AN ANALOGUE COMPUTER INVESTIGATION OF VARIOUS PERFORMANCE CRITERIA USING THE MAXIMUM PRINCIPLE

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ABSTRACT

The optimal controls for various performance criteria are investigated by means of the Pontryagin maximum principle. An analogue computer has been used to determine the optimal control solutions using the method of reverse time. The investigation considers second- and higher-order systems using the phase space as a means of representing the optimal control. Examples are given for cases in which time, error, and effort appear as the parameters in the performance functional.

CHAPTER I

INTRODUCTION TO THE PROBLEM

The theory of automatic control systems has undergone modifications during the past decade, due to the development of many elegant mathematical techniques, to cope with the type of problems arising in modern control systems design, particularly in the areas of industrial process control and space vehicle guidance. These modern methods were developed to properly formulate and to solve recent problems in optimal control because the traditional approach, based mainly on feedback amplifier theory and linear circuit analysis techniques, was becoming inadequate.

An optimal control process is one that is best in some prescribed sense, i.e. achieving the purpose of the process in minimum time, or with the minimum expenditure of effort, or with some other criterion of performance; subject to certain practical constraints on the forcing functions which are used to achieve this performance. Once the performance criteria of the system have been defined by extrinsic considerations, the optimal control is the one which minimizes these figures of merit subject to the constraints imposed.

In order to solve these optimal control problems, the relatively new and very elegant "Maximum Principle" of L. S. Pontryagin[1-4]* may be used. This principle, hypothesized in 1956 by L. S. Pontryagin, V. G. Boltyanskii, and R. V. Gamkrelidze, was based on results achieved by them in their studies of the rapidity of action of a dynamic process. In this thesis, the maximum principle has been used, with the aid of an analogue computer, to investigate the optimization of different control systems for various performance criteria. Due to the ease with which parameters and operating conditions may be changed and due to the rapid determination of system response, the analogue computer was used to act as a mathematical model of the systems to be investigated.

The main object of this thesis is to demonstrate that the results achieved by applying modern mathematical procedures and modern computational devices form a useful basis for practical control system design, independent of whether or not the computation enables the designer to realize the optimal control.

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*Superscripts indicate references in bibliography.

CHAPTER II

FORMULATION OF THE PROBLEM

In this chapter, the type of optimal control problem to be considered later is formulated, and the necessary statements of the method used to achieve the solution of this control problem are developed. A brief account of various performance criteria terminates the chapter.

I. PONTRYAGIN'S MAXIMUM PRINCIPLE

The development of statements pertaining to the maximum principle of Pontryagin is given for practical application to the optimal control problem that is formulated. Details are similar to the explanation given by Rozonoer[1] and detailed proofs of the maximum principle with references to some of the original papers may be found in 1, 4 and 5.

The Control System. It is assumed that the controlled system (machine, process, etc.) may be represented by the n first-order differential equations,

$$\dot{x}_i = f_i(x_1,--,x_n:u_1,--u_r;t), \qquad i = 1,2,--,n. \qquad (2\text{-}1)$$

or in vector form,

$$\dot{\underline{x}} = f\ (\underline{x},\underline{u},t) \qquad (2\text{-}1a)$$

The n state variables $x_1,--,x_n$ describe the behaviour of the system at any instant of time, and the system of parameters $u_1,--,u_r$ characterize the positions of the system's controlling elements. This control $\underline{u}(t)$ with components $u_k(t)$, $k = 1,2,--r$, is restricted to be in a specified closed and bounded region U,

$$\underline{u}(t) \in U \qquad (2\text{-}2)$$

which may be any closed and bounded set in r-dimensional space. At each instant of time, these controls are required to fulfill certain inequalities dictated by the physical constraints of the system, such as $|u_k(t)| \leq 1$, $k = 1,2,--r$. Current, voltage, or fuel, which are limited in some manner, may serve as the controls. In addition to the condition (2-2) stated above, it is required that $\underline{u}(t)$ be a piecewise continuous function of time (i.e. each function $u_k(t)$, which is piecewise continuous for all t under consideration, can have only a finite number of discontinuities of the first kind at the end of an interval of time). The controls which satisfy these conditions will be called admissible controls.[2]

The Control Problem. The control problem of the theory of optimum systems is the following: An admissible control $\underline{u}(t)$ must be found such that the system performance is "best" in some specific sense.

It is proposed that the criterion or functional, to be used to evaluate the system performance, be of the form,

$$I = \int_{t_o}^{t_f} F(x_1,--,x_n;u_1,--,u_r;t)dt \qquad (2\text{-}3)$$

It will be required to choose the admissible control $\underline{u}(t)$ in such a manner that it takes the system from $\underline{x}(t_o)$ to $\underline{x}(t_f)$ while minimizing the functional I. (A functional is a correspondence established between functions--or combinations of functions--and numbers).[1] This functional also encompasses the minimum time problem $(F(\underline{x},\underline{u},t)=1)$ in which the system is optimum for rapid action from some fixed initial state $\underline{x}(t_o)$ to some fixed final state $\underline{x}(t_f)$. Systems described by (2-1) with boundary conditions,

$$x_i(t_o) = x_i^{\,o},\ x_i(t_f) = x_i^{\,1},\ i = 1,2,--n,$$

are said to have fixed end points and are commonly referred to as two-point boundary value problems.

A new state variable is introduced,

$$\left.\begin{aligned} x_{n+1}(t) &= \int_0^t F(x_1,--,x_n;u_1,--,u_r;t)dt \\ x_{n+1}(0) &= x_{n+1}^{\,o} = 0, \end{aligned}\right\} \qquad (2\text{-}4)$$

adding a new differential relation to the system's differential equations (2-1),

$$\dot{x}_{n+1} = F(x_1,--,x_n;u_1,--u_r;t) \qquad (2\text{-}5)$$

Thus, the problem of optimizing a system with respect to an integral leads to that of optimizing the (n+1)st state variable $x_{n+1}(t_f)$ at the final instant of time.

In general, the control problem may be treated as the choice of an optimum control $\underline{u}(t)$ from a group of admissible controls to optimize the linear function of the final values of all co-ordinates of the system,

$$S = \sum_{k=1}^{n} c_k x_k(t_f) \qquad (2\text{-}6)$$

where c_k are constants. The optimization of the scalar product of the end points of the trajectory $\underline{x}(t_f) = x_1(t_f),--x_n(t_f)$ and the given vector $\underline{c}(t) = c_1,--c_k$ is required.

Since the classical calculus of variations does not allow us to solve this optimal control problem, due to the restrictions on the controlling elements, new mathematical methods were developed.

<u>The Maximum Principle.</u> A Hamiltonian function is introduced

$$H(\underline{x},\underline{p},\underline{u},t) = \sum_{k=1}^{n} p_k f_k(x_1,--,x_{n+1};u_1,--u_r;t) \qquad (2\text{-}7)$$

with the functions $p_k(t)$ satisfying the system of differential equations,

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1,2,--,n+1. \qquad (2\text{-}8a)$$

The original system of differential equations can be expressed as

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad i = 1,2,--,n+1. \qquad (2\text{-}8b)$$

The boundary conditions are given by

$$x_i(t_o) = x_i^{\,o}, \text{ and } p_i(t_f) = -c_i \qquad (2\text{-}8c)$$

where the vector $\underline{p}(t)$ is equal in magnitude but opposite in direction to the vector $\underline{c}(t)$, at the final time t_f.

If $\underline{u}^*(t)$ is the optimal control and $\underline{x}^*(t)$ is the optimum trajectory of the system emanating from $x_i^{\,o}$, then the maximum principle may be stated as follows: With $\underline{u}^*(t)$ the optimal control, there exists a nonzero continuous vector $\underline{p}^*(t)$ satisfying (2-8a) such that, for every instant of time $t, t_o \leq t \leq t_f$,

$$H(\underline{x}^*,\underline{p}^*,\underline{u}^*,t) \geq H(\underline{x},\underline{p},\underline{u},t)$$

with respect to all admissible controls $\underline{u}(t)$, i.e.

$$H(\underline{x}^*,\underline{p}^*,\underline{u}^*,t) = \max_{\underline{u} \in U} H(\underline{x},\underline{p},\underline{u},t) \qquad (2\text{-}9)$$

This principle also implies that $S = \sum_{k=1}^{n+1} c_k x_k(t_f)$ will be minimized for all t, when $H(\underline{x},\underline{p},\underline{u},t)$ is maximized.

In general, the maximum principle provides only necessary conditions for optimality of the control $\underline{u}^*(t)$. If the control is found which satisfies (2-9) and it causes the boundary conditions to be satisfied, the maximum principle usually gives sufficient information to uniquely define the optimal control. It will be shown that the maximum principle of Pontryagin provides a useful and quite general method of attacking the optimal control problem.

II. PERFORMANCE CRITERIA

A performance index has been defined by Anderson, et al.[6] as: "Some mathematical function of the measured response, the function being chosen to give emphasis to the system specifications of interest." Indices of performance have been used previously as an aid in control system design, but it is hoped that they may be used in the future for acceptance or rejection of a system. Many papers have been written on the various types of performance criteria,[7-9] but no general techniques have been proposed for the a priori characterization of optimum systems. Some of the general rules which can be followed in selecting a performance index have been proposed by Graham and Lathrop[8] as:

1. A general performance index should lead to systems of higher order, as well as second order, which judgment indicates are good systems when their overall response is considered. This property is called reliability.

2. A performance index should be selective. That is, the optimum value of the system parameters should be clearly

discernible.

3. The ease with which a performance index can be applied is a consideration.

The general mathematical expression to evaluate system performance is of the form,[10]

$$I = \int_{t_o}^{t_f} F(\underline{x},\underline{u},t)dt \qquad (2\text{-}10)$$

This form encompasses various performance criteria such as the following:

a) ISE (Integral squared error).

$$I = \int_0^{\infty} e^2(t)dt$$

b) ITSE (Integral of time multiplied by squared error).

$$I = \int_0^{\infty} te^2(t)dt$$

c) ISTSE (Integral of squared time multiplied by the squared value of error).

$$I = \int_0^{\infty} t^2e^2(t)dt$$

d) Control area.

$$I = \int_0^{\infty} e(t)dt$$

e) Weighted control area.

$$I = \int_0^{\infty} te(t)dt$$

f) IAE (Integral of the absolute value of error).

$$I = \int_0^{\infty} |e(t)|\, dt$$

g) ITAE (Integral of time multiplied by the absolute value of error).

$$I = \int_0^{\infty} t\,|e(t)|\, dt$$

h) ISTAE (Integral of squared time multiplied by the absolute value of error).

$$I = \int_0^{\infty} t^2 |e(t)|\, dt$$

i) Minimum time.[1,4]

$$I = \int_{t_o}^{t_f} dt = t_f - t_o$$

j) Minimum effort.[11]

$$I = \int_{t_o}^{t_f} |u|^n dt$$

k) Minimum error and effort.[11]

$$I = \int_{t_o}^{t_f} (|x_1|^m + w|u|^n)dt$$

Pontryagin's maximum principle permits the selection of a performance criterion based on relevance to actual control objectives, rather than convenience to the analysis methods. The performance criteria listed above are minimized for various systems by the use of the maximum principle.

CHAPTER III

APPLICATIONS OF THE MAXIMUM PRINCIPLE TO SECOND-ORDER SYSTEMS

The application of the maximum principle to the optimal control of linear dynamic second-order systems is investigated in this chapter. Second-order differential equations have been selected because they describe approximately the dynamics of many control process and they can be investigated conveniently on the phase plane.

I. DOUBLE INTEGRATOR PLANT $G(p) = \frac{1}{p^2}$

The system studied has a plant consisting of two integrators and a control input, u, constrained to remain within certain bounds. It is required to obtain the optimum control of the system, having the structure of Fig. 1,[12] by application of the maximum principle.

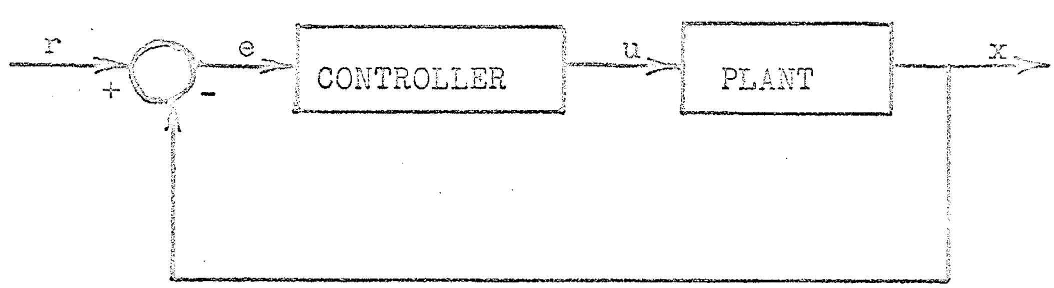


Fig. 1 - Structure of optimum feedback control system.

This system has been treated previously by many authors,[13-18] but only for a small number of performance criteria. The treatment has been extended to other performance criteria and it will be demonstrated with the use of examples.

Example 1. Integral-square error criterion.

The system considered has a plant with a transfer function $G(p) = \frac{1}{ap^2}$ between the input u and the output x (ideal torque source driving pure inertia). The desired output, or command signal, is zero and the input is subject to a saturation constraint $|u| \leq L$. It is required to obtain the controller so that the performance index

$$I = \int_0^\infty e^2(t)dt \qquad (3\text{-}1)$$

is minimized. Since the optimum controller may be characterized by the switching curve in the phase plane, it is required to find the optimum switching curve. The method used to solve this problem follows.

Let the state variables of the controlled process be the output position and velocity:

$$x_1 = x \quad x_2 = \dot{x} = \frac{dx}{dt}$$

The differential equations of the main process become

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{u}{a} \tag{3-2}$$

with the response of the system considered from the initial conditions of position and velocity to a final condition $x_1 = x_2 = 0$.

A new state variable x_3 is defined as having a derivitive equal to the integrand in the performance functional $\int_0^\infty x_1^2(t)dt$ giving

$$\dot{x}_3 = x_1^2 \tag{3-3}$$

and equations (3-2) and (3-3) are written as

$$\dot{x}_i = f_i, \quad i = 0,1,2. \tag{3-4}$$

with boundary conditions,

$$x_1(0) = x_1^{\circ}, \quad x_2(0) = x_2^{\circ}, \quad x_3(0) = 0.$$

The Hamiltonian H is:

$$H = p_1 f_1 + p_2 f_2 + p_3 f_3 \tag{3-5}$$

where the p_i's satisfy

$$\dot{p}_i = - \frac{\partial H}{\partial x_i}, \quad i = 1,2,3. \tag{3-6}$$

Thus, the Hamiltonian is written more explicitly as:

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 x_1^2 \tag{3-7}$$

and the equations for p_i are

$$\mathring{p}_1 = -2p_3 x_1 \tag{3-8}$$

$$\mathring{p}_2 = -p_1 \tag{3-9}$$

$$\mathring{p}_3 = 0 \tag{3-10}$$

Rozonoer[1] has shown that with the performance criterion of the form $\int_0^T f_3 dt$ with T fixed, certain boundary conditions are assigned to the $p_i(T)$. With ∞ substituted for T, equation (2-8c) gives the following (assuming the argument is valid for $T = \infty$),

$$p_1(\infty) = 0 \tag{3-11}$$

$$p_2(\infty) = 0 \tag{3-12}$$

$$p_3(\infty) = -1 \tag{3-13}$$

From equations (3-10) and (3-13),

$$p_3(t) = -1 \qquad 0 \leq t < \infty \tag{3-14}$$

The Pontryagin maximum principle states that the Hamiltonian will be maximized when the optimal u is chosen. Therefore, from (3-7), H will be maximized when

$$u = L\,\mathrm{sgn}(p_2) \tag{3-15}$$

where

$$\operatorname{sgn}(x) = 1 \quad \text{if } x > 0$$
$$= -1 \quad \text{if } x < 0$$

Substituting for p_3 from (3-14) and for u from (3-15), equations (3-2), (3-8), and (3-9) reduce to:

$$\mathring{x}_1 = x_2 \tag{3-16}$$
$$\mathring{x}_2 = \frac{L}{a}\operatorname{sgn}(p_2) \tag{3-17}$$
$$\mathring{p}_1 = 2x_1 \tag{3-18}$$
$$\mathring{p}_2 = -p_1 \tag{3-19}$$

with boundary conditions,

$$x_1(0) = x_1^{\,o} \tag{3-20}$$
$$x_2(0) = x_2^{\,o} \tag{3-21}$$
$$p_1(\infty) = 0 \tag{3-22}$$
$$p_2(\infty) = 0 \tag{3-23}$$

The determination of the optimum control signal may be found by solving the above equations for p_2 and substituting the latter in equation (3-15). Solution of the differential equations with the boundary conditions given traces out the optimum trajectory in the phase plane, starting from arbitrary initial conditions $(x_1^{\,o}, x_2^{\,o})$. The switching curve is represented by the switch points (where u changes sign) when the initial conditions are varied. It is assumed that as the initial conditions are varied the system will settle to the final state $x_1 = x_2 = 0$. This implies that the optimal trajectories may be found by solving the set of differential equations in reverse time,

$$\begin{aligned}
\mathring{x}_1 &= -x_2 & x_1(0) &= 0\\
\mathring{x}_2 &= -\frac{L}{a}\,\mathrm{sgn}(p_2) & x_2(0) &= 0\\
\mathring{p}_1 &= -2x_1 & p_1(0) &= 0\\
\mathring{p}_2 &= p_1 & p_2(0) &= 0
\end{aligned}$$

Since these equations are in a convenient form for analogue computation, they are simulated on an analogue computer (TR-48 Analogue Computer, Electronic Associates Inc., Long Branch, N. J.) as shown in Fig. 2. The system of differential equations is unstable, so by using various small initial conditions the optimum responses of the system may be computed. By feeding the x_1, x_2 voltages into an X,Y recorder (Model HR-97 X-Y Recorder, Houston Instruments Co., Houston, Texas), the optimal trajectories were obtained as shown in Figs. 3, 4, and 5. The switching boundary, found by drawing a curve through the switch points, is approximately,

$$x_1 + \frac{ha}{L}\,x_2|x_2| = 0 \qquad (3\text{-}24)$$

where h = 0.44. This result is similar to the one obtained by Fuller[13] where h = 0.445. In reverse time, the trajectories enter the second quadrant of the phase plane on leaving the origin. There corresponds a symmetrical trajectory entering the fourth quadrant for each such trajectory. The trajectory has a

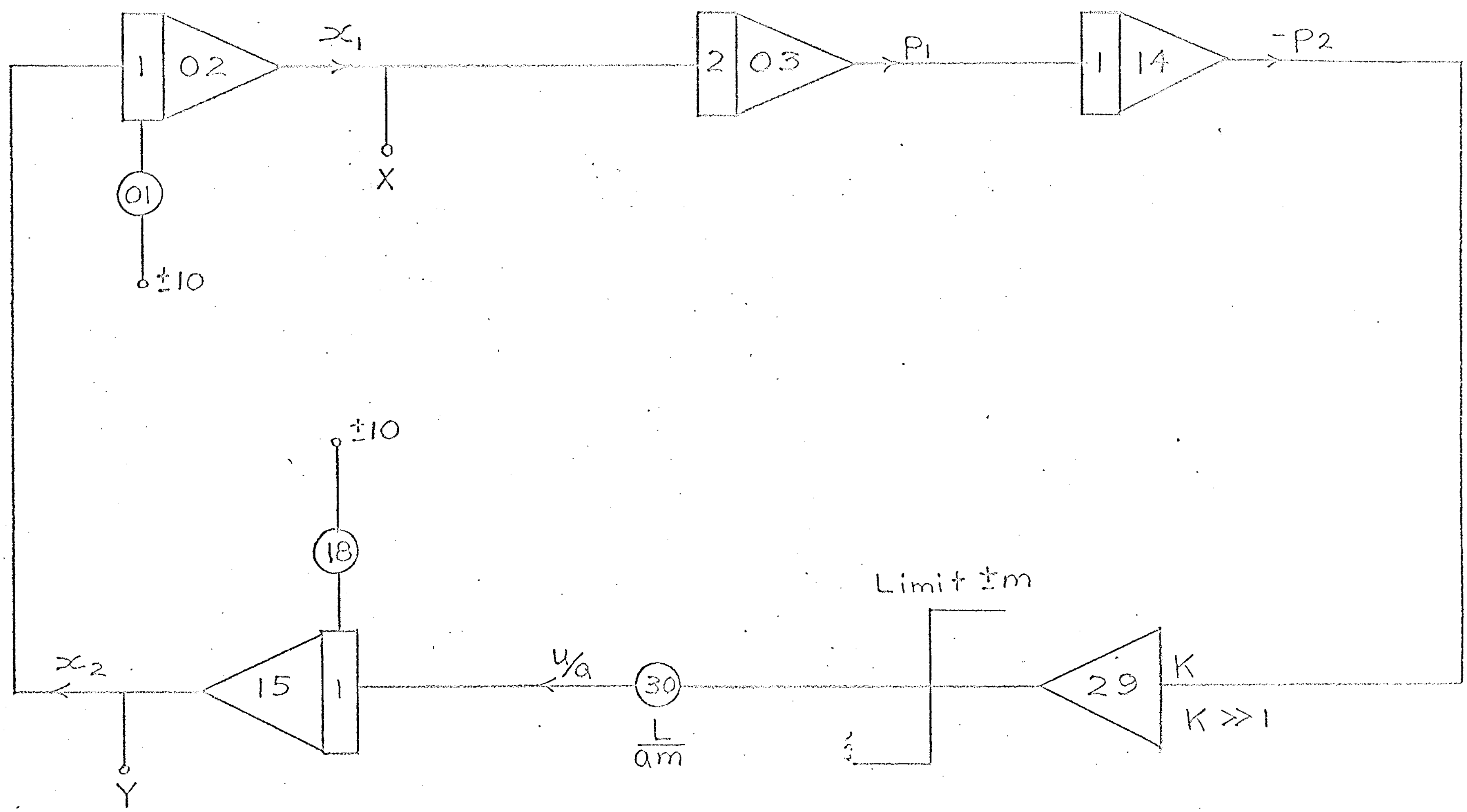


Fig. 2 Analogue computer setup for double integrator plant with integral-square error criterion.

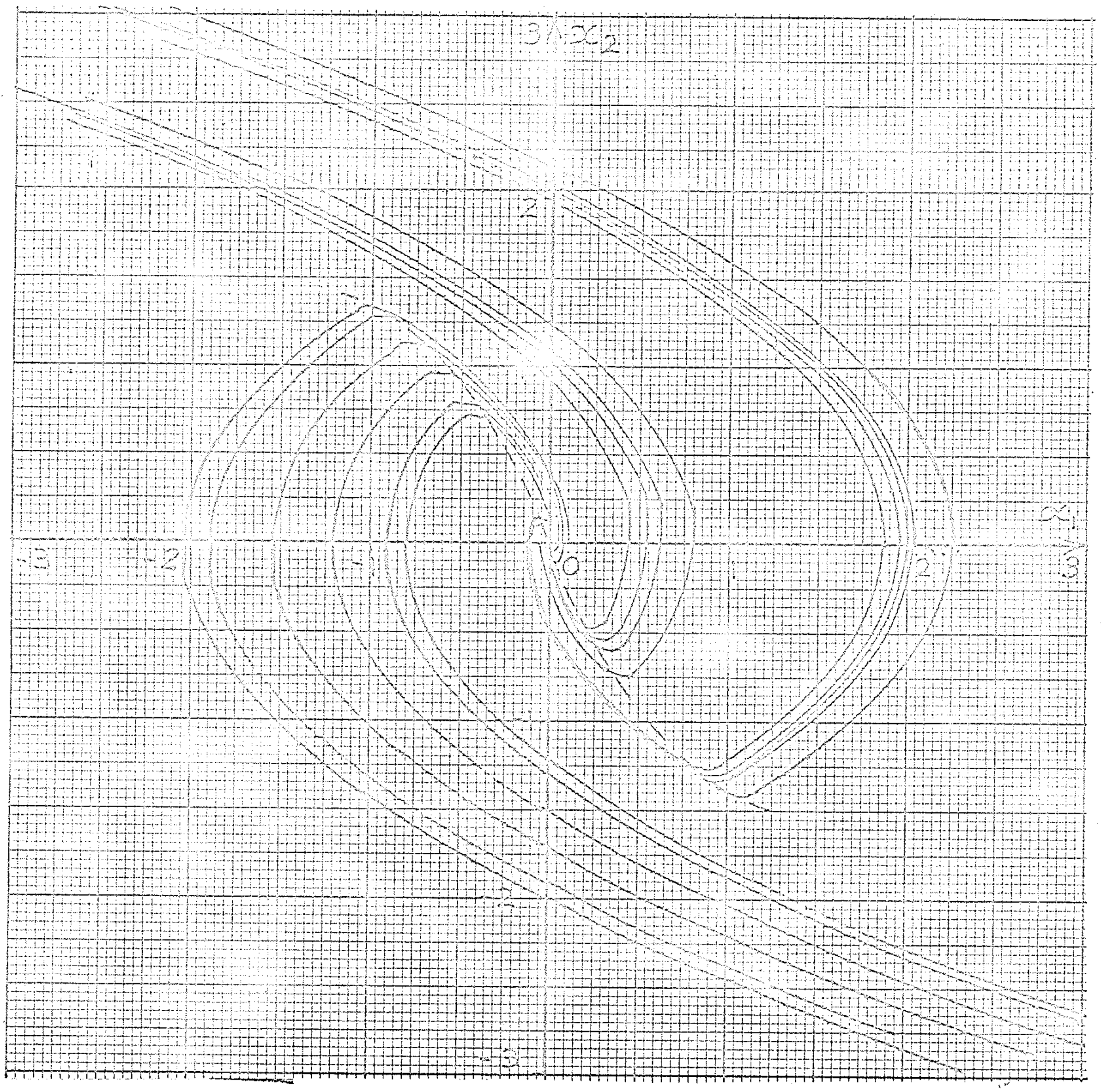


Fig. 3 Optimal trajectories for double integrator plant with integral-square error criterion ($L = 1$, $a = 1$).

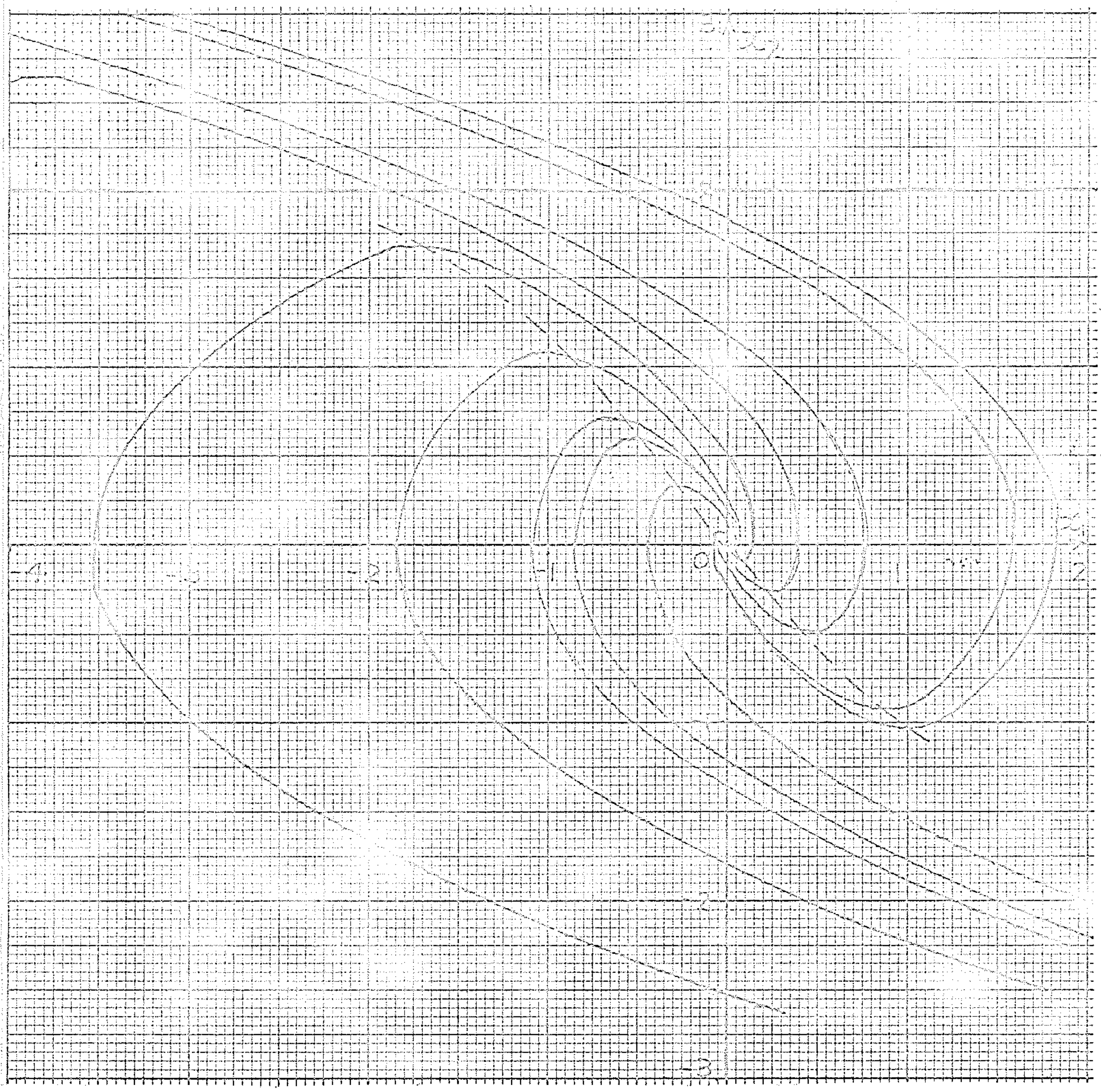


Fig. 4 Optimal trajectories for double integrator plant with integral-square error criterion ($L = 1$, $a = 2$).

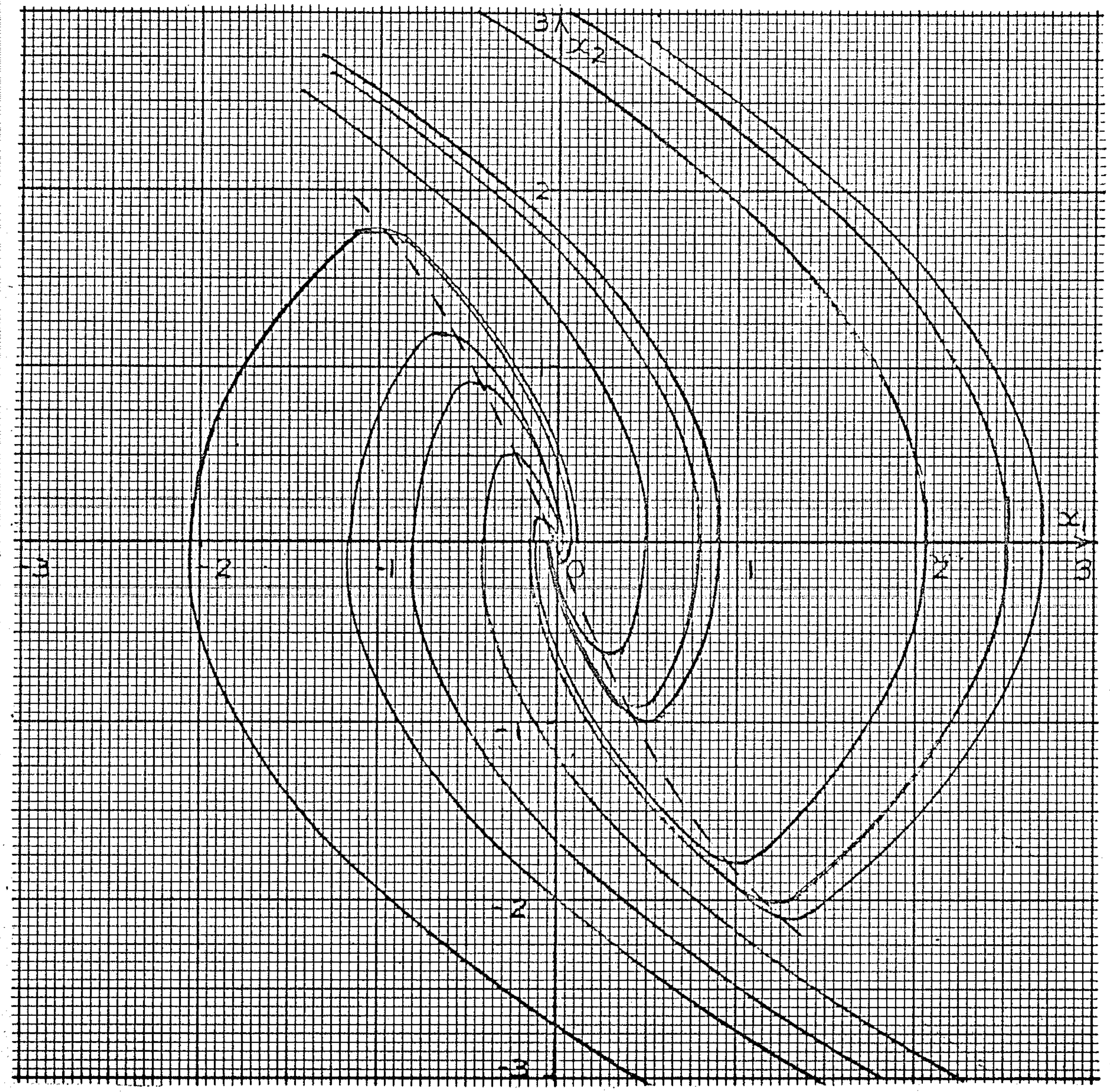


Fig. 5 Optimal trajectories for double integrator plant with integral-square error criterion (L = 2, a = 1).

general character representing the necessary conditions for optimal controls given by the maximum principle. Figures 6 and 7 show the step and ramp responses of this system.

Example 2. Minimum time criterion.

The time-optimal control problem is a classical one in the theory of control and it has been investigated by many authors.[4, 19-22] It is required to find the optimal control which will take the controlled plant from the initial state to the final state in the smallest transition time. The performance criterion takes on the form $I = t_f - t_o$ since $F(\underline{x},\underline{u},t) = 1$. An analytical treatment of the time-optimal control problem has been given by Pontryagin, et. al.,[4] and the switching boundary was found to be of the form,

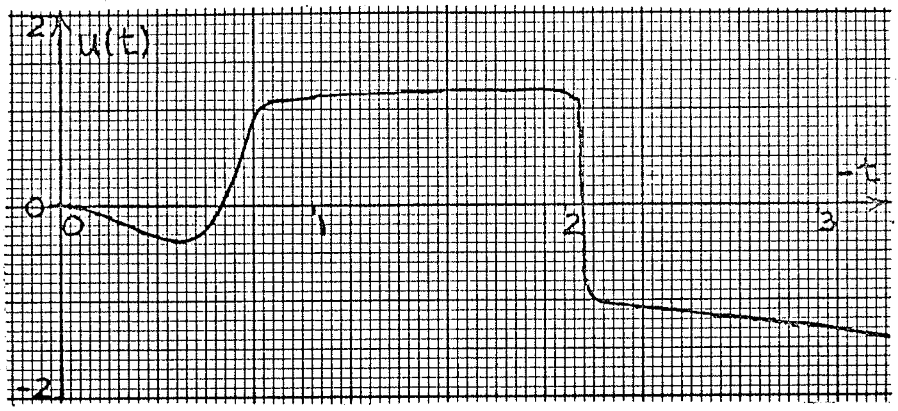
$$x_1 + 0.5x_2|x_2| = 0 \tag{3-25}$$

The following is the method used to obtain the analogue computer simulation of the same system as Ex. 1 but with the minimum time criterion. Hence,

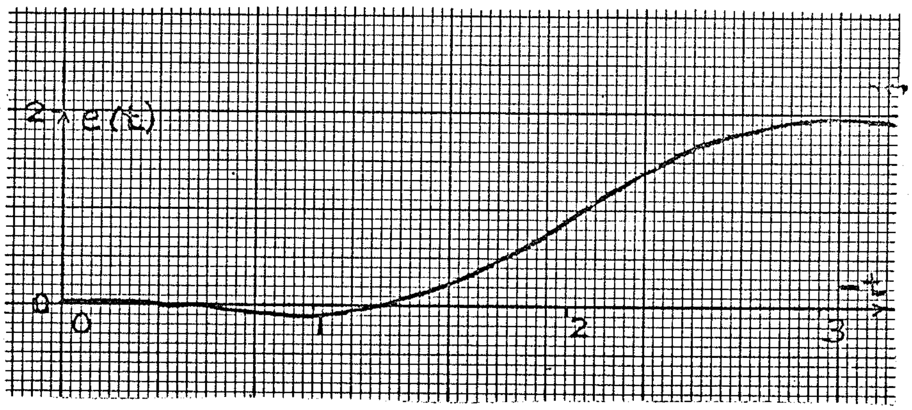
$$\mathring{x}_1 = x_2 \tag{3-26}$$
$$\mathring{x}_2 = \frac{u}{a} \tag{3-27}$$
$$\mathring{x}_3 = 1$$

and

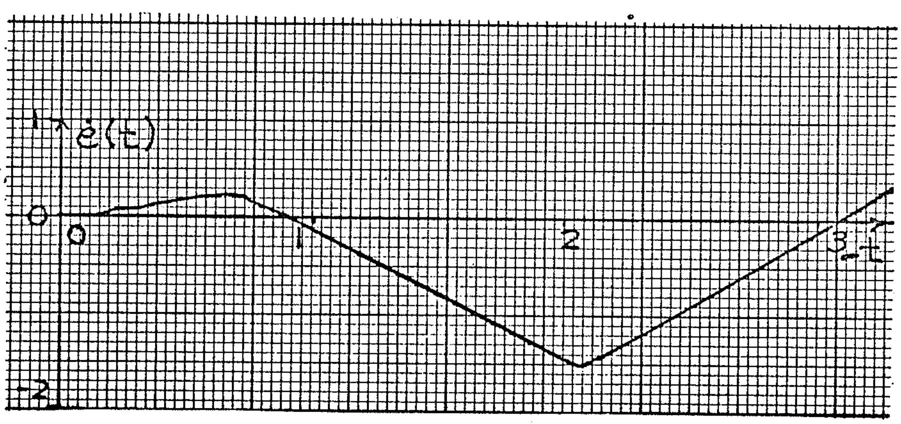
$$H = p_1x_2 + p_2\frac{u}{a} + p_3 \tag{3-29}$$



a) Control input versus time.

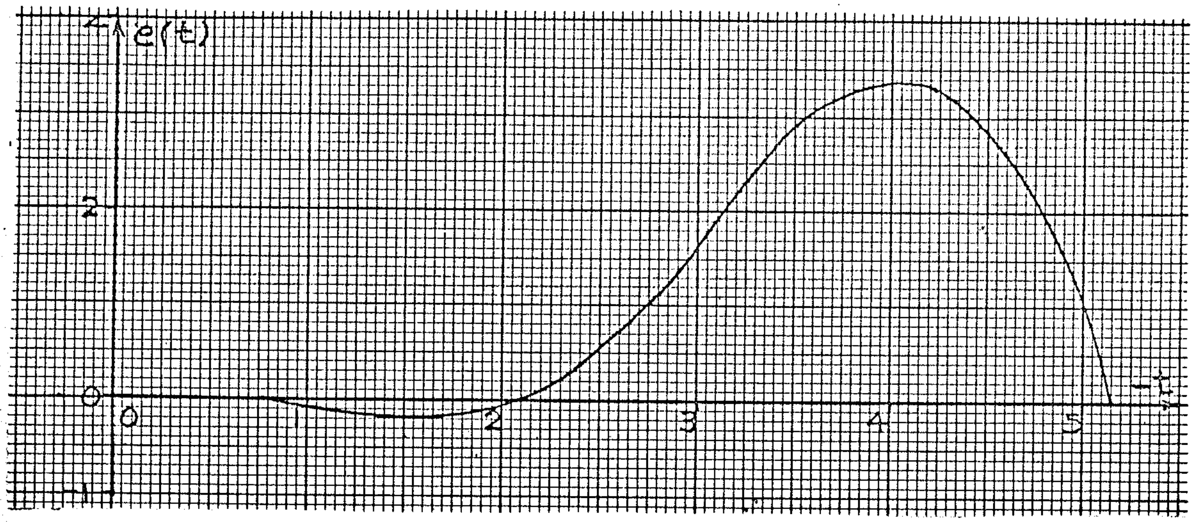


b) Error versus time.

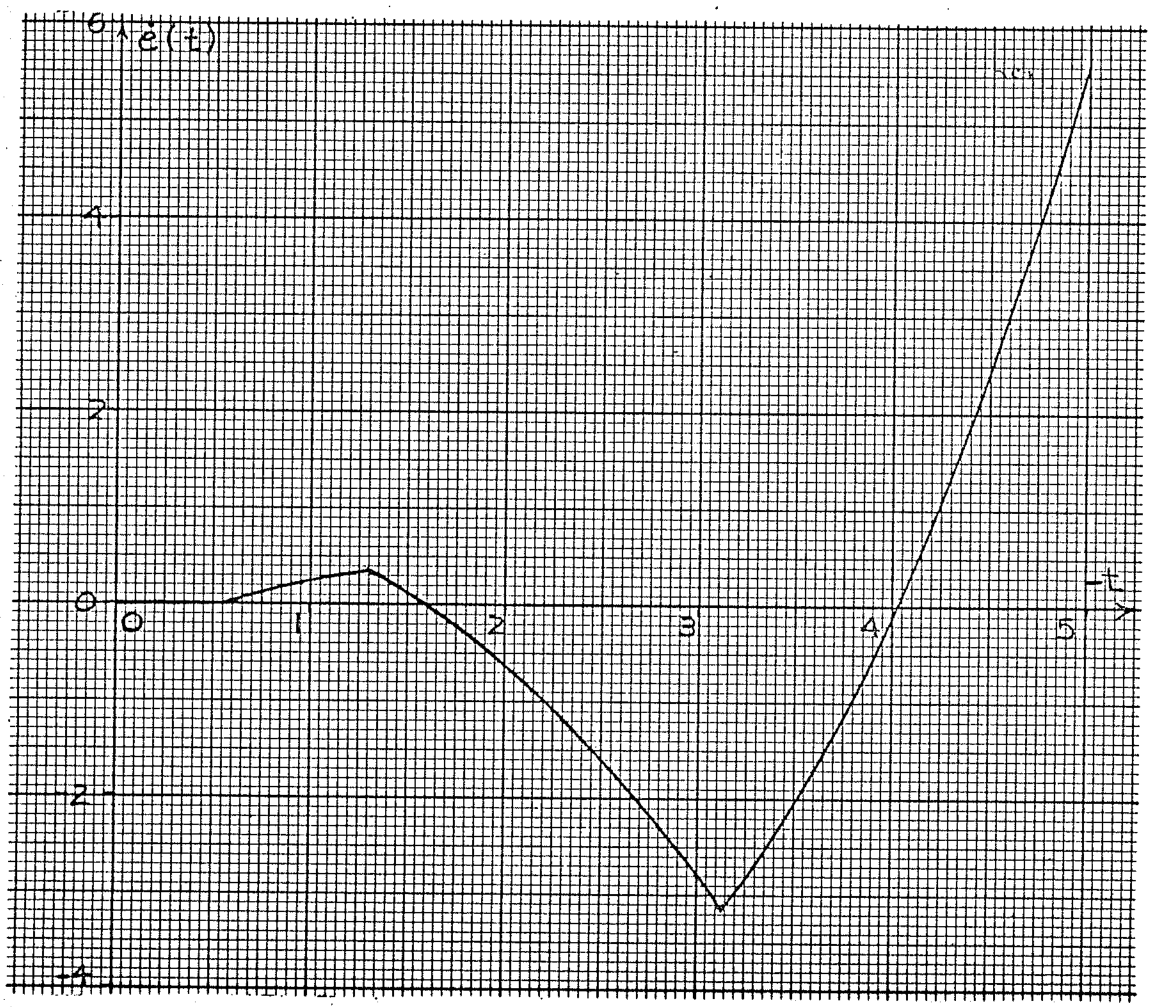


c) Error derivative versus time.

Fig. 6 Step response of system with double integrator plant and integral-square error criterion (L = 1, a = 1).



a) Error versus time.



b) Error derivative versus time.

Fig. 7 Ramp response of system with double integrator plant and integral-square error criterion (L = 1, a = 1).

with $\mathring{p}_1 = 0, \quad \mathring{p}_2 = -p_1, \quad \mathring{p}_3 = 0.$

Since there is no correspondence between $\mathring{p}_1$ and x_1, the switching points in the phase plane were not obtainable by this method. Therefore, an approximation was used:

$$I = \int_{t_0}^{t_f} |x_1|^m dt \quad m \to 0 \tag{3-30}$$

giving the following:

$$\mathring{x}_1 = x_2 \qquad x_1(0) = x_1^{\circ} \tag{3-31}$$

$$\mathring{x}_2 = \frac{u}{a} \qquad x_2(0) = x_2^{\circ} \tag{3-32}$$

$$\mathring{x}_3 = |x_1|^m \qquad x_3(0) = 0 \tag{3-33}$$

The Hamiltonian becomes,

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 |x_1|^m \tag{3-34}$$

with $$\mathring{p}_1 = -m|x_1|^{m-1} p_3 \operatorname{sgn}(x_1) \text{ for maximum } H \tag{3-35}$$

since $$\operatorname{sgn}(x) = \frac{d|x|}{dx}, \quad u = L\operatorname{sgn}(p_2)$$

$$\mathring{p}_2 = -p_1 \tag{3-36}$$

$$\mathring{p}_3 = 0 \tag{3-37}$$

In reverse time, the equations become:

$$\mathring{x}_1 = -x_2 \qquad x_1(0) = 0 \tag{3-38}$$

$$\mathring{x}_2 = -\frac{L}{a}\operatorname{sgn}(p_2) \qquad x_2(0) = 0 \tag{3-39}$$

$$\mathring{p}_1 = -m|x_1|^{m-1}\operatorname{sgn}(x_1) \qquad p_1(0) = 0 \tag{3-40}$$

$$\mathring{p}_2 = p_1 \qquad p_2(0) = 0 \tag{3-41}$$

and these are simulated on the analogue computer as shown in Fig. 8. By making m small, the switching boundary for the minimum time criterion was approximated, and it was found to be of the form of Fig. 3. With m = 0.1, the switching boundary was found to be,

$$x_1 + 0.53\frac{a}{L}x_2|x_2| = 0 \qquad (3\text{-}42)$$

which compares well with the classical result where h = 0.5. The method of attack and the analogue computer simulation for the double integrator plant with other performance criteria are given in Appendix A.

Comparison of results

The trajectories are found to be families of parabolas of the form $x_1 + kx_2|x_2| = C$ with their vertices on and symmetrical with the x_1 axis. Optimal trajectories are given by the equation $x_1 + \frac{ha}{L}x_2|x_2| = 0$ with u = -1 above the trajectory and u = +1 below the trajectory, i.e.,

$$u(t) = -\mathrm{sgn}(x_1 + \frac{ha}{L}x_2|x_2|) \qquad (3\text{-}43)$$

This type of control is called "bang-bang" control (A control u(t) is said to be "bang-bang" on $t_0 \leq t \leq t_f$ if $|u(t)|$ = constant (almost everywhere) with a possible finite number of sign changes (at most (n-1) changes of sign for an nth order plant with real

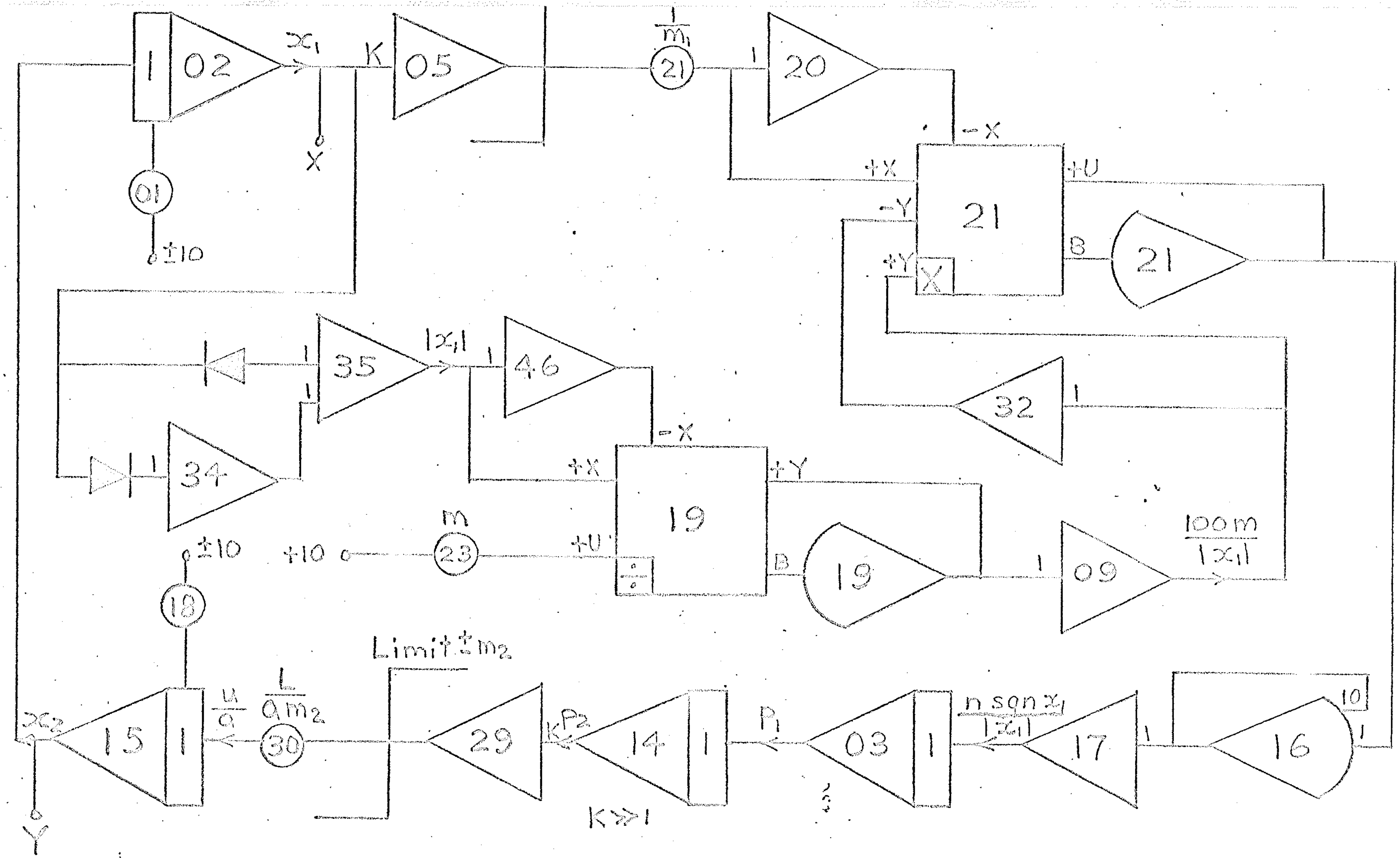


Fig. 8 Analogue computer setup for double integrator plant with minimum time criterion.

poles) on $t_0 \leq t \leq t_f$).[20] For each performance criterion, the value of h and the time for the system to go from ($x_1 = 0, x_2 = 0$) to ($x_1 = 1$, $x_2 = 0$) have been evaluated. The results are tabulated below.

TABLE I

PARAMETERS FOR VARIOUS PERFORMANCE CRITERIA

INDEX	h	TIME (sec.)
Minimum Time	0.53	2.30
IAE	0.49	2.32
Control Area	0.52	2.36
ISE	0.44	2.68
Weighted Control Area	0.48	2.57
ITAE	0.55	3.28
ITSE	0.42	3.42
ISTAE	0.45	2.84
ISTSE	0.44	3.69

From the table, it can be seen that the system with the minimum time criterion is optimum for rapid action, It may also be observed that by minimizing one performance index, many other performance indices may be approximately minimized at the same time. Thus, the optimum controls have been found for a double integrator plant with various performance criteria.

II. DAMPING ADDED. $G(p) = \frac{1}{p(p+b)}$

The second-order system with damping added was studied for the integral-square error and minimum time as performance criteria. Calculations are given here for the integral-square error criterion[14] with the results for the minimum time criterion in Appendix B.

The system as given in Fig. 1 has the plant transfer function $G(p) = \frac{1}{p(p+b)}$ (i.e. armature-controlled d-c motor) with the control input again restrained to be $|u| \leq L$. With $x_1 = x$ and $x_2 = \mathring{x}_1 = \frac{dx}{dt}$, the differential equation describing the system is

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} = u \qquad (3\text{-}44)$$

giving

$$\mathring{x}_1 = x_2 \qquad x_1(0) = x_1^{\,o} \qquad (3\text{-}45)$$

$$\mathring{x}_2 = u - bx_2 \qquad x_2(0) = x_2^{\,o} \qquad (3\text{-}46)$$

$$\mathring{x}_3 = x_1^{\,2} \qquad x_3(0) = 0 \qquad (3\text{-}47)$$

The Hamiltonian function becomes

$$H = p_1x_2 + p_2(u-bx_2) + p_3x_1^{\,2} \qquad (3\text{-}48)$$

with

$$u = L\,\mathrm{sgn}(p_2) \text{ to maximize } H \qquad (3\text{-}49)$$

and

$$\mathring{p}_1 = -2p_3x_1 \qquad (3\text{-}50)$$

$$\mathring{p}_2 = -p_1 + bp_2 \qquad (3\text{-}51)$$

$$\mathring{p}_3 = 0 \qquad (3\text{-}52)$$

In order to compute the response in reverse time, the following equations are set up on the analogue computer as shown in Fig. 9.

$$\mathring{x}_1 = -x_2 \qquad x_1(0) = 0 \tag{3-53}$$

$$\mathring{x}_2 = -L\,\mathrm{sgn}(p_2) + bx_2 \qquad x_2(0) = 0 \tag{3-54}$$

$$\mathring{p}_1 = -2x_1 \qquad p_1(0) = 0 \tag{3-55}$$

$$\mathring{p}_2 = p_1 - bp_2 \qquad p_2(0) = 0 \tag{3-56}$$

The optimal trajectories are given in Fig. 10.

The switching boundary for the system with the plant $G(p) = \frac{1}{p(p+b)}$ and integral-square criterion satisfies an equation given by Brennan and Roberts.[14] This equation is:

$$x_1 + \frac{h}{L}x_2|x_2|(1-0.701|x_2| + 0.426|x_2|^2 - 0.115|x_2|^3) = 0 \tag{3-57}$$

with $h = 0.445$. When the velocity is small, the optimal trajectories are similar to the double integrator plant. They ultimately become asymptotic to the horizontal line $x_2 = \frac{1}{b}$ from either outside or inside this line. The optimal trajectories for the minimum time criterion as given in Appendix B have the switching boundary that satisfies the equation given by Chang[22] as:

$$bx_1 + x_2 = \frac{u}{b}\ln(x_2 + \frac{u}{b}) \tag{3-58}$$

Curves (Figs. B-2, B-3) are given for b=1 (real roots, stable system, root locations: 0,-1) and b=-1 (real roots, unstable system, root locations: 0,1).

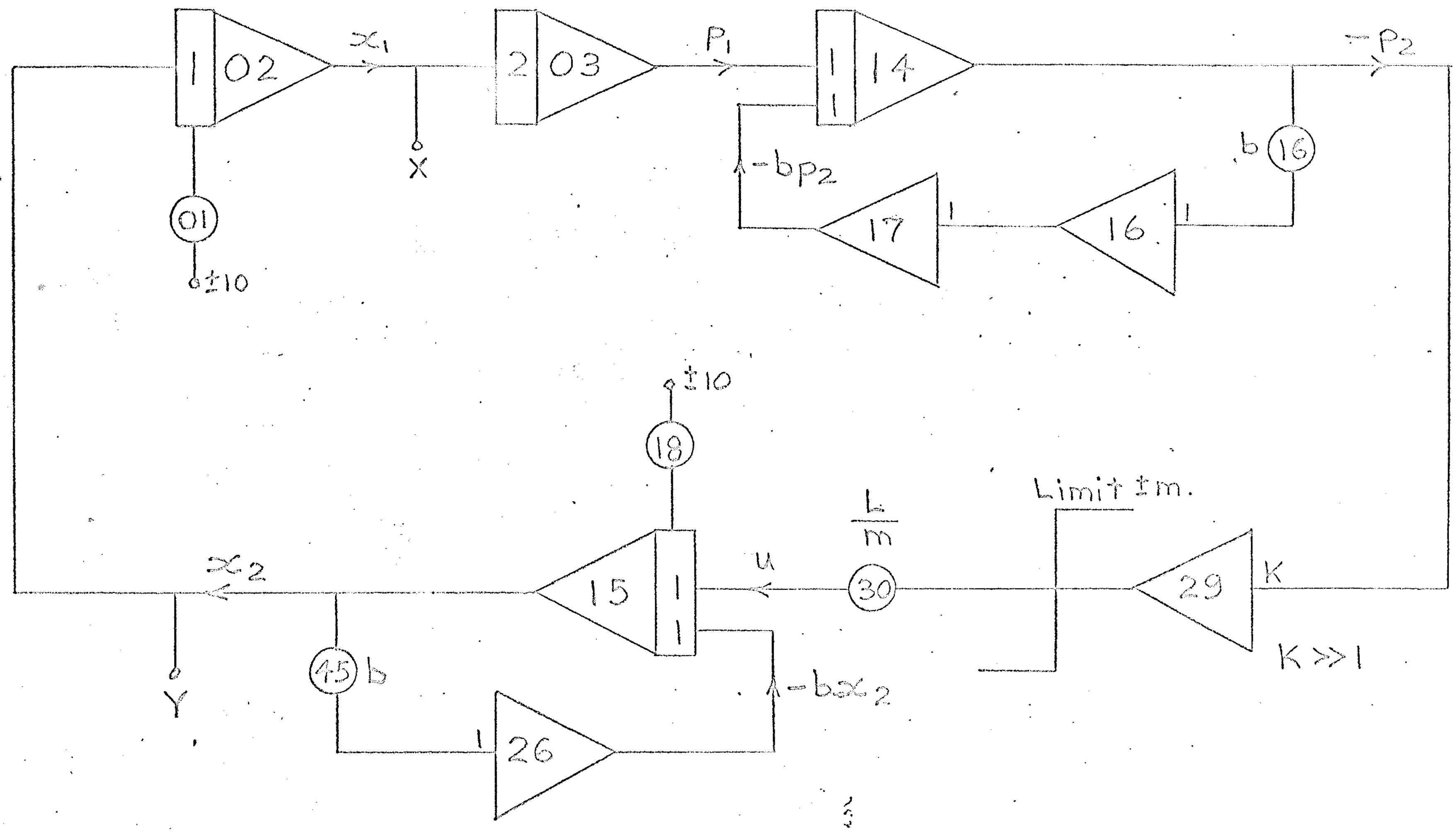
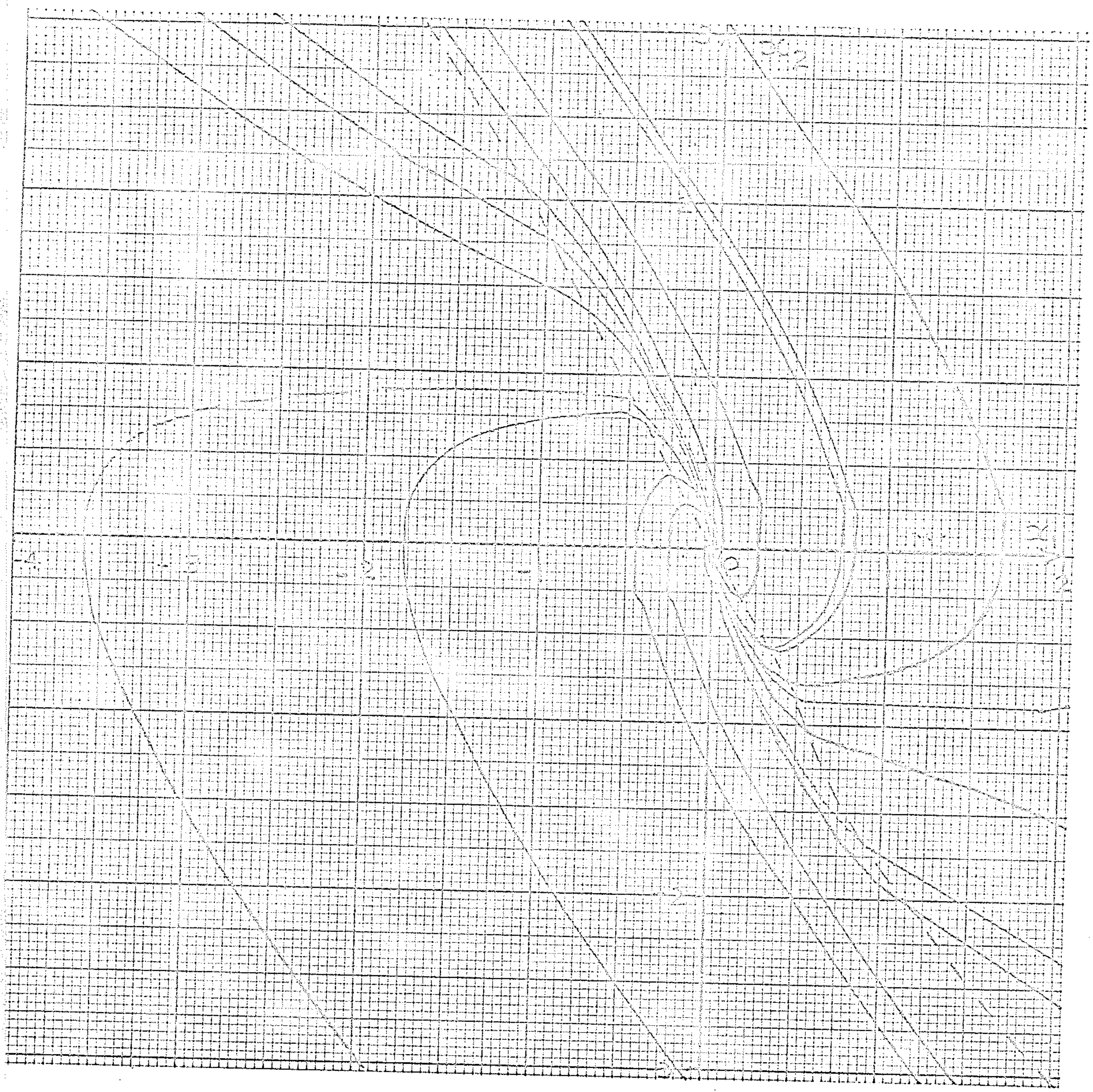


Fig. 9 Analogue computer setup for plant $G(p) = \frac{1}{p(p+b)}$ with integral-square error criterion.



ig. 10. Optimal trajectories for plant $G(p) = \frac{1}{p(p+b)}$ with integral-square error criterion ($L = 1$, $b = 1$).

III. GENERAL SYSTEM. $G(p) = \frac{1}{p^2 + bp + c}$

The control system of Fig. 1 was investigated with the plant of the form $G(p) = \frac{1}{p^2 + bp + c}$ with integral-square error and minimum time as the performance criteria. Calculations (similar to previous ones) with analogue computer setups are given in Appendix C. Also given are the optimal trajectories with the switching boundaries for various values of the damping coefficient ξ. ($0 \leq \xi \leq 1$). The results may be described as follows[22,24]:

As ξ increased from zero to one, the optimal trajectories changed from circular arcs ($\xi = 0$) to spiral arcs ($0 < \xi < 1$) to curves similar to the double integrator plant with damping added. The optimal trajectories also indicate the form of the switching boundary. With minimum time as the performance criterion, the switching boundary is of the same form as that for the integral-square error criterion.

CHAPTER IV

INVESTIGATION OF ADDITIONAL CRITERIA

In the previous chapter, performance criteria of time and error were investigated. The system of Fig. 1 with a plant $G(p) = \frac{1}{p^2+bp+c}$ and with a performance index of minimum effort is now treated using the maximum principle of Pontryagin. This requires that the system must be moved from its initial state to some final state in a certain time with minimum effort. Consider the performance criterion for minimum effort to be of the form,

$$I = \int_{t_o}^{t_f} |u|^n dt \qquad (4\text{-}1)$$

When the previous techniques of the maximum principle are applied to this type of functional, the Hamiltonian is

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3|u|^n \qquad (4\text{-}2)$$

with

$$\mathring{p}_1 = p_2c \qquad (4\text{-}3)$$

Since there is no correspondence between $\mathring{p}_1$ and x_1, the switching points of the optimal trajectories cannot be obtained by analogue computer simulation.

An error term is introduced into the functional and the investigation is turned to the problem of minimizing error and effort for second-order systems considering functionals of the form,[10, 11, 24].

$$I = \int_{t_o}^{t_f} (|x_1|^m + w|u|^n)dt \tag{4-4}$$

It is required to minimize I by the optimum choice of u, given the initial values of position and velocity. The coefficient w is a weighting constant.

Optimum control treated previously has been bang-bang control. Two additional classes of control are investigated; the continuous mode of control and the bang-bang with coasting mode of control.[10]

Class A - Continuous Mode of Control.

A continuous mode of control appears for functionals of the form,

$$I_{A-1} = \int_{t_o}^{t_f} (|x_1|^m + wu^2)dt \tag{4-5}$$

The system of Fig. 1 with a plant $G(p) = \frac{1}{p^2+bp+c}$ and a performance criterion,

$$I = \int_{t_o}^{t_f} (|x_1| + wu^2)dt \tag{4-6}$$

is considered by application of the maximum principle:

$$\mathring{x}_1 = x_2$$
$$\mathring{x}_2 = u-bx_2-cx_1 \tag{4-7}$$
$$\mathring{x}_3 = |x_1| + wu^2$$

giving the Hamiltonian,

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3(|x_1| + wu^2) \tag{4-8}$$

with

$$\mathring{p}_1 = cp_2-p_3\operatorname{sgn}(x_1)$$
$$\mathring{p}_2 = -p_1 + bp_2 \tag{4-9}$$
$$\mathring{p}_3 = 0$$

Differentiating H with respect to u, and with $p_3 = -1$, the Hamiltonian reaches a maximum when $u = p_2/2w$. If the control input is constrained, i.e. $|u| \leq L$, then the optimal control (continuous mode) as given in Fig. 11 is

$$u = -L \quad \text{if } \frac{p_2}{2w} \leq -L$$
$$u = \frac{p_2}{2w} \quad \text{if } -L \leq \frac{p_2}{2w} \leq L$$
$$u = +L \quad \text{if } \frac{p_2}{2w} > L$$

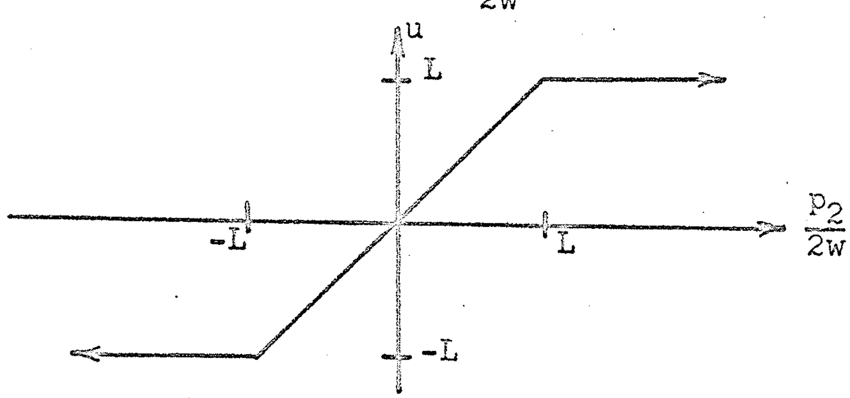


Fig. 11 - Optimal control for class A.

In reverse time, the equations are

$$
\begin{aligned}
\mathring{x}_1 &= -x_2 & x_1(0) &= 0 \\
\mathring{x}_2 &= -u+bx_2+cx_1 & x_2(0) &= 0 \\
\mathring{p}_1 &= -cp_2-\mathrm{sgn}(x_1) & p_1(0) &= 0 \\
\mathring{p}_2 &= p_1-bp_2 & p_2(0) &= 0
\end{aligned}
\qquad (4\text{-}11)
$$

which are simulated[25] on the analogue computer as shown in Fig. 12. The analogue computer simulation with calculations and curves for

$$I_{A-2} = \int_{t_o}^{t_f} (x_1^{\,2} + wu^2)dt \qquad (4\text{-}12)$$

is given in Appendix D.

Three cases were considered with $w = 1$ and $|u| \leq 1$ as follows:

1). $b = c = 0$, $G(p) = \frac{1}{p^2}$, Figs. 13, D-2.

2). $b \neq 0$, $c = 0$, $G(p) = \frac{1}{p(p+1)}$, Figs. 14, D-3.

3). $b \neq 0$, $c \neq 0$, underdamped $\frac{b}{2} < c$, $G(p) = \frac{1}{p^2+p+1}$, Figs. 15, D-4.

These figures for the three cases show the optimal trajectories with the switch curve between positive u and negative u and the boundary curves between the saturated region ($u = \pm 1$) and the linear region ($|u| \leq 1$) of Fig. 11. (Boundary curves and the switch

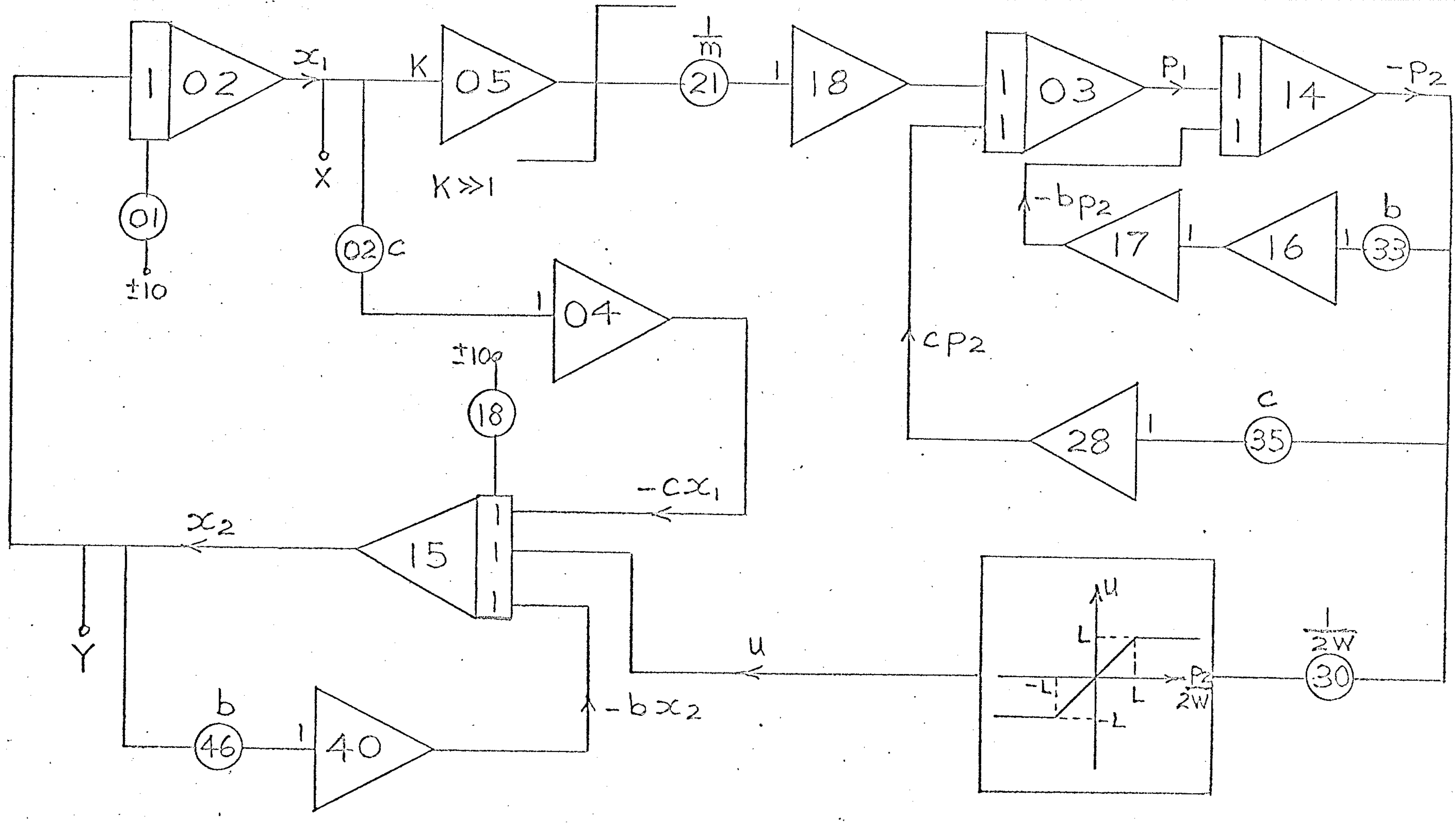
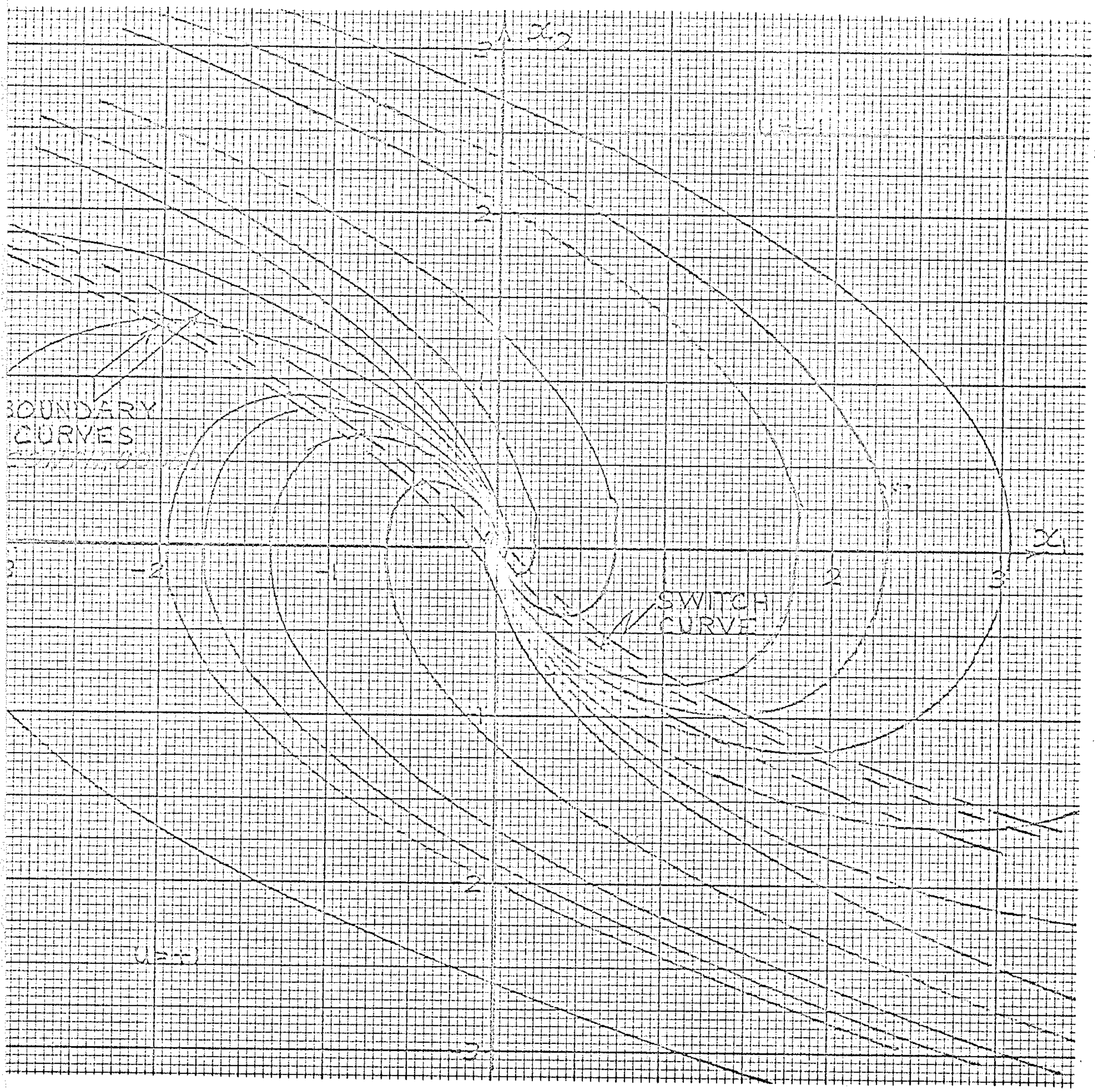


Fig. 12 Analogue computer setup for general system with performance index I_{A-1}.



ig. 13 Optimal trajectories for the system with performance index I_{A-1}. (Case 1).

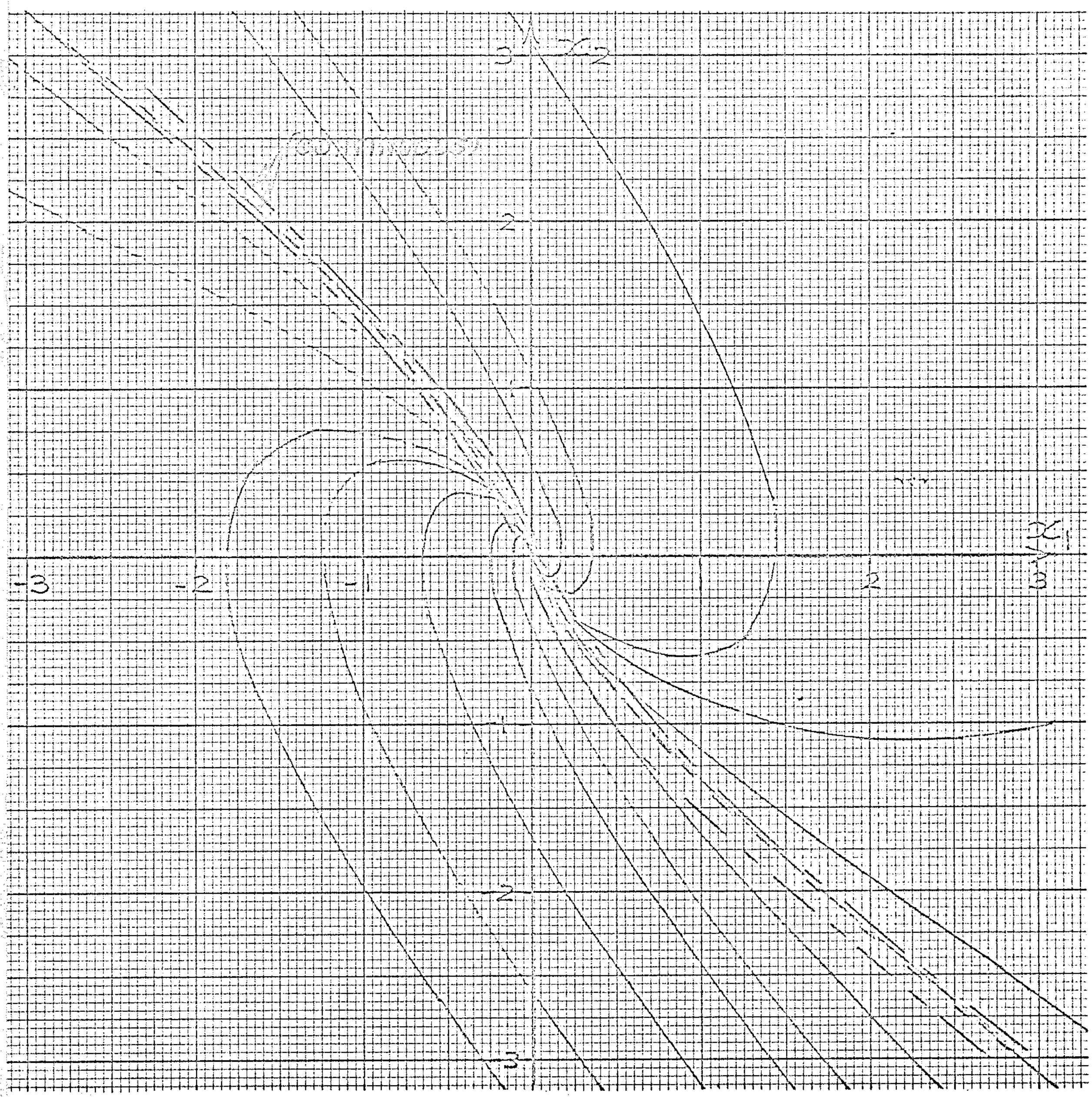


Fig. 14 Optimal trajectories for the system with performance index I_{A-1}. (Case 2).

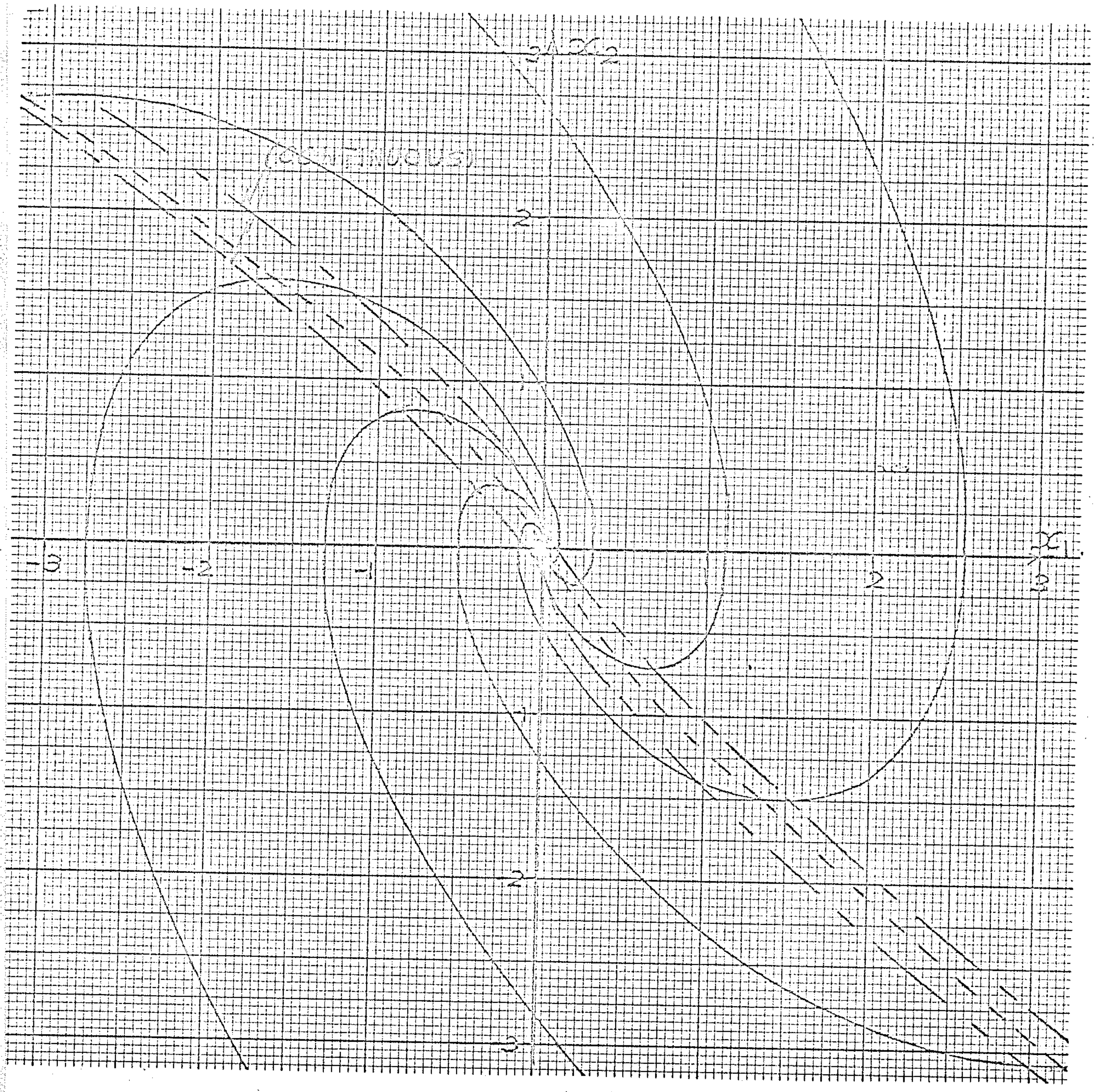


Fig. 15. Optimal trajectories for the system with performance index I_{A-I}. (Case 3).

curve were obtained by monitoring u on the analogue computer, and marking the appropriate points). The curves for the two functionals considered are of similar form indicating the type of optimal control required. As w is decreased, less weight is given to the magnitude of the force and the optimal control approaches the bang-bang control. If w is increased, a heavy penalty is imposed on u forcing u to be small.[10]

Class B - Bang-bang with Coasting Mode of Control.

A bang-bang with coasting mode of control appears for functionals of the form,

$$I_B = \int_{t_o}^{t_f} (|x_1|^m + w|u|)dt \tag{4-13}$$

The system of Fig. 1 with the plant $G(p) = \dfrac{1}{p^2+bp+c}$ and the functional,

$$I_{B-1} = \int_{t_o}^{t_f} (|x_1| + w|u|)dt \tag{4-14}$$

is considered by application of the maximum principle as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u-bx_2-cx_1 \\ \dot{x}_3 &= |x_1| + w|u| \end{aligned} \tag{4-15}$$

giving the Hamiltonian,

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3(|x_1| + w|u|) \tag{4-16}$$

with

$$\mathring{p}_1 = p_2 c - p_3 \operatorname{sgn}(x_1)$$
$$\mathring{p}_2 = -p_1 + bp_2 \qquad (4\text{-}17)$$
$$\mathring{p}_3 = 0$$

Differentiating H with respect to u, and with $p_3=-1$, the Hamiltonian reaches a maximum when $\operatorname{sgn}(u) = \operatorname{sgn}(p_2)$. When the control input is constrained, i.e. $|u| \leq L$, then the optimal control (bang-bang with coasting) as given in Fig. 16 is

$$u = 0 \quad \text{if} \left|\frac{p_2}{w}\right| < 1$$
$$|u| = L \quad \text{if} \left|\frac{p_2}{w}\right| > 1 \qquad (4\text{-}18)$$

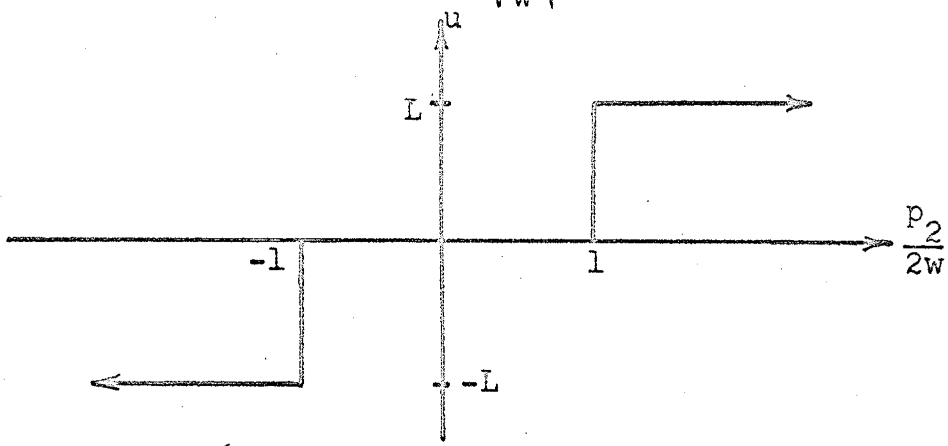


Fig. 16 - Optimal control for class B.

In reverse time, the equations are

$$\mathring{x}_1 = -x_2 \qquad x_1(0) = 0$$
$$\mathring{x}_2 = -u + bx_2 + cx_1 \qquad x_2(0) = 0$$
$$\mathring{p}_1 = -\operatorname{sgn}(x_1) - cp_2 \qquad p_1(0) = 0 \qquad (4\text{-}19)$$
$$\mathring{p}_2 = p_1 - bp_2 \qquad p_2(0) = 0$$

which are simulated on the analogue computer as shown in Fig. 17. The analogue computer simulation with calculations and curves for

$$I_{B-2} = \int_{t_o}^{t_f} (x_1^2 + w|u|)dt \qquad (4\text{-}20)$$

is given in Appendix D.

Three cases were considered with $w = 1$ and $|u| \leq 1$ as follows:

1) $b = c = 0$, $G(p) = \frac{1}{p^2}$, Figs. 18, D-6.
2) $b \neq 0$, $c = 0$, $G(p) = \frac{1}{p(p+1)}$, Figs. 19, D-7.
3) $b \neq 0$, $c \neq 0$, underdamped $\frac{b}{2} < c$, $G(p) = \frac{1}{p^2+p+1}$, Figs. 20, D-8.

The system trajectories for the cases considered above demonstrate that they are limited to the application of maximum positive or negative force ($|u| = L$), or to zero force (coasting, $u = 0$) as in Fig. 16. As w is increased, the area of the region $u = 0$ (coasting) increases, and when w decreases, the region for zero force decreases since less weight is placed on the application of force. Generally, when the system is damped, the time to reach the origin increases and more force will be required.[10]

The classes of optimal control considered in this chapter imply that the character of u in the functional basically determines the optimal control criterion.

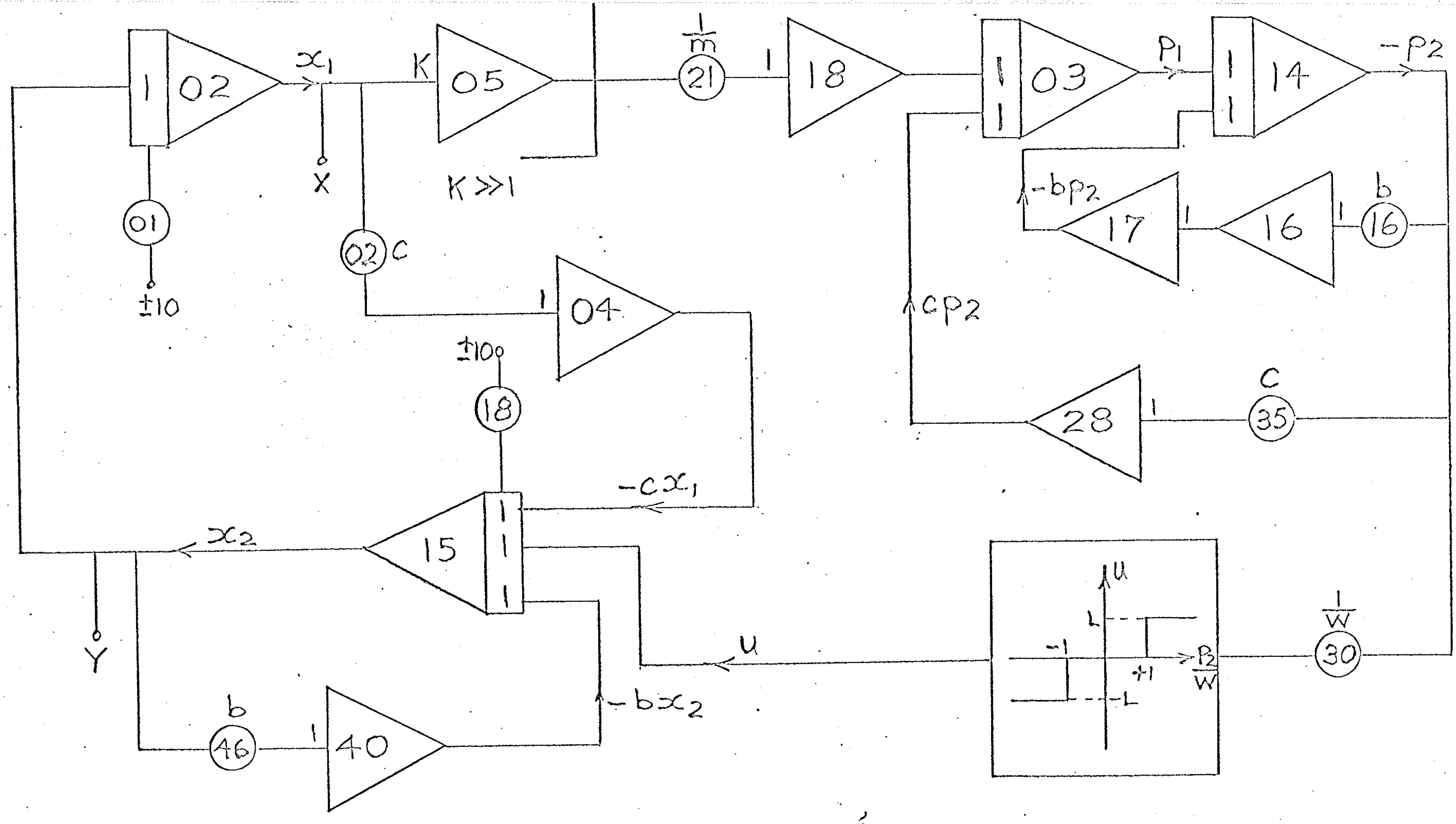
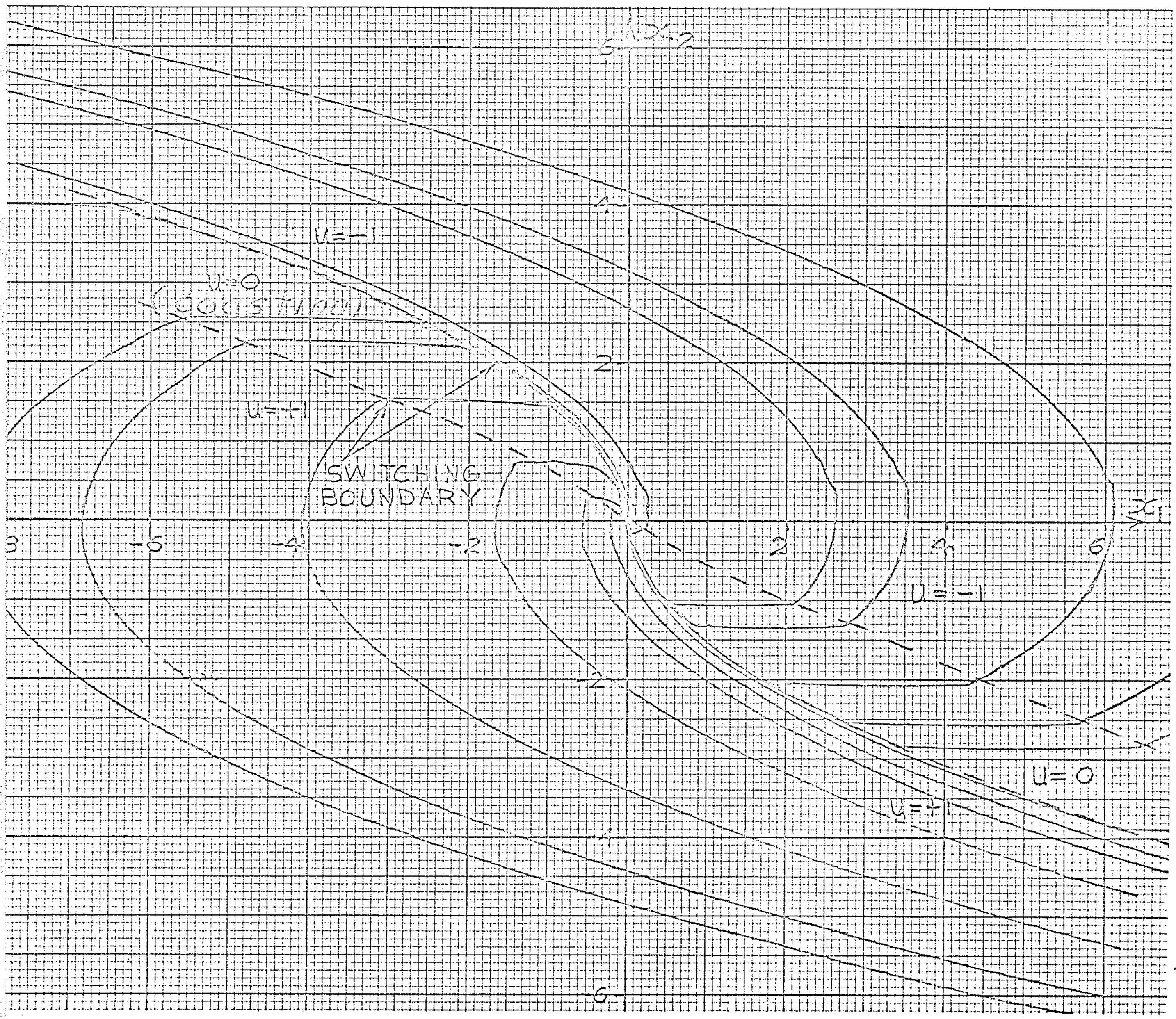


Fig. 17 Analogue computer setup for the general system with performance index I_{B-1}.



'ig. 18. Optimal trajectories for the system with performance index I_{B-1}. (Case 1).

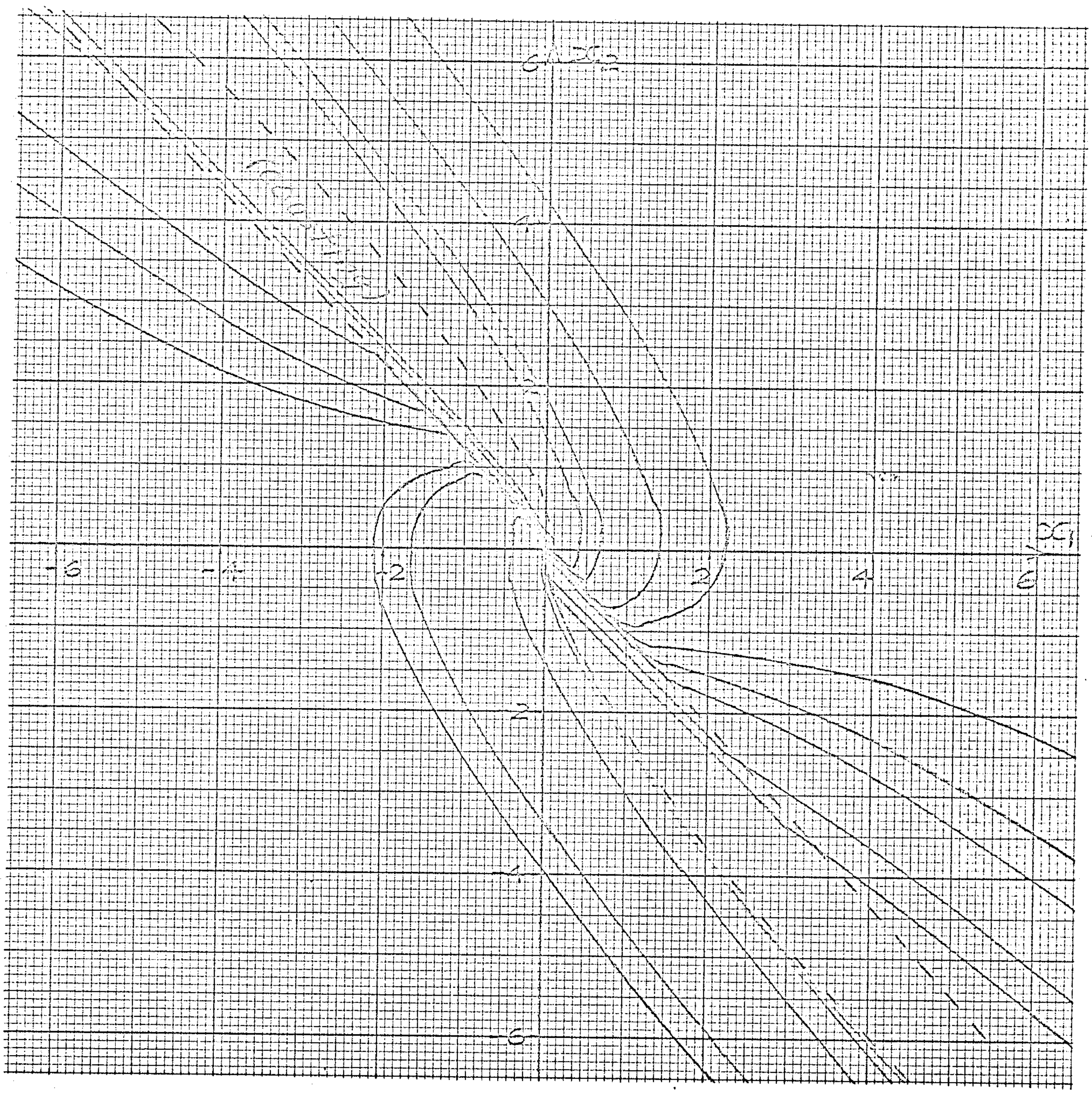


Fig. 19 Optimal trajectories for the system with performance index I_{B-1}. (Case 2).

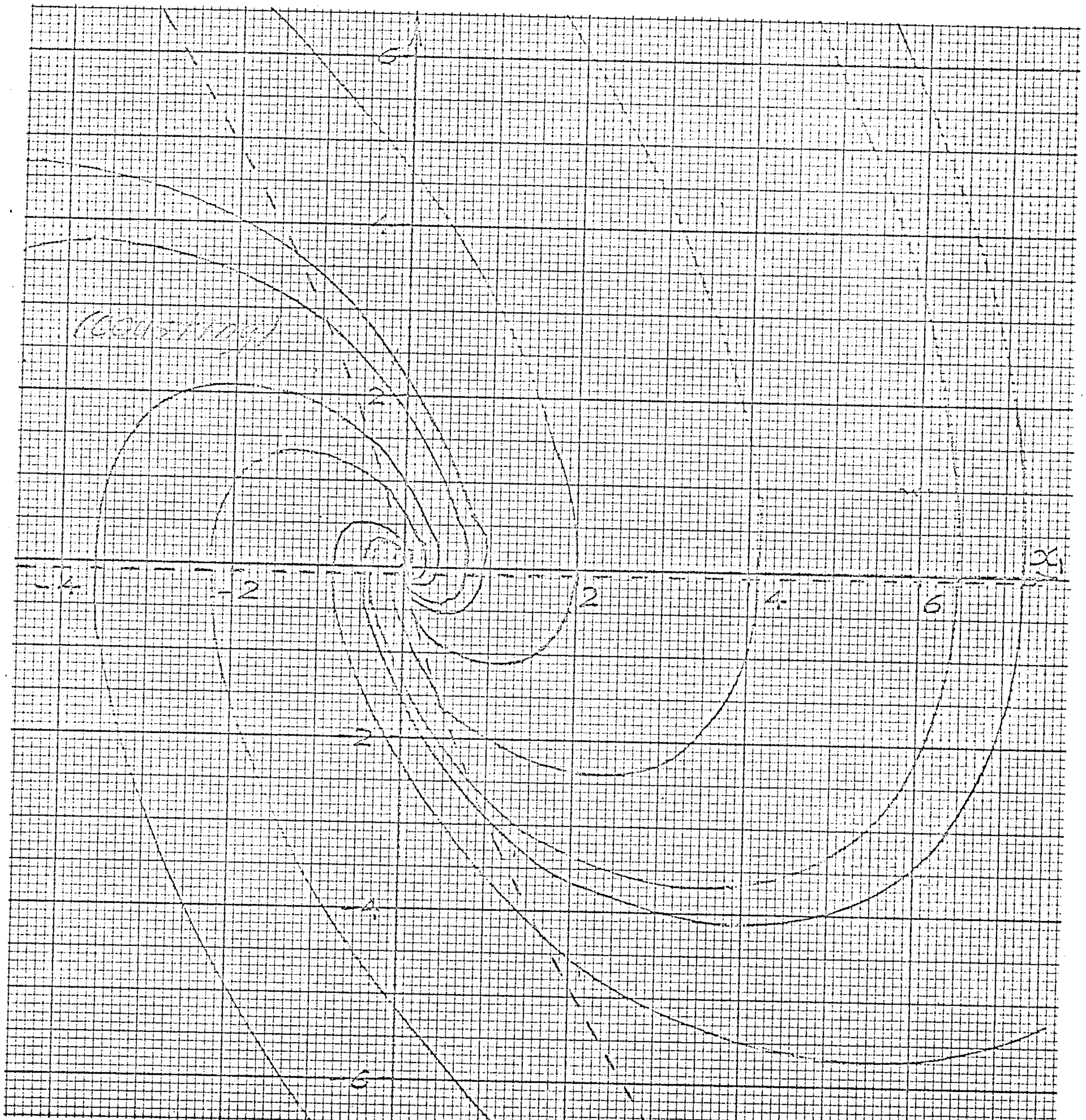


Fig. 20 Optimal trajectories for the system with performance index I_{B-1}. (Case 3).

CHAPTER V

HIGHER-ORDER SYSTEMS

Practical extension of the application of the maximum principle to determine the optimal control for higher-order systems is limited primarily by the difficulty of representing a surface in a phase space of higher dimensions.[26-29] For third-order systems, a phase space with position, velocity, and acceleration as the co-ordinates is required. In principle, this is a straight-forward extension of the phase plane analysis useful for problems in second-order systems. As the system equations become more complex, a difficulty arises in the expression of the switching surfaces.

The system of Fig. 1 is considered with a plant $G(p) = \frac{1}{p^3}$ and the control input is constrained i.e. $|u| \leq L$. This plant was selected because it may be used as a first approximation to other higher-order transfer functions. Integral-square error and minimum time are used as the performance criteria, with the following results:

a) Integral-square Error Criterion.

The solution according to the maximum principle is:

$$\dot{x}_1 = x_2 \qquad \dot{x}_3 = u$$
$$\dot{x}_2 = x_3 \qquad \dot{x}_4 = x_1^2 \qquad (5\text{-}1)$$

and $$H = p_1x_2 + p_2x_3 + p_3u + p_4x_1^2 \tag{5-2}$$

with
$$\mathring{p}_1 = -2p_4x_1 \qquad \mathring{p}_3 = -p_2$$
$$\mathring{p}_2 = -p_1 \qquad \mathring{p}_4 = 0 \tag{5-3}$$

and, for maximum H, $$u = L\operatorname{sgn}(p_3) \tag{5-4}$$

In reverse time, with $p_4 = -1$,

$$\begin{aligned}
\mathring{x}_1 &= -x_2 & x_1(0) &= 0\\
\mathring{x}_2 &= -x_3 & x_2(0) &= 0\\
\mathring{x}_3 &= -L\operatorname{sgn}(p_3) & x_3(0) &= 0\\
\mathring{p}_1 &= -2x_1 & p_1(0) &= 0\\
\mathring{p}_2 &= p_1 & p_2(0) &= 0\\
\mathring{p}_3 &= p_2 & p_3(0) &= 0
\end{aligned} \tag{5-5}$$

These equations are simulated on the analogue computer as shown in Fig. 21.

b) Minimum Time Criterion.

The solution according to the maximum principle is:

$$\mathring{x}_1 = x_2 \qquad \mathring{x}_3 = u$$
$$\mathring{x}_2 = x_3 \qquad \mathring{x}_4 = |x_1|^m \quad m \to 0 \tag{5-6}$$

where $$H = p_1x_2 + p_2x_3 + p_3u + p_4|x_1|^m \tag{5-7}$$

with
$$\mathring{p}_1 = -m|x_1|^{m-1}p_4\operatorname{sgn}(x_1) \qquad \mathring{p}_3 = -p_2$$
$$\mathring{p}_2 = -p_1 \qquad \mathring{p}_4 = 0 \tag{5-8}$$

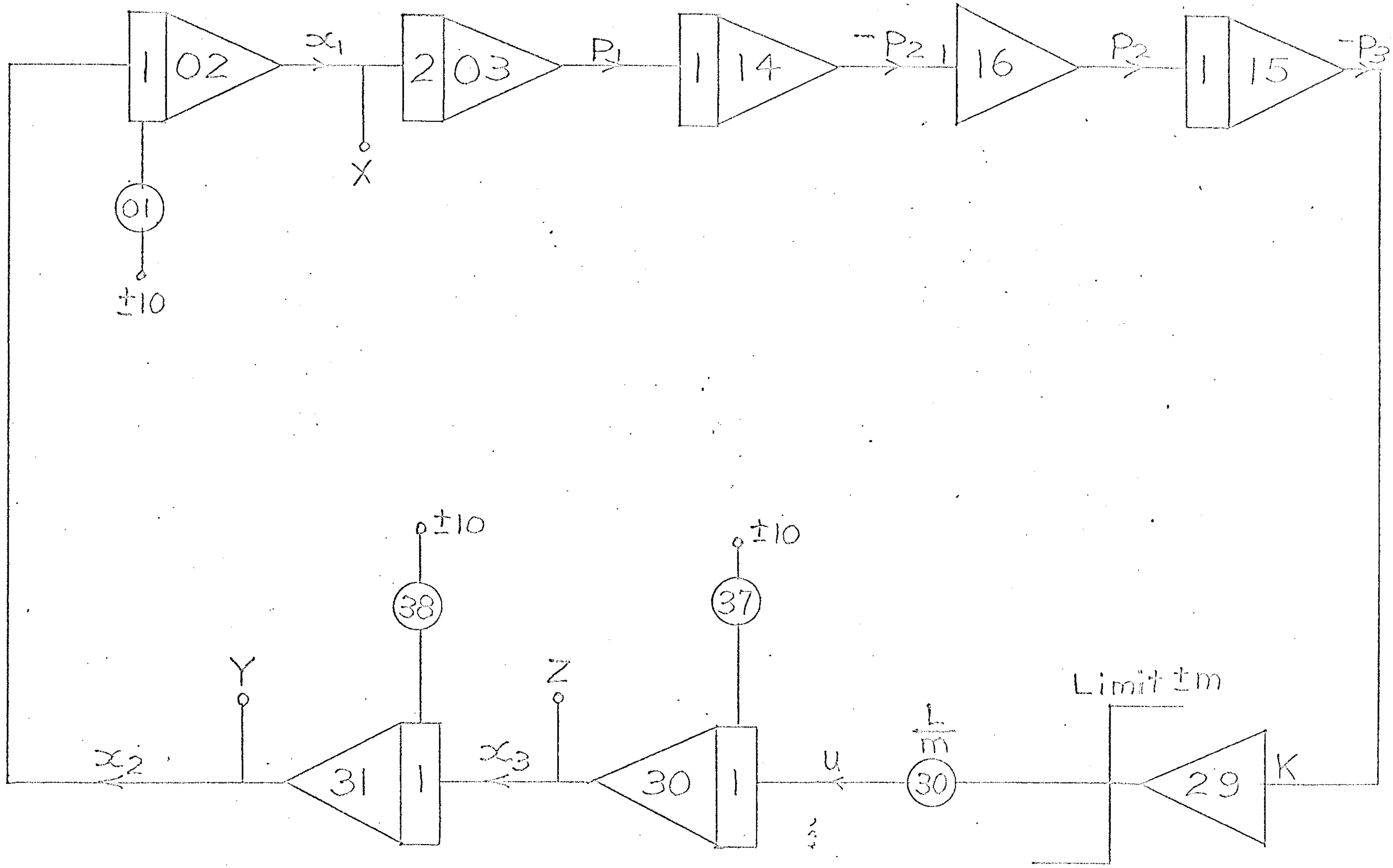


Fig. 21 Analogue computer setup for the system $G(p) = \frac{1}{p^3}$ and integral-square error criterion.

and $$u = L\,\mathrm{sgn}(p_3) \tag{5-9}$$

In reverse time, with $p_4 = -1$,

$$\begin{aligned}
\mathring{x}_1 &= -x_2 & x_1(0) &= 0 \\
\mathring{x}_2 &= -x_3 & x_2(0) &= 0 \\
\mathring{x}_3 &= -L\,\mathrm{sgn}(p_3) & x_3(0) &= 0 \\
\mathring{p}_1 &= -m|x_1|^{m-1}\mathrm{sgn}(x_1) & p_1(0) &= 0 \\
\mathring{p}_2 &= p_1 & p_2(0) &= 0 \\
\mathring{p}_3 &= p_2 & p_3(0) &= 0
\end{aligned} \tag{5-10}$$

These equations are simulated on the analogue computer as in Fig. 22.

An attempt was made to fill in the phase space with optimal trajectories for better comprehension, but it was found that the only reasonable curve that could be obtained was the so-called zero trajectory. This trajectory is given in Fig. 23 for the integral-square error and the minimum time criterion. These zero trajectories are considered as the switching surfaces and are given approximately by:[30]

$$F = x_1 + \frac{x_3^3}{3} + wx_3x_2 + w\left(\frac{x_3^2}{2} + wx_2\right)^{3/2} = 0 \tag{5-11}$$

where w is determined as

$$\begin{aligned}
w &= +1 \qquad \text{for} \quad (x_2 + \tfrac{1}{2}x_3|x_3|) > 0 \\
w &= -1 \qquad \text{for} \quad (x_2 + \tfrac{1}{2}x_3|x_3|) < 0
\end{aligned} \tag{5-12}$$

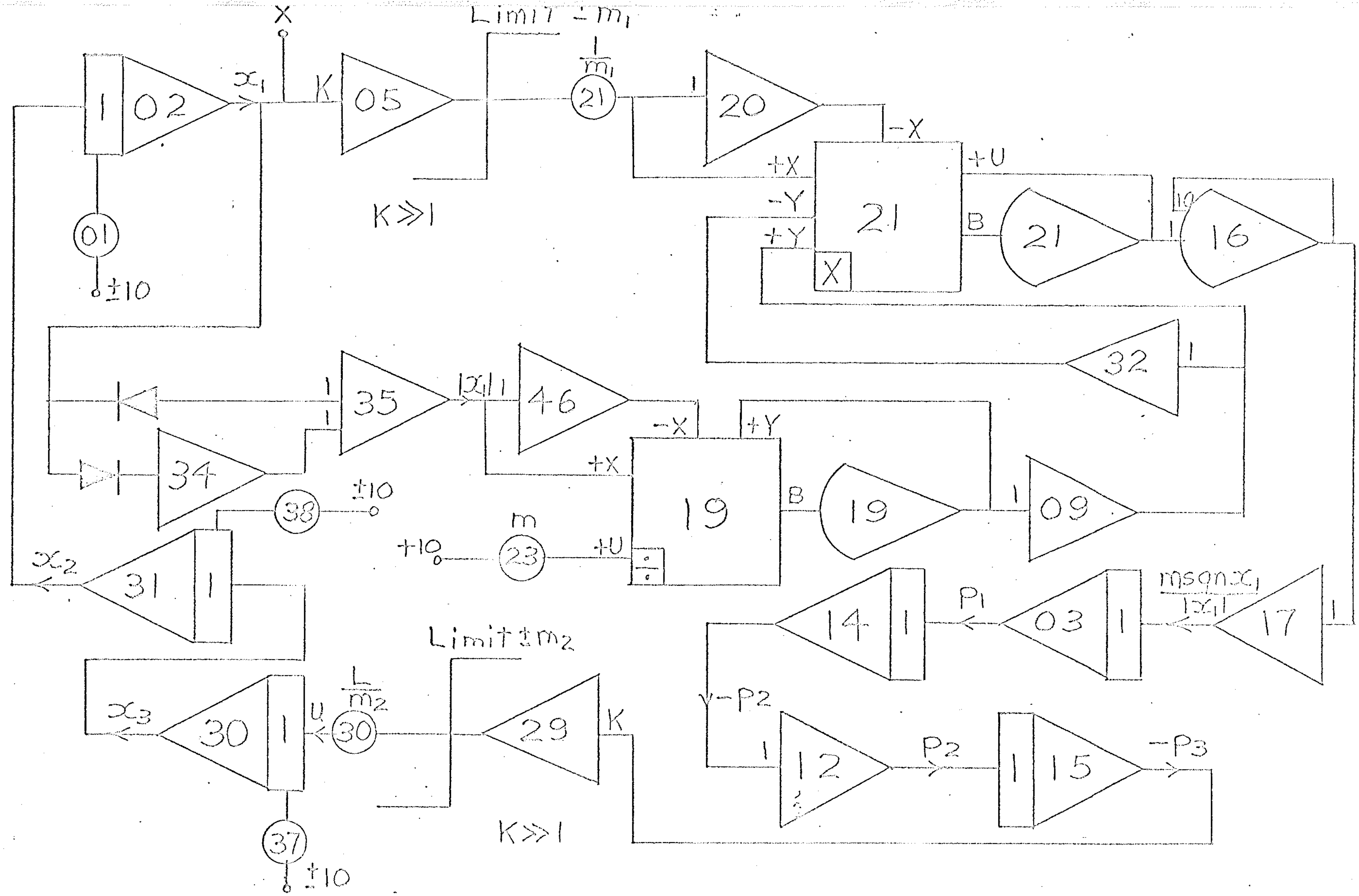
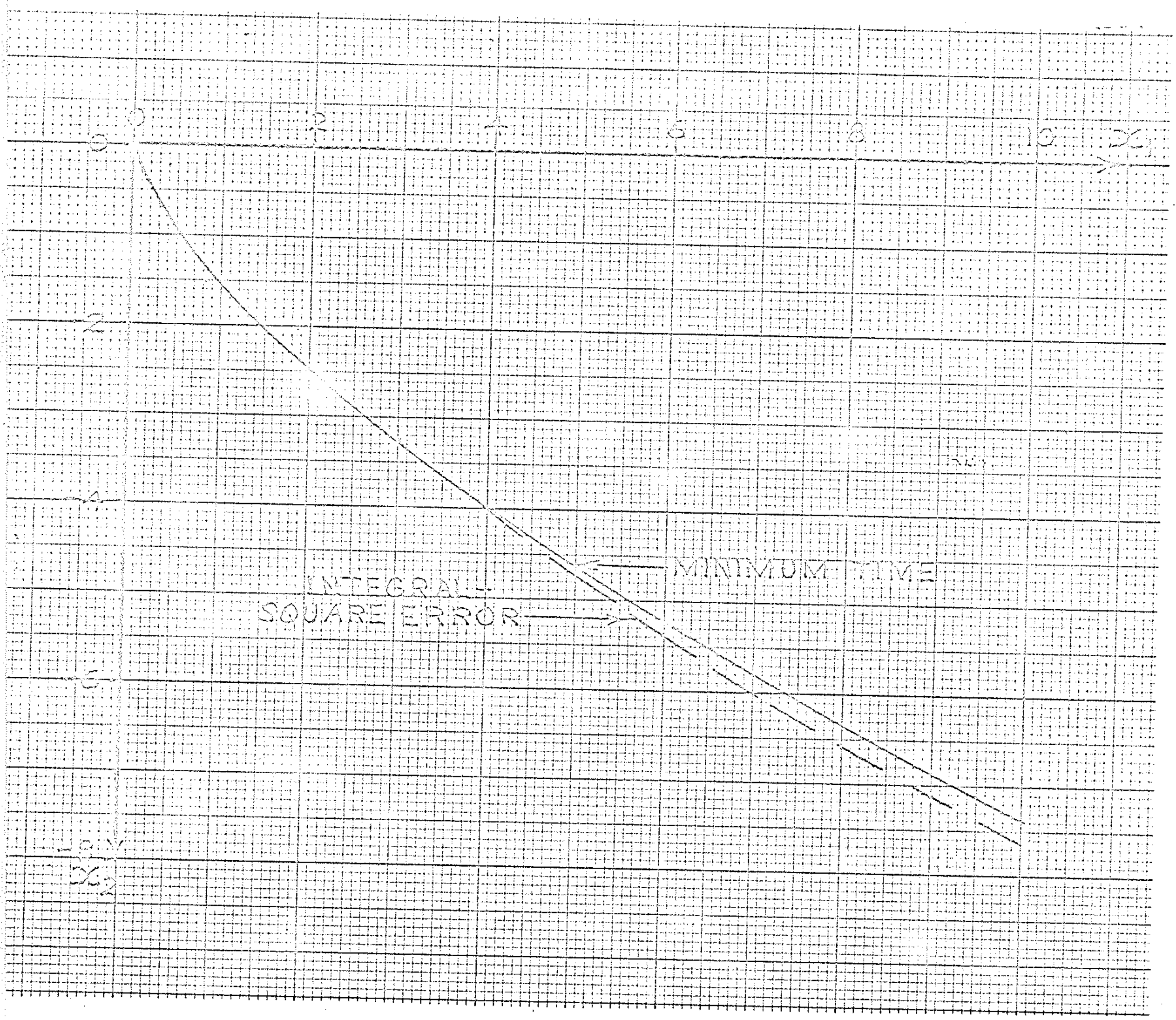


Fig. 22 Analogue computer setup for the system $G(p) = \frac{1}{p^3}$ and minimum time criterion.



g. 23 Optimal trajectories for the system $G(p) = \frac{1}{p^3}$.

The curves given are for $w = -1$, where curves for $w = +1$ are anti-symmetrical with these curves. Therefore, the control function is

$$u = -\operatorname{sgn}F = -\operatorname{sgn}\left(x_1 + \frac{x_3^3}{3} + wx_3x_2 + w\left(\frac{x_3^2}{2} + wx_2\right)^{3/2}\right) = 0 \qquad (5\text{-}13)$$

The co-ordinates of the curves as found from the analogue computer are given in Table II.

TABLE II

CO-ORDINATES OF TRAJECTORIES (w=-1)

	Integral-square error		Minimum time	
x_3	x_2	x_1	x_2	x_1
(+)	(-)	(+)	(-)	(+)
1.0	0.73	0.38	0.60	0.27
2.0	2.18	1.68	2.31	1.80
3.0	4.43	4.55	5.06	5.72
4.0	7.48	9.63	7.81	10.50

Table III gives the times taken on the trajectory for various changes in x_3.

TABLE III

TRAJECTORY TIMES

x_3	Integral-square error Time (sec.)	Minimum time Time (sec.)
0.0 to 1.0	2.16	1.30
1.0 to 2.0	1.00	1.20
2.0 to 3.0	0.86	1.10
3.0 to 4.0	0.76	0.92
Total	4.78	4.52

From Table III, it can be observed that the system with the minimum time criterion is slightly faster than the system with the integral-square error criterion indicating that it is optimum for rapid action.

The extension of Pontryagin's maximum principle to higher-order systems is seen to be very straight-forward; however, difficulties in interpretation arise due to the three or more dimensions of the necessary state space.

CHAPTER VI

CONCLUSIONS

An analogue computer, using reverse time techniques, has been used to determine the optimal controls for various performance criteria. Since the controlled input is constrained, the optimum control system often behaves like a relay servo.

For systems characterized by second-order differential equations, the phase plane provided a fairly simple method to observe the optimal trajectories and switching boundaries. When the system is of higher order, with more than two state variables, the analysis by phase space thecniques becomes increasingly complex since the switching boundary will be a surface in n-dimensional space.

The fact that the Pontryagin maximum principle exists to compute the optimal control does not mean necessarily that the results will enable the designer to realize this control, but it provides the highest possible performance with which the performance of actual control systems can be compared. In actual physical problems, one usually wishes to optimize many performance indices at the same time; the maximum principle may therefore provide a system optimum in every reasonable engineering sense. The efforts expended on optimal control are justified since they present a basis for practical control system design.

Pontryagin's maximum principle provides a quite useful and convenient method of solving optimal control problems, as may be observed in Appendix E, where it is compared to the techniques of Newton, Gould, and Kaiser.[31]

In conclusion, areas considered to require further study are:

1) Investigation of the analytic solutions of the optimum control to obtain exact expressions for the optimum controller.

2) Study of the synthesis of these optimum controllers.

3) Applications of the maximum principle to sampled-data systems and systems with stochastic inputs.

APPENDICES

APPENDIX A

CALCULATIONS FOR THE DOUBLE INTEGRATOR PLANT FOR VARIOUS PERFORMANCE CRITERIA

1). Integral of time multiplied by squared error. (ITSE)

$$I = \int_0^\infty te^2(t)dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = tx_1^{\ 2}$$

$$H = p_1x_2 + p_2\frac{u}{a} + p_3tx_1^{\ 2}$$

$$u = L\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -2p_3tx_1$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -2tx_1 \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

as in Fig. A-1.

The optimal trajectories are of the form shown in Fig. 3, with the switching boundary described by $x_1 + 0.42\frac{a}{L}x_2|x_2| = 0$.

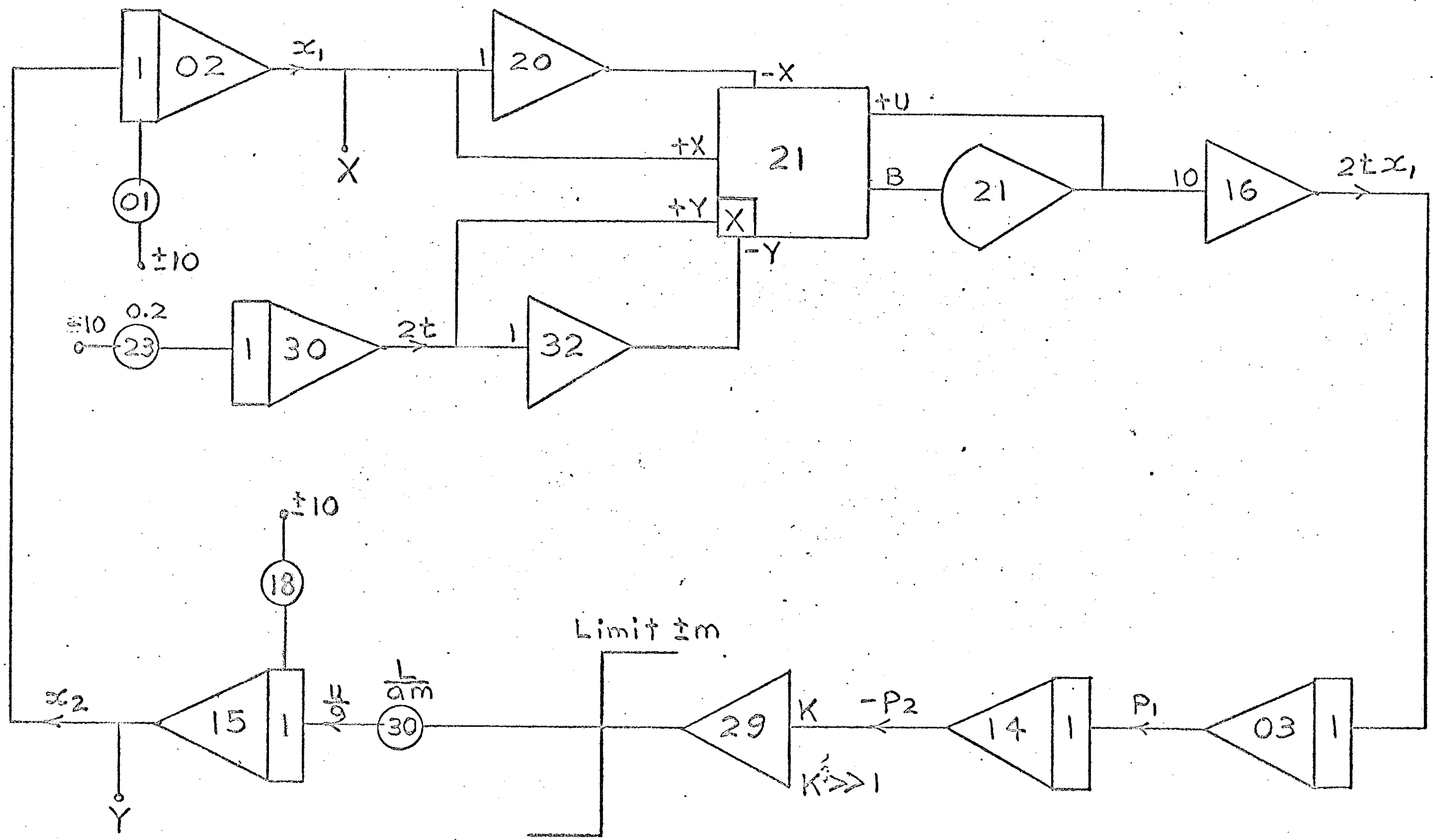


Fig. A-1 Analogue computer setup for the double integrator plant and ITSE criterion.

2). <u>Integral of squared time multiplied by squared value of error.</u> (ISTSE)

$$I = \int_0^\infty t^2 e^2(t)\,dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = t^2 x_1^2$$

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 t^2 x_1^2$$

$$u = L\,\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -2p_3 t^2 x_1$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -2t^2 x_1 \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

as in Fig. A-2.

The switching boundary is described by $x_1 + 0.43\frac{a}{L}x_2|x_2| = 0$.

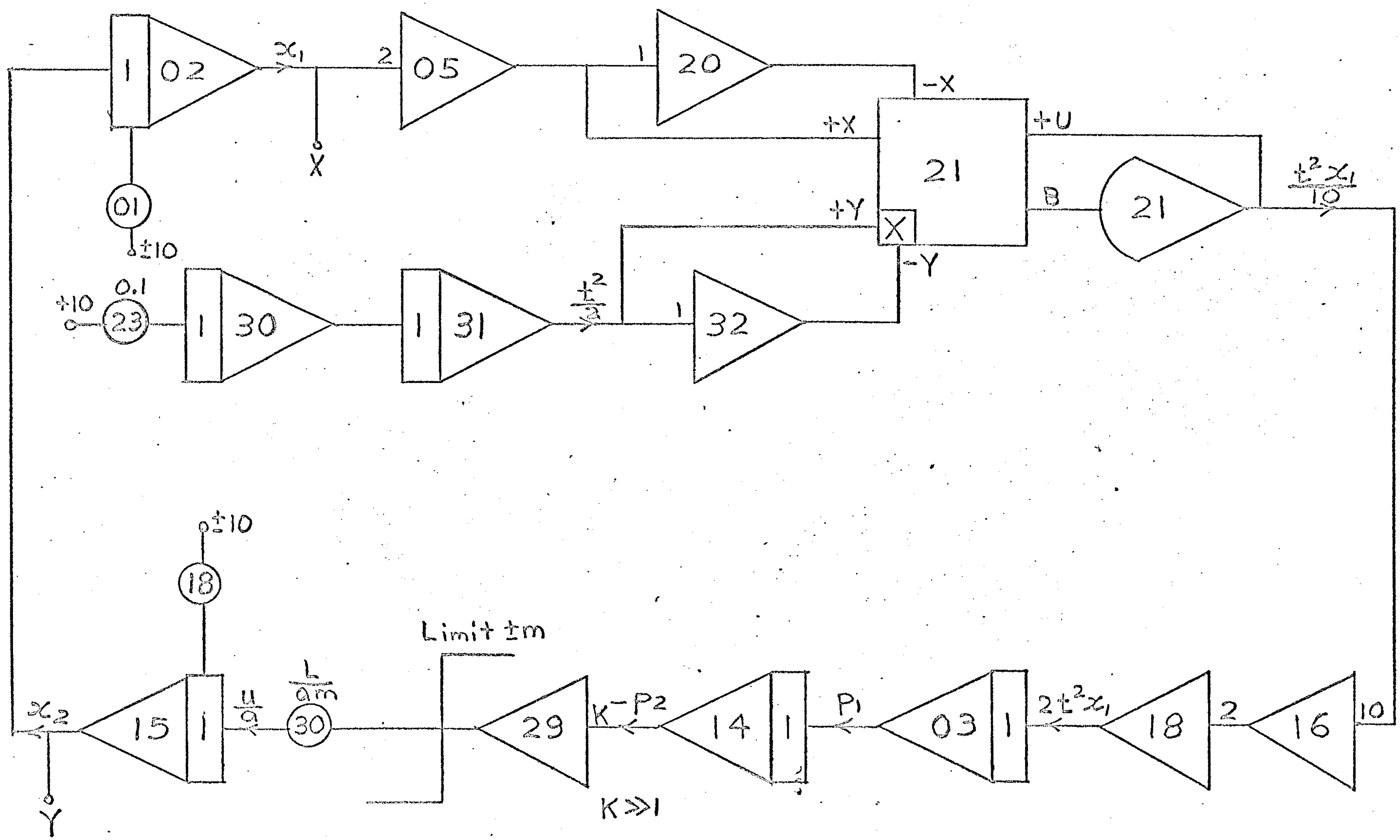


Fig. A-2 Analogue computer setup for the double integrator plant and ISTSE criterion.

3). Integral of absolute value of error.

$$I = \int_0^{\infty} |e(t)|\, dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = |x_1|$$

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 |x_1|$$

$$u = L\,\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -p_3\,\mathrm{sgn}(x_1)$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time

$$\begin{aligned} \dot{x}_1 &= -x_2 & x_1(0) &= 0 \\ \dot{x}_2 &= -\tfrac{L}{a}\mathrm{sgn}(p_2) & x_2(0) &= 0 \\ \dot{p}_1 &= -\mathrm{sgn}(x_1) & p_1(0) &= 0 \\ \dot{p}_2 &= + p_1 & p_2(0) &= 0 \end{aligned}$$

as in Fig. A-3.

The switching boundary is described by $x_1 + 0.49\frac{a}{L}x_2|x_2| = 0$.

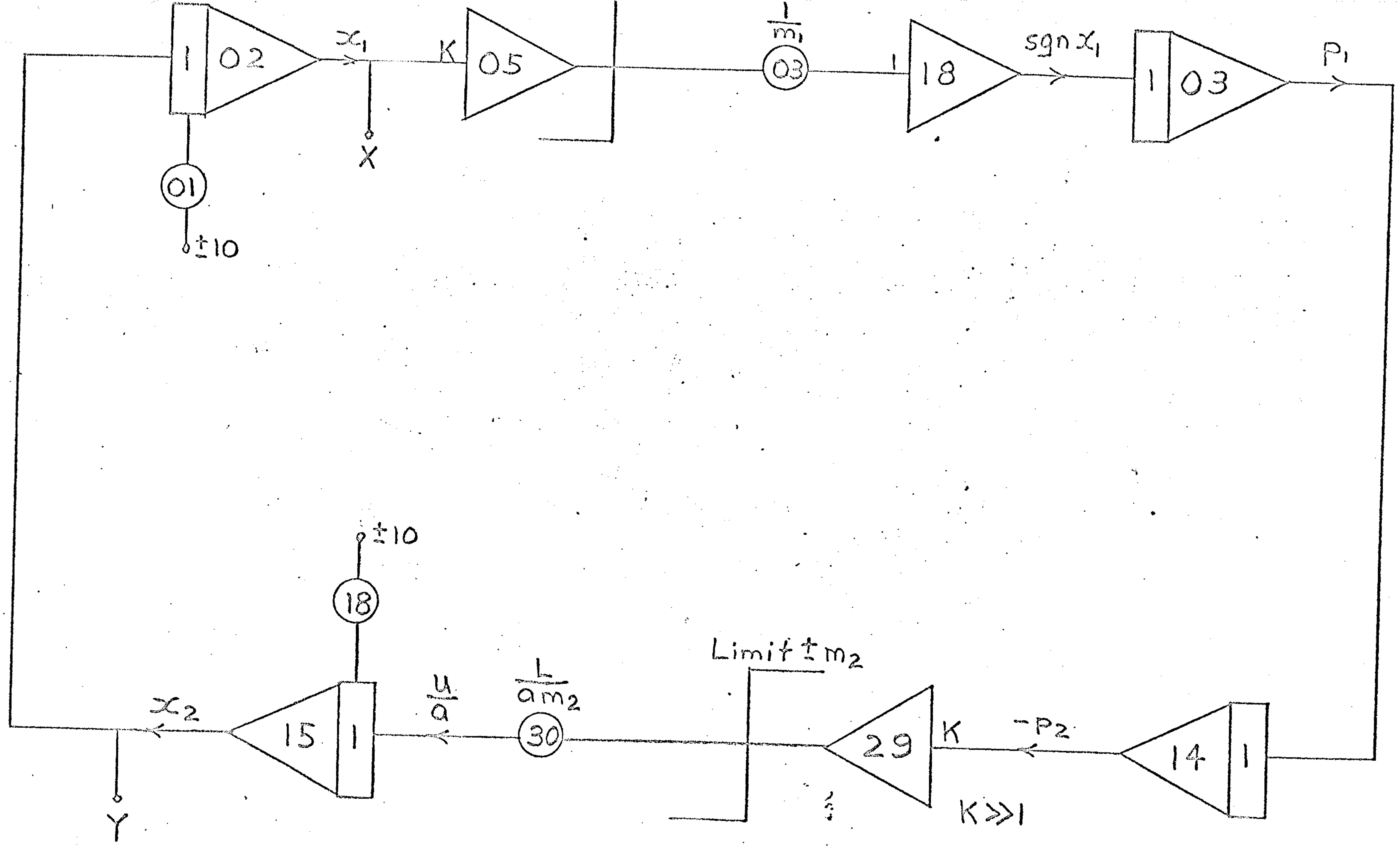


Fig. A-3 Analogue computer setup for the double integrator plant and IAE criterion.

4). <u>Integral of time multiplied by the absolute value of error.</u> (ITAE)

$$I = \int_0^\infty t\,|e(t)|\,dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = t\,|x_1|$$

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 t\,|x_1|$$

$$u = L\,\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -p_3 t\,\mathrm{sgn}(x_1)$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -t\,\mathrm{sgn}(x_1) \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

as in Fig. A-4.

The switching boundary is $x_1 + 0.55\frac{a}{L}x_2|x_2| = 0$.

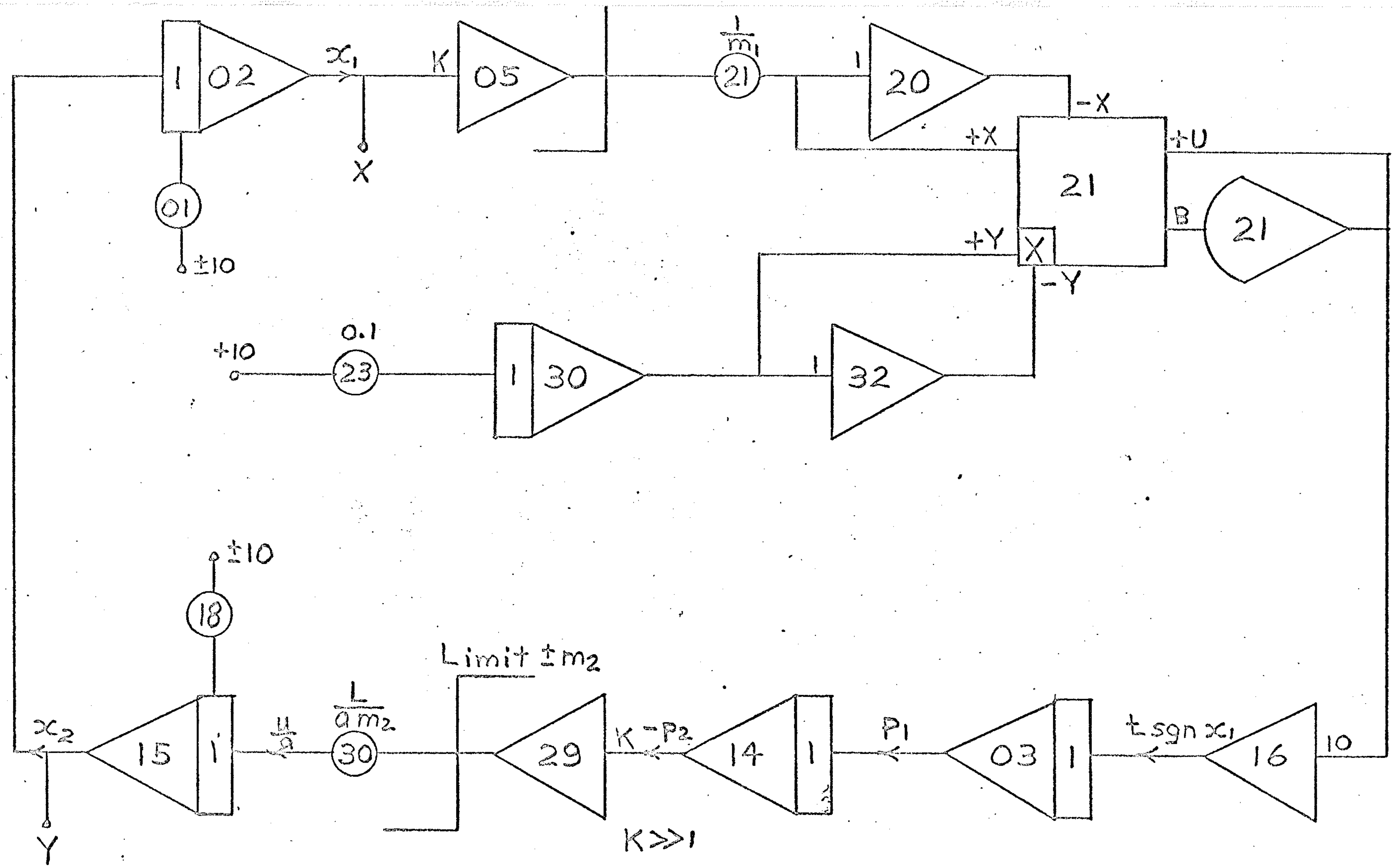


Fig. A-4 Analogue computer setup for the double integrator plant and ITAE criterion.

5). <u>Integral of squared time multiplied by the absolute value of error.</u> (ISTAE)

$$I = \int_0^\infty t^2 |e(t)| \, dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = t^2 |x_1|$$

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 t^2 |x_1|$$

$$u = L\,\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -p_3 t^2 \mathrm{sgn}(x_1)$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -t^2 \mathrm{sgn}(x_1) \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

as in Fig. A-5.

The switching boundary is approximately $x_1 + 0.45\frac{a}{L}x_2|x_2| = 0$.

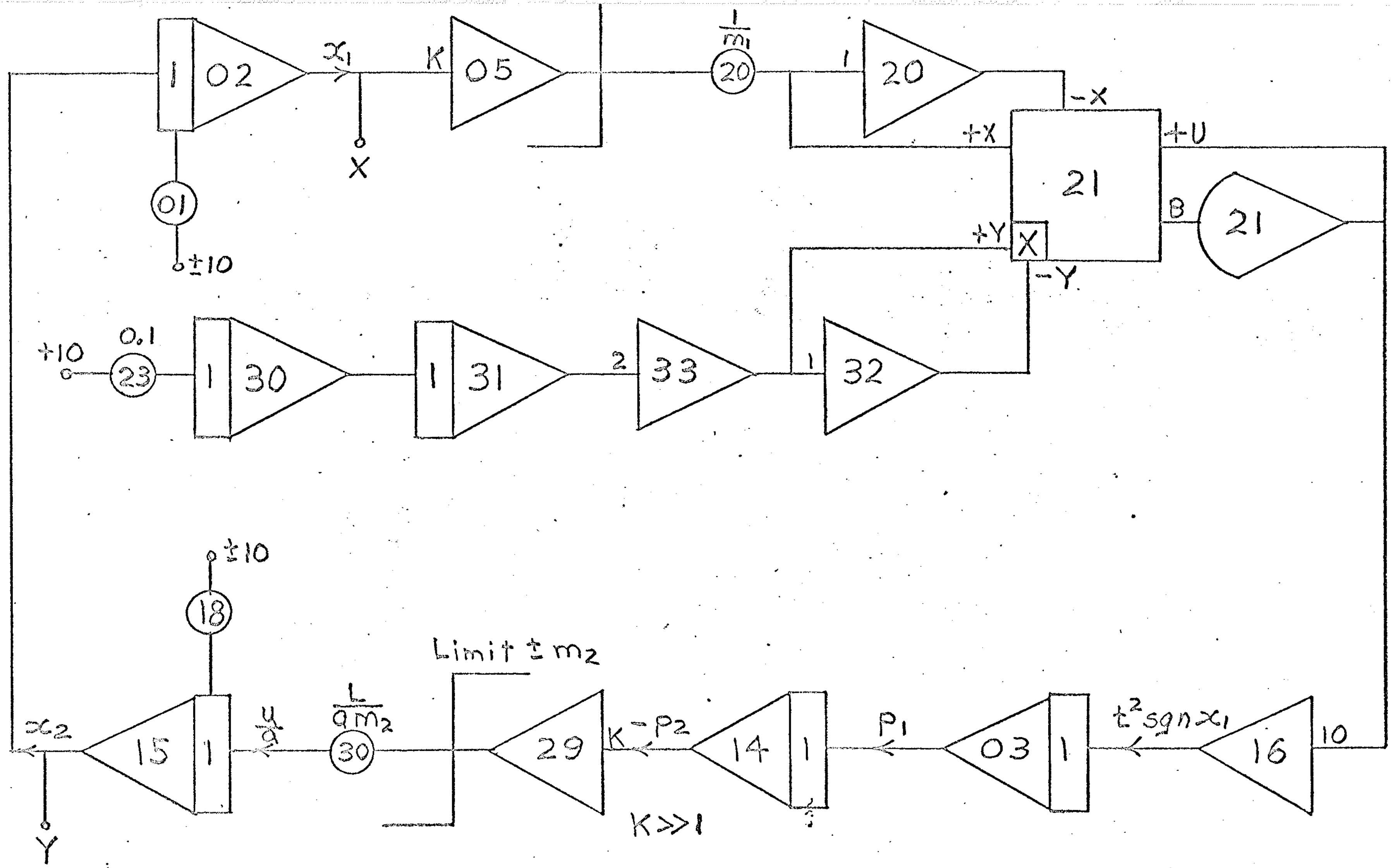


Fig. A-5 Analogue computer setup for the double integrator plant and ISTAE criterion.

6). Control area.

$$I = \int_0^{\infty} e(t)dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = x_1{}^m \quad m>1$$

$$H = p_1x_2 + p_2\frac{u}{a} + p_3x_1{}^m$$

$$u = L\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -mp_3x_1{}^{m-1}$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -mx_1{}^{m-1} \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

$$m = 1.125$$

as in Fig. A-6.

The switching boundary was found to be $x_1 + 0.52\frac{a}{L}x_2|x_2| = 0$.

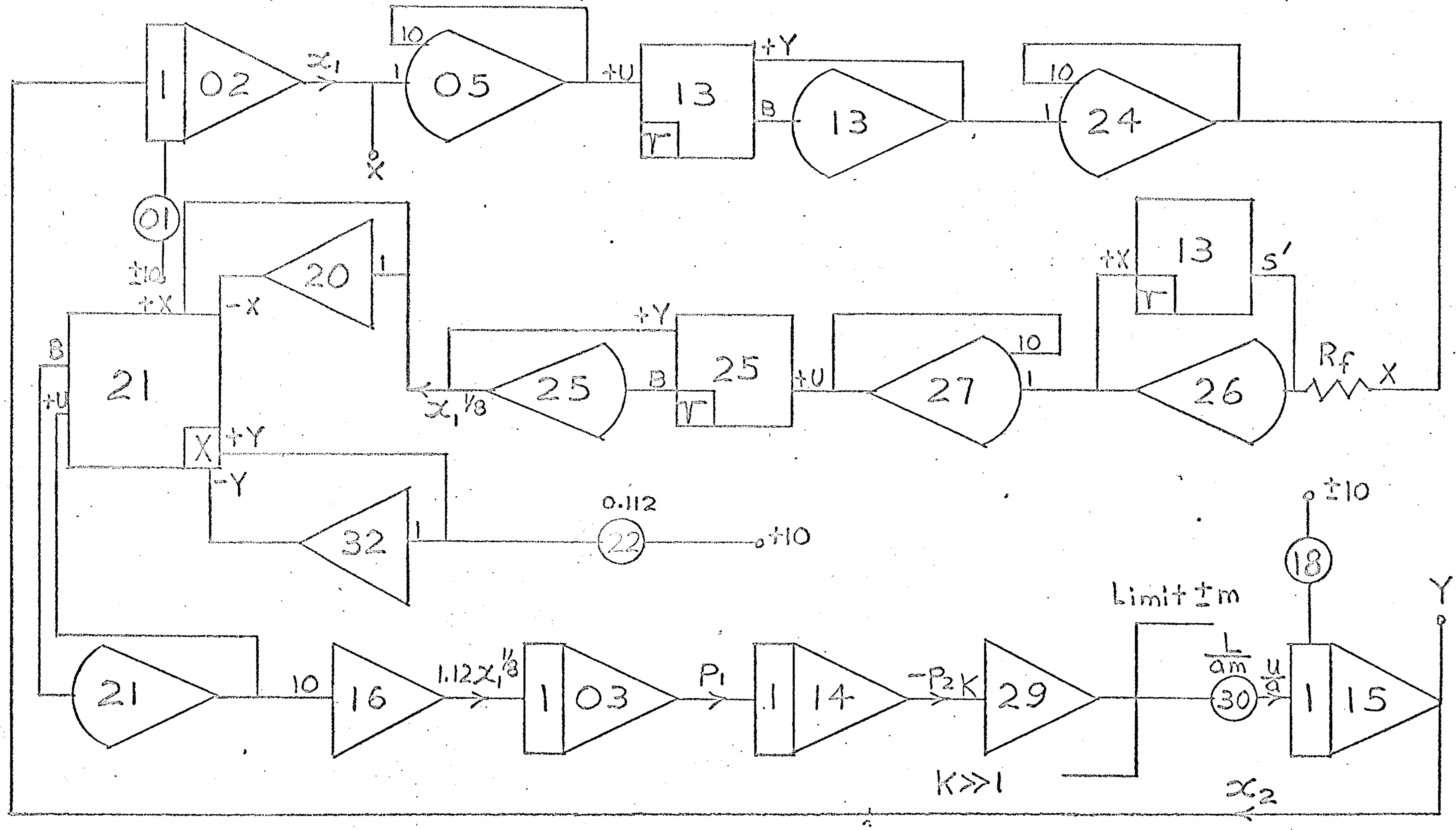


Fig. A-6 Analogue computer setup for the double integrator plant and control area criterion.

7). <u>Weighted control area.</u>

$$I = \int_0^\infty te(t)dt$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{a}$$

$$\dot{x}_3 = tx_1^m \quad m \to 1$$

$$H = p_1 x_2 + p_2 \frac{u}{a} + p_3 t x_1^m$$

$$u = L\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -p_3 t m x_1^{m-1}$$

$$\dot{p}_2 = -p_1$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -\frac{L}{a}\mathrm{sgn}(p_2) \qquad x_2(0) = 0$$

$$\dot{p}_1 = -tmx_1^{m-1} \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 \qquad p_2(0) = 0$$

as in Fig. A-7.

The switching boundary is approximately $x_1 + 0.48\frac{a}{L}x_2|x_2| = 0$.

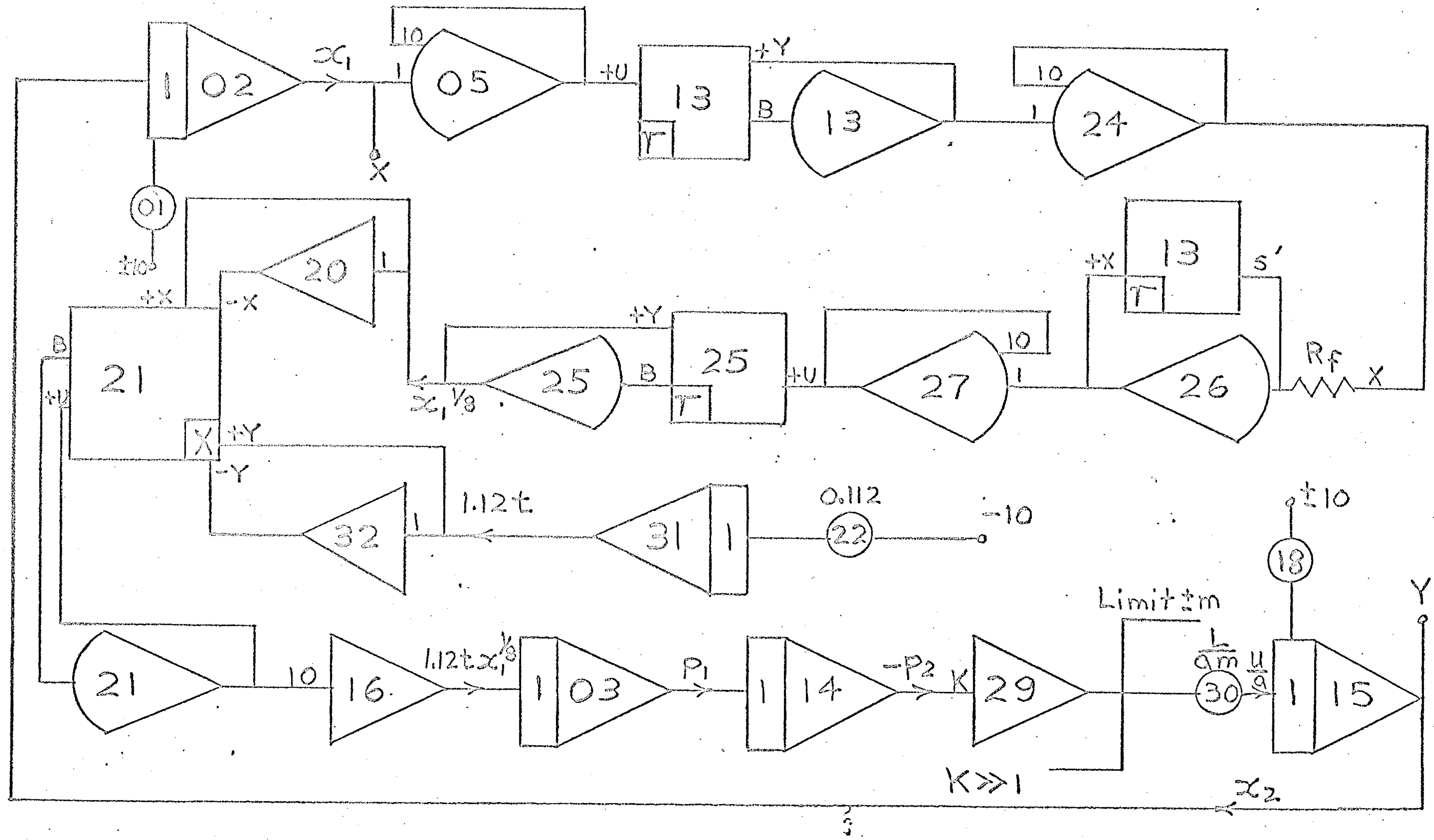


Fig. A-7 Analogue computer setup for the double integrator plant and weighted control area criterion.

APPENDIX B

CALCULATIONS FOR THE PLANT $G(p) = \frac{1}{p(p+b)}$ FOR THE MINIMUM TIME CRITERION

$$I = \int_{t_0}^{t_f} |x_1|^m dt \qquad m \to 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - bx_2$$

$$\dot{x}_3 = |x_1|^m$$

$$H = p_1 x_2 + p_2(u - bx_2) + p_3 |x_1|^m$$

$$u = L\,\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = -mp_3 |x_1|^{m-1} \mathrm{sgn}(x_1)$$

$$\dot{p}_2 = -p_1 + bp_2$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -u + bx_2 \qquad x_2(0) = 0$$

$$\dot{p}_1 \cong \frac{m}{|x_1|}\mathrm{sgn}(x_1) \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1 - bp_2 \qquad p_2(0) = 0$$

as shown in Fig. B-1.

Optimal trajectories for different parameters are given in Figs. B-2, B-3.

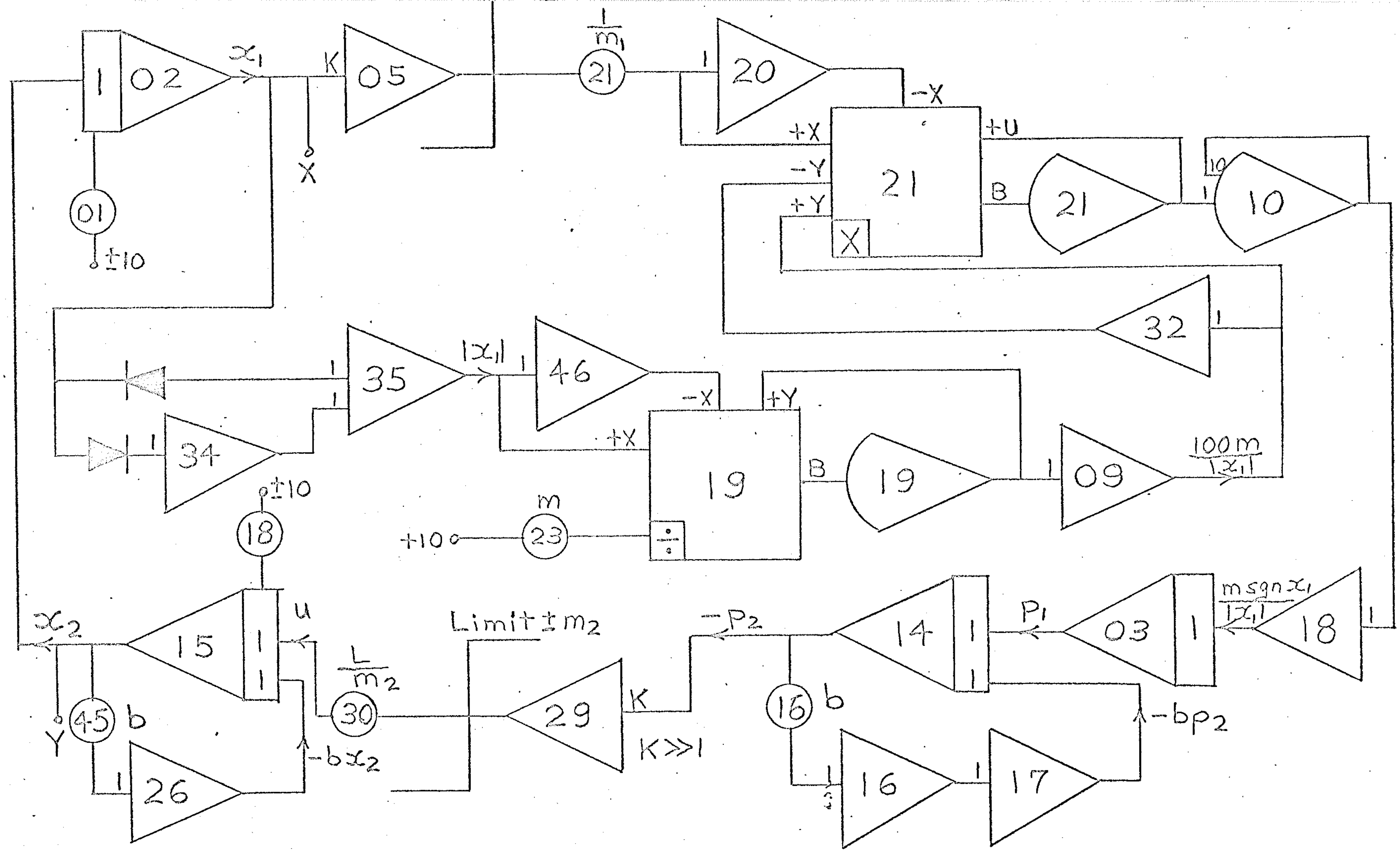


Fig. B-1 Analogue computer setup for the plant $G(p) = \frac{1}{p(p+b)}$ and minimum time criterion.

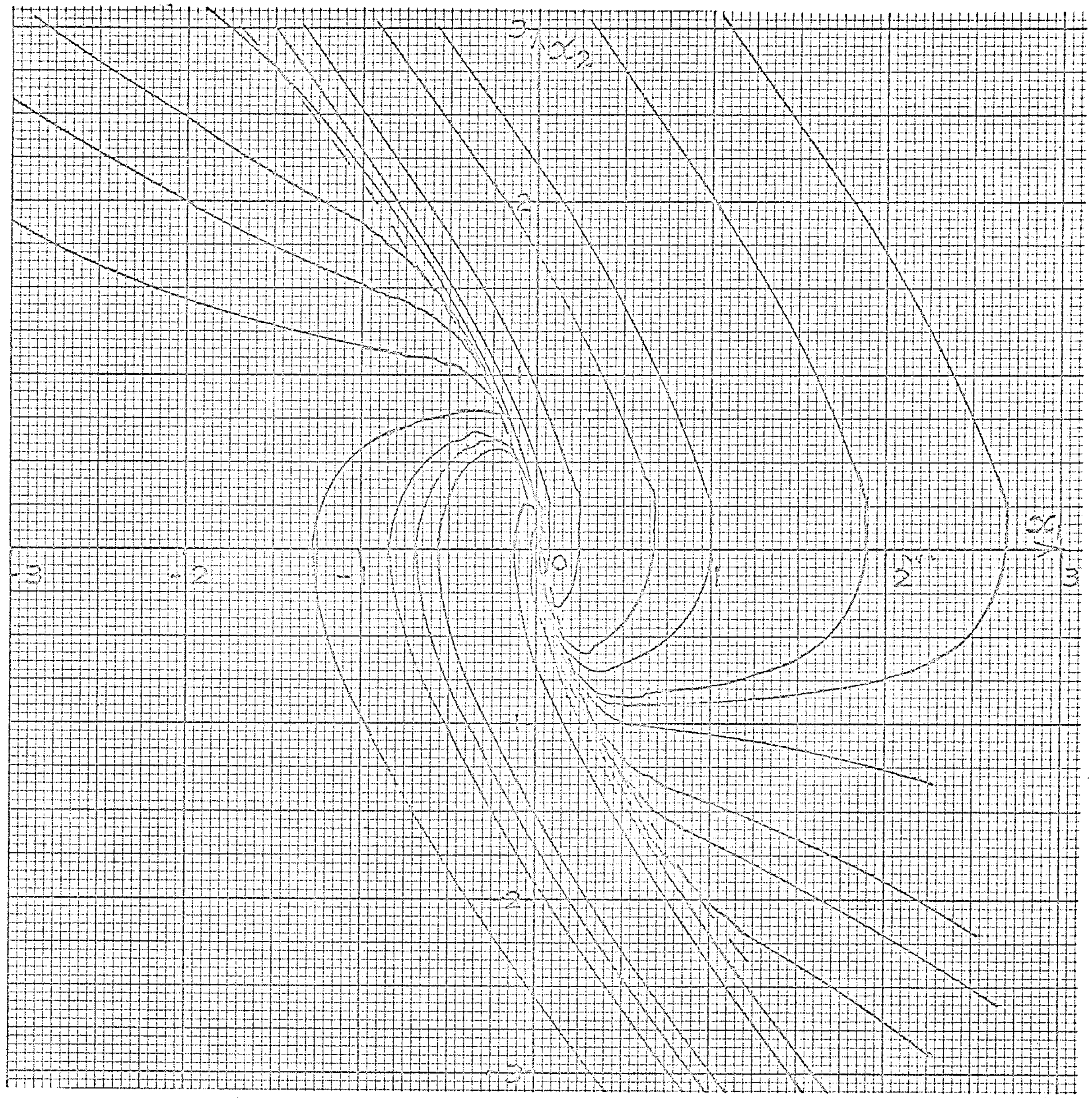


Fig. B-2 Optimal trajectories for the plant $G(p) = \frac{1}{p(p+1)}$ and minimum time criterion ($L = 1$).

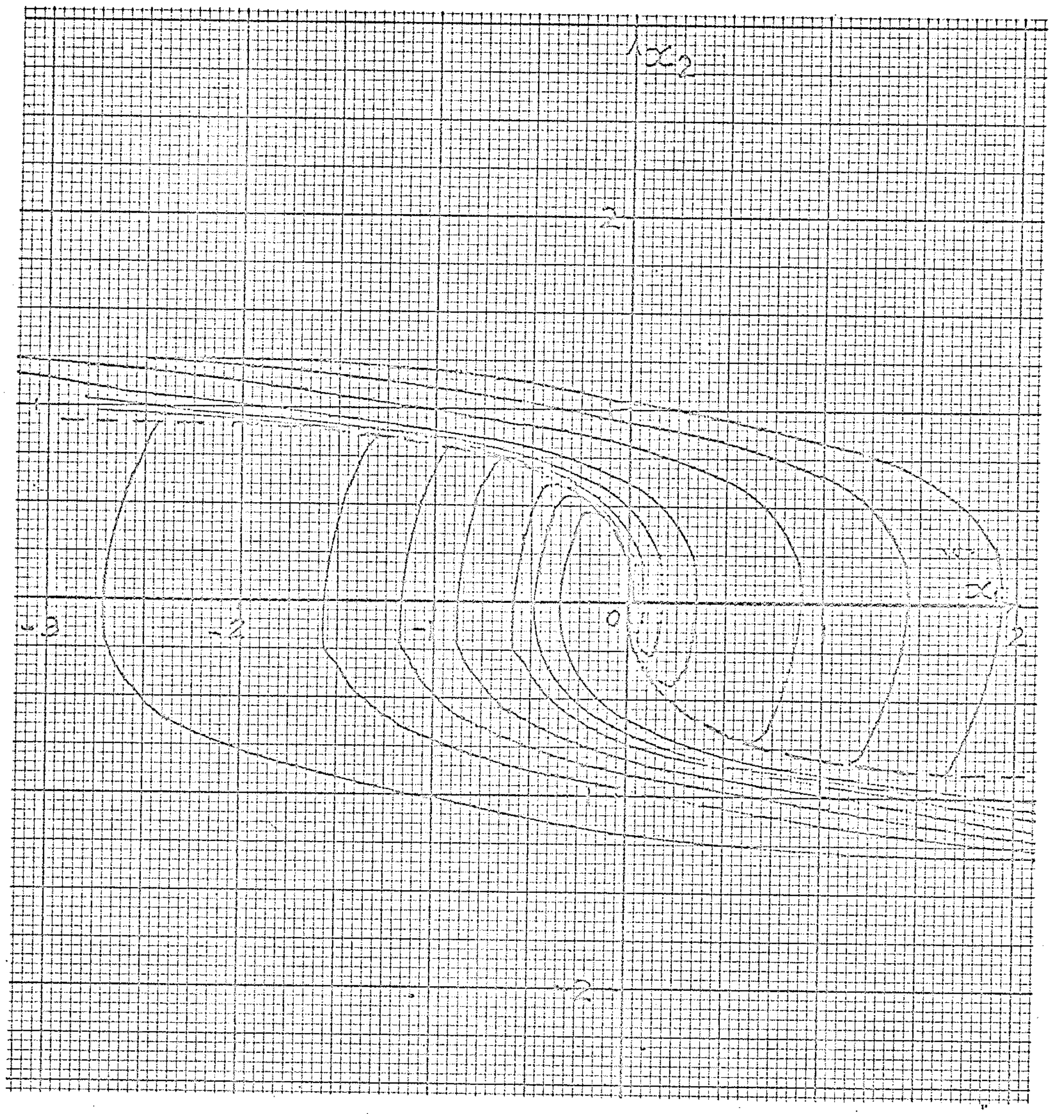


Fig. B-3 Optimal trajectories for the plant $G(p) = \frac{1}{p(p-1)}$ and minimum time criterion ($L = 1$).

APPENDIX C

CALCULATIONS AND DATA FOR THE GENERAL SYSTEM SECOND-ORDER

1). Integral-square error criterion.

$$I = \int_0^\infty e^2(t)dt$$

$$G(p) = \frac{1}{p^2+bp+c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u-bx_2-cx_1$$

$$\dot{x}_3 = x_1^{\ 2}$$

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3x_1^{\ 2}$$

$$u = L\mathrm{sgn}(p_2)$$

$$\dot{p}_1 = p_2c-2p_3x_1$$

$$\dot{p}_2 = -p_1 + p_2b$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -L\mathrm{sgn}(p_2)+bx_2+cx_1 \qquad x_2(0) = 0$$

$$\dot{p}_1 = -cp_2-2x_1 \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1-bp_2 \qquad p_2(0) = 0$$

as given in Fig. C-1.

Optimal trajectories for different parameters are given in Figs. C-2, C-3, and C-4.

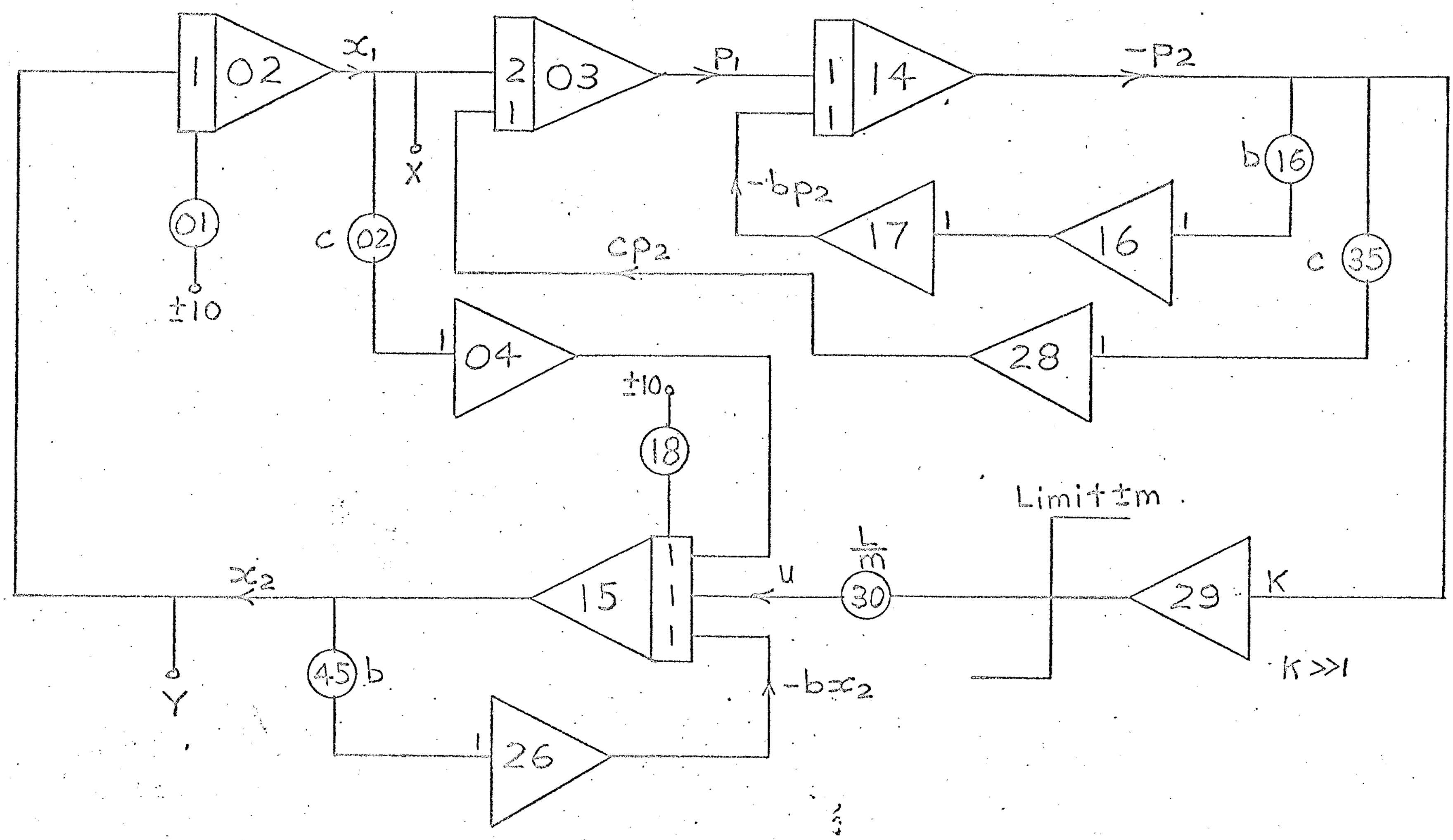


Fig. C-1. Analogue computer setup for general system with integral-square error criterion.

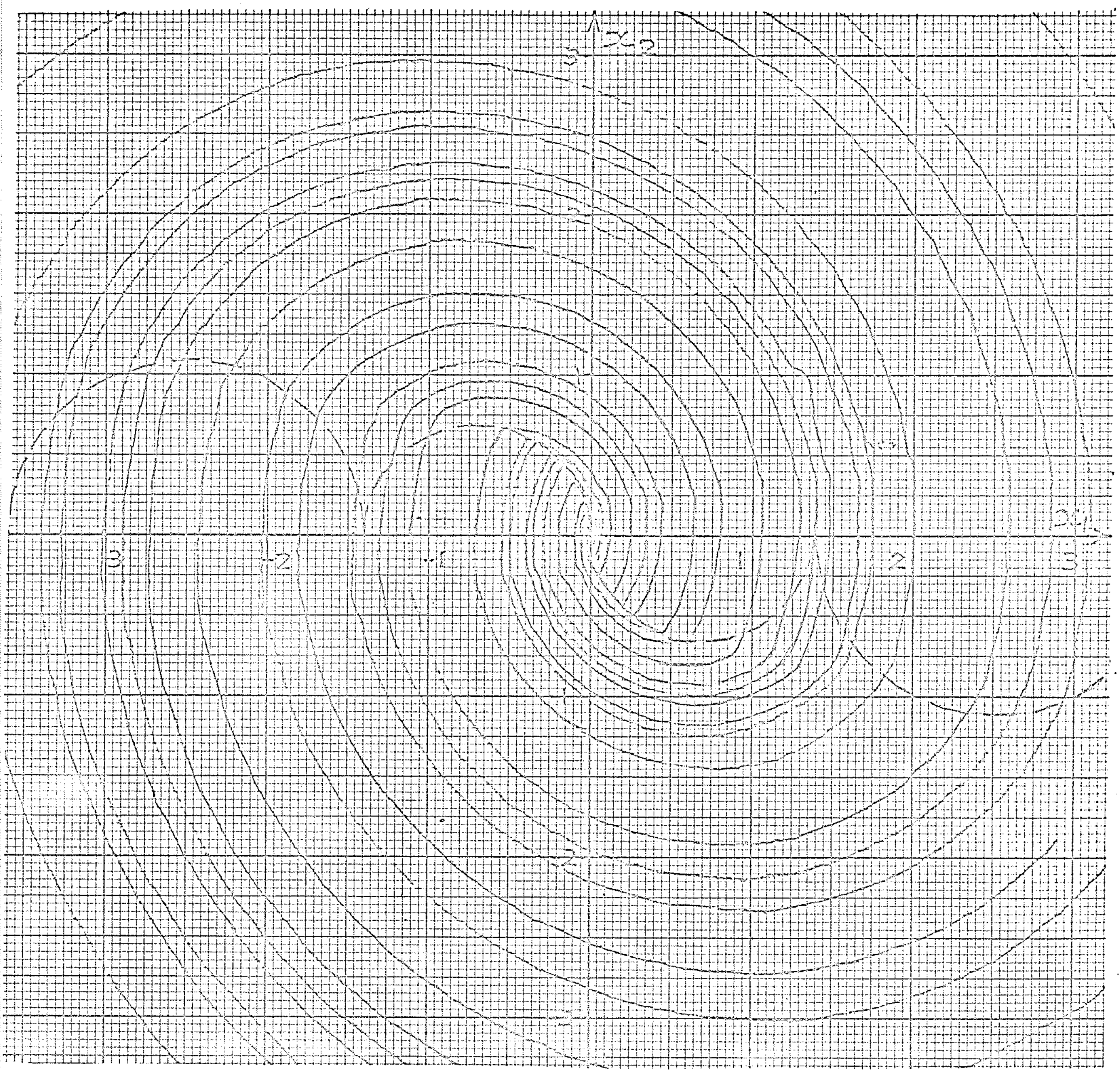


Fig. C-2 Optimal trajectories for the system $G(p) = \frac{1}{p^2+1}$ with integral-square error criterion ($\xi = 0$).

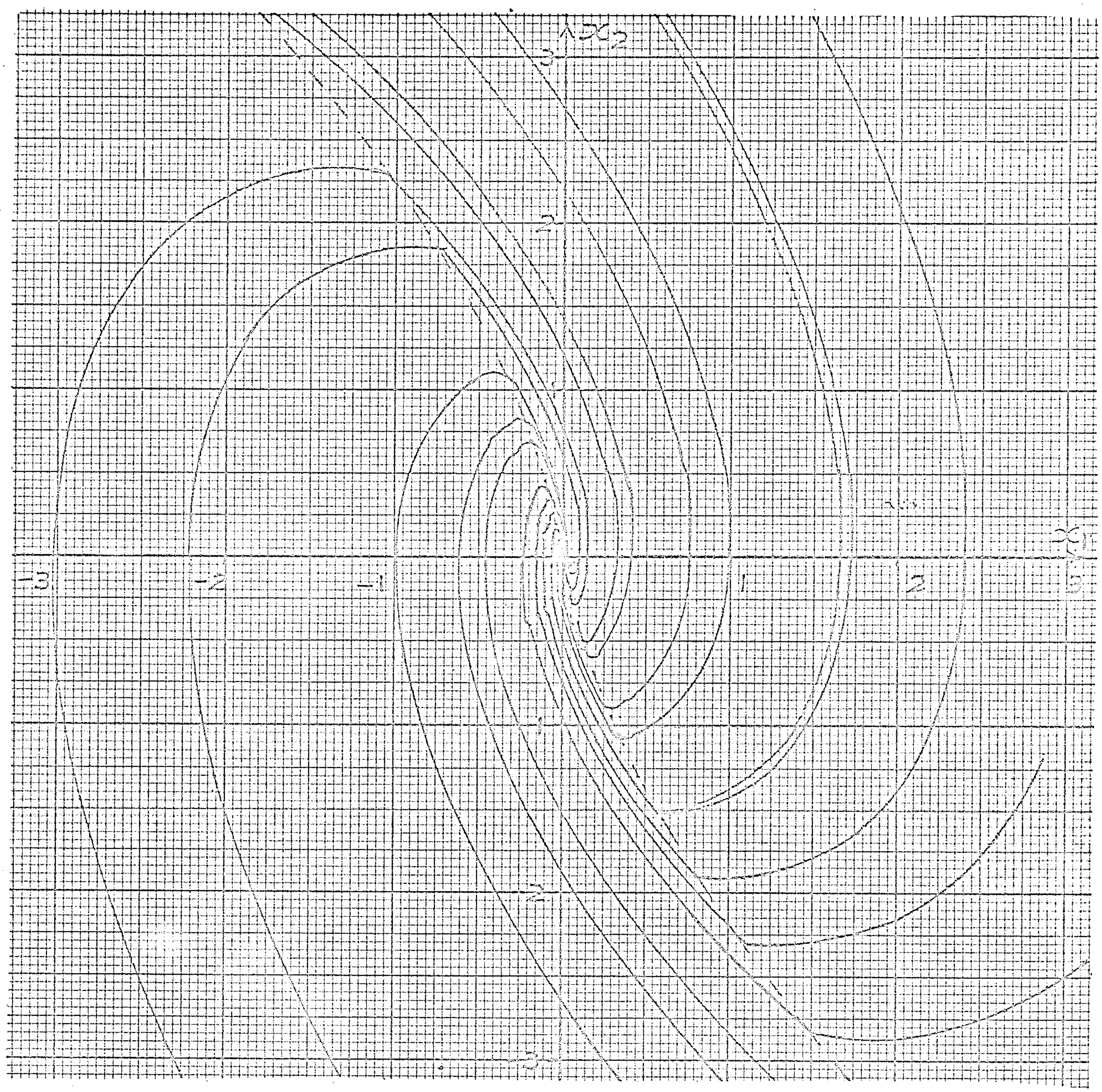


Fig. C-3 Optimal trajectories for the system $G(p) = \frac{1}{p^2+p+1}$ with integral-square error criterion ($\zeta = 0.5$).

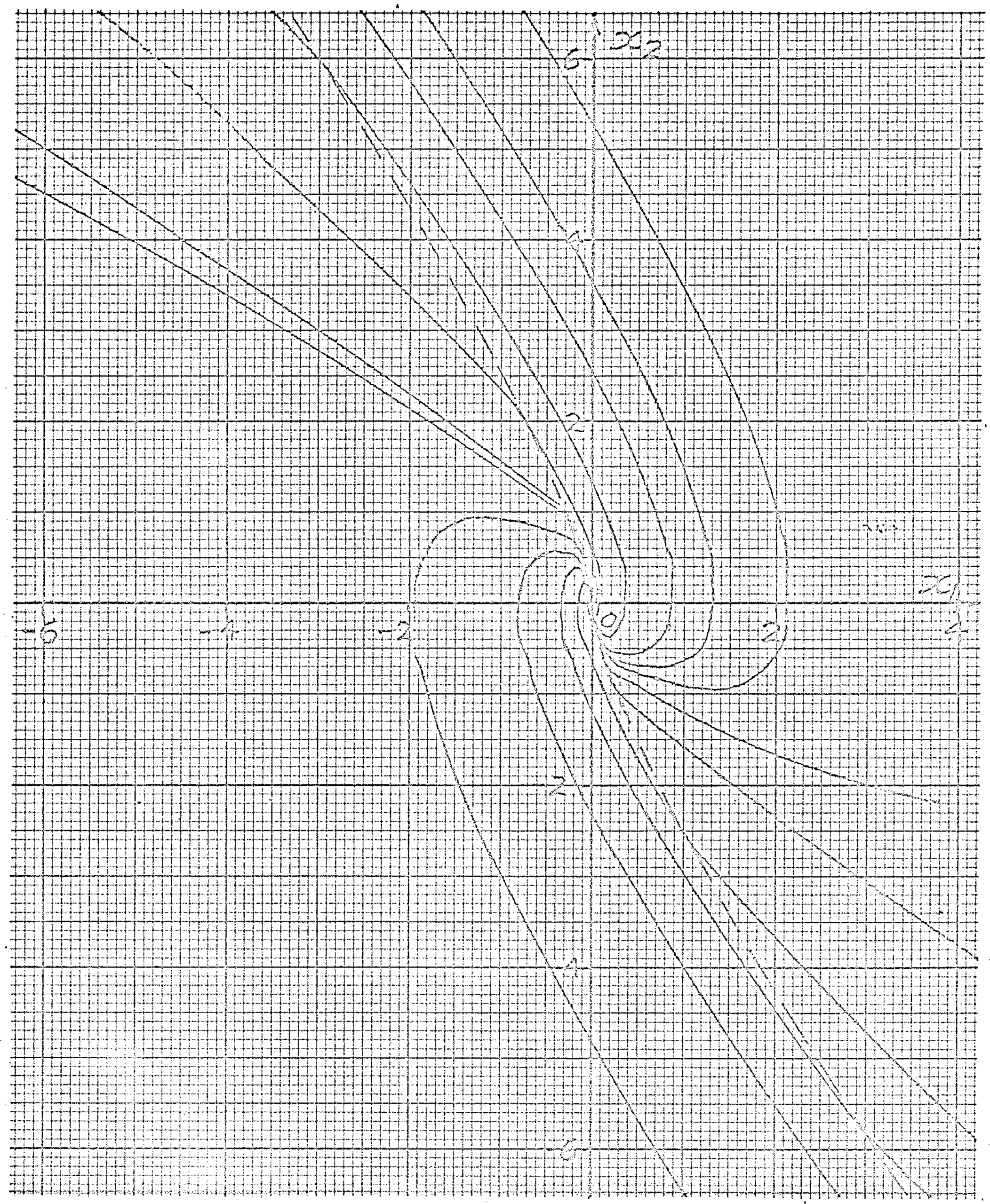


Fig. C-4 Optimal trajectories for the system $G(p) = \dfrac{1}{p^2+2p+1}$ with integral-square error criterion ($\zeta = 1.0$).

2). <u>Minimum time criterion.</u>

$$I = \int_{t_o}^{t_f} |x_1|^m dt \qquad m \to 0$$

$$G(p) = \frac{1}{p^2+bp+c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u-bx_2-cx_1$$

$$\dot{x}_3 = |x_1|^m$$

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3|x_1|^m$$

$$u = L\operatorname{sgn}(p_2)$$

$$\dot{p}_1 = p_2c-mp_3|x_1|^{m-1}\operatorname{sgn}(x_1)$$

$$\dot{p}_2 = -p_1 + p_2b$$

$$\dot{p}_3 = 0$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -L\operatorname{sgn}(p_2)+bx_2+cx_1 \qquad x_2(0) = 0$$

$$\dot{p}_1 = -p_2c-m|x_1|^{m-1}\operatorname{sgn}(x_1) \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1-bp_2 \qquad p_2(0) = 0$$

as simulated in Fig. C-5.

The optimal trajectories with the switching boundary is given in Fig. C-6.

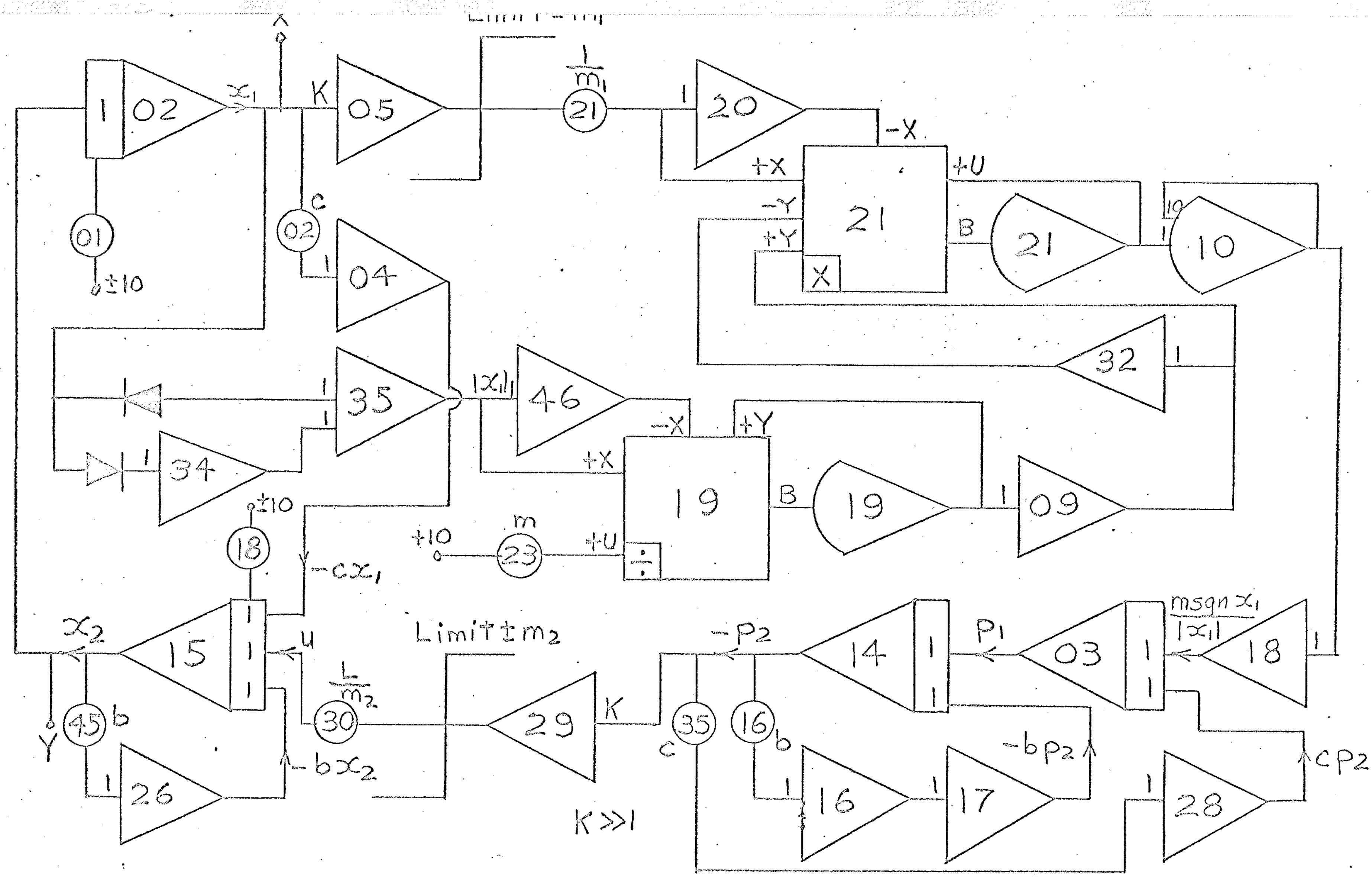


Fig. C-5 Analogue computer setup for the general system with minimum time criterion.

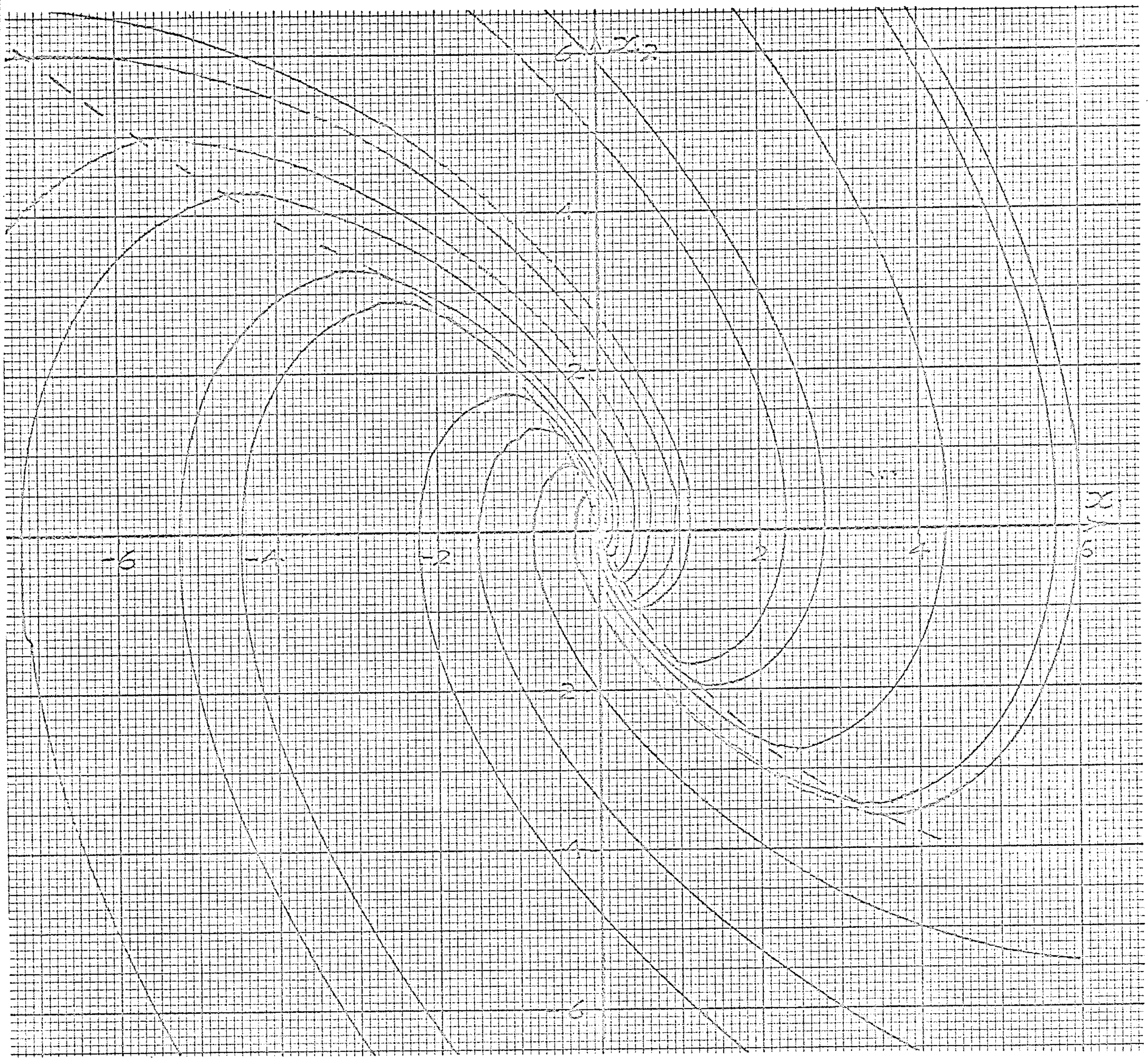


Fig. C-6 Optimal trajectories for the system $G(p) = \frac{1}{p^2+p+1}$ with minimum time criterion.

APPENDIX D

CALCULATIONS AND DATA FOR MINIMIZING ERROR AND EFFORT

1). $$I_{A-2} = \int_{t_o}^{t_f} (x_1^2 + wu^2)dt \qquad |u| \leq L$$

$$G(p) = \frac{1}{p^2+bp+c}$$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u-bx_2-cx_1$$

$$\dot{x}_3 = x_1^2 + wu^2$$

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3(x_1^2 + wu^2)$$

with $$\dot{p}_1 = cp_2 + 2x_1 \qquad \dot{p}_2 = -p_1 + bp_2$$

$$\dot{p}_3 = 0$$

and
$$u = -L \quad \text{if} \quad \frac{p_2}{2w} < -L$$
$$u = \frac{p_2}{2w} \quad \text{if} \quad -L \leq \frac{p_2}{2w} \leq L$$
$$u = +L \quad \text{if} \quad \frac{p_2}{2w} > L$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$
$$\dot{x}_2 = -u+bx_2+cx_1 \qquad x_2(0) = 0$$
$$\dot{p}_1 = -cp_2-2x_1 \qquad p_1(0) = 0$$
$$\dot{p}_2 = p_1-bp_2 \qquad p_2(0) = 0$$

which are set up on the analogue computer as in Fig. D-1. Results are given in Figs. D-2, D-3, and D-4.

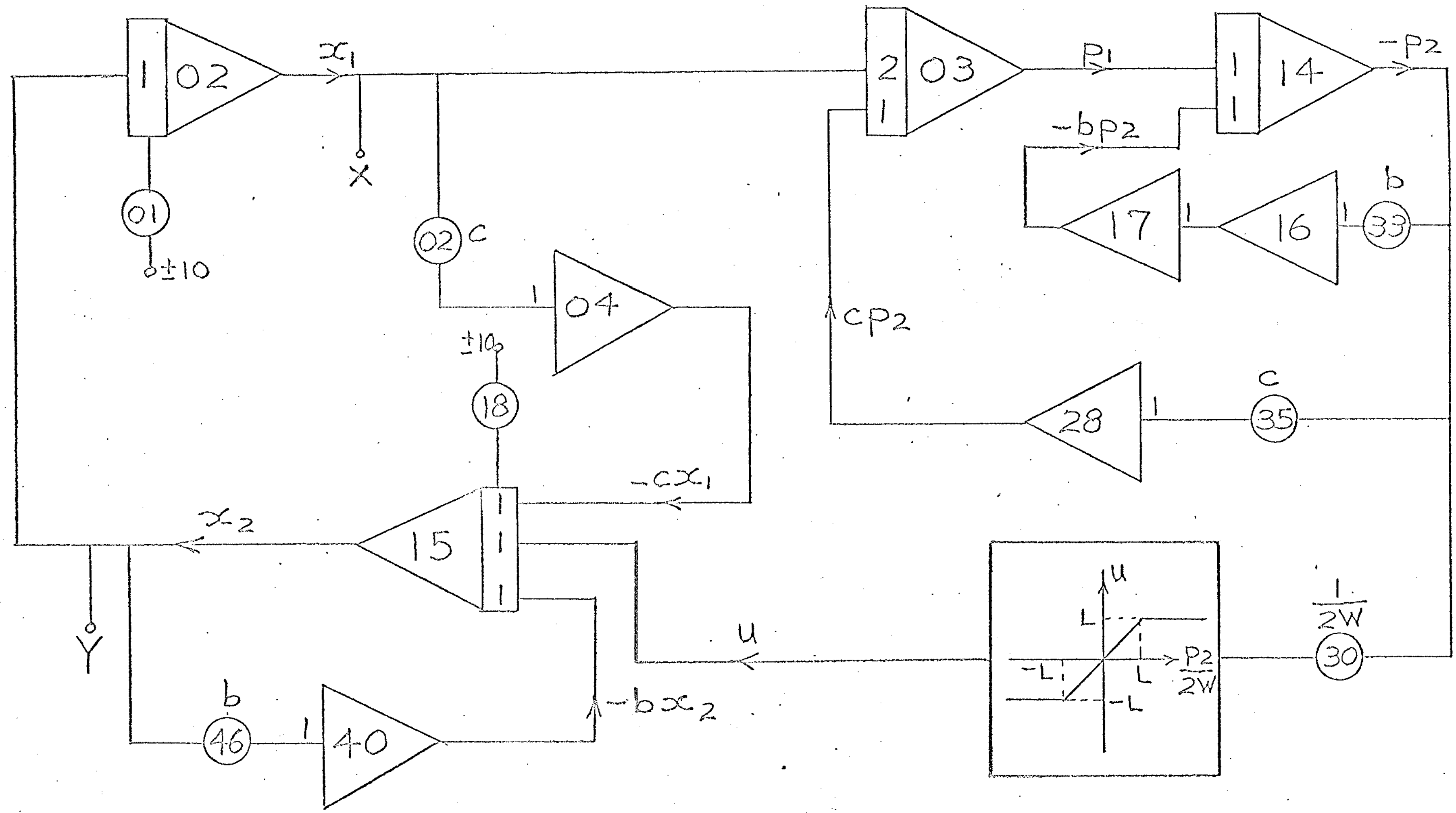


Fig. D-1 Analogue computer setup for the system with performance index I_{A-2}.

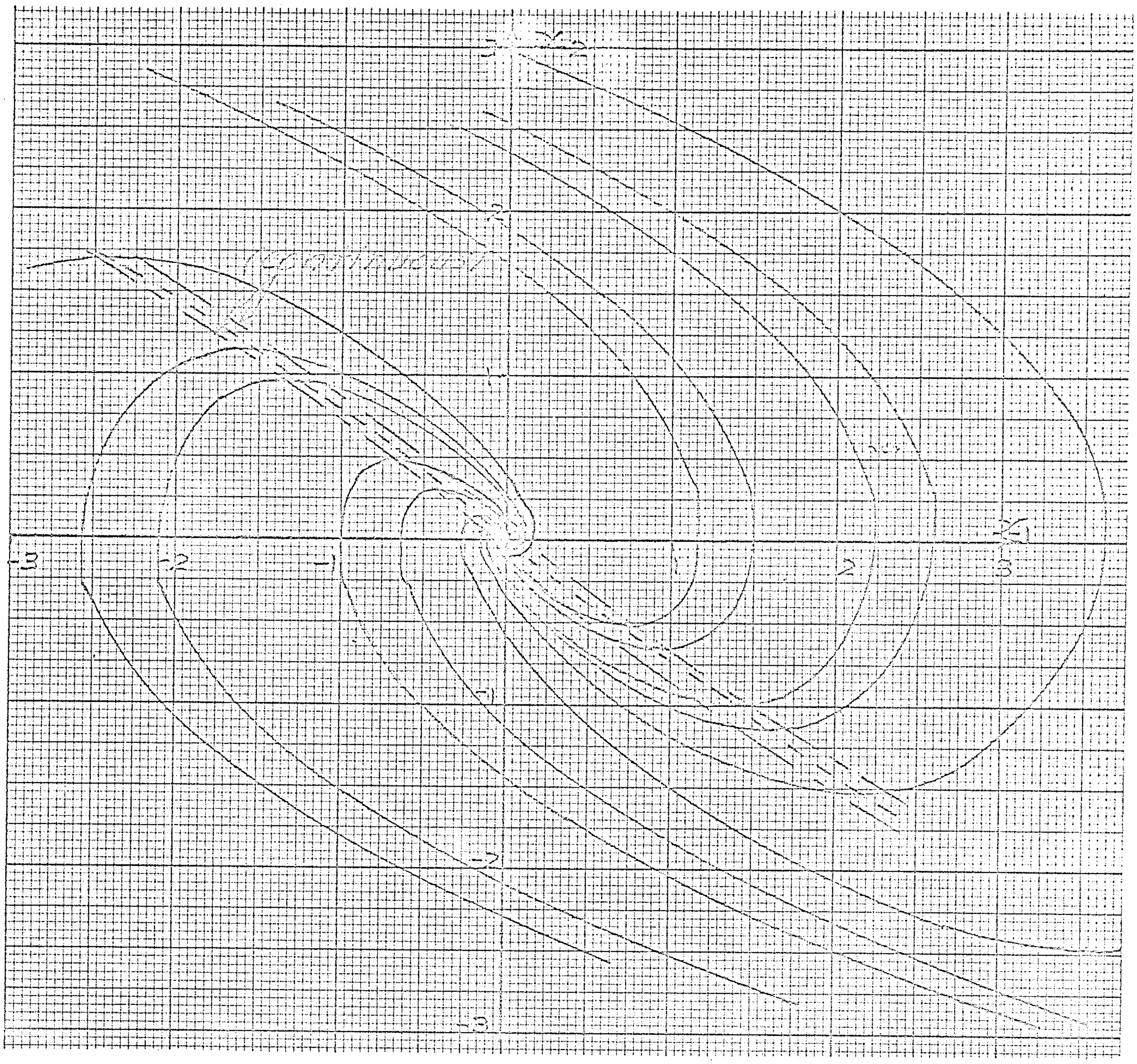


Fig. D-2 Optimal trajectories for the system with performance index I_{A-2}. (Case 1).

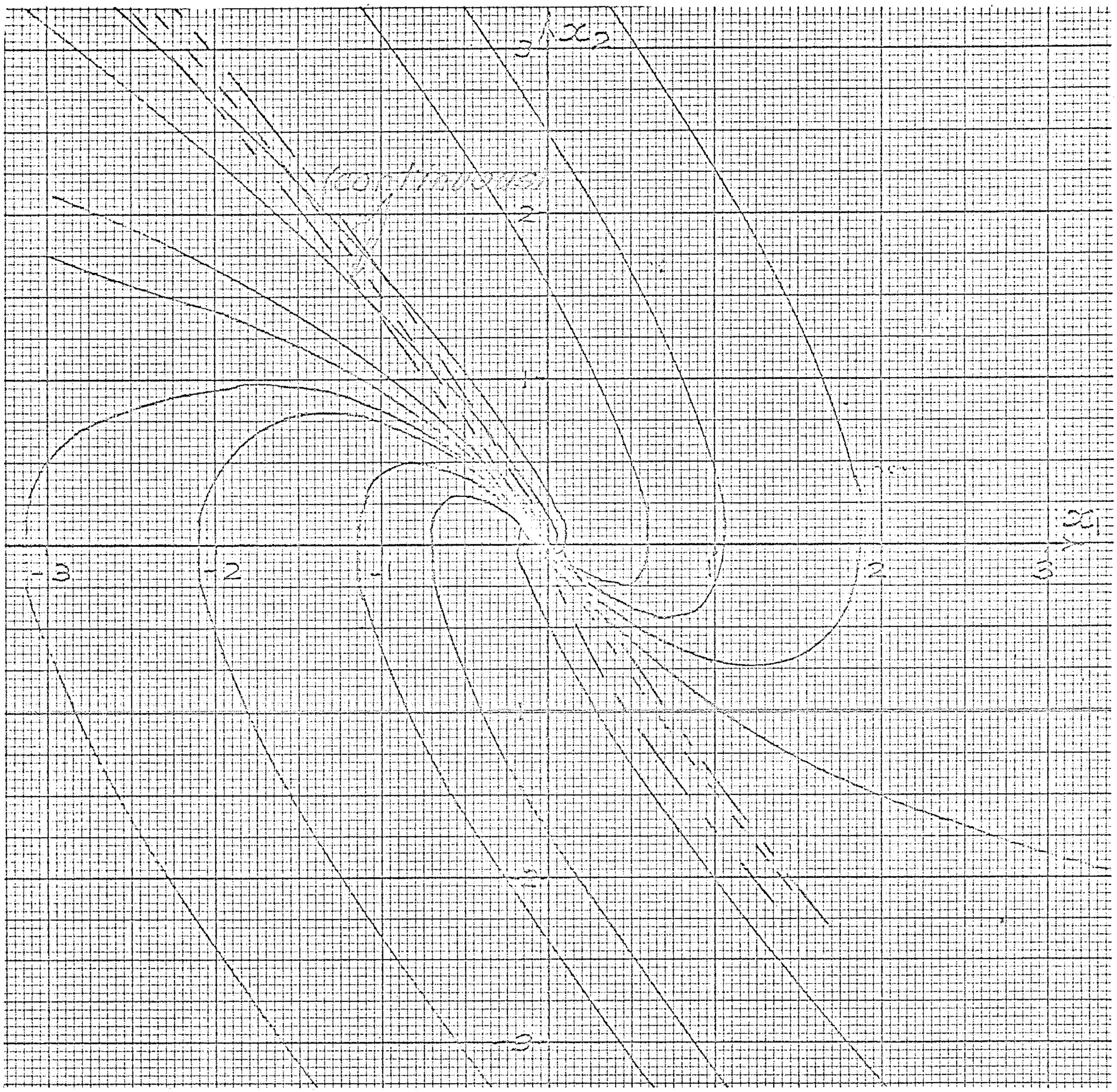


Fig. D-3 Optimal trajectories for the system with performance index I_{A-2}. (Case 2).

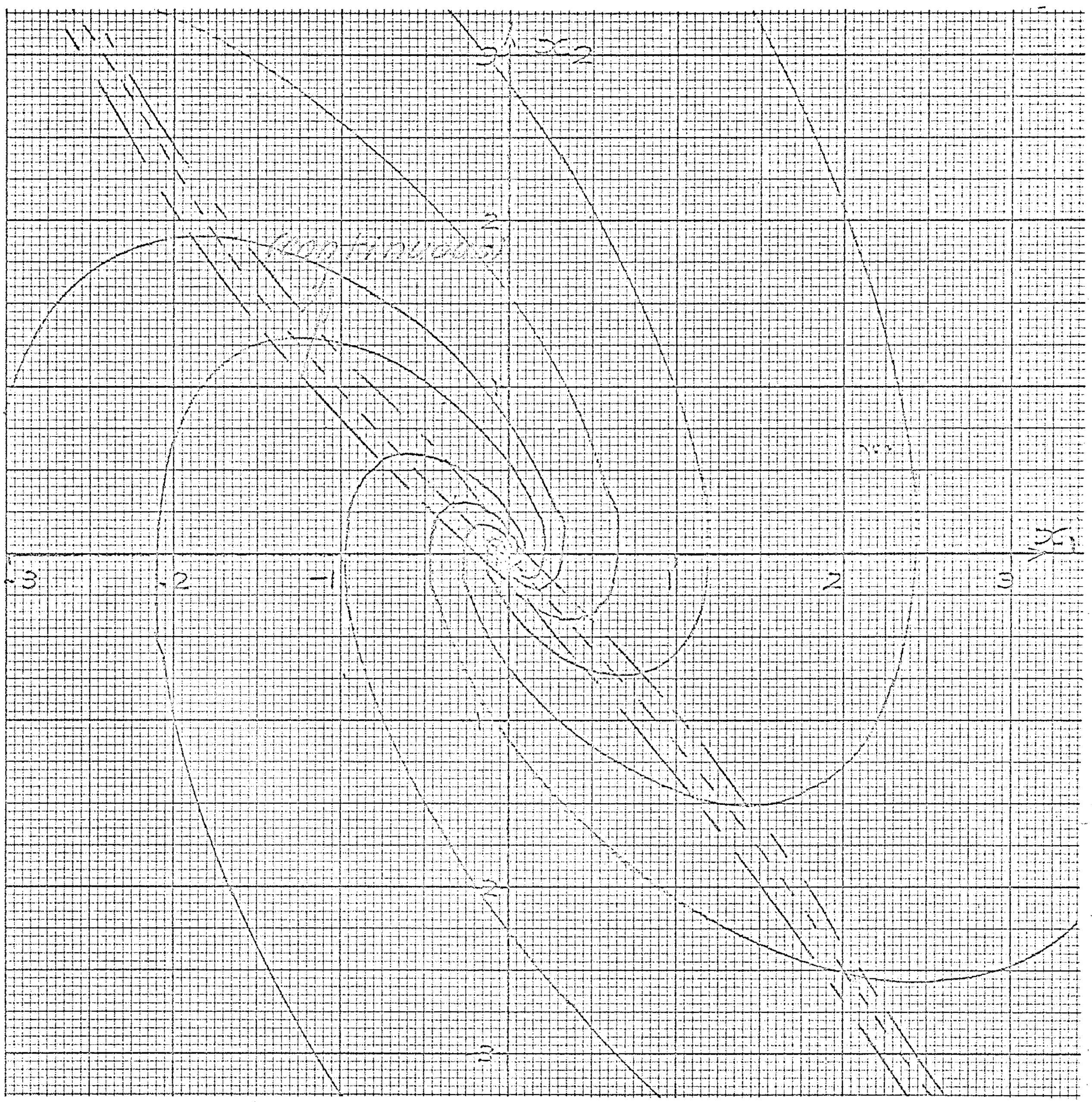


Fig. D-4 Optimal trajectories for the system with performance index I_{A-2}. (Case 3).

2). $$I_{B-2} = \int_{t_o}^{t_f} (x_1^2 + w|u|)dt \qquad |u| \leq L$$

$$G(p) = \frac{1}{p^2+bp+c}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u-bx_2-cx_1$$

$$\dot{x}_3 = x_1^2 + w|u|$$

$$H = p_1x_2 + p_2(u-bx_2-cx_1) + p_3(x_1^2 + w|u|)$$

with
$$\dot{p}_1 = cp_2 + 2x_1$$

$$\dot{p}_2 = -p_1 + bp_2$$

$$\dot{p}_3 = 0$$

and
$$u = 0 \quad \text{if} \quad \left|\frac{p_2}{2w}\right| < 1$$

$$|u| = L \quad \text{if} \quad \left|\frac{p_2}{2w}\right| > 1$$

In reverse time,

$$\dot{x}_1 = -x_2 \qquad x_1(0) = 0$$

$$\dot{x}_2 = -u+bx_2+cx_1 \qquad x_2(0) = 0$$

$$\dot{p}_1 = -cp_2-2x_1 \qquad p_1(0) = 0$$

$$\dot{p}_2 = p_1-bp_2 \qquad p_2(0) = 0$$

which are set up on the analogue computer as in Fig. D-5.

Results are given in Figs. D-6, D-7, and D-8.

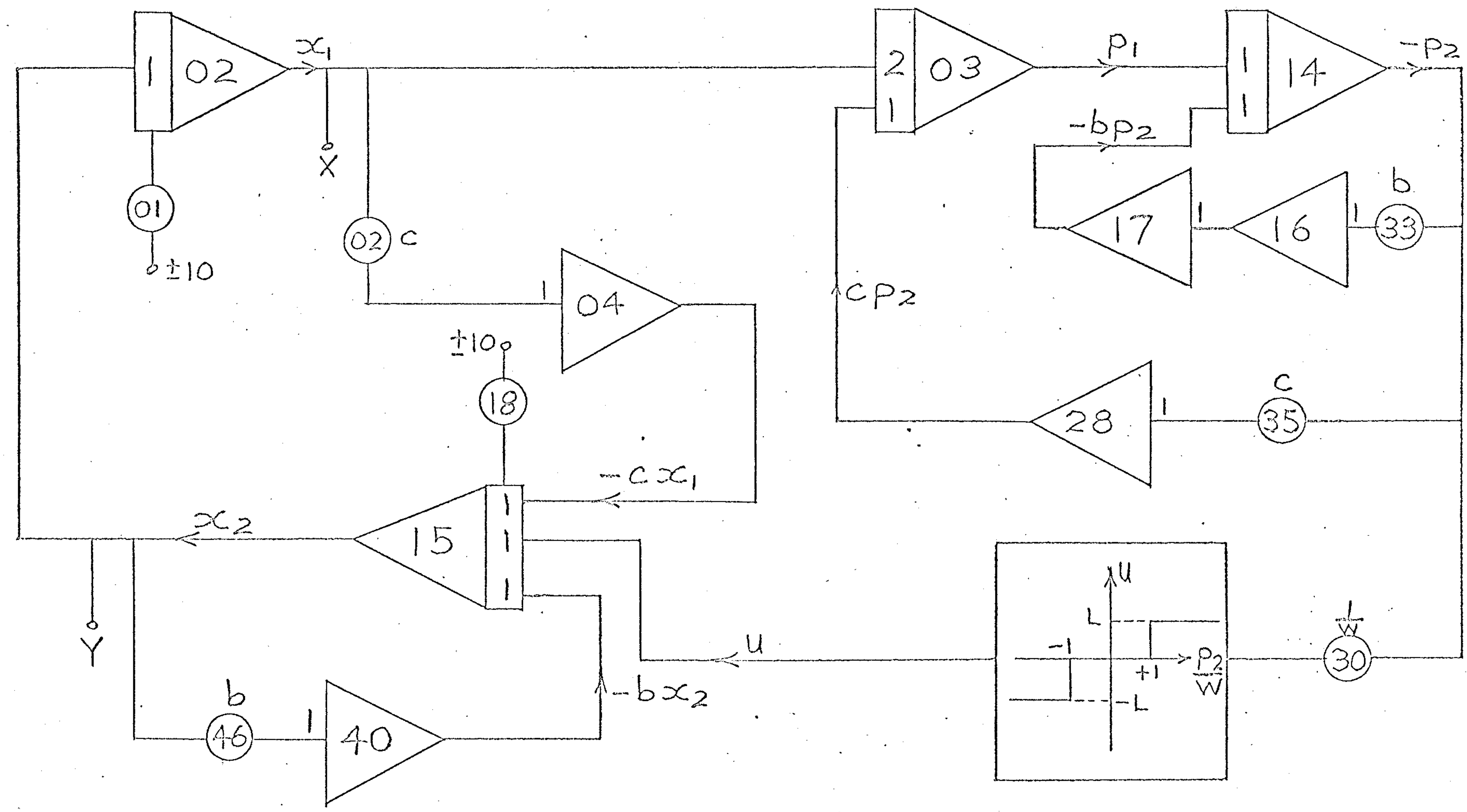


Fig. D-5 Analogue computer setup for the system with performance index I_{B-2}.

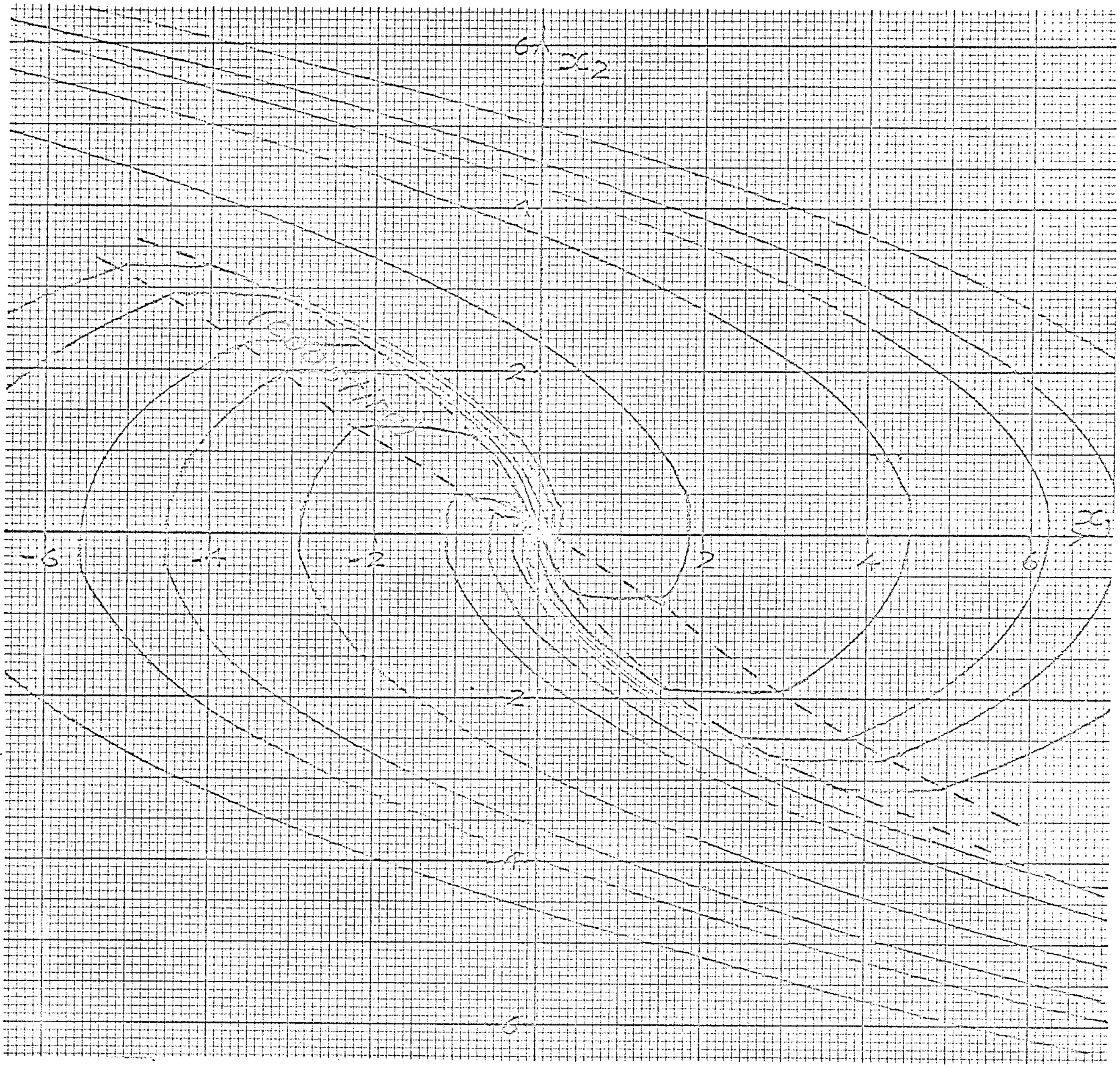
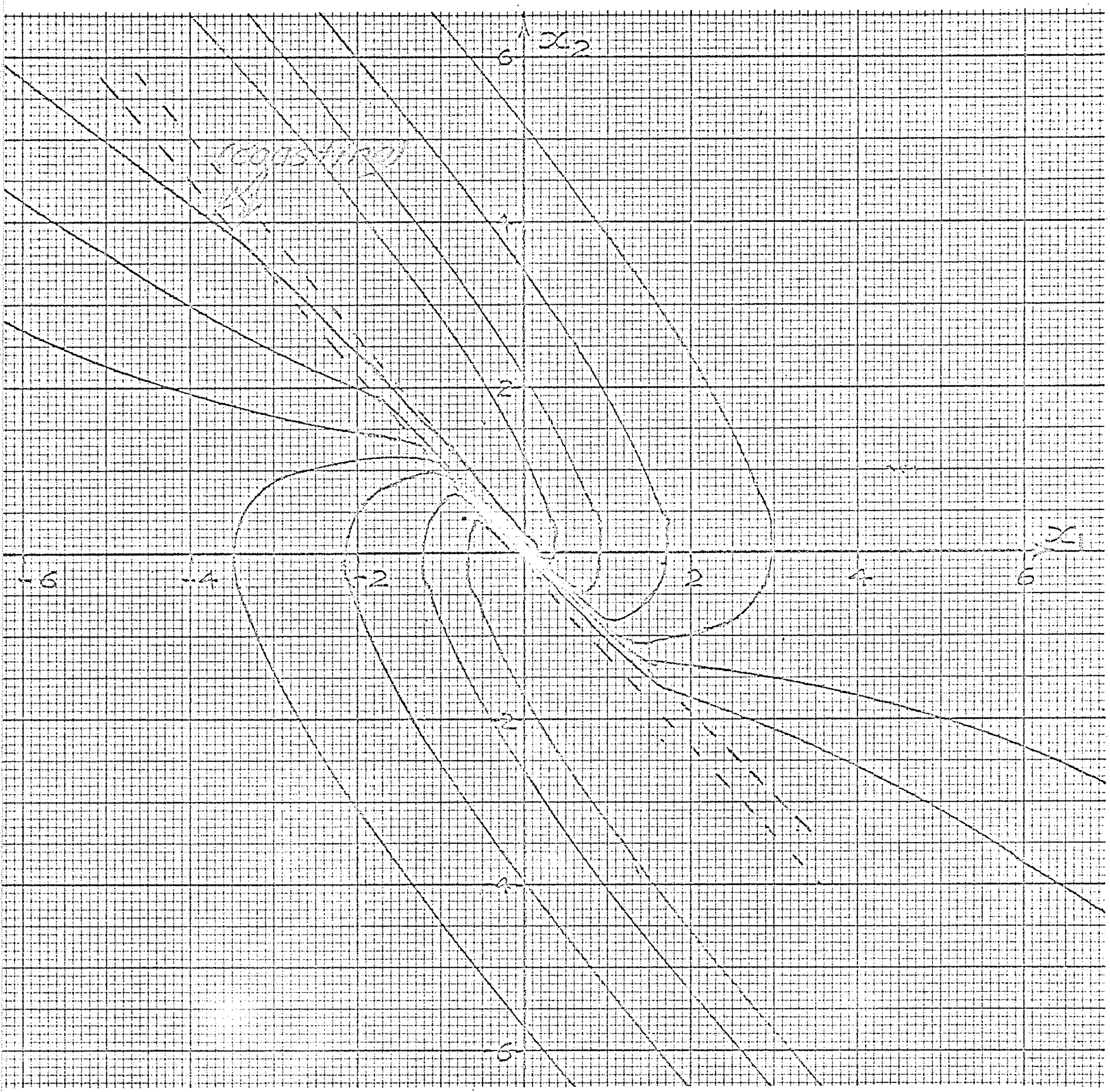


Fig. D-6 Optimal trajectories for the system with performance index I_{B-2}. (Case 1).



ig. D-7 Optimal trajectories for the system with performance index I_{B-2}. (Case 2).

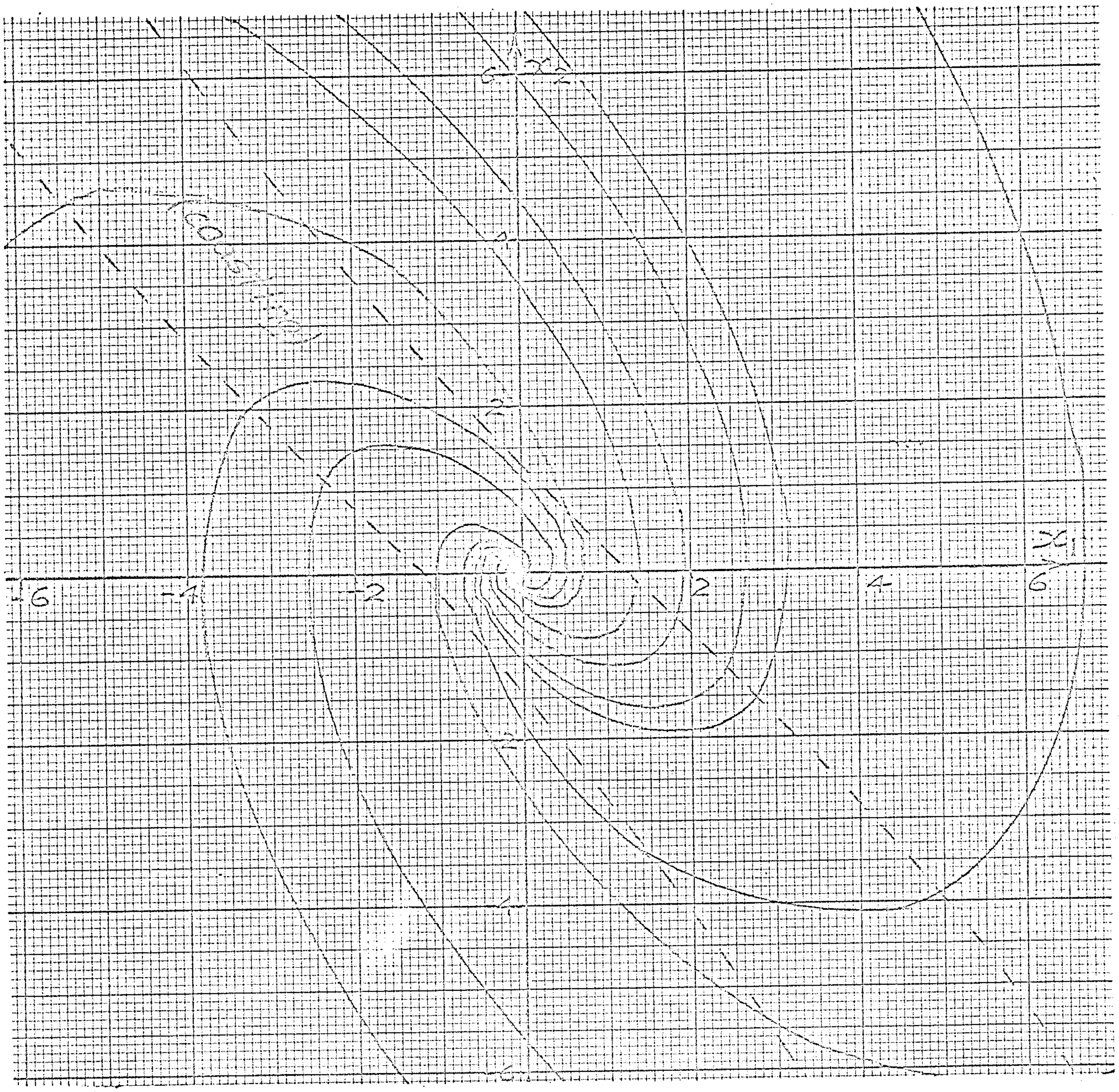


Fig. D-8 Optimal trajectories for the system with performance index I_{B-2}. (Case 3).

APPENDIX E

COMPARISON OF TWO METHODS USED TO MINIMIZE INTEGRAL-SQUARE ERROR

Pontryagin's techniques to minimize integral-aquare error were described in the major portion of this thesis. The approach of Newton, Gould, and Kaiser[31] is "to replace the saturation elements by a linear model which represents the behaviour of the saturating elements as long as the saturating signal (any signal within the fixed elements whose response exhibits saturation when the value of the signal falls outside a certain range known as the linear range) is within its linear range."

A control system of standard form,[31] as in Fig. E-1, is considered:

Given data: $G_f(s) = \frac{1}{s^2}$ $i(t) = v(t)$

$H_f(s) = 1$ $v(t) = N\,\delta_{-1}(t)$

A step input is used because it gives the same response as the decay from the initial conditions on x_1.

Required: Find the compensation $G_c(s)$ that minimizes the integral-square error $I_e = \int_{-\infty}^{+\infty} {y_e}^2(t)dt$ subject to the constraint that the integral-square value I_f of input m(t) to the fixed elements, $I_f = \int_{-\infty}^{\infty} m^2(t)dt$, shall not exceed a specified upper limit M. ($|m(t)| \leq M$).

Solution: The standard form of control system is converted to the cascade configuration,[31] as in Fig. E-2, where,

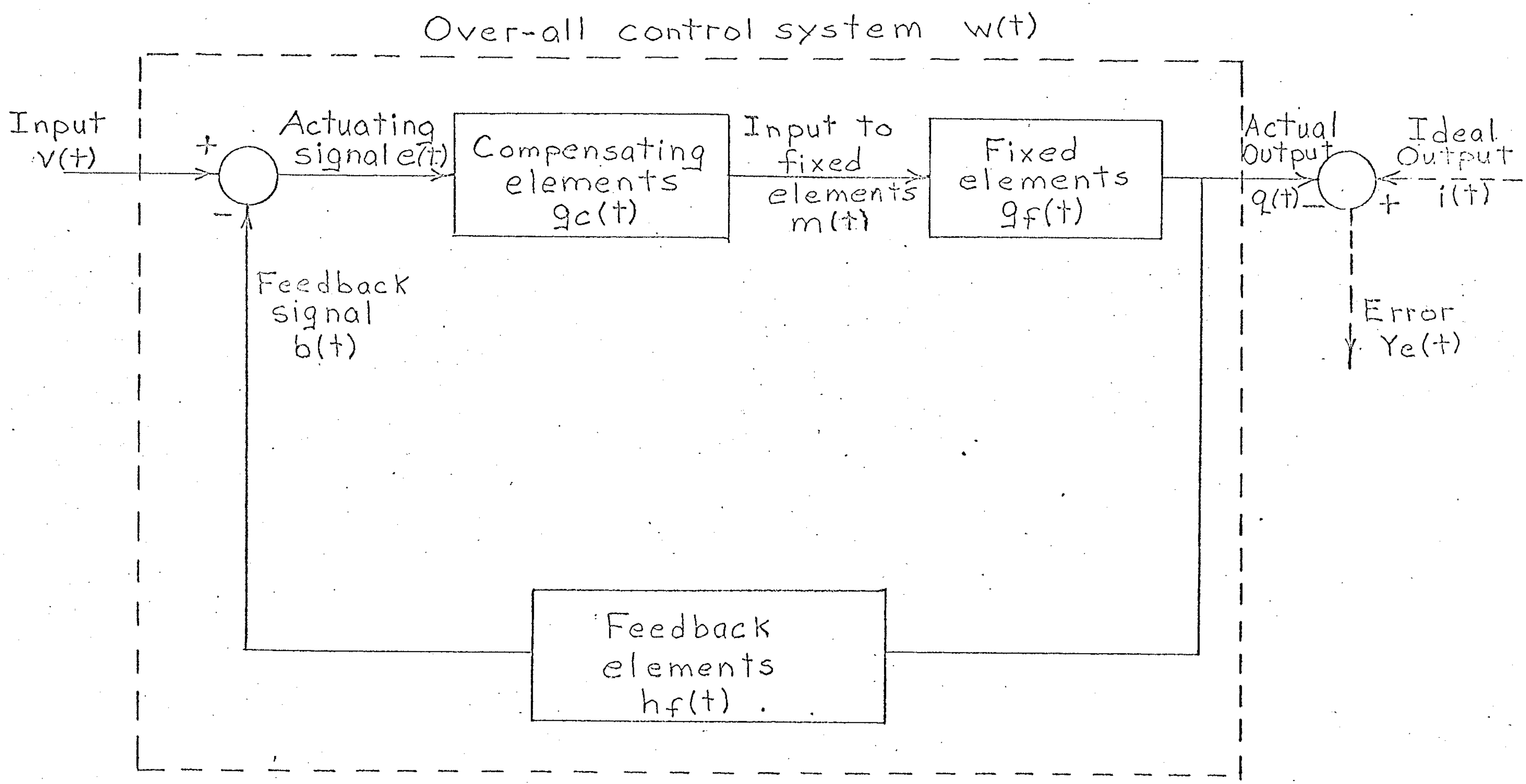


Fig. E-1 Standard form of control system. (Newton, Gould, and Kaiser).

$$W(s) = W_c(s)G_f(s)$$

with $$W_c(s) = \frac{G_c(s)}{1+G_c(s)G_f(s)H_f(s)}$$

The solution for optimum $W_c(s)$ is given by $W_{cm}(s) = \dfrac{\left[\dfrac{\Gamma(s)}{\Delta^-(s)}\right]_+}{\Delta^+(s)}$

where $$\Gamma(s) = G_f(-s)I_{vi}(s)$$

$$\Delta(s) = \rho I_{vv}(s) + G_f(s)G_f(-s)I_{vv}(s)$$

and, in general, $I_{xy}(s) = X(-s)Y(s)$.

ρ is a Lagrangian multiplier which will be set to satisfy the constraint on m. $\Delta^+(s)$ is that factor of $\Delta(s)$ containing only LHP poles and zeros; $\Delta^-(s)$ is that factor of $\Delta(s)$ containing only RHP poles and zeros so that $\Delta^+(s)\Delta^-(s) = \Delta(s)$.

Similarly $\left[\dfrac{\Gamma(s)}{\Delta^-(s)}\right]_+$ is that part of the partial fraction expansion of $\left[\dfrac{\Gamma(s)}{\Delta^-(s)}\right]$ containing only LHP poles.

From the given specifications,

$$I_{vi}(s) = I_{vv}(s) = -\frac{N^2}{s^2}$$

and $$\Gamma(s) = -\frac{N^2}{s^4}.$$

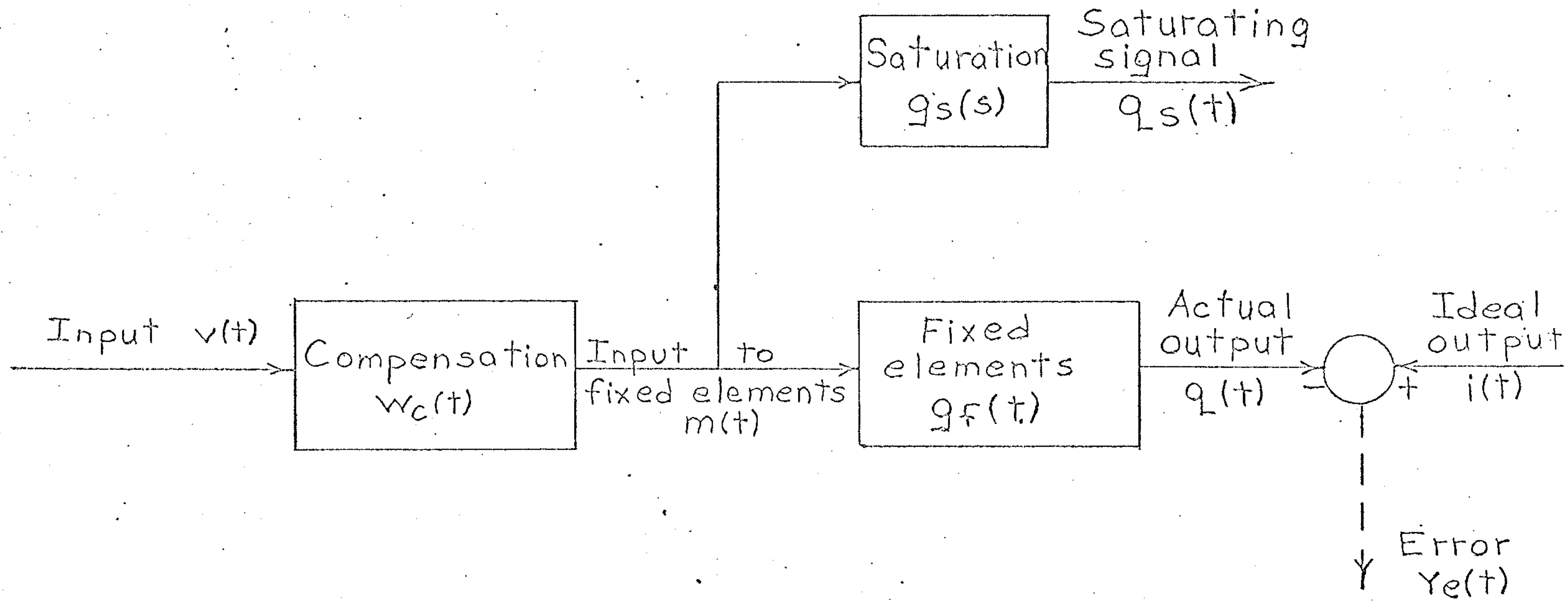


Fig. E-2 Cascade form of control system. (Newton, Gould, and Kaiser).

$$\Delta(s) = \rho I_{vv}(s) + G_f(s)G_f(-s)I_{vv}(s)$$

$$= \frac{N^2}{-s^2}(\rho + \frac{1}{s^4})$$

$$= N^2\rho\left(\frac{s^4+\frac{1}{\rho}}{-s^6}\right)$$

Let $\frac{1}{\rho} = a^4$, so that

$$\Delta^-(s) = \frac{(s^2 - \sqrt{2}\,as + a^2)}{(-s)(-s)^2}\,N^2\rho$$

$$\Delta^+(s) = \frac{(s^2 + \sqrt{2}\,as + a^2)}{(s)(s^2)}$$

$$\left[\frac{\Gamma(s)}{\Delta^-(s)}\right]_+ = \frac{1}{\sqrt{\rho}\,s}$$

Therefore,

$$W_{cm}(s) = \frac{s^2}{\sqrt{\rho}(s^2 + \sqrt{2}\,as + a^2)}$$

Now $y_e(s) = i(s) - W(s)v(s)$

$$W(s) = W_{cm}(s)G_f(s) = \frac{1}{\sqrt{\rho}(s^2 + \sqrt{2}\,as + a^2)}$$

$$y_e(s) = \frac{N}{s}\left\{1-W(s)\right\}$$

$$= N\,\frac{s + \sqrt{2}\,a}{s^2 + \sqrt{2}\,as + a^2}$$

$$I_e = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} I_{yy}(s)\,ds$$

$$= \frac{3N^2}{2\sqrt{2}}\,\rho^{1/4}$$

The integral-square error will be a minimum when ρ is as small as possible.

Now,

$$q(s) = G_f(s)m(s)$$

$$\frac{q(s)}{v(s)} = W(s) = \frac{G_c G_f}{1+G_c G_f}$$

$$\frac{m(s)}{v(s)} = \frac{W(s)}{G_f(s)}$$

Therefore,

$$m(s) = \frac{N}{\sqrt{\rho}} \quad \frac{1}{s^2 + \sqrt{2}\,as + a^2}$$

$$I_f = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} m(s)m(-s)ds \leq M$$

$$= \frac{N^2}{2\sqrt{2}\,\rho^{3/4}} \leq M$$

Since ρ must be as small as possible the equality sign is used in I_f, giving

$$\rho^{3/4} = \frac{N^2}{2\sqrt{2}\,M}$$

Now,

$$G_c(s) = \frac{W(s)}{G_f(s)-G_f(s)W(s)}$$

$$= \frac{1}{\sqrt{\rho}} \quad \frac{s}{s + \sqrt{2}\,a}$$

Substituting for ρ gives

$$G_c(s) = \frac{M^{1/3}}{N^{2/3}} \quad \frac{s}{1 + \frac{N^{2/3}}{2M^{1/3}}s}$$

which is of the form

$$G(s) = \frac{Ks}{1 + Ts}$$

This system is simulated on the analogue computer as shown in Fig. E-3 with A $\left\{A=\frac{1}{T}\right\}$ and B(B=AK) adjusted so that $|m(t)| \leq 1$ (which sets the limit M). Results in the phase plane for various step inputs are given in Fig. E-4 along with the results of the maximum principle. The step response is given in Fig. E-5 with the corresponding results by the maximum principle in Fig. E-6. For each trajectory, the time for the system to settle to zero error was recorded, as in Table IV.

TABLE IV

TRAJECTORY TIME COMPARISON

Step magnitude	Time (sec.)	
	Newton, Gould, and Kaiser	Pontryagin
1	6.36	2.96
2	10.40	3.76
3	12.00	4.48
4	15.00	4.94

From the results of both methods, it may be concluded:

1). The minimization of integral-square error by the maximum principle leads to a faster response.

2). The analysis by Newton, Gould, and Kaiser permits

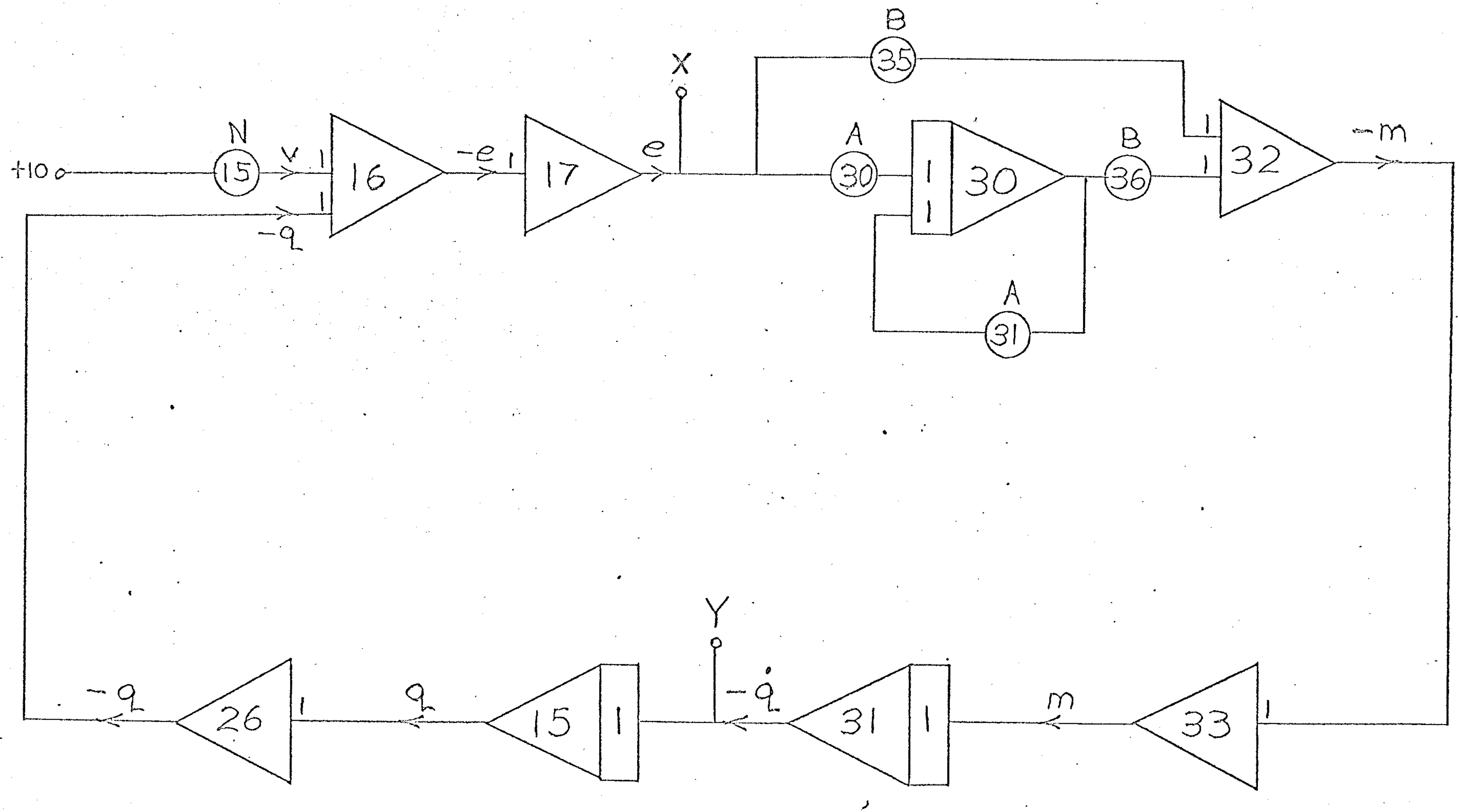
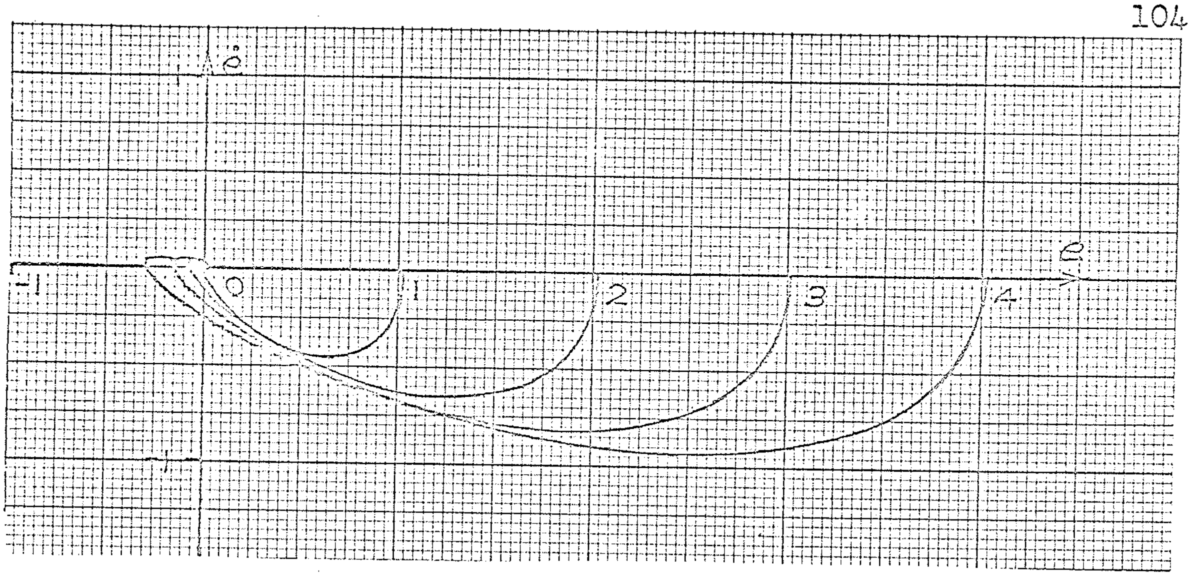
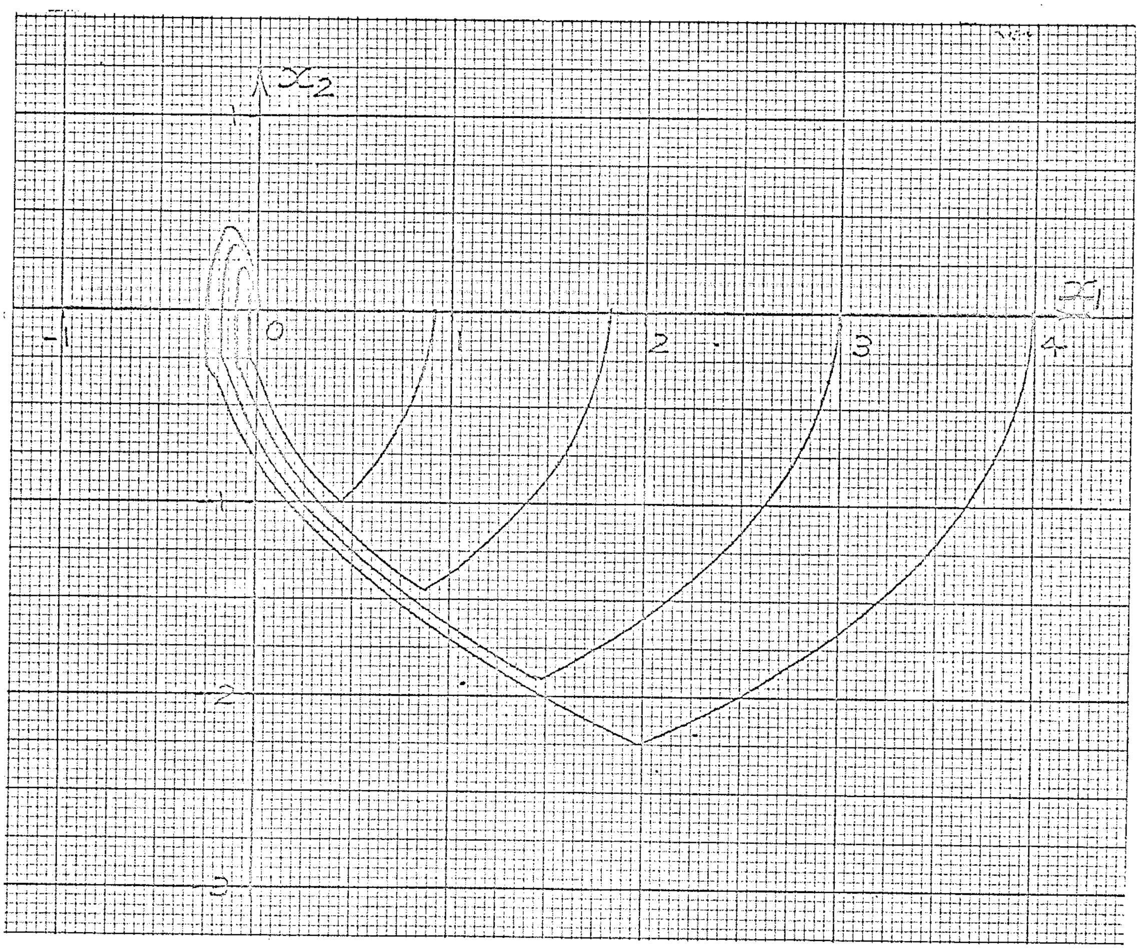


Fig. E-3 Analogue computer setup for minimization of integral-square error. (Newton, Gould, and Kaiser).

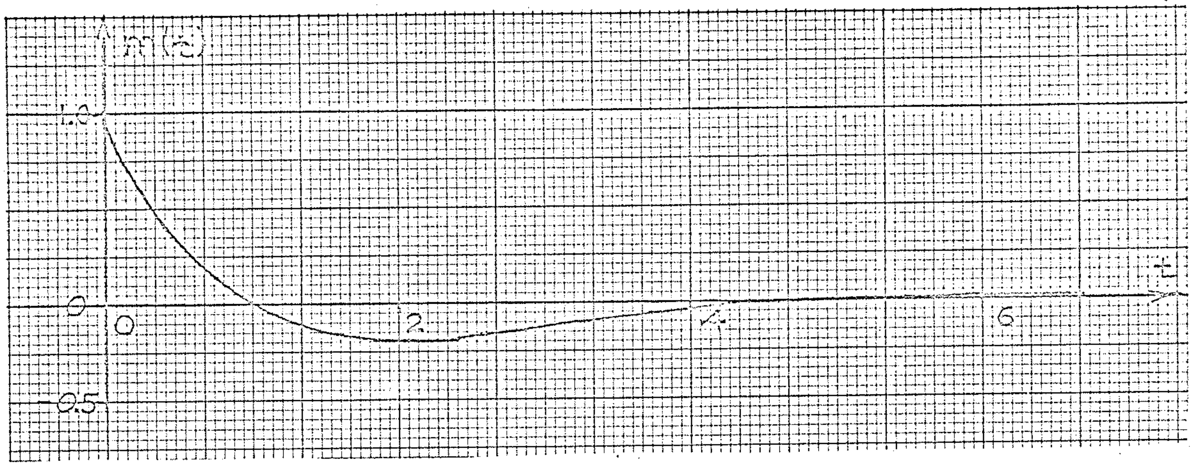


a) Newton, Gould, and Kaiser.

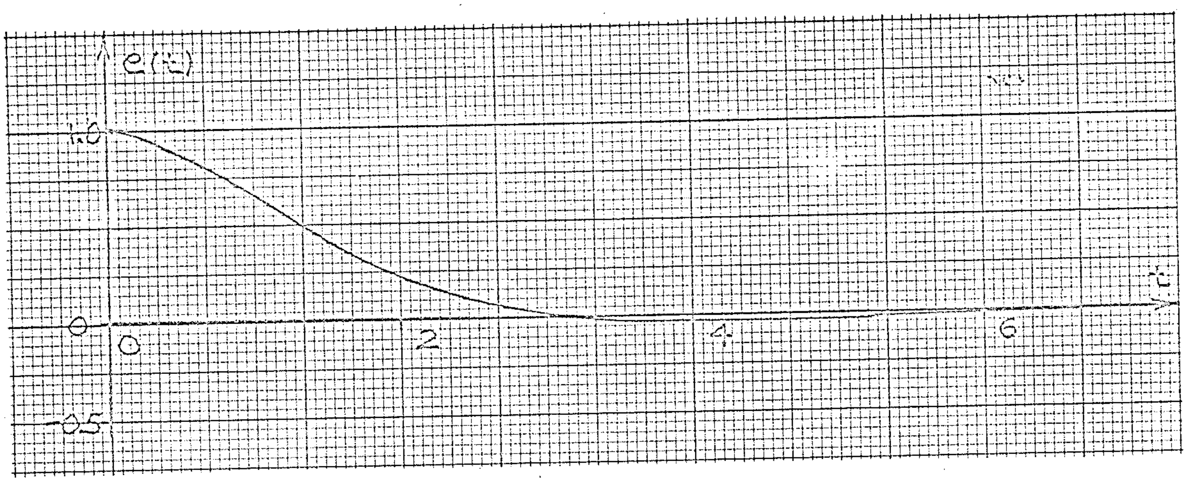


b) Pontryagin.

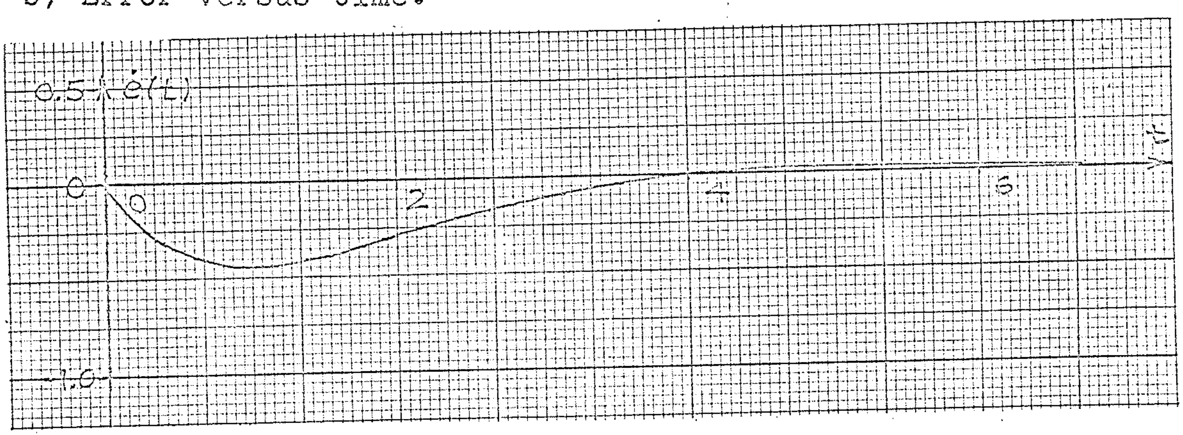
Fig. E-4 Comparison of optimal trajectories for system with plant $G(p) = \frac{1}{p^2}$ and integral-square error criterion.



a) Input to fixed elements versus time.

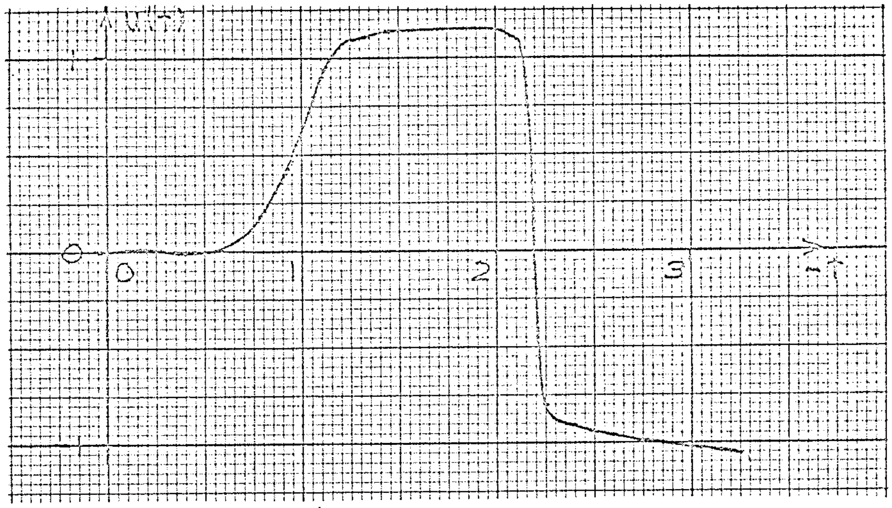


b) Error versus time.

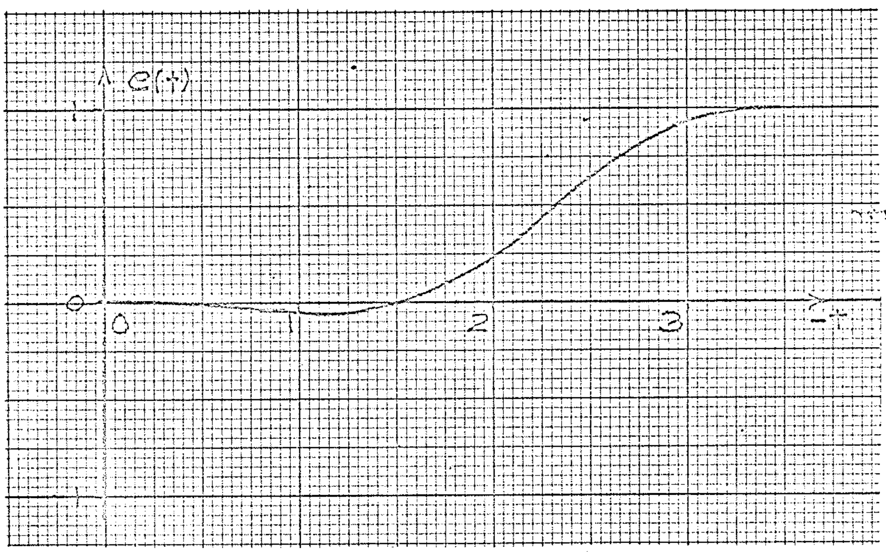


c) Error derivative versus time.

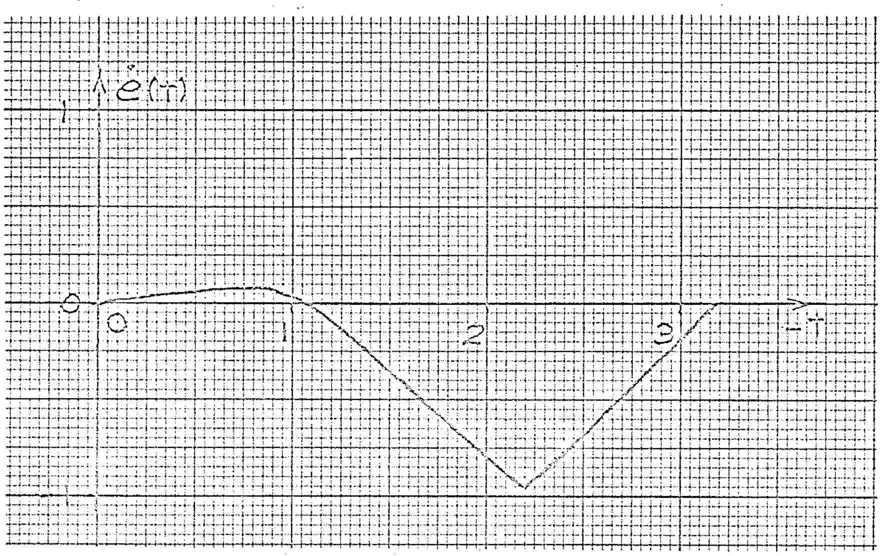
Fig. E-5 Step response of system. (Newton, Gould, and Kaiser).



a) Control input versus time.



b) Error versus time.



c) Error derivative versus time.

Fig. E-6. Step response of system. (Pontryagin)
(Time running backwards)

easy evaluation of the value of the integral-square error while the maximum principle does not lead to ready availability of this value.

3). The amount of calculations required to determine the optimum system is somewhat less when the maximum principle is used.

4). The compensation of Newton, Gould and Kaiser method is easier to realize physically than the controller resulting from Pontryagin's method.

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