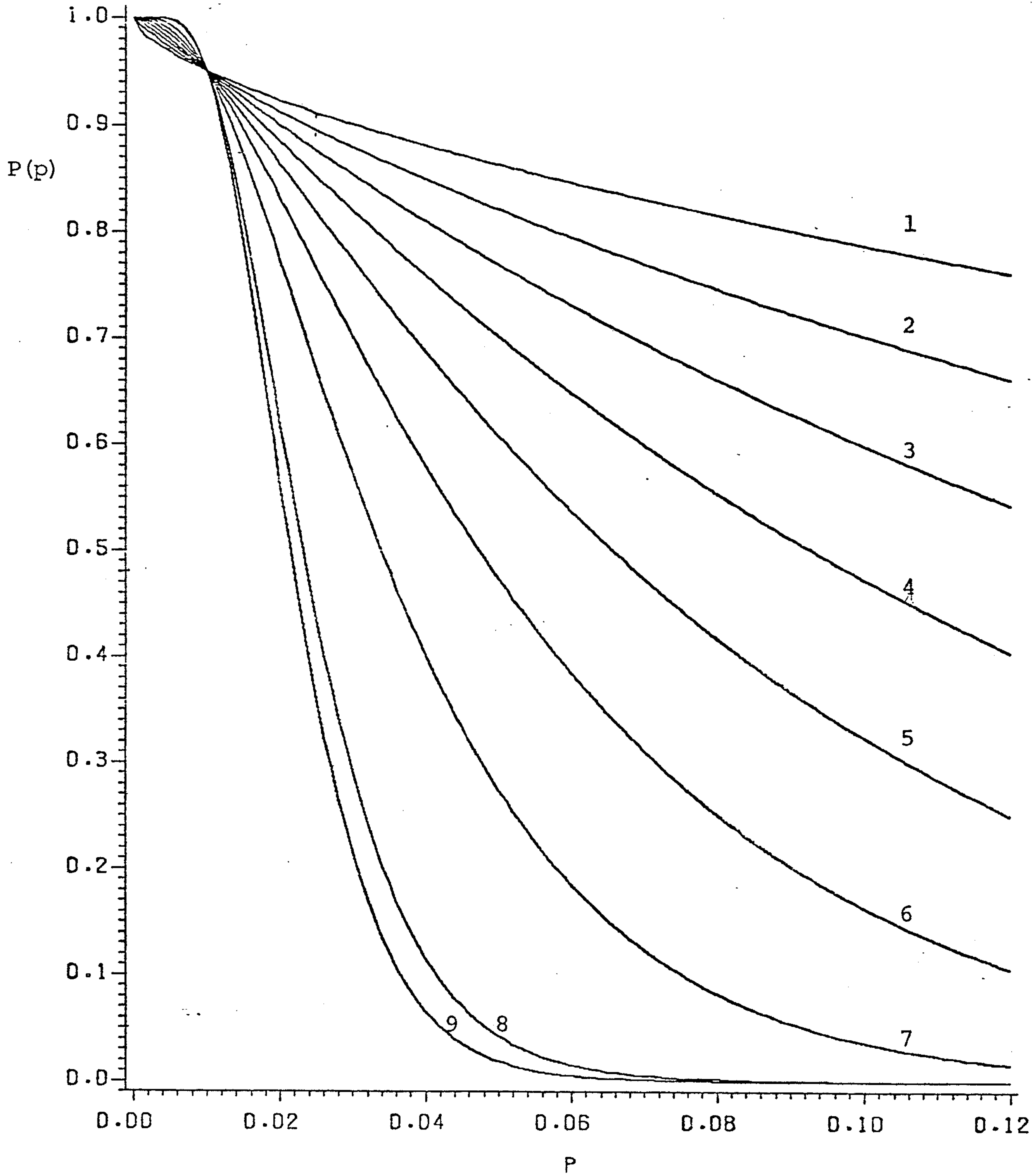
The effect of serial correlation on the acceptance sampling plans described for case I can be examined using operating characteristic (OC) curves. This is done by comparing the OC curve of a sampling plan when the data are independent with that of a corresponding OC curve of a sampling plan based on an AR process. The comparison of OC curves involves a comparison of probabilities of acceptance of lots for a given value of the proportion defective, p, falling outside the specification limits. Thus, if the two OC curves are similar, the effect of serial correlation on acceptance sampling plans can be considered negligible.

The OC curves are obtained by plotting the acceptance probabilities, P(p), against the various values of p. For case I the probability of acceptance, P(p), is given by

$1-\Phi(\sqrt{n/a_3}(k-z))$ where Φ is the distribution function of the standard normal distribution and z is such that $\int_z^{\infty}(1/\sqrt{2\pi})\exp(-x^2/2)dx = p$. When the observations are independent, the probability of acceptance is given by $1-\Phi(\sqrt{n}(k-z))$.

For case I (a), a set of OC curves for $n=30$, $\alpha=.05$, $p_1=.01$ and $\theta_1=.1$ to .8 are given in the Figure 5.1 together with the OC curve for independent case. The effect of serial correlation can be easily seen from this figure. It is observed that, for a given value of p as the value of the autocorrelation increases, the probability of acceptance also increases. For example, when $p=.04$ the probability of acceptance for independence case is .065. The corresponding probability of acceptance when $\theta_1=.2$ is .40 which is considerably larger than .065. This is true for all the values of θ_1 except when $\theta_1=.1$, in which case the OC curve is almost identical to the OC curve for independent case. This comparison indicates that if there is sufficient evidence to suggest the presence of serial correlation, then one should incorporate this knowledge of the serial correlation in the construction of an appropriate sampling plan.

For case I (b), the sampling plans have been obtained using two points on the OC curves. This is because a_2 does not involve n and consequently a closed form solution for n is possible. As in the previous case (Case I (a)) the probability of acceptance, P(p), is given by $1-\Phi(\sqrt{n/a_2}(k-z))$. But it is seen that $n=[(z_\alpha + z_\beta)/(z_2 - z_1)]^2 a_2$. Thus the



1-θ_1=.8, 2-θ_1=.7, 3-θ_1=.6, 4-θ_1=.5, 5-θ_1=.4, 6-θ_1=.3, 7-θ_1=.2, 8-θ_1=.1
9-θ_1=0.

Figure 5.1. Operating characteristic curves for Case I(a) with
n=30, p_1=.01 and α=.05.

probability of acceptance is given by $P(p)=1-\Phi[\{(z_\alpha+z_\beta)/(z_2-z_1)\}(k-z)]$. But note that $[(z_\alpha+z_\beta)/(z_2-z_1)]^2$ is the sample size required in the sampling plan when the observations are independent. Hence $P(p)=1-\Phi\sqrt{n}(k-z)$ which is identical to the independent case. That is, the OC curves for the two cases become identical and in consequence, for Case I (b), the effect of serial correlation on the acceptance sampling plans based on two points on the OC curve cannot be investigated by comparing the OC curves.

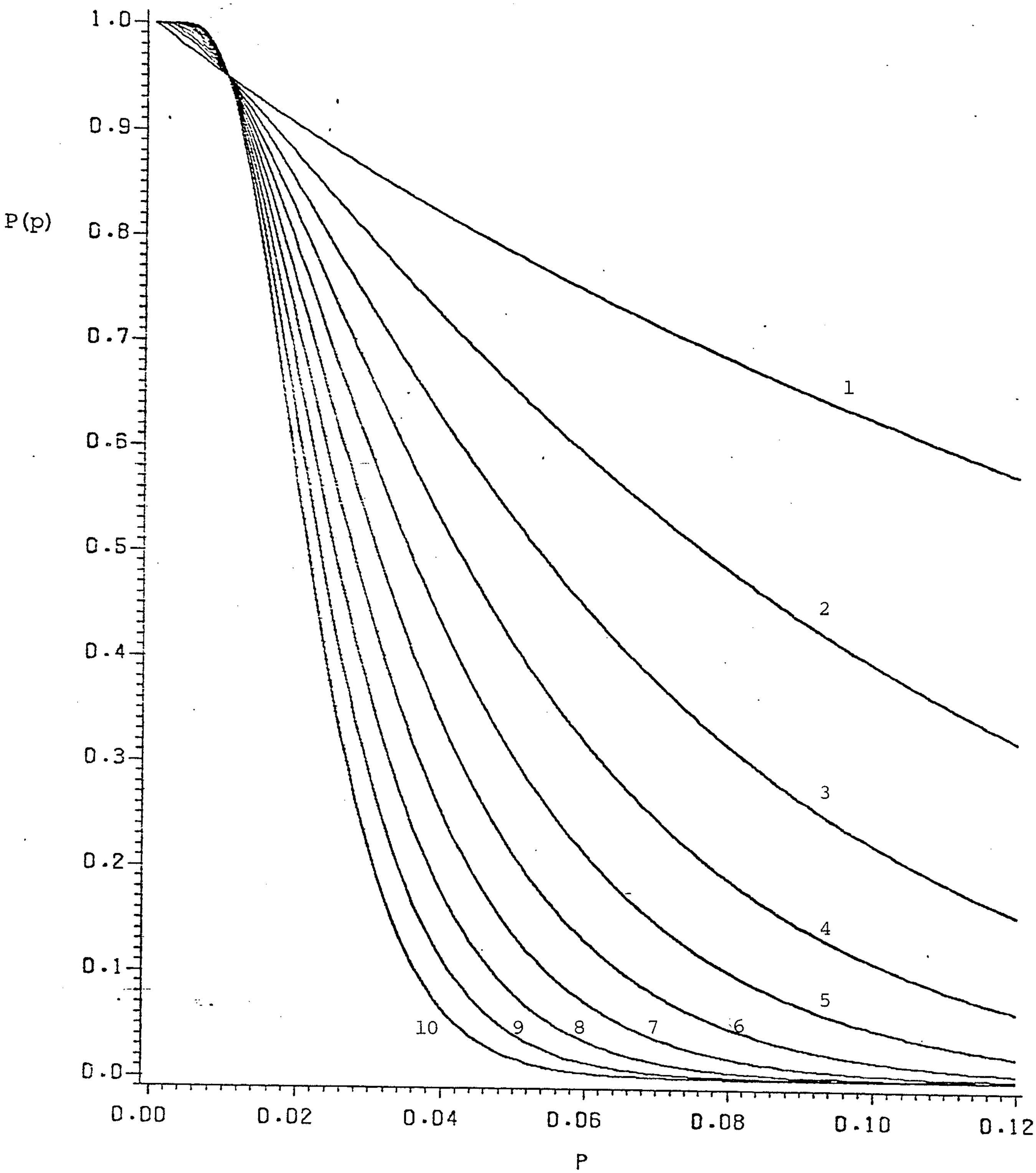
However, it has been shown that the necessary sample size for Case I (b) is $n=[(z_\alpha+z_\beta)/(z_2-z_1)]^2 a_2$ with $a_2=1$ when the serial correlation is not present. This gives us a tool to examine the effect of serial correlation for Case I (b). This is done by comparing the sample sizes for various values of θ_1 with the corresponding sample size for independent case. The following table gives the necessary sample sizes.

It can be seen from the following table of sample sizes, that for independent case a sample of size 10 is required, but for highly correlated data very large samples are required to achieve the same values of α, β, p_1 and p_2. For example, when $\theta_1=0.9$, a sample of size 19 times that for the independent is necessary.

Sample sizes for the sampling plan with $p_1=.01$, $p_2=.06$, $\alpha=.05$ and $\beta=.10$

Serial correlation	a_2	Sample size (n)
0.0	1.0000	10
0.1	1.2222	13
0.2	1.5000	15
0.3	1.8571	19
0.4	2.3333	24
0.5	3.0000	30
0.6	4.0000	40
0.7	5.6667	60
0.8	9.0000	90
0.9	19.0000	190

As has been indicated earlier, the effect of serial correlation cannot be examined for Case I(b) using OC curves when the sampling plan is derived using two points on the OC curve. However, if the sampling plan is constructed using only one point on the OC curve (i.e., p_1 and α), then the OC curves can be used to examine the effect of serial correlation. A set of such OC curves are given in Figure 5.2. Once again it can be seen that if a large serial correlation is involved, the erroneous assumption of independence may lead to more frequent rejection of lots when in fact such lots are of acceptable quality.



1-θ_1=.9, 2-θ_1=.8, 3-θ_1=.7, 4-θ_1=.6, 5-θ_1=.5, 6-θ_1=.4, 7-θ_1=.3, 8-θ_1=.2, 9-θ_1=.1, 10-θ_1=0.

Figure 5.2. Operating characteristic curves for Case I(b) with $n=30$, $p_1=.01$ and $\alpha=.05$.

5.3.2 Case II

(a) σ unknown,

In this case we consider the case where X_t is expressed as a finite linear combination of the error terms. In order to derive the sampling plans when σ^2 is unknown, the distribution of the statistics

$$t = (\bar{X}-\mu)/(\hat{\sigma}\sqrt{a_3/n}) \text{ and } t^* = (\bar{X}-\mu)/\hat{\sigma}^*\sqrt{a_3/n})$$

must be known. Here $\hat{\sigma}^2 = \sum_{t=1}^{n}(X_t-\mu)^2/n$ and $\hat{\sigma}^* = \sum_{t=1}^{n}(X_t-\bar{X})^2/n$ are the estimators of σ^2 when μ is known and unknown, respectively.

The distributional properties of t and t* are not well known for small samples. The distributional properties of t or t* can be studied by looking at the distribution of the sample mean. In order to investigate the distributional properties of the sample mean , when sampling from an AR(1) process with unknown variance 10,000 random samples of sizes n= 5,10,...,45,50 were generated for each of θ_1= .1, .2, .3, .4, .5, .6 , .7, .8, .9. The probabilities that the sample mean lies outside 3 standard deviation (3 s.d.) limits (i.e., $3\hat{\sigma}^*\sqrt{a_3/n}$) and 2 standard deviation (2 s.d.) limits (i.e., $2\hat{\sigma}^*\sqrt{a_3/n}$) were then computed. The results for various n and θ_1 combinations are given in Tables 5.1 and 5.2.

The accuracy of the simulation was checked by comparing the simulated probabilities with the theoretical probabilities for the normal distribution. It is well known that the probabilities of the sample mean from a normal distribution falling outside 3 s.d. and 2 s.d. limits are

0.0027 and 0.0456, respectively. The simulated probabilities for the normal distribution are also given in Tables 5.1 and 5.2, and they are quite close to the theoretical probabilities.

Examination of Table 5.1 leads to the following observations.

(1) When θ_1 coefficient is small (i.e., $\theta_1 \leq .2$) and for large n (i.e., $n \geq 20$) the probabilities are close to .0027. This indicates that the normal approximation for the sample mean based on large samples and small θ_1 is quite good, and the effect of serial correlation can be considered negligible. The procedure explained for Case I may then be used to obtain sampling plans for these values of θ_1 and n values by considering the observations to be independently normally distributed.

(2) For any other combination of θ_1 and n, the probabilities are not close to .0027. Hence a conjecture about the distributional properties of t or t* cannot be put forward for those combinations of θ_1 and n.

Examination of Table 5.2 indicates that almost all the probabilities are not close to 0.0456, which is the probability that the sample mean lies outside 2 s.d. limits when the underlying probability distribution is normal. Once again this indicates that t or t* for serially correlated data may not follow a normal distribution.

It has been shown that when $X_0=0$, $\bar{X}$ has a normal distribution. It is, therefore, reasonable to conjecture that t or t* have a t-distribution with (n-1) degrees of freedom. If this conjecture is true, then the probabilities that the sample mean lies outside 3 s.d. and 2 s.d. limits should be close to the theoretical probabilities obtained from a t-distribution with (n-1) degrees of freedom. These theoretical probabilities for various sample sizes are also given in Tables 5.1 and 5.2.

The theoretical probabilities from t-distribution with (n-1) degrees of freedom are always greater than the probabilities given in Table 5.1. This suggests that the distribution of t* may have a distribution with thinner tails than that of the t-distribution.

The same pattern does not exist in Table 5.2. For small n and θ_1 values, probabilities tend to be larger than the theoretical probabilities. But as n increases, this pattern disappears and for all the values of θ_1 (except for $\theta_1 = .1$ and .2) theoretical probabilities are larger.

From this simulation study, conclusive evidence about the distributional properties of t and t* based on small samples cannot be provided. As such the construction of even approximate sampling plans for serially correlated data when σ is unknown may not be feasible. Attempts should, therefore, be made to find the theoretical probability distribution of t and t* for small samples.

As has already been noted, for an AR process X_t can be expressed as an infinite linear combination of the e_t's. It is noted that the sample mean will be approximately normally distributed with mean, μ, and the variance of the sample mean is given by (5.3). In this case, the distribution of the sample mean for small samples is still unknown. This section of the paper examines the small sample properties of the sample mean when X_t follows an AR(1) process.

For an AR(1) process, $X_t = \sum_{i=0}^{\infty} \theta_1^i e_{t-i}$. In this case, it can be shown that the variance formula in (5.3) reduces to $\sigma_{\bar{X}}^2 = \sigma^2 a_1/n$ where $a_1 = [(1+\theta_1)/(1-\theta_1) - \{2\theta_1(1-\theta_1^n)/n(1-\theta_1)^2\}]$. Note that for large n, a_1 becomes $(1+\theta_1)/(1-\theta_1)$ which is the same as a_2 in section 5.2. Once again a simulation study is used to examine the behaviour of the sample mean. The probabilities that the sample mean lies outside 3 s.d. and 2 s.d. limits were obtained, and the results are given in Tables 5.3 and 5.4. The following observations can be made by examination of the results in Table 5.3.

When the coefficient θ_1 is small (i.e., $\theta_1 \leq .4$), almost all the probabilities are close to .0027 regardless of the sample size. This indicates that the normal approximation for the sample mean based on small samples is very good and the effect of serial correlation may be ignored if $\theta_1 \leq .4$. The same conclusion can be reached from the results given in Table 5.4. However, for the situation with $\theta_1 \geq .5$, the serial correlation cannot be ignored.

The effect of serial correlation can also be examined by comparing the producer's risk values, (α), for various levels of θ_1 with the α value for the independent case. For this purpose, the acceptability constant is derived for several sample sizes and α=.05, p_1=.01. The values of producer's risk for several n and θ_1 combinations are given in the following table.

Values of producer's risk (α)

$n\backslash\theta_1$	.1	.2	.3	.4	.5	.6	.7	.8	.9
15	.0671	.0866	.1085	.1329	.1599	.1896	.2224	.2582	.2964
20	.0673	.0873	.1098	.1349	.1628	.1938	.2285	.2674	.3107
30	.0677	.0881	.1111	.1369	.1656	.1978	.2342	.2762	.3257
40	.0678	.0884	.1117	.1379	.1670	.1997	.2369	.2804	.3332
50	.0679	.0887	.1122	.1384	.1678	.2009	.2386	.2828	.3376
60	.0680	.0888	.1124	.1388	.1684	.2016	.2396	.2844	.3405

The results of the above table indicate that except for θ_1=.1, all other α values are much larger than .05, which is the value of producer's risk for normally independently distributed data. This observation is true irrespective of the sample size. This table also indicates that values of α remains fairly constant for a fixed value of θ_1. This further indicates that the effect of the sample size is negligible and taking a large sample in the presence of serial correlation may not be helpful. Rather, serial correlation if presented should be incorporated in designing a proper acceptance sampling scheme.

(b) σ unknown and large n

It has been shown in section 2 that for large n, variance of the sample mean is approximated by $\sigma_{\bar{X}}^2 = a_2\sigma_e^2/n$ where $a_2=[1/\{\prod_{j=1}^{p}(1-m_j)^2\}]$. Furthermore, using large sample theory it can be shown that $\bar{X} \sim N(\mu,\sigma_e^2 a_2/n)$. Hence, applying the procedure described for Case I (b) one can obtain the sampling plan as

$$n = [(z_\alpha+z_\beta)/(z_2-z_1)]^2\ a_2$$
$$k = z_1 - z_\alpha[(z_2-z_1)/(z_\alpha+z_\beta)] \text{ or}$$
$$k = z_2 + z_\alpha[(z_2-z_1)/(z_\alpha+z_\beta)].$$

5.4 Sampling Plans for AR(1) and AR(2) Processes

In this section sampling plans for an AR(1) and AR(2) will be discussed to illustrate the application of the procedures given in the preceding section.

5.4.1 AR(1) Process

(a) σ known

When the process follows an AR(1) it has been shown that X_t can be expressed as a finite linear combination of the e_t's assuming $X_0=0$. Moreover, it can be shown that the sample mean is normally distributed with mean μ and variance $\sigma_{\bar{X}}^2= a_3\sigma_e^2/n$ where $a_3 = \sum_{i=1}^{n} [(1-\theta_1^i)/(1-\theta_1)]^2/n$. The variance of X_t (i.e., σ^2) and σ_e^2 are related by $\sigma_e^2 = (1-\theta_1^2)\sigma^2/(1-\theta_1^{2t})$. A reasonable approximation for σ_e^2 is $(1-\theta_1^2)\sigma^2$. Thus variance of the sample mean can be expressed as

$$\sigma_{\bar{X}}^2= [(1-\theta_1^2)\sigma^2 a_3]/n$$

$$= a^*\sigma^2/n$$

where $a^* = [(1+\theta_1)/(1-\theta_1)]\sum_{i=1}^{n}(1-\theta_1^i)^2/n$.

The procedure explained for Case I (a) may then be used to obtain the sampling plans.

(b) σ unknown, n large

It has been shown for large samples that $\sigma_{\bar{x}}^2 = [(1+\theta_1)/(1-\theta_1)]\sigma^2/n = a_2\sigma^2/n$ where $a_2 = (1+\theta_1)/(1-\theta_1)$. Then applying the formula (5.11) one can obtain

$$n = [(z_\alpha+z_\beta)/(z_2-z_1)][(1+\theta_1)/(1-\theta_1).$$

k is then obtained either from (5.9) or (5.10) which is not a function of a_2.

5.4.2 AR(2) Process

(a) σ known

Depending on the nature of the roots, m_1 and m_2, of the characteristic equation, Vasilopoulos and Stamboulis (1978) have derived the following formulas for the variance of the sample mean.

(i) m_1 and m_2 are real and distinct

$$\sigma_{\bar{x}}^2 = \sigma^2 a_1', \text{ where}$$

$$a_1' = \{\frac{m_1(1-m_2)^2}{(m_1-m_2)(1+m_1m_2)}\lambda(m_1,n) - \frac{m_2(1-m_1)^2}{(m_1-m_2)(1+m_1m_2)}\lambda(m_2,n)\}$$

and $\lambda(m,n) = (1+m)/(1-m) - [2m(1-m^2)/n(1-m)^2]$.

(ii) m_1 and m_2 are real and equal

$\sigma_{\bar{x}}^2 = \sigma^2 a_1''/n$, where

$$a_1'' = \{\frac{1+m}{1-m} + \frac{2m(1-m^2)}{n(1-m)^2}[\frac{1+(1+m)^2(1-m^n)-n(1-m^2)(1+m^n)}{1+m^2)(1-m^n)}]\}.$$

(iii) m_1 and m_2 are complex conjugates

$\sigma_{\bar{X}}^2 = \sigma^2 a_1'''/n$, where

$$a_1''' = \{Y(d,u) + 2d[(W(d,u,n) + Z(d,u,n)]/n\}$$

and
$$Y(d,u) = \frac{1-d^4 + 2d(1-d^2)\cos u}{(1+d^2)(1+d^2-2d\cos u)},$$

$$W(d,u,n) = \frac{2d(1+d^2)\sin u-(1+d^4)\sin 2u-d^{n+4}\sin(n-2)u}{(1+d^2)(1+d^2-2d\cos u)^2\sin u},$$

$$Z(d,u,n) = \frac{2d^{n+3}\sin(n-1)u-2d^{n+1}\sin(n+1)u+d^4\sin(n+2)u}{(1+d^2)(1+d^2-2d\cos u)^2\sin u},$$

$d^2 = -\theta_2$ and $u = \cos^{-1}(\theta_1/2d)$.

In this case, the sampling plan is found by first obtaining k using the equation (5.8) and then n is obtained either from (5.6) or (5.7). The appropriate values of a_1', a_1'' and a_1''' should be substituted in (5.6) or (5.7) depending on the nature of the roots, m_1 and m_2.

(b) σ known, n large

Using the theory of large samples it has been shown in section 5.2 that the variance of the sample mean for an AR(2) process is given by $\sigma_{\bar{X}}^2 = \sigma^2 a_2/n$ where

$$a_2 = \frac{(1-m_1m_2)(1+m_1)(1+m_2)}{(1+m_1m_2)(1-m_1)(1-m_2)}.$$

One can then obtain n from (5.11) and k may be obtained from either from (5.9) or (5.10).

Example 1

310 observations of hourly readings of viscosity of a chemical process is given in Box and Jenkins(1976, p.529). Identification techniques suggest that this data set follows an autoregressive process of order 1. Furthermore, the estimates of θ_1, σ^2, and σ_e^2 are .86, .362 and .093, respectively. Thus, the appropriate model becomes

$$Y_t = 1.32 + 0.86Y_{t-1} + e_t$$

where e_t's are assumed to be independently, normally distributed random variables with zero mean and variance, .093, and 1.32 is the estimate of the mean viscosity of the process (see Box and Jenkins, 1976, p.196). Since $\theta_1=.86<1$ this process is stationary with mean 1.32.

Now suppose that an appropriate sampling plan is required to examine whether the viscosity of the chemical is of an acceptable level. Let the maximum viscosity be 10 under certain conditions, and that 1% is considered satisfactory, and 6% is not. Let us take producer's risk, α, to be 0.05 and the consumer's risk, β, to be 0.10. We then have U=10, $p_1=.01$, $p_2=.06$, $\alpha=.05$ and $\beta=.10$. The corresponding standard normal scores are $z_1=z_{.01}=2.326$, $z_2=z_{.06}=1.555$, $z_\alpha=z_{.05}=1.645$, $z_\beta=z_{.10}=1.282$. The required sample size is then given by

$$n = [(z_\alpha+z_\beta)/(z_2-z_1)]^2[(1+\theta_1)/(1-\theta_1)]$$

$$= \left(\frac{1.645+1.282}{2.326-1.555}\right)^2 \left(\frac{1+.86}{1-.86}\right)$$

$$= (14.41)(13.29) = 192.$$

The two values of k are

$$k = z_1 - z_\alpha\left(\frac{z_2 - z_1}{z_\alpha + z_\beta}\right)$$

$$= 2.326 - 1.645\left(\frac{2.326-1.555}{1.645+1.282}\right)$$

$$= 2.326 - 1.645(.263)$$

$$= 1.893$$

and

$$k = 1.555 + 1.282(.263)$$

$$= 1.892.$$

The average of k is (1.893+1.892)/2=1.893. Hence the sampling plan is n=192, k=1.893. That is, a random sample of 192 observations is taken from the above process and the process is accepted if $\bar{X} + k\sigma \le U$.

Now suppose that the observations were independently normally distributed. We then have $n = \{(z_\alpha+z_\beta)/(z_2-z_1)\}^2 = 15$, and k=1.893 would remain the same.

In this example the observations from the chemical process are highly, serially correlated (θ_1 =.86) and one must take 13 times as many samples as the case when the data are assumed to be independent. It is also observed that, under the assumption of independence lots may be accepted more often than warranted.

<u>Example 2</u>

Box and Jenkins (1976) on page 32 of their book have reported a series of 70 consecutive yields from a batch chemical process. Once again the identification techniques suggest an autoregressive process of order 2 would fit the data. Estimates of the model coefficients are $\hat{\theta}_1$=-.32, $\hat{\theta}_2$=.18, $\hat{\mu}$=58.3 and $\hat{\sigma}^2$=139.80. The model could then be expressed as

$$Y_t = 58.3 - .32Y_{t-1} + .18Y_{t-2} + e_t.$$

Once again e_t's are assumed to be independently normally distributed with zero mean and variance σ_e^2. The corresponding characteristic equation is $m^2+.32m-.18=0$ and respective roots are m_1=.29345 and m_2=-.61345. Since both roots are less than one in absolute value the process is stationary.

Once again assume that an acceptance sampling plan is required for this process. Consider that the yield of the chemical process must be at least 55 units (i.e., L=55). Suppose that the proportions below 55 are taken as p_1= .005 and p_2= .05 with respective risks α=.05 and β=.10. Assume also that σ^2 is known to be 139.80. Then using the results in section 5.3, the required sample size is given by

$$n = [(z_\alpha+z_\beta)/(z_2-z_1)]^2 a_2$$

where $a_2 = [(1-m_1m_2)(1+m_1)(1+m_2)]/[(1+m_1m_2)(1-m_1)(1-m_2)]$.

Hence the sample size is

$$n = [(1.645+1.282)/(2.326-2.576)]^2(.63) = 87.$$

The values of k are

$$k = 2.326 - 1.645(.0854) = 2.186$$

and

$$k = 2.576 + 1.282(.0854) = 2.685.$$

The average value of k is 2.435. The required sampling plan is n=87, k=2.435. If one does not assume the data to be serially correlated the sampling plan is n=138, k=2.435.

In this example, it is seen that the erroneous assumption of independence would require 57% more samples than the case where data are serially correlated. Thus, in this case, investigation into the validity of the assumption of independence is worthwhile because of the reduction in sample size. This means savings in resources such as money, labour, time etc. This would be particularly important when the cost of testing is prohibitively high or in the case of destructive inspection.

It can be seen from the above examples that for highly correlated data the assumption of independence cannot be ignored. Therefore, engineers and quality control practitioners have to examine the presence of serial correlation before any attempt is made to construct sampling plans or control charts. Durbin-Watson statistic may be used to detect the dependence of the data, and if the serial correlation is found to be significant it should be incorporated in designing an appropriate sampling plan.

Table 5.1. The 3 s.d. tail probabilities of the sample mean for an AR(1) process - σ unknown.

$n\backslash\theta_1$	Normal	t	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	.0024	.0132	.0070	.0108	.0114	.0108	.0126	.0126	.0126	.0127	.0122
10	.0028	.0079	.0050	.0042	.0036	.0044	.0030	.0033	.0019	.0014	.0009
15	.0029	.0059	.0039	.0047	.0040	.0026	.0018	.0014	.0009	.0003	.0001
20	.0027	.0050	.0048	.0037	.0025	.0027	.0015	.0017	.0006	.0002	.0000
25	.0026	.0045	.0036	.0031	.0026	.0021	.0007	.0007	.0000	.0000	.0000
30	.0028	.0042	.0030	.0031	.0016	.0017	.0012	.0007	.0002	.0000	.0000
35	.0030	.0039	.0024	.0028	.0025	.0010	.0011	.0001	.0000	.0000	.0000
40	.0025	.0038	.0029	.0030	.0013	.0019	.0006	.0004	.0001	.0000	.0000
45	.0025	.0037	.0029	.0025	.0015	.0013	.0004	.0007	.0000	.0000	.0000
50	.0026	.0036	.0030	.0022	.0018	.0014	.0004	.0004	.0000	.0000	.0000

Table 5.2. The 2 s.d. tail probabilities of the sample mean from an AR(1) process - σ unknown

$n\backslash\theta_1$	Normal	t	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	.0455	.0574	.0771	.0807	.0925	.0880	.0899	.0966	.0958	.1026	.0943
10	.0456	.0495	.0617	.0609	.0538	.0565	.0496	.0463	.0367	.0297	.0212
15	.0464	.0486	.0535	.0542	.0506	.0443	.0390	.0306	.0254	.0164	.0093
20	.0451	.0474	.0534	.0508	.0486	.0431	.0341	.0285	.0185	.0100	.0032
25	.0464	.0475	.0458	.0469	.0403	.0343	.0301	.0218	.0137	.0070	.0022
30	.0468	.0472	.0485	.0476	.0388	.0349	.0289	.0200	.0115	.0040	.0060
35	.0457	.0469	.0476	.0469	.0430	.0348	.0286	.0196	.0120	.0036	.0009
40	.0462	.0468	.0499	.0464	.0381	.0363	.0279	.0195	.0117	.0027	.0001
45	.0472	.0446	.0481	.0479	.0363	.0343	.0279	.0182	.0101	.0028	.0001
50	.0445	.0465	.0494	.0460	.0380	.0334	.0260	.0192	.0076	.0025	.0002

Table 5.3

The 3 s.d. tail probabilities of the sample mean from an AR(1) process σ known

$n\backslash\theta_1$	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	.0020	.0023	.0020	.0010	.0003	.0003	.0000	.0000	.0000
10	.0024	.0024	.0025	.0019	.0011	.0005	.0003	.0000	.0000
15	.0024	.0025	.0026	.0024	.0017	.0014	.0006	.0001	.0000
20	.0031	.0035	.0030	.0024	.0020	.0009	.0011	.0004	.0000
25	.0028	.0031	.0032	.0025	.0019	.0016	.0009	.0005	.0000
30	.0027	.0027	.0024	.0024	.0025	.0011	.0019	.0005	.0000
35	.0027	.0025	.0023	.0025	.0019	.0016	.0018	.0010	.0000
40	.0025	.0030	.0022	.0025	.0011	.0024	.0021	.0011	.0004
45	.0023	.0026	.0026	.0029	.0024	.0020	.0020	.0006	.0000
50	.0025	.0027	.0030	.0026	.0021	.0019	.0011	.0009	.0002

Table 5.4

The 2.s.d tail probabilities of the sample mean from AR(1) process σ known

n/θ_1	.1	.2	.3	.4	.5	.6	.7	.8	.9
5	.0444	.0415	.0355	.0256	.0176	.0097	.0023	.0000	.0000
10	.0492	.0456	.0418	.0351	.0329	.0208	.0118	.0039	.0039
15	.0478	.0430	.0416	.0406	.0344	.0304	.0249	.0091	.0012
20	.0434	.0464	.0453	.0420	.0363	.0372	.0280	.0153	.0012
25	.0457	.0442	.0452	.0415	.0399	.0334	.0307	.0221	.0051
30	.0445	.0441	.0458	.0434	.0392	.0341	.0340	.0239	.0076
40	.0453	.0462	.0464	.0454	.0143	.0395	.0372	.0296	.0143
45	.0484	.0459	.0455	.0460	.0430	.0373	.0383	.0304	.0165
50	.0479	.0468	.0454	.0451	.0454	.0373	.0382	.0327	.0181

CHAPTER SIX

SAMPLING PLANS FOR BULK MATERIAL-
A BAYESIAN APPROACH

6.1 Introduction

Standard quality control techniques deal with the inspection of discrete units of products. A random sample of items is drawn from the lot and one or more quality characteristic(s) is measured on each item in the sample. The measurement may be an attribute (defective or non-defective) or a variable (length, time to failure, etc.). The information collected from the sample is used to construct the appropriate quality control charts or acceptance sampling plans. Such procedures are well-documented in standard statistical quality control texts and literature (Duncan, 1986; Schilling, 1982).

However, the product may consist of material in bulk form, such as a truck-load of cement, coal, milk or wool. Sampling from such products is known as bulk sampling. Bicking (1967) describes bulk sampling as follows:

" Bulk materials are essentially continuous and do not consist of populations of discrete, constant, identifiable unique units or items that may be drawn into the sample. Rather, the ultimate sampling units must be created, at the time of sampling by means of sampling device. The size and the form of the units depend upon the particular device employed, how it is used, the nature, condition, and structure of the material and other factors."

The bulk material may be liquid, solid or gaseous and the sample is constructed or created by some special scheme at the time of sampling. For example, if we are inspecting a shipment of coal or cartons of milk, the sampling units may be a shovel-full of coal or a carton of milk. Two types of bulk materials are recognized in industry:

Type A- Bulk materials that may not be readily subdivided into primary sampling units, such as coal, ore, paper, textile fibre, etc.

Type B- Bulk materials that may be subdivided into primary sampling units, such as bales of wool, bagged fertilizer, etc.

The primary objective of this chapter is to construct an appropriate sampling plan for Type B bulk material and extend the technique to the Type A case.

6.2 Model

Suppose a random sample of n_1 primary units, (such as n_1 bales from a shipment of wool) is drawn from a lot containing N primary units and n_2 measurements are made on each primary unit. Let X_{ij} be the jth measurement of the ith primary unit. We may model X_{ij} as

$$X_{ij} = \mu + \alpha_i + e_{ij} , \quad (6.1)$$

where μ = general process or lot mean,

α_i = between batches variability,

e_{ij} = random errors,

$i = 1, 2, \ldots, n_1$; $j = 1, 2, \ldots, n_2$.

We assume that e_{ij} and α_i are independently normally distributed with zero means and variances σ_1^2 and σ_2^2 respectively.

The acceptance sampling plans for the process mean μ are usually derived using the sample mean $\bar{X} = \Sigma\, \Sigma\, X_{ij}/n$. It is easy to show that

$$\sigma_{\bar{X}}^2 = (1 - n_1/N)\sigma_2^2/n_1 + \sigma_1^2/n; \; n = n_1 n_2.$$

If N is large (which it usually is), we may approximate Var($\bar{X}$) by

$$\sigma_{\bar{X}}^2 = \sigma_2^2/n_1 + \sigma_1^2/n \text{ (Cochran, 1963, pp.275).}$$

Consider with replacement sampling at both stages. Then using the fact that $\mathrm{Var}(\bar{X}) = E_1[V_2(\bar{X})] + V_1[E_2(\bar{X})]$ where E_1 and E_2 are the expectations taken at stage 1 and stage 2 and V_1 and V_2 are variances at stage 1 and stage 2, respectively. Recall that $\bar{X} = \{\sum_i \bar{X}_i\}/n_1$. Then

$$E_2(\bar{X}) = \{\sum_i E(\bar{X}_i)\}/n_1 = \{\sum_i \mu_i\}/n_1,$$

where μ_i is the mean of the ith primary selection.

Now $V_1[E_2(\bar{X})] = \{\sum_i \mathrm{Var}(\mu_i)\}/n_1^2 = \sigma_2^2/n_1$.

Also $V_2(\bar{X}) = \{\sum_i V_2(\bar{X}_i)\}/n_1^2 = \{\sum_i \sigma_{1i}^2/n_2\}/n_1^2$ where

$$\sigma_{1i}^2 = \{\sum_j (X_{ij} - \mu_i)^2\}/N.$$

Hence $E_1[V_2(\bar{X})] = \{\sum_i \sigma_{1i}^2/N\}/n_1n_2 = \sigma_1^2/n_1n_2$, where

$$\sigma_1^2 = \{\sum_i \sum_j (X_{ij} - \mu_i)^2\}/N.$$

Thus, $\sigma_{\bar{X}}^2 = \sigma_2^2/n_1 + \sigma_1^2/n_1n_2$.

The variances σ_1^2 and σ_2^2 are generally unknown and have to be estimated in order to construct an acceptance sampling plan. Quality control engineers and practitioners usually use the components of variation estimators $(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2)$ from the analysis of variance table. Let

$$\bar{X}_i = \sum_j X_{ij}/n_2,$$

$$S_1 = \sum_i \sum_j (X_{ij} - \bar{X}_i)^2,$$

$$S_2 = n_2 \sum_i (\bar{X}_i - \bar{X})^2,$$

$\upsilon_1 = n_1(n_2-1)$, $\upsilon_2 = n_1-1$, $m_1 = S_1/\upsilon_1$, $m_2 = S_2/\upsilon_2$.

The analysis of variance table for the model (6.1) is given by

ANOVA TABLE

Sources	Df	SS	MSS	E(MSS)
Between sampling units	υ_2	S_2	$m_2 = S_2/\upsilon_2$	$\sigma_1^2 + n_2\sigma_2^2$
Within sampling units (Error)	υ_1	S_1	$m_1 = S_1/\upsilon_1$	σ_1^2
Total	n_1n_2-1	$\sum_i \sum_j (X_{ij}-\bar{X})^2$		

(Box and Tiao, 1973).

Thus $\tilde{\sigma}_1^2 = m_1$, $\tilde{\sigma}_2^2 = (m_2 - m_1)/n_2$. Note that these are the moment estimators and are unbiased. However, if $m_1 > m_2$, $\tilde{\sigma}_2^2 < 0$ is unacceptable. To avoid this possibility we will obtain Bayes estimators of σ_1^2 and σ_2^2. Bayes procedure always gives acceptable estimators of the variance components because of the restrictions placed on the prior distributions, and this will be discussed in the next section.

6.3 Bayes Estimators

Let $(x_1, x_2, \ldots, x_n)$ be a random sample from a population characterized by the probability density function $f(x|\theta)$ where θ may be a real-valued or vector parameter. Let $g(\theta)$ be the prior distribution of θ and $L(\theta|\underline{x})$ be the likelihood function. Then by the famous theorem of Bayes (1763) we have

$$\Pi(\theta|\underline{x}) \propto L(\theta|\underline{x})\,g(\theta), \quad \theta \in \Omega. \tag{6.2}$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)$ and Ω is the range of θ.

In a state of in-ignorance, Jeffreys (1961) suggested the following rule for the prior distribution of θ:

(i) If $\Omega = (-\infty, \infty)$, assume θ to be uniformly distributed or $g(\theta)$ = constant,

(ii) If $\Omega = (o, \infty)$, assume $\log\theta$ to be uniformly distributed or $g(\theta) \propto 1/\theta$.

Jeffreys' rule often leads to a family of 'improper' priors, improper in the sense that $\int g(\theta)d\theta \neq 1$. Jeffreys contended that this is not a serious limitation since it is not intended to make a prior probability statement about θ. $g(\theta)$ is merely used as a weight function in the likelihood function to obtain the posterior distribution of θ.

Our parameters of interest are $\theta = (\sigma_1^2, \sigma_2^2, \mu)$, σ_1, $\sigma_2 > 0$, $-\infty < \mu < \infty$. The joint prior for $(\sigma_1^2, \sigma_2^2, \mu)$ may be written as

$$g(\sigma_1^2,\sigma_2^2,\mu) = p(\sigma_1^2,\sigma_2^2)h(\mu|\sigma_1^2,\sigma_2^2)$$
$$\approx p(\sigma_1^2,\sigma_2^2)h(\mu)$$

since prior information about the numerical values of σ_1^2 and σ_2^2 is not likely to affect one's prior belief about μ.

Assume σ_1^2 and σ_2^2 being independently distributed, Jeffreys' rule for prior leads to

$$g(\sigma_1^2,\sigma_2^2,\mu) \propto p_1(\sigma_1^2)p_2(\sigma_2^2)h(\mu)$$
$$\propto 1/\sigma_1^2\sigma_2^2. \tag{6.3}$$

It may be noted that the prior (6.3) belongs to the inverted gamma family

$$p(\sigma_i) \propto \{\exp(-a_i/\sigma_i^2)\}/(\sigma_i^2)^{b_i+1} \quad \text{when } a_i=b_i=0,\ i = 1,2.$$

It follows from (6.1) that the X_{ij}'s are normally distributed with mean zero and variance $\sigma_1^2 + \sigma_2^2$. After some algebra one obtains the likelihood function

$$\ell(r_1,r_2,\mu|x) \propto r_1^{-\upsilon_1/2}(r_1+n_2r_2)^{-(\upsilon_2/2+1)}$$
$$\cdot[\exp-1/2\{(\upsilon_1m_1/r_1)+(\upsilon_2m_2+n_1n_2(\bar{x}-\mu)^2)/(r_1+n_2r_2\}]$$

where $r_1 = \sigma_1^2$, $r_2 = \sigma_2^2$.

Combining the prior (6.3) with the likelihood we obtain the joint posterior distribution of $\theta = (r_1, r_2, \mu)$ as defined in (6.2) and we have

$$\Pi(r_1,r_2,\mu|x) \propto r_1^{-(\upsilon_1/2+1)}r_2^{-1}(r_1+n_2r_2)^{-(\upsilon_2/2+1)}$$
$$\cdot[\exp-1/2\{\upsilon_1m_1/2 + (\upsilon_2m_2+n_1n_2(\bar{x}-\mu)^2)/(r_1+n_2r_2\}].$$

We may integrate out (r_1, r_2, μ) in turn and derive the corresponding marginal posteriors. Under squared-error loss function, Bayes' estimator of θ is the posterior expectation of θ and the posterior expectations of (r_1, r_2, μ) involve complicated double integrals which cannot be expressed in closed forms. We will use instead a Bayesian approximation of the posterior expectations due to Lindley (1980) which does not require special tables or iterative process and is much easier to work with.

6.4 Bayesian Approximation

Lindley (1980) developed a multidimensional linear Bayes estimator of an arbitrary function as an approximation to an asymptotic expansion of the ratio of two integrals which cannot be expressed in a simple form. Lindley approximated the posterior expectation of an arbitrary function $u(\theta)$ by

$$E\{u(\theta)|\underline{x}\} = \frac{\int_\Omega u(\theta)v(\theta)\exp\{L(\theta)\}d\theta}{\int_\Omega v(\theta)\exp\{L(\theta)\}d\theta}$$

$$= [u+1/2\{\sum_i \sum_j (u_{ij}+2u_i\rho_j)\sigma_{ij} + \sum_i \sum_j \sum_k \sum_\ell L_{ijk}\sigma_{ij}\sigma_{k\ell}u_\ell\}]_{\hat{\theta}} + 0(1/n) \qquad (6.4)$$

which is the Bayes estimator of $u(\theta)$ under the squared-error loss function, and

$\theta = (\theta_1, \theta_2, \ldots, \theta_m)$,

$\hat{\theta}$ is the maximum likelihood estimator of θ,

$v(\theta)$ is the prior distribution function,

$\rho \equiv \rho(\theta) = \log v(\theta)$,

$u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$,

$L_{ijk} = \partial^3 L/\partial\theta_i\partial\theta_j\partial\theta_k$,

$\rho_j = \partial r/\partial\theta_j$,

σ_{ij} is the (i,j)th element in the inverse of the matrix $\{-L_{ij}\}$ and all functions are to be evaluated at $\hat{\theta}$.

An important property of the expansion (6.4) is that, at least for small samples, such an estimator is more efficient than the corresponding maximum likelihood counterpart in the sense that

$$\mathrm{Var}\{u(\theta)|\underline{x}\} < \hat{\mathrm{Var}}\{u(\hat{\theta})\}$$

(Sinha(1986)), where $\hat{\mathrm{Var}}$ implies estimated variance.

For m = 3, (6.4) reduces to

$$\begin{aligned} u^* &= E(u|\underline{x}) \\ &= u + (u_1a_1+u_2a_2+u_3a_3+a_4+a_5) +1/2[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ &\quad + B(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}) + C(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33})], \end{aligned} \tag{6.5}$$

where all functions are evaluated at the maximum likelihood estimates $(\hat{r}_1, \hat{r}_2, \hat{\mu})$ and

$$a_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13}$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23}$$

$$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33}$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}$$

$$a_5 = 1/2(u_{11}\sigma_{11} + u_{22}\sigma_{22} +u_{33}\sigma_{33})$$

$$A = \sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{23}L_{231}+\sigma_{22}L_{221}+\sigma_{33}L_{331}$$

$B = \sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{23}L_{232}+\sigma_{22}L_{222}+\sigma_{33}L_{332}$

$C = \sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{23}L_{233}+\sigma_{22}L_{223}+\sigma_{33}L_{333}$

(Sinha and Sloan,1985).

For the prior in (6.3) $\rho_1 = -1/r_1$, $\rho_2 = -1/r_2$, $\rho_3 = 0$. To obtain the Bayes estimator of r_1, put $u = r_1$, $u_1 = 1$, $u_2 = u_3 = 0$, $u_{ij} = 0$, $i,j = 1,2,3$, $a_4 = a_5 = 0$, $a_1=-(\sigma_{11}/r_1+\sigma_{12}/r_2)$. Substituting in (6.5)

$$r_1^* = E(r_1|\underline{x})=[r_1-(\sigma_{11}/r_1+\sigma_{12}/r_2)+1/2(A\sigma_{11}+B\sigma_{12}+C\sigma_{31})]_{\hat{\theta}}$$

Similarly

$$r_2^* = [r_2-(\sigma_{21}/r_1+\sigma_{22}/r_2)+1/2(A\sigma_{21}+B\sigma_{22}+C\sigma_{23})]_{\hat{\theta}}$$

$$\mu^* = [\mu-(\sigma_{31}/r_1+\sigma_{32}/r_2)+1/2(A\sigma_{31}+B\sigma_{32}+C\sigma_{33})]_{\hat{\theta}}$$

where $\hat{\theta} = (\hat{r}_1, \hat{r}_2, \hat{\mu})$.

For the posterior variance of r_1^*, we have

$$\mathrm{Var}(r_1|\underline{x}) = E(r_1^2|\underline{x}) - \{E(r_1|\underline{x})\}^2.$$

Let $u = r_1^2$, $u_1 = 2r_1$, $u_{11} = 2$, $u_2 = u_3 = 0$, $u_{ij} = 0$, $i,j = 2,3$, $a_4 = 0$ and $a_5 = \sigma_{11}$.

$E(r_1^2|\underline{x}) = r_1^2-2r_1(\sigma_{11}/r_1+\sigma_{12}/r_2)+r_1(A\sigma_{11}+B\sigma_{12}+C\sigma_{13})+\sigma_{11}$ and

$\mathrm{Var}(r_1|\underline{x}) = \sigma_{11}-[1/2(A\sigma_{11}+B\sigma_{12}+C\sigma_{13})-(\sigma_{11}/r_1+\sigma_{12}/r_2)]^2.$

Similarly

$$\mathrm{Var}(r_2|\underline{x}) = \sigma_{22} - [1/2(A\sigma_{12}+B\sigma_{22}+C\sigma_{23})-(\sigma_{12}/r_1+\sigma_{22}/r_2)]^2,$$

$$\mathrm{Var}(\mu|\underline{x}) = \sigma_{33} - [1/2(A\sigma_{31}+B\sigma_{32}+C\sigma_{33})-_{13}/r_1+\sigma_{23}/r_2)]^2. \quad (6.6)$$

Again, all functions are to be evaluated at the maximum likelihood estimates

$$\hat{r}_1 = m_1,\ \hat{r}_2 = [(\frac{n_1-1}{n_1})m_2 - m_1]/n_2,\ \hat{\mu} = \bar{x} \quad \text{if } (\frac{n_1-1}{n_1})m_2 > m_1$$

and $\hat{r}_1 = (\upsilon_1 m_1 + \upsilon_2 m_2)/n_1 n_2$, $\hat{r}_2 = 0$, $\hat{\mu} = \bar{x}$ if $(\frac{n_1-1}{n_1})m_2 < m_1$ (Herbach, 1959).

If $(\frac{n_1-1}{n_1})m_2 < m_1$ and we accept $\hat{r}_2 = 0$ as the maximum likelihood estimator of r_2, then $\rho_2 = -1/\hat{r}_2 = \infty$ and Lindley's expansion will not work. In such a case, one may use Jeffreys' invariant prior

$$g(r_1, r_2) \propto \frac{1}{r_1(r_1 + n_2 r_2)} \tag{6.7}$$

(Box and Tiao, 1973) and the corresponding Bayes estimators are given by

$$r_1^* = [r_1 - \{\frac{\sigma_{11}(2r_1 + n_2 r_2) + n_2\sigma_{12}r_1}{r_1(r_1 + n_2 r_2)}\} + 1/2(A\sigma_{11} + B\sigma_{12} + C\sigma_{13})]_{\hat{\theta}}$$

$$r_2^* = [r_2 - \{\frac{\sigma_{21}(2r_1 + n_2 r_2) + n_2\sigma_{22}r_1}{r_1(r_1 + n_2 r_2)}\} + 1/2(A\sigma_{21} + B\sigma_{22} + C\sigma_{23})]_{\hat{\theta}}$$

$$\mu^* = [\mu - \{\frac{\sigma_{31}(2r_1 + n_2 r_2) + n_2\sigma_{32}r_1}{r_1(r_1 + n_2 r_2)}\} + 1/2(A\sigma_{31} + B\sigma_{32} + C\sigma_{33})]_{\hat{\theta}}.$$

Posterior variances of these estimators are given by

$$\mathrm{Var}(r_1|\underline{x}) = \sigma_{11} - [1/2(A\sigma_{11} + B\sigma_{12} + C\sigma_{13}) - \{\frac{(2r_1 + 2r_2)\sigma_{11} - r_1 n_2 \sigma_{12}}{r_1(r_1 + n_2 r_2)}\}]^2$$

$$\mathrm{Var}(r_2|\underline{x}) = \sigma_{22} - [1/2(A\sigma_{21} + B\sigma_{22} + C\sigma_{23}) - \{\frac{(2r_1 + n_2 r_2)\sigma_{21} - r_1 n_2 \sigma_{12}}{r_1(r_1 + n_2 r_2)}\}]^2$$

$$\mathrm{Var}(\mu|\underline{x}) = \sigma_{33} - [1/2(A\sigma_{31} + B\sigma_{32} + C\sigma_{33}) - \{\frac{(2r_1 + n_2 r_2)\sigma_{31} - r_1 n_2 \sigma_{33}}{r_1(r_1 + n_2 r_2)}\}]^2.$$

6.5 Numerical Examples

(a) In order to estimate the average percent potassium bitartrate content in a shipment of argol, ten argol bags were randomly selected and two measurements of percent potassium bitartrate content from each bag were obtained (Tanner and Lerner, 1951). In this case, σ_1^2 represents the "within bag" variability and σ_2^2 is the "between bag" variability. The following is the summary of various computations:

$m_1 = .3124$, $m_2 = .7385$, $\bar{X} = 89.292$, $n_1 = 10$, $n_2 = 2$, $\upsilon_1 = 10$, $\upsilon_2 = 9$.

$$\sigma_{ij} = \begin{pmatrix} .0195188 & -.0097594 & 0 \\ -.0097594 & .0269677 & 0 \\ 0 & 0 & .0332325 \end{pmatrix}$$

$L_{111}=724.10584$, $L_{222}=544.9303$, $L_{333}=0$, $L_{112}=136.23257$, $L_{221}=272.46515$, $L_{331}=45.27349$, $L_{113}=0$, $L_{223}=0$, $L_{332}=136.23257$, $L_{123}=0$.

$A = 20.32689$, $B = 16.56379$, $C = 0$.

The maximum likelihood estimates are $\hat{r}_1=.3124$, $\hat{r}_2=.176125$, $\hat{\mu}=89.292$.

The Bayes estimators and their posterior variances have been computed for the prior distributions given in (6.3) and (6.7). The results are summarized in Table 6.1. In this table under method of estimation (1) and (2) refers to the prior distributions given in (6.3) and (6.7), respectively. The estimates of variances are given in the parentheses.

Table 6.1

Estimates of σ_1^2 and σ_2^2

Method of Estimation

Parameter	MLE	Moment	Bayes (1)	Bayes (2)
σ_1^2	.3124	.3124	.4229	.3675
	(.0195)	(.0195)	(.0073)	(.0165)
σ_2^2	.1761	.2131	.1784	.2651
	(.0269)	(.0352)	(.0268)	(.0191)

Examination of the results given in the Table 6.1 indicates that the posterior variances of $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are less than the correponding maximum likelihood counterpart. This verifies Sinha's (1986) result that $\mathrm{Var}(u(\theta)) < \hat{\mathrm{Var}}(u(\hat{\theta}))$.

Now consider constructing an appropriate sampling plan for the acceptance/rejection of the shipment of argol. For this purpose we use posterior variance estimates of σ_1^2 and σ_2^2 based on the prior distribution (6.3) to estimate the variance of the sample mean. Thus

$$\overset{*}{\sigma}{}^2_{\bar{x}} = \frac{.0268}{10} + \frac{.0073}{10(2)}$$

$$= .00268 + .000365 = .003045.$$

The estimated standard deviation of the sample mean $=\sqrt{.003045}$ $= .055182$.

Now assume that the shipment of argol should contain at most 90 percent of potassium bitartrate (i.e., the upper specification limit is 90 percent). The value of the test statistic is $Z = (90 - 89.292)/(.055182) = 12.83$. Let producer's risk, α, be .05 and $z_{.05} = 1.645$. Since $12.83 > 1.645$ the shipment of argol is considered unacceptable.

(b) In this example we illustrate the estimation of the components of variance in (6.1) when $m_2 < m_1$. Box and Tiao (1973, pp.247) generated 30 observations from the model (6.1) such that $n_1 = 6$, $n_2 = 5$, $\sigma_1 = 16$, $\sigma_2 = 4$. The various computations associated with this example are:

$m_1 = 14.9459$, $m_2 = 8.3363$, $\bar{X} = 5.6656$, $\upsilon_1 = 24$, $\upsilon_2 = 5$.

$$\sigma_{ij} = \begin{pmatrix} 11.972876 & -2.3945876 & 0 \\ -2.3945876 & 5.8346566 & 0 \\ 0 & 0 & .4448588 \end{pmatrix}$$

$L_{111} = .0414905$, $L_{222} = .1771732$, $L_{333} = 0$, $L_{112} = .0070869$, $L_{221} = .0354347$, $L_{331} = .1684273$, $L_{113} = 0$, $L_{223} = 0$, $L_{332} = .0630998$, $L_{123} = 0$.

$A=.7444959$, $B=.9769629$, $C = 0$. Note that in this case $m_2 < m_1$ and thus the maximum likelihood estimates of σ_1^2 and σ_2^2 are 13.3461 and 0, respectively.

As indicated earlier Lindley's expansion does not work for the prior distribution in (6.3) and as such Bayes estimators of σ_1^2 and σ_2^2 are obtained by using Jeffreys' invariant prior in (6.7). The results are given in

Table 6.2. Once again the corresponding estimated variances of variance components are given in the parentheses.

Table 6.2

Estimates of σ_1^2 and σ_2^2

Parameter	Actual	MLE	Bayes
σ_1^2	16	13.3461	15.7362
		(11.9729)	(6.2605)
σ_2^2	4	0	.1317
		(5.8347)	(5.8173)

We note that the posterior estimates are close to the actual values than the corresponding maximum likelihood estimates. The posterior variances are also less than the corresponding maximum likelihood counterpart.

The above discussion of estimation of variance components relate to the situation where the sampling units (primary sampling units) of the bulk material are easily identifiable (i.e., for Type B bulk material). The same estimation procedure can be extended to Type A bulk material. In this case, the primary sampling units have to be artificially created. For an example suppose the percentage of butter fat content in a truck load of milk is required. In this situation there are no apparent first stage sampling units, but several bottles (n_1) of the same size can be considered as primary sampling units. One can then obtain n_2 subsamples from each sampled bottle and the model (6.1) can be applied to this situation as well. The model and the estimation procedure are both unchanged.

CHAPTER SEVEN

BAYESIAN CONTROL CHARTS FOR BINOMIAL P

7.1 Introduction

The construction of control charts for a binomial proportion, p, has been discussed in section 2.5. This procedure requires that the sample proportion defective, x/n, where x is the number of defective items in a sample of size n to be approximately normally distributed with mean, p, and variance p(1-p)/n. Furthermore, p is assumed to be a constant and does not differ from lot to lot. However, according to Mood's theorem given in Chapter 1, it has been noted that inference concerning a lot when p is fixed becomes inappropriate. Deming (1982) and Schilling (1985) have noted that if p varies from lot to lot, the quality of outgoing lots can be improved by applying a suitable sampling plan.

The assumption of constant p has been questioned by Hald (1982, pp. 129-131). He has presented several examples to illustrate that p does not remain fixed in practice. In fact, Hald's examples show that p has a beta distribution and it is suggested that this distribution of p should be incorporated in setting an appropriate control chart for p. The control charts of the form $\bar{p} \pm 3s_p$ has been suggested by Hald where $\bar{p} = \sum p_i/k$, and p_i is the proportion defective in the sample i, k is the number of samples each of size n, s_p is the standard deviation of p_i's. He further recommends the use of simple graphical techniques (e.g., histogram) to identify the probability distribution of p.

One statistical method to incorporate the variability in p is to use the Bayesian approach assuming p has a certain prior probability distribution. Hald (1982) has used this approach to develop acceptance sampling by attributes assuming p has a beta prior distribution. However, the Bayes procedure has not yet been applied to the construction of control charts for p. In this chapter, the Bayesian procedure will be used to develop control charts for p. Credible intervals and highest posterior density (HPD) intervals for p will also be given. The Bayesian control charts can then be used instead of traditional control charts based on the assumption that p is fixed. These intervals can also be used in quality assurance plans as proposed by Hahn (1969, 1970).

In many quality control situations, it is also important for a quality control practitioner to know how a process would perform in the future so that necessary precautions can be taken if there is evidence to indicate that the process will be out of control in the near future. This can be achieved by predicting a future value of proportion defective, p, based on the information contained in the previous samples. But prediction of a future observation and the construction of prediction intervals have not been considered by many statisticians and practitioners. Hahn (1969, 1970) and Whitmore (1986) have considered prediction limits when sampling from a normal population. The factors for calculating prediction intervals of the form

$\bar{X} \pm rs$ have been given by Hahn. In this case, $\bar{X}$ is the sample mean, s is the sample standard deviation and r is a constant which depends on the sample size and Hahn (1969) has tabulated r for several values of the sample size. He also suggests that prediction intervals can be effectively used for process control purposes such as control limits and performance limits in quality assurance charts.

The above discussion of prediction limits by Hahn (1969, 1970) and Whitmore (1986) is related to quality variables which are normally distributed. However, prediction and prediction interval construction for proportion defective, p, has not been completely investigated. In this chapter, the classical and Bayesian prediction intervals for p will be developed using the well known predictive distribution approach. Furthermore, robustness of the posterior distribution and the credible intervals to departures from a specified prior distribution is also examined in this chapter.

7.2 Prior Distributions for p

Several prior distributions for p have been proposed in the literature. One of the widely used priors for p is the Jeffreys' invariant prior. Under state of in-ignorance Jeffreys (1961) proposes the non-informative prior

$$g(p) \propto |I(p)|^{1/2}$$

where I(p) is the Fisher's information based on a single observation. If the probability function of x for fixed p is given by

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,\ldots,n, \tag{7.1}$$

then I(p) for (7.1) is given by

$$|I(p)|^{1/2} = |-E\{\partial^2 \log f(x|p)/\partial^2 x^2\}|^{1/2} = 1/\sqrt{p(1-p)}$$

(Box and Tiao (1973), pp.41-42; Martz and Waller (1982), pp.224). Thus Jeffreys' non-informative prior distribution for p is
given by

$$g(p) \propto 1/\sqrt{p(1-p)}, \quad 0 < p < 1. \tag{7.2}$$

Assume that k random samples, each of size n, are available. Let x_i be the number of defective items in the ith sample. The total number defective in k samples is given by $z=\sum x_i$. Then combining the likelihood function with the prior given in (7.2), the posterior distribution for p is obtained as

$$\begin{aligned} \Pi(p|x) &\propto L(x|p)\ g(p) \\ &\propto p^{z-.5}\ (1-p)^{nk-z-.5} \\ &= K\ p^{z-.5}\ (1-p)^{nk-z-.5}, \quad 0 < p < 1. \end{aligned} \tag{7.3}$$

where K is the appropriate normalizing constant. It is easy to see that $K = 1/\beta(z+.5,\ nk-z+.5)$ where β is the well known beta function.

Another widely used prior distribution for p is the beta distribution of the first kind (Hald (1982), pp.130-133; Martz and Waller (1982) , pp.226). Its probability density function is given by

$$g(p) = \frac{1}{\beta(a,b)}\ p^{a-1}\ (1-p)^{b-1} \tag{7.4}$$

$0 < p < 1$, $a,b > 0$. The shape parameters of the distribution in (7.4) are a and b. Once again combining the likelihood with the prior (7.4), the posterior for p is given by

$$\Pi(p|z) = K^{*}p^{a+z-1}(1-p)^{nk+b-z-1} \tag{7.5}$$

where $K^{*} = 1/\beta(a+z,\ nk+b-z)$.

It may be noted that Jeffreys' prior belongs to the general family of distributions given in (7.4) when a = .5 and b = .5. Several other prior distributions for p have been proposed by Martz and Waller (1982 , pp.256-280).

The proportion defective p can then be estimated using the posterior distribution (7.5). Under squared error loss function, the Bayes estimator of p is given by the posterior expectation of p. For posterior distribution (7.5), the Bayes estimator of p is then given by

$$p^{*} = E(p|z) = (a+z)/(nk+a+b).$$

For Jeffreys' prior the posterior expectation of p is given by $(z+.5)/(nk+1)$.

It should be noted that in order to obtain the Bayes estimate of p using the beta prior given in (7.4), the parameters a and b of the prior distribution must be known, or they have to be estimated using data from previous samples. Usually, a and b are estimated by the method of moments as follows. Let $p_i = x_i/n$ be the estimate of the proportion defective of the sample i, where x_i is the number of defectives in the ith sample. Then $\bar{p} = \Sigma\, p_i/k$ is the mean proportion defective for k such samples. The sample variance of the p_i's is given by $s_p^2 = \Sigma\,(p_i - \bar{p})^2/(k-1)$.

The moment estimators of a and b are then given by

$$\hat{a} = n\bar{p}(\overline{pq} - s_p^2)/(ns_p^2 - \overline{pq}) \quad \text{and}$$

$$\hat{b} = \hat{a}(1-\bar{p})/\bar{p}$$

(Hald (1982, p. 130)). One can now use these estimates of a and b to obtain the Bayes estimate of p.

It should be noted that for a given sample, the above procedure may lead the estimate of a to be negative. To avoid this difficulty one may use the prior distribution proposed by Hartigan (1964) (i.e., $1/[p(1-p)]$), Jeffreys or uniform prior because these prior distributions do not have any parameters to be estimated.

7.3 Bayes Intervals for p

Consider an interval (p_1, p_2) such that

$$\int_{p_1}^{p_2} \Pi(p|z)\,dp = 1-\alpha. \tag{7.6}$$

It is called a $100(1-\alpha)\%$ percent credible interval (Edwards, Lindman, Savage (1963)). An equal tail $100(1-\alpha\%)$ percent credible interval (p_1, p_2) for p is an interval such that

$$\int_0^{p_1} \Pi(p|z)\,dp = \int_{p_2}^{1} \Pi(p|z)\,dp = \alpha/2.$$

An interval (p_1, p_2) minimizing the width $w=(p_2-p_1)$ subject to (7.6) and that requires $\Pi(p_1|z) = \Pi(p_2|z)$ is called a shortest credible interval (Sinha and Howlader, 1983).

One can also consider a Bayes interval for p given in (7.6) in such a way that the posterior probability for every point inside the interval be greater than that for every point outside. Consequently, such an interval includes values of the parameter which are more probable and excludes less probable ones. This type of interval is known as highest posterior density (HPD) interval (Box and Tiao, 1973).

Furthermore, if the posterior density function is unimodel but not necessarily symmetric, the shortest credible interval and the HPD interval are identical. If the posterior is unimodel and symmetric, shortest and equal tail credible intervals and HPD interval are the same (Sinha and Howlader, 1983).

The $100(1-\alpha)\%$ percent equal tail credible interval for p, based on the beta prior distribution can be computed easily using SAS FUNCTION BETAINV or IMSL SUBROUTINE MDBETA. If one does not have access to above the programs, p_1 and p_2 may also be obtained using the relationship

$$F = (nk+b-z)p/[(1-p)(a+z)]$$

where F has a F-distribution with 2a+2z and 2nk+2b-2z degrees of freedom (Martz and Waller, 1982, p.259). Hence p_1 and p_2 are given by

$$p_1 = (a+z)F(r,s;\alpha/2)/[(a+z)F(r,s;\alpha/2)+(nk+b-z)]$$

and

$$p_2 = (a+z)F(r,s;1-\alpha/2)/[(a+z)F(r,s;1-\alpha/2)+(nk+b-z)]$$

where $r=2a+2z$, $s=2nk+2b-2z$ and $F(r,s;\alpha)$ is the αth percentile of the F-distribution with r and s degrees of freedom. Note that for Jeffreys' prior $a=b=.5$ and $r=1+2z$, $s=2nk+1-2z$.

The posterior distribution given in (7.5) is unimodel and therefore the shortest credible and HPD intervals are identical. Thus, HPD interval for p can be obtained by the simultaneous solution of

$$P(p_1 < p < p_2) = 1-\alpha$$

and

$$\Pi(p_1 | z) = \Pi(p_2 | z)$$

from which one can obtain

$$(p_1/p_2)^{a+z-1} = [(1-p_2)/(1-p_1)]^{nk+b-z-1}$$

and

$$\int_{p_1}^{p_2} \frac{1}{\beta(a+z,nk+b-z)} p^{a+z-1}(1-p)^{nk+b-z-1} dp = 1-\alpha.$$

HPD interval for p can then be obtained by solving the above equations for p_1 and p_2. Note that the solution of these equations involves the incomplete beta function and therefore a closed form solution for p_1 and p_2 is not possible. An iterative procedure may be used to obtain the HPD interval for p.

Bayes control charts similar to classical p-charts can also be constructed. It can be shown that the expectation of p^* is a/(a+b) and the variance of p^* is

$$nab/[(a+b)^2(n+a+b)(a+b+1)].$$

Thus, 3 standard deviation (3 s.d.) control limits are given by

$$\text{upper control limit} = p^* + 3\sqrt{nab/[(a+b)^2(n+a+b)(a+b+1)}$$

and

$$\text{lower control limit} = p^* - 3\sqrt{nab/[(a+b)^2(n+a+b)(a+b+1)]}.$$

The 2 s.d. limits are similar. One must note that the above limits are not strictly Bayesian control limits but they are the Bayesian equivalent to classical control limits.

7.4 Prediction Intervals for p

Consider the problem of estimating a future value of p based on a pervious sample of n observations. Let y be the number of defective items in a future sample of size m. Then $\hat{p}=x/n$ may be used as a predictor of $p'=y/m$. Note that $\hat{p}$ is an unbiased predictor for p' in the sense that $E(p'-\hat{p})=0$. The variance of $(p'-\hat{p})$ is $p(1-p)[1/m+1/n]$. Assuming $(p'-\hat{p})$ to be approximately normally distributed with zero mean and variance $p(1-p)[1/m+1/n]$, $100(1-\alpha)\%$ prediction limits for p are given by

$$\text{lower prediction limit} = \hat{p} - z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})[1/m+1/n]}$$

$$\text{upper prediction limit} = \hat{p} + z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})[1/m+1/n]}.$$

The 2 s.d. prediction limits are similar.

Now consider the construction of prediction interval for p based on the Bayes approach. This can be achieved by obtaining the conditional distribution of a future observation, y, given the number defective, z, in k previous samples. This conditional distribution which is also the posterior distribution of y given z is called the predictive distribution of y. To obtain the predictive distribution of y, consider the conditional joint probability density function of y and p given by

$$h(y,p|z) = f(y|p,z)\Pi(p|z)$$

$$= f(y|p)\Pi(p|z). \qquad (7.7)$$

The predictive distribution $h(y|z)$ is then obtained by integrating p out from (7.7) as

$$h(y|z) = \int f(y|p)\Pi(p|z)dp.$$

It can be shown that for the prior in (7.4)

$$h(y|z) = \frac{n!\ \beta(a+z+y,\ n+nk+b-y-z)}{y!\ (n-y)!\ \beta(a+z,nk+b-z)}, \qquad (7.8)$$

$y = 0,1,2,...,n.$

Under the squared error loss function, the Bayes estimator for a future y is given by the posterior expectation, $E(y|z)$,

$$\tilde{y} = \sum_{y=1}^{n} \frac{n!\ \beta(a+z+y,\ nk+n+b-z-y)}{(y-1)!(n-y)!\ \beta(a+z,\ nk+b-z)}.$$

When a=b=.5 (i.e., for Jeffreys' prior) it is easy to compute $E(y|z)$ for various values of n and k. For any other member of this family of priors, estimates of a and b must be used to obtain the predicted value of y. Once $\tilde{y}$ is known the predicted value of the proportion defective of the lot is $\tilde{p}=\tilde{y}/n$ and $\tilde{p}$ may be defined as the Bayes predictor for p.

As indicated earlier HPD intervals can only be given when the posterior or the predictive distribution is unimodel. If the posterior is strictly a decreasing or increasing function, HPD intervals do not exist. Thus, before computing prediction intervals for p, one has to examine the behaviour of $h(y|z)$. For example, for Jeffreys' prior, when n=30, z=2 it is observed that $h(y|z)$ is unimodel. However, for any other prior this may not be the case and in that case

HPD prediction intervals for p cannot be given.

7.5 Numerical Example

Hald (1982, pp. 133) has reported quality distribution (i.e., proportion defective) of 150 lots of about 700 items in each lot. In this section this data set is used to examine the applicability of the beta prior and Jeffreys' prior to model the prior information and then to construct appropriate Bayesian intervals for p.

For this data set $\bar{p}$ = .01655, s_p^2 = .00007818 and s_p =.00884. The variance of $\hat{p}$ = x/n using binomial assumption is .000023 and the standard deviation of $\hat{p}$ is .00482. The variance of $\hat{p}$ under binomial assumption is significantly smaller than s_p^2 which gives rise to some doubts about the assumption of constant proportion defective. Hald (1982) has advocated that in such a situation control limits based on the binomial distribution should not be used. He further suggested that $\bar{p} \pm 3s_p$ may be used instead of binomial limits.

In order to fit a beta distribution, the parameters a and b have to be estimated from the data. The moment estimates of a and b are 4.88 and 289.98, respectively. A χ^2-goodness of fit test is performed assuming the underlying process distribution is a beta distribution with a=4.88, b=289.98. The results of the χ^2-goodness of fit test are given in the Table 7.1.

TABLE 7.1

χ^2- goodness of fit test for Hald's data

Class	Observed frequency	Expected frequency
0-.007	20	9.74
.007-.009	12	11.46
.009-.011	16	15.18
.011-.013	18	17.19
.013-.015	16	17.42
.015-.017	15	16.29
.017-.019	10	14.31
.019-.021	10	11.92
.021-.023	7	9.61
.023-.025	5	7.46
.025-.029	7	9.78
>.029	14	9.60

The value of the χ^2 test statistic is 17.07. The critical value at 99% level of confidence with 9 degrees of freedom is $\chi^2_{9,.01} = 21.67 > 17.07$. This indicates that the beta distribution a=4.88, b=289.98 is a reasonable fit for the above data set. The frequency histogram and the fitted beta distribution are given in Figure 7.1, which indicate that the fit is quite good for practical purposes.

Several prior distributions have been examined to see whether a better fit could be obtained. In this case various combinations of a and b around 4.88 and 289.98 have been used and it was concluded that the fit obtained earlier is reasonable.

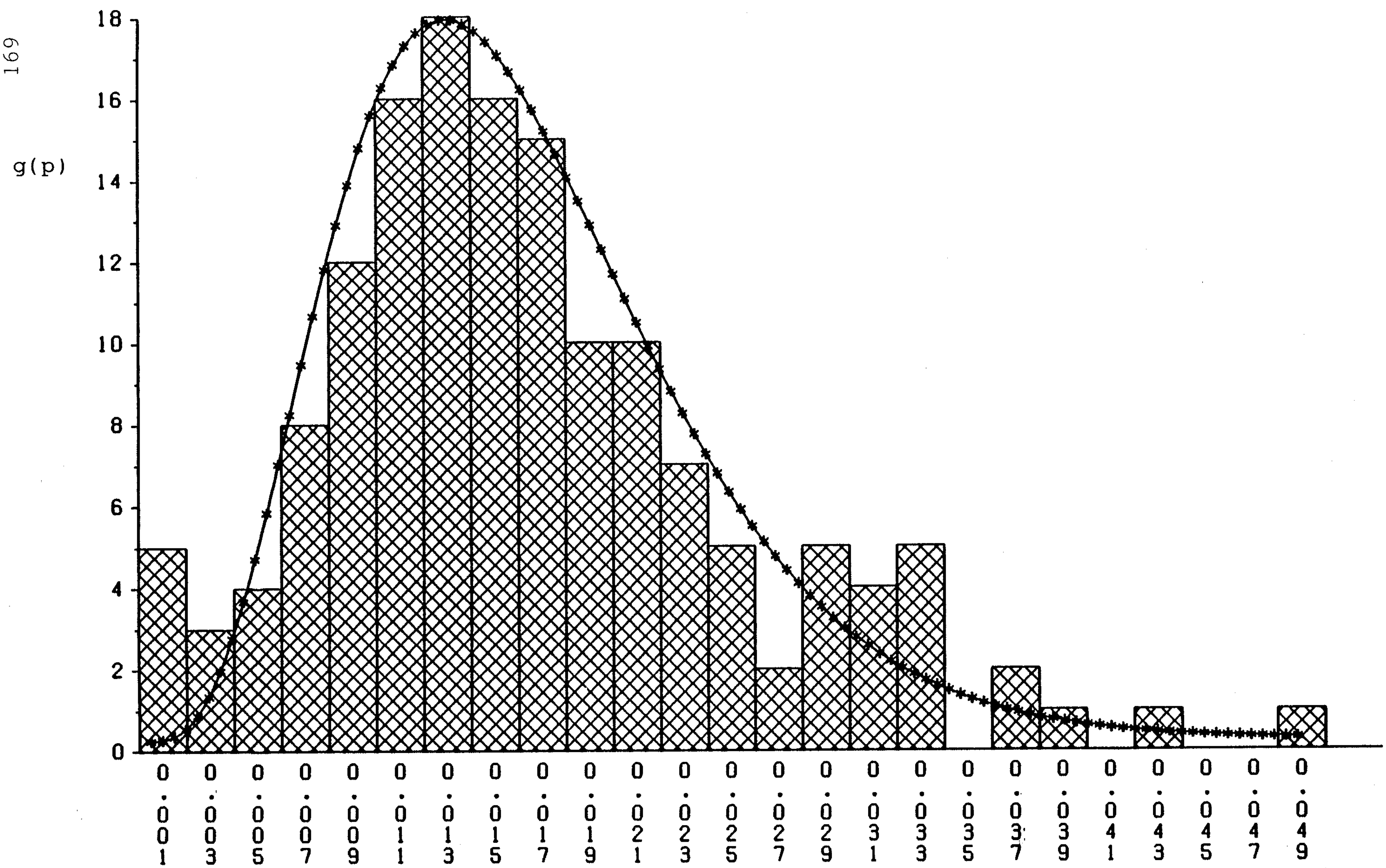


Figure 7.1. Histogram and the beta prior distribution for Hald's data

This example can also be used to examine the robustness of the posterior distribution and the credible intervals. This is done by changing the parameters a and b around 4.88 and 289.98. Several values of a and b used in this section are given in Table 7.2. Note that the uniform prior is obtained when a = b = 1.

Posterior distributions for several combinations of a and b considered are plotted in Figures 7.2(a) and 7.2(b). Examination of these figures indicates the posterior distributions for Jeffreys' and uniform priors are quite different from the rest of the posteriors. This further indicates that Jeffreys' prior behaves almost like the uniform prior and it is quite clear that these two priors are not applicable for the process which generates these observations. It seems that in most cases, these priors may not be appropriate. This is because, in quality control situations, one deals with proportion defective values which are clustered in a given range with almost all values very close to zero. In this example these values range from .001 to .049. This is why in this case the beta distribution with a = 4.88, b = 289.98 is preferred over either the Jeffreys' or the uniform prior. All other posterior distributions are clustered together and the change in a and b seems to have no effect on the posterior distributions. This indicates that the posterior is quite robust for moderate changes in the shape parameters.

95 percent equal tail credible intervals for p are given in Table 7.2. From this table it is clear that the credible intervals and their lengths do not vary at all for

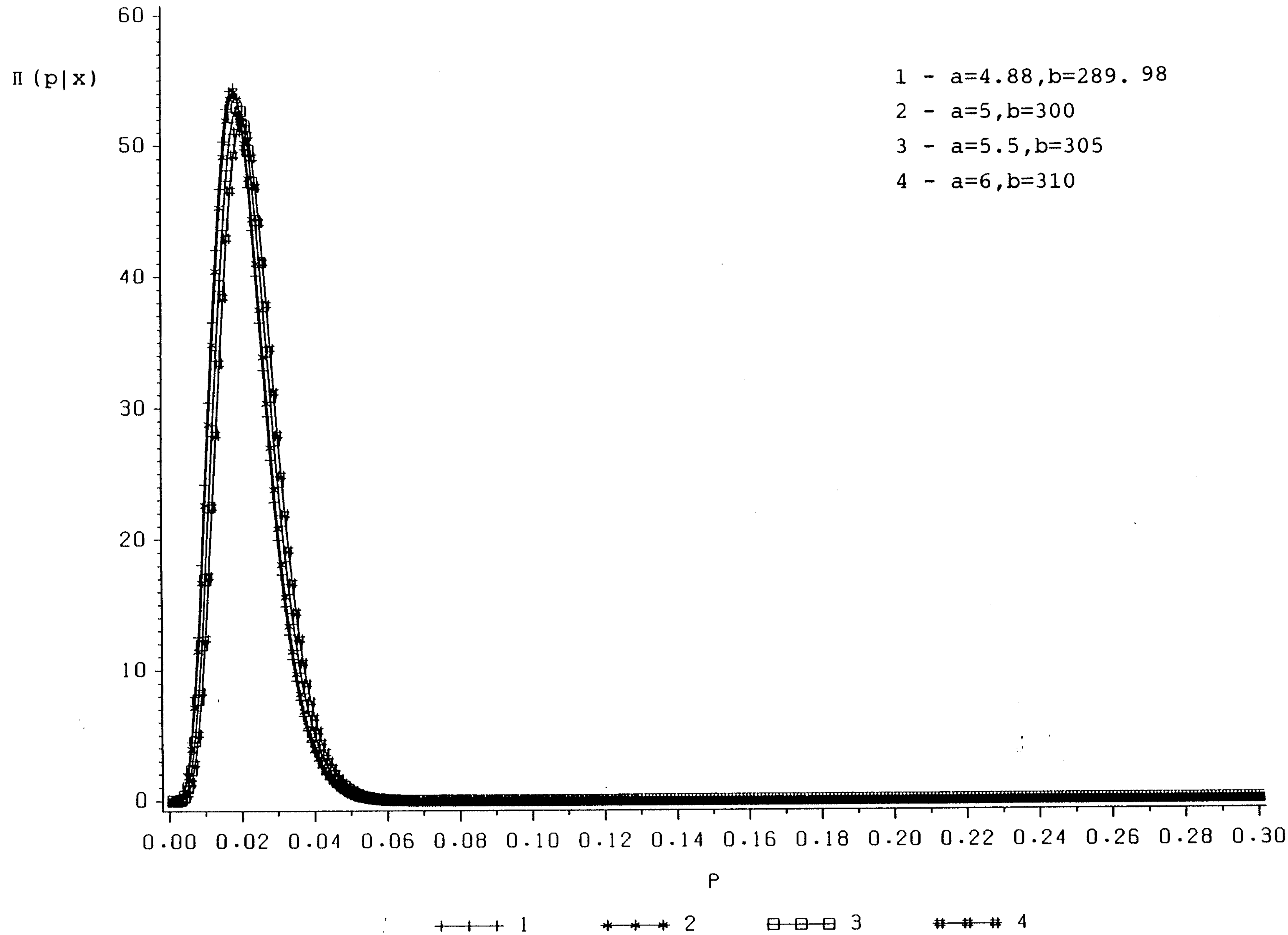


Figure 7.2(a). Posterior distributions for n=30 and x=2.

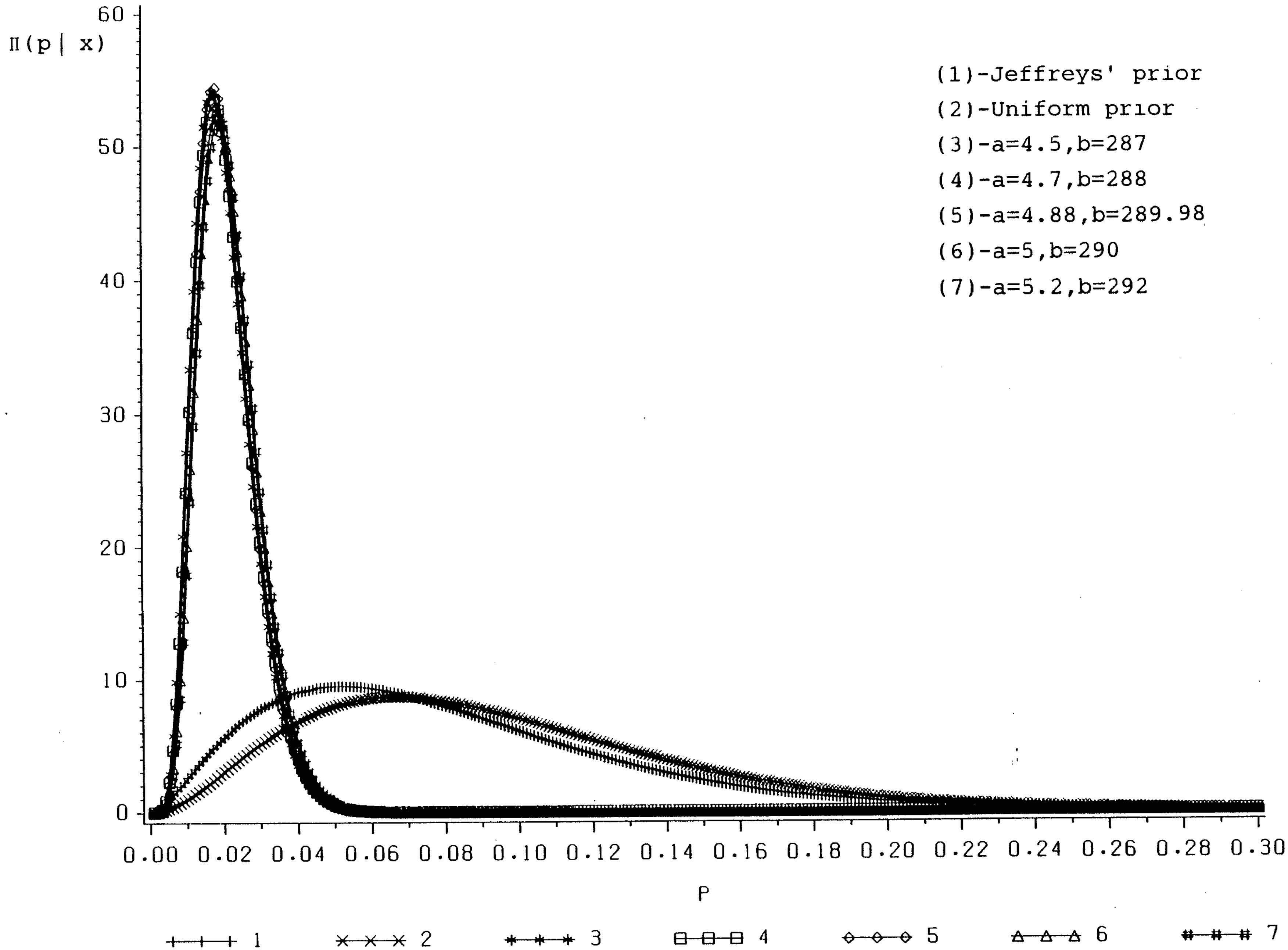


Figure 7.2(b). Posterior distributions for n=30 and x=2.

slight changes in a and b. We note that the behaviour of Jeffreys' and uniform priors are quite different to the rest of the priors.

Table 7.2

95 percent equal tail credible intervals

prior	Bayes Estimate	LL	UL	Width
Jeffreys'	.0806	.0141	.1929	.1829
Uniform	.0938	.0204	.2142	.1938
4.5,287	.0202	.0079	.0382	.0303
4.7,287	.0208	.0082	.0390	.0308
4.7,288	.0208	.0082	.0389	.0307
4.88,289.98	.0206	.0083	.0383	.0300
5,300	.0209	.0087	.0387	.0300
5,290	.0215	.0087	.0399	.0312
5.2,292	.0220	.0091	.0404	.0313
5.5,305	.0220	.0093	.0401	.0308
6,290	.0245	.0107	.0439	.0332
6,310	.0231	.0101	.0414	.0313

A plot of posterior distributions for various values of b keeping a=4.88 is given in Figure 7.3(a). It can be seen that change in b has very little effect on the posterior distribution. A similar plot for b=289.98 and several values of a is given in Figure 7.3(b). From this figure it is clear that the posterior distribution is quite sensitive to small changes in a when b is kept constant.

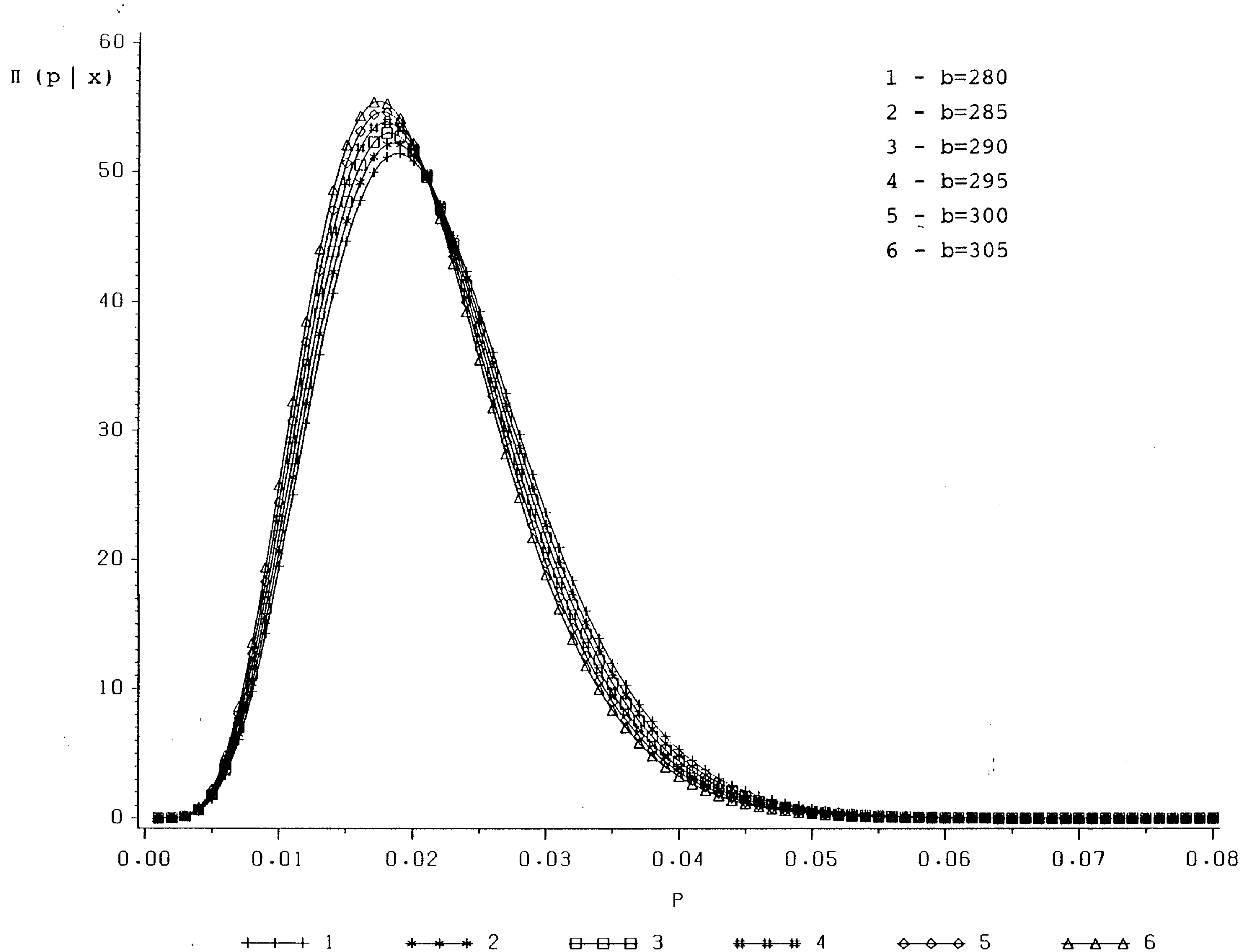


Figure 7.3(a).Posterior distributions for several values of b when a=4.88

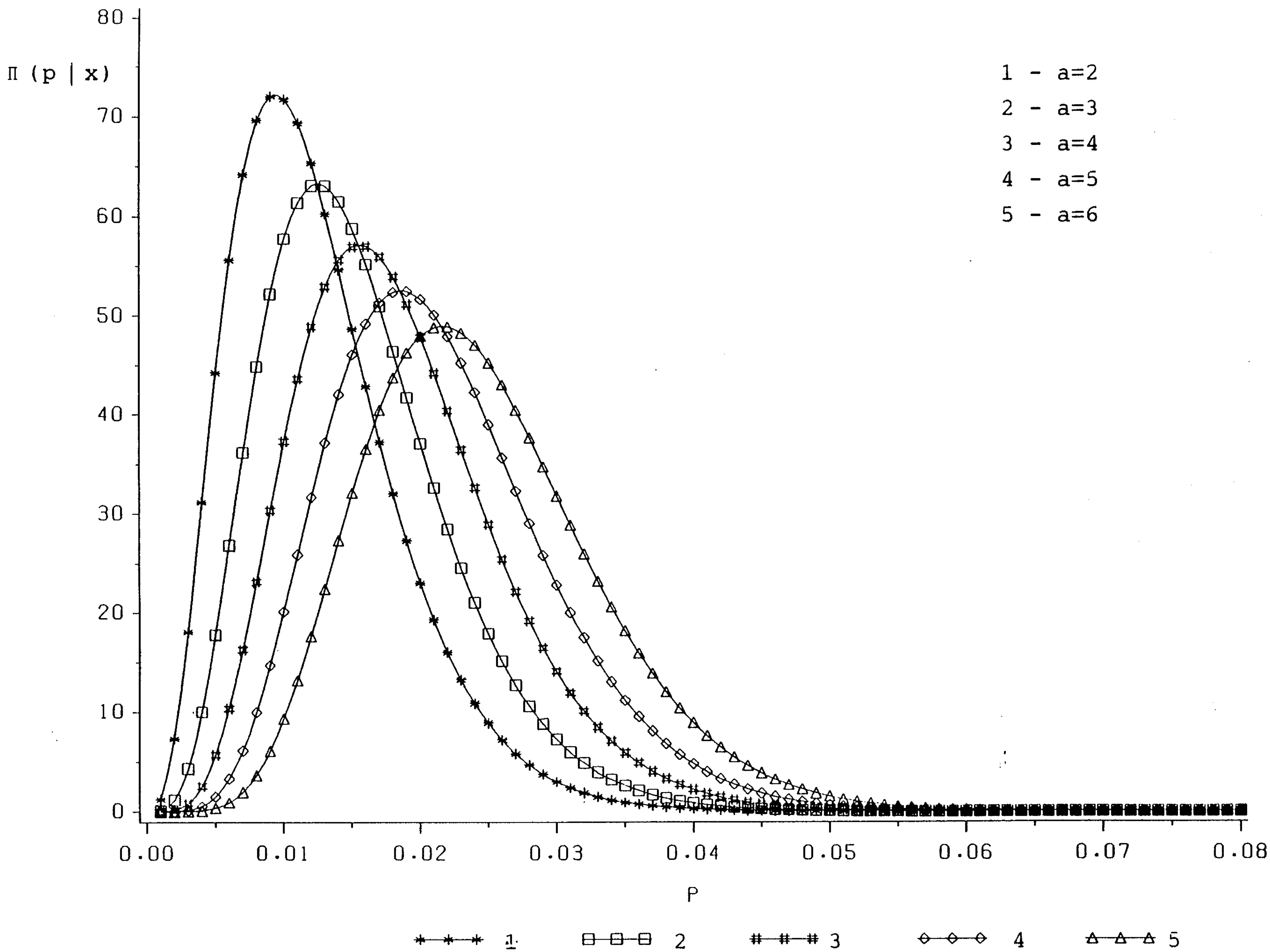


Figure 7.3(b). Posterior distributions for several values of a when b=289.98

It is also of interest to examine how sensitive the posterior is to the changes in the sample size. Such a behaviour of the posterior can be seen from the plot given in Figure 7.4. Once again, it is obvious that the sample size has very little effect on the posterior distribution. In summary one can then conclude that the posterior distribution is very robust to the changes in the parameters of the prior distribution and the sample size.

To illustrate the construction of equivalent Bayes intervals and control charts (see comments on page 165), consider the following ten samples, each of size 30, which have been generated from this process.

Sample	x_i	p'	p_i
1	1	.01811	.033
2	2	.02118	.067
3	2	.02118	.067
4	3	.02426	.100
5	5	.03042	.167
6	3	.02426	.100
7	2	.02118	.067
8	1	.01811	.033
9	3	.02426	.100
10	3	.02426	.100

The standard deviation of p' is estimated to be .002255. The Bayes estimate of p based on the ten samples is .0502 and the grand mean of p_i's is .0834. The 3-s.d. control

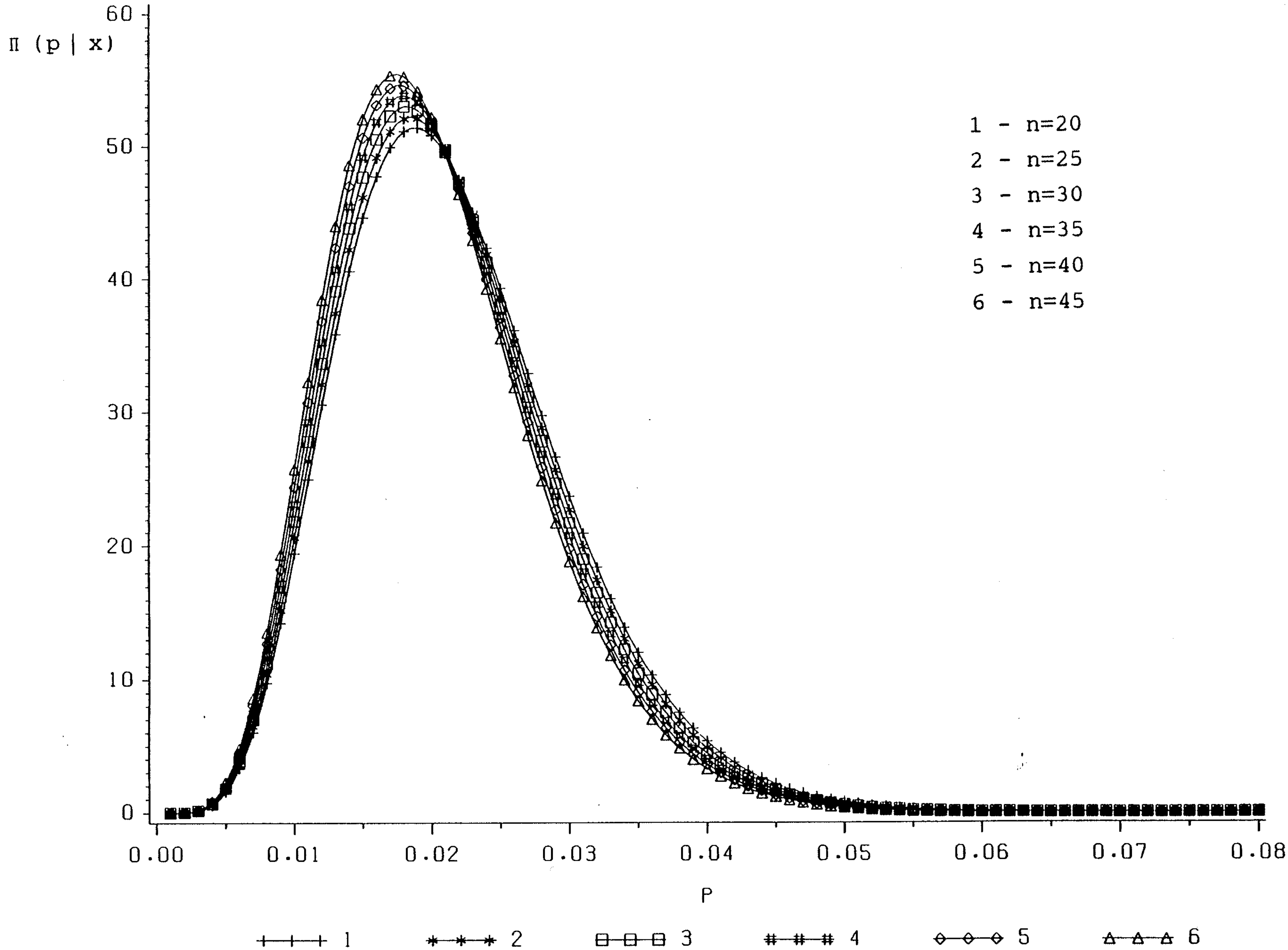


Figure 7.4. Posterior distributions for sevaral values of n when a=4.88 and b=289.98

limits are then given by

$$\text{upper control limit} = .0502 + 3(.002255) = .0569$$

and

$$\text{lower control limit} = .0502 - 3(.002255) = .0434.$$

Similarly 2-s.d. control limits are

$$\text{upper control limit} = .0547$$

and

$$\text{lower control limit} = .0457.$$

For comparison purposes for the classical p-chart, the 3-s.d. limits are (0, .2348) and the 2-s.d. limits are (0, .1843).

The Bayes estimate of p is given by

$$p^* = (4.88+25)/(300+4.88+289.88)$$
$$= 0.0502.$$

For Jeffreys' prior

$$p^* = 25.5/301 = 0.0847$$

which is, as expected, considerably larger than the estimate based on the beta prior.

Equal tail credible intervals for p are computed using SAS FUNCTION BETAINV. For beta prior 95 percent equal tail credible interval for p is (.00827, .03834) and the corresponding interval for Jeffreys' prior is (.01411, .19709). As anticipated, the width of the credible interval for Jeffreys' prior is larger than (about 6.1 times) the width of the credible interval for beta prior.

The HPD interval for p is (.00715, .03708) and for Jeffreys' prior the corresponding interval is (.00573,

.17485). As anticipated, the length of the HPD interval based on Jeffreys' prior is larger than that of the beta prior. Also notice that equal tail credible intervals are wider than the corresponding HPD intervals.

The prediction intervals for p can be computed using the predictive distribution given in (7.8). 95 percent equal tail prediction interval for p is given by (0, .067). For Jeffreys' prior, the interval becomes (0, .2). It should be noted that these are approximate intervals because of the discrete nature of the predictive distribution.

The HPD prediction interval for p based on the beta prior cannot be given because $h(y|z)$ when $a=4.88$, $b=289.98$, $n=30$, $z=25$ is strictly a decreasing function. But for Jeffreys' prior this is not the case and the HPD prediction interval is (0, .1667). See Figures 7.5(a) and 7.5(b) for plots of the predictive distributions. The classical prediction interval is (0, .2233).

The Bayes chart and the classical standard p-chart are given in Figures 7.6(a) and 7.6(b), respectively. The information in the Bayes chart is presented using box and whisker plot developed by Tukey (1977). A box and whisker chart is plotted for each sample. In this case the box and whisker chart is a graphical representation of the posterior distribution for the proportion defective. The bottom and the top whiskers of the chart correspond to the probabilities .005 and .995, respectively. Similarly, the bottom of the box and the top of the box correspond to the probabilities .025

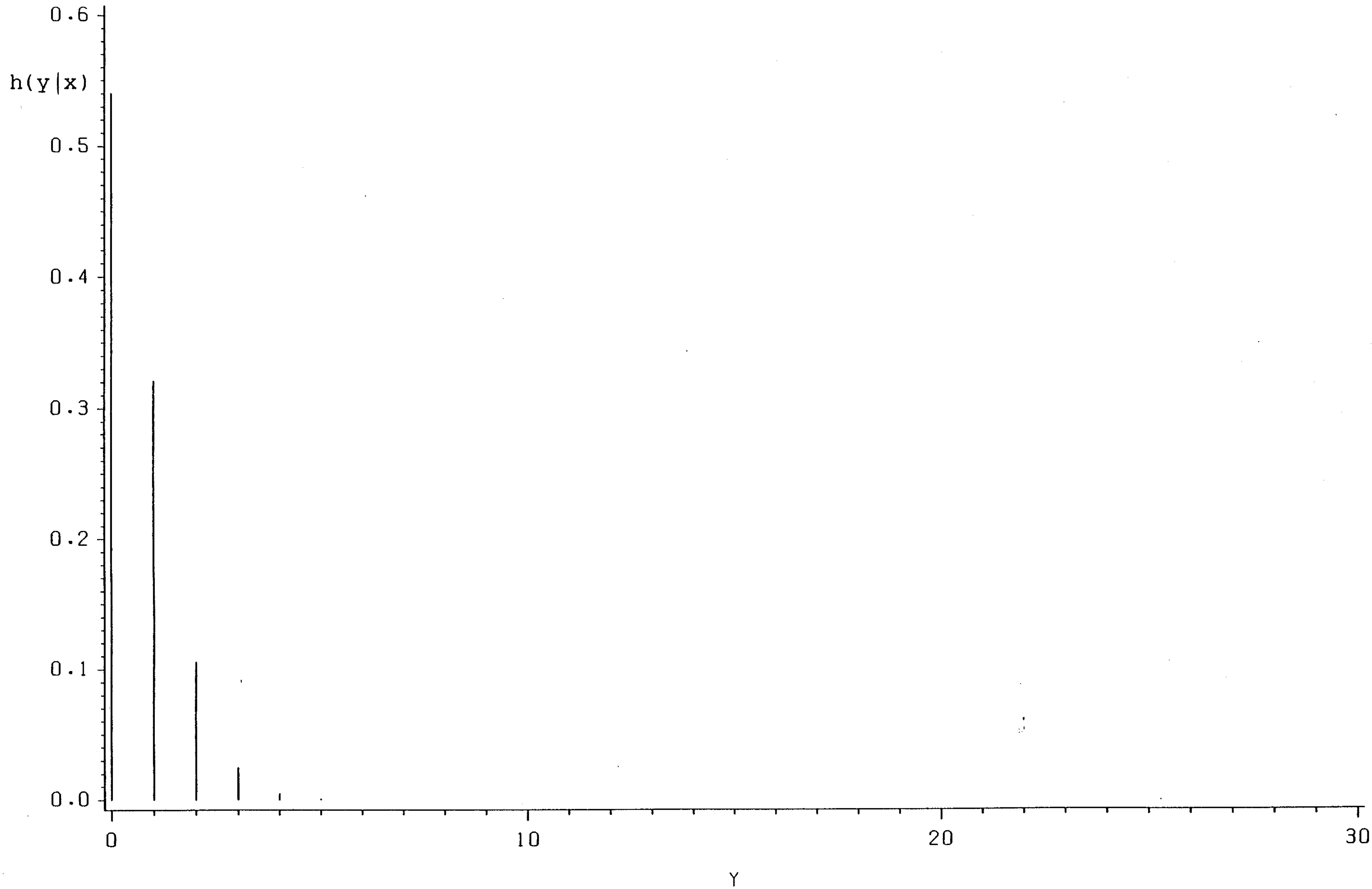


Figure 7.5(a). Predictive distribution for beta prior when a=4.88,b=289. 98

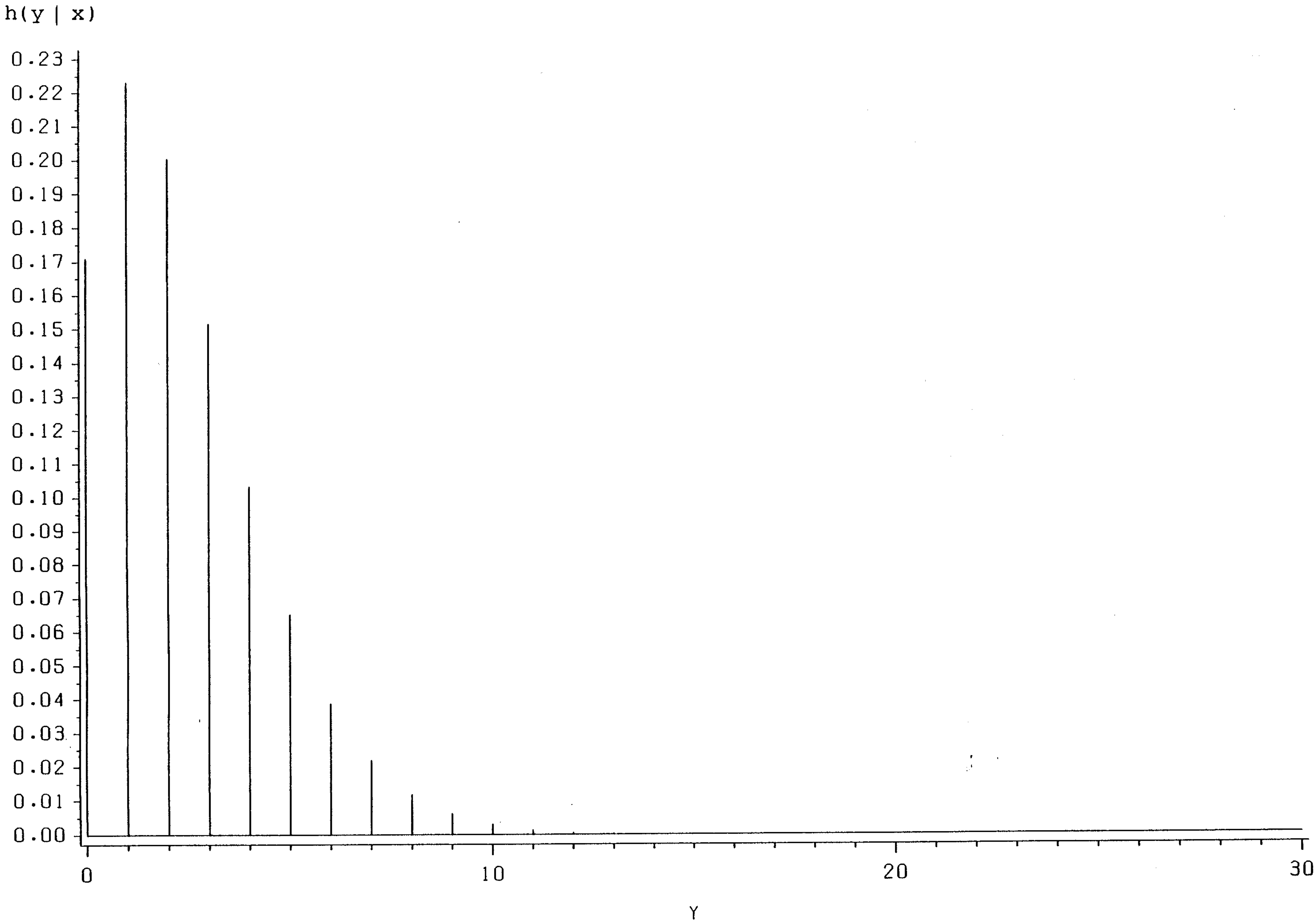


Figure 7.5(b). Predictive distribution for Jeffreys' prior.

and .975, respectively. The dash near the centre of the box is the posterior mean (the Bayes estimate) of the proportion defective.

Examination of the box-whisker chart reveals that one point (sample five) is outside 3 s.d. upper limits indicating possible out of control situation. This is not the case in the standard p-chart. In the p-chart every point is within the 2 s.d. limits. The points in the p-chart appear to be random and there is no indication whatsoever that the process is out of control. The Bayes chart presented in this chapter also contains more information and the Bayes control limits are much narrower than the classical limits. This gives an additional incentive to a manufacturer to improve the process so that the products will be of acceptable quality. Bayes chart thus allows a much better control and more understanding of the overall process. In that sense Bayes charts can be considered superior to classical charts.

7.6 Conclusions

In this study we have examined the applicability of non-informative and beta prior distributions to describe the prior information of proportion defective. It has been observed that Jeffreys' and uniform prior may not be suitable to model the process distribution. If prior information based on previous samples are not available, non-informative priors considered in this study may be used. However, in many practical situations data from previous samples are

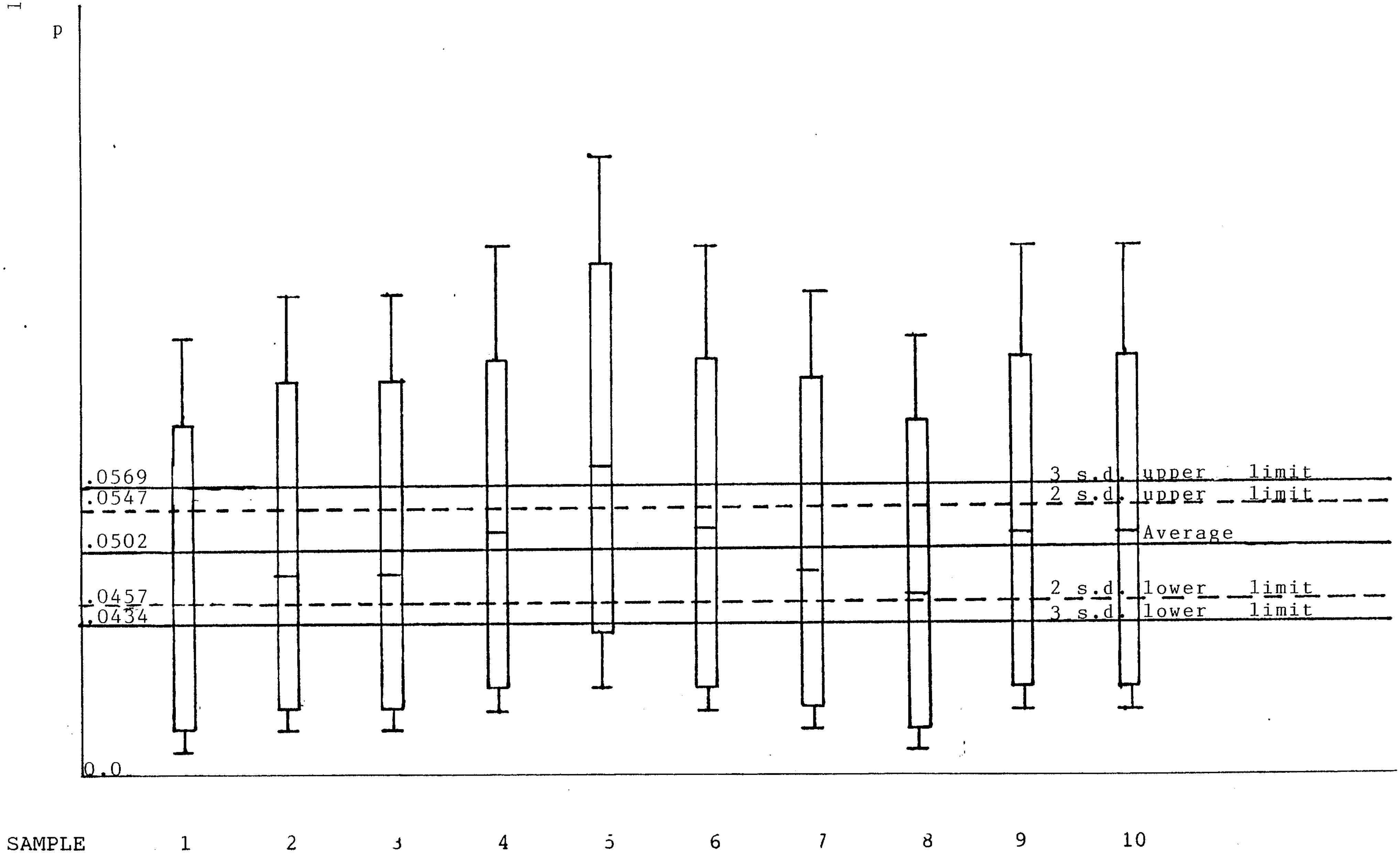


Figure 7.6(a). Bayesian Control Chart.

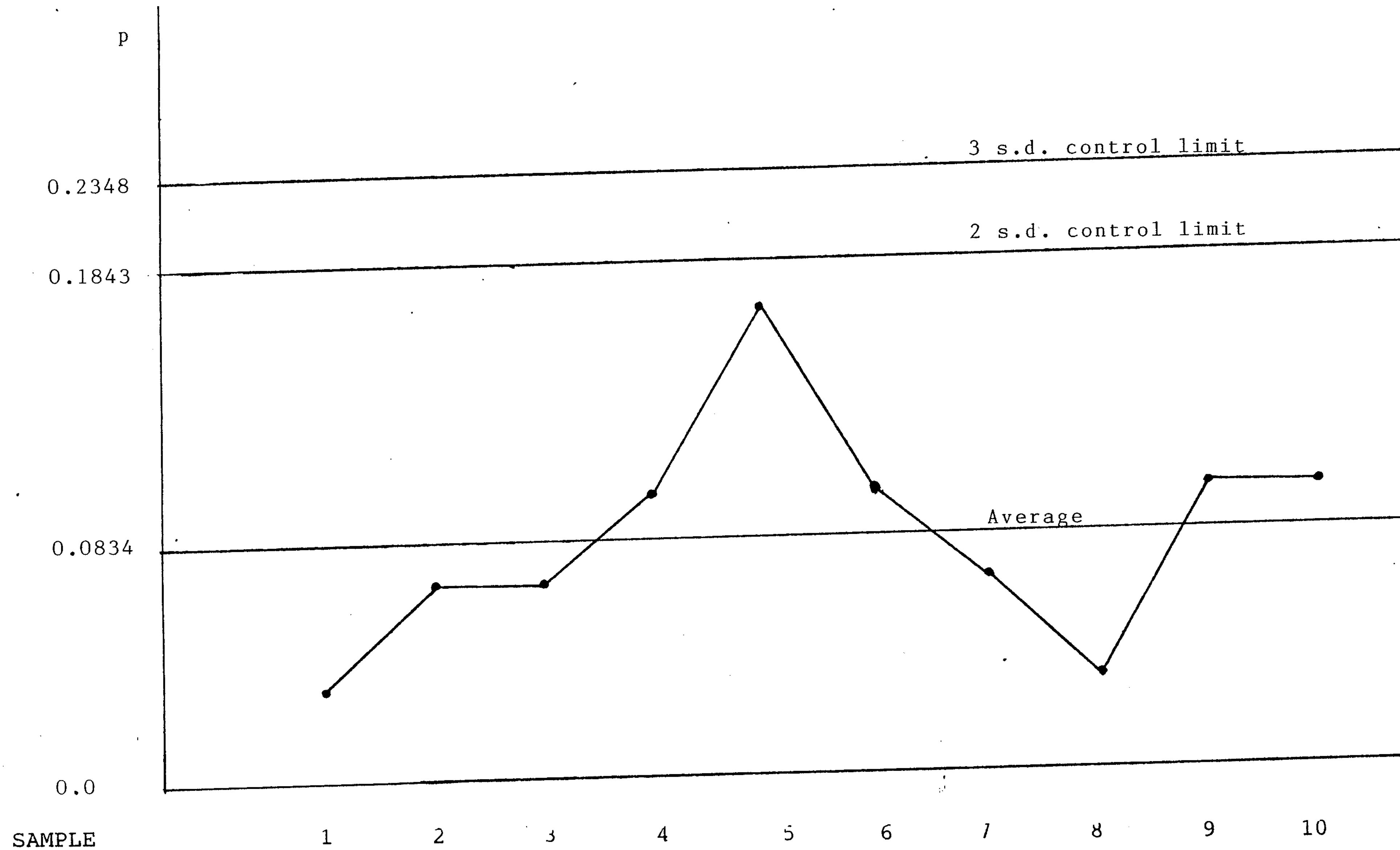


Figure 7.6(b). Standard p-chart.

available. In such a situation, one should attempt to identify the prior distribution and use this distribution to construct appropriate credible intervals for p. The process distribution may then be used to construct appropriate Bayes control charts for p. To identify the process distribution one may use various statistical tools such as goodness of fit tests, probability plots and empirical Bayes techniques. It should be noted that because the Bayes intervals are narrower there will be frequent rejection of lots more than in the case of classical control charts. This may seem to be a disadvantage to a manufacturer, but the use of Bayes control charts gives an additional incentive to the manufacturer to improve his process so that the overall quality of the final product may be improved.

7.7 Future Direction of Research

In this section, we suggest some areas for future research in topics covered in this thesis. As has been indicated, control charts by variables are based on the assumption that the quality measurements are normally distributed. In chapter 3, it has been observed that this assumption may not be reasonable because the control limit constants are sensitive to departures from normality. Tables of constants required to construct control charts for Tukey's λ-family distributions are given in chapter 3. However, robustness of control charts based on commonly used distributions in life testing and reliability studies such as

Weibull, logistic, lognormal, Pareto and Rayleigh has not yet been examined. Control limit constants for these distributions may also be tabulated. As suggested in this thesis, practitioners should first investigate the form of the underlying process distribution; should the quality variable belong to one of these probability distributions, such tables of constants would be valuable to practitioners.

The same conclusion can be made for sampling plans by variables. The assumption of normality is essential for the use of standard sampling plans (e.g., MIL-STD-414), but no sufficient attention has been given in the past to provide sampling plans for non-normal distributions. MIL-STD-414 type sampling plans may be constructed at least for the widely used distributions given in this section.

In chapter 5, sampling plans based on serially correlated data have been considered, but the discussion was limited to stationary processes. Further research is needed to examine the effect of serial correlation in sampling plans based on non-stationary processes. Another possible area of future research is to examine the behaviour of acceptance sampling plans for serially correlated periodic data.

In chapter 6, a method of estimation of variance components for type B bulk material has been proposed. However, there is another common practice in bulk sampling which is known as composite sampling. In this case, an initial sample of n_1 primary units are selected and the initial sample is mixed to form a composite sample. A second sample of size n_2 from this composite is selected and this second sample is then analyzed. This is a common method in most laboratory tests because large quantities of bulk material cannot be used in testing and therefore the initial sample has to be reduced to a smaller quantity by composite sampling. A satisfactory method to obtain variance components in composite sampling is presently not available and the Bayes procedure discussed in chapter 6 may be extended for this case.

It is well known that the sampling plans given in standard literature are based on a single quality characteristic. However, there are situations where several quality characteristics are measured on the same unit (e.g., length and diameter of a steel bar), but at present individual sampling plans for each quality variable have to be used. This procedure may lead to an acceptance of a lot based on one variable, but a rejection based on the other variable. Further research is, therefore, needed to construct sampling plans to incorporate more than one quality variable into a single sampling plan.

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APPENDIX

TABLE I

VALUES OF THE RATIO $[\Gamma_{n,n;\alpha}/\Gamma_{n,n;1-\beta}]$.

n	$\alpha=.05$ $\beta=.10$	$\alpha=.05$ $\beta=.15$	$\alpha=.10$ $\beta=.10$	$\alpha=.10$ $\beta=.15$
2	0.0914	0.1054	0.1367	0.1577
3	0.1536	0.1731	0.2071	0.2333
4	0.2045	0.2272	0.2612	0.2901
5	0.2465	0.2711	0.3043	0.3347
6	0.2817	0.3076	0.3398	0.3710
7	0.3119	0.3386	0.3698	0.4014
8	0.3382	0.3653	0.3955	0.4273
9	0.3613	0.3887	0.4181	0.4498
10	0.3819	0.4095	0.4379	0.4696
11	0.4004	0.4281	0.4557	0.4871
12	0.4172	0.4448	0.4717	0.5029
13	0.4324	0.4600	0.4862	0.5173
14	0.4465	0.4739	0.4995	0.5303
15	0.4594	0.4868	0.5117	0.5422
16	0.4713	0.4986	0.5229	0.5532
17	0.4825	0.5096	0.5334	0.5634
18	0.4929	0.5198	0.5431	0.5728
19	0.5026	0.5293	0.5522	0.5817
20	0.5117	0.5383	0.5608	0.5899
21	0.5203	0.5468	0.5688	0.5977
22	0.5284	0.5547	0.5763	0.6049

Table I continued.

23	0.5361	0.5622	0.5835	0.6118
24	0.5434	0.5693	0.5902	0.6184
25	0.5503	0.5761	0.5966	0.6245
26	0.5569	0.5825	0.6027	0.6304
27	0.5632	0.5886	0.6086	0.6359
28	0.5693	0.5944	0.6141	0.6413
29	0.5750	0.6000	0.6194	0.6464
30	0.5805	0.6054	0.6245	0.6512
31	0.5858	0.6105	0.6293	0.6558
32	0.5909	0.6154	0.6339	0.6503
33	0.5957	0.6201	0.6385	0.6646
34	0.6004	0.6246	0.6428	0.6687
35	0.6049	0.6390	0.6469	0.6726
36	0.6093	0.6332	0.6509	0.6764
37	0.6135	0.6373	0.6547	0.6801
38	0.6176	0.6412	0.6584	0.6836
39	0.6215	0.6449	0.6621	0.6871
40	0.6253	0.6486	0.6655	0.6904
41	0.6290	0.6522	0.6689	0.6936
42	0.6326	0.6556	0.6721	0.6966
43	0.6360	0.6589	0.6753	0.6996
44	0.6394	0.6621	0.6784	0.7025
45	0.6426	0.6653	0.6814	0.7054
46	0.6458	0.6683	0.6842	0.7081
47	0.6488	0.6712	0.6871	0.7108
48	0.6519	0.6742	0.6898	0.7133

Table I continued.

49	0.6548	0.6769	0.6924	0.7158
50	0.6576	0.6797	0.6950	0.7183
51	0.6604	0.6823	0.6975	0.7207
52	0.6631	0.6848	0.6999	0.7229
53	0.6657	0.6874	0.7024	0.7252
54	0.6683	0.6898	0.7047	0.7274
55	0.6708	0.6922	0.7069	0.7296
56	0.6724	0.6946	0.7091	0.7317
57	0.6756	0.6968	0.7113	0.7337
58	0.6779	0.6991	0.7135	0.7357
59	0.6802	0.7012	0.7155	0.7376
60	0.6825	0.7034	0.7175	0.7395
61	0.6847	0.7055	0.7195	0.7414
62	0.6868	0.7075	0.7214	0.7432
63	0.6889	0.7095	0.7233	0.7449
64	0.6909	0.7114	0.7252	0.7467
65	0.6929	0.7133	0.7270	0.7484
66	0.6949	0.7152	0.7287	0.7501
67	0.6968	0.7170	0.7305	0.7517
68	0.6987	0.7188	0.7322	0.7533
69	0.7005	0.7206	0.7339	0.7549
70	0.7024	0.7223	0.7355	0.7564
71	0.7041	0.7240	0.7371	0.7579
72	0.7059	0.7257	0.7387	0.7594
73	0.7076	0.7273	0.7403	0.7609
74	0.7093	0.7289	0.7418	0.7623

Table I continued.

75	0.7109	0.7305	0.7433	0.7637
76	0.7125	0.7320	0.7447	0.7651
77	0.7141	0.7335	0.7462	0.7664
78	0.7157	0.7350	0.7476	0.7678
79	0.7172	0.7365	0.7489	0.7691
80	0.7187	0.7379	0.7503	0.7703
81	0.7202	0.7393	0.7517	0.7716
82	0.7217	0.7407	0.7530	0.7728
83	0.7231	0.7421	0.7543	0.7740
84	0.7246	0.7434	0.7556	0.7752
85	0.7259	0.7447	0.7568	0.7764
86	0.7273	0.7460	0.7581	0.7775
87	0.7288	0.7473	0.7593	0.7787
88	0.7300	0.7486	0.7605	0.7798
89	0.7313	0.7498	0.7617	0.7809
90	0.7326	0.7511	0.7628	0.7820
91	0.7339	0.7523	0.7639	0.7831
92	0.7351	0.7534	0.7651	0.7841
93	0.7364	0.7546	0.7662	0.7852
94	0.7376	0.7558	0.7673	0.7862
95	0.7888	0.7569	0.7684	0.7872
96	0.7399	0.7580	0.7694	0.7882
97	0.7411	0.7591	0.7705	0.7892
98	0.7423	0.7612	0.7715	0.7901
99	0.7434	0.7613	0.7725	0.7911
100	0.7445	0.7623	0.7735	0.7920

TABLE II

VALUES OF THE RATIO $[(1-p_1')^n-(p_2')^n]/[(1-p_1'')^n-(p_2'')^n]$

	$p_1=.01$				$p_1=.02$		
n	$p_2=.02$	$p_2=.03$	$p_2=.04$	$p_2=.05$	$p_2=.03$	$p_2=.04$	$p_2=.05$
2	1.010	1.021	1.031	1.042	1.010	1.021	1.032
3	1.015	1.031	1.047	1.063	1.015	1.031	1.047
4	1.020	1.041	1.063	1.085	1.020	1.041	1.063
5	1.026	1.052	1.079	1.107	1.026	1.052	1.079
6	1.031	1.062	1.095	1.129	1.031	1.063	1.096
7	1.036	1.073	1.112	1.153	1.036	1.074	1.113
8	1.041	1.084	1.129	1.176	1.041	1.085	1.129
9	1.046	1.095	1.146	1.201	1.047	1.096	1.147
10	1.052	1.106	1.164	1.225	1.052	1.107	1.165
11	1.057	1.118	1.182	1.250	1.057	1.118	1.183
12	1.062	1.289	1.199	1.276	1.063	1.129	1.201
13	1.068	1.140	1.218	1.302	1.068	1.141	1.219
14	1.073	1.152	1.237	1.329	1.073	1.153	1.238
15	1.078	1.164	1.256	1.356	1.079	1.164	1.257
16	1.084	1.175	1.275	1.384	1.084	1.176	1.277
17	1.089	1.187	1.295	1.412	1.089	1.188	1.296
18	1.095	1.199	1.314	1.441	1.095	1.201	1.316
19	1.100	1.212	1.335	1.471	1.101	1.213	1.336
20	1.106	1.224	1.355	1.501	1.107	1.225	1.375
21	1.112	1.236	1.376	1.532	1.112	1.238	1.378
22	1.117	1.249	1.397	1.563	1.118	1.250	1.399
23	1.123	1.261	1.418	1.595	1.124	1.263	1.421

24	1.129	1.274	1.439	1.628	1.129	1.276	1.443
25	1.134	1.287	1.462	1.661	1.135	1.289	1.465
26	1.139	1.300	1.484	1.695	1.141	1.302	1.487
27	1.146	1.314	1.507	1.730	1.146	1.315	1.510
28	1.151	1.327	1.530	1.766	1.152	1.329	1.533
29	1.157	1.340	1.554	1.802	1.158	1.342	1.557
30	1.163	1.354	1.577	1.839	1.164	1.356	1.581
31	1.169	1.368	1.601	1.877	1.169	1.369	1.605
32	1.175	1.382	1.626	1.915	1.176	1.384	1.629
33	1.181	1.396	1.651	1.954	1.182	1.398	1.655
34	1.186	1.409	1.676	1.994	1.188	1.412	1.681
35	1.193	1.424	1.702	2.035	1.194	1.427	1.706
36	1.199	1.438	1.728	2.077	1.199	1.441	1.732
37	1.205	1.453	1.754	2.119	1.206	1.456	1.759
38	1.211	1.468	1.781	2.163	1.212	1.471	1.786
39	1.217	1.483	1.808	2.208	1.218	1.486	1.814
40	1.223	1.498	1.836	2.253	1.224	1.501	1.842
41	1.229	1.513	1.864	2.299	1.231	1.516	1.870
42	1.235	1.528	1.893	2.346	1.237	1.532	1.899
43	1.242	1.544	1.922	2.394	1.243	1.547	1.928
44	1.248	1.559	1.951	2.443	1.249	1.563	1.957
45	1.254	1.575	1.981	2.494	1.256	1.579	1.988
46	1.261	1.591	2.011	2.545	1.262	1.595	2.018
47	1.267	1.608	2.042	2.597	1.269	1.611	2.049
48	1.274	1.624	2.073	2.650	1.275	1.628	2.081
49	1.279	1.640	2.105	2.705	1.282	1.645	2.113
50	1.286	1.675	2.137	2.760	1.288	1.661	2.146

TABLE III

VALUES OF $[\chi^2_{2n-2,\alpha}/\chi^2_{2n-2,1-\beta}]$

n	$\alpha=.05$ $\beta=.10$	$\alpha=.05$ $\beta=.15$	$\alpha=.10$ $\beta=.10$	$\alpha=.10$ $\alpha=.15$
2	28.4332	18.4331	21.8543	14.1681
3	8.9202	6.9432	7.3141	5.6931
4	5.7127	4.7314	4.8294	3.9998
5	4.4439	3.8025	3.8290	3.2763
6	3.7629	3.2867	3.2860	2.8702
7	3.3355	2.9557	2.9426	2.6075
8	3.0406	2.7235	2.7042	2.4222
9	2.8238	2.5508	2.5281	2.2836
10	2.6571	2.4166	2.3920	2.1755
11	2.5244	2.3089	2.2834	2.0885
12	2.4160	2.2204	2.1944	2.0167
13	2.3255	2.1460	2.1200	1.9563
14	2.2488	2.0826	2.0566	1.9047
15	2.1826	2.0277	2.0020	1.8599
16	2.1250	1.9798	1.9542	1.8207
17	2.0742	1.9373	1.9122	1.7860
18	2.0291	1.8995	1.8747	1.7550
19	1.9888	1.8656	1.8411	1.7271
20	1.9524	1.8349	1.8108	1.7019
21	1.9194	1.8070	1.7833	1.6789
22	1.8893	1.7816	1.7581	1.6579
23	1.8617	1.7582	1.7351	1.6386

Table III continued.

24	1.8363	1.7366	1.7139	1.6208
25	1.8129	1.7166	1.6942	1.6043
26	1.7911	1.6981	1.6760	1.5890
27	1.7709	1.6808	1.6591	1.5746
28	1.7520	1.6646	1.6432	1.5613
29	1.7343	1.6495	1.6284	1.5487
30	1.7178	1.6353	1.6145	1.5369
31	1.7022	1.6219	1.6014	1.5258
32	1.6875	1.6093	1.5890	1.5153
33	1.6736	1.5973	1.5773	1.5054
34	1.6605	1.5860	1.5662	1.4960
35	1.6481	1.5753	1.5558	1.4871
36	1.6362	1.5651	1.5458	1.4786
37	1.6250	1.5554	1.5363	1.4705
38	1.6143	1.5461	1.5273	1.4628
39	1.6041	1.5373	1.5186	1.4554
40	1.5943	1.5288	1.5104	1.4483
41	1.5850	1.5207	1.5025	1.4416
42	1.5761	1.5130	1.4950	1.4361
43	1.5675	1.5055	1.4877	1.4289
44	1.5593	1.4984	1.4808	1.4230
45	1.5514	1.4915	1.4741	1.4172
46	1.5438	1.4849	1.4676	1.4117
47	1.5365	1.4786	1.4615	1.4064
48	1.5294	1.4724	1.4555	1.4013
49	1.5226	1.4665	1.4497	1.3963

Table III continued.

50	1.5161	1.4608	1.4442	1.3915
51	1.5098	1.4553	1.4388	1.3869
52	1.5037	1.4500	1.4336	1.3824
53	1.4978	1.4448	1.4286	1.3781
54	1.4920	1.4398	1.4238	1.3739
55	1.4865	1.4350	1.4191	1.3699
56	1.4811	1.4303	1.4145	1.3659
57	1.4759	1.4257	1.4101	1.3621
58	1.4708	1.4213	1.4058	1.3584
59	1.4659	1.4172	1.4016	1.3548
60	1.4612	1.4128	1.3976	1.3513
61	1.4566	1.4088	1.3936	1.3479
62	1.4521	1.4048	1.3898	1.3446
63	1.4477	1.4010	1.3861	1.3414
64	1.4435	1.3973	1.3823	1.3382
65	1.4393	1.3936	1.3789	1.3352
66	1.4353	1.3901	1.3755	1.3322
67	1.4313	1.3866	1.3721	1.3293
68	1.4275	1.3833	1.3689	1.3265
69	1.4238	1.3799	1.3657	1.3237
70	1.4201	1.3767	1.3626	1.3210
71	1.4166	1.3737	1.3596	1.3184
72	1.4131	1.3706	1.3566	1.3158
73	1.4097	1.3676	1.3537	1.3133
74	1.4064	1.3647	1.3509	1.3108
75	1.4031	1.3618	1.3481	1.3084

Table III continued.

76	1.3999	1.3590	1.3454	1.3061
77	1.3969	1.3563	1.3427	1.3038
78	1.3938	1.3536	1.3402	1.3015
79	1.3908	1.3512	1.3376	1.2993
80	1.3879	1.3485	1.3351	1.2972
81	1.3851	1.3459	1.3327	1.2951
82	1.3823	1.3435	1.3303	1.2929
83	1.3796	1.3411	1.3280	1.2909
84	1.3769	1.3387	1.3257	1.2889
85	1.3743	1.3364	1.3235	1.2870
86	1.3717	1.3341	1.3213	1.2851
87	1.3692	1.3319	1.3191	1.2832
88	1.3667	1.3297	1.3170	1.2814
89	1.3643	1.3276	1.3149	1.2796
90	1.3619	1.3255	1.3129	1.2778
91	1.3596	1.3234	1.3109	1.2760
92	1.3573	1.3214	1.3089	1.2743
93	1.3550	1.3194	1.3070	1.2726
94	1.3528	1.3174	1.3051	1.2710
95	1.3507	1.3155	1.3033	1.2694
96	1.3485	1.3136	1.3014	1.2678
97	1.3464	1.3118	1.2997	1.2662
98	1.3443	1.3099	1.2979	1.2647
99	1.3423	1.3082	1.2961	1.2632
100	1.3403	1.3064	1.2944	1.2617

VITAE

Karunarathnage Piyasena Hapuarachchi was born in Pubbiliya, Kobeigane, Sri Lanka on the 21st of May 1947. He completed his high school education in 1966 and was admitted to the Vidyalankara University of Sri Lanka in 1967. He received his B.Sc. (Hon) in January, 1971. Since 1972 he was a statistician to the government of Sri Lanka until he was admitted to the University of Southampton in 1976. He received a M.Sc. (Social Science) in Social Statistics in 1980 from the University of Southampton. He was then admitted to the University of Manitoba in 1980 and received a M.Sc. in statistics in 1982. In 1982, he was admitted to the Ph.D. program in statistics. He was a sessional lecturer in statistics from 1982-1985. He is currently employed as an Assistant Professor at the University of Winnipeg.