ON COMPACTIFICATIONS DETERMINED BY $C^{\#}(X)$

by

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ABSTRACT

The subalgebra $C^{\#}(X)$ was first defined by Nel and Riordan as being those members f of $C^{*}(X)$ such that for every maximal ideal of C(X) there exists a real number r such that $f-\mathbf{r}$ belongs to M. Since then, many different characterizations of $C^{\#}(X)$ have been found. In this thesis we describe $C^{\#}(X)$ for various topological spaces; these examples were found in papers written by Nel and Riordan, Stefani and Zanardo, and Choo. We also investigate when $C^{\#}(X)$ determines a compactification of X. Stefani and Zanardo have shown that if X is either pseudocompact, locally compact, 0-dimensional or srongly 0-dimensional then $C^{\#}(X)$ determines a compactification of X. We examine further conditions on X, as given by Melvin Henriksen, that allow $C^{\#}(X)$ to determine the Freudenthal compactification of X.

If $C^{\#}(X)$ determines a compactification γX of X we show that the zero sets of the members of $C^{\#}(X)$ form of Wallman base for γX. Also, compactifications determined by $C^{\#}(X)$ for arbitrary Tychonoff spaces X turn out to be perfect compactifications. Finally we give a characterization of the zero sets of the members of $C^{\#}(X)$ for arbitrary Tychonoff spaces.

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CHAPTER 1

BASIC PROPERTIES OF $C^{\#}(X)$

All hypothesized topological spaces will be assumed to be Tychonoff spaces. Notation and terminology will coincide with the ones used in [GJ]. Results indicated by (**) are new.

1.1 Definition Let X be a Tychonoff space. The subset $C^{\#}(X)$ of $C(X)$ is the set of all continuous real-valued functions f such that for every maximal ideal M in $C(X)$, there exists a real number r such that $f-\mathbf{r} \in M$.

It can easily be verified that $C^{\#}(X)$ is both a subalgebra and a sublattice of $C(X)$.

Before giving a few charaterizations of $C^{\#}(X)$ we note some of its basic properties. In what follows $C_K(X)$ will denote the set of all continuous functions with compact support and $C_F(X)$ the set of all continuous functions with finite image.

1.2 Proposition $C^{\#}(X)$ contains all constant functions.

Proof: This follows immediately from the fact that all maximal ideals of $C(X)$ contains the zero function $\mathbf{0}$. **QED**

The set $\{x \in X: f(x) = 0\}$ will be called the zero set $Z(f)$ of the function f. The cozero set of a function f will represent the set of points in the domain on which f does not vanish. We will denote the cozero set of f by $Cz(f)$. Also if A is a subset of the space X then we will denote the closure of A in X by $cl_X A$.

1.3 Proposition a) $C_K(X)$ is contained in $C^{\#}(X)$.

b) $C_F(X)$ is contained in $C^{\#}(X)$.

Proof: a) Let M be a maximal ideal of $C(X)$. Suppose $f \in C_K(X)$ but f does not belong to M. Then there exists a $g \in M$ such that $Z(f) \cap Z(g) = \varnothing$. Consequently $Z(g) \subseteq Cz(f)$. But if $f \in C_K(X)$ then $cl_X Cz(f)$ is compact. Consequently $Z(g)$ is compact. Now $Z(g)$ meets every zero set in $Z[M]$ and $Z[M]$ has the finite intersection property. Consequently $Z[M]$ has a non-empty intersection. Let $p \in \cap\{Z(h): h \in M\}$. Then $Z(f-\mathbf{f(p)})$meets every zero set of $Z[M]$. Hence $f-\mathbf{f(p)} \in M$ and so $f \in C^{\#}(X)$.

b) Let $f \in C^{\#}(X)$ and M be a maximal ideal in $C(X)$. Suppose $f[X] = \{r_1,r_2,\ldots,r_n\}$. Then $(f-\mathbf{r_1})(f-\mathbf{r_2}) \cdots (f-\mathbf{r_n}) = \mathbf{0}$, an element of M. Since M is a prime ideal then $f-\mathbf{r_i} \in M$ for some i; hence $f \in C^{\#}(X)$. **QED**

1.4 Proposition [SZ1] If S is C-embedded in X and $f \in C^{\#}(X)$ then $f|_S \in C^{\#}(S)$.

Proof: Since S is C-embedded in X then there exists a homomorphism ψ which maps $C(X)$ onto $C(S)$ defined by $\psi(f) = f|_S$. Let M be a maximal ideal in $C(S)$. Suppose $M = M^p$ where $p \in \beta S$. Then $\psi^{\leftarrow}[M] = \{\psi^{\leftarrow}[\{f \in C(S): p \in cl_S Z(f)\}] = \{f \in C(X): p \in cl_{\beta X} Z(f|_S)\}$. Note that since X is C-embedded in X then $cl_{\beta X} S = \beta S$ (6.9 [GJ]). Let $M' = \{f \in C(X): p \in cl_{\beta X} Z(f)\}$. We would like to show that $M' = \psi^{\leftarrow}[M]$.

We first claim that if S is a C-embedded subset of X and Z is a zero set of X then $cl_{\beta X}(S \cap Z) = cl_{\beta X}S \cap cl_{\beta X}Z$. Suppose x does not belong to $cl_{\beta X}(S \cap Z)$. The claim is established if we can show that x does not belong to $cl_{\beta X}S \cap cl_{\beta X}Z$. Suppose $x \in cl_{\beta X}Z$. Then there exists a zero set Z_0 of X such that $x \in cl_{\beta X}Z_0$ and $Z_0 \cap (Z \cap S) = \varnothing$. Let $g \in C(X)$ such that $Z(g) = Z \cap Z_0$. Then $Z(g) \cap S = \varnothing$ and so $1/g|_S \in C(S)$. If we let $f = 1/g|_S$ and f_0 be the extension of f over X then $f_0g[S] = \{1\}$ and $f_0g[Z(g)] = \{0\}$. Hence S and Z(g) are contained in disjoint zero sets of X. It follows from 6.5 of [GJ] that $cl_{\beta X}S \cap cl_{\beta X}Z(g) = \varnothing$. Now since $x \in cl_{\beta X}Z(g)$ then x does not belong to $cl_{\beta X}S$; thus x does not belong to $cl_{\beta X}S \cap cl_{\beta X}Z$ and so $cl_{\beta X}(S \cap Z) = cl_{\beta X}S \cap cl_{\beta X}Z$.

The following argument now completes the proof. If $f \in \psi^{\leftarrow}[M]$ then $p \in cl_{\beta S}Z(f|_S) = cl_{\beta X}Z(f|_S) \subseteq cl_{\beta X}Z(f)$. Hence $f \in M'$ and $\psi^{\leftarrow}[M] \subseteq M'$. Suppose $f \in M'$. Then $p \in cl_{\beta X}Z(f)$.

Since $p \in \beta S$ then:

$p \in cl_{\beta X}Z(f) \cap \beta S$

$= cl_{\beta X}Z(f) \cap cl_{\beta X}S$

$= cl_{\beta X}(Z(f) \cap S)$ (by the above claim)

$= cl_{\beta S}Z(f|_S)$.

Hence $f \in \psi^{\leftarrow}[M]$. It then follows that $M' \subseteq \psi^{\leftarrow}[M]$ and so $M' = \psi^{\leftarrow}[M]$. Thus $\psi^{\leftarrow}[M]$ is a maximal ideal in C(X).

Let $f \in C^{\#}(X)$. Then there exists an $r \in \mathbf{R}$ such that $f-\mathbf{r} \in \psi^{\leftarrow}[M]$. Since $\psi(f-\mathbf{r}) \in \psi \circ \psi^{\leftarrow}[M] = M$, then $(f|_S)-\mathbf{r} \in M$; thus $f|_S \in C^{\#}(S)$.
QED

Let $\mathbf{R}^*$ represent the extended real line (i.e. the one-point compactification of $\mathbf{R}$). If f is function mapping a space X into $\mathbf{R}^*$ let the f^* denote the extension of f over βX.

1.5 Proposition [SZ2] If $f \in C^{\#}(X)$, then $f[X]$ is compact. Hence $C^{\#}(X)$ is contained in $C^{*}(X)$.

Proof: We first show that $f[X]$ is closed in $\mathbf{R}$. Suppose $r \in cl_{\mathbf{R}} f[X] - [X]$. Let $Z_n = f^{\leftarrow}[\,[r-1/n, r+1/n]\,]$ and $S = \{Z_n\}_{n \in N}$; then S has the finite intersection property. Let Z[M] be a z-ultrafilter containing S. Since r is not contained in f[X] then the collection S has empty intersection in X. By hypothesis $f \in C^{\#}(X)$ and so $f-\mathbf{k} \in M$ for some k in $\mathbf{R}$; thus $Z(f-\mathbf{k}) \in Z(M)$. It follows that $Z(f-\mathbf{k}) \cap f^{\leftarrow}[[r-1/n, r+1/n]] \neq \varnothing$ for all n; hence k = r. This is a contradiction since $f^{\leftarrow}(r)$ does not meet X; consequently $cl_{\mathbf{R}} f[X] - f[X] = \varnothing$.

We now show that $f[X]$ is compact. Suppose $x \in \beta X - X$. Since $f[X]$ is closed in $\mathbf{R}$ then either $f^*(x) \in f[X]$ or $f^*(x) = \infty$. In the latter case, for every $n \in \mathbf{N}$, $x \in f^{*\leftarrow}[\mathbf{R}^* - [n,n]]$. Consider the maximal ideal M^p in C(X). There exists an r in $\mathbf{R}$ such that $(f-\mathbf{r}) \in M^p$, and so $x \in cl_{\beta X} Z(f-\mathbf{r})$ (7.3, [GJ]). Then for every n, there exists a $y_n \in [f^{*\leftarrow}[\mathbf{R}^* - [n,n]]] \cap Z(f-\mathbf{r})$. For each n, $f^*(y_n) = r$ and $\{f^*(y_n)\}_n$ converges to ∞ (since f^* is continuous and x is a limit point of $\{y_n\}_n$). This is a contradiction. Consequently $f^*(x) \in f[X]$, $f[X] = f^*[\beta X]$, hence $f[X]$ is compact. **QED**

Recall that if M^p is a maximal ideal of C(X) then $M^p = \{f \in C(X): p \in cl_{\beta X} Z(f)\}$. Let $p \in X$ and let σ_p be a homomorphism from C(X) onto the field of real numbers $\mathbf{R}$ defined by $\sigma_p(f) = f(p)$. Then M^p is the kernel of σ_p. It follows that the residue class ring $C(X)/M^p$ is isomorphic to $\mathbf{R}$ (see 4.6 of [GJ]). Let $\psi: C(X)/M^p \to \mathbf{R}$ be this isomorphism defined by $\psi(M^p + f) = f(p)$. Note that for every maximal ideal M^p the residue class ring $C(X)/M^p$ contains a canonical copy of $\mathbf{R}$, namely $M_{\mathbf{R}} = \{M^p + \mathbf{r}: r \in \mathbf{R}\}$. A maximal ideal M^p is said to be <u>real</u> if $C(X)/M^p$ is isomorphic to $\mathbf{R}$. We will say that the coset $M^p + f$ is real if $M^p + f \in M_{\mathbf{R}}$. In 5.6 of [GJ] it is shown that if p belongs to the realcompactification υX of X then M^p is real. In fact $\{p \in \beta X: M^p \text{ is real}\} = \upsilon X$. If M^p is a real maximal ideal then the z-ultrafilter $Z[M^p]$ is called a <u>real z-ultrafilter.</u>

Proofs of the equivalence of the following statements are found in part in [NR], [C1], [SZ2] and [Isa].

1.6 <u>Theorem</u> For $f \in C(X)$ then following are equivalent:

1) $f \in C^{\#}(X)$

2) For every C-embedded subset S of X, f[S] is compact.

3) $f \in C^*(X)$ and for every C-embedded copy D of $\mathbf{N}$ in X f[D] is closed, hence finite.

4) $f \in C^*(X)$ and f[Z] is closed for every zero set Z in X.

5) $f \in C^*(X)$ and for every $p \in \beta X - \upsilon X$ there is a neighbourhood of p in βX on which f^β is constant.

6) $f \in C^*(X)$ and, for every $r \in \mathbf{R}$, $cl_{\beta X} Z(f-\mathbf{r}) = Z(f^\beta - \mathbf{r})$.

7) (**) $f \in C^*(X)$ and, for every open subset U of βX, $f[U \cap X] = f^\beta[U]$.

8) M + f is real for any maximal ideal M in C(X).

Proof: 1) => 2) Let S be a C-embedded subset of X. By proposition 1.4, for every $f \in C^{\#}(X)$, $f|_S \in C^{\#}(S)$. By proposition 1.5, if $f|_S \in C^{\#}(X)$, $f|_S[S]$ is compact.

2) => 3) Let D be a C-embedded copy of $\mathbf{N}$ in X. Then by 1.5, f[D] is compact and consequently is closed. Suppose f[D] is infinite. Let $\{r_n: n \in \mathbf{N}\}$ be a discrete converging subsequence of f[D] with only finitely many repeating elements (0.13, [GJ]). This will contradict our hypothesis as we

shall now see. Now $f^{\leftarrow}[\{r_n\}_n]$ is clopen in D, and so is C-embedded in D, hence is C-embedded in X. Also $f^{\leftarrow}[\{r_n\}_n]$ is homeomorphic to **N**. But the image of $f^{\leftarrow}[\{r_n\}_n]$ is not compact since $f \circ f^{\leftarrow}[\{r_n\}_n]$ doesn't contain its limit point. Consequently f[D] is finite.

3) => 4). Suppose f[Z] is not closed for some zero set Z in X. Let $r \in cl_{\mathbf{R}}f[Z]$. Let $\{x_n\}_{n \in \mathbf{N}}$ be a sequence in Z such that the $\lim_{n \to \infty} f(x_n) = r$. Let $g \in C(X)$ such that $Z = Z(g)$ and $h = 1/((f-\mathbf{r})^2+g^2)$. Now $h \in C(X)$ and is unbounded on $\{x_n: n \in \mathbf{N}\}$. Consequently, by 1.20 of [GJ], $\{x_n: n \in \mathbf{N}\}$ contains a copy D of **N**, C-embedded in X. The fact that f[D] is not finite is a contradiction of our hypothesis. Consequently f[Z] is closed.

4) =>1) Let M be any maximal ideal of C(X). Then $M = M^p = \{f \in C(X): p \in cl_{\beta X}Z(f)\}$ for some p in βX (7.3 of [GJ]). Suppose that f satisties condition 4 and that $f-\mathbf{f^{\beta}(p)}$ is not contained in M^p. Then there exists a zero set Z in $Z[M^p]$ such that $p \in cl_{\beta X}Z$ and $Z \cap Z(f-\mathbf{f^{\beta}(p)}) = \varnothing$. Note that $f^{\beta}(p) \in f^{\beta}[cl_{\beta X}Z] = cl_{\mathbf{R}}f^{\beta}[Z] = cl_{\mathbf{R}}f[Z] = f[Z]$ (since f is bounded and $cl_{\mathbf{R}}[Z]$ is compact). Then there exists a $y \in Z$ such that $f^{\beta}(p) = f(y)$. Then y belongs to Z and to $Z(f-\mathbf{f^{\beta}(p)})$. This contradicts the fact that $Z \cap Z(f-\mathbf{f^{\beta}(p)})$ is empty. Consequently $f-\mathbf{f^{\beta}(p)} \in M^p$, which implies $f \in C^{\#}(X)$.

3) => 5). Suppose that for every C-embedded copy D of **N** in X, f[D] is finite. Let $f \in C^{\#}(X)$ and let $x_0 \in \beta X - \upsilon X$. Suppose f^{β} not constant on any neighbourhood of x_0. Let $g \in C(\beta X)$ such that $x_0 \in Z(g) \subseteq \beta X - \upsilon X$ (1.53, [Wa]). We will construct the following sequence. Choose $x_1 \in g^{\leftarrow}[(-1,1)] \cap X$ such that $f(x_1) \neq f(x_0)$ and $x_{n+1} \in f^{\beta\leftarrow}[(x_0) - \delta_n, f(x_0) + \delta_n)] \cap g^{\leftarrow}[(-1/n,1/n)] \cap X$, where $\delta_n = |f(x_n) - f(x_0)|$, and $f(x_{n+1}) \neq f(x_0)$ for all n. Let $D = \{x_n: n \in \mathbf{N}\}$. D is a C-embedded copy of **N**, since g maps D homeomorphically onto a closed set in **R** (1.19 [GJ]). The fact that f[D] is not finite is a contradiction. Hence

such a sequence cannot be constructed, and so f^β must be constant on some neighbourhood of x_0.

5) => 6) Obviously $cl_{\beta X}Z(f-\mathbf{r}) \subseteq Z(f^\beta-\mathbf{r})$. Let $p \in Z(f-\mathbf{r})$. We will consider two cases. First suppose $p \in Z(f^\beta-\mathbf{r}) - \upsilon X$. By 8D(1) of [GJ] this implies $p \in cl_{\beta X}Z(f-\mathbf{r})$. Now suppose $p \in Z(f^\beta-\mathbf{r})-\upsilon X$. By hypothesis f^β is constant on a neighbourhood of p, so $Z(f^\beta-\mathbf{r})$ is a neighbourhood of p. It follows that $p \in cl_{\beta X}Z(f-\mathbf{r})$, and that $Z(f^\beta-\mathbf{r}) \subseteq cl_{\beta X}Z(f-\mathbf{r})$.

6) => 1) Suppose $f \in C^*(X)$ and that for every $r \in \mathbf{R}$ $Z(f^\beta-\mathbf{r}) = cl_{\beta X}Z(f-\mathbf{r})$. Let M be any maximal ideal of C(X). Then $M = M^p = \{f \in C(X): p \in cl_{\beta X}Z(f)\}$ for some p in βX (7.4 of [GJ]). Now $p \in Z(f^\beta - \mathbf{f^\beta(p)})$ and, since $Z(f^\beta - \mathbf{f^\beta(p)}) = cl_{\beta X}Z(f - \mathbf{f^\beta(p)})$, $p \in cl_{\beta X}Z(f - \mathbf{f^\beta(p)})$. Consequently $f - \mathbf{f^\beta(p)} \in M^p$ and $f \in C^\#(X)$.

6) => 7). Suppose $f \in C^*(X)$ and for every $r \in \mathbf{R}$, $cl_{\beta X}Z(f-\mathbf{r}) = Z(f^\beta-\mathbf{r})$. Let U be an open subset of βX. That $f[U \cap X] \subseteq f^\beta[U]$ is obvious. If $r \in f^\beta[U]$, then $Z(f^\beta-\mathbf{r}) \cap U$ is non-empty and since $Z(f^\beta-\mathbf{r}) = cl_{\beta X}Z(f-\mathbf{r})$ it follows that $cl_{\beta X}Z(f-\mathbf{r}) \cap U$ is non-empty. Since U is open then $U \cap Z(f-\mathbf{r})$ is non-empty. Consequently $r \in f[U \cap X]$.

7) => 6) Suppose that for every open subset U of βX, $f[U \cap X] = f^\beta[U]$. We are required to show that $cl_{\beta X}Z(f-\mathbf{r}) = Z(f^\beta-\mathbf{r})$. Suppose $x \in Z(f^\beta-\mathbf{r}) - cl_{\beta X}Z(f-\mathbf{r})$. Let U be an open neighbourhood of x such that $U \cap cl_{\beta X}Z(f-\mathbf{r}) = \varnothing$. Then r is not contained in $f[U \cap X]$. But since U meets $Z(f^\beta-\mathbf{r})$ then $r \in f^\beta[U]$. This contradicts the given condition, and so our implication follows.

1) => 8) If $f \in C^\#(X)$ then for any maximal ideal M of C(X) there exists a real number r such that $f-\mathbf{r} \in M$. Hence $M + (f-\mathbf{r}) = M$ and $M + f = M + \mathbf{r}$.

8) => 1) Suppose that $M^p + f$ is real for each $p \in \beta X$. Thus $M^p + f = M^p + \mathbf{r}$ where $r \in \mathbf{R}$. Then $f-\mathbf{r} \in M^p$; hence $f \in C^\#(X)$. **QED**

In [Fr], Frolik defines a Z-mapping from a space X to a space Y as being a map which sends the zero sets of X to closed sets of Y. By condition 4 of 1.6, $C^{\#}(X)$ consists precisely of those members of $C^*(X)$ that are Z-mappings. In [Isa], Isawata establishes the following proposition for the family of WZ-mapping of which $C^{\#}(X)$ forms a subfamily. For completeness we include a proof of the special case.

1.7 Proposition [Isa] If $f \in C^{\#}(X)$, then f is open iff f^{β} is open.

Proof: (=>) Suppose f^{β} is open and let U be an open subset of X. Let U^* be an open subset of βX such that $U = U^* \cap X$. Then by 1.6 (7), $f[U] = f[U^* \cap X] = f^{\beta}[U^*]$, which is open by hypothesis.

($\Leftarrow$) Let f be open and let U be an open subset of βX. Then by 7) of 1.6, $f^{\beta}[U] = f[U \cap X]$. But $f[U \cap X]$ is open in **R** so $f^{\beta}[U]$ is open. Thus f^{β} is an open map. **QED**

There is an interesting characterization of $C^{\#}(X)$ which depends only on maximal ideals in $C(X)$ and $C^*(X)$.

1.8 Theorem [C2] $C^{\#}(X)$ is the largest subring of $C^*(X)$ satisfying:

1) $C^{\#}(X)$ contains all constant functions.

2) $M^p \cap C^{\#}(X) = M^{*p} \cap C^{\#}(X)$ for every $p \in \beta X$.

Proof: Suppose G is a subring of $C^*(X)$ satisfying conditions 1 and 2. We will first show that G is contained in $C^{\#}(X)$. Let $g \in G$ and let $r \in \mathbf{R}$. It will suffice to show that $Z(g^{\beta}-\mathbf{r}) \subseteq cl_{\beta X}Z(g-\mathbf{r})$. Let $p \in Z(g^{\beta}-\mathbf{r})$. Then $g^{\beta}(p) = r$ and so $g-\mathbf{r} \in M^{*p} \cap G$. Condition 2 implies that $g-\mathbf{r} \in M^p \cap G$, and so $p \in cl_{\beta X}Z(g-\mathbf{r})$. Hence $Z(g^{\beta}-\mathbf{r}) = cl_{\beta X}Z(g-\mathbf{r})$. Then, by 6) of theorem 1.6, $g \in C^{\#}(X)$, consequently $G \subseteq C^{\#}(X)$.

For the converse, we know that $C^{\#}(X)$ satisfies condition 1. We now show that it satisfies condition 2. Let $p \in \beta X$. Recall that $M^p = \{f \in C(X): p \in cl_{\beta X}Z(f)\}$ and that $M^{*p} = \{f \in C^*(X): f^{\beta}(p) = 0\}$. If $g \in M^p \cap C^{\#}(X)$, then $p \in cl_{\beta X}Z(g) = Z(g^{\beta})$ (by 1.6). Now $g^{\beta}(p) = 0$, and so $g \in M^{*p} \cap C^{\#}(X)$. Thus $M^p \cap C^{\#}(X) \subseteq M^{*p} \cap C^{\#}(X)$. Conversely, let $g \in M^{*p} \cap C^{\#}(X)$. Then $g^{\beta}(p) = 0$, hence $p \in Z(g^{\beta}) = cl_{\beta X}Z(g)$. Consequently $g \in M^p \cap C^{\#}(X)$, and so $M^{*p} \cap C^{\#}(X) \subseteq M^p \cap C^{\#}(X)$. We have thus shown that $M^p \cap C^{\#}(X) = M^{*p} \cap C^{\#}(X)$.
QED

If f belongs to C(X) the unique extension of f to a member of $C(\upsilon X)$ will be denoted by f^{υ}. The following theorem shows that $C^{\#}(X)$ does not distinguish the space X from its realcompactification υX.

1.9 Theorem(**) The mapping defined by $\psi(f) = f^{\upsilon}$ induces an isomorphism from $C^{\#}(X)$ onto $C^{\#}(\upsilon X)$.

Proof: This map is well known to be a one-to-one ring homomorphism from $C^*(X)$ into $C(\upsilon X)$. We would like to show that every f in $C^{\#}(X)$ extends to f^{υ} in $C^{\#}(\upsilon X)$ and that, for every f in $C^{\#}(\upsilon X)$, $f|_X$ belongs to $C^{\#}(X)$. Since X is C-embedded in υX, it follows from proposition 1.4 that $f|_X \in C^{\#}(X)$ whenever $f \in C^{\#}(\upsilon X)$.

Let $f \in C^{\#}(X)$. Then by 5) of 1.6, for any $p \in \beta X - \upsilon X$, f^{υ} is constant on a neighbourhood of p. Since $f^{\beta} = (f^{\upsilon})^{\beta}$, then by 1.6, $f^{\upsilon} \in C^{\#}(\upsilon X)$. Thus the mapping $\psi: C^{\#}(X) \rightarrow C^{\#}(\upsilon X)$ defined by $\psi(f) = f^{\upsilon}$ is an isomorphism. **QED**

1.10 Proposition(**) The space X is pseudocompact iff $C(X) = C^{\#}(X)$.

Proof: Suppose X is pseudocompact. Every maximal ideal in C(X) is real in C(X) when and only when X is pseudocompact (5.8 (b) of [GJ]). If M is a maximal ideal of C(X) then since M is real C(X)/M is a canonical copy of **R**. Consequently for every f in C(X) there exists r $\in$ **R** such that M + f = M + **r**. Hence f–**r** $\in$ M; as M was arbitrary f $\in$ $C^{\#}(X)$. As f was arbitrary C(X) = $C^{\#}(X)$. Conversely if C(X) = $C^{\#}(X)$ then C(X) = $C^{*}(X)$; thus X is pseudocompact. **QED**

1.11 <u>Proposition</u> [NR] For f $\in$ $C^{*}(X)$, the following are equivalent:

1) f $\in$ $C^{\#}(X)$.

2) Every z-ultrafilter on X has a member on which f is constant.

3) For every z-ultrafilter U on X, the family $f^{\#}U$ of all closed sets in **R** whose preimage under f belongs to U is a z-ultrafilter.

Proof: 1) => 2) Let f $\in$ $C^{\#}(X)$. Let U be a z-ultrafilter on X. Let $Z^{\leftarrow}[U]$ = {f $\in$ C(X): Z(f) $\in$ U}. Note that $Z^{\leftarrow}[U]$ is a maximal ideal in C(X) (2.5 (b) of [GJ]). By hypothesis (and 1.6) there exists an r in **R** such that f–**r** $\in$ $Z^{\leftarrow}[U]$; that is Z(f–**r**) $\in$ U. Consequently f is constant on some member of U.

2) => 3) Let U be a z-ultrafilter on X and let f $\in$ C(X) such that f is constant on some member of U. Let $f^{\#}U$ = {A : A is a closed subset of **R** and $f^{\leftarrow}[A]$ $\in$ U}. We will first show that $f^{\#}U$ is a z-filter. Closed subsets are zero sets in **R** and so $f^{\leftarrow}[A]$ is a zero set for every A in $f^{\#}U$. Let B_1 and B_2 belong to $f^{\#}U$. Then $f^{\leftarrow}[B_1]$ and $f^{\leftarrow}[B_2]$ belong to U. Note that $f^{\leftarrow}[B_1] \cap f^{\leftarrow}[B_2] = f^{\leftarrow}[B_1 \cap B_2] \in$ U, and that $B_1 \cap B_2$ is a closed subset of **R**. Therefore $B_1 \cap B_2$ $\in$ $f^{\#}U$. Let A $\in$ $f^{\#}U$ and let M be closed subset subset of **R** such that A $\subseteq$ M. Evidently $f^{\leftarrow}[A] \subseteq f^{\leftarrow}[M]$, so $f^{\leftarrow}[M]$ meets every member of U. Consequently $f^{\leftarrow}[M]$, a zero set of X, belongs to U. Thus M $\in$ $f^{\#}U$, and we have shown that $f^{\#}U$ is a z-filter.

Now let S be a closed subset of **R** such that S meets every member of $f^{\#}U$. By hypothesis f is constant on some member of U. Then there exists an r in **R** such that for some V in U, $V \subseteq Z(f-\mathbf{r})$. Consequently $Z(f-\mathbf{r}) \in U$. Thus $\{r\} \in f^{\#}U$, and $\{r\} \subseteq S$. Now $S \in f^{\#}U$ since S is a superset of $\{r\}$. Consequently $f^{\#}U$ is a z-ultrafilter.

3) => 1) Let M be a maximal ideal in C(X). Then Z[M] is a z-ultrafilter on X, and $f^{\#}Z[M]$ is a z-ultrafilter on **R**. We will show that if $V \in Z[M]$ then $cl_{\mathbf{R}}f[V] \in f^{\#}Z[M]$. Let $V \in Z[M]$ and $W \in f^{\#}Z[M]$. Then $f^{\leftarrow}[W] \in Z[M]$ and $V \cap f^{\leftarrow}[W]$ is non-empty. Now $f[V \cap f^{\leftarrow}[W]] \subseteq f[V] \cap f \circ f^{\leftarrow}[W] \subseteq f[V] \cap W$, so $f[V] \cap W$ is non-empty. Hence f[V] meets every member of $f^{\#}Z[M]$ and so $cl_{\mathbf{R}}f[V]$ is a zero set in **R** which meets all members of $f^{\#}[M]$. This implies that $cl_{\mathbf{R}}f[V] \in f^{\#}[M]$.

Since f is bounded it follows that $cl_{\mathbf{R}}f[V]$ is compact. As $f^{\#}Z[M]$ has the finite intersection property it follows that $\cap\{A:A \in f^{\#}Z[M]\}$ is non-empty. Let $r \in \cap\{f^{\#}Z[M]\}$. Then $\{r\}$ is a zero set which meets every member of $f^{\#}Z[M]$, so $\{r\} \in f^{\#}Z[M]$. Now $f^{\leftarrow}[\{r\}] = Z(f-\mathbf{r}) \in Z[M]$, so $f-\mathbf{r} \in M$. Thus $f \in C^{\#}(X)$. **QED**

In [NR], Nel and Riordan seem to imply that $f \in C(X)$ suffices for the above proposition to be true. However the condition that f be bounded is required as the following example shows. Let X = **R** and define $f:\mathbf{R} \to \mathbf{R}$ by $f(x) = x$. Evidently f satisfies condition 3, but the fact that f is not bounded implies that f does not belong to $C^{\#}(X)$.

1.12 <u>Corollary</u> [SZ2] If X is discrete then $C^{\#}(X) = C_F(X)$.

Proof: It follows from 1.3 that $C_F(X) \subseteq C^{\#}(X)$. Let $f \in C^{\#}(X)$ and suppose X is infinite. We construct the sequence $A = \{x_n\}_n$ from f[X] such that all x_n are distinct. Now $f^{\leftarrow}[A]$ is infinite, countable and discrete. Consequently it is a C-embedded copy of **N**. But $f \circ f^{\leftarrow}[A]$ is not finite. This contradicts 1.6 (3). Hence $f[X]$ is finite, $f \in C_F(X)$ and so $C_F(X) = C^{\#}(X)$. **QED**

1.13 Corollary [SZ2] Let $f \in C^{\#}(X)$ and let C be a compact subset of $\beta X - \upsilon X$. Then $f^{\beta}[C]$ is finite. In particular if X is locally compact and realcompact and $f \in C^{\#}(X)$ then $f^{\beta}[\beta X - X]$ is finite.

Proof: Let $f \in C^{\#}(X)$ and C be a compact subset of $\beta X - \upsilon X$. If $x \in C$ then f^{β} is constant on an open neighbourhood D_x (in βX) of x (1.6 (5)). Then $\{D_x: x \in C\}$ forms an open cover of the compact set C, and hence has a finite subcover $\{D_{xi}: i = 1 \text{ to } n\}$. It follows that $f^{\beta}[C]$ is finite. If X is locally compact and realcompact then X is an open subset of βX (6.9, [GJ]). Thus $\beta X - X$ is a compact subset of $\beta X - \upsilon X$, and hence $f^{\beta}[\beta X - \upsilon X]$ is finite. **QED**

1.14 Corollary [SZ2] If $f \in C^{\#}(X)$ and f^{β} is constant on a subset E of $\beta X - \upsilon X$, then f^{β} is constant on an open neighbourhood in βX of E.

Proof: Let $f \in C^{\#}(X)$, let $E \subseteq \beta X - \upsilon X$, and suppose $f^{\beta}[E] = \{r\}$ where $r \in \mathbf{R}$. By 1.6 (5) f^{β} is constant on a neighbourhood of every point of E. It follows immediately that f^{β} is constant on an open neighbourhood of E. **QED**

1.15 Proposition [SZ2] Let X be locally compact and realcompact. Then $f \in C^{\#}(X)$ iff $f \in C^*(X)$ and every connected component of $\beta X - X$ has an open neighbourhood in βX on which f^{β} is constant.

Proof: (=>) By corollary 1.13 $f^{\beta}[\beta X - X]$ is finite. Let E be a connected component of $\beta X - X$. Evidently f^{β} is constant on E (since E is connected). Consequently, by corollary 1.14, f^{β} is constant on an open neighbourhood of E.

($\Leftarrow$) This follows immediately from theorem 1.6 (5). **QED**

We will say that a topological space is ultranormal if, for any pair of disjoint closed subsets F_1 and F_2 in X, there exists a clopen subset A containing F_1 and missing F_2.

In [Do] Dominguez defines and investigates $C^{\#}(X,E)$ for nonarchimedean fields E. Such rings are "nonarchimedean analogues" of $C^{\#}(X)$. He proves that if $f \in C^{\#}(X,E)$ and X is realcompact then f is a closed E-valued function. The following proposition verifies that the corresponding result holds for the real-valued functions of $C^{\#}(X)$.

1.16 Proposition(**) Let $f \in C^{\#}(X)$ and assume X is ultranormal. If B is closed in X then f[B] is compact. (In particular, f is a closed map.)

Proof: Let $f \in C^{\#}(X)$ and let B be a closed subset of X. Suppose f[B] is not closed. Let $r \in f^{\beta}[cl_{\beta X}B]-f[B]$, and $p \in cl_{\beta X}B \cap f^{\beta\leftarrow}(r)$. Then $f^{\beta}(p) = r$ and $f^{\beta\leftarrow}(r) \cap B$ is empty, i.e. $Z(f-\mathbf{r}) \cap B = \varnothing$. Since X is ultranormal we can find a clopen set A such that $Z(f-\mathbf{r}) \subseteq A$ and $B \subseteq X-A$. Let χ_A be the characteristic function on A; thus $\chi_A[A] = \{1\}$ and $\chi_A[X-A] = \{0\}$. Then $B \subseteq Z(\chi_A)$, so we have $\chi_A \in M^p = \{f \in C(X): p \in cl_{\beta X}Z(f)\}$ (as $p \in cl_{\beta X}B \subseteq cl_{\beta X}(X-A)$). Thus $Z(\chi_A) \in Z[M^p]$. Also, since $f \in C^{\#}(X)$ and $r = f^{\beta}(p)$ it follows that $f-\mathbf{r} \in M^p$ (here we use the fact that $Z(f^{\beta}-\mathbf{r}) = cl_{\beta X}Z(f-\mathbf{r})$ by 1.6 (6)). But $Z(f-\mathbf{r}) \cap Z(\chi_A)$ is empty. This contradicts the fact that $Z(f-\mathbf{r})$ and $Z(\chi_A)$ both belong to $Z[M^p]$. Thus $f[B] = f[cl_{\beta X}B]$ and so f[B] is compact. **QED**

We also have another condition that we can impose on X which forces the ring $C^{\#}(X)$ to coincide with the set of closed maps in $C^*(X)$.

1.17 Theorem [He] If X is realcompact and $f \in C^{\#}(X)$ then the frontier in X of Z(f) (denoted $Fr_X Z(f)$) is compact and f is closed.

Proof: Let $f \in C^{\#}(X)$. If Z(f) is compact, then obviously $Fr_X Z(f)$ is compact. Suppose Z(f) is not compact. We will show that $Fr_X Z(f) = Fr_{\beta X} Z(f) \subseteq X$. Suppose $p \in (\beta X - X) \cap Fr_{\beta X} Z(f)$. Then by 1.6 there exists a neighbourhood of p in βX, say N, such that f^β is constant on N. Since $p \in Fr_{\beta X} Z(f)$ the interior of N meets both $\beta X - cl_{\beta X} Z(f)$ and Z(f), and consequently $f^\beta(p) = 0$. Then there exists an $x \in \beta X - cl_{\beta X} Z(f) = \beta X - Z(f^\beta)$ (1.6) such that $f^\beta(x) = 0$. This is a contradiction: no such point can exist outside of $Z(f^\beta)$. Therefore $Fr_{\beta X} Z(f) \subseteq X$ and so $Fr_X Z(f) = Fr_{\beta X} Z(f)$. It follows that $Fr_X Z(f)$ is compact. In 1.3 of [Isa], Isawata shows that if f is a Z-mapping and $Fr_X Z(f)$ is compact, then f is closed. The theorem follows. **QED**

1.18 Corollary [He] Let X be normal and metacompact. If $f \in C^{\#}(X)$ and every closed discrete subspace of X is realcompact then f is closed.

Proof: Morita, in [Mor], shows that if every closed discrete subspace of a normal metacompact space X is realcompact then so is X. Consequently by the above theorem f is closed. **QED**

A space X is called δ-normally separated if whenever $Z \in Z(X)$ and A is closed in X and $Z \cap A$ is empty then Z and A are completely separated in X.

Let D(X) denote the set of all closed functions in $C^*(X)$.

1.19 Proposition(**) Let X be a δ-normally separated space or realcompact. Then X is countably compact iff $C(X) = C^{\#}(X)$.

Proof: In [C1] Choo shows that X is countably compact iff every continuous function on X is bounded and closed, i.e. $C(X) = D(X)$. Suppose X is δ-normally separated. In [Ze], Zenor shows that a Tychonoff space is countably compact iff it is pseudocompact and δ-normally separated. If S is countably compact then X is pseudocompact and $C(X) = C^{\#}(X)$, by 1.10. Conversely if $C(X) = C^{\#}(X)$ then, by 1.10, X is pseudocompact and, as noted above, X is countably compact. Suppose now that X is realcompact. Theorem 1.17 states that if X is realcompact then $C^{\#}(X) \subseteq D(X)$. Now $D(X) \subseteq C^{\#}(X)$ since for every f in D(X) f is bounded and f sends zero sets to closed sets. Thus $C^{\#}(X) = D(X) = C(X)$. Conversely if $C(X) = C^{\#}(X)$ then, by 1.10, X is pseudocompact. Pseudocompact realcompact spaces are compact (5H (2), [GJ]) and so X is countably compact. **QED**

Let F(X) denote the subalgebra of C(X) consisting of those functions f in C(X) for which there exists a compact subset K in X (K depending on f) such that $f[X-K]$ is finite.

In [Do] Dominguez shows that for a metacompact locally compact realcompact space the nonarchimedean analogue of $C^{\#}(X)$, namely $C^{\#}(X,E)$, is in fact just F(X,E) (the nonarchimedean analogue of F(X)). Proposition 1.20 shows a similar result is true for $C^{\#}(X)$ not only under these conditions but also if X is ultranormal.

1.20 Proposition(**) Let X be a metacompact locally compact space. If X is either realcompact or ultranormal then $C^{\#}(X) = F(X)$.

Proof: We will first show that $F(X) \subseteq C^{\#}(X)$. Let $f \in F(X)$ and let D be a C-embedded copy of $\mathbf{N}$ in X. (If there does not exist such a D then S is pseudocompact (1.21, [GJ]) and $C(X) = C^{\#}(X)$, by 1.10. Hence $F(X) \subseteq C^{\#}(X)$.) Let K be compact subset of S such that $f[X-K]$ is finite. Since K is compact D

$\cap$ K is finite. Then $f[D] = f[D \cap K] \cup f[D \cap (X - K)]$ is finite. It follows from 1.6 (3) that $f \in C^{\#}(X)$ and finally that $F(X) \subseteq C^{\#}(X)$.

Now we show that $C^{\#}(X) \subseteq F(X)$. Let $f \in C^{\#}(X)$ and assume that X is realcompact, metacompact and locally compact. Then by 1.17 f is closed and, by 1.5, f[X] is compact. In theorem 2 of [No], it is shown that if $f{:}Z \to Y$ is a closed continuous map from a metacompact locally compact Hausdorff space Z to a compact space Y, then there exists a compact subset $K \subseteq Z$ such that $f[Z-K]$ is finite. It follows from this that $f \in F(X)$. Consequently $C^{\#}(X) = F(X)$.

If X is ultranormal then by 1.16 f[X] is compact. Hence it follows just as in the case above that $C^{\#}(X) = F(X)$. **QED**

We now investigate $C^{\#}(X)$ for some familiar topological spaces.

1.21 $C^{\#}(\mathbf{R}^n)$ for $n \in \mathbf{N}$. $\mathbf{R}^n$ is realcompact and locally compact. Furthermore $\mathbf{R}^n$ is paracompact and consequently metacompact (= weakly paracompact). Then by proposition 1.20 $C^{\#}(\mathbf{R}^n)$ is the set of all functions in C(X) such that f is finite on the complement of some compact subset of $\mathbf{R}^n$. If $n > 1$ then by 6L (3) of [GJ] $\beta\mathbf{R}^n - \mathbf{R}^n$ is connected; thus it follows from 1.15 that $f^{\beta}[\beta\mathbf{R}^n - \mathbf{R}^n]$ takes on a single value. In the case of $\mathbf{R}$, $f \in C^{\#}(\mathbf{R})$ implies that f is constant on $\{x: x \leq a\}$ and constant on $\{x: x \geq b\}$ for some $a, b \in \mathbf{R}$. (This follows from the fact that f has only a finite number of values on the complement of a bounded subset of $\mathbf{R}$, hence f must be constant on both of the connected "tail ends" of $\mathbf{R}$).

We also note that, by proposition 1.15, $f^{\beta}[\beta\mathbf{R} - \mathbf{R}]$ takes on at most two values since $\beta\mathbf{R} - \mathbf{R}$ has only two connected components (6.10, [GJ]).

1.22 $C^{\#}(\mathbf{R}^{\aleph_0})$ is exactly the set of all constant functions [SZ1]

Proof: Let $f \in C^{\#}(\mathbf{R}^{\aleph_0})$. Let $E_n = \{\langle x_i\rangle_{i \in \mathbf{N}} \in \mathbf{R}^{\aleph_0}: x_i = 0 \text{ for } i > n\}$. Let $f_n = f|_{E_n}$. Note that E_n is closed in the normal space $\mathbf{R}^{\aleph_0}$ and hence is C-embedded in $\mathbf{R}^{\aleph_0}$ (see 3D of [GJ]). Thus by 1.4 $f_n \in C^{\#}(E_n)$. It is clear that E_n is homeomorphic to $\mathbf{R}^n$. Then, by proposition 1.20, there exists a compact subset in E_n such that f_n is finite on its complement. Consequently for every $n \geq 2$ we can find an $a_n > 0$ and $r_n \in \mathbf{R}$ such that, for the compact subset $K_n = \{\langle x_i\rangle_{i = 1 \text{ to } n} \in E_n: |x_i| \leq a_n, 1 \leq i \leq n\}$, $f_n[E_n - K_n] = \{r_n\}$. Such an a_n exists for each $n \geq 2$ since f_n will be finite on the complement of a bounded subset of E_n consequently on the unbounded connected part of E_n. Therefore the continuous function f_n must be constant there. We can choose a_n such that it is the smallest so that $f_n[E_n - K_n]$ is constant. (We can justify this by the following argument. Observe that f_n will be constant on $\Pi_{j = 1 \text{ to } n}(\mathbf{R} - [a_{(j)1}, a_{(j)2}])$. We let $a_n = \max\{|a_{(j)1}| : j = 1 \text{ to } n, i = 1,2\}$. Clearly $|x_i| \leq a_n$ for all $i = 1$ to n, and a_n is the smallest number such that $f_n[E_n - K_n] = \{r_n\}$.) Note that $f_{n+1}|_{E_n} = f_n$ for all n. Then it follows that $K_n \subseteq K_{n+1}$ and $r_{n+1} = r_n = r_2 = r$ for all $n \geq 2$. We will now show that $f(\langle x_i\rangle_{i \in \mathbf{N}}) = r$ for all $\langle x_i\rangle_{i \in \mathbf{N}} \in \mathbf{R}^{\aleph_0}$. Let $\langle x_i\rangle_{i \in \mathbf{N}} \in \mathbf{R}^{\aleph_0}$. Since f is continuous then, for every $\varepsilon > 0$, there exists an open neighbourhood A of $\langle x_i\rangle_{i \in \mathbf{N}}$ such that $|f(\langle x_i\rangle) - f(\langle y_i\rangle)| < \varepsilon$ for all $\langle y_i\rangle \in A$. Let $A = \cap_{1 \leq i \leq m}\pi_i^{\leftarrow}A_i$ where A_i is an open neighbourhood of x_i in $\mathbf{R}$, $1 \leq i \leq m$. We now specifically choose from A the point $\langle y_i\rangle_{i \in \mathbf{N}}$ such that $y_i = x_i$ for $1 \leq i \leq m$, $|y_{m+1}| > a_m$ and $y_i = 0$ for $i > m+1$. Then $\langle y_i\rangle \in E_{m+1} - K_{m+1}$, $\langle y_i\rangle \in A$ and $f(\langle y_i\rangle) = f_{m+1}(\langle y_i\rangle) = r$. We now have the following situation: for every $\langle x_i\rangle_{i \in \mathbf{N}} \in \mathbf{R}^{\aleph_0}$, and for every $\varepsilon > 0$, and for any open neighbourhood A of $\langle x_i\rangle_i$, there exists $\langle y_i\rangle \in A$ such that $f(\langle x_i\rangle) = r$. Consequently $|f(\langle x_i\rangle) - f(\langle y_i\rangle)| = |f(\langle x_i\rangle) - r| < \varepsilon$ and so $f(\langle x_i\rangle) = r$. **QED**

1.23 $C^{\#}(\mathbf{N}) = C_F(\mathbf{N})$. This is a special case of 1.12.

In [NR] Nel and Riordan state that $C^{\#}(\mathbf{Q}) = C_F(\mathbf{Q})$. However this cannot be true as the following proposition will show.

1.24 Proposition(**) If X is 0-dimensional and first countable then $C_F(X) = C^{\#}(X)$ iff X is discrete.

Proof: We have already shown in 1.12 that if X is discrete then $C^{\#}(X) = C_F(X)$. Suppose that X is 0-dimensional and first countable and that $C_F(X) = C^{\#}(X)$. Suppose X is not discrete. Let q be a non-isolated point of X. As X is first countable, there exists a discrete sequence $\{q_n: n = 0 \text{ to } \infty\}$ in X that converges to q. Let $H = \{q_n: n = 0 \text{ to } \infty\} \cup \{q\}$. Let $\{M_i: i = 0 \text{ to } \infty\}$ be a collection of neighbourhoods of q such that $q_0 \in M_0$, $q_n \in M_n$ but $\{q_0, q_1, ..., q_{n-1}\} \subseteq X - M_n$ and $\cap\{M_i: i = 0 \text{ to } \infty\} = \{q\}$. Let N_0 be a clopen subset of X such that $q_0 \in N_0$, $N_0 \subseteq M_0$ and $N_0 \cap H - \{q_0\} = \varnothing$. Let N_n be a clopen subset of X such that $q_n \in N_n$, $N_n \subseteq M_n$ and $N_n \cap [\cup\{N_k: k = 0 \text{ to } n-1\} \cup H - \{q_n\}] = \varnothing$. We will define a function f such that $f(x) = 0$ if $x \in \{q\} \cup X - \cup\{N_k: k = 0 \text{ to } \infty\}$, and $f(x) = 1/(n+1)$ if $x \in N_n$. It can easily be verified that f is a continuous function. Let F be a non-empty closed subset of X. If $\{k \in \mathbf{N}: F \cap N_k \neq \varnothing\}$ is infinite then f[F] contains the point zero ($q \in F$ since F is closed) and hence f[F] is closed in $\mathbf{R}$. If $\{k \in \mathbf{N}: F \cap N_k \neq \varnothing\}$ is finite then f[F] is finite and hence is again closed in f[X]. Consequently by 1.6 (4) $f \in C^{\#}(X)$. But f(X) is not finite. This contradicts the fact that $C_F(X) = C^{\#}(X)$. It follows that no point of X is the limit of a sequence of distinct points; thus X is discrete. **QED**

Since $\mathbf{Q}$ is both 0-dimensional and first countable but is not discrete then by the above proposition $C_F(\mathbf{Q}) \neq C^{\#}(\mathbf{Q})$.

CHAPTER 2

Compactifications determined by $C^{\#}(X)$

In this section we study how, and when, $C^{\#}(X)$ can be used to compactify X.

We will say that γX is a compactification of X if γX is a compact space and X is dense in γX. Let K(X) denote the set of all compactifications of X. Its elements will actually be equivalence classes of compactifications. A compactification αX is said to be equivalent to a compactification γX of X if there exists a homeomorphism $h:\alpha X \to \gamma X$ such that $h(x) = x$ for each x in X. In what follows we will not distinguish between equivalent compactifications of X. Also note that in this section, when we say that a set is a subalgebra of C(X), we will mean it is also a sublattice of C(X).

If A is a subalgebra of C(X), we will say that A separates points of X if for any two points x_1 and x_2 of X there exists a function f in A such that $f(x_1) = 0$ and $f(x_2) = 1$. Also, A is said to separate points and closed sets of X if given any closed subset F of X and a point x not contained in F there exists a function f in A such that $f(x) \notin cl_{\mathbf{R}}f[F]$.

We start with a brief discussion on the general theory of compactifications and their associated subalgebras of $C^*(X)$. Lemma 2.1 and theorem 2.2 and 2.5 are well known results.

2.1 Lemma If K is a compact Hausdorff space and A is a subalgebra of C(K) that contains the constant functions, then the following are equivalent:

1) A separates points and closed sets of K.

2) A separates points of K.

Proof: It is obvious that 1) implies 2). We now show that 2) implies 1). Let F be closed subset of K and $x \in K-F$. For every $y \in F$ there exists an $f_y \in A$ such that $f_y(x) = 0$ and $f_y(y) = 1$. Now $\{f_y^{\leftarrow}[(1/2,2)]: y \in F\}$ forms an open cover of F. Let $\{f_{y_i}^{\leftarrow}[(1/2,2)]: y_i \in F, i = 1 \text{ to } n\}$ be a finite subcover and $g = \mathbf{2}[(f_{y1} \vee f_{y2} \vee \cdots \vee f_{yn}) \vee \mathbf{0}] \wedge \mathbf{1}$. Then $g(x) = 0$ and if $y \in F$ then $g(y) = 1$; thus $g[F] = \{1\}$. Hence $g(x) \notin cl_{\mathbf{R}}g[F] = \{1\}$. Since g belongs to A it follows that A separates points and closed sets of K. **QED**

2.2 <u>Theorem</u> Let A be a subalgebra of C(X) that contains the constant functions, and separates the points and closed sets of X. Then,

1) there is a compactification $\gamma_A X$ of X with these properties:

1a) For every f in A there exists an f^γ in $C(\gamma_A X)$ such that $f^\gamma|_X = f$.

1b) Let $A^\gamma = \{f^\gamma: f \in A\}$. Then A^γ separates points of $\gamma_A X$.

2) if αX is a compactification of X with the properties:

2a) For every f in A there exists an f^α in $C(\alpha X)$ such that $f^\alpha|_X = f$

2b) Let $A^\alpha = \{f^\alpha: f \in A\}$. Then A^α separates points of $\gamma_A X$,

then αX and $\gamma_A X$ are equivalent compactifications of X. In other words, $\gamma_A X$ is uniquely determined (up to equivalence) by properties 1a) and 1b).

Proof: 1) We give a general outline of the construction of $\gamma_A X$. The <u>evaluation function determined by A</u> is the function $e_A: X \to \Pi_{f \in A} Y_f$ defined by $[e_A(x)]_f = f(x)$. In 1.5 of [Wa] it is shown that, 1) since every f in A is continuous so is e_A, 2) since A separates points, e_A is one-to-one, and 3) since A separates points and closed sets of X, e_A is open. The following diagram summarizes the construction of $\gamma_A X$ and is reminiscent of one method of construction of the Stone-Čech compactifications (see 1.9 of [Wa]).

Let $T = \Pi_{f \in A} cl_{\mathbf{R}} f[X]$.

$$\begin{tikzcd} X \arrow[r, "\text{dense emb. } e_A"] \arrow[dr, "f \in A"'] & cl_T e_A[X] = \gamma_A X \arrow[r, "\text{emb. } \tau"] \arrow[d, "\pi_f \circ \tau |\gamma"] & T \arrow[dl, "\pi_f"] \\ & cl_{\mathbf{R}} f[X] & \end{tikzcd}$$

Since e_A is continuous, one-to-one and open it is a dense embedding of X into the closure (in T) of $e_A[X]$ which itself is embedded in T (by τ). The compact space $cl_T e_A[X]$ is the compactification $\gamma_A X$ of X we seek. We let π_f represent the projection map from T to $cl_{\mathbf{R}} f[X]$. Every function f in A extends to a function f^γ in $C(\gamma X)$ where $f^\gamma = \pi_f \circ \tau|_{\gamma X}$. Let A^γ denote the set $\{f^\gamma \in C(\gamma X)\colon f^\gamma|_X = f \in A\}$. Now A^γ contains all constant functions and will separate points of γX (otherwise the mapping τ will not be one-to-one). This establishes part 1) of the theorem.

We now prove part 2). To do so, we identify X with its homeomorph $e_A[X]$. Let αX be a compactification of X and suppose the properties 2a) and 2b) are satisfied. We are required to construct a homeomorphism h mapping αX onto $\gamma_A X$ such that, for all x in X, $h(x) = x$. Let T be as described above and recall that $\gamma_A X = cl_T e_A[X]$. We define the mapping $h\colon \alpha X \to T$ by $[h(p)]_f = f^\alpha(p)$. Note that, if π_f is the projection map then, by 8.8 of [Wa], h is a continuous function. If $x \in X$ then $[h(x)]_f = f^\alpha(x) = f(x) = e_A(x)$. It follows that $h[\alpha X]$ is a compact set which contains $e_A[X]$ and so $cl_T e_A[X] = h[\alpha X]$, that is, $\gamma_A X = h[\alpha X]$. We now show that h is one-to-one. If $h(p) = h(q)$ then $[h(p)]_f = f^\alpha(p) = [h(q)]_f = f^\alpha(q)$ for all f in A. Since A^α separates points of $\gamma_A X$ (by property 2b) then it follows that $p = q$; thus h is one-to-one. We have thus constructed a homeomorphism h from αX onto $\gamma_A X$ such that $h(x) = x$ for all x in X; hence αX is equivalent to $\gamma_A X$. **QED**

2.3 Definition If A is a subalgebra of $C^*(X)$ which contains the constant functions and separates points and closed sets of X, then we will call $\gamma_A X$ the compactification of X determined by A.

2.4 Definition We say that a subalgebra A of $C^*(X)$ is uniformly closed if it is closed in the topology of uniform convergence on $C^*(X)$. By the uniform closure of a subalgebra A we will mean the subalgebra A together with the limit of every uniformly converging sequence of A. The uniform closure of A will be denoted by uA.

2.5 Theorem Let A be a subalgebra of C(X) containing the constant functions and separating points and closed subsets of X. Then:

1) $\{f|_X : f \in uA^\gamma\} = uA$ (where A^γ is as previously described).

2) A is uniformly closed iff $A^\gamma = C(\gamma_A X)$.

Proof: 1) Let $g|_X$ belong to the set $S = \{f|_X : f \in uA^\gamma\}$. Then g is the limit of some uniformly converging sequence $\{g_n\}_{n \in I}$ in A^γ. It follows that $g|_X$ is the limit of the uniformly converging sequence $\{g_n|_X\}_{n \in I}$ in A. Consequently $g|_X$ belongs to uA and $S \subseteq uA$. Now suppose $f \in uA$. Then f is the limit of some uniformly converging sequence $\{g_n\}_{n \in I}$ in A. Suppose the sequence $\{g^\gamma_n\}_{n \in I}$ converges to the function h in uA^γ. Then the functions h and f must be identical on X (since X is dense in γX). Consequently $h|_X = f$ and $f \in S$.

We now prove part 2). Suppose $A = uA$. Since A contains the constant functions and separates points and closed sets of X then, by 2.2, A determines a (unique up to equivalence) compactification γX of X and, by 2.1, the set A^γ separates points of γX. It then immediately follows from the Stone-Weierstrass theorem (9.7 28 [Ro]) that $A^\gamma = C(\gamma_A X)$.

The converse follows from part 1). **QED**

We have thus shown that we can associate to each compactification γX of X a unique uniformly closed subalgebra A which contains the constant functions and separates points and closed sets of X. If a subalgebra A contains the constant functions and separates points and closed sets of X then (by the Stone-Weierstrass theorem) A^γ is dense in $C(\gamma_A X)$ and $uA^\gamma = C(\gamma_A X)$. The compactification of X determined by A will be the one associated to uA.

We now specifically direct our attention to the conditions that are required on X so that the subalgebra $C^\#(X)$ determines a compactification of X and to what the nature of such a compactification would be.

2.6 Lemma [SZ2] If X is either pseudocompact, locally compact or 0-dimensional then $C^\#(X)$ determines a compactification of X.

Proof: By 2.2, in each case we need only show that $C^\#(X)$ separates points and closed sets of X. Suppose X is pseudocompact. Then $C^\#(X) = C^*(X)$ (by 1.10); hence $C^\#(X)$ separates points and closed sets.

Suppose X is locally compact. We know that $C_K(X) \subseteq C^\#(X)$ (1.3). We will show that $C_K(X)$ separates points and closed sets of X. Let $x \in X$ and let A be a closed subset of X such that x is not in A. Since X is locally compact there exist a compact neighbourhood B of x such that $B \cap A = \varnothing$. Let S be an open subset such that $x \in S \subseteq B$. There exists an $f \in C(X)$ such that $f[X-S] = \{0\}$ and $f(x) = 1$. The cozero set of f, $Cz(f)$, is contained in S, thus $cl_X(Cz(f)) \subseteq$

B and is compact. It then follows that $f \in C_K(X)$ and that $C_K(X)$ separates points and closed sets of X. Consequently $C^{\#}(X)$ determines a compactifications of X.

Thirdly, let X be 0-dimensional. It follows from 1.3 that $C_F(X) \subseteq C^{\#}(X)$. Since X is 0-dimensional then X has a base of clopen sets. Let $x \in X$ and $A \subseteq X$ such that $x \in X-A$. Now $X-A$ can be written in the form $\cup_{i \in I} O_i$ where each O_i is a clopen subset of X. Let $x \in O_j$ for some j in the index set I. Then there exists an $f \in C(X)$ such that $f[O_j] = \{0\}$ and $f[X-O_j] = \{1\}$. It follows that $f \in C_F(X)$, $C_F(X)$ separates points and closed sets and that $C^{\#}(X)$ determines a compactification of X. **QED**

In the last part of the preceding proof, we showed that $C_F(X)$ (uniquely) determines a compactification of the 0-dimensional space X. We will denote this compactification by ζX. In [Ba] it is shown that if X is 0-dimensional then ζX is obtained by identifying the components of $\beta X-X$. This compactification of X is also briefly discussed in 10.24 of [Wa].

A space X is <u>strongly 0-dimensional</u> iff for every pair A, B of completely separated sets of X there exists a clopen set U such that $A \subseteq U \subseteq X-B$.

2.7 <u>Proposition</u> [SZ2] Let X be a topological space.

1) If X is pseudocompact then $C^{\#}(X)$ determines βX.

2) If X is strongly 0-dimensional then $C^{\#}(X)$ determines βX (which equals ζX by 3.34 and 10.24 of [Wa]).

3) If X is 0-dimensional and realcompact then $C^{\#}(X)$ determines ζX.

Proof: 1) If X is pseudocompact then $C^{\#}(X) = C(X) = C^*(X)$ (2.5). Consequently $C^{\#}(X)$ determines βX (since $C^*(X)$ determines βX).

2) X is strongly 0-dimensional iff $uC_F(X) = C^*(X) \subseteq uC^\#(X)$ (16A 2, [GJ]) and so $C^\#(X)$ determines βX.

3) Let $f \in C^\#(X)$. Let $x \in \beta X - X$ and C be a component of x in $\beta X - X$. If $y \in C$ then by 1.6 (5) there exists an open neighbourhood in βX of y on which f^β is constant. It follows that, if f^β is not constant on C, then C can be expressed as the disjoint union of open sets hence C is not connected. Since this is a contradiction f^β must be constant on every component of $\beta X - X$. Let τ be the mapping from βX to ζX which identifies the components of $\beta X - X$ into points of $\zeta X - X$ and fixes the points of X. Let g be a mapping from ζX to $\mathbf{R}$ such that $g(x) = f(x)$ for any x in X and $g[\tau[C]] = f^\beta[C]$ for every component C in $\beta X - X$. As f^β is constant on each such C, g is well-defined. Note that $f^\beta = g \circ \tau$, hence by 9.4 of [Wi], g is continuous and g is an extension of X over ζX. Consequently every function f in $C^\#(X)$ extends over ζX. Recall that ζX is uniquely determined by $uC_F(X)$. If $f \in C^\#(X)$ then $f = g|_X$ where $g|_X \in uC_F(X)$. It follows that $C^\#(X) \subseteq uC_F(X)$, and since $uC_F(X) \subseteq uC^\#(X)$, $uC^\#(X) = uC_F(X)$. Thus $C^\#(X)$ determines ζX. **QED**

2.8 <u>Theorem</u>(**) If X is a locally compact realcompact space such that $\beta X - X$ is connected then $C^\#(X)$ determines the one-point compactification of X.

Proof: As X is locally compact then, by 2.6, $C^\#(X)$ determines a compactification γX of X. Let $f \in C^\#(X)$. Since $\beta X - X$ is connected it follows from 1.15 that f^β is constant on $\beta X - X$. Let τ be the canonical map from βX onto γX. Then $f^\beta = f^\gamma \circ \tau$. Consequently $f^\gamma \circ \tau[\beta X - X] = f^\gamma[\gamma X - X]$ (6.12, [GJ]). Thus $\gamma X - X$ cannot have two points otherwise $\{f^\gamma : f \in C^\#(X)$ could not separate them. It then follows that $C^\#(X)$ determines the one-point compactification of X. **QED**

2.9 Examples 1) Since $\mathbf{Q}$ and $\mathbf{N}$ are strongly 0-dimensional then by 2.7 $C^\#(\mathbf{Q})$ (respectively $C^\#(\mathbf{N})$) determines $\beta\mathbf{Q}$ (respectively $\beta\mathbf{N}$).

2) $C^\#(\mathbf{R}^n)$ determines the one-point compactification of $\mathbf{R}^n$ for all $n > 1$.

Proof: Since $\mathbf{R}^n$ is locally compact and realcompact and since $\beta\mathbf{R}^n - \mathbf{R}^n$ is connected for all $n > 1$ (6L (3), [GJ]) then this is just a special case of 2.8. **QED**

3) $C^\#(\mathbf{R})$ determines the 2-point compactification of $\mathbf{R}$.

Proof: Suppose $C^\#(\mathbf{R})$ determines the compactification of $\gamma\mathbf{R}$ (2.6). Recall that $\beta\mathbf{R}-\mathbf{R}$ is the union of two homeomorhpic connected sets (6.10, [GJ]). Therefore by 1.15, if $f \in C^\#(\mathbf{R})$ then f^β takes on at most two values on $\beta\mathbf{R}-\mathbf{R}$, say $\{r_1,r_2\}$. Let τ be the canonical map from $\beta\mathbf{R}$ onto $\gamma\mathbf{R}$. Now τ fixes the points of $\mathbf{R}$. If f^γ is the extension of f to $\gamma\mathbf{R}$ then $\{r_1,r_2\} = f^\beta[\beta\mathbf{R}-\mathbf{R}] = f^\gamma\circ\tau[\beta\mathbf{R}-\mathbf{R}] = f^\gamma[\gamma\mathbf{R}-\mathbf{R}]$ (6.12, [GJ]). Since $f^{\beta\leftarrow}(r_1)$ is a component of $\beta\mathbf{R}-\mathbf{R}$, and since $\tau\circ f^{\beta\leftarrow}(r_1) = \tau\circ(f^\gamma\circ\tau)^{\leftarrow}(r_1) = \tau\circ\tau^{\leftarrow}\circ f^{\gamma\leftarrow}(r_1) = f^{\gamma\leftarrow}(r_1)$ then $f^{\gamma\leftarrow}(r_1)$ is connnected in $\gamma\mathbf{R}-\mathbf{R}$. Consequently no more than two points in $\gamma\mathbf{R}-\mathbf{R}$ can be separated by $\{f^\gamma: f \in C^\#(\mathbf{R})\}$. We now show that $\gamma\mathbf{R}$ is not the one-point compactification of $\mathbf{R}$. Suppose it were, and let $\{q\} = \gamma\mathbf{R}-\mathbf{R}$. Let f be a function mapping $\mathbf{R}$ to $\mathbf{R}$ defined as follows: $f(x) = 1$ if $x > 1$, $f(x) = x$ if $x \in [0,1]$ and $f(x) = 0$ if $x < 0$. Obviously f is bounded and continuous on $\mathbf{R}$. Let A be a closed subset of $\mathbf{R}$. Let $A_1 = A \cap [0,1]$ and $A_2 = A \cap \mathbf{R}-[0,1]$. Then $f[A] = f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$, obviously the union of two closed sets in $\mathbf{R}$. Hence, by 1.6 (4), $f \in C^\#(\mathbf{R})$. By 1.6 (6) $cl_{\beta\mathbf{R}}Z(f-1) = Z(f^\beta-1)$ and $cl_{\beta\mathbf{R}}Z(f) = Z(f^\beta)$. Then f^β has two values on $\beta\mathbf{R}-\mathbf{R}$. This contradicts the fact that $f^\beta[\beta\mathbf{R}-\mathbf{R}] = f^\gamma\circ\tau[\beta\mathbf{R}-\mathbf{R}] = f^\gamma(q)$, a singleton. It follows that $\gamma\mathbf{R}$ is the two-point compactification of $\mathbf{R}$. **QED**

In [He] Melvin Henriksen establishes various conditions on X for which $C^{\#}(X)$ determines a compactification of X. We consider a few of these here. First recall the following definition.

2.10 Definition A space is called rimcompact if it has a base of open sets with compact frontiers.

It is shown in [Mor] that every rimcompact space has a compactification ϕX such that $\phi X - X$ is 0-dimensional, and whenever γX is a compactification of X with $\gamma X - X$ 0-dimensional, there is a continuous map of ϕX onto γX leaving X pointwise fixed.

2.11 Definition ϕX is called the Freudenthal compactification of X.

From [Di] we state the following two lemmas:

2.12 Lemma [Di] Let X be rimcompact. Suppose $f \in C^*(X)$ and for each real number r, $Fr_X Z(f-\mathbf{r})$ is compact. Then f has a unique extension to f^{ϕ} in $C(\phi X)$.

2.13 Lemma [He] If X is rimcompact and realcompact, then every $f \in C^{\#}(X)$ has a (unique) extension $f^{\phi} \in C(\phi X)$.

Proof: Since X is realcompact, it follows from 1.17 that $Fr_X Z(f-\mathbf{r})$ is compact for all $r \in \mathbf{R}$. Consequently by 2.12 every f in $C^{\#}(X)$ has an extension to $f^{\phi} \in C(\phi X)$. **QED**

Note that this lemma does not imply that, if X is rimcompact and realcompact, then $C^{\#}(X)$ determines ϕX as 2.21 will show. It is possible that there are "not enough functions" in $\{f^{\phi}: f \in C^{\#}(X)\}$ to separate points of ϕX.

2.14 Definition If B is a subalgebra of $C^{\#}(X)$ a maximal stationary set S of B is a subset of X maximal with respect to the property that every f in B is constant on S. (A simple application of Zorn's lemma shows that every subalgebra B has a maximal stationary set).

2.15 Definition A subring A of $C^{*}(X)$ is called algebraic if it contains the constant functions and those members f in $C^{*}(X)$ such that $f^2 \in A$.

From 16.29 to 16.32 of [GJ] we obtain:

2.16 Lemma If X is compact and A is an algebraic subring of $C^{*}(X)$, then every maximal stationary set of A in X is connected and the uniform closure of A, namely uA, is $\{f \in C(X)$: f is constant on every connected stationary set of A$\}$.

2.17 Notation If γX is a compactification of X, $C_{\#}(\gamma X)$ will denote $\{f \in C(\gamma X): f|_X \in C^{\#}(X)\}$.

2.18 Lemma If X is rimcompact realcompact then $C_{\#}(\phi X)$ is an algebraic subring of $C(\phi X)$.

Proof: Let $f^{\phi} \in C(\phi X)$. Suppose $(f^{\phi})^2 \in C_{\#}(\phi X)$. Then $(f^{\phi})^2|_X = f^2 \in C^{\#}(X)$. Then for any maximal ideal M of C(X) there exists an $r \in \mathbf{R}$ such that $f^2 - \mathbf{r} \in M$. That is, either $f - (\mathbf{r})^{1/2}$ or $f + (\mathbf{r})^{1/2}$ belongs to M (since M is prime). Consequently $f \in C^{\#}(X)$ and $f^{\phi} \in C_{\#}(\phi X)$. Hence $C_{\#}(\phi X)$ is an algebraic subring of $C(\phi X)$. **QED**

2.19 Theorem [He] If X is realcompact and $C^{\#}(X)$ determines a compactification γX of X, then X is rimcompact and $\gamma X = \phi X$.

Proof: Let $x_0 \in X$ and let V be an open neighbourhood of x_0. By hypothesis $C_\#(\gamma X)$ separates points of γX. Then $C^\#(X)$ separates points and closed sets of X (2.2). There exists an $f \in C^\#(X)$ such that $f(x_0) = 0$ and $f[X-V] = \{1\}$.

We will first show that X is rimcompact. Let $g = (f - \mathbf{1/2}) \vee \mathbf{0}$; then $Z(g) = \{x \in X: f(x) \leq 1/2\}$. Consequently $x_0 \in \mathrm{int}_X Z(g)$. Since X is realcompact it follows from 1.17 that $\mathrm{Fr}_X Z(g)$ is compact and Z(g) is contained in V. Thus X has an open base of sets with a compact boundary; so X is rimcompact.

Since X is both rimcompact and realcompact then by 2.13 every $f \in C^\#(X)$ extends to $f^\phi \in C(\phi X)$ and by 2.18 $C_\#(\phi X)$ is an algebraic subring of $C(\phi X)$.

It remains to show that $C_\#(\phi X)$ separates points of ϕX. Let S be a maximal stationary set of $C_\#(\phi X)$. That is, S is a subset of ϕX maximal with respect to the property that every f^ϕ in $C_\#(\phi X)$ is constant on S. By 2.16, S is connected. Suppose S contains more than one point. Since $C^\#(X)$ determines a compactification then $C^\#(X)$ separates points in X. Consequently $C_\#(\phi X)$ separates points in X and so $|S \cap X| \leq 1$ and S meets $\phi X - X$. Let $K = S \cap (\phi X - X)$. Suppose $S - K = \{p\}$ where $p \in X$ and let $x \in K$. Since ϕX is a Hausdorff space there exists disjoint open neighbourhoods N_x and N_p in ϕX such that $x \in N_x$ and $p \in N_p$. Since K is 0-dimensional there exists a clopen neighbourhood of x in K, say M such that $M \subseteq N_x$. Now let A be an open subset of ϕX such that $M = A \cap K$ and such that p does not belong to A. Then $M = A \cap S$. Hence M is open in S. Let B be a closed subset of ϕX such that $M = B \cap K$. Since $M \subseteq \phi X - N_p$ we obtain the following chain of equalities: $M = (\phi X - N_p) \cap (B \cap K) = (\phi X - N_p) \cap B \cap (K \cup \{p\}) = (\phi X - N_p) \cap B \cap S$. Then M is also closed in S which in turn means that S is not connected, a contradiction. We conclude that if S meets X, S must be a singleton.

Obviously if S is contained in $\phi X - X$ then, again, S is a singleton. Hence $C_{\#}(\phi X)$ separates points of ϕX and so $C^{\#}(X)$ determines the compactification ϕX of X. **QED**

2.20 Notation Let R(X) denote the set of points of X which fail to have a compact neighbourhood.

Note that X–R(X) is open in X, for if $x \in X - R(X)$, there exists a compact neighbourhood V containing x. Suppose the interior of V meets R(X). Then there exists y in R(X) such that y belongs to the interior of V. Consequently y has a compact neighbourhood and this is a contradiction. So X–R(X) is open in X.

2.21 Example In [He] Henriksen constructs an example of a rimcompact realcompact space X such that $C^{\#}(X)$ does not separate points of ϕX and so does not determine ϕX. Let $W^* = \{\sigma: \sigma \leq \omega_1\}$, the set of all ordinals which do not exceed the first uncountable ordinal. Let $W = \{\sigma: \sigma < \omega_1\} = W^* - \{\omega_1\}$. Now W^* is compact and if f is in C(W) f is constant on a tail of W (see 5.12, [GJ]). We will consider the space $X = [0,1] \times W^*$. Its topology will be the natural product topology, together with all subsets of $[0,1] \times W$. We first show that X is rimcompact. Let $x \in X$. If $x \in [0,1] \times W$ then x is isolated and hence has a clopen neighbourhood. Let $t_0 \in [0,1]$. Then (t_0, ω_1) has a basic open neighbourhood X such that $V \cap ([0,1] \times \{\omega_1\}) = (t_0 - \varepsilon, t_0 + \varepsilon) \times \{\omega_1\}$. Obviously $Fr_X V = \{t_0 - \varepsilon, t_0 + \varepsilon\}$. It follows that X is rimcompact. Let $T = [0,1] \times \{\omega_1\}$, and let $N = (x,y) \times (t,\omega_1]$ be an open rectangle containing some point p in T. The closed interval $[t, \omega_1]$ is uncountable and one can easily find an open cover for $cl_X N$ which will not have a finite subcover. Consequently a point of

T does not have a compact neighbourhood while all other points of X are isolated; hence T = R(X). Now X–R(X) is realcompact (8.18, [GJ]) so X is the disjoint union of a realcompact discrete space and R(X), which is homeomorphic to the compact space [0,1]. Then X is realcompact (8.16, [GJ]).

Hence X is both rimcompact and realcompact. Let $g \in C^*(X)$.We will show that if g separates two points of R(X) then g cannot belong to $C^\#(X)$. Suppose that for $0 \leq r < s \leq 1$, $g(r,\omega_1) \neq g(s,\omega_1)$. Since [r,s] is connected there exists a countably infinite increasing discrete subset $S = \{x_n\}_{n \in \mathbf{N}}$ of [r,s] such that $f(x_n,\omega_1) \neq g(x_m,\omega_1)$ for $n \neq m$. Since any f in C(W) is constant on a tail of W, then for each n in **N** there exists some ordinal α_n such that g is constant on $\{x_n\} \times [\alpha_n,\omega_1]$, since each X-neighbourhood of (x_n,ω_1) contains a co-countable subset of $\{x_n\} \times W^*$. Let α be the supremum of $\{\alpha_n\}_{n \in \mathbf{N}}$. Since W has no countable cofinal set, it follows that $\alpha < \omega_1$. Then $g(x_n,\alpha) = g(x_n,\omega_1)$ for each n. It follows that $\{(x_n,\alpha)\}_{n \in \mathbf{N}}$ is countably infinite C-embedded discrete subspace of X such that $g(x_n,\alpha) \neq g(x_m,\alpha)$ for $n \neq m$. Then by 1.6 (3), g cannot be in $C^\#(X)$. Therefore R(X) is a maximal stationary set of $C^\#(X)$.

2.22 <u>Theorem</u> [He] If X is a realcompact space that is 0-dimensional at each point of R(X) then $C^\#(X)$ determines ϕX; i.e. $uC_\#(\phi X) = C(\phi X)$.

Proof: Recall that $C_K(X) \subseteq C^\#(X)$ (1.3). We will first show that $C_K(X)$ separates points and closed sets. Let A be any closed set in X. Let $x \in X$–R(X) such that x is not in A. Let B be a compact neighbourhood of x. Let C be an open neighbourhood of x such that $C \subseteq B$ and $C \cap A$ is empty. Then there exists an $f \in C(X)$ such that $f(x) = 1$ and $f[X–C] = \{0\}$. Now $Cz(f) \subseteq C$, so $cl_X CZ(f) \subseteq B$ and f has compact support i.e. $f \in C_K(X)$. We have shown that $C_K(X)$ separates points of X–R(X) from disjoint closed sets. If $x \in R(X)$, then, by hypothesis, x has a neighbourhood base of clopen sets. Let V be a clopen neighbourhood of x such that $V \cap A$ is empty. Clearly $C_F(X)$ separates x and A, and since $C_F(X) \subseteq C^{\#}(X)$ (1.3) it follows that $C^{\#}(X)$ separates points and closed sets of X. Thus $C^{\#}(X)$ determines a compactification of X. By theorem 2.19 $C^{\#}(X)$ determines ϕX.

We now show that $uC_{\#}(\phi X) = C(\phi X)$. That $uC_{\#}(\phi X) \subseteq C(\phi X)$ is obvious. Recall that $uC_{\#}(\phi X) = \{f \in C(\phi X)$: f is constant on every connected stationary set of $C_{\#}(\phi X)\}$ (2.16). Now $C_{\#}(\phi X)$ separates points of ϕX and so stationary sets of ϕX are singletons. Thus $C(\phi X) \subseteq uC_{\#}(\phi X)$ and $uC_{\#}(\phi X) = C(\phi X)$. **QED**

2.23 Corollary [He] If X is a realcompact such that $cl_{\phi X}(\phi X - X)$ is 0-dimensional then $uC_{\#}(\phi X) = C(\phi X)$.

Proof: It is easily verified that $X \cap cl_{\phi X}(\phi X - X) = R(X)$. Now apply 2.22. **QED**

CHAPTER 3

ON ZERO SETS OF $C^{\#}(X)$

In this section we show that a compactification of X determined by $C^{\#}(X)$ is actually a space of ultrafilters of zero sets of $C^{\#}(X)$. We also characterize the zero sets of $C^{\#}(X)$ under specific conditions. We begin by stating and proving the two following lemmas which are instrumental in proving theorem 3.3. This theorem establishes a direct relation between the zero sets of $C^{\#}(X)$ and the compactification determined by $C^{\#}(X)$. We will denote the set of all zero sets of $C^{\#}(X)$ by $Z^{\#}(X)$.

3.1 Lemma(**) Let X be Tychonoff, let $f \in C^{\#}(X)$, and supppose that f has an extension f^{γ} to the compactification γX of X. Then $cl_{\gamma X}Z(f) = Z(f^{\gamma})$.

Proof: Suppose $p \in Z(f^{\gamma}) - X$. Let $x \in \tau^{\leftarrow}(p)$ where $\tau: \beta X \to \gamma X$ is the canonical map. Then $\tau(x) = p$ so $f^{\gamma}(\tau(x)) = f^{\gamma}(p)$; hence $(f^{\gamma} \circ \tau)(x) = 0$. But by the uniqueness of f^{β}, $f^{\beta} = f^{\gamma} \circ \tau$ so $x \in Z(f^{\beta})$. Consequently:

$$\tau^{\leftarrow}[Z(f^{\gamma}) - X] \subseteq Z(f^{\beta}) - X. \quad (*)$$

We now obtain our result from the following chain of equalities and containments:

$$\begin{aligned} Z(f^{\gamma}) - X &= \tau \circ \tau^{\leftarrow}[Z(f^{\gamma}) - X] \quad \text{(as } \tau \text{ is onto)} \\ &\subseteq \tau[Z(f^{\beta})] \quad \text{(by *)} \\ &= \tau[cl_{\beta X}Z(f)] \quad (1.6) \\ &= cl_{\gamma X}(\tau[Z(f)]) \quad \text{(as } \tau \text{ is continuous and closed)} \\ &= cl_{\gamma X}Z(f). \end{aligned}$$

Then $Z(f^{\gamma}) \subseteq cl_{\gamma X}Z(f)$ which implies that $Z(f^{\gamma}) = cl_{\gamma X}Z(f)$. **QED**

Note that in the above proof, it was not necessary that $C_{\#}(\gamma X)$ separate points of γX. Thus $C^{\#}(X)$ need not determine γX for the lemma to hold true. In 2.21 we have given an example of a rimcompact realcompact space, X, where (by 2.13) every function of $C^{\#}(X)$ extends over γX. But for this particular example $C^{\#}(X)$ failed to determine γX. One might conjecture that if $cl_{\gamma X}Z(f) = Z(f^{\gamma})$ for all f in $C^{\#}(X)$, then $C^{\#}(X)$ determines γX. Example 2.21 shows that this conjecture fails, and that a stronger hypothesis is required.

3.2 Lemma(**) Suppose $C^{\#}(X)$ determines the compactification γX. If Z_1 and Z_2 are disjoint zero sets from $Z^{\#}(X)$, then $cl_{\gamma X}Z_1 \cap cl_{\gamma X}Z_2$ is empty.

Proof: Let f and g belong to $C^{\#}(X)$ such that $Z(f) \cap Z(g)$ is empty. Suppose $p \in cl_{\gamma X}Z(f) \cap cl_{\gamma X}Z(g)$. Then by 3.1 $p \in Z(f^{\gamma}) \cap Z(g^{\gamma})$ and $f^{\gamma}(p) = g^{\gamma}(p) = 0$. Suppose τ is the natural map from βX onto γX and let $x \in \tau^{\leftarrow}(p)$. Then $f^{\beta}(x) = f^{\gamma}\circ\tau(x) = f^{\gamma}(p) = 0$. Similarly $g^{\beta}(x) = 0$. Recall that βX is a compactification of X where disjoint zero sets of X have disjoint closures in βX (1.14 [Wa]). Consequently $cl_{\beta X}Z(f) \cap cl_{\beta X}Z(g)$ is empty and by 1.6 (6) we conclude that $Z(f^{\beta}) \cap Z(g^{\beta}) = \varnothing$. This contradicts the fact that $f^{\beta}(x) = g^{\beta}(x) = 0$; hence $cl_{\gamma X}Z(f) \cap cl_{\gamma X}Z(g)$ must be empty. **QED**

3.3 Theorem(**) If $C^{\#}(X)$ determines the compactification γX of X, then every point in γX is the limit of a unique ultrafilter of zero sets in $Z^{\#}(X)$ and the collection $Z^{\#}(X)$ forms a Wallman base for γX.

Proof: If $Z^{\#}(X)$ is to be a Wallman base we must verify that it satisfies the following conditions (see 19L of [Wi]): $Z^{\#}(X)$ must be a base for closed sets of X such that 1) if F is a closed set and x is a point not in F there exist a Z in $Z^{\#}(X)$ such that $x \in Z$ and Z does not meet F, 2) $Z^{\#}(X)$ is closed under finite unions and finite intersections, i.e. $Z^{\#}(X)$ is a lattice, and 3) if Z_1 and

Z_2 are disjoint zero sets in $Z^\#(X)$ then for some zero sets C_1 and C_2 in $Z^\#(X)$ $Z_1 \subseteq X-C_1$, $Z_2 \subseteq X-C_2$ and $(X-C_1) \cap (X-C_2)$ is empty. Since $C^\#(X)$ determines a compactification of X then $C^\#(X)$ separates points and closed sets; thus condition 1 is satisfied. Let F be a closed subset of X and $x \in X-F$. Since $C^\#(X)$ separates points and closed sets of X then there exists a function f in $C^\#(X)$ such that $f(x) = 1$ and $f[F] = \{0\}$. Thus $F \subseteq Z(f)$ and x does not belong to $Z(f)$. Hence $Z^\#(X)$ forms a base for closed sets of X (3.2, [GJ]). That $Z^\#(X)$ is closed under finite intersections and unions follows quickly from the fact that $C^\#(X)$ forms a subalgebra and sublattice of $C^*(X)$. We can show that $Z^\#(X)$ satisfies condition 3 by the following argument. Let f and g $\in C^\#(X)$ such that $Z(f) \cap Z(g)$ is empty. Then by 3.1 and 3.2 $Z(f^\gamma) \cap Z(g^\gamma)$ is empty. Since γX is normal we can find two disjoint open sets O_1 and O_2 in γX such that $Z(f^\gamma) \subseteq O_1$ and $Z(g^\gamma) \subseteq O_2$. Since $C_\#(\gamma X)$ separates points of the compact space γX it can easily be shown that $C_\#(\gamma X)$ separates points and closed (hence compact) sets of γX. If $p \in \gamma X-O_1$ then there exists a function $h^\gamma{}_p \in C_\#(\gamma X)$ such that $h^\gamma{}_p(p) = 0$ and $h^\gamma{}_p(Z(f^\gamma)) = \{1\}$. The collection $\{h^\gamma{}_p{}^\leftarrow[(-1/2,1/2)]: p \in \gamma X-O_1\}$ forms an open cover of the compact set $\gamma X-O_1$. Let $\{h^\gamma{}_{pi}{}^\leftarrow[(-1/2,1/2)]: i = 1 \text{ to } n, p_i \in \gamma X-O_1\}$ be a finite subcover of $\gamma X-O_1$. Let $q = h^\gamma{}_{pi} \wedge \cdots \wedge h^\gamma{}_{pn}$. Since $C^\#(X)$ forms a sublattice of $C^*(X)$ then $q \in C_\#(\gamma X)$ and so $q = h^\gamma$ for some $h \in C^\#(X)$. Evidently $h^\gamma[Z(f^\gamma)] = \{1\}$. Now if $x \in Z(f^\gamma)$ then there exists a function $t^\gamma{}_x \in C_\#(\gamma X)$ such that $t^\gamma{}_x(x) = 2$ and $t^\gamma{}_x[\gamma X-O_1] = \{0\}$. As before $\{t^\gamma{}_x{}^\leftarrow[(1,3)]: x \in Z(f^\gamma)\}$ forms an open cover of $Z(f^\gamma)$ so there exists a finite subcover $\{t^\gamma{}_{xi}{}^\leftarrow[(1,3): i = 1 \text{ to } m, x_i \in Z(f^\gamma)\}$. If we let $b = h^\gamma \wedge t^\gamma{}_{xi} \wedge \cdots \wedge t^\gamma{}_{xm}$ then $b = t^\gamma$ for some $t \in C^\#(X)$ (as above), $Z(f^\gamma) \subseteq Cz(t^\gamma)$, and $\gamma X-O_1 \subseteq Z(t^\gamma)$. By following a similar procedure we can find a function $k^\gamma \in C_\#(\gamma X)$ such that $Z(g^\gamma) \subseteq Cz(k^\gamma)$ and $\gamma X-O_2 \subseteq Z(k^\gamma)$. Obviously $Cz(t^\gamma) \cap Cz(k^\gamma)$ is empty. Then $Z(f) \subseteq X-Z(t)$, $Z(g) \subseteq X-Z(k)$, and $Cz(k) \cap Cz(t) = \varnothing$. Thus $Z^{\#}(X)$ satisfies condtion 3 and hence forms a Wallman base in X.

We will now show that if $Z^{\#}(X)$ is the set of all zero sets of $C^{\#}(X)$ then each point of γX is the limit of a $Z^{\#}(X)$-ultrafilter. Let $U^{\#}_p = \{Z(f-\mathbf{f}^{\gamma}(\mathbf{p})): f \in C^{\#}(X)\}$ ($p \in \gamma X$). We claim that $U^{\#}_p$ is a $Z^{\#}(X)$-ultrafilter. Let f and g $\in C^{\#}(X)$. Observe that, by 3.1, $Z(f^{\gamma}-\mathbf{f}^{\gamma}(\mathbf{p})) \cap Z(g^{\gamma}-\mathbf{g}^{\gamma}(\mathbf{p})) = cl_{\gamma X}(Z(f-\mathbf{f}^{\gamma}(\mathbf{p}))) \cap cl_{\gamma X}(Z(g-\mathbf{g}^{\gamma}(\mathbf{p})))$. Since this intersection contains the point p then, by 3.2, $Z(f-\mathbf{f}^{\gamma}(\mathbf{p})) \cap Z(g-\mathbf{g}^{\gamma}(\mathbf{p}))$ is non-empty. Since $Z(f-\mathbf{f}^{\gamma}(\mathbf{p})) \cap Z(g-\mathbf{g}^{\gamma}(\mathbf{p})) = Z(|f| + |g|) - (|\mathbf{f}| + |\mathbf{g}|)(\mathbf{p}))$ it follows that $U^{\#}_p$ is closed under finite intersections. We now show that $U^{\#}_p$ is closed under supersets and does not contain the empty set. Let $Z(g-\mathbf{g}^{\gamma}(\mathbf{p})) \in U^{\#}_p$, $h \in C^{\#}(X)$ and suppose that $Z(g-\mathbf{g}^{\gamma}(\mathbf{p})) \subseteq Z(h)$. Then $p \in Z(g-\mathbf{g}^{\gamma}(\mathbf{p})) = cl_{\gamma X}(Z(g-\mathbf{g}^{\gamma}(\mathbf{p})) \subseteq cl_{\gamma X}Z(h) = Z(h^{\gamma})$. Consequently $Z(h) = Z(h-\mathbf{h}^{\gamma}(\mathbf{p})) \in U^{\#}_p$. Since $\{Z(f^{\gamma}-\mathbf{f}^{\gamma}(\mathbf{p}): f \in C^{\#}(X)\}$ doesn't contain the empty set and since $p \in Z(f^{\gamma}-\mathbf{f}^{\gamma}\mathbf{p})) = cl_{\gamma X}Z(f-\mathbf{f}^{\gamma}(\mathbf{p}))$ (by 3.1) then no member of $U^{\#}_p$ is empty. Thus $U^{\#}_p$ is a $Z^{\#}(X)$-filter. For simplicity the terms "$Z^{\#}$-filter" (respectively "$Z^{\#}$-ultrafilter") will denote a filter (respectively an ultrafilter) on the lattice $Z^{\#}(X)$. Let $h \in C^{\#}(X)$ and suppose $Z(h) \cap Z(g-\mathbf{g}^{\gamma}(\mathbf{p}))$ is non-empty for all g in $C^{\#}(X)$. Then $Z(h) \cap Z(h-\mathbf{h}^{\gamma}(\mathbf{p}))$ is non-empty and so $h^{\gamma}(p) = 0$. This implies that $Z(h) \in U^{\#}_p$; hence $U^{\#}_p$ is a $Z^{\#}$-ultrafilter. Note that $U^{\#}_p$ will be free or fixed depending on whether $p \in \gamma X-X$ or X respectively.

Note that $\cap\{cl_{\gamma X}Z(f-\mathbf{f}^{\gamma}(\mathbf{p})): f \in Z^{\#}(X)\} = \cap\{Z(f^{\gamma}-\mathbf{f}^{\gamma}(\mathbf{p})): f \in C^{\#}(X)\} = \{p\}$ since $C_{\#}(\gamma X)$ separates points of γX. Also if p, q $\in \gamma X$, such that $p \neq q$, then again there exists f, g $\in C^{\#}(X)$ such that $f^{\gamma}(p) \neq g^{\gamma}(q)$ and hence $Z(f-\mathbf{f}^{\gamma}(\mathbf{p}))$ does not belong to $U^{\#}_q$. It follows that $U^{\#}_p \neq U^{\#}_q$. Hence there is a one-to-one correspondence between points of γX and the ultrafilters of $Z^{\#}(X)$.

Let $\omega_X(Z^\#)$ denote the set of all $Z^\#$-ultrafilters. Let $Z \in Z^\#(X)$; we denote by Z^ω the set of members of $\omega_X(Z^\#)$ which contain the zero set Z. Then $\{Z^\omega : Z \in Z^\#(X)\}$ forms a base for the closed sets of $\omega_X(Z^\#)$ (19L (1) [Wi]). It is also shown in 19L of [Wi] that if $Z^\#$ forms a Wallman base for X then $\omega_X(Z^\#)$ is a compactification of X. In what follows we show that $\omega_X(Z^\#)$ is homeomorphic to γX. Define the map $\tau : \gamma X \to \omega_X(Z^\#)$ by $\tau(p) = U^\#_p$. As noted in the preceding paragraph, τ is a bijection. Let $f \in C^\#(X)$.

$$
\begin{aligned}
\tau^{\leftarrow}[Z(f)^\omega] &= \{p \in \gamma X : \tau(p) \in Z(f)^\omega\} \\
&= \{p \in \gamma X : U^\#_p \in Z(f)^\omega\} \\
&= \{p \in \gamma X : Z(f) \in U^\#_p\} \\
&= \{p \in \gamma X : Z(f) \in \{Z(g - \mathbf{g}^\gamma(\mathbf{p})) : g \in C^\#(X)\}\} \\
&= \{p \in \gamma X : cl_{\gamma X} Z(f^\gamma) = cl_{\gamma X} Z(g - \mathbf{g}^\gamma(\mathbf{p})) \text{ for some } g \in C^\#(X)\} \\
&= \{p \in \gamma X : Z(f^\gamma) = cl_{\gamma X} Z(g - \mathbf{g}^\gamma(\mathbf{p})) \text{ for some } g \in C^\#(X)\} \\
&= \{p \in \gamma X : Z(f^\gamma) = Z(g^\gamma - \mathbf{g}^\gamma(\mathbf{p})) \text{ for some } g \in C^\#(X)\} \quad \text{(by 3.1).} \\
&= Z(f^\gamma), \text{ a basic closed set of } \gamma X \text{ (since } C^\#(X) \text{ determines } \gamma X\text{).}
\end{aligned}
$$

Thus τ is continuous. As τ is a bijection, it follows that $\tau[Z(f^\gamma)] = Z(f)^\omega$ and so $\tau^{\leftarrow}$ is continuous. Hence τ is a homeomorphism mapping γX onto $\omega_X(Z^\#)$. Then if γX is a compactification determined by $C^\#(X)$ γX is actually $\omega_X(Z^\#)$.

QED

3.4 Proposition(**) If $C^\#(\upsilon X)$ determines a compactification γX then $C^\#(X)$ determines γX.

Proof: Suppose $C^\#(\upsilon X)$ determines the compactification γX. Then for every $f \in C^\#(\upsilon X)$ f extends to $f^\gamma \in C(\gamma X)$ and the set $S^\gamma = \{f^\gamma : f \in C^\#(\upsilon X)\}$ separates points of γX (2.2). By proposition 1.9, $C^\#(X)$ is isomorphic to $C^\#(\upsilon X)$. Then every f in $C^\#(X)$ extends to f^γ via f^υ and $\{f^\gamma : f \in C^\#(X)\}$ separates points of γX. Consequently $C^\#(X)$ determines γX. **QED**

3.5 Definition If U is an open subset of X and γX is a compactification of X, then $Ex_{\gamma X}U$ is defined to be $\gamma X - cl_{\gamma X}(X-U)$. The set $Ex_{\gamma X}U$ is often called the extension of U in γX.

3.6 Definition Let U be an open subset of X. A compactification γX of X is a perfect compactification with respect to the open set U if $cl_{\gamma X}(Fr_X U) = Fr_{\gamma X}(Ex_{\gamma X}U)$. A compactification of X is perfect if it is perfect with respect to every open subset of X.

In lemma 1 of [Sk], Skljarenko gives the following characterization of a perfect compactification:

3.7 Proposition A compactification γX of the space X is perfect with respect to the open set $U \subseteq X$ iff, for every set $A \subseteq U$, $cl_{\gamma X}A \cap cl_{\gamma X}(Fr_X U) = \varnothing$ implies that $cl_{\gamma X}A \cap cl_{\gamma X}(X-U) = \varnothing$.

If X is realcompact and $C^{\#}(X)$ determines a compactification then we know that this compactification must be the Freudenthal compactification ϕX (see 2.19). It is well known that this is a perfect compactification of X (see 6.39 of [Is]). However we do not yet know the nature of a compactification determined by $C^{\#}(X)$ for an arbitrary Tychonoff space X. The following lemma shows that such a compactification must be perfect.

3.8 Lemma(**) If $C^{\#}(X)$ determines the compactification γX of X then γX is a perfect compactification of X.

Proof: We apply the characterization given in 3.7. Let U be an open subset of X. Let $A \subseteq U$. Suppose $cl_{\gamma X}A \cap cl_{\gamma X}(Fr_X U) = \varnothing$. We need to show that this implies that $cl_{\gamma X}A \cap cl_{\gamma X}(X-U) = \varnothing$. Since $C^{\#}(X)$ determines γX, then by using an argument similar to the one used in the proof of 2.2, $C_{\#}(\gamma X)$ separates disjoint closed sets of γX. Hence there exists a function f^{γ} in $C_{\#}(\gamma X)$ such that $f^{\gamma}[A] = \{0\}$ and $f^{\gamma}[Fr_X U] = \{1\}$. Let g be a function on X such that $g(x) = f(x)$ if $x \in cl_X U$, and $g(x) = 1$ if $x \in X-U$. Note that $g \in C^*(X)$ (7.6 [Wi]). Since $f \in C^{\#}(X)$ it follows from the definition of g and 1.6 (3) that $g \in C^{\#}(X)$. Then g extends to $g^{\gamma} \in C(\gamma X)$. Consequently $g^{\gamma}[A] = \{0\}$ and $g^{\gamma}[X-U] = \{1\}$; thus $A \subseteq g^{\gamma\leftarrow}[\{0\}]$ and $X-U \subseteq g^{\gamma\leftarrow}[\{1\}]$. Then $cl_{\gamma X}A \cap cl_{\gamma X}(X-U) = \varnothing$. Hence γX is a perfect compactification of X. **QED**

In theorems 1 and 2 of [Sk] we have the following characterizations of perfect compactifications. We will use of these in the proof of proposition 3.15.

3.9 Proposition Let γX be a compactification of X. Then the following are equivalent:

1) γX is a perfect compactification of X.
2) If U and V are disjoint open sets of X, then $Ex_{\gamma X}(U \cup V) = Ex_{\gamma X}U \cup Ex_{\gamma X}V$.
3) Let $g: \beta X \to \gamma X$ be the (unique) extension of the identity map on X. Then $g^{\leftarrow}(p)$ is a connected subset of βX for each p in γX.

From 2.2 in [Dia] we have the following easy result:

3.10 Lemma Let γX be a compactification of X. If W is an open subset of γX such that $Fr_{\gamma X}W \subseteq X$ then $W = Ex_{\gamma X}(W \cap X)$.

Let X be an arbitrary topological space. Suppose $C^{\#}(\upsilon X)$ determines a compactification of υX; then it must be the Freudenthal compactification

$\phi\upsilon X$ of υX (by 2.19). Obviously, in this particular case, $C^{\#}(X)$ also determines $\phi\upsilon X$. The following two lemmas will lead to the proof that if υX is rimcompact and $C_{\#}(\beta X)$ separates points of υX and separates the connected components of $\beta X-\upsilon X$ then $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X.

3.11 Lemma(**) Let X be a topological space such that $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X. Then the natural map τ from βX onto $\phi\upsilon X$ maps distinct components of $\beta X-\upsilon X$ to distinct points of $\phi\upsilon X-\upsilon X$.

Proof: Suppose $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X and let τ be the natural map from βX onto $\phi\upsilon X$. If C is a connected component in $\beta X-\upsilon X$ then $\tau[C]$ is connected in $\phi\upsilon X-\upsilon X$. Since $\phi\upsilon X-\upsilon X$ is 0-dimensional $\tau[C]$ must be a singleton. Now suppose $p \in \phi\upsilon X-\upsilon X$. Since $\phi\upsilon X$ is a perfect compactification of X (3.8), then by 3) of 3.9, $\tau^{\leftarrow}(p)$ is connected in βX. Consequently $\tau^{\leftarrow}(p)$ is contained in a component of $\beta X-\upsilon X$. It then follows that $\tau[C] = \{p\}$ and hence τ maps distinct components of $\beta X-\upsilon X$ to distinct points of $\phi\upsilon X$. **QED**

3.12 Lemma(**) Let X be a topological space such that υX is rimcompact. Then if $f \in C^{\#}(X)$ and C is a connected component of $\beta X-\upsilon X$ then $f^{\beta}[C]$ is a singleton.

Proof: Suppose υX is rimcompact and $f \in C^{\#}(X)$. Then, by 2.13, f extends to $f^{\phi\upsilon}$ in $C(\phi\upsilon X)$. If C is a component in $\beta X-\upsilon X$ and τ is the natural map from βX onto $\phi\upsilon X$ then $f^{\beta}[C] = f^{\beta}\circ\tau[C]$. But since $\phi\upsilon X-\upsilon X$ is 0-dimensional $\tau[C]$ is a singleton. Hence $f^{\beta}[C]$ is a singleton. **QED**

3.13 Proposition(**) Let X be a topological space. Then $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X iff $C_{\#}(\upsilon X)$ separates points of υX and $C_{\#}(\beta X)$ separates the connected components of $\beta X-\upsilon X$, i.e. if C is a connected component of $\beta X-\upsilon X$ then $C = \cap\{Z(f^{\beta}-\mathbf{r_f}): f \in C^{\#}(X)\}$ where $\{r_f\} = f^{\beta}[C]$.

Proof: Let X be a topological space and suppose $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X. It follows immediately that $C_{\#}(\upsilon X)$ separates points of X. If $p \in \phi\upsilon X-\upsilon X$ then $\{p\} = \cap\{cl_{\phi\upsilon X}Z(f-\mathbf{f^{\phi}(p)}): f \in C^{\#}(X)\} = \cap\{Z(f^{\phi}-\mathbf{f^{\phi}(p)}): f \in C^{\#}(X)\}$ (by 3.1 and 3.3). Let τ be the natural map from βX onto $\phi\upsilon X$. By lemma 3.11 $\tau^{\leftarrow}(p)$ is a component, say C, in $\beta X-\upsilon X$. Obviously $C^{\#}(\upsilon X)$ will determine $\phi\upsilon X$ and so, by 2.19, υX is rimcompact. By lemma 3.12 $f^{\beta}[C]$ is a singleton, say $\{r_f\}$. Then:

$$\begin{aligned}
C = \tau^{\leftarrow}(p) &= \tau^{\leftarrow}[\cap\{cl_{\phi\upsilon X}Z(f-\mathbf{f^{\phi}(p)}): f \in C^{\#}(X)\}] \\
&= \cap\{\tau^{\leftarrow}[cl_{\phi\upsilon X}Z(f-\mathbf{f^{\phi}\circ\tau\circ\tau^{\leftarrow}(p)})]: f \in C^{\#}(X)\} \\
&= \cap\{\tau^{\leftarrow}[cl_{\phi\upsilon X}Z(f-\mathbf{f^{\beta}[C]})]: f \in C^{\#}(X)\} \\
&\supseteq \cap\{cl_{\beta X}\tau^{\leftarrow}[(Z(f-\mathbf{f^{\beta}[C]})]: f \in C^{\#}(X)\} \\
&= \cap\{Z(f^{\beta}-\mathbf{f^{\beta}[C]}): f \in C^{\#}(X)\}
\end{aligned}$$

Since $C \subseteq \cap\{Z(f^{\beta}-\mathbf{f^{\beta}[C]}): f \in C^{\#}(X)\}$ then

$C = \cap\{Z(f^{\beta}-\mathbf{f^{\beta}[C]}: f \in C^{\#}(X)\} = \cap\{Z(f^{\beta}-\mathbf{r_f}): f \in C^{\#}(X)\}$ which is what we were required to prove.

We proceed to prove the converse. Suppose C is a component in $\beta X-\upsilon X$. If $y \in C$ and $f \in C^{\#}(X)$ then by 1.6 (5) f^{β} is constant on some open neighbourhood of y. If f^{β} is not constant on C then C can be expressed as the disjoint union of open sets; this contradicts the fact that C is a component of y. It follows that, for any $f \in C^{\#}(X)$, f^{β} is constant on C. To show that $C^{\#}(X)$ determines $\phi\upsilon X$ we need only show that $C^{\#}(\upsilon X)$ separates points and closed sets of υX. It will follow from 2.19 that the compactification of υX determined by $C^{\#}(\upsilon X)$ is $\phi\upsilon X$, and from 3.4, that $C^{\#}(X)$ determines $\phi\upsilon X$. Let

F be a closed subset of υX and $F^* = cl_{\beta X}F$. Then $F = F^* \cap \upsilon X$. Let $x \in \upsilon X - F$ and $y \in F$. Then by hypothesis there exists a function $f_y \in C_\#(\upsilon X)$ such that $f_y(y) = 1$ and $f_y(x) = 0$. If y belongs to some component C of $\beta X - \upsilon X$ then since $C_\#(\beta X)$ separates components of βX from the points of υX then there exists a function $f_y \in C_\#(\beta X)$ such that $f_y(y) = 1$ $(= f_y[C])$ and $f_y(x) = 0$. Let $\{f_y^{\leftarrow}[(1/2,2)]: y \in F^*\}$ form an open cover of F^*; then there exists an integer n such that the collection of sets $\{f_{yi}^{\leftarrow}[(1/2,2)]: y_i \in F^*, i = 1 \text{ to } n\}$ forms a finite subcover of F^*. Let $g = \mathbf{2}[(f_{yi} \wedge f_{y2} \wedge \cdots \wedge f_{yn}) \vee \mathbf{0}] \wedge \mathbf{1}$. Then $g(x) = 0$, and if $y \in F^*$, $g(y) = 1$; hence $g[F^*] = \{1\}$. Since $C^\#(X)$ forms a sublattice, $g \in C_\#(\beta X)$; hence $g|_{\upsilon X} \in C^\#(\upsilon X)$. It thus follows that $C^\#(\upsilon X)$ separates points and closed sets of υX. Then $C^\#(\upsilon X)$ and $C^\#(X)$ both determine $\phi\upsilon X$. **QED**

3.14 Theorem(**) Suppose $C^\#(X)$ determines a compactification of X. Then $C^\#(X)$ determines βX iff the maximal ideal space $M(C^*(X))$ is homeomorphic to $M(C^\#(X))$.

Proof: (=>) Suppose $C^\#(X)$ determines βX. We know that βX is homeomorphic to the maximal ideal space $M(C^*(X))$ (1.24 [Wa]). We would like to show that βX is homeomorphic to $M(C^\#(X))$. Let $\omega_X(Z^\#)$ denote the set of all $Z^\#$-ultrafilters. If Z^ω denotes the members of $\omega_X(Z^\#)$ which contain the zero set Z in $Z^\#(X)$ then $\{Z^\omega: Z \in Z^\#(X)\}$ forms a base for the closed sets of $\omega_X(Z^\#)$ (19L (1) [Wi]). Since $C^\#(X)$ determines the compactification βX then by 3.3 every point q of βX is the limit of a unique $Z^\#$-ultrafilter $U^\#_q$ in $\omega_X(Z^\#)$. Define the map $\tau:\beta X \to \omega_X(Z^\#)$ by $\tau(p) = U^\#_p$. Now τ is a homeomorphism if it sends and pulls back basic closed sets to basic closed sets. Let $f \in C^\#(X)$.

$$\begin{aligned}\tau^{\leftarrow}[Z(f)^\omega] &= \{p \in \beta X: \tau(p) \in Z(f)^\omega\}\\ &= \{p \in \beta X: U^\#_p \in Z(f)^\omega\}\end{aligned}$$

$= \{p \in \beta X: Z(f) \in U^{\#}_p\}$

$= \{p \in \beta X: Z(f) \in \{Z(g - \mathbf{g^{\beta}(p)}): g \in C^{\#}(X)\}\}$

(recall the definition of $U^{\#}_p$ in the proof of 3.3)

$= \{p \in \beta X: cl_{\beta X}Z(f) = cl_{\beta X}Z(g - \mathbf{g^{\beta}(p)})$ for some $g \in C^{\#}(X)\}$

$= \{p \in \beta X: Z(f^{\beta}) = Z(g^{\beta} - \mathbf{g^{\beta}(p)})$ for some $g \in C^{\#}(X)\}$ (by 1.6 (6))

$= Z(f^{\beta})$, a basic closed set of βX (3.2 [GJ]).

Also $\tau[Z(f^{\beta})] = \{U^{\#}_p \in \omega_X(Z^{\#}): p \in Z(f^{\beta})\}$. Since $Z(f) \in U^{\#}_p$ for every $p \in Z(f^{\beta})$ then $\tau[Z(f^{\beta})] \subseteq Z(f)^{\omega}$. If $U^{\#}_x \in Z(f)^{\omega}$ for some $x \in \beta X$ then $Z(f) \in U^{\#}_x$, hence $x \in Z(f^{\beta})$. Consequently $U^{\#}_x \in \tau[Z(f^{\beta})]$ and so $Z(f)^{\omega} \subseteq \tau[Z(f^{\beta})]$. It follows that $\tau[Z(f^{\beta})] = Z(f)^{\omega}$, hence τ is a homeomorphism mapping βX onto $\omega_X(Z^{\#})$.

We now show that $M(C^{\#}(X))$ is homeomorphic to $\omega_X(Z^{\#})$. We first show that the maximal ideals of $C^{\#}(X)$ are exactly the subsets in $C^{\#}(X)$ of form $M^{\#p} = \{f \in C^{\#}(X): p \in cl_{\beta X}Z(f)\}$ $(p \in \beta X)$ by showing that $Z(M^{\#p})$ is a $Z^{\#}$-ultrafilter. It is clear that $Z(M^{\#p})$ is closed under supersets in $Z^{\#}(X)$ and that $Z(M^{\#p})$ does not contain the empty set. Since disjoint zero sets in X are completely separated they would have disjoint closures in βX. Thus, since p is in the closure of every Z in $Z(M^{\#p})$, no two members of $Z(M^{\#p})$ can be disjoint (by 1.14 of [Wa]). To show that $Z(M^{\#p})$ is maximal, suppose that a zero set Z meets every member of $Z(M^{\#p})$. If p is not in $cl_{\beta X}Z$, since $C^{\#}(X)$ determines βX then $C^{\#}(X)$ separates p from $cl_{\beta X}Z \cap X$. That is, there exists a zero set $Z' \in Z^{\#}[X]$ such that $p \in Z'$ and $Z' \cap cl_{\beta X}Z \cap X = \varnothing$. But then $Z' \cap X$ is in $Z(M^{\#p})$ and misses Z, which is a contradiction. Thus, $M^{\#p}$ is a maximal ideal. (Note that this might not be the case if $C^{\#}(X)$ does not determine a compactification). A closed base for $M(C^{\#}(X))$ is $\{C(f): f \in C^{\#}(X)\}$ where $C(f) = \{M^{\#p} \in M(C^{\#}(X)): f \in M^{\#p}\}$ (7.11 [GJ]). Define $\tau: \omega_X(Z^{\#}) \to M(C^{\#}(X))$ by $\tau(U^{\#}_p) = M^{\#p}$. Since $f \in M^{\#p}$ iff $Z(f) \in U^{\#}_p$ then $\tau[Z(f)^{\omega}] = C(f)$. Similarly $\tau^{\leftarrow}[C(f)] = Z(f)^{\omega}$; thus τ is a homeomorphism since it maps and pulls back basic closed sets to basic closed sets. Hence $M(C^{\#}(X))$ is homeomorphic to $\omega_X(Z^{\#})$. It follows that βX is homeomorphic to $M(C^{\#}(X))$; hence $M(C^*(X))$ is homeomorphic to $M(C^{\#}(X))$.

($\Leftarrow$) Suppose that $C^{\#}(X)$ determines the compactification γX of X and suppose that $M(C^*(X))$ is homeomorphic to $M(C^{\#}(X))$. We can show as above, by using 3.1, that the maximal ideals of $C^{\#}(X)$ are precisely of form $M^{\#p} = \{f \in C^{\#}(X): p \in cl_{\gamma X}Z(f)\}$ $(p \in \gamma X)$. Also, exactly as in the previous paragraph, we have that $\omega_X(Z^{\#})$ is homeomorphic to $M(C^{\#}(X))$. Since βX is homeomorphic to $M(C^*(X))$ then βX is homeomorphic to $M(C^{\#}(X))$ and hence to $\omega_X(Z^{\#})$. But we have shown in 3.3 that γX is homeomorphic to $\omega_X(Z^{\#})$. Thus γX is homeomorphic to βX hence $C^{\#}(X)$ determines βX. **QED**

In what follows we investigate which subsets of X are possible candidates for the zero sets of $C^{\#}(X)$.

3.15 Proposition(**) Let X be a topological space. If F is a closed G_δ of βX such that $Fr_{\beta X}F \subseteq int_{\beta X}\upsilon X$, then F is a zero set of a member of $C_{\#}(\beta X)$.

Proof: Let F be a closed G_δ of βX such that $Fr_{\beta X}F \subseteq int_{\beta X}\upsilon X$. Then there exists a function $g \in C^*(X)$ such that $F = Z(g^\beta)$. In what follows, the βX-interior of F may or may not be empty. Now $(\beta X\text{-}int_{\beta X}F) \cap cl_{\beta X}(\beta X-\upsilon X)$ is compact. Let $S = g^\beta[\beta X-int_{\beta X}F) \cap cl_{\beta X}(\beta X-\upsilon X)]$. Then S is a closed subset of $\mathbf{R}$ and misses $\{0\}$ since $Fr_{\beta X}Z(g^\beta) \subseteq int_{\beta X}\upsilon X$. Hence there exists $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \cap S$ is empty. Let $N = (-\varepsilon,\varepsilon)$; then $Z(g^\beta) \subseteq g^{\beta\leftarrow}[N] \subseteq \beta X$. Furthermore $g^{\beta\leftarrow}[N] \cap g^{\beta\leftarrow}[S] = \varnothing$. Let $h^\beta = \varepsilon/\mathbf{2}$. Clearly $Z(|h^\beta \wedge g^\beta|) = Z(g^\beta)$.

We will show that $|h^\beta \wedge g^\beta| \in C_{\#}(\beta X)$; as $|h \wedge g|^\beta = |h^\beta \wedge g^\beta|$ this is equivalent to showing that $|h \wedge g| \in C^{\#}(X)$. It will be sufficient to show that $|h^\upsilon \wedge g^\upsilon| \in C_{\#}(\upsilon X)$ since $|h \wedge g|^\upsilon = |h^\upsilon \wedge g^\upsilon|$ and if $|h^\upsilon \wedge g^\upsilon| \in C^{\#}(\upsilon X)$

then $|h \wedge g| \in C^{\#}(X)$ (1.9). By 1.6 (7), it will suffice to show that, for every open subset U of βX, $|h^{\upsilon} \wedge g^{\upsilon}|[U \cap \upsilon X] = |h^{\beta} \wedge g^{\beta}|[U]$. We proceed to do this.

Let U be an open subset of βX. That $|h^{\upsilon} \wedge g^{\upsilon}|[U \cap \upsilon X] \subseteq |h^{\beta} \wedge g^{\beta}|[U]$ is clear. We now show the opposite inclusion holds. Let $p \in U$. We wish to show that $|h^{\beta} \wedge g^{\beta}|$ maps p into $|h^{\beta} \wedge g^{\beta}|[U \cap \upsilon X]$. We consider two cases.

Case 1: If $p \in U \cap \upsilon X$ then $|h^{\beta} \wedge g^{\beta}|(p) = |h^{\upsilon} \wedge g^{\upsilon}|(p) \in |h^{\upsilon} \wedge g^{\upsilon}|[U \cap \upsilon X]$.

Case 2: Let $p \in U \cap (\beta X - \upsilon X)$. Note that $\beta X - \upsilon X$ can be expressed as the disjoint unions of $\mathrm{int}_{\beta X}Z(g^{\beta}) \cap (\beta X - \upsilon X)$ and $(\beta X - \upsilon X) \cap Cz(g^{\beta})$ (see page 2 for the notation Cz). Thus in this case either $p \in \mathrm{int}_{\beta X}Z(g^{\beta}) \cap (\beta X - \upsilon X)$ or $p \in (\beta X - \upsilon X) \cap Cz(g^{\beta})$. If $p \in \mathrm{int}_{\beta X}Z(g^{\beta})$, $g^{\beta}(p) = 0$. If $p \in (\beta X - \upsilon X) \cap Cz(g^{\beta})$ then g^{β} maps p into S; thus $|g^{\beta}(p)| > \varepsilon/2$. Let us first consider the subcase where $g^{\beta}(p) = 0$. It follows that $|h^{\beta} \wedge g^{\beta}|(p) = 0$ and since $U \cap Z(g^{\beta}) \cap (\beta X - \upsilon X)$ is non-empty and $\mathrm{Fr}_{\beta X}Z(g^{\beta}) \subseteq \mathrm{int}_{\beta X}\upsilon X$ then $U \cap \mathrm{int}_{\beta X}Z(g^{\beta})$ is non-empty, meets υX, and is an open subset of βX. So there exists an x in $U \cap \upsilon X$ such that $|h^{\beta} \wedge g^{\beta}|(x) = 0$. Consequently whether we have case 1 or the first subcase of case 2, $|h^{\beta} \wedge g^{\beta}|$ maps p into $|h^{\beta} \wedge g^{\beta}|[U \cap \upsilon X]$.

We now consider the subcase of case 2 where $|g^{\beta}(p)| > \varepsilon/2$. Then $|h^{\beta} \wedge g^{\beta}|(p) = \varepsilon/2$. We want to show that there exists an x in $U \cap \upsilon X$ such that $|h^{\beta} \wedge g^{\beta}|(x) = \varepsilon/2$. Suppose not. Then $U \cap \upsilon X \subseteq g^{\beta\leftarrow}[(-\varepsilon/2,\varepsilon/2)] \cap \upsilon X$.

If W is a subset of υX, by $\mathrm{Ex}_{\beta\upsilon X}W$ we will mean $\beta X - \mathrm{cl}_{\beta X}(\upsilon X - W)$. To arrive at a contradiction we first claim that:

$$\mathrm{Ex}_{\beta\upsilon X}[g^{\beta\leftarrow}[(-\varepsilon/2,\varepsilon/2)] \cap \upsilon X] \cap (\beta X - \upsilon X) = \mathrm{int}_{\beta X}Z(g^{\beta}) \cap (\beta X - \upsilon X).$$

Let $M = g^{\beta\leftarrow}[(-\varepsilon/2,0) \cup (0,\varepsilon/2)]$. Then $M \subseteq \mathrm{int}_{\beta X}\upsilon X$ (by definition of ε) and $g^{\beta\leftarrow}[(-\varepsilon/2,\varepsilon/2)] = \mathrm{Fr}_{\beta X}Z(g^{\beta}) \cup \mathrm{int}_{\beta X}Z(g^{\beta}) \cup M$. For simplicity we will denote $Z(g^{\beta})$ by Z.

We first make the following two useful observations:

1) $\upsilon X-[(\mathrm{int}_{\beta X}Z) \cup M] = \mathrm{Fr}_{\beta X}Z \cup (\upsilon X-[Z \cup M])$. This follows from the fact that $\mathrm{Fr}_{\beta X}Z \subseteq \mathrm{int}_{\beta X}\upsilon X$ and that $\mathrm{Fr}_{\beta X}Z \cap (\upsilon X-[Z \cup M])$ is empty.

2) $\mathrm{Ex}_{\beta\upsilon X}M = M$. To prove this we will show that $\mathrm{Fr}_{\beta X}M \subseteq \upsilon X$ and then apply 3.10. Recall that $M = g^{\beta\leftarrow}[(-\varepsilon/2,0) \cup (0,\varepsilon/2)]$. Note that $M \subseteq g^{\beta\leftarrow}[[-\varepsilon/2,\varepsilon/2]]$, a closed subset of βX. Hence $\mathrm{cl}_{\beta X}M \subseteq g^{\beta\leftarrow}[[-\varepsilon/2,\varepsilon/2]]$. Observe that $g^{\beta\leftarrow}[[-\varepsilon/2,\varepsilon/2]] \cap (\beta X-\upsilon X) = (\mathrm{int}_{\beta X}Z) \cap (\beta X-\upsilon X)$ and that $\mathrm{cl}_{\beta X}M \subseteq \beta X-\mathrm{int}_{\beta X}Z$ (since $M \cap Z = \varnothing$); hence $\mathrm{cl}_{\beta X}M \subseteq \upsilon X$ and so $\mathrm{Fr}_{\beta X}M \subseteq \upsilon X$.It follows quickly from 3.10 that $\mathrm{Ex}_{\beta\upsilon X}(M \cap \upsilon X) = M$. The following chain of equalities now establishes the claim:

$\mathrm{int}_{\beta X}Z \cap (\beta X-\upsilon X)$

$= ((\mathrm{int}_{\beta X}Z) \cup M) \cap(\beta X-\upsilon X)$ (since $M \subseteq \mathrm{int}_{\beta X}\upsilon X$)

$= \mathrm{Ex}_{\beta\upsilon X}(\upsilon X \cap (\mathrm{int}_{\beta X}Z)) \cup \mathrm{Ex}_{\beta\upsilon X}(\upsilon X \cap M) \cap (\beta X-\upsilon X)$

(by 3.10, the above observation and the hypothesis on Z)

$= [\mathrm{Ex}_{\beta\upsilon X}((\mathrm{int}_{\beta X}Z) \cup M) \cap \upsilon X] \cap (\beta X-\upsilon X)$ (by 3.9 (2))

$= (\beta X-\mathrm{cl}_{\beta X}(\upsilon X-[(\mathrm{int}_{\beta X}Z \cup M])) \cap (\beta X-\upsilon X)$

$= \beta X-\mathrm{cl}_{\beta X}[\mathrm{Fr}_{\beta X}Z \cup (\upsilon X-(Z \cup M))] \cap (\beta X-\upsilon X)$

(by observation 1)

$= [\beta X-\mathrm{cl}_{\beta X}(\mathrm{Fr}_{\beta X}Z)] \cap [\beta X-\mathrm{cl}_{\beta X}(\upsilon X-(Z \cup M))] \cap (\beta X-\upsilon X)$

$= ([\beta X-\mathrm{cl}\beta X(\mathrm{Fr}_{\beta X}Z)] \cap (\beta X-\upsilon X)) \cap (\beta X-\mathrm{cl}_{\beta X}(\upsilon X-(Z \cup M)) \cap (\beta X-\upsilon X))$

$= (\beta X-\upsilon X) \cap (\mathrm{Ex}_{\beta\upsilon X}(Z \cup M) \cap \beta X-\upsilon X)$ (since $\mathrm{Fr}_{\beta X}Z \subseteq \mathrm{int}_{\beta X}\upsilon X$)

$= \mathrm{Ex}_{\beta\upsilon X}(Z \cup M) \cap (\beta X-\upsilon X)$

$= \beta X-\mathrm{cl}_{\beta X}(\upsilon X-(Z \cup M)) \cap \upsilon X) \cap (\beta X-\upsilon X)$

$= \mathrm{Ex}_{\beta\upsilon X}(g^{\beta\leftarrow}[(-\varepsilon/2,\varepsilon/2)] \cap \upsilon X) \cap (\beta X-\upsilon X)$

Thus the claim is established.

Since $U \cap \upsilon X \subseteq g^{\beta\leftarrow}[(-\varepsilon/2,\varepsilon/2)] \cap \upsilon X$ then

$$\mathrm{Ex}_{\beta\upsilon X}(U \cap \upsilon X) \subseteq \mathrm{Ex}_{\beta\upsilon X}[g^{\beta\leftarrow}(-\varepsilon/2,\varepsilon/2)] \cap \upsilon X.$$

Consequently, by the above claim, $U \cap (\beta X - \upsilon X) \subseteq \mathrm{int}_{\beta X} Z(g^\beta) \cap (\beta X - \upsilon X)$; hence $g^\beta(p) = 0$. This is a contradiction since $|g^\beta(p)| > \varepsilon/2$. Consequently the supposition we made fails; hence there exists an x in $U \cap \upsilon X$ such that $|g^\beta(p)| \geq \varepsilon/2$. Then $|h^\beta \wedge g^\beta|(x) = \varepsilon/2$, and so $|h^\beta \wedge g^\beta|$ sends p to a point in $|h^\beta \wedge g^\beta|[U \cap \upsilon X]$. It follows that $|h^\beta \wedge g^\beta|[U] \subseteq |h^\upsilon \wedge g^\upsilon|[U \cap \upsilon X]$, and hence $|h^\upsilon \wedge g^\upsilon| \in C^\#(\upsilon X)$ (by 1.6 (7)). Thus F, as described, is the zero set $Z(|h^\beta \wedge g^\beta|)$ of $C_\#(\beta X)$. **QED**

3.16 Corollary(**) Let X be a topological space. A closed subset F of X whose βX-frontier is a closed G_δ of βX contained in $\mathrm{int}_{\beta X}\upsilon X$ is a zero set of $C^\#(X)$.

Proof: Let F be a closed subset of X such that the $\mathrm{Fr}_{\beta X}F$ is a closed G_δ and $\mathrm{Fr}_{\beta X}F \subseteq \mathrm{int}_{\beta X}\upsilon X$. If $F^* = \mathrm{cl}_{\beta X}F$ then $F = F^* \cap X$. There exists a collection $\{O_i : i \in \mathbf{N}\}$ of open sets in βX such that $\mathrm{Fr}_{\beta X}F = \cap_{i \in \mathbf{N}} O_i$. Then $F^* = \cap\{\mathrm{int}_{\beta X}F^* \cup O_i : i \in \mathbf{N}\}$ and so F^* is a closed G_δ of βX. Hence there exists a g in $C^\#(X)$ such that $Z(g^\beta) = F^*$. Now $F^* \cap X = Z(g^\beta) \cap X = Z(g) = F$, hence F is a zero set of $C^\#(X)$ (by 3.15). **QED**

Given the above two statements one might suspect that, for an arbitrary space X and $f \in C^\#(X)$, $\mathrm{Fr}_{\beta X}Z(f)$ is disjoint from $\beta X - \mathrm{int}_{\beta X}\upsilon X$. However the following example shows that, in general, this is not the case.

3.17 Example(**) Let us define a function $f: \mathbf{Q} \to \mathbf{R}$ as follows: $f(x) = 0$ if $x \leq 0$, $f(x) = 1/(n+1)$ if $x \in (1/[2^{1/2}(n+1)], 1/[2^{1/2}n])$, $n = 1,2,3,\ldots$, and $f(x) = 1$ if $x > 1/(2^{1/2})$. It can easily be verified that f is a well-defined continous closed function on $\mathbf{Q}$. Hence by 1.6 (4) $f \in C^\#(\mathbf{Q})$. Note that $Z(f) = (-\infty, 0] \cap \mathbf{Q}$. Recall that $\mathbf{Q} = \upsilon\mathbf{Q}$; Since $\beta\mathbf{Q} - \mathbf{Q}$ is dense in $\beta\mathbf{Q}$ (see 6.10 of [GJ]) then $\mathrm{int}_{\beta\mathbf{Q}}\upsilon\mathbf{Q}$ is empty. Also note (by 1.6 (5)) that $\mathrm{Fr}_{\beta\mathbf{Q}}Z(f)$ cannot meet $\beta\mathbf{Q} - \mathbf{Q}$ (since if $p \in \mathrm{Fr}_{\beta\mathbf{Q}}Z(f) \cap (\beta\mathbf{Q} - \mathbf{Q})$ and every $g \in C^\#(\mathbf{Q})$ is constant on a neighbourhood of p then $f^\beta(x) = 0$ on some point outside of $Z(f^\beta)$). Hence $Fr_{\beta\mathbf{Q}}Z(f) \subseteq \mathbf{Q}$. Since $Z(f)$ is closed in $\mathbf{Q}$ then $cl_{\beta\mathbf{Q}}Z(f) \cap \mathbf{Q} = Z(f)$ and hence $Fr_{\beta\mathbf{Q}}Z(f) \subseteq Z(f)$. Obviously 0 is the limit of the sequence $\{1/n\}_{n \in \mathbf{N}} \subseteq \beta\mathbf{Q}-Z(f^\beta)$; thus $0 \in Fr_{\beta\mathbf{Q}}Z(f)$. We have thus exhibited a zero set of $C^\#(\mathbf{Q})$ whose $\beta\mathbf{Q}$-frontier meets $\beta\mathbf{Q}-int_{\beta\mathbf{Q}}\upsilon\mathbf{Q}$.

3.18 Theorem(**) Let X be a realcompact locally compact space. Then the zero sets of $C^\#(X)$ are exactly the closed G_δ's of X whose βX-frontiers are contained in X.

Proof: Let X be a realcompact locally compact space and suppose F is a closed G_δ of X such that $Fr_{\beta X}F \subseteq int_{\beta X}X$. Let $F^* = cl_{\beta X}F$ and $\{O_i: i \in \mathbf{N}\}$ be a collection of open sets of X such that $F = \cap_{i \in \mathbf{N}}O_i$. If $B_i = (int_{\beta X}F) \cup O_i$ then $F^* = \cap_{i \in \mathbf{N}}B_i$ (since $Fr_{\beta X}F \subseteq O_i$ for all i). By 3.15 F^* is a zero set member of $C_\#(\beta X)$ hence there exists a $g \in C^\#(X)$ such that $F^* = Z(g^\beta)$. Thus $F \in Z^\#(X)$.

If Z(f) is a zero set of $C^\#(X)$ then Z(f) is a closed G_δ of X (1.10 [GJ]) and $Fr_{\beta X}Z(f) \subseteq int_{\beta X}X$ (1.17). In this case, $int_{\beta X}X = X$ since X is locally compact (3.15 [GJ]). Hence all zero sets of $C^\#(X)$ are closed G_δ's whose βX-frontiers are contained in X; thus the theorem is proved. **QED**

3.19 Corollary(**) If X is locally compact realcompact space, then the collection of closed G_δ's whose βX-frontier is contained in X forms a base for the closed sets of X.

Proof: Since X is locally compact then $C^\#(X)$ determines a compactification of X (22.6). Then, by 3.3, the collection $Z^\#(X)$ forms a base for the closed sets of X. Thus the corollary follows immediately from 3.18. **QED**

APPENDIX A

SUMMARY

1.1 Definition Let X be a Tychonoff space. The subset $C^{\#}(X)$ of $C(X)$ is the set of all continuous real-valued functions f such that for every maximal ideal M in C(X), there exists a real number r such that $f-\mathbf{r} \in M$.

1.2 Proposition $C^{\#}(X)$ contains all constant functions.

1.3 Proposition a) $C_K(X)$ is contained in $C^{\#}(X)$.

b) $C_F(X)$ is contained in $C^{\#}(X)$.

1.4 Proposition [SZ1] If S is C-embedded in X and $f \in C^{\#}(X)$ then $f|_S \in C^{\#}(S)$.

1.5 Proposition [SZ2] If $f \in C^{\#}(X)$, then $f[X]$ is compact. Hence $C^{\#}(X)$ is contained in $C^{*}(X)$.

1.6 Theorem For $f \in C(X)$ then following are equivalent:

1) $f \in C^{\#}(X)$

2) For every C-embedded subset S of X, f[S] is compact.

3) $f \in C^*(X)$ and for every C-embedded copy D of $\mathbf{N}$ in X f[D] is closed, hence finite.

4) $f \in C^*(X)$ and f[Z] is closed for every zero set Z in X.

5) $f \in C^*(X)$ and for every $p \in \beta X - \upsilon X$ there is a neighbourhood of p in βX on which f^{β} is constant.

6) $f \in C^*(X)$ and, for every $r \in \mathbf{R}$, $cl_{\beta X}Z(f-\mathbf{r}) = Z(f^{\beta} - \mathbf{r})$.

7) (**) $f \in C^*(X)$ and, for every open subset U of βX, $f[U \cap X] = f^{\beta}[U]$.

8) M + f is real for any maximal ideal M in C(X).

1.7 Proposition [Isa] If $f \in C^{\#}(X)$, then f is open iff f^{β} is open.

1.8 Theorem [C2] $C^{\#}(X)$ is the largest subring of $C^*(X)$ satisfying:

1) $C^{\#}(X)$ contains all constant functions.

2) $M^p \cap C^{\#}(X) = M^{*p} \cap C^{\#}(X)$ for every $p \in \beta X$.

1.9 Theorem(**) The mapping defined by $\psi(f) = f^{\upsilon}$ induces an isomorphism from $C^{\#}(X)$ onto $C^{\#}(\upsilon X)$.

1.10 Proposition(**) The space X is pseudocompact iff $C(X) = C^{\#}(X)$.

1.11 Proposition [NR] For $f \in C^*(X)$, the following are equivalent:

1) $f \in C^{\#}(X)$.

2) Every z-ultrafilter on X has a member on which f is constant.

3) For every z-ultrafilter U on X, the family $f^{\#}U$ of all closed sets in $\mathbf{R}$ whose preimage under f belongs to U is a z-ultrafilter.

1.12 Corollary [SZ2] If X is discrete then $C^{\#}(X) = C_F(X)$.

1.13 Corollary [SZ2] Let $f \in C^{\#}(X)$ and let C be a compact subset of $\beta X - \upsilon X$. Then $f^{\beta}[C]$ is finite. In particular if X is locally compact and realcompact and $f \in C^{\#}(X)$ then $f^{\beta}[\beta X - X]$ is finite.

1.14 Corollary [SZ2] If $f \in C^{\#}(X)$ and f^{β} is constant on a subset E of $\beta X - \upsilon X$, then f^{β} is constant on an open neighbourhood in βX of E.

1.15 Proposition [SZ2] Let X be locally compact and realcompact. Then $f \in C^{\#}(X)$ iff $f \in C^*(X)$ and every connected component of $\beta X - X$ has an open neighbourhood in βX on which f^{β} is constant.

1.16 Proposition(**) Let $f \in C^{\#}(X)$ and assume X is ultranormal. If B is closed in X then f[B] is compact. (In particular, f is a closed map.)

1.17 Theorem [He] If X is realcompact and $f \in C^{\#}(X)$ then the frontier in X of Z(f) (denoted $Fr_X Z(f)$) is compact and f is closed.

1.18 Corollary [He] Let X be normal and metacompact. If $f \in C^{\#}(X)$ and every closed discrete subspace of X is realcompact then f is closed.

1.19 Proposition(**) Let X be a δ-normally separated space or realcompact. Then X is a countably iff $C(X) = C^{\#}(X)$.

1.20 Proposition(**) Let X be a metacompact locally compact space. If X is either realcompact or ultranormal then $C^{\#}(X) = F(X)$.

1.21 $C^{\#}(\mathbf{R}^n)$ for $n \in \mathbf{N}$.

1.22 $C^{\#}(\mathbf{R}^{\aleph_0})$ is exactly the set of all constant functions [SZ1]

1.23 $C^{\#}(\mathbf{N}) = C_F(\mathbf{N})$.

1.24 Proposition(**) If X is 0-dimensional and first countable then $C_F(X) = C^{\#}(X)$ iff X is discrete.

CHAPTER II

2.1 Lemma If K is a compact Hausdorff space and A is a subalgebra C(K) that contains the constant functions, then the following are equivalent:

1) A separates points and closed sets of K.

2) A separates points of K.

2.2 Theorem Let A be a subalgebra of C(X) that contains the constant functions, and separates the points and closed sets of X. Then,

1) there is a compactification $\gamma_A X$ of X with these properties:

1a) For every f in A there exists an f^γ in $C(\gamma_A X)$ such that $f^\gamma|_X = f$.

1b) Let $A^\gamma = \{f^\gamma : f \in A\}$. Then A^γ separates points of $\gamma_A X$.

2) if αX is a compactification of X with the properties:

2a) For every f in A there exists an f^α in $C(\alpha X)$ such that $f^\alpha|_X = f$

2b) Let $A^\alpha = \{f^\alpha : f \in A\}$. Then A^α separates points of $\gamma_A X$,

then αX and $\gamma_A X$ are equivalent compactifications of X. In other words, $\gamma_A X$ is uniquely determined (up to equivalence) by properties 1a) and 1b).

2.3 Definition If A is a subalgebra of $C^*(X)$ which contains the constant functions and separates points and closed sets of X, then we will call $\gamma_A X$ the compactification of X determined by A.

2.4 Definition We say that a subalgebra A of $C^*(X)$ is uniformly closed if it is closed in the topology of uniform convergence on $C^*(X)$. By the uniform closure of a subalgebra A we will mean the subalgebra A together with the limit of every uniformly converging sequence of A. The uniform closure of A will be denoted by uA.

2.5 Theorem Let A be a subalgebra of C(X) containing the constant functions and separating points and closed subsets of X. Then:

1) $\{f|_X : f \in uA^\gamma\} = uA$ (where A^γ is as previously described).

2) A is uniformly closed iff $A^\gamma = C(\gamma_A X)$.

2.6 Lemma [SZ2] If X is either pseudocompact, locally compact or 0-dimensional then $C^\#(X)$ determines a compactification of X.

2.7 Proposition [SZ2] Let X be a topological space.

1) If X is pseudocompact then $C^\#(X)$ determines βX.

2) If X is strongly 0-dimensional then $C^\#(X)$ determines βX (which equals ζX by 3.34 and 10.24 of [Wa]).

3) If X is 0-dimensional and realcompact then $C^\#(X)$ determines ζX.

2.8 Theorem(**) If X is a locally compact realcompact space such that $\beta X - X$ is connected then $C^\#(X)$ determines the one-point compactification of X.

2.9 Examples

2.10 Definition A space is called rimcompact if it has a base of open sets with compact frontiers.

2.11 Definition ϕX is called the Freudenthal compactification of X.

From [Di] we state the following two lemmas:

2.12 Lemma [Di] Let X be rimcompact. Suppose $f \in C^*(X)$ and for each real number r, $Fr_X Z(f-\mathbf{r})$ is compact. Then f has a unique extension to f^ϕ in $C(\phi X)$.

2.13 Lemma [He] If X is rimcompact and realcompact, then every $f \in C^\#(X)$ has a (unique) extension $f^\phi \in C(\phi X)$.

2.14 Definition If B is a subalgebra of $C^\#(X)$ a maximal stationary set S of B is a subset of X maximal with respect to the property that every f in B is constant on S. (A simple application of Zorn's lemma shows that every subalgebra B has a maximal stationary set).

2.15 Definition A subring A of $C^*(X)$ is called algebraic if it contains the constant functions and those members f in $C^*(X)$ such that $f^2 \in A$.

2.16 Lemma If X is compact and A is an algebraic subring of $C^*(X)$, then every maximal stationary set of X is connected and the uniform closure of A, namely uA, is $\{f \in C(X)$: f is constant on every connected stationary set of A$\}$.

2.17 Notation If γX is a compactification of X, $C_\#(\gamma X)$ will denote $\{f \in C(\gamma X)$: $f|_X \in C^\#(X)\}$.

2.18 Lemma If X is rimcompact realcompact then $C_\#(\phi X)$ is an algebraic subring of $C(\phi X)$.

2.19 Theorem [He] If X is realcompact and $C^\#(X)$ determines a compactification γX of X, then X is rimcompact and $\gamma X = \phi X$.

2.20 Notation Let R(X) denote the set of points of X which fail to have a compact neighbourhood.

2.21 Example In [He] Henriksen constructs an example of a rimcompact realcompact space X such that $C^\#(X)$ does not separate points of ϕX and so does not determine ϕX.

2.22 Theorem [He] If X is a realcompact space that is 0-dimensional at each point of R(X) then $C^{\#}(X)$ determines ϕX; i.e. $uC_{\#}(\phi X) = C(\phi X)$.

2.23 Corollary [He] If X is a realcompact such that $cl_{\phi X}(\phi X - X)$ is 0-dimensional then $uC_{\#}(\phi X) = C(\phi X)$.

CHAPTER III

3.1 Lemma(**) Let X be Tychonoff, let $f \in C^{\#}(X)$, and supppose that f has an extension f^{γ} to the compactification γX of X. Then $cl_{\gamma X}Z(f) = Z(f^{\gamma})$.

3.2 Lemma(**) Suppose $C^{\#}(X)$ determines the compactification γX. If Z_1 and Z_2 are disjoint zero sets from $Z^{\#}(X)$, then $cl_{\gamma X}Z_1 \cap cl_{\gamma X}Z_2$ is empty.

3.3 Theorem(**) If $C^{\#}(X)$ determines the compactification γX of X, then every point in γX is the limit of a unique ultrafilter of zero sets in $Z^{\#}(X)$ and the collection $Z^{\#}(X)$ forms a Wallman base for γX.

3.4 Proposition(**) If $C^{\#}(\upsilon X)$ determines a compactification γX then $C^{\#}(X)$ determines γX.

3.5 Definition If U is an open subset of X and γX is a compactification of X, then $Ex_{\gamma X}U$ is defined to be $\gamma X - cl_{\gamma X}(X-U)$. The set $Ex_{\gamma X}U$ is often called the extension of U in γX.

3.6 Definition Let U be an open subset of X. A compactification γX of X is a perfect compactification with respect to the open set U if $cl_{\gamma X}(Fr_X U) = Fr_{\gamma X}(Ex_{\gamma X}U)$. A compactification of X is perfect if it is perfect with respect to every open subset of X.

3.7 Proposition A compactification γX of the space X is perfect with respect to the open set $U \subseteq X$ iff, for every set $A \subseteq U$, $cl_{\gamma X}A \cap cl_{\gamma X}(Fr_X U) = \varnothing$ implies that $cl_{\gamma X}A \cap cl_{\gamma X}(X-U) = \varnothing$.

3.8 Lemma(**) If $C^{\#}(X)$ determines the compactification γX of X then γX is a perfect compactification of X.

3.9 Proposition Let γX be a compactification of X. Then the following are equivalent:

1) γX is a perfect compactification of X.
2) If U and V are disjoint open sets of X, then $Ex_{\gamma X}(U \cup V) = Ex_{\gamma X}U \cup Ex_{\gamma X}V$.
3) Let g: $\beta X \to \gamma X$ be the (unique) extension of the identity map on X. Then $g^{\leftarrow}(p)$ is a connected subset of βX for each p in γX.

3.10 Lemma Let γX be a compactification of X. If W is an open subset of γX such that $Fr_{\gamma X}W \subseteq X$ then $W = Ex_{\gamma X}(W \cap X)$.

3.11 Lemma(**) Let X be a topological space such that $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X. Then the natural map τ from βX onto $\phi\upsilon X$ maps distinct components of $\beta X - \upsilon X$ to distinct points of $\phi\upsilon X - \upsilon X$.

3.12 Lemma(**) Let X be a topological space such that υX is rimcompact. Then if $f \in C^{\#}(X)$ anc C is a connected component of $\beta X - \upsilon X$ then $f^{\beta}[C]$ is a singleton.

3.13 Proposition(**) Let X be a topological space. Then $C^{\#}(X)$ determines the compactification $\phi\upsilon X$ of X iff $C_{\#}(\upsilon X)$ separates points of υX and $C_{\#}(\beta X)$ separates the connected components of $\beta X - \upsilon X$, i.e. if C is a connected component of $\beta X - \upsilon X$ then $C = \cap\{Z(f^{\beta} - \mathbf{r_f}): f \in C^{\#}(X)\}$ where $\{r_f\} = f^{\beta}[C]$.

3.14 Theorem(**) Suppose $C^{\#}(X)$ determines a compactification of X. Then $C^{\#}(X)$ determines βX iff the maximal ideal space $M(C^{*}(X))$ is homeomorphic to $M(C^{\#}(X))$.

3.15 Proposition(**) Let X be a topological space. If F is a closed G_{δ} of βX such that $Fr_{\beta X}F \subseteq int_{\beta X}\upsilon X$, then F is a zero set of a member of $C_{\#}(\beta X)$.

3.16 Corollary(**) Let X be a topological space. A closed subset F of X whose βX-frontier is a closed G_{δ} of βX contained in $int_{\beta X}\upsilon X$ is a zero set of $C^{\#}(X)$.

3.17 Example(**)

3.18 Theorem(**) Let X be a realcompact locally compact space. Then the zero sets of $C^{\#}(X)$ are exactly the closed G_δ's of X whose βX-frontiers are contained in X.

3.19 Corollary(**) If X is locally compact realcompact space, then the collection of closed G_δ's whose βX-frontier is contained in X forms a base for the closed sets of X.

APPENDIX B

INDEX OF SYMBOLS

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