# NETWORK RESPONSES DUE TO PARAMETER CHANGES

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by:

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ABSTRACT

This thesis discusses the various techniques of obtaining the sensitivity of network due to a parameter change. The idea of sensititivity is extended to approximate a change in a given network function when the change in some parameter is known. Various well established techniques are compared and their merits discussed. A new view is presented by use of three dimensional plots to disucss sensitivities in the real frequency domain.

# LIST OF SYMBOLS

## Chapter II

2.1

$Z(s)$ – General Network impedance function

$s_1, s_2 \ldots s_k \ldots s_n$ – Zeros of the above network function

$\delta z(s)$ – change at a branch impedance.

$A_k$ – residue of the partial fraction expansion of the reciprocal of network impedance function at the zero in question.

2.2

$H(j\omega)$ – A general network function in the real frequency domain.

$\omega$ – radian frequency.

$e_1, e_2 \ldots e_n$ – tolerances of elemental values of network.

$P$ – Numerator of H(jw).

$Q$ – Denominator of H(jw).

$\delta_i$ – actual deviation of elemental values at a network.

$A$ – Function of a complex variable.

$|A|$ – magnitude of A

$\phi$ – phase of A.

$G(s)$ – the voltage gain of a network.

Chapter III

3.1 $S_K^T$ – Sensitivity of network with respect to a variable K.

$T(s,x)$ - Network function.

$K$ - parameter of the network function.

$A(s) + KB(s)$ - numerator of T(s).

$C(s) + KD(s)$ - denominator of T(s).

$x$ elemental value of network.

$H(s,x)$ - general form of both numerator and denominator of T(s,x).

$q(s)$ - polynominal of H(s,x) not containing x.

$p(s)$ - polynominal of H(s,x) containing x with x factored from it.

$G$ - the gain as defined for root locus by W. R. Evans[6].

3.2 $-p_i, z_i$ - general pole and zero locations.

$S_x^{z_i}$ - sensitivity of pole with respect to x.

$g$ - the constant appearing in front of T(s) when T(s) is factored form.

$V$ - sum of tree admittance products of a network.

$W(1,2)$ - sum of two tree admittance products.

$z_{11}, z_{12}, z_{22}$ - open circuit impedance parameters of a two-p rt network.

3.3 $W$ - is the closed loop gain of system with negative feedback and an open loop gain of G.

$k_j$ - residue of W at pole location $S = S_j$.

$\bar{G}(s)$ - voltage gain of a network.

$f$ - superscript, meaning farads.

$\Omega$ - superscript, meaning ohms.

$h$ - superscript, meaning henrys.

$\Pi \mathbb{P}$ - product of pole vectors in the s plane to pole in question.

$\Pi \mathbb{Z}$ - product of zero vector in the s plane to pole in question.

$y_{11}, y_{12}, y_{22}$ - the short circuit admittance. parameters of a two port network.

$\chi_o$ - nominal value of the element x.

$\Delta\chi$ - small change in the elemental value of x

$\Delta\alpha$ the change in the arguement of x to $\Delta$ x.

Chapter IV

4.1

$H(j\omega)$ - network function

$|H|$ - magnitude of H(jw)

$\phi(j\omega)$ - arguement of H(jw)

$S_x^{|H|}$ - sensitivity of $|H(jw)|$ due to a change in x

$\dot{S}_x^{\phi}$ - sensitivity of $\phi$ due to a change in x

$H(R)$ - a resistive network function.

$a+Rd$ - numerator of H(R)

$c+Rd$ - denominator of H(R).

$f_0, f_1, f_2$ - complex variable function of w used to describe $H(j\omega, x)$

Chapter V

$F$ - a differentiable function

$\Delta F$ - a small change in F due to $\Delta x$.

# Chapter I

## INTRODUCTION AND MOTIVATION

Many articles papers and theses in synthesis are written to realize required network functions and time response phenomena. These give rise to many methods of designing filters, compensators, and other passive network devices. However, few articles describe the difficulties in realizing these networks in practice.

One of the problems that arises is inherent error which occurs in the manufacture of inductors, capacitors, and resistors. For example, the most common types of resistors vary as much as 10% or 20% of their nominal values. Three questions immediately arise from the use of such circuit elements:

1) What will be the resultant response of a network with elements which are not exactly those specified by the designer? What happens to the response when the value of a resistor, inductor or capacitor is 10% or 20% in error from the specified value?

2) How accurate must we choose certain elements in a network? Some may have more effect on the desired response than others. Can certain elements be allowed to be chosen less carefully than others?

3) Which synthesis technique is the best to use in order to realize a desired response? There are many synthesis techniques devised today so that the designer is able to choose from two or more networks. In manufacturing it may be advantageous from an economic point of view to use several more elements which need to be less exact than a simple network which needs very exact values.

Thus the purpose of this thesis is to present techniques already in use to answer the above questions and to propose some new ideas on this matter. The shortcomings of the techniques are illustrated by use of examples.

Chapter II deals with some of the earliest techniques developed. These have many obvious shortcomings. Chapter III uses techniques developed in the field of Control Systems and specializes these for linear, time-invariant, passive, lumped, finite, and bilateral networks. This chapter deals with network functions in the complex frequency domain. Chapter IV is an attempt to solve problems in the real frequency domain by a direct approach. Chapter V extends the idea of sensitivity discussed in the preceding two chapters to more than one element. The author also wishes to present the different techniques established in various papers in such a manner that it can be of use to the reader.

# Chapter II

## EARLY CONTRIBUTIONS

### 2.1 Displacement of Zeros Due to Incremental Parameter Changes

Probably the earliest contribution to this problem was made by A. Papoulis in 1955.[1] The following is a review of his work to show the development of this field as well as the need for research in this area.

Consider an impedance function $Z(s)$ with zeros at $s_1, s_2, \ldots\ldots s_k \ldots\ldots s_n$.

$$\text{i.e.} \quad Z(s)\Big|_{s=s_k} = 0 \qquad \ldots\ldots\ldots (2.1.1)$$

If we now add $\delta z(s)$ in any branch, the zeros of the impedance function will move to the new positions at $s_1^*, s_2^*, \ldots\ldots s_k^* \ldots\ldots s_n^*$

Since the zeros of the network are also the roots of the determinant of the system these new zeros can be found by considering the impedance or the admittance looking in at this point where $\delta z(s)$ is added.

Thus we can write

$$Z(s) + \delta Z(s)\Big|_{s=s_k^*} = 0 \qquad \ldots\ldots\ (2.1.2)$$

Assume now that zero in question is of multiplicity

Then,

$$\frac{(s-s_k)^m}{Z(s)}\Bigg|_{s=s_k^*} = A_k \neq 0 \qquad \ldots\ldots\ (2.1.3)$$

We have from (2.1.2)

$$Z(s)\Big|_{s=s_k^*} = -\delta Z(s)\Big|_{s=s_k^*} \qquad \ldots\ldots\ (2.1.4)$$

From this we can write the following relationships:

$$(s_k^* - s_k)^m = \frac{-\delta Z(s_k^*)\ (s_k^* - s_k)^m}{Z(s_k^*)} \qquad \ldots\ldots\ (2.1.5)$$

$$(s_k^* - s_k) = \sqrt[m]{\frac{-\delta Z(s_k^*)\,(s_k^* - s_k)^m}{Z(s_k^*)}} \qquad \ldots\ldots\ (2.1.6)$$

Define: $$F(s) = \frac{-\delta Z(s)\ (s-s_k)^m}{Z(s)} \qquad \ldots\ldots\ (2.1.7)$$

Superscripts indicate reference numbers in the Bibliography

Then $F(s_k) = -\delta Z(s_k) \quad A_k$ ....... (2.1.8)

$F'(s) = \frac{1}{Z(s)}$ has a singularity of order $m$ at $s = s_k$.

Thus $F(s)$ will have no singularity at this point since the factors contributing to this singularity have been cancelled. $F(s)$ will therefore be analytic in some region around $s = s_k$.

Thus if $s_k^*$ is close to $s_k$ then $F(s_k^*)$ can be expanded in a Taylor series about $s_k$ as follows:

$$F(s_k^*) = F(s_k) + F'(s_k)(s_k^* - s_k) + \cdots \qquad \text{....... (2.1.9)}$$

If $(s_k^* - s_k)$ is sufficiently small then the following approximation can be made:

$$F(s_k^*) \cong F(s_k), \qquad \text{....... (2.1.10)}$$

so that

$$(s_k^* - s_k) = \sqrt[m]{\delta Z(s) \quad A_k} \qquad \text{....... (2.1.11)}$$

$A_k$ is the residue of the partial fraction expansion of $\frac{1}{Z(s)}$ at $s = s_k$.

Equation (2.1.11) thus becomes very useful in determining the initial direction of motion of zeros by incremental changes in parameters. However it has a drawback because of tedious calculations necessary for the residue of each zero. Another short-coming is that this technique allows variations only in the small and shows only the initial direction of movement of zeros due to parameter changes.

## 2.2 Bounds on Frequency Response of a Network

The following development was done as an application to networks by W. C. Yengst[2] based on work done by B. R. Myers[3] in Control Systems in 1959. It enables one to work out the maximum and minimum bounds on $H(j\omega)$ due to many simultaneous changes in parameters of a network.

Let us consider a network where the element values differ at most from those required by the amounts $e_1, e_2, \cdots e_n$. Then the network function can be written as

$$H(s, e_1, e_2 \cdots e_n) = \frac{P(s, e_1, e_2, \cdots e_n)}{Q(s, e_1, e_2, \cdots e_n)} \quad \cdots\cdots\cdots \quad (2.2.1)$$

$$= \frac{P_0(s) + \sum_i e_i P_i(s) + \sum_{i \neq j} e_i e_j P_{ij}(s) + \cdots}{Q_0(s) + \sum e_i Q_i(s) + \sum_{i \neq j} e_i e_j P_{ij}(s) + \cdots} \quad \cdots\cdots\cdots \quad (2.2.2)$$

where the $P's$ and $Q's$ are polynominal in $s$.

From Myers' Theorem[4] we have

$$P(j\omega, e_1, e_2 \cdots e_n) = P_0(j\omega) + \sum e_i P_i(j\omega) + \sum_{i \neq j} e_i e_j P_{ij}(j\omega)$$

$$+ \cdots + \sum_{i \neq j \neq \cdots \neq n} [e_1, e_2, \cdots e_n] P_{ij \cdots n}(j\omega) \quad \cdots\cdots\cdots \quad (2.2.3)$$

where the $e_i$'s are real numbers.

For a specified $\omega = \omega_0$, $P(j\omega_0, e_1, \cdots e_n)$ falls in the interior of the polygon of $P$ calculated at the extreme values of

$$e_i = \pm \delta_i \quad \text{for} \quad i = 1, 2, \cdots n .$$

If there are $n$ $e_i$'s there will be $2^n$ extreme values of the polygon.

For example consider $A(j\omega, e_1, e_2)$ with only two variables. We must plot $A(j\omega_0, e_1, e_2)$ in the complex plane for the extreme values of

a) $e_1 = -\delta_1$, $\qquad e_2 = -\delta_2$

b) $e_1 = -\delta_1$, $\qquad e_2 = \delta_2$

c) $e_1 = \delta_1$, $\qquad e_2 = -\delta_2$

d) $e_1 = \delta_1$, $\qquad e_2 = \delta_2$

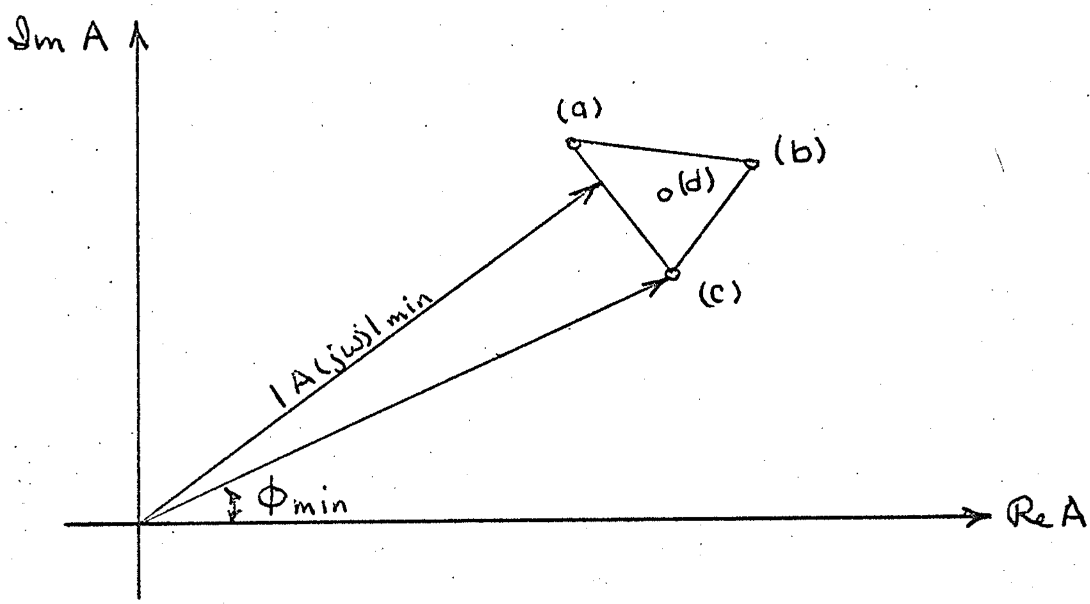


Fig. 2.2.1

Here we can pick out for this example

$|A(j\omega)|_{min}$ and $\phi_{min}$

Thus for a ratio of two polynominals as in the case of network functions, the maximum and minimum limits may be found.

i.e. $$|H(j\omega)|_{max} = \frac{|P(j\omega)|_{max}}{|Q(j\omega)|_{min}} \quad \ldots\ldots\ldots \quad (2.2.4)$$

$$|H(j\omega)|_{min} = \frac{|P(j\omega)|_{min}}{|Q(j\omega)|_{max}} \quad \ldots\ldots\ldots \quad (2.2.5)$$

Similarly for the phase of $H(j\omega) = \phi_H(j\omega)$

$$\phi_H(j\omega)_{max} = \phi_P(j\omega)_{max} - \phi_Q(j\omega)_{min} \quad \ldots\ldots\ldots \quad (2.2.6)$$

$$\phi_H(j\omega)_{min} = \phi_P(j\omega)_{min} - \phi_Q(j\omega)_{max} . \quad \ldots\ldots\ldots \quad (2.2.7)$$

Although this seems like a good approach to obtaining errors in network response, it has the following short-comings:

1) The calculations are laborious for any more than two simultaneous variations. Also they must be done for each frequency separately.

2) The maximum and minimum values certainly enclose the limits but in some cases are poor guides since in some cases, these limits seem to be quite large. The following example will illustrate this point:

Consider the voltage gain of the following circuit:

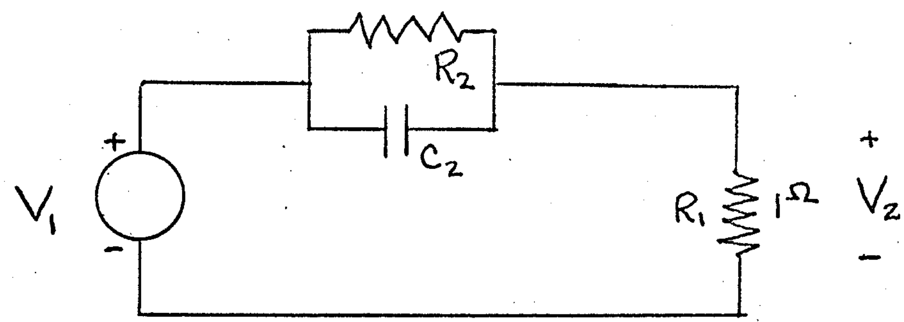


Fig. 2.2.2.

$$G(j\omega) = \frac{V_2}{V_1}(j\omega) = \frac{1+j\omega R_2 C_2}{(R_2+1)+j\omega R_2 C_2} \qquad \ldots\ldots\ldots (2.2.8)$$

The nominal values for $R_2$ and $C_2$ are:

$R_2 = 0.5$ ohms , $C_2 = 1$ farad

This gives

$$G(s) = \frac{s+2}{s+3} \qquad \ldots\ldots\ldots (2.2.9)$$

and the following Bode plot

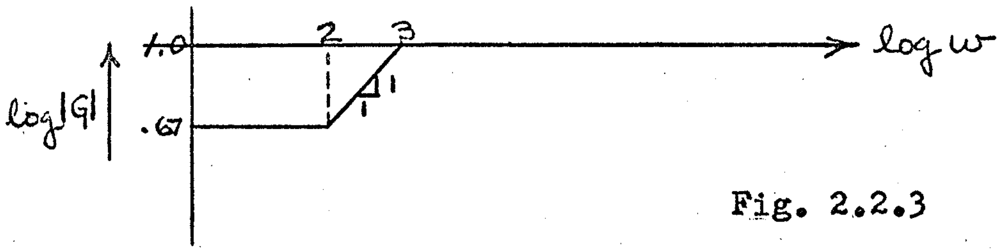


Fig. 2.2.3

Using the technique described on pages 8 - 10 and varying $R_2$ and $C_2$ by 10%, the data was plotted in Figure 2.2.4.

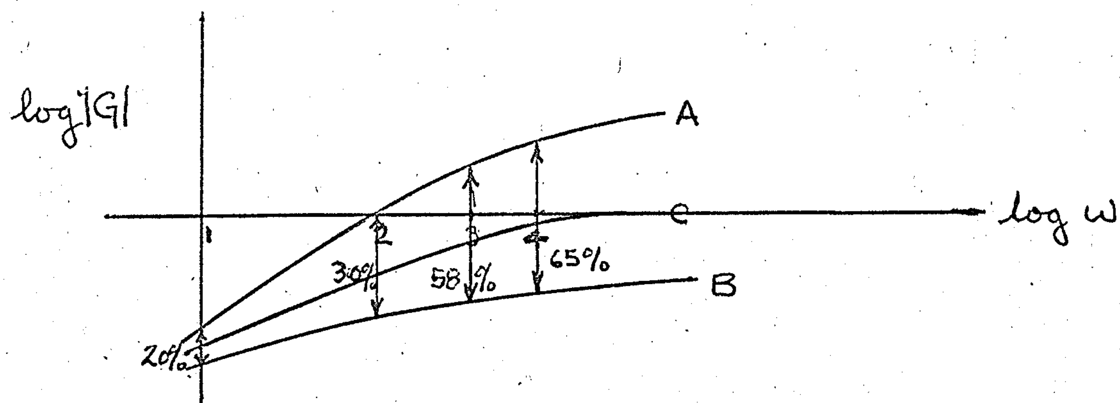


Fig. 2.2.4

C - Required curve

A,B - Bounds given from data.

The percentage error increases with frequency hence becoming quite large for $\omega > 4$. This is due to the fact that the polygon becomes larger with increasing frequency. This gives safe limits in which the resulting gain is to occur but from figure 2.2.4 we see that this does not give a measure of the magnitude of the actual error. We notice that for high frequencies, $C_2$ shunts out $R_2$ and the gain becomes unity, independent of any errors in $R_2$ or $C_2$.

# Chapter III

## SENSITIVITY BY ROOT LOCUS METHODS

### 3.1 General

Bode[5] has defined the term sensitivity as:

$$S_K^T \triangleq \frac{\partial(\ln T(s))}{\partial(\ln K)} = \frac{\partial T/T}{\partial K/K} \qquad \ldots\ldots\ (3.1.1)$$

where $T(s)$ is the network function and $K$ is a variable parameter in the system. It could be almost any parameter such as capacitance, resistance, temperature, etc.

Bode has shown that network functions can be written in general as follows:

$$T(s) = \frac{A(s) + K\,B(s)}{C(s) + K\,D(s)} \qquad \ldots\ldots\ (3.1.2)$$

where A, B, C and D are polynomials in $s$. The proof can be simplified somewhat if we let K be an element value in the circuit such as resistance, inductance, or capacitance.

Proof:

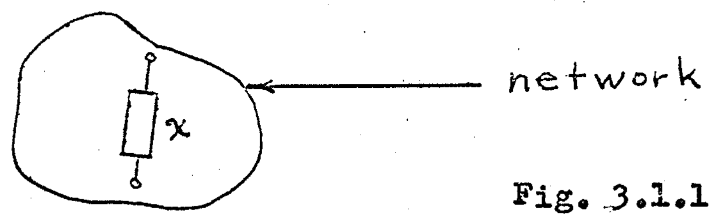


Fig. 3.1.1

$T(s)$ is a ratio of tree admittance or impedance products or of the compliments of the trees considered. Since $x$ either appears once in a tree or not at all and, since these products are summed, $x$ is never raised to any power higher than one. Separating the numerator and denominator into two parts and factoring out the $x$ we obtain the form.

$$T(s) = \frac{A(s) + x\,B(s)}{C(s) + x\,D(s)} \qquad \text{....... (3.1.3)}$$

The following analysis will examine the effect on the poles and zeros of the network function $T(s)$ by varying $x$. The results of this analysis are useful since many synthesis specifications require prescribed pole-zero configurations.

We note that both the numerator and denominator are of the form

$$H(s,x) = q(s) + x\,p(s) \qquad \text{....... (3.1.4)}$$

By examining the roots of $H(s,\chi)=0$ we can determine the poles and zeros of $T(s)$ using $\chi$ as a variable parameter.

$$\text{i.e. } q(s) + \chi\, p(s) = 0 \qquad \ldots\ldots\ldots (3.1.5)$$

$$\text{or} \quad \chi\, \frac{p(s)}{q(s)} = -1 \qquad \ldots\ldots\ldots (3.1.6)$$

Equation (3.1.6) is the familiar root locus as formulated in detail by W. R. Evans[6] with $G = \chi \frac{p(s)}{q(s)}$ . Many methods have been established to facilitate the plotting of the locus of $G=-1$ .

## 3.2 Pole and Zero Sensitivity by Root Locus

In general we can write

$$T(s) = g\,\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{1}^{n}(s+p_i)} \qquad \ldots\ldots\ (3.2.1)$$

noting that $z_i, p_i,$ and $g$ may be functions of the parameter $x$.

$$S_x^T = \frac{d\ln T}{d\ln x} = x\,\frac{d\ln T}{dx} \qquad \ldots\ldots\ (3.2.2)$$

$$\ln T = \ln g + \sum_1^m \ln(s+z_i) - \sum_1^n \ln(s+p_i) \qquad \ldots\ldots\ (3.2.3)$$

$$\frac{d\ln T}{dx} = \frac{1}{g}\frac{\partial g}{\partial x} + \sum_1^m \frac{\frac{\partial z_i}{\partial x}}{s+z_i} - \sum_1^n \frac{\frac{\partial p_i}{\partial x}}{s+p_i} \qquad \ldots\ldots\ (3.2.4)$$

so that

$$S_x^T = \frac{x}{g}\frac{\partial g}{\partial x} + \sum_1^m \frac{x\frac{\partial z_i}{\partial x}}{s+z_i} - \sum_1^n \frac{x\frac{\partial p_i}{\partial x}}{s+p_i} \qquad \ldots\ldots\ (3.2.5)$$

we now define zero sensitivity as

$$S_x^{z_i} = x\frac{\partial z_i}{\partial x} = \frac{\partial z_i}{\partial \ln x} \qquad \ldots\ldots\ (3.2.6)$$

i.e. the change of zero location per relative parameter variation and define pole sensitivity as

$$S_x^{p_i} = x\frac{\partial p_i}{\partial x} = \frac{\partial p_i}{\partial \ln x} \qquad \ldots\ldots\ (3.2.7)$$

so that

$$S_x^T = \frac{\partial \ln g}{\partial \ln x} + \sum_1^m \frac{S_x^{z_i}}{s+z_i} - \sum_1^n \frac{S_x^{p_i}}{s+p_i} \qquad \ldots\ldots\ (3.2.8)$$

Before we apply these definitions let us consider the following theorem which will allow us to find certain pole zero sensitivities by inspection.

Theorem[7]

If there is a single element of value x connecting two otherwise separate parts of a network, then the sensitivity of the roots of the polynomial $V$[8] of the overall network with respect to the x is zero.

Proof:

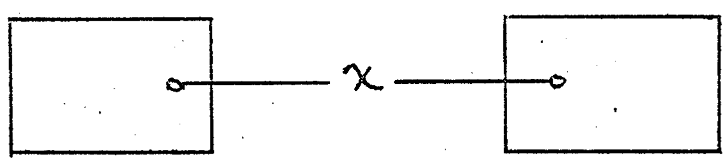


Fig. 3.2.1

- x must appear in all the trees of the network.
- The sum of tree admittance products can be written as

$$V = x\, W(1,2),$$

where $W(1,2)$ is the sum of the 2-tree admittance products not containing x.

- Roots of $W(1,2)$ are roots of $\chi$.

$\therefore$ Roots of $V$ are independent of x.

Thus for a network of the type shown in Figure 3.2.2

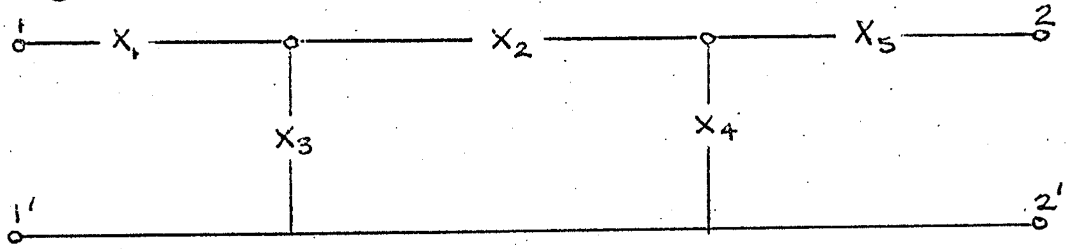


Fig. 3.2.2

the poles of $z_{11}$, $z_{12}$ and $z_{22}$ are independent of $X_1$ and $X_5$ since

$$z_{11} = \frac{W_{1,1'}(Y)}{V(Y)} \qquad \text{.......} \; (3.2.9)$$

and

$$z_{12} = z_{21} = \frac{W_{12,12'}(Y) - W_{12',1'2}(Y)}{V(Y)} \qquad \text{.......} \; (3.2.10)$$

$$z_{22} = \frac{W_{2,2'}(Y)}{V(Y)} \qquad \text{.......} \; (3.2.11)$$

Another typical example is the bridged-T network of Figure 3.2.3

i.e.

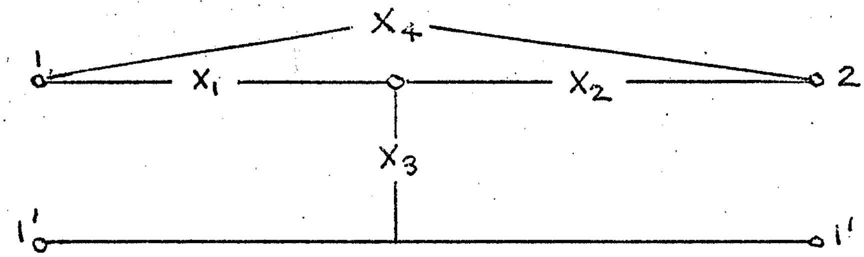


Fig. 3.2.3.

Here the poles of the $z$ parameters are independent of $x_3$, since here $x_3$ is the only linking element in the network.

This theorem may be applied to any of the network function that has a $V$ product in the numerator or denominator. This theorem saves labour in finding the pole or zero sensitivities.

### 3.3 Sensitivity of Zeros

We have defined the sensitivity of zeros as

$$S_x^{z_i} = \frac{\partial z_i}{\partial x/x} \qquad \text{....... (3.3.1)}$$

Let us use a more general symbol, namely $s_i$ to let the notation include both pole and zero sensitivity.

i.e.

$$S_x^{s_i} = \frac{\partial s_i}{\partial x/x} = x\,\frac{\partial s_i}{\partial x} \qquad \text{....... (3.3.2)}$$

The root locus of both numerator and denominator is defined by

$$H(s,x) = q(s) + x\;p(s) = 0 \qquad \text{....... (3.3.3)}$$

so that

$$S_x^{s_j} = (-1)\,x\,\frac{\partial H/\partial x}{\partial H/\partial s_j} \qquad \text{....... (3.3.4)}$$

everywhere along the root locus, where $s_j$ is position of the roots along the locus. From equations (3.3.3) and (3.3.4) we obtain

$$S_x^{s_j} = -\left.\frac{x\;p(s)}{q'(s) + x\;p'(s)}\right|_{s=s_j} \qquad \text{....... (3.3.5)}$$

where the prime notation denotes the first derivative with respect to S .

If we now consider a function

$$W = \frac{G}{1+G} \qquad \ldots\ldots\ (3.3.6)$$

where

$$G = x\, p(s)/q(s)$$

then

$$W(s) = \frac{x\ p/q}{1 + x\, p/q} = \frac{x\, p}{q + x\, p} . \qquad \ldots\ldots\ (3.3.7)$$

The residue of W for $s = s_j$ i.e. $k_j$ is given by

$$k_j = \frac{x\ p}{q' + x\, p'}\Bigg|_{s=s_j} , \qquad \ldots\ldots\ (3.3.8)$$

thus

$$S_x^{s_j} = -k_j \qquad \ldots\ldots\ (3.3.9)$$

This method of finding pole-zero sensitivity shall be referred to as the residue-method of pole-zero sensitivity in later discussions.

The following example will help to illustrate some of the development previously introduced.

Let us assume that the following circuit has been synthesized for a voltage transfer function.

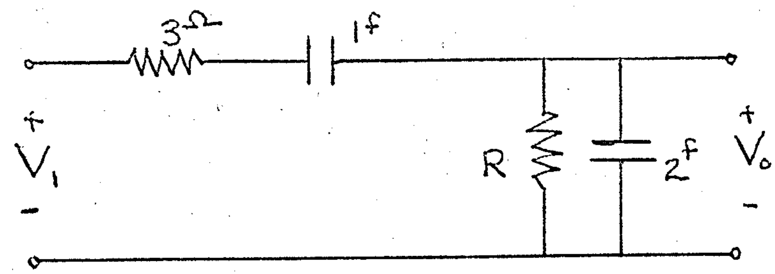


Fig. 3.3.1

Then the voltage gain is

$$\bar{G}(s) = \frac{V_o}{V_1} = \frac{Rs}{3s+1+Rs(6s+3)} \qquad \ldots\ldots\ (3.3.10)$$

If the nominal value of $R = 1$ ohm,

$$\bar{G}(s)\Big|_{R=1} = \frac{s}{6s^2+6s+1} = \frac{s}{(s+.789)(s+.211)} \qquad \ldots\ldots\ (3.3.11)$$

If we consider the denominator of $\bar{G}$ as $H$ we obtain

$$H(s,R) = 3s+1+Rs(6s+3) \qquad \ldots\ldots\ (3.3.12)$$

and

$$G(s) = \frac{x\,p(s)}{q(s)} = \frac{RS(6s+3)}{3s+1}$$

$$= \frac{2Rs(s+\frac{1}{2})}{(s+\frac{1}{3})} \qquad \ldots\ldots\ (3.3.13)$$

The root locus for $G$ is shown in Figure 3.3.2.

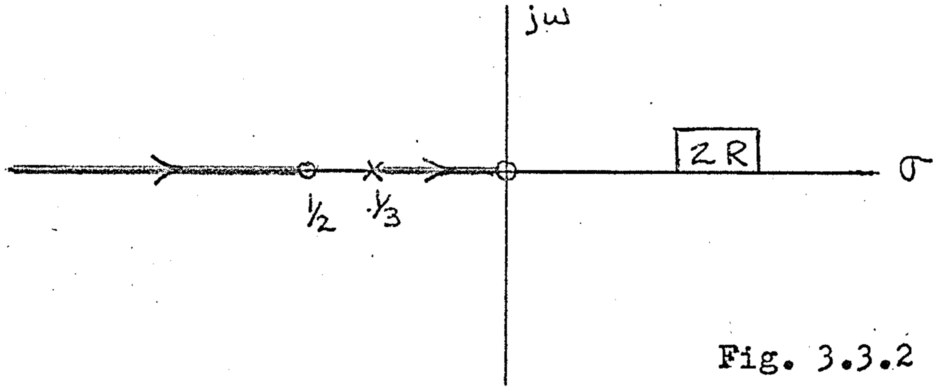


Fig. 3.3.2

We notice that the root locus could be calibrated in $R$ and that the roots of $H$ could be examined for various values of $R$. If we let $R$ be equal to one ohm, then

$$H(s) = (s+.789)(s+.211) \qquad \ldots\ldots\ (3.3.14)$$

This gives the location of the poles of $W(s)$ as shown in Figure 3.3.3.

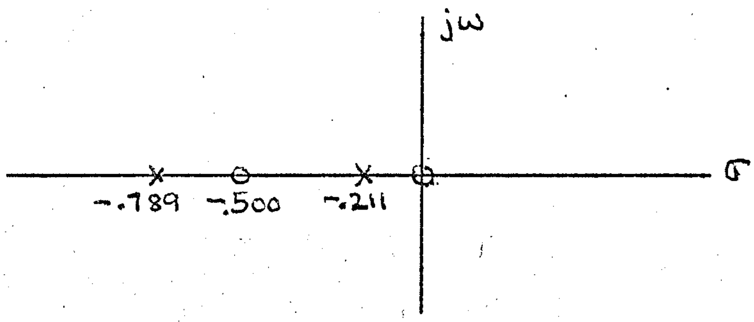


Fig. 3.3.3

The sensitivity of the pole at $s=-.211$ is given by

$$S_R^{-.211} = -k_{-.211} = -2\left[\frac{\Pi\ \mathbb{P}}{\Pi\ \mathbb{Z}}\right]$$

$$= -2\left[\frac{.211\,\angle 180^\circ\ \ .289}{.578}\right] = .211 \qquad \text{....... (3.3.15)}$$

For more complicated $G$ graphical means could be used and tools such as a 'Spirule' may be employed.

The sensitivity of the pole at $s=-.789$ is given by

$$S_R^{-.789} = -k_{-.789} = -2\left[\frac{\Pi\ \mathbb{Z}}{\Pi\ \mathbb{P}}\right]$$

$$= -2\left[\frac{.789\,\angle 180^\circ\ \ .289\,\angle 180^\circ}{.578\,\angle 180^\circ}\right] = .789 \qquad \text{....... (3.3.16)}$$

Suppose we allow to change by 10%, then the approximate change of pole position are calculated as follows:

$$\Delta p_i \cong S_R^{-p_i}\ \frac{\Delta R}{R} \qquad \text{....... (3.3.17)}$$

thus

$$\Delta p_{-.211} \cong .211 \times .10 = .0211 \qquad \text{....... (3.3.18)}$$

and

$$\Delta p_{-.789} \cong .789 \times .10 = .0798 \qquad \text{....... (3.3.19)}$$

From the previous example the following observations are made:

1) The pole furthest to the left is more sensitive to a change in $R$ than the one on the right. This result allows us to compare the effect on specific poles or zeros and enables us to design utilizing sensitivities where in some cases it may be necessary to have some poles or zeros may be compared with variations in other element values.

2) $$G(s) = \frac{-y_{12}}{y_{22}} = \frac{-y_{21}}{y_{22}} \qquad \text{.......} \ (3.3.20)$$

Noting that

$$y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0} , \qquad \text{.......} \ (3.3.21)$$

we see that the output is shorted in finding $G(s)$. Therefore the numerator is independent of $R$ (or any shunt elements).

3) A change in $R$ causes a corresponding change in gain.

4) The locus has to be calibrated for different values of $R$.

## 3.4 Evaluation of $S_x^{p_i}$ without Evaluation of $x$ Itself.

The pole sensitivity may be expressed as

$$S_x^{p_i} = \frac{\partial p_i}{\frac{\partial x}{x}} \cong \frac{\Delta p_i}{\frac{\Delta x}{x}} = \frac{\Delta p_i}{j \frac{\Delta x}{jx}} \qquad \ldots\ldots\ (3.4.1)$$

Assuming we have a root locus of $\frac{p(s)}{q(s)} = -x$ of some shape as shown in Figure 3.4.1.

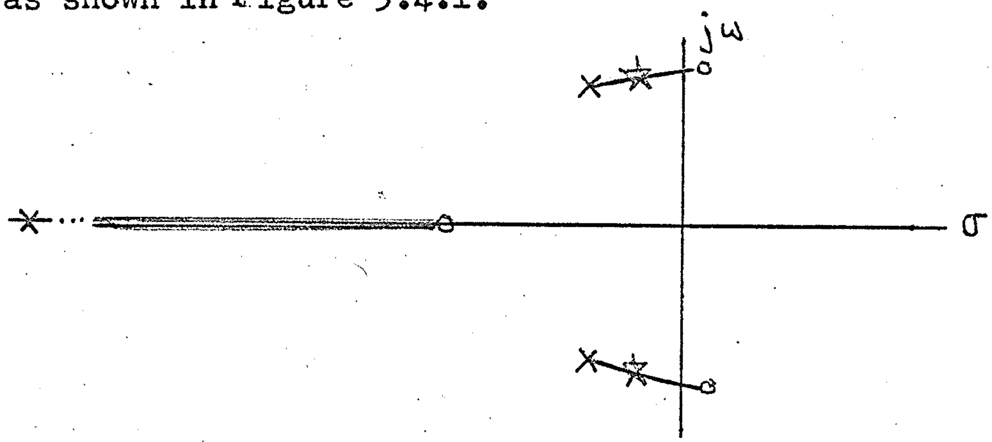

Fig. 3.4.1

suppose we wish to find the pole sensitivity as the locations marked (☆) without evaluating $x$ corresponding to these pole locations.

Allow $p_i$ to move at a perpendicular small distance from the root locus of $\Delta p_i$.

For a particular point on the root locus such as $p_i$, the following equation is satisfied,

$$\frac{p(p_i)}{q(p_i)} = -x_o \ , \qquad \ldots\ldots\ (3.4.2)$$

where $x_0$ is a real number.

For the new point

$$\frac{p(pi+\Delta pi)}{q(pi+\Delta pi)} = -(x_0+\Delta x) \qquad \cdots\cdots\cdots (3.4.3)$$

If we examine the $x$ plane of the above equation we find that $\Delta x$ leaves $x_0$ perpendicularly due to conformality*.
(see figure 3.4.2)

i.e.

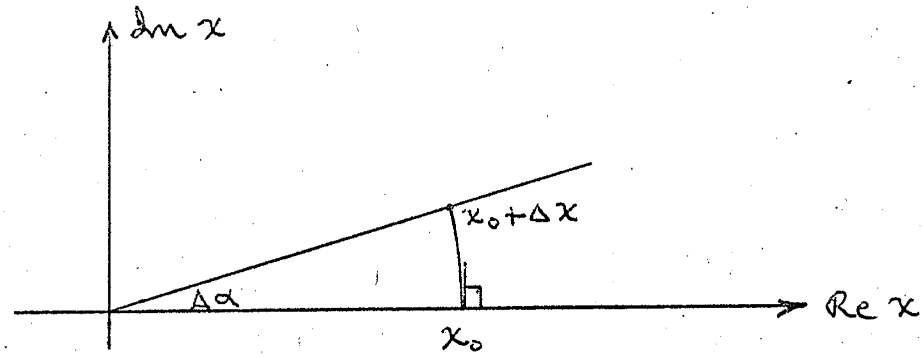


Fig. 3.4.2

Since $\Delta x \cong j\Delta\alpha \cdot x_0$ ....... (3.4.4)

Then

$$\Delta\alpha \cong \Delta x / j x_0 \qquad \cdots\cdots\cdots (3.4.5)$$

Substituting equation (3.4.5) into (3.4.1) we obtain

$$S_x^{pi} \cong \frac{\Delta pi}{\frac{j\Delta x}{jx}} = \frac{\Delta pi}{j\Delta\alpha} \qquad \cdots\cdots\cdots (3.4.6)$$

*See appendix for proof.

where $\Delta\alpha$ is the angle of $\chi$ at new location of the pole.

Equation (3.4.6) allows us to establish a graphical procedure for evaluating the pole sensitivity.

We move $p_i$ by a known distance $\Delta p_i$. The angle of $\chi$ at this new location can be calculated in the following manner:

$$\chi = \frac{-p(s)}{q(s)} \qquad \ldots\ldots\ldots (3.4.7)$$

$$\angle\chi = \pi + \sum \angle \text{pole vectors to } p+\Delta p_i - \sum \angle \text{zero vectors to } p+\Delta p_i \qquad \ldots\ldots\ldots (3.4.8)$$

and $\Delta\alpha = \angle\chi$ ....... (3.4.9)

Finally having $\Delta p_i$ and $\Delta\alpha$ we substitute into

$$S_{\chi}^{p_i} = \frac{\Delta p_i}{j\Delta\alpha} \qquad \ldots\ldots\ldots (3.4.10)$$

Let us consider the same example done previously.

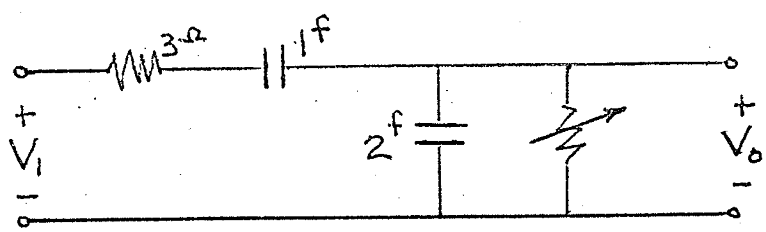


Fig. 3.4.3

Recall that

$$\bar{G}(s) = \frac{V_o}{V_i}(s) = \frac{Rs}{3s+1+Rs(6s+3)} \quad \ldots\ldots\ldots (3.4.11)$$

Where the root locus of the denominator of $\bar{G}(s)$ is shown in Figure (3.4.4)

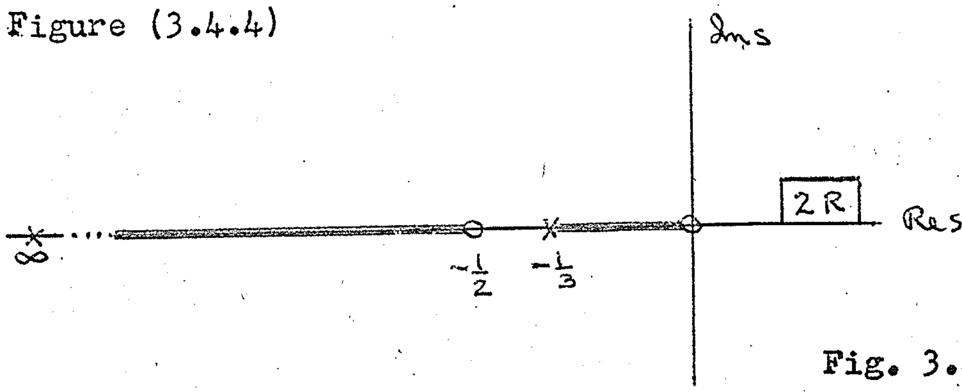


Fig. 3.4.4

Suppose that a pole is needed at $s = -.211$.

Proceeding by the graphical method to determine $S_R^{-.211}$ we obtain: $S_R^{p_i} = \frac{\Delta s}{j\Delta\alpha}$

where $\Delta\alpha$ is the angle of $x$.

$\Delta\alpha = \pi + \sum_i \angle \text{zero vectors} - \sum_i \angle \text{pole vectors}$

for $\Delta p_i$ let us choose it be $+j.05$.

$$\Delta\alpha = \pi + \tan^{-1}\frac{.05}{.122} - \left(\pi - \tan^{-1}\frac{.05}{.122}\right) - \tan^{-1}\frac{.05}{.289}$$

$$= -.144 \text{ radians}$$

$$\therefore S_R^{-.211} \triangleq \frac{-j.05}{-j.144} \quad .347$$

This result compares favorably with that obtained by the residue method. Although it is an approximation it is less laborious since the root locus does not have to be calibrated. This feature allows a circuit designer to move certain poles and zeros in order to find a less sensitive position.

## 3.5 Response Due to Pole Changes in the Real Frequency Domain

Let us plot on a Bode diagram the effect of pole changes for a 10% increase in $R$ using the previous example where $R$ had the value of one ohm.

This gives $\bar{G}(s)$ the pole-zero configuration of Figure 3.5.1

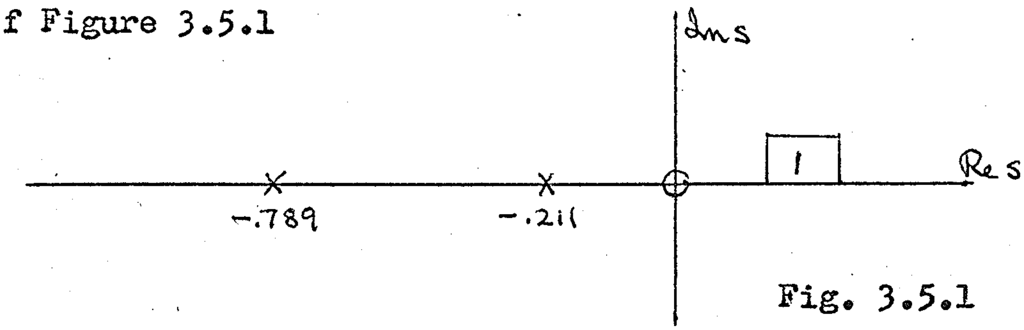


Fig. 3.5.1

Using the values of sensitivity as obtained by the residue method of pole-zero sensitivity viz.

$$S_R^{-.211} = .211 \qquad \ldots\ldots\ (3.5.1)$$

and

$$S_R^{-.789} = .789 \qquad \ldots\ldots\ (3.5.2)$$

we obtain the pole-zero configuration of $\bar{G}_n(s)$ shown in Figure (3.5.2)

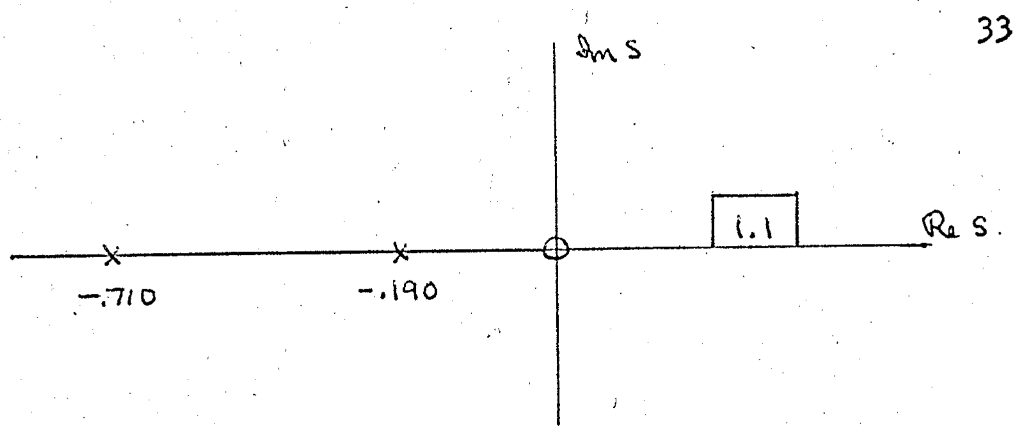


Fig. 3.5.2

where

$$\overline{G}_n(s) = \frac{1.1\, s}{(s+.190)(s+.710)} \qquad \text{.......} \quad (3.5.3)$$

Bode plots of $\overline{G}(s)$ and $\overline{G}_n(s)$ are illustrated in Figure (3.5.3).

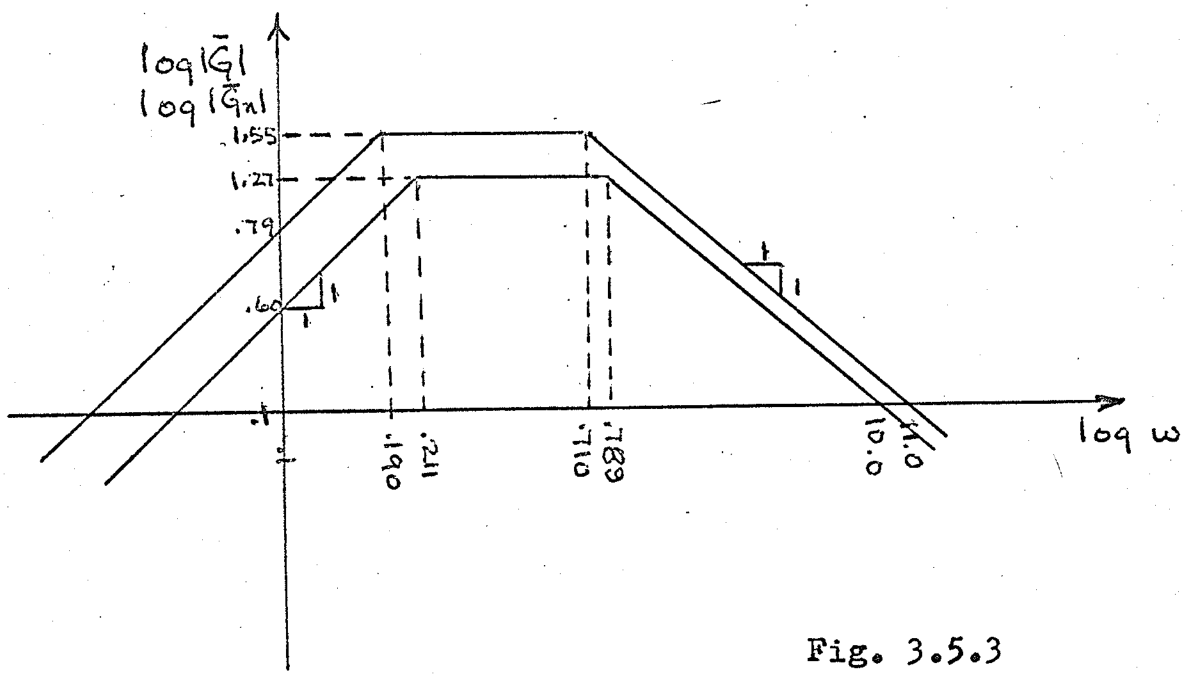


Fig. 3.5.3

From the new pole position we can compare the frequency responses due to an increase in R . Since many synthesis specifications are given in terms of such diagrams (Bode plots) this may be a desirable result. The following chapter will try to obtain this same result more directly.

# Chapter IV

# SENSITIVITY OF NETWORK FUNCTIONS IN THE REAL FREQUENCY DOMAIN

## 4.1 General

The following development is considered to be original. Although it offers no simple method for calculating sensitivity and error, it does give a different point of view and is useful in certain special cases which are discussed. For some simple circuits, quick direct answers are obtainable in the real frequency domain.

Assume we have a network function

$$H(j\omega) = |H(j\omega)|\, e^{j\phi(j\omega)} = |H| e^{j\phi} \qquad \ldots\ldots\ (4.1.1)$$

Let us now consider the sensitivity function

$$S_x^{H(j\omega)} = \frac{x\, d \ln H(j\omega)}{dx} \qquad \ldots\ldots\ (4.1.2)$$

$$= x \frac{d}{dx}(\ln |H| + j\phi) \qquad \ldots\ldots\ (4.1.3)$$

$$= \frac{d \ln |H(j\omega)|}{d \ln x} + j \frac{d\phi}{d \ln x} \qquad \ldots\ldots\ (4.1.4)$$

$$S_x^{H(j\omega)} = S_x^{|H|} + j\, \dot{S}_x^{\phi} \qquad \ldots\ldots\ (4.1.5)$$

A knowledge of $S_x^{|H|}$ usually implies a knowledge of $\dot{S}_x^{\phi}$ since we can write

$$\dot{S}_x^{\phi} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{S_x^{|H|}(y)}{y-\omega}\, dy \qquad \text{....... (4.1.6)}$$

provided the function is non-minimum phase.
Thus the phase sensitivity $\dot{S}_x^{\phi} = \frac{d\,\phi}{d\ln x}$ may be found from the magnitude sensitivity

$$S_x^{|H|} = d\ln |H| \Big/ d\ln x \qquad \text{....... (4.1.7)}$$

Using the definition for magnitude sensitivity

$$S_x^{|H|} = \frac{\partial \ln |H(j\omega)|}{\partial \ln x} \qquad \text{....... (4.1.8)}$$

we note that for any particular $\omega_o$, $S_x^{|H(j\omega_o)|}$ is the slope of $|H(j\omega)|$ versus $x$ on a log-log plot.
Thus for all $\omega$ we construct a three dimensional plot as shown in Figure (4.1.1) and $S_x^{|H(j\omega)|}$ defines surface.

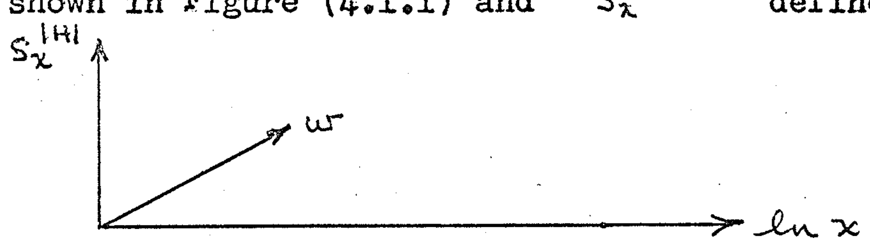


Fig. 4.1.1

This reduces to a two dimensional problem for any particular value of $x$ say $x_i$. In general this may be difficult to apply but there are a few special cases which are noteworthy.

Case (1)

"Resistance Only" Networks

This case is the simplest of all to analyze since the general form of the network function is not a function of $\omega$. "Resistance only" networks arise in the design and analysis of attenuator pads and potentiometer networks. For the resistance only case

$$H(R) = \frac{a + R\,b}{c + R\,d} \quad \ldots\ldots\ldots (4.1.9)$$

where $a, b, c$ and $d$ are constants.

We now plot $\ln H$ versus $\ln R$ with the help of asymptotic lines as sketched in Figure (4.1.2)

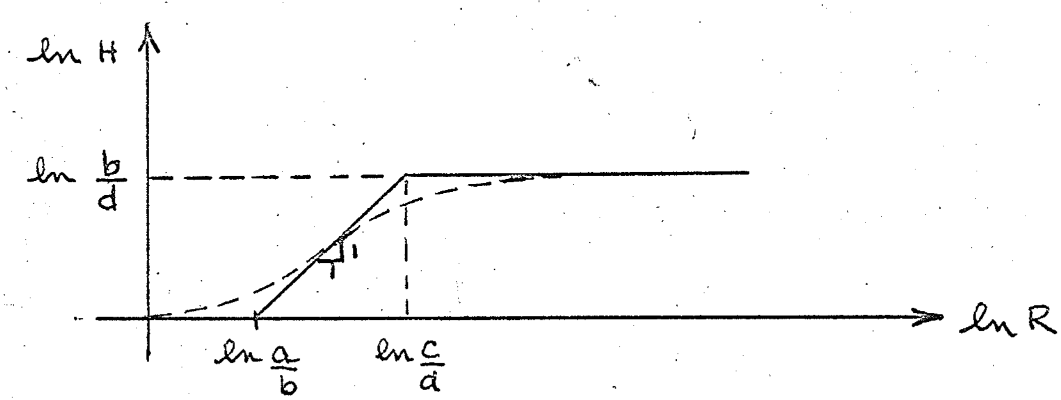


Fig. 4.1.2

The sensitivity of the network is given in Figure (4.1.3)

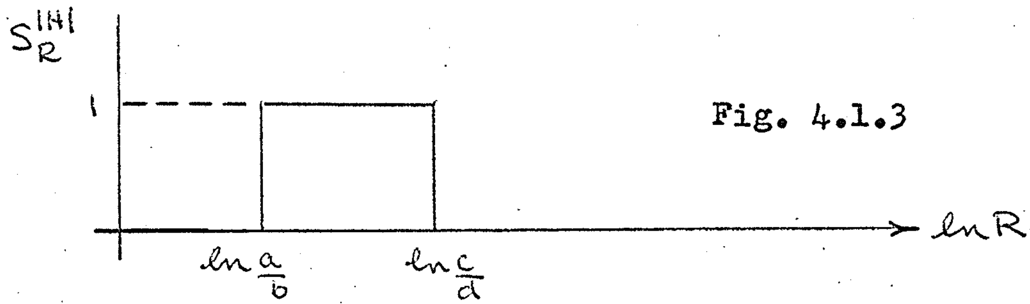


Fig. 4.1.3

In this case, $H$ is most sensitive to changes in $R$ in the range $\frac{a}{b} < R < \frac{c}{d}$

Case (2)

"Reactance Only" Networks

The general form of a network function is

$$H(s) = \frac{A(s) + x\ B(s)}{C(s) + x\ D(s)} \qquad \ldots\ldots\ (4.1.10)$$

and it is well known that only even or odd powers of may be present in the numerator and denominator. Thus

$$H(j\omega, x) = f_0 \left(\frac{x + f_1}{x + f_2}\right) \qquad \ldots\ldots\ (4.1.11)$$

where $f_0$, $f_1$ and $f_2$ are functions of $\omega$. $f_0$ is imaginary and $f_1$ and $f_2$ are real. For a particular frequency $\omega_0$ we plot asymptotes of $H(j\omega_0, x)$ on a log-log plot as shown in Figure (4.1.4)

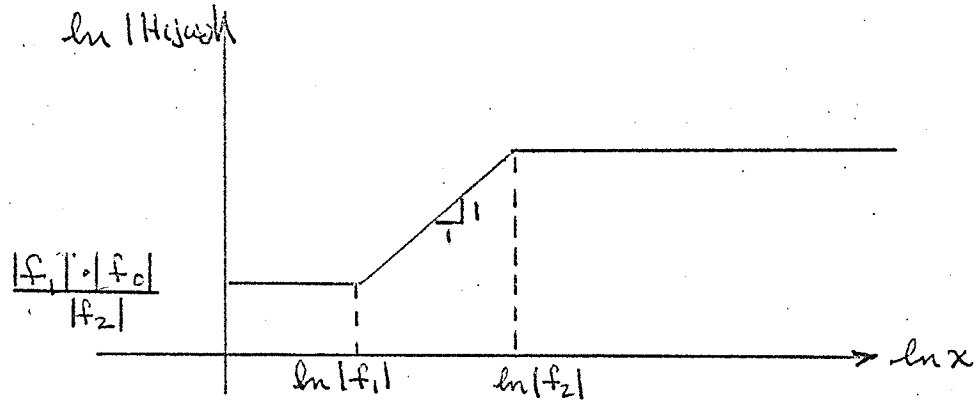


Fig. 4.1.4

For all $\omega$ this defines a surface. To find the sensitivity, we merely take the slope. In three dimensions we have the shape illustrated in Figure (4.1.5)

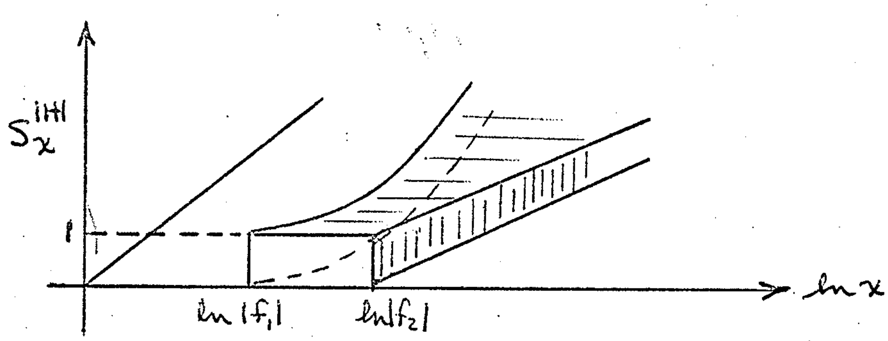


Fig. 4.1.5

By choosing a nominal value for $x$ , we obtain the sensitivity of $|H|$ as a function of frequency.

The following example will illustrate the method for reactive networks:

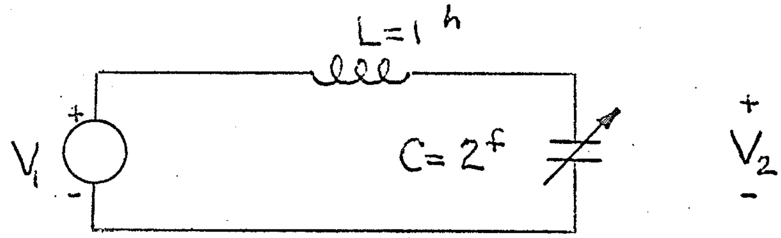


Fig. 4.1.6

The voltage gain for the network in Figure (4.1.6) is given by

$$\frac{V_2}{V_1} = \bar{G}(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega} \qquad \ldots\ldots\ (4.1.12)$$

$$= \frac{1}{1-\omega^2 C} = -\frac{1}{\omega^2} \cdot \frac{1}{C - \frac{1}{\omega^2}} \qquad \ldots\ldots\ (4.1.13)$$

$$\bar{G}(j\omega_0) = \frac{1}{\omega_0^2} \cdot \frac{1}{C - \frac{1}{\omega_0^2}} \qquad \ldots\ldots\ (4.1.14)$$

Figure (4.1.7) shows a plot of the asymptotes of $|\bar{G}|$ .

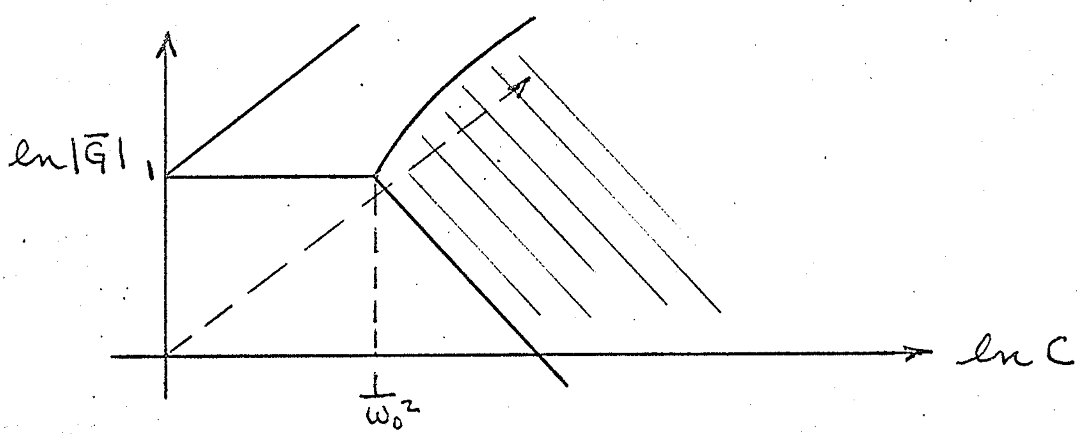


Fig. 4.1.7

Plotting the sensitivity of $|\bar{G}|$ with respect to $\omega$ we obtain the plot of Figure (4.1.8)

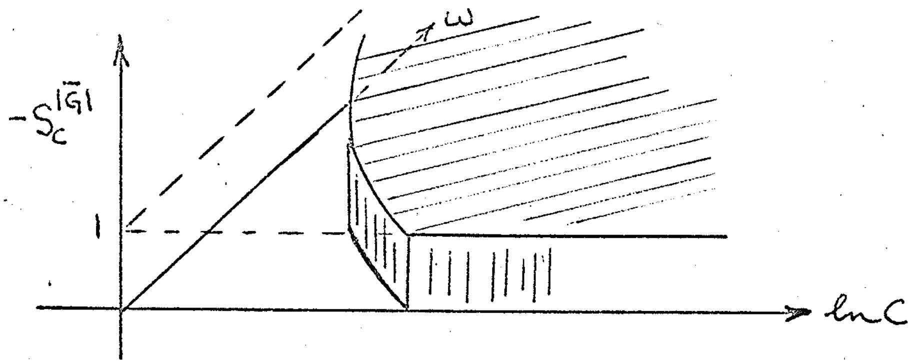


Fig. 4.1.8

For a nominal value of $C=2$ farads we obtain the sketch of Figure (4.1.9)

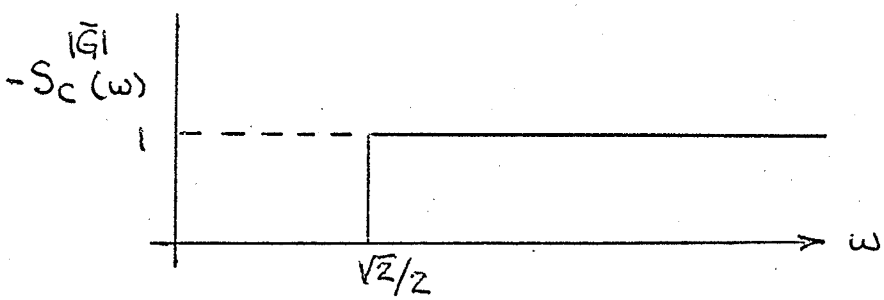


Fig. 4.1.9

The example illustrates that sensitivity of the gain increases with frequency. It should be noted that at the poles of the network function, sensitivity has very little meaning since it deals with the infinitesimal. $S_x^{|G|}$ does not exist at $\omega=\frac{\sqrt{2}}{2}$ . This problem can be overcome by dealing with the reciprocal of the network function. The logarithmic derivative therefore exists at this point since it is now a zero of the function.

Case (3)

General

The network function

$$H(j\omega, x) = f_0(j\omega)\left(\frac{x + f_1(j\omega)}{x + f_2(j\omega)}\right) \quad \text{.......} \quad (4.1.15)$$

may be rewritten in the form

$$H(j\omega, x) = (\mathrm{Re}\, f_0 + j\,\mathrm{Im}\, f_0)\left(\frac{x + \mathrm{Re}\, f_1 + j\,\mathrm{Im}\, f_1}{x + \mathrm{Re}\, f_2 + j\,\mathrm{Im}\, f_2}\right) \quad \text{.......} \quad (4.1.16)$$

Special cases arise when either the real or imaginary parts of $f_1$ and $f_2$ are zero. These can be handled easily by asymptotic approximations. To illustrate such a case, let us consider the voltage gain of the network of Figure (4.1.10).

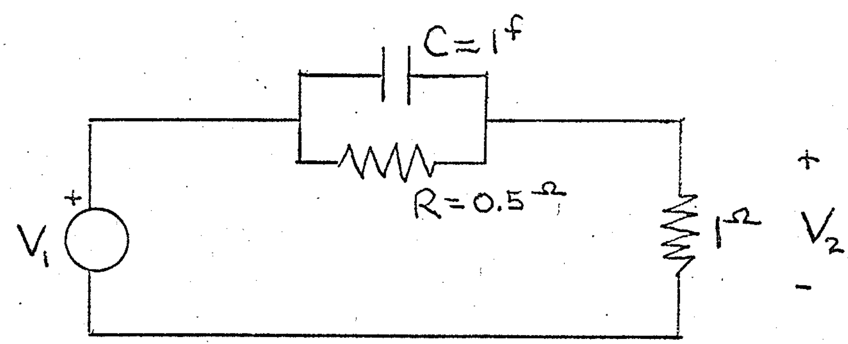


Fig. 4.1.10

i.e. $\bar{G}_{12}(j\omega) = \frac{V_2}{V_1}(j\omega) = \frac{C + 2/j\omega}{C + 3/j\omega}$ ....... (4.1.17)

We plot for a particular

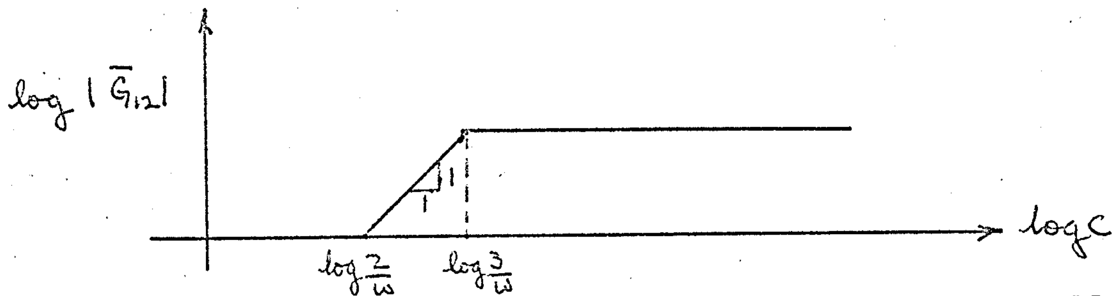


Fig. 4.1.11

Since sensitivity is the slope on a log-log plot we obtain the sensitivity of $|\bar{G}|$ as function of log C as shown in Figure (4.1.12)

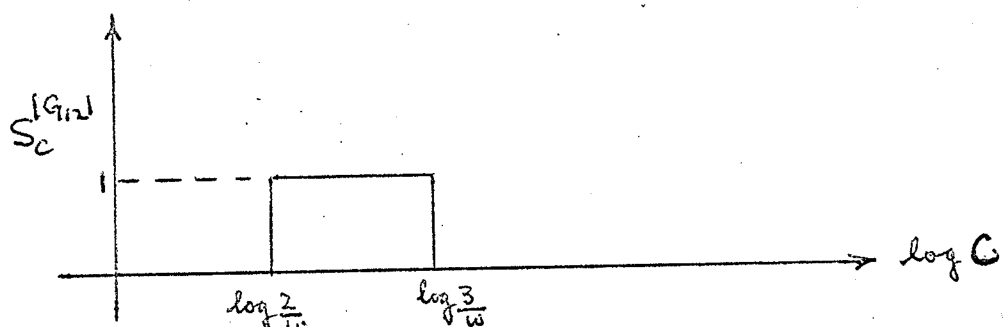


Fig. 4.1.12

If we choose C to equal 1 farad we obtain the sensitivity as a function of frequency (Figure (4.1.13)).

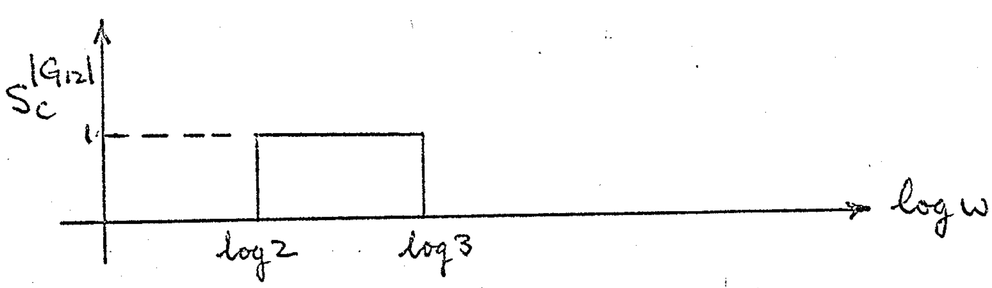


Fig. 4.1.13

In general, sensitivity calculations are not this simple. It may be necessary to use a digital computer to obtain the sensitivity of the magnitude of a network function. The following approach may be useful for such an analysis:

We have in general

$$H = f_0 \cdot \left[ \frac{x + f_1}{x + f_2} \right] = \frac{x + \mathrm{Re}\, f_1 + j\, \mathrm{Im}\, f_1}{x + \mathrm{Re}\, f_2 + j\, \mathrm{Im}\, f_2} \cdot f_0 \quad \cdots\cdots\cdots \quad (4.2.18)$$

now,

$$H = |f_0| \frac{\sqrt{(x + \mathrm{Re}\, f_1)^2 + (\mathrm{Im}\, f_1)^2}}{\sqrt{(x + \mathrm{Re}\, f_2)^2 + (\mathrm{Im}\, f_2)^2}} \quad \cdots\cdots\cdots \quad (4.1.19)$$

and $\ln|H| = \ln|f_0| + \frac{1}{2} \ln \left[ (x + \mathrm{Re}\, f_1)^2 + (\mathrm{Im}\, f_1)^2 \right]$

$$- \frac{1}{2} \ln \left[ (x + \mathrm{Re}\, f_1)^2 + (\mathrm{Im}\, f_2)^2 \right] \quad \cdots\cdots\cdots \quad (4.1.20)$$

$$\ln|H| = \ln|f_0| + \frac{1}{2} \ln \left[ x^2 + 2x\, \mathrm{Re}\, f_1 + |f_1|^2 \right]$$

$$- \frac{1}{2} \ln \left[ x^2 + 2x\, \mathrm{Re}\, f_2 + |f_2|^2 \right]. \quad \cdots\cdots\cdots \quad (4.1.21)$$

So that

$$S_x^{|H|} = \frac{\partial \ln|H|}{\partial \ln x} = \frac{x\, \partial \ln|H|}{\partial x} \quad \cdots\cdots\cdots \quad (4.1.22)$$

$$= \frac{x^2 + 2\mathrm{Re}\, f_1 x}{x^2 + 2\mathrm{Re}\, f_1 + |f_1|^2} - \frac{x^2 + 2\mathrm{Re}\, f_2\, x}{x^2 + 2\mathrm{Re}\, f_2 + |f_2|^2} \quad \cdots\cdots\cdots \quad (4.1.23)$$

$$= \frac{1}{1 + \dfrac{|f_1|^2}{x^2 + 2\mathrm{Re}\, f_1 x}} - \frac{1}{1 + \dfrac{|f_2|^2}{x^2 + 2\mathrm{Re}\, f_2 x}} \quad \cdots\cdots\cdots \quad (4.1.24)$$

We have thus obtained the sensitivity directly as a function of $\omega$ without plotting it first with respect to a variable $x$ .

Let us now examine the example considered before.

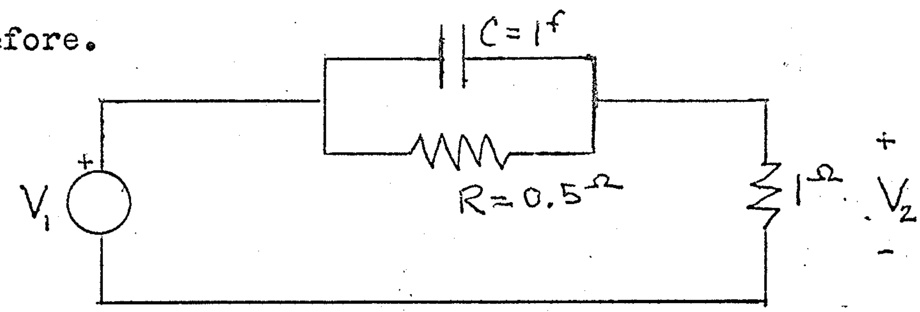


Fig. 4.1.14

The voltage of network shown in Figure (4.1.14) is given by

$$\bar{G}_{12}(j\omega) = \frac{V_2}{V_1}(j\omega) = \frac{C + 2/j\omega}{C + 3/j\omega} \qquad \ldots\ldots\ldots (4.1.25)$$

Here we have

$$|f_1| = 2/\omega \quad , \quad |f_2| = 3/\omega$$

$$x = C = 1 \text{ farad}$$

and $\quad \mathrm{Re}\, f_1 = \mathrm{Re}\, f_2 = 0$

Utilizing equation (4.1.24) we can write the sensitivity of $|H|$ as a function of $C$ and $\omega$ as

$$S_C^{|H|} = \frac{1}{1 + 4/\omega^2} - \frac{1}{1 + 9/\omega^2} \qquad \ldots\ldots\ldots (4.1.26)$$

$$S_c^{|H|} = \frac{\omega^2}{4+\omega^2} - \frac{\omega^2}{9+\omega^2} \qquad \ldots\ldots\ldots (4.1.27)$$

A plot of the sensitivity of each term in equation (4.1.26) is shown in Figure (4.1.15)

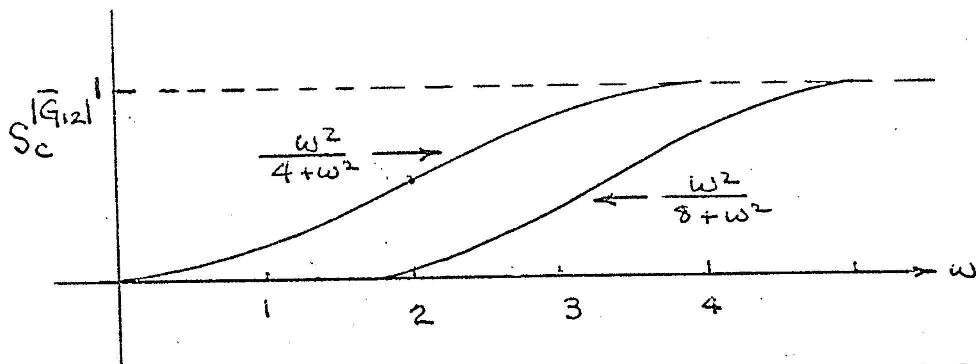


Fig. 4.1.15

Subtracting these two curves we obtain $S_c^{|\bar{G}_{12}|}$ as in Figure (4.1.16)

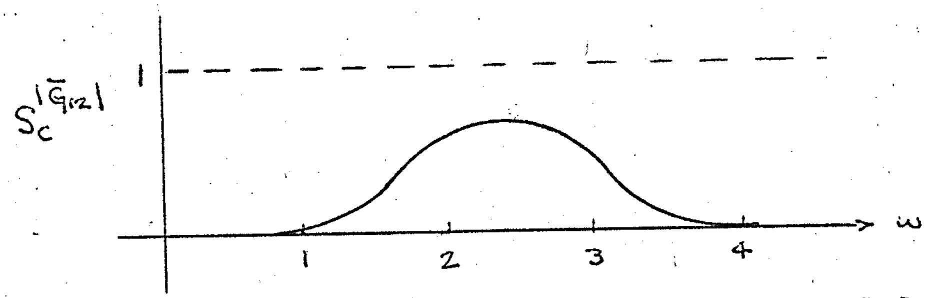


Fig. 4.1.16

We have shown previously that

$$S_c^{|H|} = \frac{\omega^2}{4+\omega^2} - \frac{\omega^2}{9+\omega^2} \qquad \ldots\ldots\ldots (4.1.28)$$

To obtain the phase sensitivity we solve the integral

$$\overset{\circ}{S}{}_c^{\phi}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{S_c^{H}(y)}{y-\omega}\, dy \qquad \text{.......} \ (4.1.29)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y^2}{(4+y^2)(y-\omega)}\, dy - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y^2}{(9+y^2)(y-\omega)}\, dy \qquad \text{.......} \ (4.1.30)$$

Performing a partial fraction expansion we obtain

$$\overset{\circ}{S}{}_c^{\phi} = \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\frac{4y+4\omega}{4+\omega^2}}{y^2+4}\, dy + \int_{-\infty}^{\infty} \frac{\frac{\omega^2}{4+\omega^2}}{y-\omega}\, dy\right]$$

$$- \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \frac{\frac{9y+9\omega}{9+\omega^2}}{y^2+9}\, dy + \int_{-\infty}^{\infty} \frac{\frac{\omega^2}{\omega^2+9}}{y-\omega}\, dy\right] \qquad \text{.......} \ (4.1.31)$$

$$= \frac{1}{\pi(4+\omega^2)}\left[\int_{-\infty}^{\infty} \frac{4y}{y^2+4}\, dy + \int_{-\infty}^{\infty} \frac{4\omega}{y^2+4} + \int_{-\infty}^{\infty} \frac{\omega^2}{y-\omega}\, dy\right]$$

$$- \frac{1}{\pi(9+\omega^2)}\left[\int_{-\infty}^{\infty} \frac{9y}{y^2+9}\, dy + \int_{-\infty}^{\infty} \frac{9\omega}{y^2+9}\, dy + \int_{-\infty}^{\infty} \frac{\omega^2}{y-\omega}\, dy\right] \qquad \text{.......} \ (4.1.32)$$

We now proceed to take the Cauchy principal part in the following manner:

$$= \frac{1}{\pi(4+\omega^2)} \left[ \lim_{\epsilon\to\infty} 2\ln(y^2+4)\Big|_{-\epsilon}^{\epsilon} + \lim_{\epsilon\to\infty} 2\omega\tan^{-1}\frac{y}{2}\Big|_{-\epsilon}^{\epsilon} + \omega^2 \lim_{\epsilon\to\infty}\left(-\frac{1}{\omega}\ln(y-\omega)\right)\Big|_{-\epsilon}^{\epsilon} \right]$$

$$- \frac{1}{\pi(9+\omega^2)} \left[ \lim_{\epsilon\to\infty} \frac{9}{2}\ln y^2+9 \Big|_{-\epsilon}^{\epsilon} + \lim_{\epsilon\to\infty} \frac{9}{3}\omega\tan^{-1}\frac{y}{2}\Big|_{-\epsilon}^{\epsilon} + \omega^2 \lim_{\epsilon\to\infty}\left(-\frac{1}{\omega}\ln y-\omega\right)\Big|_{-\epsilon}^{\epsilon} \right] \quad \text{.......} \quad (4.1.33)$$

$$= \frac{1}{\pi(4+\omega^2)} 2\omega\left(\frac{\pi}{2}+\frac{\pi}{2}\right) - \frac{1}{\pi(9+\omega^2)} 3\omega\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \quad \text{.......} \quad (4.1.34)$$

$$= \frac{2\omega}{4+\omega^2} - \frac{3\omega}{9+\omega^2} \quad \text{.......} \quad (4.1.35)$$

Plotting the terms of equation (4.1.35) we obtain Figure (4.1.17)

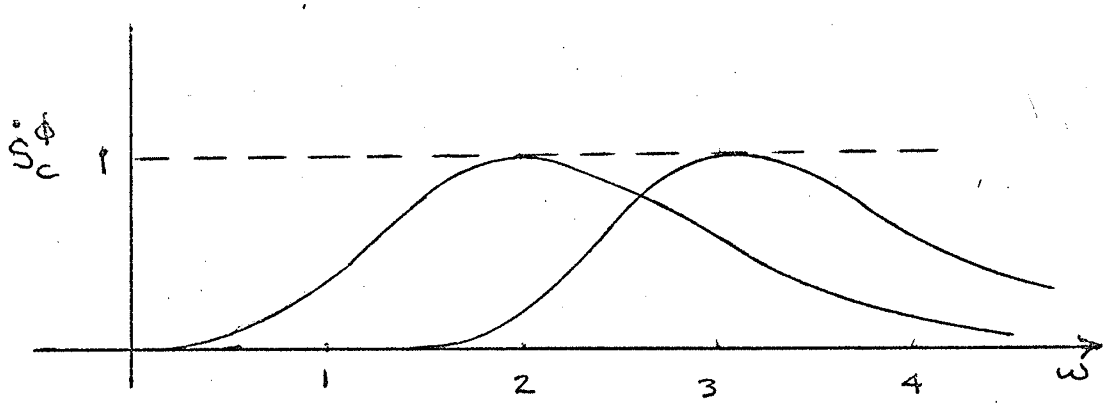


Fig. 4.1.17

Subtracting these curves, we find the resultant phase sensitivity yields the following result as given in Figure (4.1.18)

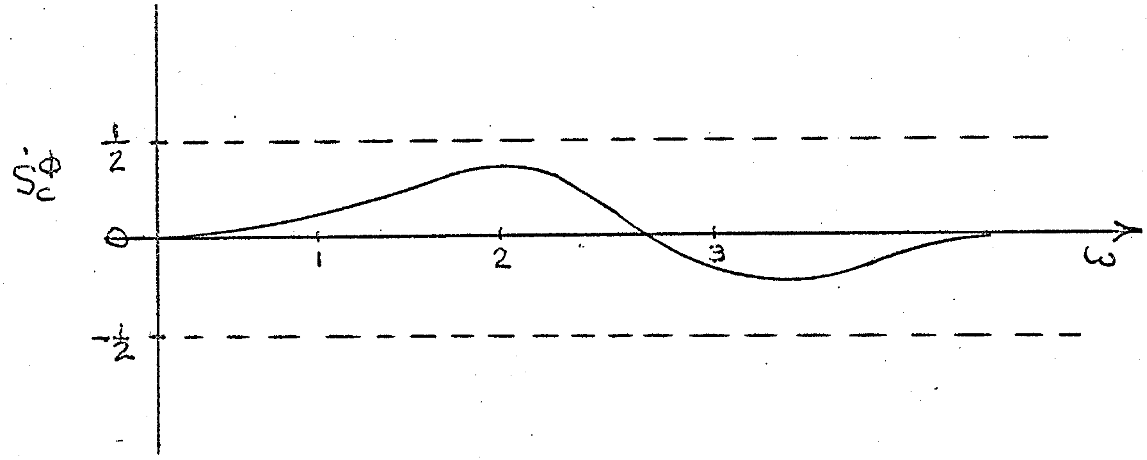


Fig. 4.1.18

Thus we have obtained $\hat{S}_c^{\phi} = \frac{\partial \phi}{\partial \ln c}(\omega)$ analytically. For a more complicated circuit, digital means may be necessary.

## Chapter V

## MULTIPARAMETER SENSITIVITY

Sensitivity allows us to make approximate calculations to predict the behaviour of networks due to element changes.

i.e. $$S_x^F = \frac{\partial F/F}{\partial x/x} \qquad \text{....... (5.1)}$$

so that $$\frac{\Delta F}{F} \simeq \left(S_x^F\right)\left(\frac{\Delta x}{x}\right) \qquad \text{....... (5.2)}$$

If $\Delta x$ and $\Delta F$ are sufficiently small, good results are obtained.

We notice that $\frac{\Delta x}{x}$ can be interpreted as the percent change in $x$ and that $\frac{\Delta F}{F}$ as the percent change in the function.

For variations of more than one parameter,

$$\frac{\Delta F}{F} = S_{x_1}^F \frac{\Delta x_1}{x_1} + S_{x_2}^F \frac{\Delta x_2}{x_2} + \cdots \qquad \text{....... (5.3)}$$

Synthesis techniques to minimize multiparameter sensitivity are discussed by S. C. Lee[9]. His objective is quite different from this thesis since the author tries to develop new synthesis techniques by comparison of sensitivity only.

## Chapter VI

## OBSERVATIONS AND CONCLUSIONS

All of the techniques discussed have their limits in predicting the overall response due to parameter changes. Each method does have its own merits and gives different information. Thus the designer may choose the method or methods which give the most information as well as those which facilitate calculation.

The first method discussed (that of Papoulis) gives the initial direction of motion of poles and zeros due to parameter changes. The second method in Chapter I gives a bound on the frequency response and may possibly give better answers to some problems than in the example cited. This example was chosen to show the shortcomings of this method.

The methods of Chapter III are probably the best for finding sensitivity and lend well to the use of graphical techniques which give limited accuracy. The root locus in general may be difficult to plot precisely. A "Spirule" or other root locus plotting devices would be a valuable asset in obtaining sensitivities of various network functions.

Chapter IV discusses a direct procedure for calculating the approximate frequency response change due to parameter changes. It has two advantages: firstly, it offers a visual aid for discussing sensitivity as a three dimensional plot, and secondly the two special cases of resistive and reactive networks facilitate calculation of the new frequency response.

Appendix

THE CONFORMALITY OF THE FUNCTION $\frac{p(s)}{q(s)} = -x$.

We have the locus of all points in the plane for which $\frac{p(s)}{q(s)} = -x$ for all values of $x$. We must show that at some point $s = p_i$ for $x_0$ in the range $0 < x_0 < \infty$ that the mapping from the $s$ plane into the $x$ plane is conformal in some small neighbourhood of $p_i$, that is $p_i + \Delta p_i$.

Proof:

From complex variables, it is well known that if $p/q$ is an analytic function and $\frac{d}{ds}\left(\frac{p}{q}\right) \neq 0$ in some region $R$, then the mapping is conformal.

Since $p$ and $q$ are polynomials in $s$, $\frac{d}{ds}\left(\frac{p}{q}\right)$ exists at all points except $q = 0$ (pole on the root locus).

We also require that $\frac{d}{ds}\left(\frac{p}{q}\right) \neq 0$ in some region $R$.

i.e. $$\frac{d}{ds}\left(\frac{p}{q}\right) = \frac{qp' - pq'}{q^2} \neq 0$$

This means that $q$ must remain finite (cannot be at a pole on the root locus) and that $qp' \neq pq'$

or $$\frac{p}{q} \neq \frac{p'}{q'}$$

The only way this can occur is if $p = 0$ or if $p$ and $q$ are constants. However this is the trivial case. If $p = 0$ this implies that we cannot be at a zero on the root locus.

Therefore the mapping is conformal anywhere along the root locus except at the end points.

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