THE "EXTENDED IDENTITY PROBLEM" FOR GROUPS
WITH ONLY ONE DEFINING RELATION

BY

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THE WRITER WISHES TO EXPRESS SINCERE
APPRECIATION TO DR. N.S. MENDELSOHN
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FOR HIS INTEREST AND DIRECTION.

INTRODUCTION

Let I be a well-ordered system and let $S=\{s_i;\ i\in I\}$ be a set of elements indexed by I. Corresponding to each s_i, we define another set of elements $\bar{S}=\{s_i^{-1};\ i\in I\}$. A word is a finite string of elements in $S\cup\bar{S}$, $W=a_1a_2\ldots a_t$, where a_i is in $S\cup\bar{S}$. For example, $W=s_{e_2}s_{e_4}s_{e_4}s_{e_6}^{-1}s_{e_6}s_{e_9}^{-1}$, where e_i is in I, is a word. The empty word, that is, the word consisting of no elements at all, is denoted by 1. If $W=a_1a_2\ldots a_t$, we represent $a_t^{-1}a_{t-1}^{-1}\ldots a_2^{-1}a_1^{-1}$ by W^{-1}, where $(s_i^{-1})^{-1}$ is defined to be s_i. The set of all such W's is denoted by M.

A word $W=a_1a_2\ldots a_t$ is <u>reduced as written</u> if $a_ia_{i+1}\neq a_ia_i^{-1}$ or $a_ia_{i+1}\neq a_{i+1}^{-1}a_{i+1}$, where $i=1,2,\ldots,t-1$. For example $W=s_{e_1}^{-1}s_{e_2}s_{e_4}s_{e_4}$ is reduced as written, and $W_1=s_{e_2}s_{e_3}s_{e_3}^{-1}$ is not reduced as written. A word, $W=a_1a_2\ldots a_t$, is <u>cyclically reduced</u> if it is reduced as written, and $a_1\neq a_t^{-1}$.

Define $s_is_i^{-1}$ and $s_j^{-1}s_j$ to be freely equal to 1. Two words W_2 and W_3 are said to be freely equal to each other if by cancelling such $s_is_i^{-1}$ and $s_j^{-1}s_j$ in W_2, we arrive at W_3. This relationship is denoted by "$\equiv$". For example, $s_i s_i^{-1} \equiv 1$, $s_{e_4} s_{e_8} s_{e_8}^{-1} s_{e_9} s_{e_1}^{-1} s_{e_1} \equiv s_{e_4} s_{e_9} s_{e_1}^{-1} s_{e_1}$

$$\equiv s_{e_4} s_{e_9}$$

It is not difficult to prove that "$\equiv$" is an equivalent relation. If W is in M, denote the equivalent class of W to be $[W]$. Thus every $[W]$ contains a unique reduced word.

The <u>length</u> of a word is the number of elements that are in the word when it is reduced as written. For example, the length of $W = s_{e_4} s_{e_7} s_{e_1} s_{e_1}^{-1} s_{e_3}^{-1}$ is three. The length of the empty word is zero.

We introduce an operation $*$ between elements in M to be: if W_1, W_2 are in M, then $W_1 * W_2 = W_1 W_2$. It can be verified that under this operation all equivalent classes $[W]$ from a group.

(1) $[W_1]\,[W_2] = [W_1 W_2]$

(2) $([W_1]\,[W_2])\,[W_3] = [W_1]\,([W_2]\,[W_3])$

(3) $[W^{-1}]$ is the inverse of $[W]$

(4) $[1]$ is the idendity element.

We call it the free group F generated by $S = \{s_i;\ i \in I\}$.

It is also known that every subgroup of a free group is free.[1]

Let I be a well-ordered indexed system and suppose G is a group generated by $X = \{x_i;\ i \in I\}$,then if F is the

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1. Marshcal Hall, Jr., <u>The Theory of Groups</u>, pg. 91

free group generated by $S=\{s_i;\ i\in I\}$, there exists a unique epimorphism E, mapping F onto G, such that $E(s_i)= x_i$. Thus G is isomorphic to F/K, where K is the kernel of E. If $k=a_1a_2\ldots a_t$ is in K, where a_i is in $S\cup\overline{S}$, then its image is the idendity, that is, $E(k)=b_1b_2\ldots b_t = g = 1$, where b_i is in $X\cup\overline{X}$ and $E(a_i) = b_i$. All such g's in G will be called <u>relations</u> between generators of G. Choose a subset of K, $T =\{k_1,k_2,\ldots,k_j,\ldots\}$, such that they form a minimal set of generators of K. Corresponding to T we get $R=\{r_1,r_2,\ldots,r_j,\ldots$ where r_i is in G. We say that R is a set of defining relations for G. Any element in K is a finite products of the k_j's and their conjugates and inverses, since K is the smallest normal subgroup of F generated by the k_j's. That is to say, if f is in K, then f can be expressed as ; $f\equiv\prod_{i=1}^{n} t_ik_{e_i}^{\epsilon}t_i^{-1}$ where k_{e_i} is in T, t_i is in F and $\epsilon = +1$ or -1. Suppose g is in G where $g = b_1b_2\ldots b_t$, b_i is in $X\cup\overline{X}$, and g=1, this implies that the pre-image of g is in K, that is, there exists f in K, such that E(f)=g. Since f is in K, we have

$$f\equiv\prod_{i=1}^{n} t_ik_{e_i}^{\epsilon}t_i^{-1} \quad \text{and } E(f)\equiv\prod_{i=1}^{n} E(t_i)E^{\epsilon}(k_{e_i})E(t_i^{-1})$$

$$g\equiv\prod_{i=1}^{n} g_ir_{e_i}^{\epsilon}g_i^{-1} \qquad \text{where } r_{e_i} \text{ is in } R$$

Therefore we can state that any relation in G can be expressed as a finite products of the r_i's and their conjugates and inverses. A group is completely determined by its set of defining relations, since F/K is completely determined and F/K is isomorphic to G.

The "word problem" or the "idendity problem" for a group G given by a set of generators and a finite set of defining relations is that of deciding whether any two given words are equal or not on account of the defining relations. In other words, it is to decide whether any given word is the idendity or not. It is a very difficult problem. In fact, Novikov in 1955 showed that the "word problem" cannot be solved. That is, there is no algorithm for one to decide whether any word in any given group with defining relations is equal to the idendity or not. The unsolvability of the "word problem" in the most general case leads us to investigate the questions of under what special cases our problem is solvable. It is obvious that the "word problem" is solvable in free groups. Magnus has shown in 1932 that the "word problem" can be solved for groups with only one defining relation.[2] In fact, Magnus even can solve the "extended idendity problem" which states that if G is generated by a countable set of generators X with only one defining relation, and if X'is a subset of X, then we can decide whether any given word can be expressed as a word in X' or not. Thus our "word problem" or "idendity problem" is just a special case of the "extended idendity problem"

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2. Magnus, W. 1932 Das Identitatsproblem fur Gruppen mit einer definierenden Relation Math. Ann.106,pg.259-307

by taking X' to be the empty subset of X. Magnus' algorithm for the solution of the "extended idendity problem" depends very strongly on the fundamental theorem of the "Freiheitssatz" and some knowledge on the free product of two groups with an amalgamated subgroup. This thesis is to prove and restate the algorithm for solving the "extended idendity problem" for groups with only one defining relation in a self-contained manner. Chapter 1 is to be devoted to Magnus' fundamental theorem of the "Freiheitssatz" and the "Hauptform" of the "Freiheitssatz". Chapter 2 is mainly about the existence of the free product of two groups with an amalgamated subgroup. Chapter 3 is chiefly concerned with the solution of the "extended idendity problem".

CHAPTER I

THE 'FREIHEITSSATZ'

The 'Freiheitssatz' states: if we are given generators $a_1, a_2, \ldots, a_n, a_{n+1} = x$ and a relation between them, $R(a_1, a_2, \ldots, a_n, x) = 1$, and if $W(a_1, a_2, \ldots, a_n) = 1 \not\equiv 1$ follows from $R(a_1, a_2, \ldots, a_n, x) = 1$, (that is to say, if x can be eliminated from $R(a_1, a_2, \ldots, a_n, x) = 1$), then $R \equiv TS(a_1, a_2, \ldots, a_n)T^{-1}$, where T is a word containg x and $W(a_1, a_2, \ldots, a_n) = 1$ follows from $S(a_1, a_2, \ldots, a_n) = 1$. We can state the 'Freiheitssatz' in another way; if G is a group generated by $(a_1, a_2, \ldots, a_n, a_{n+1})$ with a defining relation $R(a_1, a_2, \ldots, a_n, a_{n+1}) = 1$, where R is cyclically reduced and contains all the a_i, $i = 1, 2, \ldots, n+1$, then any set of the a_i's with number less than n+1 generates a free subgroup of G. Before we give the formal proof of the 'Freiheitssatz' we need three important lemmas.

Lemma 1.1 If G is a group generated by a system of generators which can be divided into four classes of generators;

$a_1, a_2, \ldots, a_m$ represented by a

$b_1, b_2, \ldots, b_n$ represented by b

$x_1, x_2, \ldots, x_r$ represented by x

$y_1, y_2, \ldots, y_s$ represented by y such that there are two finite systems of relations between them ;

(A) $P_u(a,b,x)=1$ $u=1,2,\ldots,j$

(B) $Q_v(b,x,y)=1$ $v=1,2,\ldots,k$

satisfying the following conditions:

(1) a cannot be eliminated from (A), that is, there is no relation between b_i's and x_i's deducible from (A).

(2) x cannot be eliminated from (A)

(3) y cannot be eliminated frome(B)

then there is no relation between a and b which can be obtained from the systems of relations of (A) and (B) together.

Proof: Suppose to the contrary that we have a relation $R(a,b)$ between a and b obtained from (A) and (B) together. Thus we have the following idendity:

$R(a,b) \equiv \prod_{i=1}^{h} T_i K_i^{(j)} T_i^{-1}$, where $j=1,2$, $K_i^{(1)}$ is a word in (a,b,x) and is a relation obtained from (A), and $K_i^{(2)}$ is a word in (b,x,y) and is a relation obtained from (B).

That is, $K_i^{(1)} = K_i^{(1)}(a,b,x) \equiv \prod_{i=1}^{d} M_i P_{u_i} M_i^{-1}$ and $K_i^{(1)} = 1$,

$$K_i^{(2)} = K_i^{(2)}(b,x,y) \equiv \prod_{i=1}^{k} N_i Q_{v_i} N_i^{-1} = 1$$

We name the following:

1. $F_i \equiv T_i K_i^{(j)} T_i^{-1}$, the i^{th} factor of the representation.

2. T_i, the conjugation word.

3. $K_i^{(j)}$, the kernel, which is taken to be cyclically reduced.

   Therefore, the kernel may be defined as what remains when we write a factor cyclically.

There are infinitely many such representations of R(a,b): we choose among them one which satisfies the following conditions:

I. Every factor is reduced as written.

II. It contains a minimal number of factors.

III. Among all such possible representations satisfying (I) and (II), choose one that has a minimal number of the a_i's and y_i's.

From the assumptions of our lemma and from (I),(II),(III), we can draw the following conclusions:

a). Every kernel contains either the a_i's or the y_i's, but never both of them. We call them the "characteristic elements".

b). There exist kernels containing the y_i's.

c). Between two adjacent factors having a_i's in their kernels, the conjugating words cannot cancel all the y_i's in each other.

Proof: Let $F_1 \equiv T_1 K_1^{(1)} T_1^{-1}$; $F_2 \equiv T_2 K_2^{(1)} T_2^{-1}$

where T_1, T_2 both contain some y_i's and where we suppose that the y_i's in T_1^{-1}, and T_2 are cancelled out by each other. Therefore we can write F_1, F_2 as;

$F_1 \equiv T_1 K_1^{(1)} T_1^{-1} \equiv MS_1 K_1^{(1)} S_1^{-1} M^{-1}$, where S_1 does not contain any y_i's but M contains all the y_i's in T_1.

$F_2 \equiv T_2 K_2^{(1)} T_2^{-1} \equiv MS_2 K_2^{(1)} S_2^{-1} M^{-1}$, where S_2 does not contain any y_i's but M contains all the y_i's in T_2.

$$
\begin{aligned}
F_1 F_2 &\equiv MS_1 K_1^{(1)} S_1^{-1} M^{-1} MS_2 \; K_2^{(2)} S_2^{-1} M^{-1} \\
&\equiv MS_1 K_1^{(1)} S_1^{-1} S_2 K_2^{(1)} S_2^{-1} M^{-1} \\
&\equiv MK_3^{(1)} M^{-1}, \text{ where } K_3^{(1)} \equiv S_1 K_1^{(1)} S_1^{-1} S_2 K_2^{(1)} S_2^{-1}
\end{aligned}
$$

Thus we can reduce the number of factors in the representation. This contradicts (II), which states that the representation has the minimal number of factors. Similarly, we can claim that between two adjacent factors having y_i's in their kernels, the conjugating words cannot cancel all the a_i's in each other.

d). Any characteristic elements in a kernel of a factor can never be cancelled out by the kernel of its adjacent factor;

Proof: Suppose the contrary; then the kernels of the two adjacent factors must be from the same system of relations, say, from (B). We have

$$F_1 \equiv T_1 K_1^{(2)} T_1^{-1} \quad , \qquad F_2 \equiv T_2 K_2^{(2)} T_2^{-1}$$

and since some y_i's in $K_1^{(2)}$ are cancelled by $K_2^{(2)}$, we must have $T_1 \equiv T_2$, and $F_1 F_2 \equiv T_1 K_1^{(2)} K_2^{(2)} T_1^{-1}$

$$\equiv T_1 K_3^{(2)} T_1^{-1}$$

Therefore the number of factors in the representation can be reduced. This contradicts (II).

e). For any two adjacent factors, not more that one half of the a_i's or y_i's of a factor can be cancelled out by the conjugating word of its neighbouring factor.

Proof: Suppose to the contrary that we have F_1 and F_2 as adjacent factors, and more than one half of the y_i's in F_2 are cancelled out by the conjugating word of F_1. We must have $F_2 \equiv T_2 K_2^{(2)} T_2^{-1}$

$$\equiv M\overline{M}$$

where M contains more y_i's than $\overline{M}$, and M contains the same number of a_i's or more than that of M.

We also have, $F_1 \equiv M S_1 K_1^{(j)} S_1^{-1} M^{-1}$, thus

$$
\begin{aligned}
F_1F_2 &\equiv MS_1K_1^{(j)}S_1^{-1}M^{-1}M\overline{M} \\
&\equiv MS_1K_1^{(j)}S_1^{-1}\overline{M} \\
&\equiv M\overline{M}(\overline{M}^{-1}S_1K_1^{(j)}S_1^{-1}\overline{M}) \\
&\equiv F_2DK_1^{(j)}D^{-1} \qquad \text{where } D\equiv \overline{M}^{-1}S_1
\end{aligned}
$$

Here $F_2DK_1^{(j)}D^{-1}$ contains a fewer number of a_i's and y_i's than does F_1F_2. This contradicts (III), which states that our representation has the minimal number of the a_i's and y_i's.

Now we have

$$
\begin{aligned}
R(a,b) &\equiv \prod_{i=1}^{h} T_iK_i^{(j)}T_i^{-1} \\
&\equiv F_1F_2\cdots F_h
\end{aligned}
$$

where the right hand side satisfies (I),(II),(III), and (a),(b),(c),(d),(e) . The following should be noticed:

1). In the right hand side, there exist factors containing the y_i's.

2) Owing to the left hand side, all the y_i's in the right hand side must be cancelled out.

3). Because of (c),(d),(e), the only way we can cancel out the y_i's in a factor is that exactly one half of them are cancelled out by the conjugating word of its left adjacent factor and the other half by the conjugating word of its right adjacent factor. We will call such a factor permutable, owing to our following claim:

If F_2 is permutable, then $F_1F_2F_3 \equiv F_2F_1'F_3 \equiv F_1F_3'F_2$, where F_1' , F_3' are conjugates of F_1 and F_3 respectively, and the numbers of a_i's and the numbers of y_i's in $F_1F_2F_3$, $F_2F_1'F_3$, $F_1F_3'F_2$ are the same.

Proof: Let $F_2 \equiv MS\overline{M}$, where M and $\overline{M}$ contain the same number of a_i's and the same number of y_i's, and S does not contain any a_i's nor y_i's, then

$$F_1 \equiv MS_1K_1^{(j)}S_1^{-1}M^{-1} \quad , \quad F_3 \equiv \overline{M}^{-1}S_3K_3^{(j)}S_3^{-1}\overline{M} \text{ , and}$$

$$\begin{aligned}
F_1F_2F_3 &\equiv (MS_1K_1^{(j)}S_1^{-1}M^{-1})(MS\overline{M})(\overline{M}^{-1}S_3K_3^{(j)}S_3^{-1}\overline{M}) \\
&\equiv MS_1K_1^{(j)}S_1^{-1}SS_3K_3^{(j)}S_3^{-1}\overline{M} \\
&\equiv (MS\overline{M})(\overline{M}^{-1}S^{-1}S_1K_1^{(j)}S_1^{-1}S\overline{M})(\overline{M}^{-1}S_3K_3^{(j)}S_3^{-1}\overline{M}) \\
&\equiv F_2F_1'F_3 \quad \text{, where } F_1' \equiv (M^{-1}S\overline{M})^{-1}F_1(M^{-1}S\overline{M}) \\
&\equiv (MS_1K_1^{(j)}S_1^{-1}M^{-1})(MSS_3K_3^{(j)}S_3^{-1}S^{-1}M^{-1})(MSM) \\
&\equiv F_1F_3'F_2 \quad \text{, where } F_3' \equiv (\overline{M}^{-1}S^{-1}M^{-1})^{-1}F_3(\overline{M}^{-1}S^{-1}M^{-1}
\end{aligned}$$

Here it is obvious that F_1', F_3' are not permutable. Also, If F_2 is permutable, then F_1 and F_3 are not permutable.

Among our possible representations satisfying (I),(II),(III), we choose one that also satisfies the following;

IV. It possesses a minimal number of permutable factors.

V. Among those satisfying IV, choose the representation whose first factor which is permutable has the smallest index.

We will show that there does not exist any permutable factor in our representation, thus a contradiction arises and our lemma is proved. Before proving it in the general case, we will give an example to illustrate that our claim is true. Suppose $R(a,b) \equiv F_1F_2F_3F_4F_5F_6F_7F_8F_9$, where F_4, and F_8 are permutable. Since F_4 is permutable, we permute F_4 with F_3 and get $F_1F_2F_4F_3'F_5F_6F_7F_8F_9$ (1)

As far as permutability is concerned, we have to consider F_2, F_4, F_3', F_5, the rest remaining the same. We claim that F_5 in (1) has to be permutable. It is known that F_3' is not permutable. If F_2 is permutable, then F_4 is not permutable, this implies that F_5 has to be permutable, otherwise (V) would be violated. If F_4 is permutable, then F_2 is not permutable, this implies again that F_5 has to be permutable, otherwise (V) would be violated. If both F_2 and F_4 are not permutable, then F_5 has to be permutable, otherwise (IV) would be contradicted. Therefore F_5 must be permutable in (1). We permute F_5 with F_3' and get $F_1F_2F_4F_5F_3''F_6F_7F_8F_9$ (2)

Here, F_5 is no longer permutable. Using a similar argument, we know that F_6 has to be permutable. We permute F_6 with

F_3'' and get $F_1F_2F_4F_5F_6F_3'''F_7F_8F_9F$ (3)

In (3), F_6 is no longer permutable, and so is F_5. Using a similar argument again, we conclude that F_7 must be permutable, but this is impossible since F_8 is given to be permutable. Thus a contradiction arises, therefore the representation $F_1F_2F_3F_4F_5F_6F_7F_8F_9$ does not have any permutable factor. Now we come to the proof of the general case. Let $R(a,b) \equiv F_1F_2 \dots F_{r-3}F_{r-2}F_{r-1}F_rF_{r+1}F_{r+2} \dots F_h$ (1')

Suppose that F_r is the first permutable factor in (1'); permute F_r with F_{r-1} and we get

$$F_1F_2 \dots F_{r-3}F_{r-2}F_rF'_{r-1}F_{r+1}F_{r+2} \dots F_h \quad (2')$$

As far as permutability is concerned, we have to consider F_{r-2}, F_r , F'_{r-1} , F_{r+1} , where the rest remains the same. We claim that F_{r+1} is permutable in (2'). We know that F'_{r-1} is not permutable. If F_{r-2} is permutable, then F_r is not permutable, but F_{r+1} has to be permutable, otherwise (V) will be violated. If F_{r-2} is not permutable, and also F_r is not permutable, then F_{r+1} has to be permutable, otherwise (IV) would be contradicted. If F_{r-2} is not permutable, but F_r is permutable, then F_{r+1} has to be permutable, otherwise (V) would be violated. Therefore F_{r+1} is permutable in (2'). Now permute F_{r+1} with F'_{r-1}, and we get

$$F_1F_2 \dots F_{r-3}F_{r-2}F_rF_{r+1}F''_{r-1}F_{r+2} \dots F_h \quad (3')$$

In (3') F_{r+1} is not permutable, so is F''_{r-1} , Using a similar argument, we can conclude that F_{r+2} has to be permutable. If in (1'), F_r is the only permutable factor, we repeat the process, and after h-r steps we come to $F_1F_2\ldots F_{r-3}F_{r-2}F_rF_{r+1}F_{r+2}\ldots F_h\overline{F}_{r-1}$, where the only possible permutable factor is F_r. If F_r is permutable, then (V) is violated. If there were no permutable factor, (IV) is contradicted. Therefore, lemma 1.1 is proved. If in (1') F_{r+t} is the second permutable factor, after t-1 steps, we get $F_1F_2\ldots F_{r-3}F_{r-2}F_rF_{r+1}\ldots F_{r+t-2}\overline{F}_{r-1}F_{r+t-1}F_{r+t}\ldots F_h$, where F_{r+t-1} must be permutable, since F_{r+t} is permutable, we have a contradiction. Therefore $R(a,b)\equiv F_1F_2\ldots F_h$ does not have any permutable factor. Consequently our lemma is proved.

<u>Lemma 1.1'</u> (an extension of lemma 1.1 which is important for the proof of the 'Hauptform' of the 'Freiheitssatz').

If G is a group generated by a system of generators which can be divided into four classes of generators :

$a_1, a_2, \ldots, a_s$ represented by a

$b_1, b_2, \ldots, b_t$ represented by b

$x_1, x_2, \ldots, x_m$ represented by x

$y_1, y_2, \ldots, y_n$ represented by y

and two finite systems of relations between them:

(A) $P_u(a,b,x)=1 \qquad u=1,2,\ldots,k$

(B) $Q_v(b,x,y)=1 \qquad v=1,2,\ldots,d$

such that:

1. From (A), a system of relations between the a_i's and b_i's follows, but no relation for the b_i's alone.
2. All the a_i's cannot be eliminated from (A)
3. All the y_i's cannot be eliminated from (B)

then the following is true; if there is a relation between a and b resulting from (A) and (B) together, then this relation must have resulted from (A) alone.

Proof: The proof follows the same argument as in lemma1.1. We have $R(a,b) \equiv \prod_{i=1}^{n} T_i K_i^{(j)} T_i^{-1}$, since there cannot be any y_i's in the factors of the right hand side, we must have $R(a,b) \equiv \prod_{i=1}^{n} T_i K_i^{(1)} T_i^{-1}$, where $K_i^{(1)}$ are relations from system (A)

Lemma 1.2 If G is a group generated by a system of generators which can be divided into three classes of generators;

$a_1, a_2, \ldots, a_m$ represented by a

$x_1, x_2, \ldots, x_r$ represented by x

$y_1, y_2, \ldots, y_s$ represented by y

and two finite systems of relations between them;

$(\bar{A})$ $\quad \bar{P}_u(a,x)=1 \qquad u=1,2,\ldots,t$

$(\bar{B})$ $\quad \bar{Q}_v(a,y)=1 \qquad v=1,2,\ldots,n$

such that;

1. All the x_i's cannot be eliminated from $(\bar{A})$
2. All the y_i's cannot be eliminated from $(\bar{B})$

then there is no relation between the a_i's alone from $(\bar{A})$ and $(\bar{B})$ together.

Proof; Suppose to the contrary that we have a relation $R(a)=1$ between the a_i's obtained from $(\bar{A})$ and $(\bar{B})$ together.

$$R(a) \equiv \prod_{i=1}^{h} T_i K_i^{(j)} T_i^{-1} = 1 \quad , \text{ where } j=1 \text{ or } 2 \text{ and}$$

$$K_i^{(1)} = K_i^{(1)}(a,x) \equiv \prod_{i=1}^{d} M_i \bar{P}_{u_i} M_i^{-1} = 1$$

$$K_i^{(2)} = K_i^{(2)}(a,y) \equiv \prod_{i=1}^{k} N_i \bar{Q}_{v_i} N_i^{-1} = 1$$

From all possible representations of $R(a)=1$, we choose one that satisfies the following conditions.

I' Every factor is reduced as written.

II' It contains the minimal number of factors.

III' Among all possible representations satisfying

(I') and (II') choose one that has the minimal number of x_i's and y_i's.

From the assumptions of our lemma and (I'),(II'),(III'), we have the following conclusions:

a') Every kernel contains either the x_i's or y_i's, but not both. We call them the characteristic elements.

b') There exist kernels containing the x_i's and there exist kernels containing the y_i's.

c') Between two adjacent factors having x_i's in their kernels, the conjugating words cannot cancel all the y_i's in each other, otherwise (II') will be contradicted. Similarly, between two adjacent factors having y_i's in their kernels, the conjugating words cannot cancel all the x_i's in each other.

d') Any characteristic elements in a kernel can never be cancelled out by its adjacent kernel, otherwise (II') will be violated.

e') For any two adjacent factors, not more than one half of the characteristic elements of a factor can be cancelled out by the conjugating word of its adjacent factor.

We have $R(a) \equiv \prod_{i=1}^{h} T_i K_i^{(j)} T_i^{-1} \equiv F_1 F_2 \cdots F_h$

1. There exist factors in the right hand side containing the x_i's and y_i's.

2. From the left hand side, it follows that all the x_i's and y_i's must be cancelled out.

3. Because of (c'), (d'), (e'), the only way we can cancel out the characteristic elements in a factor is that one half of them are cancelled out by the conjugating word of its left adjacent factor and the other half by the conjugating word of its right adjacent factor. Using an argument similar to that used in our proof of lemma 1.1, we can show that such permutable factor does not exist in our representation. Thus a contradiction arises, and our lemma is proved.

<u>Lemma 1.3</u> If G is a group generated by a system of generators which can be divided into two classes of generators

$d_1, d_2, \ldots, d_t$ represented by d, and

$p_0, p_1, p_2, \ldots p_h$

and a system of h-k+1 ($k>0$) relations between them:

$$Q_0 \quad (d; p_0, p_1, \ldots, p_k) = 1$$

$$Q_1 \quad (d; p_1, p_2, \ldots, p_{k+1}) = 1$$

$$\vdots$$

$$Q_{h-k}(d; p_{h-k}, p_{h-k+1}, \ldots, p_h) = 1$$

where in $Q_v(d;p_v,p_{v+1},\dots,p_{v+k})=1$, $v=0,1,\dots,h-k$, p_v and p_{v+k} appear in Q_v when cyclically reduced. We call such a system of relations ,(q). If $R(d;p_0,p_1,\dots,p_s)=1$, $0\leqslant s< k$, is a relation from (q), then there must be at least one relation from (q), say, $Q_v(d;p_v,\dots,p_{v+k})=1$ where p_v or p_{v+k} can be eliminated.

Proof; If $h-k=0$, our system of relations (q) has only one relation, $Q_0(d;p_0,p_1,\dots,p_k)=1$, from which we have $R(d;p_0,p_1,\dots,p_s)=1$, where $0\leqq s<k$, then our lemma is trivially true, since $R(d;p_0,p_1,\dots,p_s)=1$ obviously says that p_k is eliminated from Q_0.

Let us now assume that our assertion is true for all (q) having less than $h-k+1$ relations, and show that our claim is true for (q) having $h-k+1$ relations. We have to consider two cases

1. $h-k\leqslant s$
2. $h-k> s$

Case 1. $h-k\leqslant s$, we will make use of lemma 1.1.

We identify	p_0	with a
	$d,p_1,\dots,p_s$	with b
	$p_{s+1},\dots,p_k$	with x
	$p_{k+1},\dots,p_h$	with y

	a	b	x	y
(A)	Q_0 $(p_0,$	$d, p_1, p_2, \ldots, p_s,$	$p_{s+1}, \ldots, p_k)=1$	
	Q_1 (	$d, p_1, p_2, \ldots, p_s,$	$p_{s+1}, \ldots, p_k,$	$p_{k+1})=1$
(B)	$\vdots$			
	$Q_{h-k}($	$d, \quad p_{h-k}, \ldots, p_s,$	$p_{s+1}, \ldots, p_k,$	$p_{k+1}, p_{k+2}, \ldots, p_h)=1$

By lemma 1.1 we have the following;

1. Either a, identified as p_0, is eliminated from (A) or

2. x, identified as $p_{s+1}, \ldots, p_k$, is eliminated from (A), or

3. y, identified as $p_{k+1}, \ldots, p_h$, is eliminated from (B)

Both (1) and (2) imply that lemma 1.3 is true. Statement (3) tells us that there is a relation between $d, p_1, p_2, \ldots, p_s, \ldots, p_k$

$$
\begin{aligned}
\text{from}\quad & Q_1 \;(d;\; p_1 \;, \ldots, p_{k+1})=1 \\
& Q_2 \;(d;\; p_2 \;, \ldots, p_{k+2})=1 \\
& \vdots \\
& Q_{h-k}(d;\; p_{h-k}, \ldots, p_h \;)=1
\end{aligned}
$$

This system of relations satisfies all the assumptions in lemma 1.3 with less than h-k+1 relations; therefore by our hypothesis, there exists at least one $Q_i=1$, $i=1,2,\ldots,$or h-k in which p_i, or p_{i+1} can be eliminated. Thus lemma 1.3 is true.

Case 2 $h-k>s$

We identify $p_0, p_1, \ldots, p_s$ with a

$d_1, d_2, \ldots, d_t$ with b

$p_{s+1}, \ldots, p_{s+k}$ with x

$p_{s+k+1}, \ldots, p_h$ with y

		a	b	x	y
	Q_0	$(p_0, \ldots, p_s,$	$d,$	$p_{s+1}, \ldots, p_k \quad)$	$=1$
	Q_1	$(p_1, \ldots, p_s,$	$d,$	$p_{s+1}, \ldots, p_{k+1})$	$=1$
(A)	$\vdots$				
	Q_s	$(\quad p_s,$	$d,$	$p_{s+1}, \ldots, p_{k+s})$	$=1$
	Q_{s+1}	(	$d,$	$p_{s+1}, \ldots, p_{k+s},$	$p_{k+s+1})=1$
(B)	$\vdots$				
	Q_{h-k}	(	$d,$	$\ldots,$	$p_h)=1$

By applying lemma 1.1

1. Either x, identified as $p_{s+1}, \ldots, p_{s+k}$, is eliminated from (A), or
2. a, identified as $p_0, p_1, \ldots, p_s$, is eliminated from (A) or
3. y, identified as $p_{k+s+1}, \ldots, p_h$ is eliminated from (B).

Since (A) sastisfies the assumptions of our lemma with number of relations less than h-k+1, therefore by our hypothesis, statement (1) implies that lemma 1.3 is true. Similarly, statement (3) also implies that lemma 1.3 is true. If we set $p_i{}'=p_{k+s-i}$, $i=0,1,2,\ldots,k+s$

$$Q_j{}'=Q_{s-j} \quad , \; j=0,1,2,\ldots,s$$

we have;

$$Q_0'(d;p_0',p_1',\ldots,p_k') \quad = \quad Q_s \quad (d;p_s,p_{s+1},\ldots,p_{s+k}) =1$$

$$Q_1'(d;p_1',\ldots,p_{k+1}' \;) \quad = \quad Q_{s-1}(d;p_{s-1},\ldots,p_{s+k-1} \;) =1$$

$$\vdots$$

$$Q_s' \;(d;p_s',\ldots,p_{s+k}') \quad = \quad Q_0 \quad (d,p_0,p_1,\ldots,p_k \quad) =1$$

then statement (2) says that there exists a relation between $d,p_0',p_1',\ldots,p_s'$, where $0\leqslant s<k$, from $(q')=(Q_0',Q_1',\ldots,Q_s')$.

Since (q') satisfies all the assumptions of lemma 1.3 with the number of relations less than h-k+1, by our hypothesis, there must be at least one Q_v' from (q') from which p_v', or p_{v+k}' can be eliminated. That is to say, there must be a Q_t from (A) from which p_t or p_{t+k} can be eliminated. Therefore statement (2) also implies that lemma 1.3 is true. Thus our lemma is proved.

The proof of the 'Freiheitssatz' has to be divided into two cases.

Case 1. We suppose that $R(a,x)=1$ is cyclically reduced and x truly appears in r, and we show that $a^n=1$ resulting from $R(a,x)=1$ is impossible.

Case 2. We suppose that $R(a_1,a_2,\ldots,a_n,x)=1$, where R is cyclically reduced and contains at least two different a_i's and x. We will show that $W(a_1,a_2,\ldots,a_n)=1 \neq 1$ resulting from $R(a_1,a_2,\ldots,a_n,x)=1$ is impossible.

Case 1. We suppose to the contrary that $a^n=1$ follows from $R(a,x)=1$. The following should be noted:

1. The exponent-sum of x in $R(a,x)=1$ has to be zero.

Pr0Of: Since $a^n=1$ follows from $R(a,x)=1$, we have $a^n \equiv \prod_{i=1}^{h} T_i R^{e_i}(a,x) T_i^{-1}$, where $e_i=+1,-1$

We let $e_1+e_2+ \ldots +e_h=q$, and the exponent-sum of x in $R(a,x)$ be m, and the exponent-sum of a in $R(a,x)$ be k, then we have the following equations:

$$n=q\cdot k$$

$$0=q\cdot m$$

from which we can conclude that $m=0$, that is, the exponent-sum of x in R is zero.

2. If we set $x^k a x^{-k} = b_k$, where $k = \ldots,-2,-1,0,1,2,\ldots$ then any word in which x has exponent-sum zero can be expressed in terms of the b_k's. For example, $xa^2x^{-2}ax^2ax^{-1}a^4$

$$\equiv (xa^2x^{-1})(x^{1-2}ax^{-(1-2)})(x^{(1-2)+2}ax^{-(1-2+2)})$$
$$(x^{(1-2+2-1)}a^4)$$
$$\equiv b_1^2 b_{-1} b_1 b_0^4$$

3. Similarly, we can show that if $W(a,x)$ has exponent-sum t in x, then $W(a,x) \equiv \overline{W}(\ldots,b_k,\ldots)x^t$. For example, $ax^2a^{-3}x^{-1}a^2x \equiv b_0 b_2^{-3} b_1^2 x^2$

4: Since $R(a,x)=1$ has exponent-sum zero in x, we can express $R(a,x)$ as a word in the b_k's. Therefore $R(a,x)=\overline{R}(\ldots b_j \ldots)$ where j runs through a <u>fixed</u> set of integers. For example $R(a,x)=x^{-1}ax^2a^4x^3ax^{-4}=b_{-1}b_1^4b_4$, here j runs through $-1, 1$, and 4. Since $b_{-1}b_1^4b_4=1$, we also have $a^tb_{-1}b_1^4b_4a^{-t}=$, where t is any integer, that is, $b_{-1+t}b_{1+t}^4b_{4+t}=\overline{R}(\ldots b_{j+t}\ldots)=1$

Thus, if $R(a,x)=1$ has exponent-sum zero in x,

$R(a,x)= \overline{R}(\ldots b_j \ldots)=\overline{R}_0(\ldots b_j \ldots)=1$

$= \overline{R}_t(\ldots b_{j+t}\ldots)=1$, where j is a fixed set of integers and t is any integer, positive or negative.

5. Since $a^n=1$ is a relation from $R(a,x)=1$, we have the idendity;

$$a^n \equiv \prod_{i=1}^{h} T_i(a,x)R^{e_i}(a,x)T_i^{-1}(a,x)=1, \text{ where } e_i=+1,-1.$$

It should be noted that the representation of a^n on the right hand side is not unique. Owing to the fact that the exponentsum of x on both sides is zero, we can express the idendity in terms of the b_k's, thus we have;

$$b_0^n \equiv \prod_{i=1}^{h} \overline{T}_i(\ldots b_{j_i} \ldots)\, x^{t_i}\, \overline{R}^{e_i}(\ldots b_j \ldots)\, x^{-t_i}\overline{T}_i^{-1}(\ldots b_{j_i}.$$

$$\equiv \prod_{i=1}^{h} \overline{T}_i(\ldots b_{j_i} \ldots)\, \overline{R}_{t_i}^{e_i}(\ldots b_{j+t_i} \ldots)\, \overline{T}_i^{-1}(\ldots b_{j_i} \ldots)$$

where t_i is the exponent-sum of x in $T_i(a,x)$, and $T_i(a,x)=\overline{T}_i(\ldots b_{j_i} \ldots)x^{t_i}$, and $R(a,x)=\overline{R}(\ldots b_j \ldots)$. Therefore $b_0^n=1$ is a consequence (a relation resulting from) of a finite set of relations, $\overline{R}_{t_1}, \overline{R}_{t_2}, \ldots, \overline{R}_{t_h}$.

If $a^n \equiv \prod_{i=1}^{n} W_i(a,x)\, R^{e_i}(a,x)\, W_i^{-1}(a,x)$, we will have

$$b_0^n \equiv \prod_{i=1}^{n} \overline{W}_i(\ldots k_i \ldots)x^{s_i}\overline{R}^{e_i}(\ldots b_j \ldots)x^{-s_i}\overline{W}^{-1}(\ldots k_i \ldots)$$

$$\equiv \prod_{i=1}^{n} \overline{W}_i(\ldots k_i \ldots)\, \overline{R}_{s_i}^{e_i}(\ldots b_{j+s_i} \ldots)\, \overline{W}^{-1}(\ldots k_i \ldots)$$

which tells us that $b_0^n=1$ is a consequence of

another set of relations, $\overline{R}_{s_1}, \overline{R}_{s_2}, \ldots, \overline{R}_{s_n}$.
Therefore $b_0^b=1$ can be a consequence of an unlimited number of ysstems of finitely many relations. The following should be noted:

1. All these relations are isomorphic in the sense that $\overline{R}_n$ can be made identical to $\overline{R}_{t+n}$, by replacing each b_k in $\overline{R}_n$ by b_{k+n}.
2. The length of $\overline{R}_t$ is less than that of R. In fact length of $\overline{R}_t$ + total number of 'a' in R =length of R

We will show that $a^n=1$ being a consequence of $R(a,x)=1$ is impossible by making use of lemma 1.3 and by mathematical induction. Our claim is obviously true if $R(a,x)=x^m=1$, $m\neq 0$. Assume that our claim is true for all those R' whose length is less than that of R, then we want to show that it is also true for R. Before we give the proof of our assertion, we will give an example to show how the proof is carried out. Suppose the contrary that $a^n=1$ is a consequence of $R(a,x)=x^{-1}ax^2a^4x^3ax^{-4}=1$. Thus $\overline{R}(b_{-1},b_1,b_4)= b_{-1}b_1^4b_4$, and suppose that $a^n \equiv \prod_{i=1}^{4} T_i(a,x)\ R^{e_i}(a,x)\ T_i^{-1}(a,x)$, then we have the idendity;

$$b_0^n \equiv \prod_{i=1}^{4} \overline{T}_i(\ldots b_{j_i}\ldots)\ x^{t_i}\ \overline{R}^{e_i}(\ldots b_j \ldots)\ x^{-t_i}\ \overline{T}_i^{-1}(\ldots b_{j_i}\ldots)=1$$

$$b_0^n \equiv \prod_{i=1}^{4} \overline{T}_i(\ldots b_{j_i}\ldots)\ \overline{R}_{t_i}^{e_i}\ (\ldots b_j \ldots)\ \overline{T}^{-1}(\ldots b_{j_i}\ldots)$$

Suppose that $t_1=3$, $t_2=1$, $t_3=6$, $t_4=4$, then $b_0^n=1$ is a consequence of $\overline{R}_1$, $\overline{R}_3$, $\overline{R}_4$, $\overline{R}_6$. Choose a system of relations from $\{\overline{R}_t\}$ containing $\overline{R}_1$, $\overline{R}_3$, $\overline{R}_4$, $\overline{R}_6$, say, we take

$$R_0 \equiv \overline{R}(b_0,b_1,b_2,b_3,b_4,b_5) = b_0b_2^4b_5 = \overline{R}_1$$
$$R_1 \equiv \overline{R}(b_1,b_2,b_3,b_4,b_5,b_6) = b_1b_3^4b_6 = \overline{R}_2$$
$$R_2 \equiv \overline{R}(b_2,b_3,b_4,b_5,b_6,b_7) = b_2b_4^4b_7 = \overline{R}_3$$
$$R_3 \equiv \overline{R}(b_3,b_4,b_5,b_6,b_7,b_8) = b_3b_5^4b_8 = \overline{R}_4$$
$$R_4 \equiv \overline{R}(b_4,b_5,b_6,b_7,b_8,b_9) = b_4b_6^4b_9 = \overline{R}_5$$
$$R_5 \equiv \overline{R}(b_5,b_6,b_7,b_8,b_9,b_{10}) = b_5b_7^4b_{10} = \overline{R}_6$$
$$R_6 \equiv \overline{R}(b_6,b_7,b_8,b_9,b_{10},b_{11}) = b_6b_8^4b_{11} = \overline{R}_7$$
$$R_7 \equiv \overline{R}(b_7,b_8,b_9,b_{10},b_{11},b_{12}) = b_7b_9^4b_{12} = \overline{R}_8$$

This system of relations satisfies all the assumptions of lemma 1.3, with d equals the empty set. Since $b_0^n=1$ results from the above system of relations, by lemma 1.3 we must have one of the R_i in which b_i, or b_{i+5} can be eliminated. Since the length of R_i is less than that of R, by our hypothesis, b_i and b_{i+5} can never be eliminated from R_i, thus a contradiction arises, consequently $a^n=1$ can never be resulted from $R(a,x)=1$

Suppose $b_0^n \equiv \prod_{i=1}^{3} \overline{W}_i(\ldots k_i \ldots)\ \overline{R}_{s_i}^{e_i}(\ldots b_j \ldots)\ \overline{W}_i^{-1}(\ldots k_i \ldots)$,

where $s_1 = -2$, $s_2 = 1$, $s_3 = 2$, that is, $b_0^n = 1$ is a consequence of $\overline{R}_2$, $\overline{R}_1$, $\overline{R}_{-2}$. Choose a system of relations from $\{\overline{R}_t\}$ containing $\overline{R}_{-2}$, $\overline{R}_1$, $\overline{R}_2$, say, we take

$$R_0 \equiv \overline{R}(b_{-3}, b_{-2}, b_{-1}, b_0, b_1, b_2) = b_{-3} b_{-1}^4 b_2 = \overline{R}_{-2}$$

$$R_1 \equiv \overline{R}(b_{-2}, b_{-1}, b_0, b_1, b_2, b_3) = b_{-2} b_0^4 b_3 = \overline{R}_{-1}$$

$$R_2 \equiv \overline{R}(b_{-1}, b_0, b_1, b_2, b_3, b_4) = b_{-1} b_1^4 b_4 = \overline{R}_0$$

$$R_3 \equiv R(b_0, b_1, b_2, b_3, b_4, b_5) = b_0\, b_2\, b_5 = \overline{R}_1$$

$$R_4 \equiv \overline{R}(b_1, b_2, b_3, b_4, b_5, b_6) = b_1\, b_3^4\, b_6 = \overline{R}_2$$

$$R_5 \equiv \overline{R}(b_2, b_3, b_4, b_5, b_6, b_7) = b_2\, b_4^4\, b_7 = \overline{R}_3$$

Lemma 1.2 will be make use of, identifying

a with b_0 , b_1 , b_2 , b_3, b_4

x with b_5 , b_6 , b_7

y with b_{-3}, b_{-2}, b_{-1}

$(\overline{A})$ is the system of relations containing R_3, R_4, R_5

$(\overline{B})$ is the system of relations containing R_0, R_1, R_2

Since from $(\overline{A})$ and $(\overline{B})$ a relation for the a_i's follows $(b_0^n = 1)$, by lemma 1.2, we must have x eliminated from $(\overline{A})$ or y eliminated from $(\overline{B})$. We will make use of lemma 1.3 to show that y cannot be eliminated from $(\overline{B})$, that is , there is no relation between the a_i's from $(\overline{B})$.

$$(\bar{B}) \qquad \begin{array}{ll} R_0 & \bar{R}(b_{-3},b_{-2},b_{-1},b_0,b_1,b_2)=1 \\ R_1 & \bar{R}(b_{-2},b_{-1},b_0,b_1,b_2,b_3)=1 \\ R_2 & \bar{R}(b_{-1},b_0,b_1,b_2,b_3,b_4)=1 \end{array}$$

Identify	p_0	with	b_4
	p_1	with	b_3
	p_2	with	b_2
	p_3	with	b_1
	p_4	with	b_0
	p_5	with	b_{-1}
	p_6	with	b_{-2}
	p_7	with	b_{-3}
	$Q_v=1$	with	$R_{2-v}=1$, $v=0,1,2.$

To say that all the y_i's are eliminated from $(\bar{B})$ means that there is a relation between b_0,b_1,b_2,b_3,b_4, which follows from $(\bar{B})$, that is to say, a relation between p_0,p_1,p_2,p_3,p_4, from

$$\begin{aligned} &Q_0(p_0,p_1,p_2,p_3,p_4,p_5)=1 \\ &Q_1(p_1,p_2,p_3,p_4,p_5,p_6)=1 \\ &Q_2(p_2,p_3,p_4,p_5,p_6,p_7)=1 \end{aligned}$$

By lemma 1.3, there must exist an $Q_i=1$, from which p_i or p_{i+5} can be eliminated. But by hypothesis, this is impossible. Therefore y cannot be eliminated from $(\bar{B})$.

Similarly, by identifying p_i with b_i, $i=0,1,2,3,4,5,6,7.$, and $Q_v=1$ with $R_{3+v}=1$, $v=0,1,2.$, we can show that x cannot be eliminated from $(\bar{A})$. Thus $a^n=1$ cannot be a consequence of $R(a,x)=1$.

Now we come to the general proof. From the system of relations $\bar{R}_t(\ldots b_{j+t}\ldots)=1$, where $R=\bar{R}_0(\ldots b_j\ldots)=1$, j runs through a fixed set of integers, and suppose M_1 is the maximum of j, and M_0 is the minimum of j, then $M_1-M_0=L$, and L has to be greater than zero, otherwise x does not appear in R, we choose a finite system from which $b_0^n=1$ is a consequence:

$$R_0 \equiv \bar{R}(b_{-N}, b_{-N+1}, \ldots, b_{-N+L}) = 1$$

$$R_1 \equiv \bar{R}(b_{-N+1}, b_{-N+2}, \ldots, b_{-N+L+1}) = 1$$

$$\vdots$$

$$R_{M-L+N} \equiv \bar{R}(b_{M-L}, b_{M-L+1}, \ldots, b_M) = 1, \text{ where } N\geq 0, M\geq 0, L>0.$$

We have to have $M>-N$, otherwise x does not appear in R. Without loss of generality, take $M-L\geq 0$, for if $b_0^n=1$ does not follow from the system with a larger value of M, it surely does not follow from that system with a lesser number of relations. We have the following two cases to consider:

a). $N=0$, and $M-L>0$

b). $N>0$, and $M-L>0$

<u>Case a</u> : We have $N=0$, $M-L>0$, and

$$
\begin{aligned}
R_0 &\equiv \overline{R}(b_0, b_1, b_2, \ldots, b_L) = 1 \\
R_1 &\equiv \overline{R}(b_1, b_2, b_3, \ldots, b_{L+1}) = 1 \\
&\vdots \\
R_{M-L} &\equiv \overline{R}(b_{M-L}, \ldots, b_M) = 1
\end{aligned}
$$

This system of relations satisfies all the assumptions of lemma 1.3, and since $b_0^n=1$ follows from this system of relations, we must have one of the $R_i=1$ from which b_i or b_{i+L} can be eliminated, but this is not possible. Therefore $b_0^n=1$ is not the consequence of $R(a,x)=1$

<u>Case b</u> : We have $N>0$, $M-L>0$ and

$$
\begin{aligned}
R_0 &\equiv \overline{R}(b_{-N}, b_{-N+1}, \ldots, b_{-N+L}) = 1 \\
R_1 &\equiv \overline{R}(b_{-N+1}, b_{-N+2}, \ldots, b_{-N+L+1}) = 1 \\
&\vdots \\
R_N &\equiv \overline{R}(b_0, b_1, \ldots, b_L) = 1 \\
&\vdots \\
R_{M-L+N} &\equiv \overline{R}(b_{M-L}, b_{M-L+1}, \ldots, b_L) = 1
\end{aligned}
$$

Identify a with $b_0, b_1, \ldots, b_{L-1}$

x with $b_L, b_{L+1}, \ldots, b_M$

y with $b_{-N}, b_{-N+1}, \ldots, b_{-1}$

then $(\bar{A})$ consists of $R_N, R_{N+1}, \ldots, R_{M-L+N}$

$(\bar{B})$ consists of $R_0, R_1, R_2, \ldots, R_{N-1}$

Since from $(\bar{A})$ and $(\bar{B})$ a relation for the a_i's follows, (that is, $b_0^n=1$), by lemma 1.2, we must have x eliminated from $(\bar{A})$ or y eliminated from $(\bar{B})$. Since the system of relations $(\bar{A})$ satisfies the assumptions of lemma 1.3, if a relation between $b_0, b_1, b_2, \ldots, b_{L-1}$ follows from $(\bar{A})$, (that is to say, if x is eliminated from $(\bar{A})$,) then by lemma 1.3 there is at least one relation, say R_{N+i}, $i=0,1,\ldots,$or $M-L$ from which b_i or b_{L+i} can be eliminated. But this is impossible by our hypothesis. Therefore x cannot be eliminated from $(\bar{A})$. Suppose y is eliminated from $(\bar{B})$, that is, a relation between $b_0, b_1, \ldots, b_{L-1}$ follows from $(\bar{B})$. We identify; $p_0, p_1, p_2, \ldots, p_s$ with $b_0, b_1, b_2, \ldots, b_{L-1}$

$p_{s+1}, \ldots, b_k$ with b_{-1}

$p_{k+1}, \ldots, p_h$ with $b_{-2}, b_{-3}, \ldots, b_{-N}$

$Q_v=1$ with $P_{N-v-1}=1$, $v=0,1,\ldots,N-1$

By lemma 1.3 again, and by our hypothesis, y cannot be eliminated from $(\bar{B})$. Since x cannot be eliminated from $(\bar{A})$

and y cannot be eliminated from $(\overline{B})$, a contradiction arises, implying that $a^n=1$ cannot be a consequence of $R(a,x)=1$.

Case 2; We suppose that $R(a_1,a_2,\ldots a_n,x)=1$, where R is cyclically reduced and contains at least two different a_i's and x. We will show that $W(a_1,a_2,\ldots,a_n)=1 \not\equiv 1$, resulting from $R(a_1,a_2,\ldots,a_n,x)$ is impossible.

The following should be noted:

1. The exponent-sum of x in R need not be zero, because the exponent-sum of the a_i's in W may be zero for each i.
2. If there exists an a_t such that the exponent-sum of a_t in R is zero, we rearrange the order of the a_i's and put a_t to be a_1.
3. If there does not exist any a_v whose exponent-sum in R is zero, we will make the following substitution:

   suppose a_1 in R has exponent-sum $s_1 \neq 0$

   a_2 in R has exponent-sum $s_2 \neq 0$,

   we put $a_1 = b_1^{s_2}$, $a_2 = b_1^{-s_1} b_2$.

   After substituting, we have;

   $R(a_1,a_2,\ldots,a_n,x) = \overline{R}(b_1,b_2,a_3,\ldots,a_n,x) = 1$

   $W(a_1,a_2,\ldots,a_n) = \overline{W}(b_1,b_2,a_3,\ldots,a_n) = 1$

where b_1 in $\bar{R}$ has exponent-sum zero. Since R is cyclically reduced, it is obvious that the only element in $\bar{R}$ in which we can do some cancellation is b_1; there are no cancellations between $b_2, a_3, \ldots, a_n$ and x. Therefore $\bar{R}$ contains x if and only if R does. Consequently our substitution does not affect the proof of our claim.

4. Since W is a consequence of R, we have the following idendity;

$$W(a_1,\ldots,a_n) \equiv \prod_{i=1}^{h} T_i(a_1,a_2,\ldots,a_n,x)\ R^{e_i}(a_1,\ldots,a_n,x)\ T_i^{-1}$$

After substituting $a_1 = b_1^{s_2}$, $a_2 = b_1^{-s_1} b_2$, we have

$$\bar{W}(b_1,b_2,a_3,\ldots,a_n)$$
$$\equiv \prod_{i=1}^{h} \bar{T}_i(b_1,b_2,a_3,\ldots,a_n,x)\ \bar{R}^{e_i}(b_1,b_2,a_3,\ldots,a_n,x)\ \bar{T}_i^{-1}$$

where b_1 has exponent-sum zero in $\bar{R}$ and $\bar{W}$. Now we set

$$c_{v,k} = \begin{cases} b_1^k b_2 b_1^{-k} & \text{for } v=2 \\ b_1^k a_v b_1^{-k} & \text{for } v=3,4,\ldots,n \end{cases}$$

$$x_k = b_1^k x b_1^{-k}$$

(For the case in (2), where we have a_1 whose exponent-sum zero in R, set $a_1 = b_1$, $a_2 = b_2$), then

$\bar{W} = F(c_{2,k_2}, c_{3,k_3}, \ldots, c_{n,k_n}) = F(\ldots c_{v,k'} \ldots)$, where

$k_2, k_3, \ldots, k_n$ represented by k' are fixed sets of integers.

$$\overline{R} = P(c_{2,j_2}, c_{3,j_3}, \ldots, c_{n,j_n}; \ldots x_m \ldots) = P(\ldots c_{v,j} \ldots; \ldots x_m \ldots)$$

where $j_2, j_3, \ldots, j_n$, represented by j, and m are fixed sets of integers, and we let M_1 be the maximum of m and M_0 be the minimum of m, and $L = M_1 - M_0$.

$$\overline{T}_i = H_i(c_{2,s_{i_2}}, c_{3,s_{i_3}}, \ldots, c_{n,s_{i_n}}; \ldots x_{m_i} \ldots)\, b_1^{t_i}$$

$= H_i(\ldots c_{v,s_i'} \ldots; \ldots x_{m_i} \ldots)\, b_1^{t_i}$, where m_i and $s_{i_2}, s_{i_3}, \ldots, s_{i_n}$, represented by s_i' are fixed sets of integers, and t_i is the exponent-sum of b_1 in $\overline{T}_i$.

Thus we have the following idendity;

$$F(\ldots c_{v,k'} \ldots)$$

$$\equiv \prod_{i=1}^{h} H_i(\ldots c_{v,s_i'} \ldots; \ldots x_{m_i} \ldots)\, b_i^{t_i} P^{\pm 1}(\ldots c_{v,j} \ldots; \ldots x_m \ldots)\, b_i^{-t_i} H_i^{-1}$$

$$\equiv \prod_{i=1}^{h} H_i(\ldots c_{v,s_i'} \ldots; \ldots x_{m_i} \ldots)\, P_{t_i}^{\pm 1}(\ldots c_{v,j+t_i} \ldots; \ldots x_{m+t_i} \ldots) H_i^{-1}$$

Therefore $F(\ldots c_{v,k'} \ldots)$ is a consequence of a finite system of relations, $P_{t_1}, P_{t_2}, \ldots P_{t_h}$. Since the representation of W and consequently F is not unique, then corresponding to different representations, we get different systems with a finite number of relations, from which F is the consequence. As in case 1, from $\{P_t\}$ we pick a finite number of relations from which

F is a consequence:

$$\overline{P}_0 \equiv P(\ldots c_{v,j+N} \ldots ; x_N, \ldots, x_{N+L}) = 1$$

$$\overline{P}_1 \equiv P(\ldots c_{v,j+N+1} \ldots ; x_{N+1}, \ldots, x_{N+L+1}) = 1$$

$$\vdots$$

$$\overline{P}_{M-N} \equiv P(\ldots c_{v,j+M} \ldots ; x_M, \ldots, x_{M+L}) = 1$$

where $M \geq N$, $L \geq 0$, since $L = M_1 - M_0$, and x_{N+v}, x_{N+v+L} appear in $\overline{P}_v$ when cyclically reduced. We have to consider three cases:

1: M=N

2. L=0, M N

3. L 0), M N

<u>Case 1</u>: If M=N, then F=1 is a consequence of $\overline{P}_0 \equiv P(\ldots c_{v,j+N} \ldots ; x_N, \ldots, x_{N+L})$, which is impossible by our hypothesis, since the length of $\overline{P}_0$ is less than that of R.

<u>Case 2</u>: L=0 , $M > N$, our system of equations becomes;

$$\overline{P}_0 \equiv P(\ldots c_{v,j+N} \ldots ; x_N) = 1$$

$$\overline{P}_1 \equiv P(\ldots c_{v,j+N+1} \ldots ; x_{N+1}) = 1$$

$$\vdots$$

$$\overline{P}_{M-N} \equiv P(\ldots c_{v,j+M} \ldots ; x_M) = 1$$

We identify a with $\ldots c_{v,s}\ldots$

x with x_N

y with $x_{N+1},\ldots,x_M$

By lemma 1.2, either x is eliminated from $\overline{P}_0$ or y is eliminated from $\overline{P}_1,\overline{P}_2,\ldots,\overline{P}_{M-N}$. But x being eliminated from P_0 is impossible by our hypothesis, thus y has to be eliminated from $\overline{P}_1,\overline{P}_2,\ldots,\overline{P}_{M-N}$. If $M-N=1$, by hypothesis y cannot be eliminated from $\overline{P}_1$, thus we have a contradiction. If $M-N>1$, we repeat what we have done before until we have $(\overline{B})'$ consisting of only one relation. Therefore $W=1$ cannot be a consequence of $R=1$.

Case 3' We have $L>0$, $M-N>0$, the system of relations:

$$\overline{P}_0 \equiv P(\ldots c_{v,j+N}\ \ldots ;\ x_N\ ,\ldots,x_{N+L}\ ) =1$$

$$\overline{P}_1 \equiv P(\ldots c_{v,j+N+1}\ldots ;\ x_{N+1},\ldots,x_{N+L+1}) =1$$

$$\vdots$$

$$\overline{P}_{M-N} \equiv P(\ldots c_{v,j+M}\ \ldots ;\ x_M\ ,\ldots,x_{M+L}\ ) =1$$

We identify d with $\ldots c_{v,s}\ldots$

$p_0,p_1,\ldots,p_s$ with $x_N,x_{N+1},\ldots,x_{N+L-1}$

$p_{s+1},\ldots,p_k$ with x_{N+L}

$$p_{k+1},\dots,p_h \quad \text{with} \quad x_{N+L+1},\dots,x_M$$

$$Q_v=1 \qquad \text{with} \qquad \overline{P}_v=1 \ , \quad v=0,1,\dots,M-N$$

A relation between the d_i's follows from the system, therefore by lemma 1.3, theref must be at least a $Q_v=1$, where p_v or p_{v+L} can be eliminated, contradicting to our hypothesis. Therefore $W=1$ is not a consequence of $R=1$.

'Hauptform' of the 'Freiheitssatz': Suppose we are given generators $b_0,b_1,\dots,b_M$ and $c_1,c_2,\dots c_t$, represented by c, and a system of relations between them:

$$S_0 \ (c, \ b_0 \ ,b_1,\dots,b_K \ ) = 1$$
$$S_1 \ (c; \ b_1 \ ,b_2,\dots,b_{K+1}) = 1$$
$$\vdots$$
$$S_{M-K} \ (c; \ b_{M-K}, \ \dots \ ,b_M \ ) = 1$$

where $K\leqslant M$, and b_v and b_{v+K} appear in $S_v=1$ when cyclically reduced. If $R(c;\ b_H,b_{H+1},\dots b_L)=1$, where $0\leqq H\leqslant L\leqslant M$, is a consequence of the above system of relations, then R must be from among those in the above system of relations containing only $c,\ b_H,b_{H+1},\dots,b_L$.

<u>Proof</u> : If $K=M$, our theorem is reduced to the 'Freiheitssatz', and our claim is obviously true. We suppose that our theorem is true for all systems of relations having less than $M-K+1$ relations and show that it is true for system of relations having $M-K+1$ relations. The following cases should be considered;

1. $L<K$
2. $L\geq K$, and $L\leq M-K-1$
3. $L\geq K$, and $L>M-K-1$

Case 1. If $L<K$, by lemma 1.3 and the 'Freiheitssatz', there does not exist any relation between c, and $b_H, b_{H+1}, \ldots, b_L$, where $0\leq H\leq L<K$, from our system of relations. Thus our theorem is trivially true.

Case 2. If $L\geq K$, and $L\leq M-K-1$, we identify:

a with $b_0, b_1, \ldots, b_L, b_{L+1}, \ldots b_{M-K-1}$

b with c ; b_{M-K}

x with $b_{M-K+1}, b_{M-K+2}, \ldots, b_{M-1}$

y with b_M

(A) with $S_0, S_1, \ldots, S_{M-K-1}$

(B) with S_{M-K}

		b	a	b	x	y
	S_0	$(c;$	$b_0, \ldots, b_K)=1$			
	S_1	$(c;$	$b_1, \ldots, b_{K+1})=1$			
	$\vdots$					
(A)	S_{L-1}	$(c;$	$b_{L-1}, \ldots, b_{K+L-1})=1$			
	S_L	$(c;$	$b_L, \ldots, b_{K+L})=1$			
	$\vdots$					
	S_{M-K-1}	$(c;$	$b_{M-K-1},$	$b_{M-K},$	$b_{M-K+1}, \ldots, b_{M-1})=1$	
(B)	S_{M-K}	$(c;$		$b_{M-K},$	$b_{M-K+1}, \ldots, b_{M-1},$	$b_M)=1$

Since from (A) a system of relations between the a_i's and b_i's follows, but , by lemma 1.3 and the 'Freiheitssatz', there is no relation between the b_i's alone can be deduced, and neither from (A) alone, nor from (B) alone is there any relation between the b_i's and x_i's, therefore by lemma 1.1' our relation must be from (A), thus by our hypothesis the theorem is true.

Case 2a. $L \geqslant K$, $L > M-K-1$, and $H=0$

We identify a with $b_0, b_1, \ldots, b_{M-K-1}$

b with $c;\ b_{M-K}, b_{M-K+1}, \ldots, b_L$

x with $b_{L+1}, b_{L+2}, \ldots, b_{M-1}$

y with b_M

(A) with $S_0, S_1, \ldots, S_{M-K-1}$

(B) with S_{M-K}

Using an argument similar to that used before, we can show that our theorem is true.

Case 2b. $L \geqslant K$, $L > M-K-1$, and $H > 0$

We identify a with $b_M, b_{M-1}, \ldots, b_{K+1}$

b with $c;\ b_K$

x with $b_{K-1}, b_{K-2}, \ldots, b_2, b_1$

y with b_0

(A) with $S_{M-K}, S_{M-K-1}, \ldots, S_2, S_1$

(B) with S_0

Using an argument similar to that used before, we can show that our theorem is true.

CHAPTER II

FREE PRODUCT OF TWO GROUPS WITH AN AMALGAMATED SUBGROUP

Definition 2.1: Let G be a group and C be a set of generators for G, where $C=C_1 \cup C_2$, C_1 and C_2 not necessarily disjoint. C_1 generate a subgroup G_1, and C_2 generate a subgroup G_2. It is obvious that G is generated by the two subgroup G_1 and G_2. If R_1 is the set of defining relations for G_1 and R_2 is the set of defining relations for G_2, and if the set of defining relations for G is $R_1 \cup R_2$, then we say that G is the free product of G_1 and G_2 with amalgamated subgroup $H= G_1 \cap G_2$. In case where H is the identity, then we say that G is the free product of G_1 and G_2.

From the above definition, we can draw the following conclusions:

1. Let g_1, g_2 be in G_1 and G_2 respectively, the product $g_1 g_2$ is in H if and only if g_1 is in H and g_2 is in H.
2. If g is an element of G, then g can be represented as $A_1B_1A_2B_2 \ldots A_nB_n$, where A_1 and B_n can be the identity the A_i's in G_1, B_i's in G_2, but the A_i's and B_i's are not in H. This is called the reduced representation of g.

Because of (1), if $\bar{A}_1\bar{B}_1\bar{A}_2\bar{B}_2\ldots\bar{A}_t\bar{B}_t$ is another reduced representation of g, then $n = t$.

3. If g is an element of G, then g can be represented as $A_1B_1A_2B_2\ldots A_nB_n$, where A_1, B_n may be the identity, and the A_i's are in G_1, B_i's are in G_2, and if we can decide whether A_i, $i=1,2,\ldots,n$, is in H or not, and also we can decide whether B_i, $i=1,2,3,\ldots,n$, is in H or not, then we can get a reduced representation of g.

<u>Theorem</u> 2.1 : Let G_1' and G_2' be groups, such that G_1' contains a subgroup H_1, and G_2' contains a subgroup H_2, where H_1 and H_2 are isomorphic, then the free product of G_1 and G_2 with amalgamated subgroup H exists, where G_1 is isomorphic to G_1' and G_2 is isomorphic to G_2', and H is isomorphic to H_1 and H_2.

<u>Proof</u>: Let S_1, and S_2 be sets of coset representatives of H in G_1 and G_2 respectively, and both contain the idendity. Let B be the set of all words of the form $s_1s_2\ldots s_nh$, where $s_i\neq 1$, s_i is in S_1 or S_2, but adjacent s_i's are not in the same S_1 or S_2, and h is in H. We define multiplication between two words in B as follows:

1. $W_1 = s_1 s_2 \ldots s_n h$; $W_2 = \bar{s}_1 \bar{s}_2 \ldots \bar{s}_m \bar{h}$

if s_n and $\bar{s}_1$ are in different S_j, $j=1,2.$, we have :

$$hs_1 = s_1' h_1$$

$$h_1 \bar{s}_2 = s_2' h_2$$

$$\vdots$$

$$h_{m-2} \bar{s}_{m-1} = s_{m-1}' h_{m-1}$$

$$h_{m-1} \bar{s}_m \bar{h} = s_m' h'$$

Define $W_1 \cdot W_2 = s_1 s_2 \ldots s_n s_1' s_2' \ldots s_m' h'$

2. $W_1 = s_1 s_2 \ldots s_n h$; $W_2 = \bar{s}_1 \bar{s}_2 \ldots \bar{s}_m \bar{h}$

if s_n and $\bar{s}_1$ are in the same S_j, $j=1,2.$, and $s_n h \bar{s}_1 = 1$ in G_j, then s_{n-1}, and $\bar{s}_2$ must be in the same S_i, $i \neq j$, we have:

$$s_{n-1} \bar{s}_2 = s_2' h_1$$

$$h_1 \bar{s}_3 = s_3' h_2$$

$$\vdots$$

$$h_{m-3} \bar{s}_{m-1} = s_{m-1}' h_{m-2}$$

$$h_{m-2} \bar{s}_m \bar{h} = s_m' h'$$

Define $W_1 \cdot W_2 = s_1 s_2 \ldots s_{n-2} s_2' s_3' \ldots s_m' h'$

3. $W_1 = s_1 s_2 \ldots s_n h$; $W_2 = \bar{s}_1 \bar{s}_2 \ldots \bar{s}_m \bar{h}$

if s_n and $\bar{s}_1$ are in the same S_j, $j=1,2.$, and $s_n h \bar{s}_1 \neq 1$, then we have :

$$s_n\bar{h}s_1 = s_1'h_1$$

$$h_1\bar{s}_2 = s_2'h_2$$

$$\vdots$$

$$h_{m-2}\bar{s}_{m-1} = s_{m-1}'h_{m-1}$$

$$h_{m-1}\bar{s}_m\bar{h} = s_m'h'$$

Define $W_1 \circ W_2 = s_1 s_2 \ldots s_{n-1} s_1' s_2' \ldots s_m' h'$

4. $W_1 = h_1$; $W_2 = h_2$, where h_1 and h_2 are in H, we have:

$h_1 h_2 = h$, where h is in H

Define $W_1 \circ W_2 = h$

Since 1 is in B, we define $W_0 = 1$ and $W_0 \cdot W = W \cdot W_0 = W$ for any W in B. If $W = s_1 s_2 \ldots s_n h$, then we have:

$$h^{-1} s_n^{-1} = s_1'h_1$$

$$h_1 s_{n-1}^{-1} = s_2'h_2$$

$$\vdots$$

$$h_{n-2} s_2^{-1} = s_{n-1}'h_{n-1}$$

$$h_{n-1} s_1^{-1} = s_n'h'$$

We define $W^{-1} = s_1' s_2' \ldots s_n' h'$, then it is not difficult to verify that $W \circ W^{-1} = W^{-1} \circ W = W_0$. Thus we can claim that B is a group with the above defined multiplication if we can prove that the associative law holds. Let g be in G_j,

$j=1,2.$, and we define a mapping from B into B as follows:

$$F_g \; : \; B \longrightarrow B$$

1. $W = s_1 s_2 \ldots s_n h$ is in B. If s_n and g are in the same G_j,

we have $s_n hg = s'h'$. Define $F_g(W) = s_1 s_2 \ldots s_{n-1} s'h'$, if $s' \neq 1$

$$= s_1 s_2 \ldots s_{n-1} h' \quad , \text{ if } s'=1$$

2. If s_n and g are in different G_j's, $j=1,2.$, we have

$hg = \overline{s}'\overline{h}'$. Define $F_g(W) = s_1 s_2 \ldots s_n \overline{s}'\overline{h}'$, if $\overline{s}' \neq 1$

$$= s_1 s_2 \ldots s_n \overline{h}' \quad , \text{ if } \overline{s}'=1$$

It is easy to verify that $F_g F_{g^{-1}}$ is the identity map on B and $F_{g^{-1}} F_g$ is also the identity map on B. Thus the map F_g, for any g in G_j, is a permutation of the words of B. Let S_B be the symmetric group on B. We define a mapping T from B to S_B as follows: if $W = s_1 s_2 \ldots s_n h$, define $T(W) = F_{s_1} F_{s_2} \ldots F_{s_n} F_h$. The mapping T is one to one, since $T(W_1)$ and $T(W_2)$ represent different permutations on B if W_1 is not the same word as W_2, because if $T(W) = F_{s_1} F_{s_2} \ldots F_{s_n} F_h$, then $F_{s_1} F_{s_2} \ldots F_{s_n} F_h(W_0) = s_1 s_2 \ldots s_n h$

$$= W \; .$$

Thus T is a one to one mapping. It is obvious that T preserves product, that is, $T(W_1.W_2) = T(W_1)T(W_2)$. For any W_1, W_2, W_3 in B,

$$\begin{aligned} T[W_1\circ(W_2\circ W_3)] &= T(W_1)T(W_2\circ W_3) \\ &= T(W_1)[T(W_2)T(W_3)] \\ &= [T(W_1)T(W_2)]T(W_3) \text{ , since associative law holds in } S_B \\ &= T(W_1\circ W_2)T(W_3) \\ &= T[(W_1\circ W_2)W_3] \end{aligned}$$

Since the map T is one to one, taking the inverse image, we have $W_1\circ(W_2\circ W_3) = (W_1\circ W_2)\circ W_3$. Therefore B is a group.

Suppose G is the free product of G_1 and G_2 with amalgamated subgroup H. We set up a mapping V, mapping B to G as follows; $V(s_1 s_2 \ldots s_n h) = s_1 s_2 \ldots s_n h$. It is not difficult to verify that V is an isomorphism. Therefore B can be identify as the free product of G_1 and G_2 with amalgamated subgroup H .

Thus our theorem is proved.

CHAPTER III

THE EXTENDED IDENTITY PROBLEM

Let G be generated by $(a_1,a_2,\ldots a_n,\ldots)$ which is finite or countable, with a defining relation $R(a_1,a_2,\ldots,a_n)=1$, where $a_1,a_2,\ldots,a_n$ effectively appear in R, that is, they appear in R when R is cyclically reduced. Let $(a_{i_1},a_{i_2},\ldots,a_{i_t},\ldots)$ be a subset of our set of generators (this subset can be empty). Furthermore let $W(\ldots a_i\ldots)$ be any word of our set of generators. The extended identity problem is to find a method in order to decide in a finite number of steps whether $W(\ldots a_i\ldots)$ can be transformed in $\overline{W}$, where $\overline{W}$ is a word of our subset $(a_{i_1},a_{i_2},\ldots,a_{i_t},\ldots)$; in the case where our subset is empty, it is known as the identity problem. If $W = \overline{W}$, then $\overline{W}$ can actually be written out. On account of the 'Freiheitssatz' and theorem 2.1, we can simplify our extended identity problem in the following manner:

I. If F is a free group, we can surely solve the extended identity problem.

II. If G is generated by $(a_1,a_2,\ldots,a_n,a_{n+1},\ldots)$ with $R(a_1,a_2,\ldots,a_n)=1$, where $a_1,a_2,\ldots,a_n$ effectively appear in R, then G is the free product of G' and $\overline{G}$, where G' is generated by $a_1,a_2,\ldots,a_n$ with defining relation R, and $\bar{G}$ is the free group generated by $(a_{n+1}, a_{n+2}, \ldots)$. If one wants to solve the extended idendity problem for G, it is sufficient for one to solve the extended identity problem for G'. Since if g is in G, then $g = A_1B_1A_2B_2 \ldots A_nB_n$, where A_1 and B_n may be the identity, and the A_i's are in G' and B_i's in $\bar{G}$. If any B_i contains a generator not appearing in our given subset of generators, then we are sure that g cannot be a word of our subset. If all B_i's are from our given subset, then our work only involves the A_i's in G'.

III. Our extended identity problem is thus reduced to the case where G is generated by $(a_1, a_2, \ldots, a_n)$ with $R(a_1, a_2, \ldots, a_n)=1$. Let $(a_{i_1}, a_{i_2}, \ldots, a_{i_t})$ be a subset of $(a_1, a_2, \ldots, a_n)$, where, say, a_j does not appear in our subset. By the 'Freiheitssatz', $(a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$ generate a free subgroup of G. Therefore, if we want to find out whether our given word W in $(a_1, a_2, \ldots, a_n)$ can be a word of our subset or not, it is sufficient for us to find out whether W is a word in $(a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$ or not, because if W is a word in $(a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$, then we know at once whether if is a word of our given subset or not.

IV. To solve the extended identity problem, we have three cases to consider:

1. R consists of one generator only.
2. R consists of more than two generators in which the exponent-sum of one of them is zero.
3. R consists of more than two generators in which the exponent-sum of every generator is not zero.

It is obvious that case 1 can be solved easily. For case 2, we can restrict our work to some subgroup of G, as shown below:

V. G is generated by $(a_1, a_2, \ldots, a_n)$ with $R(a_1, a_2, \ldots, a_n)=1$.

Set $a_1 = a$

$a_2 = b$

$a_3 = c$

the rest by ... , then

G is generated by $(a, b, c, \ldots)$ with $R(a, b, c, \ldots)=1$.

Suppose a has exponent-sum zero in R; then by (II), we only have to consider our subset to be without a, or without a generator other than a, say, b. That is, our subset can be $(b, c, \ldots)$ or $(a, c, \ldots)$. We let

$$a^k b a^{-k} = b_k$$

$$a^k c a^{-k} = c_k$$

$$\ldots \qquad , \text{ where k is any integer,}$$

then $R(a,b,c,\ldots)= \bar{R}_0(b_{k_1},c_{k_2},\ldots)=1$

$$= \bar{R}_t(b_{k_1+t},c_{k_2+t},\ldots)=1, \text{ where } t \text{ is}$$

any integer, $k_1,k_2,\ldots$, are fixed sets of integers. Let H be a subgroup of G generated by $(b_k,c_k,\ldots)$ where k is any integer, then H is a subgroup of G whose words have exponent-sum zero in a, and also any word in G in which a has exponent-sum zero can be expressed as a word in H. H has an infinite number of relations, $\bar{R}_t(b_{k_1+t},c_{k_2+t},\ldots)=1$, and the length of $\bar{R}_t$ is strictly less than that of R.

If $W(a,b,c,\ldots)$ can be expressed as a word of the subset $(b,c,\ldots)$, then a in $W(a,b,c,\ldots)$ must have exponent-sum zero, that is, $W(a,b,c,\ldots)$ must be in H. The proof is very simple, for if $W(a,b,c,\ldots)=\bar{W}(b,c,\ldots)$, then $W(a,b,c,\ldots)\bar{W}^{-1}(b,c,\ldots) = 1$

$$\equiv \prod_{i=1}^{h} T_i R^e T_i^{-1} \quad , \; e= +1 \text{ or } -1$$

where a in the right hand side has exponent-sum zero, thus a in the left hand side must have exponent-sum zero. Therefore $W(a,b,c,\ldots)$ must be in H. If we want to know whether $W(a,b,c,\ldots)$ is a word of the subset $(a,c,\ldots)$ or not, we express $W(a,b,c,\ldots)$ to be freely equal to $W'(b_k,c_k,\ldots)a^u$, where u is the exponent-sum of a in W. Since W is a word of $(a,c,\ldots)$ if and only if W' is, we can solve the problem by working on W'. Therefore the extended identity problem can be simplified to two problems to be solved in H, that is, H is generated by $(b_k, c_k, \ldots)$ with $\bar{R}_t(b_{k_1+t}, c_{k_2+t}, \ldots)=1$, and the subset $(b_0, c_0, \ldots)$ or the subset $(c_k, \ldots)$.

In order to solve the extended identity problem, a thorough study of H is necessary. H has defining relations $\bar{R}_t(b_{k_1+t}, c_{k_2+t}, \ldots)=1$, where t can be any integer. Let M_1 be the maximum of k_1, and M_0 be the minimum of k_1, thus b_{M_1+t} and b_{M_0+t} effectively appear in $\bar{R}_t(b_{k_1+t}, c_{k_2+t}, \ldots)=1$. For example if H is generated by (b_k, c_k, d_k) with

$$\overset{6}{\bar{R}}_0 = b_{-1}c_0b_0b_2^2c_2^{-1}d_0=1$$

$$\bar{R}_t = b_{-1+t}c_{0+t}b_{0+t}b_{2+t}^2c_{2+t}^{-1}d_{0+t}=1$$

Here k_1 is the set of integers $(-1,0,2)$, and $M_1=2$, $M_0=-1$, and b_{2+t}, b_{-1+t} effectively appear in $\bar{R}_t=1$. We let u_0 and u_1 be any two integers where $u_0 \leq u_1$, and denote T_{u_0,u_1} to be the subgroup of H generated by $(c_k, \ldots; b_{u_0}, b_{u_0+1}, \ldots, b_{u_1})$. In our above example, if $u_0=-2$, $u_1=4$, then $T_{-2,4}$ is the subgroup generated by $(c_k, d_k;\ b_{-2}, b_{-1}, b_0, b_1, b_2, b_3, b_4)$. The particular group where we have $u_1-u_0=M_1-M_0$, will be denoted by H_a, where $a = u_1-M_1= u_0-M_0$. With the same group cited as our example, we have $M_1-M_0= 2-(-1) = 3$; thus if $u_0=1$, and $u_1=4$, then $T_{1,4}= H_2$ is generated by $(c_k,d_k; b_1,b_2,b_3,b_4)$. It should be noted that H_2 has only one defining relation, $\overline{R}_2 = b_1c_2b_2b_4^2c_4^{-1}d_2 =1$, by the 'Haupt-form' of the 'Freiheitssatz'. If $u_1-u_0 \quad M_1-M_0$, then T_{u_0,u_1} contains several H_a's as subgroups, and also by theorem 2.1, T_{u_0,u_1} is the free product of T_{u_0,u_1-1}, and $H_{u_1-M_1}$ with amalgamated subgroup generated by $(c_k,\ldots;b_{u_1-M_1+M_0},\ldots,b_{u_1-1})$, where T_{u_0,u_1-1} is generated by $(c_k,\ldots;b_{u_0},b_{u_0+1},\ldots,b_{u_1-1})$ and $H_{u_1-M_1}$ is generated by $(c_k,\ldots;b_{u_1-M_1+M_0},\ldots,b_{u_1})$. Similarly T_{u_0,u_1-1} is the free product of T_{u_0,u_1-2}, and $H_{u_1-M_1-1}$ with amalgamated subgroup $T_{u_1-1-M_1+M_0,u_1-2}$. Thus after a finite number of steps, we will get T_{u_0,u_1-s} as the free product of H_{a_1} and H_{a_2}, with amalgamated subgroup $H_{a_1} \quad H_{a_2}$. For example, if we have the same group as before, $T_{-2,4}$ generated by $(c_k,d_k,; b_{-2},b_{-1},b_0,b_1,b_2,b_3,b_4)$ is the free product of $T_{-2,3}$ generated by $(c_k,d_k;b_{-2},b_{-1},b_0,b_1,b_2,b_3)$ and H_2

generated by $(c_k, d_k;\ b_1, b_2, b_3, b_4)$ with amalgamated subgroup $T_{1,3}$ generated by $(c_k, d_k;\ b_1, b_2, b_3)$. Similarly $T_{-2,3}$ is the free product of $T_{-2,2}$ and H_1 with amalgamated subgroup $T_{0,2}$, and finally $T_{-2,2}$ is the free product of H_1 and H_0 with amalgamated subgroup $T_{-1,1}$. Because of the 'Haupt-form' of the 'Freiheitssatz' , the following should be noted;

1. H_a has only one defining relation, $\overline{R}_a = 1$
2. If $u_1 - u_0 > M_1 - M_0$, the defining relations for T_{u_0,u_1} are;

$$\overline{R}_{u_0 - M_0} = 1$$
$$\overline{R}_{u_0 - M_0 + 1} = 1$$
$$\vdots$$
$$\overline{R}_{u_1 - M_1} = 1$$

3. If $u_1 - u_0 < M_1 - M_0$, thenn T_{u_0,u_1} is a free subgroup.

We will prove the solvability of our extended identity problem by making use of mathematical induction, (inducting on the length of R) and theorem 2.1. We have shown that the problem is solvable if R contains only one generator. We assume that the problem is solvable for all groups having R', whose length is less than that of R. Thus since the length of $\overline{R}_t$ is always less than that of R, we can solve the extended identity problem in H_t. That is to say, if we are given a word, W, we should find out in which H_a W belongs. If W is in H_a, then we can find out whether W is a word of our subset or not by our hypothesis, but if we can show that W is not in any H_a, then we are sure that W cannot be a word of our given subset, since $(b_0, c_0, \ldots)$ generate a subgroup of H_a containing b_0, and also $(c_k, \ldots)$ generate a subgroup of H_a, for any integer a. Thus it only remains for us to find a method to decide whether W is in any of the H_a's or not. It can be done in the following manner; with any given W, there always exists a subgroup T_{u_0,u_1} containing all the generators appearing in W and b_0. If $u_1-u_0 \leq M_1-M_0$, then we are done. If $u_1-u_0 > M_1-M_0$, then T_{u_0,u_1} is the free product of T_{u_0,u_1-1} and $H_{u_1-M_1}$ with amalgamated subgroup $T_{u_1-1-M_1+M_0,u_1-2}$, and $W = A_1B_1A_2B_2\ldots A_nB_n$ (not necessarily reduced), where A_1 and B_n may be the identity, and the A_i's are in T_{u_0,u_1-1}, and the B_i's are in $H_{u_1-M_1}$. Since if $\bar{u}_1-\bar{u}_0=\bar{M}_1-\bar{M}_0$, we know that W is in $H_{\bar{u}_1-\bar{M}_1}$, suppose that we can decide that for any word in T_{u_0,u_1-1} , that word is in H_a, a subgroup of T_{u_0,u_1-1}, or not, then we can show that we can do the same in T_{u_0,u_1}. Since $W = A_1B_1A_2B_2\ldots A_nB_n$, we can reduce the right hand side, if we can decide whether any word in $H_{u_1-M_1}$ can be in $T_{u_1-1-M_1+M_0,u_1-2}$ or not (this we can always do), and if any word in T_{u_0,u_1-1} can be in $T_{u_1-1-M_1+M_0,u_1-2}$ or not, this can also be done, since $H_{u_1-M_1-1}$ is a subgroup of T_{u_0,u_1-1} and $H_{u_1-M_1-1}$ contains $T_{u_1-1-M_1+M_0,u_1-2}$ as a subgroup, and we can decide whether a B_i is in any of the H_a's, subgroups of T_{u_0,u_1-1}, or not by our hypothesis. If we know that B_i is in $H_{u_1-M_1-1}$, then we can know whether B_i is from $T_{u_1-1-M_1+M_0,u_1-2}$ or not: of course, if B_i is not in $H_{u_1-M_1-1}$, then we are sure that B_i can never be in $T_{u_1-1-M_1+M_0,u_1-2}$. Since we can reduce $A_1B_1A_2B_2\ldots A_nB_n$, we can decide whether W is in T_{u_0,u_1-1} or in $H_{u_1-M_1}$ or not in either or them. Consequently, we can decide whether W is in any of the H_a's or not.

VI. G is generated by $(a,b,c,\ldots)$ with $R(a,b,c,\ldots)=1$, where the exponent-sum of every generator in R is not zero. Suppose we are given W and the subset which

which does not contain b, and suppose that the exponent-sum of a in R is s_1, and the exponentsum of b in R is s_2, we set ;

$$a = \bar{a}^{s_2} \quad , \quad b = \bar{b}\,\bar{a}^{-s_1} \quad ,$$

then $R(a,b,c,\ldots) = R(\bar{a}^{s_2}, \bar{b}\,\bar{a}^{-s_1}, c, \ldots) = 1$

$= P(\bar{a},\bar{b},c,\ldots)=1$, where the exponent-sum of $\bar{a}$ in P is zero, setting

$$\bar{a}^{k}\bar{b}\,\bar{a}^{-k} = \bar{b}_k$$

$$\bar{a}^{k}c\,\bar{a}^{-k} = c_k$$

$$\ldots$$

we have $P(\bar{a},\bar{b},c,\ldots)=1$

$$= \bar{P}_0(\bar{b}_{k_1}, c_{k_2}, \ldots)=1$$

$$= \bar{P}_t(\bar{b}_{k_1+t}, c_{k_2+t}, \ldots)=1$$

It is not difficult to check that the length of $\bar{P}_t$ is strictly less than that of R. We let $\bar{G}$ be a group generated by $(\bar{a},\bar{b},c,\ldots)$ with only one relation P, $P(\bar{a},\bar{b},c,\ldots) = R(\bar{a}^{s_2}, \bar{b}\,\bar{a}^{-s_1}, c, \ldots) = 1$. Applying same argument as before, if we want to solve the extended identity problem for $\bar{G}$, it is sufficient for us to confine our work to $\bar{H}$, a subgroup of $\bar{G}$, generated by $(\bar{b}_k, c_k, \ldots)$ with defining relations $\bar{P}_t=1$. Using the same method as before, we can decide in which $\bar{H}_a$

our given word is, or whether it is in none of the H_a's. If the given word is in $\overline{H}_a$, with only one defining relation $\overline{P}_a = 1$, we can solve the problem since the length of $\overline{P}_a = 1$ is less than that of R. Thus we know that the extended identity problem can be solved in $\overline{G}$. If we can show that the solvability of the problem in $\overline{G}$ means the solvability of the problem in G, then we are done. Let K be a subgroup of $\overline{G}$ generated by $(\overline{a}^{s_2}, \overline{b}\,\overline{a}^{-s_1}, c, \ldots)$ with the defining relation P=1. It is obvious that G can be identified as K, if we identify a with $\overline{a}^{s_2}$, b with $\overline{b}\,\overline{a}^{-s_1}$, c with c,... . Thus for our given word $W(a,b,c,\ldots)$ in G, if we want to find out whether it is a word of our subset $(a,c,\ldots)$ or not, we identify $W(a,b,c,\ldots)$ with $W(\overline{a}^{s_2}, \overline{b}\,\overline{a}^{-s_1}, c, \ldots)$ in K, which is a subgroup of $\overline{G}$. Since by the 'Freiheitssatz', $(\overline{a}, c, \ldots)$ generate a free subgroup of $\overline{G}$, and since any subgroup of a free group is free, the set $(\overline{a}^{s_2}, c, \ldots)$ generate a free subgroup and it is also a free subgroup of K.

$W(\overline{a}^{s_2}, \overline{b}\,\overline{a}^{-s_1}, c, \ldots) = \overline{W}(\overline{a}, \overline{b}, c, \ldots)$, if we can find out that $\overline{W}$ is a word in $(\overline{a}, c, \ldots)$, that is $\overline{W} = W'(\overline{a}, c, \ldots)$, then W' must be a word of $(\overline{a}^{s_2}, c, \ldots)$. that is to say, W in G is a word in $(a,c,\ldots)$ if and only if W, being identified as a word of K is a word of $(\bar{a}^{s_2},c,\ldots)$ which is the same as saying that if W is identified as a word of $\bar{G}$, it must be from $(\bar{a},c,\ldots)$. Thus the extended identity problem in G can be solved.

We will now give some examples to show how the solution of the extended identity problem goes in specific cases. Suppose G is a group generated by (a,b,c) with defining relation $R=a^{-1}b^2ab^{-3}c=1$. Thus the problem is reduced to be solved in H generated by (b_k,c_k) with defining relations $\bar{R}_t = b^2_{-1+t}b^{-3}_{0+t}c_{0+t}=1$

1. Suppose we are given $W= aba^2b^{-1}a^{-2}c^4$, and the subset (b,c). It is obvious that W can never be a word of our subset, since the exponent-sum of a in W is not zero.

2. Suppose $W= a^2b^2a^{-3}ba^5ca^{-4}b^3a^2ca^{-2}$, and the given subset is (b,c). Thus we want to know if $W = \bar{W} = b^2_2b_{-1}c_4b^3_0c_2$ in H can be a word of (b_0,c_0) or not. $\bar{W}$ is in $T_{-1,2}$ generated by $(c_k;b_{-1},b_0,b_1,b_2)$; the free product of $T_{-1,1}$ generated by $(c_k;b_{-1},b_0,b_1)$ and H_2 generated by $(c_k;b_1,b_2)$ with an amalgamated subgroup $T_{1,1}$ generated

by (c_k, b_1). Therefore $\overline{W} = (b_2^2)(b_{-1}c_4b_0^3c_2)$ where b_2^2 is in H_2 and $b_{-1}c_4b_0^3c_2$ is in $T_{-1,1}$. In order to find out in which of the H_a $\overline{W}$ belongs, we have to find out whether $\overline{W}$ is in $T_{-1,1}$ or in H_2, or $\overline{W}$ belongs to neither of them. Thus we have to check whether b_2^2 is in $T_{1,1}$ or $b_{-1}c_4b_0^3c_2$ is in $T_{1,1}$. First of all, we shall check whether b_2^2 is in the amalgamated subgroup or not. b_2^2 is in H_2 generated by $(c_k; b_1, b_2)$ with only one defining relation $\overline{R}_2 = b_1^2b_2^{-3}c_2=1$. H_2 is the free product of G_1 generated by (b_1, b_2, c_2) with only one defining relation $\overline{R}_2 = b_1^2b_2^{-3}c_2 = 1$ and G_2 generated by $(\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, c_4, \ldots)$, a free subgroup. Since the exponent-sum of every generator in $\overline{R}_2$ is not zero, and we want to know whether b_2^2 in G_1 can be expressed as a word in (c_2, b_1) or not, we have to make the following substitution:

$b_1 = A^{-3}$

$b_2 = BA^{-2}$, and for convenience sake, we let

$c_2 = C$

therefore, $\bar{R}_2 = b_1^2 b_2^{-3} c_2 = P = A^{-6}(BA^{-2})^{-3}C = 1$

$$= A^{-4}B^{-1}A^2B^{-1}A^2B^{-1}C = 1$$

if we let $A^kBA^{-k} = B_k$ and $A^kCA^{-k} = C_k$, then

$$P = \bar{P}_0 = B_{-4}^{-1}B_{-2}^{-1}B_0^{-1}C_0 = 1$$

$$= \bar{P}_t = B_{-4+t}^{-1}B_{-2+t}^{-1}B_t^{-1}C_t$$

$$b_2^2 = BA^{-2}BA^{-2}$$

$$= W' = B_0B_{-2}A^{-4}$$

We want to know whether W' is a word in (A,C) or not, which is equivalent to knowing whether B_0B_{-2} in $\bar{H}$ generated by (B_k,C_k) with defining relations $\bar{P}_t=1$, can be a word of (C_k) or not. B_0B_{-2} is in $\bar{T}_{-2,0}$ generated by $(C_k;B_{-2},B_{-1},B_0)$, which is a free subgroup. Thus B_0B_{-2} cannot be a word of the (C_k). This implies that $BA^{-2}BA^{-2}$ cannot be a word in (A,C), and this in turn implies that b_2^2 cannot be in $T_{1,1}$. Now we have to investigate whether $b_{-1}c_4b_0^3c_2$ is in $T_{1,1}$ or not. First of all we have to decide whether $b_{-1}c_4b_0^3c_2$ is in H_1 or not, since H_1 contains $T_{1,1}$ as a subgroup. It is obvious that $b_{-1}c_4b_0c_2$ is in H_0, generated by $(c_k;\ b_{-1},b_0)$, thus if it is also in H_1, then $b_{-1}c_4b_0^3c_2$

is in the meet of H_0 and H_1 which is $T_{0,0}$ generated by $(c_k; b_0)$. This means that we have to find out whether $b_{-1}c_4b_0^3c_2$ in H_0 can be expressed as a word of $(c_k; b_0)$ or not. H_0 is the free product of G^1 generated by (b_{-1}, b_0, c_0) with only one defining relation $\overline{R}_0 = b_{-1}^2 b_0^{-3} c_0 = 1$ and the free subgroup G^2, generated by $(\ldots, c_{-2}, c_{-1}, c_1, c_2, \ldots)$. Now $b_{-1}c_4b_0^3c_2 = (b_{-1})(c_4)(b_0^3)(c_2)$ where b_{-1}, b_0^3 are in G^1 and c_4, c_2 are in G^2. Therefore, it remains for us to find out whether b_{-1} is a word of (b_0, c_0) or not. Since the exponent-sum of every generator in $\overline{R}_0$ is not zero, we have to make the following substitution:

$$b_0 = A^2$$

$b_{-1} = BA^3$, and for convenience sake, we let

$c_0 = C$; then

$$\begin{aligned} \overline{R}_0 = P &= BA^3BA^3A^{-6}C \\ &= \overline{P}_0 = B_0B_3C_0 \\ &= \overline{P}_t = B_tB_{3+t}C_t \quad , \text{ and} \end{aligned}$$

$$b_{-1} = BA^3 = W_1 = B_0A$$

Since we want to know whether BA^3 can be a word in (A,C) or not, it means that we want to know whether B_0 in $\overline{H}$, generated by (C_k,B_k) with defining relations $\overline{P}_t=1$ can be a word of (C_k) or not. Since B_0 is in $\overline{T}_{0,0}$ generated by (C_k,B_0), a free subgroup, it is obvious that B_0 is not a word in (C_k), which means that BA^3 B_0A^3 cannot be expressed as a word in (A,C), which in turn means that b_{-1} is not a word in (b_0,c_0), consequently $b_{-1}c_4b_0c_2$ is not in the meet of H_1 and H_0. Thus $b_{-1}c_4b_0^3c_2$ is not in H_1, therefore it cannot be in $T_{1,1}$. Thus $\overline{W} = (b_2^2)(b_{-1}c_4b_0^3c_2)$ is a reduced representation, and we can conclude that $\overline{W}$ cannot be a word of (b_0,c_0), therefore, $W= a^2b^2a^{-3}ba^5ca^{-4}b^3a^2ca^{-2}$ cannot be a word of (b,c).

Suppose G is a group generated by (a,b) with only one defining relation $R= a^{-1}b^2ab^{-3} = 1$. Thus G has a subgroup H generated by (b_k) with defining relations

$\overline{R}_t = b^2_{-1+t}b^{-3}_{0+t}=1$. We are given the subset containing a, and

$W = ab^{-2}a^{-1}b^2ab^{-1}a^2ba^{-1}b^{-1}a^{-1}b^2ab^{-2}ab^{-3}a^{-1}$; and we want to know whether W can be a word of our subset or not.

$W= W' = b_1^{-2}b_0^2b_1^{-1}b_3b_2^{-1}b_1^2b_2^{-2}b_3^{-3}a^2 = \overline{W}\, a^2$. To solve the problem

we only have to consider whether $\overline{\overline{W}}=b_1^{-2}b_0^2b_1^{-1}b_3b_2^{-1}b_1^2b_2^{-2}b_3^{-3}$ in H is the identity or not. $\overline{\overline{W}}$ is in $T_{0,3}$ generated by (b_0,b_1,b_2,b_3), the free product of $T_{0,2}$ generated by (b_0,b_1,b_2) and H_3 generated by (b_2,b_3) with an amalgamated subgroup $T_{2,2}$ generated by (b_2). $\overline{\overline{W}}=(b_1^{-2}b_0^2b_1^{-1})(b_3b_2^{-1})(b_1^2b_2^{-2})(b_3^{-3})$ where $b_1^{-2}b_0^2b_1^{-1}$, $b_1b_2^{-2}$ are in $T_{0,2}$ and $b_3b_2^{-1}$, b_3^{-3} are in H_3, and we have to decide whether $\overline{\overline{W}}$ is in $T_{0,2}$ or in H_3, or in neither of them. It is obvious that $b_1^{-2}b_0^2b_1^{-1}$ is in H_1, a subgroup of $T_{0,2}$ with only one defining relation $\overline{R}_1= b_0^2b_1^{-3}=1$, thus $b_1^{-2}b_0^2b^{-1}_1 =1$ is trivially true, therefore $\overline{\overline{W}} = (1)(b_3b_2^{-1})(b_1^2b_2^{-2})(b^{-3})$. It is also obvious that $b_1^2b_2^{-2}$ is in H_2, a subgroup of $T_{0,2}$, with only one defining relation $\overline{R}_2 = b_1^2b_2^{-3}=1$. Furthermore, $b_1^2b_2^{-2} = b_2$ is in the amalgamated subgroup T_2, thus
$\overline{\overline{W}} = (1)(b_3b_2^{-1})(b_2)(b^{-3})$ is in H_3, and it is obvmiously the identity. Therefore our given word W can be expressed as a^2.

BIBLIOGRAPHY

1. M. Hall, Jr., The Theory of Groups ,
   The Macmillan Company, New York, 1959.

2. W. Magnus, Das Identitätsproblem für Gruppen mit
   einer difinierenden Relation,
   Maths. Ann. 106, (1932), 295-307.

3. W. Magnus, Uber diskontinuierliche Gruupen mit
   einer definierenden Relation (Der Freiheitssatz),
   Crelles J. 103 (1930), 141-165.

4. B.H. Neumann, An Essay on Free Products of Groups
   with Amalgamation,
   Royal Society of London,Philosophical Transaction,
   Series A, Volume 246, (1953-1954), 503-554.