AN INVESTIGATION OF THE OCCURRENCE OF DOUBLE COMPLEX POLES IN THE DRIVING-POINT IMMITTANCE FUNCTION OF A LINEAR PASSIVE NETWORK

A Thesis

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by

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ABSTRACT

Analysis of the double tuned circuit is found to yield double complex poles of the transfer functions of this network but only for the trivial case. However, it is subsequently shown by various techniques that this pole configuration in the driving-point functions of some networks is actually possible.

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to investigate the possibility of synthesizing a driving-point immittance function containing double complex poles in the s plane by a linear, lumped, passive, bilateral, time-invariant network. It is known that active networks (ideally) can possess this property, e.g. identical amplifiers which are cascaded, but the investigation of this property as applied to passive networks has not appeared in the literature.

It is generally accepted that certain analogues exist between electronics and physics. A vibrating particle can be considered to be a harmonic oscillator provided it is sufficiently isolated. Two identical atoms are found to vibrate at the same frequency if they are isolated but are seen to depart from this mode as the distance between them decreases. This fact explains the phenomena of spectral lines in gases (where the atoms are considered to be loosely coupled) and the energy bands in solids (where the atoms are more closely coupled).

Analogous to the problem of the harmonic oscillator is the problem of the so-called double tuned circuit in electronics. Each section of the circuit may be considered by itself to be an oscillator. Considering both halves to be independent, we may adjust the resonant frequencies of these such that they are equal. It is found that when the two halves are coupled by mutual inductance, the equality of the two frequencies is destroyed as in the case of the oscillating particles. We find that double complex poles exist only in a trivial sense.

In addition, we note that for a purely reactive network, double complex poles are impossible due to realizability restrictions. However, the possiblity of a lossy network which possesses this property still exists. In any event, this is unlikely from the point of view that most processes in nature vary in a continous manner with respect to changes in parameters. There is also the question that if double complex poles do occur in passive networks, how lossy must the network be before they become likely to occur?

These are the reasons for the investigation of electrical networks to determine in general whether or not double complex poles are possible. If they are possible, then perhaps too much importance is placed on analogues between these two fields of networks and particles.

CHAPTER II

THE DOUBLE TUNED CIRCUIT

Analysis. The double tuned circuit [1] of Figure 1 is to be excited by the ideal current source i(t).

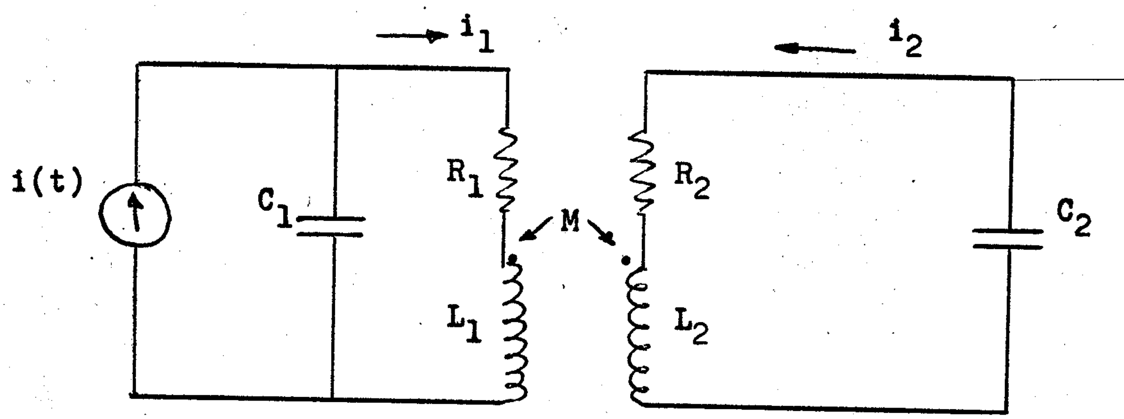

Figure 1. Circuit diagram of the double tuned network.

By representing this current source in parallel with C_1 as a voltage source, the circuit diagram of Figure 2 is obtained.

To simplify the analysis of this circuit, it will be assumed that the resonant frequency of both primary and secondary circuits is the same. We then define

$$\omega_0^2 \equiv \frac{1}{L_1C_1} = \frac{1}{L_2C_2} \quad . \tag{1}$$

[1] E. Brenner, and M. Javid, Analysis of Electric Circuits, New York: McGraw-Hill Book Co., Inc., 1959, pp. 611-617.

The foregoing assumption is valid for many circuits in practice.

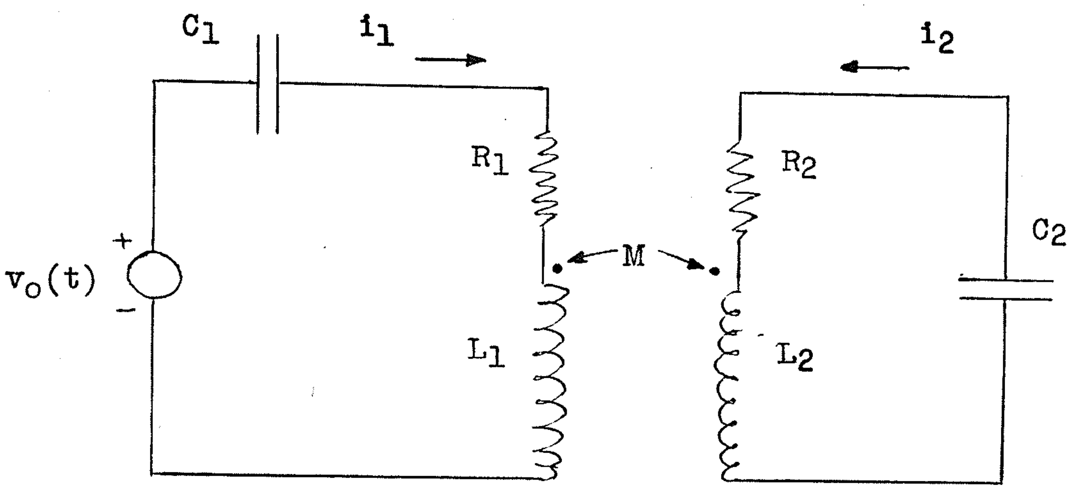

Figure 2. Equivalent circuit for the double tuned network.

Writing the mesh equations of the system in matrix form we have:

$$\begin{bmatrix} v_o(t) \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11}(p) & Z_{12}(p) \\ Z_{21}(p) & Z_{22}(p) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{2}$$

where
$$Z_{11}(p) = \frac{1}{pC_1} + R_1 + pL_1 \tag{3}$$
$$Z_{12}(p) = Z_{21}(p) = pM = pk\sqrt{L_1L_2} \tag{4}$$
$$Z_{22}(p) = \frac{1}{pC_2} + R_2 + pL_2 \tag{5}$$
and
$$v_o(t) = \frac{1}{pC_1} i(t) \tag{6}$$

Rearranging equations (3) and (5) and using equation (1), we obtain:

$$Z_{11}(p) = \frac{L_1}{p}\left[p^2 + \frac{R_1}{L_1}p + \omega_o^2\right] \tag{7}$$

and $$Z_{22}(p) = \frac{L_2}{p}\left[p^2 + \frac{R_2}{L_2}p + \omega_o^2\right] \qquad (8)$$

We now let $$Q_1 \equiv \frac{\omega_o L_1}{R_1} \qquad (9)$$

and $$Q_2 \equiv \frac{\omega_o L_2}{R_2} \qquad (10)$$

Solving equations (9) and (10) for R_1 and R_2 and substituting the results into equations (7) and (8) we have

$$Z_{11}(p) = \frac{L_1}{p}\left[p^2 + \frac{\omega_o p}{Q_1} + \omega_o^2\right] \qquad (11)$$

$$Z_{22}(p) = \frac{L_2}{p}\left[p^2 + \frac{\omega_o p}{Q_2} + \omega_o^2\right] \qquad (12)$$

Solving the mesh equations (2) for i_2, we obtain

$$i_2 = \frac{-Z_{21}v_o(t)}{Z_{11}Z_{22} - Z_{21}Z_{12}} \qquad (13)$$

Substituting equations (4), (6), (11), and (12) into equation (13) we have on rearranging:

$$i_2 = -k\omega_o^2\sqrt{\frac{L_1}{L_2}}\,\frac{p^2 i(t)}{\left(p^2 + \frac{\omega_o p}{Q_1} + \omega_o^2\right)\left(p^2 + \frac{\omega_o p}{Q_2} + \omega_o^2\right) - p^4k^2} \qquad (14)$$

The poles of the transfer functions are then the roots of the equation:

$$\left(s^2 + \frac{\omega_o s}{Q_1} + \omega_o^2\right)\left(s^2 + \frac{\omega_o s}{Q_2} + \omega_o^2\right) - s^4k^2 = 0 \qquad (15)$$

where p has been replaced by the complex variable s.

Equation (15) is not explicitly solvable but if we set $Q_1 = Q_2 = Q_o$, then we can find an analytic expression for the roots. This equation thus becomes on factoring:

$$\left[\left(s^2 + \frac{\omega_o s}{Q_o} + \omega_o^2\right) + ks^2\right]\left[\left(s^2 + \frac{\omega_o s}{Q_o} + \omega_o^2\right) - ks^2\right] = 0$$

If we denote the pole positions by s_p, then we have

$$s_p = \omega_0 \left[\frac{-1}{2Q_0(1 \pm k)} \pm \sqrt{\frac{1}{4Q_0^2(1 \pm k)^2} - \frac{1}{1 \pm k}} \right] \qquad (16)$$

For the poles to be complex, the discriminant of equation (16) must be negative and we may write:

$$s_p = \frac{-\omega_0}{2Q_0(1 \pm k)} \pm \frac{j\omega_0}{\sqrt{1 \pm k}} \sqrt{1 - \frac{1}{4Q_0^2(1 \pm k)}} \qquad (17)$$

Conclusion. For the poles to be double complex, we see that $k = 0$ is required. This means that for double complex poles of the transfer function to exist, the coefficient of coupling must be zero. But from equation (14), this condition renders $i_2 = 0$ regardless of $i(t)$. This particular system therefore consists of two parts, primary and secondary, which are completely isolated and hence is of no practical value.

This result prompted further investigation of driving-point functions in general with the condition that they contain double complex poles. The decision to consider driving-point functions rather than transfer functions was made in order to facilitate a simpler approach to the problem.

CHAPTER III

DRIVING-POINT IMMITTANCE FUNCTIONS WITH DOUBLE COMPLEX POLES

I. THE POSITIVE REAL PROPERTY

The requirements for physical realizability were first given in complete form by Otto Brune[2] in 1931. A summary of these is presented where $H_d(s)$ is the driving-point immittance function to which the quotation refers.

> " 1. The transform driving-point immittance of a physically realizable lumped parameter network must be a ratio of polynomials in s with real positive coefficients.
>
> 2. The degree of the numerator polynomial cannot differ from the degree of the denominator by more than 1.
>
> 3. The driving-point immittance function cannot have either zeros or poles in the right-half s plane; in addition, zeros and poles that occur for imaginary values of s (at $s = j\omega$) must be simple.
>
> 4. A driving-point immittance function must be a positive real function; that is, if $\mathrm{Re}[s] \geq 0$, $\mathrm{Re}\left[H_d(s)\right] \geq 0$.[3] "

Throughout this paper we shall refer to the four preceding statements as conditions one to four respectively.

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[2] O. Brune, "Synthesis of a Finite Two Terminal Network whose Driving-point Impedance is a Prescribed Function of Frequency," J. Math. and Phys., 10:191-236, 1931.

[3] M. Javid, and E. Brenner, Analysis, Transmission, and Filtering of Signals, New York: McGraw-Hill Book Co., Inc., 1963, pp. 378-379.

It can be shown that condition four is equivalent to $\mathrm{Re}\left[H_d(j\omega)\right] \geq 0$, for $0 \leq \omega \leq \infty$,[4] provided conditions one to three are satisfied.

II. SELECTION OF THE DESIRED DRIVING-POINT FUNCTION

Poles of the driving-point function.

Let us define the function $H_d(s)$ as follows:

$$H_d(s) = \frac{N(s)}{D(s)} ,$$

where the poles of $H_d(s)$ are double complex. To satisfy condition three, such poles must be chosen in the open left-half plane. Let these poles be located at $s = -a \pm jb$, where "a" and "b" are real and positive. The denominator D(s) then becomes:

$$D(s) = \left[(s+a+jb)(s+a-jb)\right]^2 \tag{18}$$

Expanding equation (18) and collecting terms we obtain:

$$D(s) = s^4+4as^3+(6a^2+2b^2)s^2+(4a^3+4ab^2)s+(a^2+b^2)^2 \tag{19}$$

Zeros of the driving-point function.

It now remains to choose the numerator, N(s), which according to condition two, can be third, fourth, or fifth degree in s.

Before choosing this numerator, we shall prove the following statement:

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[4] M. E. Van Valkenburg, Introduction to Modern Network Synthesis, New York: John Wiley & Sons, Inc., 1962, pp. 82-85.

"Let $Z(s)$ be a rational function with real coefficients and let it be defined as $Z(s) \equiv \frac{A(s)}{B(s)}$ where $A(s)$ is one degree higher than $B(s)$. If $Z(s)$ is expanded in the form $Z(s) = K_x + K_y s + K_z \frac{C(s)}{B(s)}$ where K_x, K_y, and K_z are real constants and where $C(s)$ is one degree lower than $B(s)$ and if each term in the expansion is positive real, then $Z(s)$ is positive real." Proof of this theorem is given below:

Let us examine the equation

$$Z(s) = K_x + K_y s + K_z \frac{C(s)}{B(s)} \quad . \qquad (20)$$

Since each term on the right hand side of equation (20) is positive real, it follows that $K_x > 0$. If we let $s = j\omega$ in (20), then the term $K_y j\omega$ has no real part and thus K_y is arbitrarily positive. In addition, we have:

$$K_z \mathrm{Re}\left[C(j\omega) \overset{*}{B}(j\omega) \right] \geq 0 \text{ for } 0 \leq \omega \leq \infty,$$ upon rationalization of the last term of equation (20). Calculating the real part of $Z(j\omega)$ from equation (20), we have:

$$Z(s) = \frac{K_x B(s) + K_y s B(s) + K_z C(s)}{B(s)}$$

$$Z(j\omega) = \frac{\left[K_x B(j\omega) + K_y j\omega B(j\omega) + K_z C(j\omega) \right] B^*(j\omega)}{B(j\omega) B^*(j\omega)}$$

$$Z(j\omega) = \frac{K_x |B(j\omega)|^2 + K_y j\omega |B(j\omega)|^2 + K_z C(j\omega) B^*(j\omega)}{|B(j\omega)|^2}$$

Hence $$\mathrm{Re}\left[Z(j\omega) \right] = \frac{K_x |B(j\omega)|^2 + K_z \mathrm{Re}\left[C(j\omega) B^*(j\omega) \right]}{|B(j\omega)|^2} \qquad (21)$$

Examining equation (21), we note that each term in the numerator is positive for $0 \leq \omega \leq \infty$. Therefore Z(s) is positive real and the proof is complete.

This theorem indicates that if a third degree N(s) exists such that Hd(s) is positive real, then there also exists a fourth and a fifth degree N(s) for which Hd(s) is positive real.

Let N(s) be defined by the following relation:

$$N(s) \equiv a_3 s^3 + a_2 s^2 + a_1 s + a_0 \tag{22}$$

By condition one, we require that $a_n > 0$, for n = 0, 1, 2, 3. Our function to be tested for the positive real property is then:

$$H_d(s) = \frac{a_3 s^3 + a_2 s^2 + a_1 s + a_0}{s^4 + 4as^3 + (6a^2 + 2b^2)s^2 + (4a^3 + 4ab^2)s + (a^2 + b^2)^2}$$

If this function is found to be positive real for certain numerator coefficients, then it can be realized by any of the well-known methods such as the method of Bott and Duffin.[5]

III. NETWORK SCALING

Suppose we define a new function by the relation:

$$\bar{H}_d(s) = mH_d\left(\frac{s}{n}\right)$$

[5] R. Bott, and R. J. Duffin, "Impedance Synthesis without the Use of Transformers", J. Appl. Phys., 20: 816, 1949.

Then the original $H_d(s)$ will have been frequency scaled by a factor n and magnitude scaled by a factor m. Frequency scaling (for a physically realizable $H_d(s)$) consists of dividing each capacitance and inductance by n and leaving the resistances unchanged. Pole and zero positions will be shifted by a factor n. Thus a pole or zero at $s_1 = -a_1 \pm jb_1$ will become a pole or zero at $s_2 = -na_1 \pm jnb_1$.

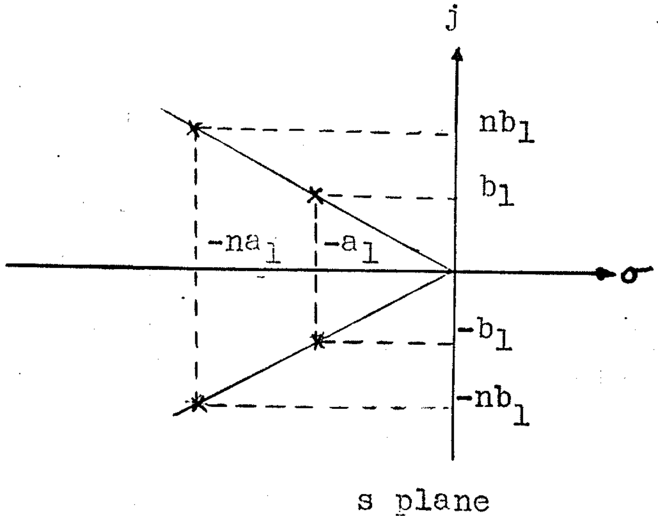

Figure 3. The effect of frequency scaling on the poles of a network function.

Therefore any investigation of the network function with a pole or zero at $s_1 = -a_1 \pm jb_1$ will apply to the frequency scaled network with a pole or zero at $s_2 = -na_1 \pm jnb_1$ as well since frequency scaling of a driving-point function results in another driving-point function which of course is positive real.

Magnitude scaling consists of multiplying each resistance and inductance by m and dividing each capacitance by m. Pole and zero locations are not affected.

The foregoing discussion applies to driving-point impedances but the same type of argument may be applied to driving-point admittances as well.

IV. NUMERATOR REDUCTION

Let us now magnitude scale equation (22) by the factor $\frac{1}{a_0}$. We then have:

$$\frac{N(s)}{a_0} = \frac{a_3 s^3}{a_0} + \frac{a_2 s^2}{a_0} + \frac{a_1 s}{a_0} + 1 \qquad (23)$$

We define: $b_3 \equiv \frac{a_3}{a_0}$, $b_2 \equiv \frac{a_2}{a_0}$, $b_1 \equiv \frac{a_1}{a_0}$. Equation (23) then becomes:

$$\frac{N(s)}{a_0} = b_3 s^3 + b_2 s^2 + b_1 s + 1 \qquad (24)$$

Replacing the s in equation (24) by $\frac{s}{\sqrt[3]{b_3}}$ we have:

$$\frac{N\left(\frac{s}{\sqrt[3]{b_3}}\right)}{a_0} = s^3 + Xs^2 + Ys + 1 \qquad (25)$$

where $X \equiv \frac{b_2}{\sqrt[3]{b_3^2}}$ and $Y \equiv \frac{b_1}{\sqrt[3]{b_3}}$. Equation (25) is then said to be in Vyshnegradskii[6] form.

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[6] M. V. Meerov, Stability of Automatic Regulating Systems, London: Butterworth & Co. Ltd., 1961 , pp. 127-133.

We have now magnitude scaled N(s) by a factor $\frac{1}{a_0}$ and frequency scaled it by a factor $\sqrt[3]{b_3}$. Equation (25) thus indicates that all values of the coefficients X and Y will exhaust all possible third degree polynomials. Without loss in generality, we set $a_3=1$, $a_2=X$, $a_1=Y$, and $a_0=1$ in equation (22) to yield

$$N(s)=s^3+Xs^2+Ys+1 \tag{26}$$

Numerator restriction.

If we consider X and Y positive (according to condition one) and apply the Routh-Hurwitz criterion to equation (26), the following Hurwitz determinant[7] is obtained:

$$\begin{vmatrix} X & 1 & 0 \\ 1 & Y & 0 \\ 0 & X & 1 \end{vmatrix} > 0$$

from which $XY-1>0$. The equation of the boundary of stability is then

$$XY=1 \tag{27}$$

This is the equation of a hyperbola in the X-Y plane and is called the Vyshnegradskii hyperbola. A plot of this curve is given in Figure 4. Referring to this plot, we see that only values of X and Y in the stability region where the numerator is Hurwitz need be considered.

[7] A. Hurwitz, "Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reelen Theilen besitzt," Math. Ann., 46:273-284, 1895.

Suppose now that X and Y have been determined such that $H_d(s)$ is positive real and we wish to determine the zeros of this function in the s plane.

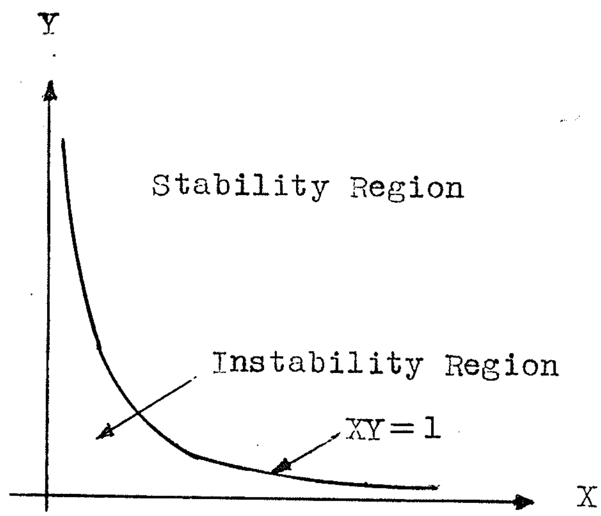

Figure 4. Vyshnegradskii curve.

The numerator can be factored into the following form:

$$N(s) = (s + s_1)(s + s_2)(s + s_3)$$

where s_1, s_2, and s_3 all have negative real parts. To find these it is necessary to solve the equation

$$s^3 + Xs^2 + Ys + 1 = 0$$

using specific values of X and Y. This procedure locates the zeros of $H_d(s)$ in the s plane.

V. TESTING FOR THE POSITIVE REAL PROPERTY

We consider $H_d(s)$ of the form:

$$H_d(s)=\frac{s^3+Xs^2+Ys+1}{s^4+4as^3+(6a^2+2b^2)s^2+(4a^3+4ab^2)s+(a^2+b^2)^2} \tag{28}$$

If we replace s by $j\omega$, rationalize, and take the real part of the resulting numerator of the right-hand side, we obtain:

$$\begin{aligned}\mathrm{Re}[\mathrm{Num.}\ H_d(j\omega)] = & -(X-4a)\omega^6+[X(6a^2+2b^2)+1-4a^3-4ab^2-4aY]\omega^4 \\ & -[X(a^2+b^2)^2+6a^2+2b^2-Y(4a^3-4ab^2)]\omega^2 \\ & +(a^2+b^2)^2\end{aligned} \tag{29}$$

Since the rationalization procedure renders the denominator real and positive for all frequencies, it need not be considered. For $H_d(s)$ to be positive real, we require

$$\mathrm{Re}[\mathrm{Num.}\ H_d(j\omega)] \geq 0 \text{ for } 0 \leq \omega < \infty. \tag{30}$$

Equation (30) will be satisfied provided that the equation

$$\mathrm{Re}[\mathrm{Num.}\ H_d(j\omega)] = 0 \tag{31}$$

has no positive roots in ω^2 of odd multiplicity. Because this equation is a cubic equation in ω^2, an explicit solution exists. If we denote the roots of (31) by r_1, r_2, and r_3, then at least one of these is real. Let this root be r_1. Observing equation (29), we see that $\mathrm{Re}[\mathrm{Num.}\ H_d(j\omega)]$ will remain positive for large ω only if

$$X \leq 4a \tag{32}$$

We also require that if r_2 and r_3 are real

$$r_2(X, Y, a, b) \leq 0 \tag{33}$$

and $$r_3(X, Y, a, b) \leq 0. \qquad (34)$$

Equations (33) and (34) then define loci in the X-Y plane, and along with equation (32), define the region in which $H_d(s)$ is positive real. Since very complicated expressions are involved, these loci were not computed. Instead, these boundaries were determined by trial and error, using Sturm's theorem.

The additional restriction given by equation (32) is shown superimposed on the Vyshnegradskii hyperbola in Figure 5. The shaded areas indicate regions of the X-Y plane where our driving-point function cannot be positive real and hence all points (X,Y) in these regions need not be tested.

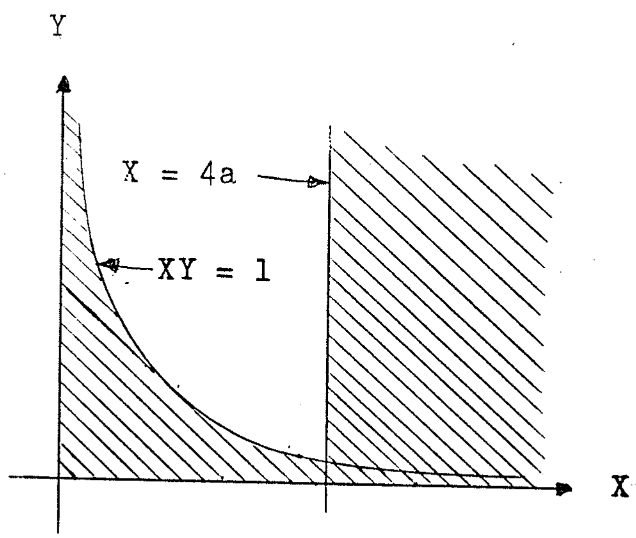

Figure 5. Regions where $H_d(s)$ cannot be positive real.

Sturm's theorem.

A computer program written in IBM Fortran was used to test specific $H_d(s)$ for the positive real property (see Appendix A). Input data consisted of the coefficients of each numerator and denominator polynomial and their respective degree in s. The output then indicated whether or not the function was positive real. The program is based on Sturm's theorem[8] which is a variation of Euclid's algorithm. Proof of this theorem is available in the literature[9].

Let us denote $\mathrm{Re}[\mathrm{Num.}\ H_d(j\omega)]$ by $P_0(x)$ where the substitution $x=\omega^2$ has been made. We then have the relation

$$P_0(x) = a_0x^r + a_1x^{r-1} + \ldots + a_{r-1}x + a_r \tag{35}$$

where r is an integer. $P_0(x)$ is defined as the first Sturm function. The second Sturm function $P_1(x)$ is found by differentiation of equation (35) with respect to x:

$$P_1(x) = ra_0x^{r-1} + \ldots + 2a_{r-2}x + a_{r-1}$$

The remaining fnctions are found by a variation of Euclid's algorithm.

e.g.
$$\frac{P_0(x)}{P_1(x)} = \beta_1 x + \beta_2 + \frac{-P_2(x)}{P_1(x)} ,$$

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[8] L. Weinberg, Network Analysis and Synthesis, New York: McGraw-Hill Book Co., Inc., 1962, pp. 240-243.

[9] L.E. Dickson, New First Course in the Theory of Equations, New York: John Wiley & Sons, Inc., 1939.

$$\frac{P_1(x)}{P_2(x)} = \beta_3 x + \beta_4 + \frac{-P_3(x)}{P_2(x)}\ , \text{ etc.}$$

where β_i, i = 1, 2, 3, . . . are all constants. This process is continued until $P_r(x)$ of degree zero is found or until the remainder is identically zero.

Let us assume that the former termination exists. Consider now a value of $x=x_1$. Each Sturm function evaluated at this x_1 is either positive, negative, or zero. If we denote V_{x_1} as the number of variations of sign of the Sturm functions at $x=x_1$, taken in the order of the subscripts (i.e. $P_0(x_1)$, $P_1(x_1)$, . . ., $P_r(x_1)$), and V_{x_2} as the number of variations of sign of the Sturm functions at $x=x_2$, also taken in the order of the subscripts, then Sturm's theorem states that the number of zeros of $P_0(x)$ is given by $|V_{x_1}-V_{x_2}|$. In our case we wish to use $x_1=0$ and $x_2=\infty$. This will give the total number of zeros of $P_0(x)$ between zero and infinity. If there are no zeros of $P_0(x)$ between these limits, then $H_d(s)$ is positive real.

Suppose that the generation of Sturm functions terminates prematurely and suppose, for example, that the last non-zero function is of degree three, that is

$$P_g(x) = b_0 x^3 + b_1 x^2 + b_2 x + b_3$$

This premature termination indicates multiple zeros of $P_0(x)$. If $P_0(x)$ has a multiple zero, then the first derivative, $P_1(x)$, has this zero also. Thus the last non-zero Sturm function is the common factor in $P_0(x)$ and $P_1(x)$. By using the Sturm test on $P_g(x)$, we can tell whether or not it contains real, positive zeros. If so, these zeros can be determined by repeated use of Sturm's theorem. If $P_0(x)$ is found to contain no real positive zeros of odd multiplicity, then $H_d(s)$ is a positive real function.

The computer program was designed to detect premature termination and to continue the Sturm test on the last non-zero Sturm function.

Specific numerical examples.

The actual testing procedure was performed in the following manner:

Without loss in generality, values of the parameters "a" and "b" were selected such that

$$a^2 + b^2 = 2.$$

This selection placed the double complex poles of $H_d(s)$ on a circle of radius $\sqrt{2}$, centred at the origin of the s plane. The range of "a" was therefore limited to $0 < a < \sqrt{2}$ since "a" and "b" are real. For different "a" in this range, a set of values of X and Y was determined such that $H_d(s)$ was positive real. The results of this testing are shown in Figures 6. 7, 8, and 9. The shaded areas of these Figures indicate, for various pole positions, the regions in the X-Y plane where $H_d(s)$ was found to be positive real.

We note from these results that as "a" decreases, the "positive real" area also decreases. However, decreasing "a" is equivalent to moving the poles of $H_d(s)$ closer to the imaginary axis, upon which they cannot exist. We deduce that as "a" decreases, the probability that an arbitrary choice of zeros will yield a positive real function also decreases.

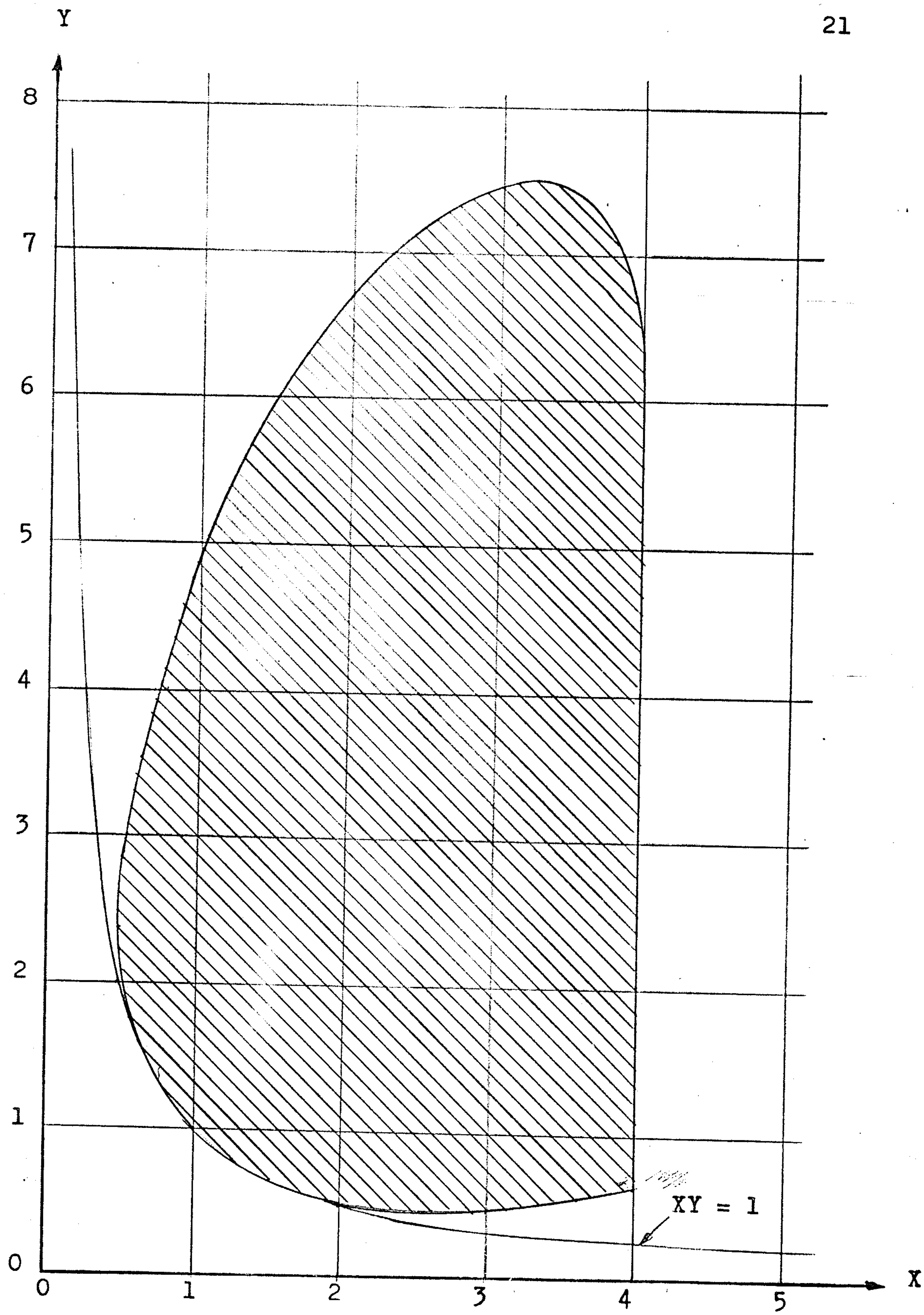

Figure 6. The positive real region of $H_d(s)$ for a = 1.00

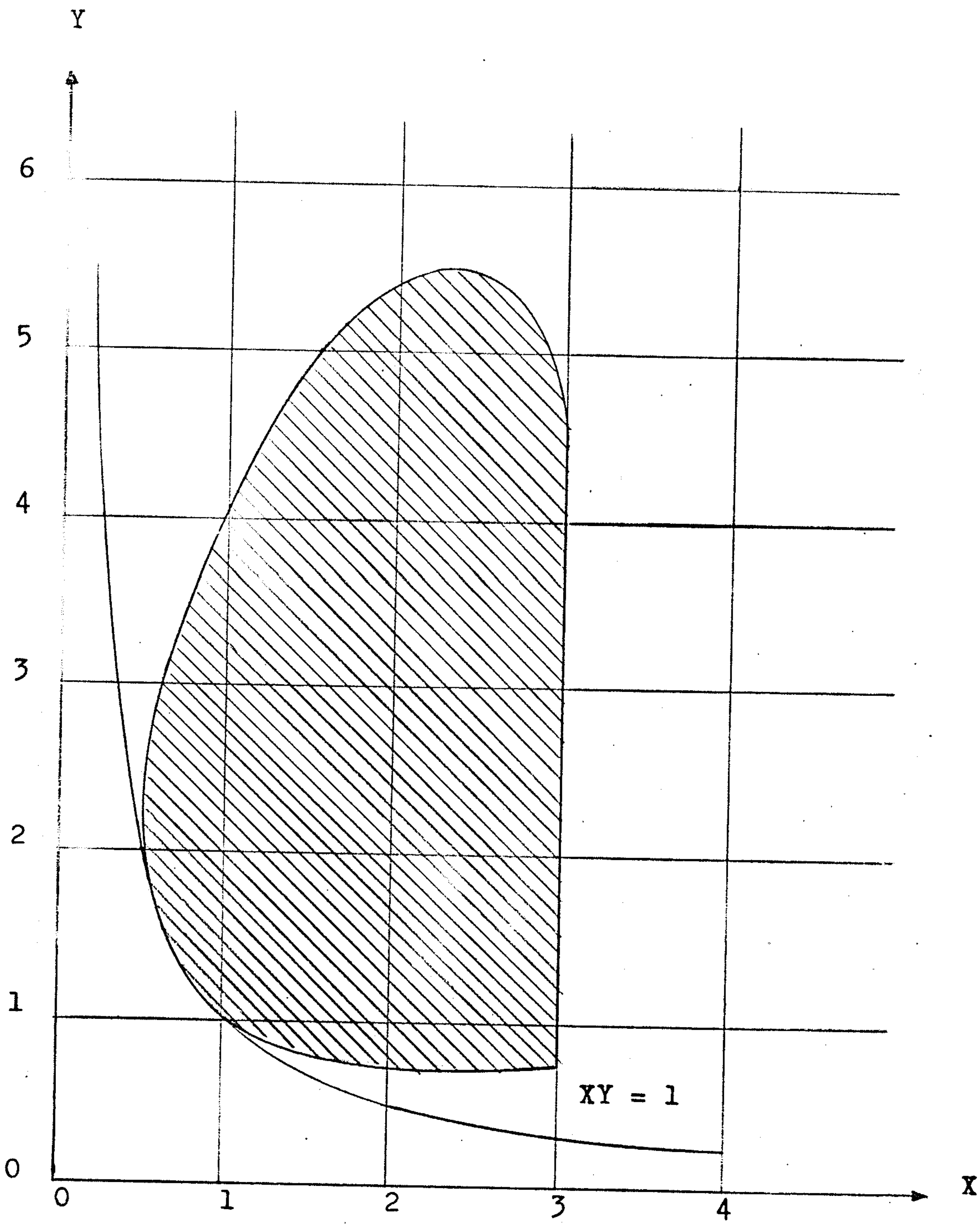

Figure 7. The positive real region of $H_d(s)$ for $a = 0.75$

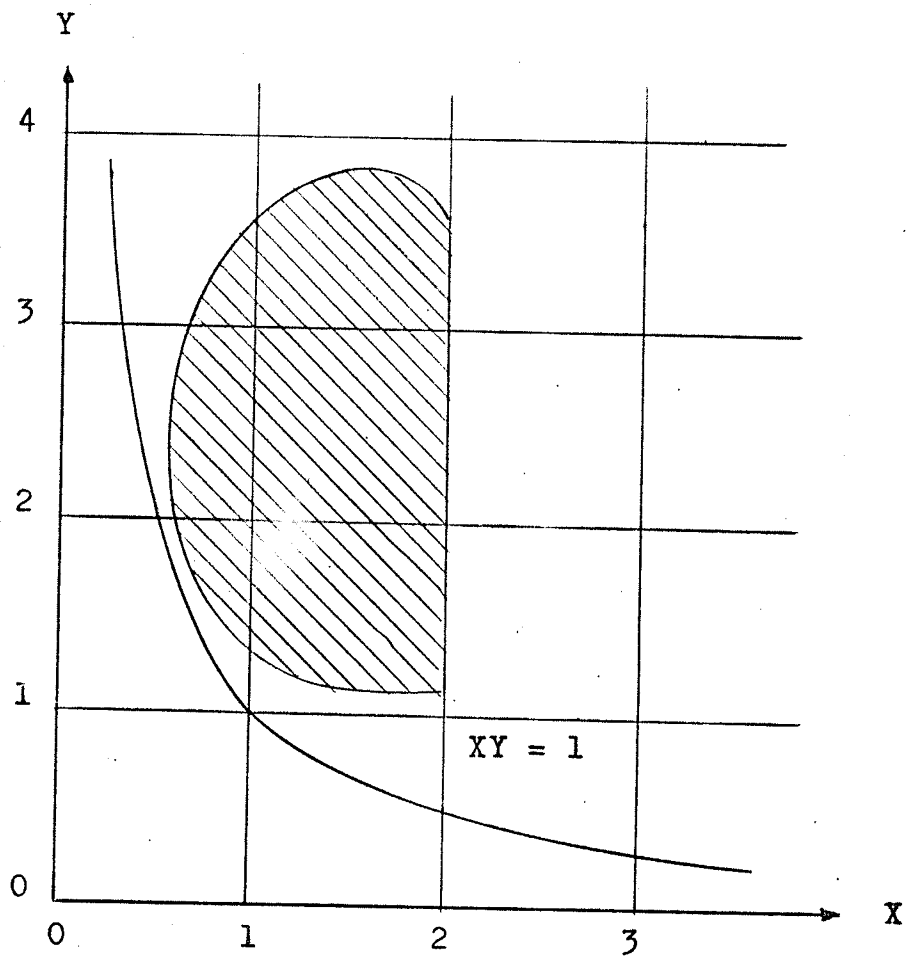

Figure 8. The positive real region of $H_d(s)$ for a = 0.50.

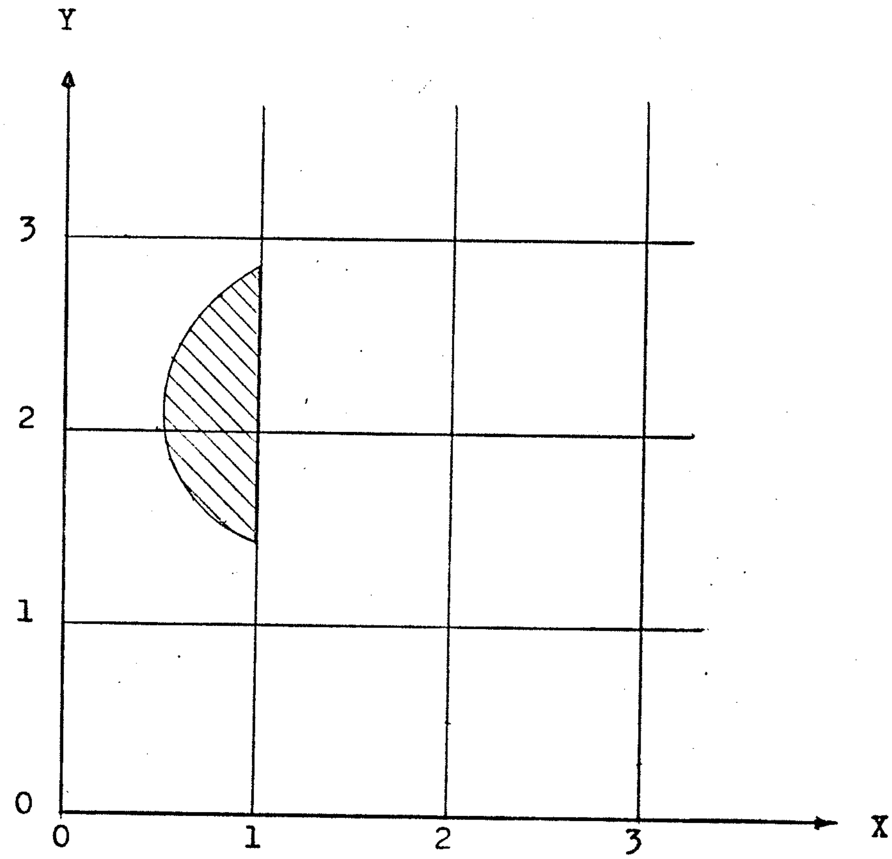

Figure 9. The positive real region of $H_d(s)$ for $a = 0.25$.

CHAPTER IV

THE ZERO RESISTANCE LOCUS

In this thesis it was our aim to investigate the possibility of a driving-point immittance function with double complex poles. This necessitated, of course, physical realizability which implied the positive real property. However, it was necessary to employ unusual procedures due to analytic problems in order to find a positive real function.

We now introduce a new concept in network synthesis, viz. the zero-resistance locus of an immittance function. Using this concept, we will show that any pole zero configuration obeying conditions one, two, and three may represent a positive real function, under certain constraints. We will also develop an alternate test for a positive real function.

Theory of the zero-resistance locus.[10]

Let us consider the driving-point impedance Z(s) and the relation

$$\mathrm{Re}[Z(s)] = 0. \qquad (36)$$

The solution to this equation is the totality of s values which satisfy the equation and these values then define a

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[10] R. W. Brockett, "A Theorem on Positive Real Functions," IRE Trans. on Circuit Theory, CT-11:301-302, June, 1964.

locus in the s plane. We shall denote this locus as the zero-resistance locus of the network function Z(s).

Equation (36) is satisfied whenever

$$Z(s)=\pm j\,|Z(s)|$$

or

$$Z(s)= Z(s)\,\angle\tfrac{\pi}{2}\pm n\pi,\ n=0,\ 1,\ 2,\ \ldots$$

However, for any excitation s,

$$\operatorname{Arg} Z(s) \equiv \sum_{\substack{\text{all}\\ \text{zeros}}}(\text{zero angles to } s)-\sum_{\substack{\text{all}\\ \text{poles}}}(\text{pole angles to } s) \qquad (37)$$

where all angles are measured with respect to the positive real axis in the s plane and in an anti-clockwise sense. Therefore the zero-resistance locus must satisfy

$$\sum_{\substack{\text{all}\\ \text{zeros}}}(\text{zero angles to } s)-\sum_{\substack{\text{all}\\ \text{poles}}}(\text{pole angles to } s)=\frac{\pi}{2}\pm n\pi, \qquad (38)$$

where n=0, 1, 2, ... If we multiply equation (38) through by two, we have:

$$2\sum_{\substack{\text{all}\\ \text{zeros}}}(\text{zero angles to } s)-2\sum_{\substack{\text{all}\\ \text{poles}}}(\text{pole angles to } s)=\pi\pm n\pi, \qquad (39)$$

where n=0, 1, 2, ... Equation (39) is just the angle condition for the root locus[11] of $[Z(s)]^2$.

Therefore to find the zero-resistance locus of Z(s), we simply double up on all poles and zeros of Z(s) and draw the root locus by well established techniques.

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[11] W. R. Evans, "Graphical Analysis of Control Systems," Trans. AIEE, 67:547-551, 1948.

For a driving-point function, numerator and denominator cannot differ in degree by more than one. Therefore the function $[Z(s)]^2$ will not differ in the number of poles and zeros by more than two. In fact, only three cases are possible for the relative degree of numerator and denominator of $[Z(s)]^2$. Either the numerator and denominator are of the same degree, or the numerator is of degree two higher than that of the denominator, or the denominator is of degree two higher than that of the numerator.

It is known that any root locus plot has the number of branches equal to the number of poles of the open loop transfer function of the system if the poles at infinity are also counted. Asymptotes of the root locus are only present where the degree of the numerator differs fron that of the denominator of the open loop transfer function. If then the numerator and denominator are of the same degree, the root locus plot of $[Z(s)]^2$ will always be bounded and can always be represented in the finite s plane.

When they exist, the angles of the asymptotes of the root locus are given by

$$\theta = \frac{\pi \pm 2n\pi}{N_p - N_z}, \quad n=0,\ 1,\ 2,\ \ldots \tag{40}$$

where θ is the angle of the asymptotes measured anti-clockwise from the positive real axis of the s plane, N_p is the number of "finite" poles of $[Z(s)]^2$, and N_z is the number of "finite" zeros of $[Z(s)]^2$.

However, we have shown that

$$N_p - N_z = \pm 2 \tag{41}$$

so that substituting equation (41) into (40) we obtain

$$\theta = \pm \frac{\pi}{2}$$

This implies that as $Re[s] \rightarrow +\infty$, the root locus of $[Z(s)]^2$ will remain bounded in the direction of the real axis. Thus there exists a line defined by $Re[s]=c$ such that the entire root locus lies to the left of it. Since the shape of the root locus does not depend on the imaginary axis we can, therefore, position this axis so that it coincides with this line. However, since the root locus plot of $[Z(s)]^2$ is the zero-resistance locus of Z(s), it is the locus of $Re[Z(s)]=0$. For Z(s) to be a positive real function, we require that $Re[Z(s)] \geq 0$ for $Re[s] \geq 0$.

We then conclude that Z(s) is positive real if and only if the zero resistance locus is confined to the closed left-half plane. This is, therefore, an alternate test for the positive real property.

Since the imaginary axis can be shifted such that the zero-resistance locus lies entirely in the left-half plane, and since such a shift will not change the shape of this locus, any pole-zero configuration complying with conditions one to four can be altered in a simple manner to represent a driving-point function (if it is not already one). The resulting function can then be synthesized by one of the known methods.

Numerical example.

Consider the function $R(s)= \frac{(s+2.5)(s^2+s+0.5)}{[(s+1+j)(s+1-j)]^2}$.

With a "Spirule", the root locus of $[R(s)]^2$ was constructed (see Figure 10). Note that the original position of the imaginary axis was such that the locus did not pass into the right-half s plane. Therefore R(s) is positive real.

Since this function is positive real, it can be synthesized by such methods as those contained in the Brune and Bott & Duffin references. The resulting networks are shown in Figures 11, 12, and 13. The computations appear in Appendix B.

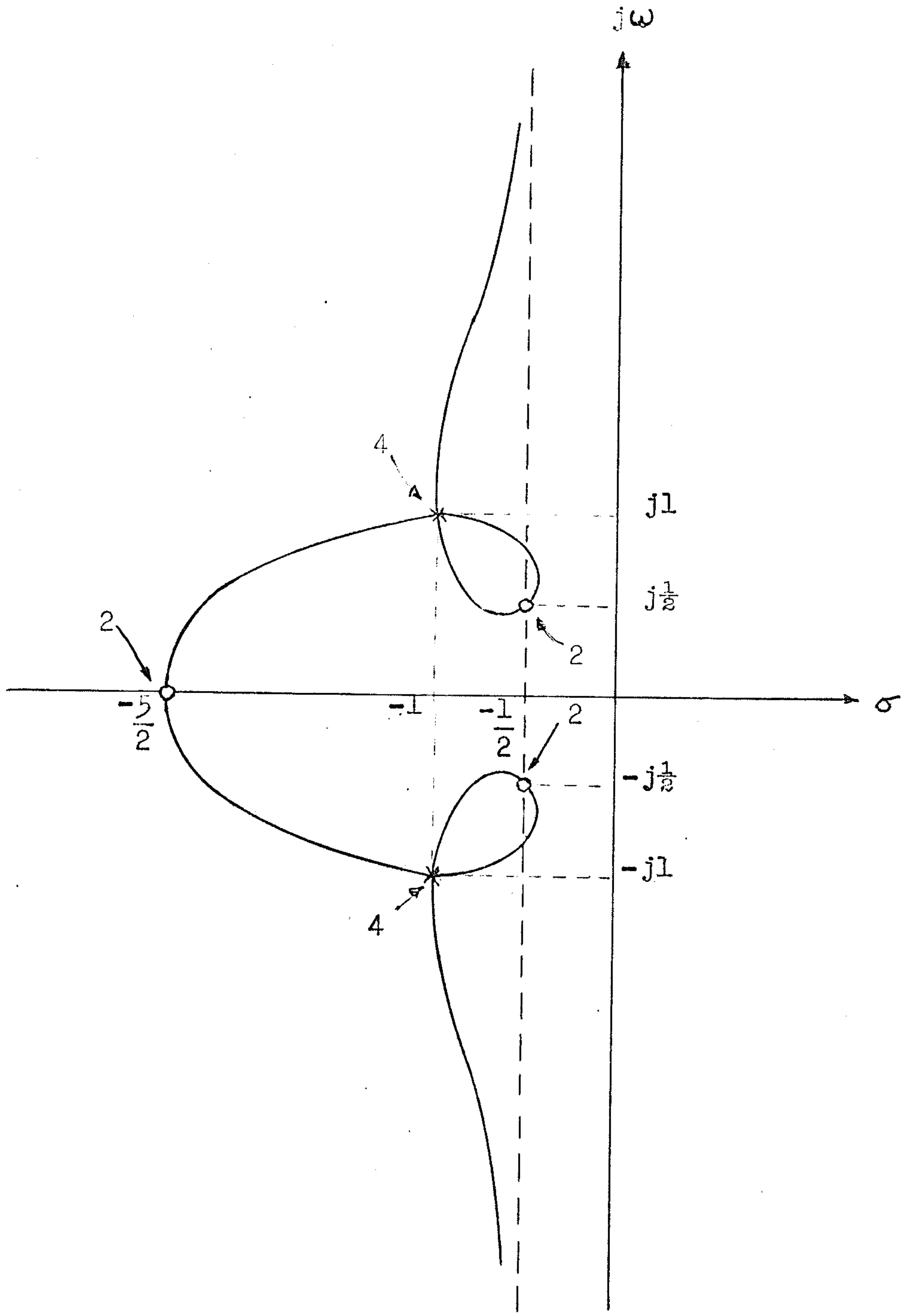

Figure 10. Zero-resistance locus of a specific impedance function.

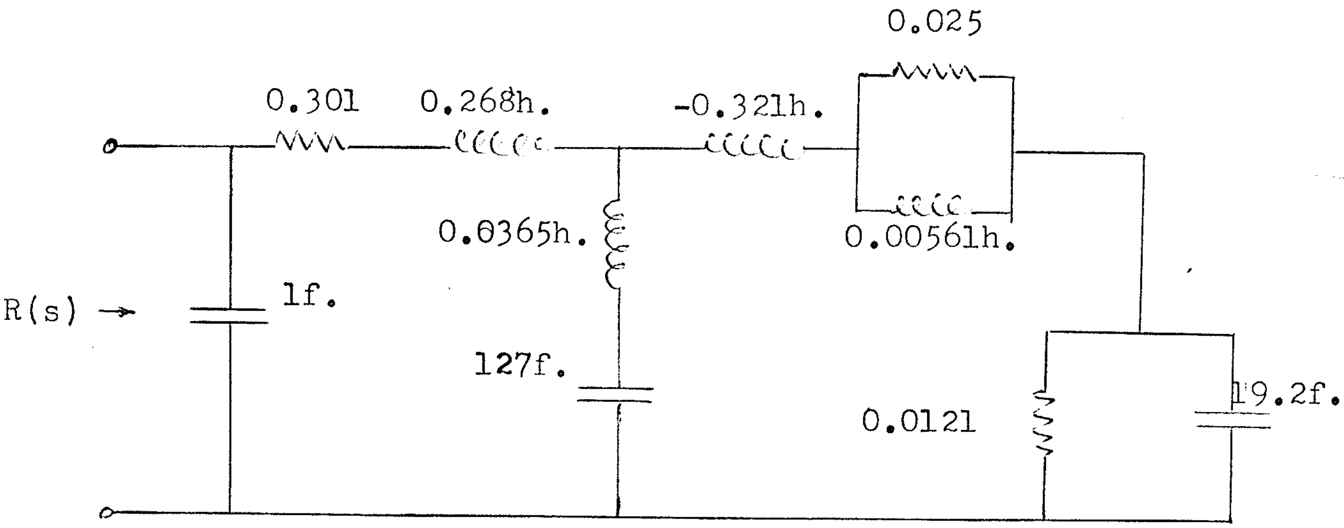

Figure 11. A network whose driving-point impedance contains double complex poles (Brune realization).

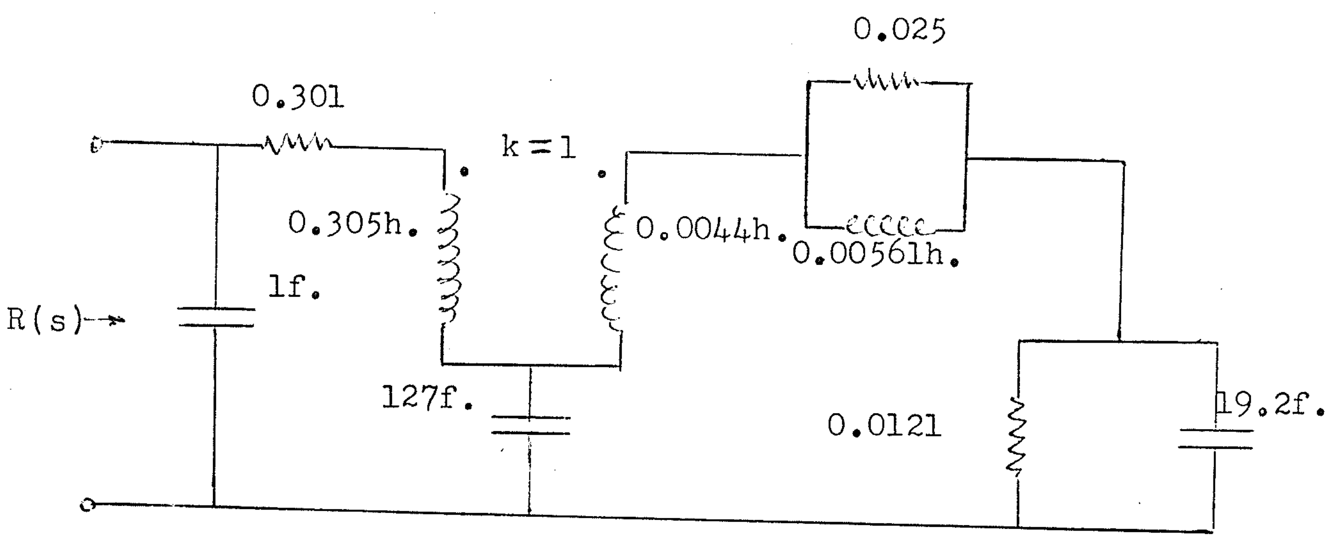

Figure 12. The equivalent circuit of Figure 8 with positive elements and a perfect transformer.

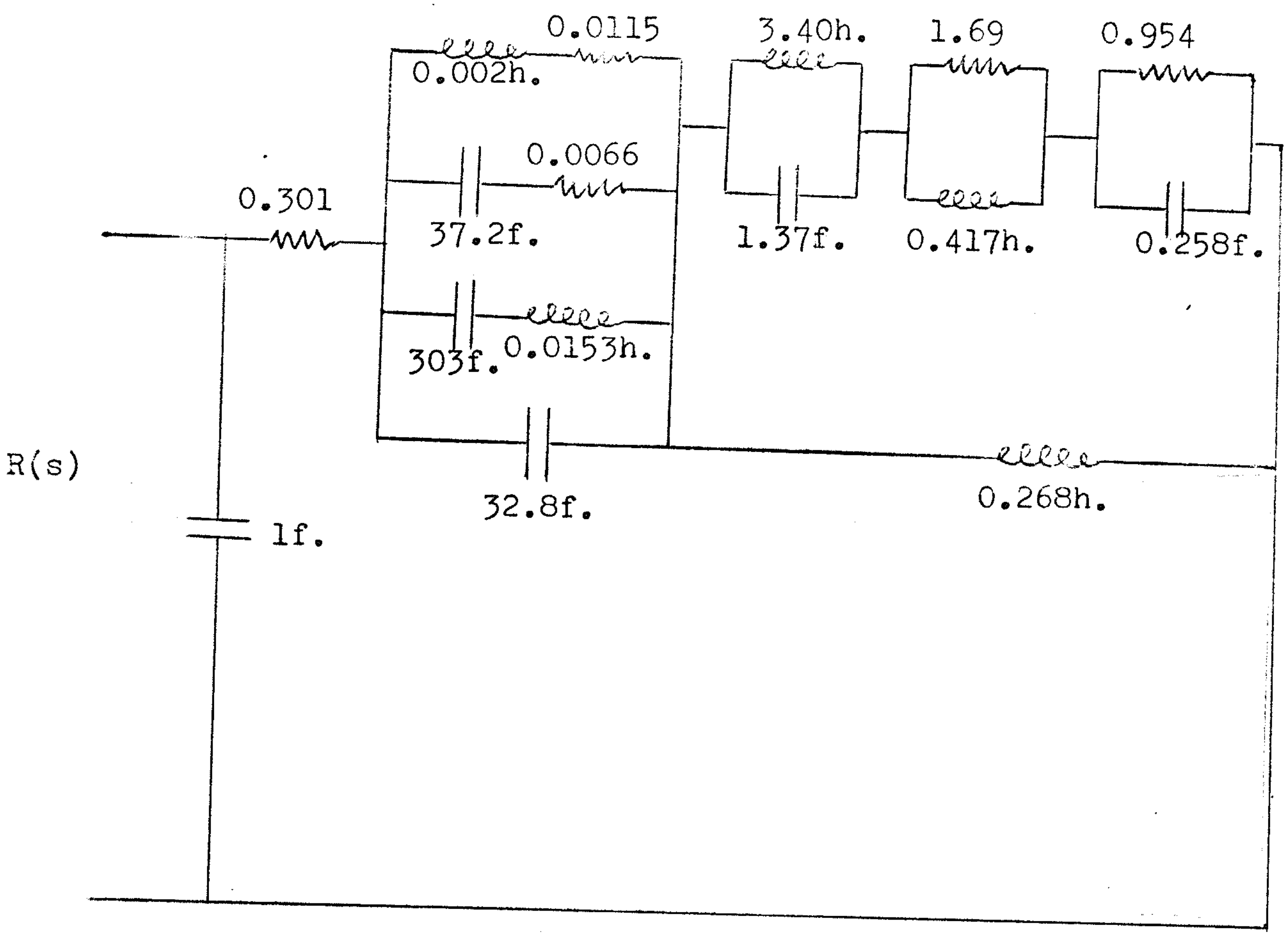

Figure 13. Bott & Duffin realization of a network whose driving-point impedance contains double complex poles.

CHAPTER V

CONCLUSIONS

The concept of the zero-resistance locus as we have seen is a very important one indeed. These loci can be used as a basis for an alternate test for the positive real property of a rational function and were used to demonstrate the fact that any pole-zero configuration can be altered in a simple manner and then synthesized as a driving-point immittance. Of course, the network synthesized does not represent the original function before shifting.

One application of these loci to network synthesis occurs in the production of minimum resistance functions from minimum reactance functions. From the zero-resistance locus of a positive real function, we can graphically determine the imaginary axis shift (series resistance or shunt conductance) required to yield a minimum resistance function. This method is particularly useful in higher order systems where mathematical computation becomes extremely laborious. Another application is to predistortion, a synthesis technique that yields lossy practical networks rather than networks containing mathematically derived (sometimes ideal) elements. The technique itself involves an s plane imaginary axis shift and hence this locus can be used to determine the maximum allowable predistortion for any given driving-point immittance.

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BIBLIOGRAPHY

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APPENDIXES

APPENDIX A

COMPUTER PROGRAM

```
C     TEST FOR POSITIVE REAL PROPERTY OF A RATIONAL POLYNOMIAL FRAC
C     DIMENSION STATEMENTS
      DIMENSION A(20),B(20),C(20),D(20),W(20),X(20),Y(20),Z(20)
      DIMENSION SUM(20),SUN(20),DEL(20),DEM(20),POLY(20)
      DIMENSION R(20,20),S(20,20)
C     FORMAT STATEMENTS
  100 FORMAT (I2,I2)
  101 FORMAT (F14.0)
  102 FORMAT (18X F14.0)
  103 FORMAT (32H TEST FOR POSITIVE REAL PROPERTY)
  104 FORMAT (34H OF A RATIONAL POLYNOMIAL FRACTION)
  105 FORMAT (F14.8)
  106 FORMAT (47H NUMBER OF REAL ZEROS BETWEEN ZERO AND INFINITY,I5
  107 FORMAT (23H NUMERATOR COEFFICIENTS)
  108 FORMAT (25H DENOMINATOR COEFFICIENTS)
  111 FORMAT (I5)
  113 FORMAT (37H THE EXPANSION TERMINATES PREMATURELY)
  114 FORMAT (44H THE FIRST STURM FUNCTION HAS MULTIPLE ZEROS)
C     READ DEGREES OF NUMERATOR AND DENOMINATOR
    1 PAUSE
      TYPE 103
      TYPE 104
      READ 100,N,M
C     READ COEFFICIENTS OF NUMERATOR AND DENOMINATOR
      TYPE 107
      NN=N+1
```

```
      DO5I=1,NN
      READ 101,A(I)
    5 TYPE 105,A(I)
      MM=M+1
      TYPE 108
      DO10I=1,MM
      READ 102,B(I)
   10 TYPE 105,B(I)
C     DETERMINATION OF POLES AND ZEROS AT INFINITY
      IF(M-N) 20,40,30
C     REMOVAL OF A POLE AT INFINITY
   20 C(1)=A(1)
      L=N
      DO25I=2,N
   25 C(I)=A(I)-A(N+1)/B(N)*B(I-1)
      DO27I=1,L
   27 D(I)=B(I)
      GO TO 45
C     REMOVAL OF A ZERO AT INFINITY
   30 C(1)=B(1)
      L=M
      DO35I=2,M
   35 C(I)=B(I)-B(M+1)/A(M)*A(I-1)
      DO37I=1,L
   37 D(I)=A(I)
      GO TO 45
   40 L=M
   45 CONTINUE
C     RATIONALIZATION PROCEDURE
      LP1=L+1
```

```
      LP2=L+2
      LX2=L*2+1
      DO47I=LP1,LX2
      C(I)=0.
   47 D(I)=0.
C     PRODUCT OF EVEN TERMS
      DO50I=1,LX2,2
      SUM(I)=0.
      DO50J=1,I,2
      K=I-J+1
      SUMX=C(J)*D(K)
   50 SUM(I)=SUM(I)+SUMX
      DO55I=1,LX2,4
   55 W(I)=SUM(I)
      DO60I=3,LX2,4
   60 X(I)=SUM(I)
C     PRODUCT OF ODD TERMS
      DO65I=2,LX2,2
      SUN(I)=0.
      DO65J=2,I,2
      K=I-J+2
      SUNX=C(J)*D(K)
   65 SUN(I)=SUN(I)+SUNX
      DO70I=2,LX2,4
   70 Y(I)=SUN(I)
      DO75I=4,LX2,4
   75 Z(I)=SUN(I)
C     COMBINATION OF TERMS
      DEL(1)=W(1)
      DO80I=5,LX2,4
```

```
   80 DEL(I)=W(I)-Z(I-1)
      DO85I=3,LX2,4
   85 DEM(I)=-X(I)+Y(I-1)
      DO90I=3,LX2,4
   90 DEL(I)=0.
      DO95I=5,LX2,4
   95 DEM(I)=0.
      POLY(1)=DEL(1)
      DO97I=3,LX2,2
   97 POLY(I)=DEL(I)+DEM(I)
C     DETERMINATION OF THE FIRST NON-ZERO TERM IN POLY(I)
      IF(POLY(LX2)) 205,210,205
  205 NMAX=LX2
      GO TO 222
  210 CONTINUE
      DO215I=2,LX2,2
      M1=LX2-I
      IF(POLY(M1)-0.) 220,215,220
  215 CONTINUE
  220 NMAX=M1
  222 NEQV=NMAX
C     CHANGE OF INDEX
      DO6I=1,NMAX,2
      MD2=(I+1)/2
    6 POLY(MD2)=POLY(I)
      NMAX=(NMAX+1)/2
      NX1=NMAX
      NX2=NMAX-2
C     GENERATION OF STURM FUNCTIONS
      DO225I=1,NMAX
```

```
225 S(1,I)=POLY(I)
226 G=-1.
    DO230I=1,NMAX
    G=G+1.
230 S(2,I)=G*POLY(I)
    NM1=NMAX-1
    DO235I=1,NM1
235 S(2,I)=S(2,I+1)
    NMAX=NMAX+1
    DO260J=3,NX1
    NMAX=NMAX-1
    NM1=NMAX-1
    NM2=NMAX-2
    DO250I=2,NM1
    R(J,1)=S(J-2,1)
250 R(J,I)=S(J-2,I)-S(J-2,NMAX)/S(J-1,NMAX-1)*S(J-1,I-1)
    DO255I=1,NM2
255 S(J,I)=-(R(J,I)-R(J,NMAX-1)/S(J-1,NMAX-1)*S(J-1,I))
    J1=J
    DO257I=1,NM2
    IF(S(J1,I)-0.) 259,257,259
257 CONTINUE
    GO TO 500
259 IF(NMAX-2) 260,270,260
260 CONTINUE
270 CONTINUE
C   DETERMINATION OF THE SIGN OF S AT ZERO
    NT=NX1-1
    NZERO=0
    DO280J=1,NT
```

```
      IF(S(J,1)-0.) 281,280,283
  281 IF(S(J+1,1)-0.) 284,290,286
  290 J=J+1
      GO TO 281
  286 NZERO=NZERO+1
  284 NZERO=NZERO
      GO TO 280
  283 IF(S(J+1,1)-0.) 287,295,289
  295 J=J+1
      GO TO 283
  287 NZERO=NZERO+1
  289 NZERO=NZERO
  280 CONTINUE
C     DETERMINATION OF THE SIGN OF S AT INFINITY
      I=NX1+1
      NINF=0
      DO310J=1,NT
      I=I-1
      IF(S(J,I)-0.) 301,310,303
  301 IF(S(J+1,I-1)-0.) 304,302,306
  302 J=J+1
      I=I-1
      GO TO 301
  306 NINF=NINF+1
  304 NINF=NINF
      GO TO 310
  303 IF(S(J+1,I-1)-0.) 307,308,309
  308 J=J+1
      I=I-1
      GO TO 303
```

```
  307 NINF=NINF+1
  309 NINF=NINF
  310 CONTINUE
C     THE TOTAL NUMBER OF REAL ZEROS BETWEEN ZERO AND INFINITY
      NTOT=NZERO-NINF
      IF(NTOT-0) 430,420,410
  410 NTOT=NTOT
      GO TO 420
  430 NTOT=-NTOT
      GO TO 420
  420 TYPE 106,NTOT
      GO TO 1
  500 TYPE 113
      TYPE 114
      TYPE 111,J1
C     REPEATED USE OF STURM'S THEOREM
      NMAX=NM2+1
      DO505I=1,NMAX
      S(1,I)=S(J1-1,I)
  505 TYPE 105,S(1,I)
      GO TO 226
      END
```

APPENDIX B

COMPUTATIONS FOR BRUNE AND BOTT & DUFFIN REALIZATIONS

The following driving-point impedance will be synthesized here by both the Brune and Bott & Duffin methods:

$$R(s) = \frac{s^3+3.5s^2+3s+1.25}{s^4+4s^3+8s^2+8s+4}$$

If we remove a 1 farad shunt capacitor and a 0.301 ohm series resistor from R(s), then we are left with a minimum function $Z_1(s)$ which can be calculated from the relation

$$Z_1(s) = \frac{1}{\frac{1}{R(s)} - s} - 0.301$$

Computing $Z_1(s)$ we find

$$Z_1(s) = \frac{0.849s^3+1.99s^2+0.968s+0.046}{0.5s^3+5s^2+6.75s+4}$$

It turns out that $Z_1(j0.464) = j0.124$. We may now proceed to synthesize $Z_1(s)$ by either the Brune or Bott & Duffin method since both require minimum functions. The network so far obtained is shown in Figure 15.

<u>Brune synthesis</u>.

Consider the minimum function $Z_1(s)$. We have the relation $Z_1(j0.464) = j0.124$. Assuming the reactance at this frequency to be due to a positive inductor, we shall remove from $Z_1(s)$ an inductor whose reactance is equal to $Z_1(s)$ at $s = j0.464$. The value of this inductor is calculated to be 0.268 henries. The remaining impedance $Z_2(s)$ is determined from $Z_2(s) = Z_1(s) - 0.268$. Computing $Z_2(s)$ we have

$$Z_2(s) = \frac{-0.134s^4-0.149s^3+0.188s^2-0.104s+0.046}{0.5s^3+5s^2+6.75s+4}$$

By this technique, we have produced a zero of $Z_2(s)$ at $s = \pm j0.464$. Thus $(s^2+0.215)$ can be factored out of the numerator to yield

$$Z_2(s) = \frac{(s^2+0.215)(-0.134s^2-0.491s+0.217)}{0.5s^3+5s^2+6.75s+4}$$

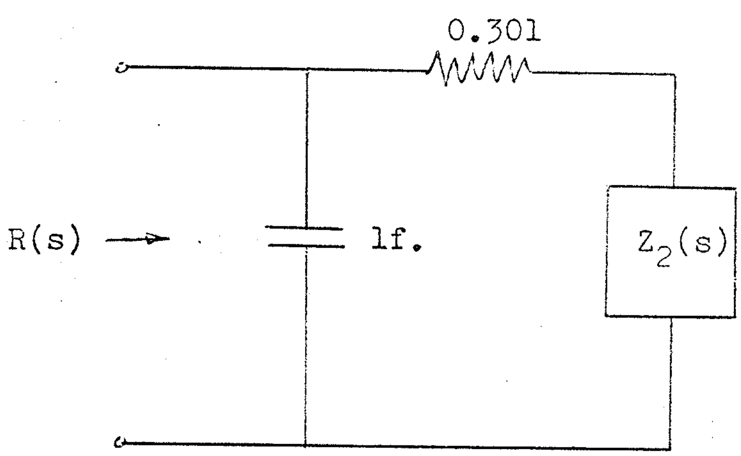

Figure 15. Production of a minimum function to be used in the Brune and Bott & Duffin methods.

These zeros on the imaginary axis can be removed by a partial fraction expansion of $Z_2^{-1}(s)$. The residue of $Z_2^{-1}(s) = Y_2(s)$ was calculated and was found to be 13.7 at these poles of $Y_2(s)$ on the imaginary axis. We then remove the quantity $\frac{(2)(13.7)s}{s^2+0.215}$ from $Y_2(s)$. This corresponds to a series L-C network. If we let

$$Y_3(s) = Y_2(s) - \frac{27.4s}{s^2+0.025}$$

then
$$Y_3(s) = \frac{4.17s+18.5}{-0.134s^2-0.491s+0.217}$$

We now remove the zero at infinity of $Y_3(s)$ (an inductor) by removing it as a pole of the inverse, $Z_3(s)$. Thus we have

$$Z_3(s) = -0.0321s + \frac{0.104s+0.217}{4.17s+18.5}$$

The first term on the right represents a negative inductor of -0.0321 henries and the other term represents a shunt R-L network in series with a shunt R-C network. The complete network for R(s) appears in Figure 11 and an equivalent network with all positive elements and a perfect transformer appears in Figure 12.

Bott & Duffin synthesis.

Consider again the minimum function $Z_1(s)$. As before, $Z_1(j0.464) = j0.124$. Let P(s) be defined as

$$R(s) = \frac{kZ_1(s) - sZ_1(k)}{kZ_1(k) - sZ_1(s)}$$

We assume the reactance at $s = j0.464$ to be due to an inductor whose reactance is equal to j0.124. The value of this inductor is 0.268 henries. Let this inductor be denoted by L_2. We then solve the equation

$$Z_1(k) = L_1k$$

The only positive root of this equation is $k = 0.396$.

We now calculate C_1 from the relation

$$C_1 = \frac{1}{kZ_1(k)}$$

and obtain $C_1 = 23.8$ farads. Computing $P(s)$ from the definition, we have

$$P(s) = \frac{(s^2+0.215)(0.053s+0.215)}{(0.849s^3+2.305s^2+1.671s+0.424)}$$

We next synthesize $Z_2(s)$ defined by $Z_2(s) = Z_1(k)P(s)$.

Thus

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{0.849s^3+2.305s^2+1.671s+0.424}{0.106(s^2+0.215)(0.053+0.215)}$$

$$= \frac{8s+18.6}{0.053s+0.215} + \frac{65.2s}{s^2+0.215}$$

$Y_2(s)$ consists of a series R-C, a series R-L, and a series L-C all connected in parallel.

We shall now synthesize $Z_3(s) = \frac{Z_1(k)}{P(s)}$. We find that

$$Z_3(s) = \frac{0.106(0.849s^3+2.305s^2+1.671s+0.424)}{(s^2+0.215)(0.053s+0.215)}$$

$$= \frac{0.732s}{s^2+0.215} + \frac{0.0899s+0.205}{0.053s+0.215}$$

Z_3 consists of a parallel L-C, a parallel R-L, and a parallel R-C all connected in series. The complete network appears in Figure 13.