SCHAUDER DECOMPOSITIONS AND BASES IN BANACH SPACES.

by

Ronald Kerman

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Abstract

A decomposition of a Banach Space B is defined as a sequence (M_i) of non-trivial subspaces of B such that for each $X \in B$ there exists a unique sequence (X_i), $X_i \in M_i$, with $X = \sum_{i=1}^{\infty} X_i$. As is shown in the introduction, a decomposition is a generalization of the well-known concept of a basis. Associated with a decomposition (M_i) of a Banach Space is a sequence (P_i) of projections. When P_i is continuous for each i, (M_i) is said to be a Schauder decomposition. The purpose of this paper is to present certain results for Schauder decompositions which are known to be true for bases. Certain results for bases alone are included.

Among the general results on decompositions are a necessary and sufficient condition for a decomposition to be a Schauder decomposition and a generalization of the Weak Basis Theorem to decompositions.

The consequences of the restriction that a decomposition be shrinking or boundedly complete or both are considered in Section 2. Prominent there are properties of a certain decomposition in the conjugate of a Banach Space having a shrinking decomposition. A recent result of Sanders characterizing the reflexive spaces among those Banach Spaces having Schauder decompositions in terms of the properties of the decomposition is included.

Finally, decompositions for which the expansions of the elements are unconditionally convergent are considered. Emphasized are the relations between the bounded completeness of the unconditional decompositions of a Banach Space and the weak sequential completeness of the space. Lastly, certain theorems on unconditional bases are presented.

Introduction

A basis for a Banach Space B is defined as a sequence (b_i) of distinct elements of B satisfying the property that for each $X \in B$ there exists a sequence of real numbers (a_i) such that $X = \sum_{i=1}^{\infty} a_i b_i$ where convergence is in the norm. Alternatively, we may think of a basis for a Banach Space B as a sequence (M_i) of distinct one-dimensional subspaces of B such that for every $X \in B$ there exists a unique sequence (X_i), $X_i \in M_i$ with $X = \sum_{i=1}^{\infty} X_i$. A natural way to generalize the concept of a basis would be to remove the restriction that each M_i be one-dimensional. Accordingly, a decomposition of a Banach Space B is defined as a sequence (M_i) of non-trivial subspaces of B such that for each $X \in B$ there exists a unique sequence $(X_i), X_i \in M_i$ with $X = \sum_{i=1}^{\infty} X_i$. The series $\sum_{i=1}^{\infty} X_i$ is called the expansion of X with respect to the decomposition (M_i) and each M_i is referred to as a coordinate space of the decomposition. A decomposition is a generalization of a basis in the sense that, given a basis (b_i) for a Banach Space B, (M_i) will be a decomposition of B where M_i is the one-dimensional subspace of B generated by b_i. Again, any Banach Space having a basis will be separable, the set of all finite linear combinations of the basis elements having rational coefficients being a countable dense subset. Giving the set S in Example 1 of Section 4 sufficiently high cardinality, the Banach Space there constructed will be inseparable and hence cannot possibly have a basis. However, it does admit a decomposition, as is shown there.

Let B be a Banach Space and L a subspace of B. Then a linear mapping $P : B \to B$ is said to be a projection of B onto L if $P^2 = P$ (i.e. P is idempotent) and $P(B) = L$. If (M_i) is a decomposition of a Banach Space B then with each M_i, $i = 1, 2, \ldots$, there is associated a projection P_i of B onto M_i defined for $X \in B$ by $P_i(X) = X_i$, where $\sum_{i=1}^{\infty} X_i$ is the expansion of X with respect to (M_i). In many cases we will use the alternative motation (M_i, P_i) for the decomposition (M_i).

Let (b_i) be a basis in a Banach Space B and let (M_i, P_i) be the decomposition of B generated by (b_i). In this case we define a sequence (β_i) of linear functionals on B by $\beta_i(X) = a_i$ where $P_i(X) = a_i b_i$. β_i is called the i^{th} coefficient functional with respect to the basis (b_i).

If a decomposition (M_i, P_i) of a Banach Space B satisfies the further property that P_i is continuous for each i then we say (M_i, P_i) is a Schauder decomposition of B. A basis (b_i) of a Banach Space B is called a Schauder basis if the decomposition associated with it is a Schauder decomposition. This is equivalent to the requirement that each of the coefficient functionals associated with (b_i) be continuous. Schauder bases have been studied for many years. The concept of a Schauder decomposition on a Banach Space has not received attention until quite recently.

The purpose of this thesis is to present results for Schauder decompositions the analogues of which are known to hold for Schauder bases. Thus Sections 1, 2, and 4 largely parallel the exposition of the theory of Schauder bases in Day [1] . These sections include some theorems which are stated for bases only. In such cases it is not known whether the corresponding result holds for decompositions.

Some general theorems about Schauder decompositions are presented in Section 1. Thus Theorem 1 of that section gives a necessary and sufficient condition for a decomposition to be a Schauder decomposition. A statement and an outline for the proof of this theorem are given in a paper of Sanders [3]. The well-known theorem, due to Banach, that every basis is a Schauder basis is then obtained as a corollary. The fact that this last theorem does not hold in general for decompositions is illustrated by an example due to Sanders. The Weak Basis Theorem states that every weak basis (i.e. a basis for which the expansion of an element need only converge in the weak sense) is a strong basis. Theorem 1.2 shows that a counterpart of the Weak Basis Theorem holds for decompositions provided that each of the coordinate spaces is closed.

If (M_i, P_i) is a decomposition of a Banach Space B, we define the operators S_n and T_n on B by $S_n = \sum_{i=1}^{n} P_i$, $T_n = I - S_n$ where I is the identity operator on B.

A Schauder decomposition (M_i, P_i) of a Banach Space B is said to be a shrinking decomposition if $\lim_n \|f\|_n = 0$ for all $f \in B^*$ where $\|f\|_n$ is the norm of f restricted to $R(T_n)$ (i.e. the range of the operator T_n). A Schauder decomposition (M_i, P_i) of a Banach Space is said to be boundedly complete if for any sequence (X_i), $X_i \in M_i$ in B for which $\sup_n \|\sum_{i=1}^{n} X_i\| < \infty$ there exists an $X \in B$ such that $X = \sum_{i=1}^{\infty} X_i$.

In Section II certain results known to hold for bases having either or both of these properties are extended to decompositions. Theorem 2.1 which is an extension of Day p. 70 Lemma 1 shows that given a Schauder decomposition (M_i, P_i) for B one can construct a certain decomposition for B* if and only if (M_i, P_i) is shrinking. Theorem 2.4 shows further that this decomposition for B* must be boundedly complete. Theorem 2.2, a result of Alaoglu, characterizes every Banach Space with a boundedly complete basis as isomorphic to a conjugate space. It is unknown whether a similar characterization holds for Banach Spaces with a boundedly complete decomposition. Theorem 2.6 states that a necessary and sufficient condition for a Banach Space B with a Schauder decomposition (M_i, P_i) to be reflexive is that M_i be reflexive for each i and that (M_i, P_i) be both shrinking and boundedly complete. This theorem is due to Sanders [4] and is a generalization of a result of James.

In Section III Phillip's Lemma and the Orlicz-Pettis Theorem are developed, the results themselves and some of their consequences being needed at several points in Section IV.

A series $\sum_{i=1}^{\infty} X_i$ in a Banach Space is said to be unconditionally convergent if the order of summation is immaterial. If B is a Banach Space with a Schauder decomposition (M_i, P_i), then we say that (M_i, P_i) is an unconditional decomposition provided that $\sum_{i=1}^{\infty} P_i(X)$ is unconditionally convergent for each $X \in B$. This property of decompositions in conjunction with the others defined above is considered in Section IV.

It is known that a Banach Space with an unconditional basis is weak sequentially complete if and only if the basis is boundedly complete. In Theorem 4.2 it is shown that the weak sequential completeness of a space with an unconditional decomposition implies that the decomposition is boundedly complete. Theorem 4.3, an adaptation of Theorem 1, p. 70 in Day [1] gives sufficient conditions on the coordinate spaces of a Banach Space B with an unconditional decomposition (M_i, P_i) in order that B be weak sequentially complete. Two examples follow to show firstly that Theorem 4.3 is indeed a partial generalization of the corresponding theorem for bases and secondly that not all the sufficient conditions of Theorem 4.3 are necessary. Finally Theorems 4.5 and 4.6 show what improvements are possible in Theorems 2.1 and 2.5 with regard to bases when the assumption that they be unconditional is added.

Throughout this paper B will denote a Banach Space and capital X's and Y's will denote variable vectors in B. Two Banach Spaces will be said to be isomorphic when there exists a mapping from one to the other which is both an isomorphism in the algebraic sense and a homeomorphism. The real number system will be denoted by R.

Section I

THEOREM 1.1

A decomposition (M_i, P_i) of a Banach Space B is a Schauder decomposition if and only if M_i is closed for each i. Furthermore, if (M_i, P_i) is a Schauder decomposition there exists a real constant K such that $\|S_n\| \leq K$ for all n whence $\|T_n\| \leq K + 1$ for all n.

PROOF:

(i) Suppose (M_i, P_i) is a Schauder decomposition of B. Now $M_i = \bigcap_{j, j\neq i} P_j^{-1}(0)$. However, since P_j is continuous for all j, P_j-1(0) is closed, whence M_i is closed.

(ii) Suppose (M_i, P_i) is a decomposition of B and that M_i is closed for all i. Define the function $\| \ \|' : B \to R$ by $\|X\|' = \sup_n \|S_n(X)\|$. This function will be well-defined since $\lim_n S_n(X) = X$ for all X. It is easily shown that $\| \ \|'$ is a norm. Further, since $S_n(X) \to X$ for each X, we get $\|X\| \leq \|X\|'$. Thus if it can be shown that $B' = (B, \| \ \|')$ is complete in the norm $\| \ \|'$ we will have B' isomorphic to B under the identity mapping on B by the interior mapping theorem. To this end let (x_k) be a Cauchy sequence in B'. Denote $S_j - S_k$ by S_{kj}. Then $\|S_{kj}(X_m - X_n)\| \leq \|S_j(X_m - X_n)\| + \|S_k(X_m - X_n)\| \leq 2\|X_m - X_n\|'$. In particular $\|P_i(X_m - X_n)\| = \|S_{i-1,i}(X_m - X_n)\| \leq 2\|X_m - X_n\|'$ Hence $(P_i(X_k))$ will be a Cauchy sequence in M_i for each i. Since the M_i are closed subspaces of a Banach Space they will be complete. Thus there exist $X^{(i)} \in M_i$ such that $P_i(x_k) \to x^{(i)}$ for each i. Let $\epsilon > 0$ be given. Since (x_k) is a Cauchy sequence in B' there exists m_ϵ such that $\|X_m - X_n\|' < \epsilon/4$ for $m > n > m_\epsilon$. This means $\|S_{kj}(X_m - X_n)\| \leq 2\|X_m - X_n\|' < \frac{\epsilon}{2}$ for $m > n > m_\epsilon$; i.e. $\|S_{kj}(X_m) - \sum_{i=k+1}^{j} P_i(X_m)\|$. Letting $n \to \infty$ we get $\|S_{kj}(X_m) - \sum_{i=k+1}^{j} P_i(X_n)\| < \epsilon/2$.

for $m > m_\epsilon$. Now $S_n(X_m) \to X_m$. Thus there exists M such that $\|S_{kj}(X_m)\| < \frac{\epsilon}{2}$ for $j > k > M$. We thus have $\|\sum_{i=k+1}^{j} X^{(i)}\| < \epsilon$ for $j > k > M$. This means that $(\sum_{i=1}^{n} X^{(i)})$ is a Cauchy sequence in B. By the completeness of B there exists $X \in B$ such that $\sum_{i=1}^{\infty} X^{(i)} = X$. We finally show that $X_k \to X$ in B'. Let $\epsilon > 0$ be given. Since (X_k) is a Cauchy sequence in B' there exists m_ϵ such that $\|S_j(X_m - X_n)\| < \epsilon$ for $m, n > m_\epsilon$, $j = 1,2,\ldots$. Fixing j and m and letting $n \to \infty$ we find

$$\|S_j(X_m) - \sum_{i=1}^{j} X^{(i)}\| \le \epsilon \text{ for } m > m_\epsilon. \quad (1)$$

Now $X = \sum_{i=1}^{\infty} X^{(i)}$ where $X^{(i)} \in M_i$ for each i. By the uniqueness of the expansion of X w.r.t. (M_i, P_i) we get $X^{(i)} = P_i(X)$. Thus, (1) may be written as $\|S_j(X_m - X)\| \le \epsilon$ for $m > m_\epsilon$ $j = 1, 2, \ldots$. Since j was arbitrary $\sup_j \|S_j(X_m - X)\| \le \epsilon$ for $m > m_\epsilon$, and so $X_k \to X$ in B'. Therefore B' is isomorphic to B. In particular, this means there exists K such that $\|X\|' \le K\|X\|$ for all $X \in B$. We show S_n is a continuous mapping and $\|S_n\|' \le 1$ for each n. For, consider arbitrary $X \in B$ such that $\|X\|' \le 1$ i.e. $\sup_k \|S_k(X)\| \le 1$. Then $\|S_n(X)\|' = \sup_k \|S_k(S_n(X))\| = \sup_k \|S_{p(k)}(X)\| \le 1$ where $p_{(k)} = \min.(k, n)$. Thus S is continuous on B' and so on B.

Now suppose $\|X\| \le 1$; then $\|S_n(X)\| \le \|S_n(X)\|' \le \|S_n\|'(\|X\|') \le K\|X\|$, whence $\|S_n\| \le K$ for each n. Since $T_n = I - S_n$ where I is the identity operator on B, we have $\|T_n\| \le \|I\| + \|S_n\| \le K + 1$. This establishes the second assertion of the theorem. The first follows easily from the fact that $P_i = S_i - S_{i-1}$.

COROLLARY 1.1

Every basis is a Schauder basis.

PROOF:

This follows from the fact that each coordinate space is a one dimensional subspace of B and hence closed.

DEFINITION 1.1

(M_i, P_i) is said to be a monotone decomposition if $\|S_n(X)\| \leq \|S_m(X)\|$ whenever $n < m$.

COROLLARY 1.2

Every Banach Space B with a Schauder decomposition (M_i, P_i) can be renormed isomorphically so that the decomposition is monotone.

PROOF:

(M_i, P_i) will be monotone if B is given the norm $\|\ \|'$ as defined in the above theorem. Choose arbitrary $X \in B$ and let $n > m$. If $S_n(X) = 0$ then $S_m(X) = 0$ and we are done. If not, consider $\|S_m(z)\|'$ where $z = \frac{S_n(X)}{\|S_n(X)\|'}$. Since $\|z\|' = 1$, $\|S_m\|' \leq 1$ and $S_m S_n = S_m$ we have that $\frac{\|S_m(X)\|'}{\|S_n(X)\|'} = \|S_m(z)\|' \leq 1$; i.e. $\underline{\|S_m(X)\|'} \leq \|S_n(X)\|'$.

While every basis is a Schauder basis it is not true in general that every decomposition is a Schauder decomposition as the following example illustrates.

EXAMPLE 1.1

Let (m) be the space of bounded sequences of real numbers with the sup. or uniform norm and let (C_o) be the subspace of (m) consisting of sequences which converge to zero. Let P be a projection from (m) onto (C_o). For a proof of the existence of a projection see Taylor.* Let $M_1 = N(P) = \{X \in (m) : P(X) = 0\}$, and let M_i be the span of e_{i-1}, $i = 2, 3, \ldots$, where e_{i-1} is that element of (m) whose $i-1^{th}$ term is 1 and whose remaining terms are 0.

We show (M_i) is a decomposition of (m). If $X = y + z$, where $z = P(X) \in (C_o)$ then clearly $Y \in N(P)$. X can be represented uniquely in this way. For, suppose $X = Y_1 + z_1 = Y_2 + z_2$. Then $z_1 - z_2 = Y_2 - Y_1$.

*[6] p. 241 Theorem 4.8-A.

But $Y_2 - Y_1$ is in the null space and so $z_2 = P(z_2) = P(z_1) = z_1$ and $Y_1 = Y_2$ whence the representation is unique. However, e_{i-1}, $i = 2, 3, \ldots,$ is a basis for (C_o). Thus to each $X \in B$ there exists a unique $Y \in N(P)$ and a unique sequence (a_i) of real numbers such that $X = Y + \sum_{i=1}^{\infty} a_i e_{i-1}$ And so the claim is established.

However Sobczyk* has shown that there is no continuous projection of the space (m) on the subspace (C_o) whence P is not continuous. Thus the kernel $N(P)$ cannot be closed. By theorem 1, (M_i) is not a Schauder decomposition. |

DEFINITION 1.2

Let B be a Banach Space. The ordered pair $((M_i), (P_i))$ where (M_i) is a sequence of non-trivial subspaces of B and P_i is a continuous projection from B onto M_i, is said to be a biorthogonal sequence in B if $M_i \cap M_j = \{0\}$ whenever $i \neq j$.

DEFINITION 1.3

Let $((M_i), (P_i))$ be a biorthogonal sequence in a Banach Space B. $((M_i), (P_i))$ is said to be basic in B if (M_i) is a Schauder decomposition for csp. $\bigcup_{i=1}^{\infty} M_i$ where csp. $\bigcup_{i=1}^{\infty} M_i$ denotes the smallest closed linear manifold containing $\bigcup_{i=1}^{\infty} M_i$.

PROPOSITION 1.1

A biorthogonal sequence $((M_i), (P_i))$ in a Banach Space B is basic iff. $(\sum_{i=1}^{n} P_i)$ is point-wise bounded on csp. $\bigcup_{i=1}^{\infty} M_i$.

PROOF:

(i) Suppose $((M_i), (P_i))$ is basic. Then given $X \in$ csp. $\bigcup_{i=1}^{\infty} M_i$ we have $\lim_n \sum_{i=1}^{n} P_i(X) = X$ whence $\sup_n \|\sum_{i=1}^{n} P_i(X)\| < \infty$.

(ii) Suppose conversely that $(\sum_{i=1}^{n} P_i)$ is point-wise bounded on csp. $\bigcup_{i=1}^{\infty} M_i$. We show that $((M_i), (P_i))$ is basic.

*[5] pp. 938-947.

Consider the set $C = \{ X \in \text{csp.} \bigcup_{i=1}^{\infty} M_i : X = \lim_n . \sum_{i=1}^{n} P_i(X). \}$. C is obviously a linear manifold. We show it is closed. It will be sufficient to show that if the sequence $(X_m) \in C$ satisfies (a) $\lim_m . X_m = X$ and (b) $\lim_n . \sum_{i=1}^{n} P_i(X_m) - X_m = 0$ for each m then $\lim_n . \sum_{i=1}^{n} P_i(X) - X = 0$. Let $\epsilon > 0$ be given.

Now $\|\sum_{i=1}^{n} P_i(X) - X\| = \|\sum_{i=1}^{n} P_i(X) - \sum_{i=1}^{n} P_i(X_m) + \sum_{i=1}^{n} P_i(X_m) - X_m + X_m - X\| \le \|\sum_{i=1}^{n} P_i\| \, \|X - X_m\| + \|\sum_{i=1}^{n} P_i(X_m) - X_m\| + \|X_m - X\|$. (1) Since the sequence of operators $(\sum_{i=1}^{n} P_i)$ is point-wise bounded on the Banach Space csp. $\bigcup_{i=1}^{\infty} M_i$ there exists $M > 1$ such that $\sup_n . \|\sum_{i=1}^{n} P_i\| < M$. Since $\lim_m . X_m = X$ there exists m_ϵ such that $\|X - X_m\| < \epsilon/3M$ for $m \ge m_\epsilon$. Finally, since $\lim_n . \sum_{i=1}^{n} P_i(X_{m\epsilon}) = X_{m\epsilon}$ there exists N such that $\|\sum_{i=1}^{n} P_i(X_{m\epsilon}) - X_{m\epsilon}\| < \epsilon/3$ for $n \ge N$.

Thus, the m in (1) being arbitrary, we can take $m = m_\epsilon$ and obtain $\|\sum_{i=1}^{n} P_i(X) - X\| < \epsilon$ for $n \ge N$.

Clearly $C \subset$ csp. $\bigcup_{i-1}^{\infty} M_i$. Consider now $X_j \in M_j$. $P_j(X_j) = X_j$ and $P_k(X_j) = 0$, $k \ne j$, by the biorthogonality of $((M_i), (P_i))$ whence $X_j = \lim_n . \sum_{i=1}^{n} P_i(X_j)$; i.e. $X_j \in C$. Therefore $C =$ csp. $\bigcup_{i=1}^{\infty} M_i$.

The uniqueness of the expansion for $X \in$ csp.$\bigcup_{i=1}^{\infty} M_i$ w.r.t. (M_i) is assured by the properties of the P_i. For suppose $X = \sum_{i=1}^{\infty} X_i = \sum_{i=1}^{\infty} Y_1$ where $X_i, Y_i \in M_i$, $i = 1,2,\ldots$. By the biorthogonality of $((M_i), (P_i))$ and the continuity of P_i, we get $X_i = P_i(X) = P(\sum_{i=1}^{\infty} X_i) = P(\sum_{i=1}^{\infty} Y_i) = P(Y_i) = Y_i$.

DEFINITION 1.4

A sequence (M_i) of non-trivial (not necessarily closed) subspaces of a Banach Space B is a weak decomposition of B provided that for each $X \in B$ there exists a unique sequence (X_i), $X_i \in M_i$ such that $X =$ weak $- \lim_n . \sum_{i=1}^{n} X_i$.

THEOREM 1.2

Suppose (M_i, P_i) is a weak decomposition of a Banach Space B and that M_i is closed for each i. Then (M_i, P_i) will be a strong Schauder decomposition of B.

PROOF:

We will show that csp. $\overset{\infty}{\underset{i=1}{\cup}} M_i = B$.

Assuming the contrary, consider $X \in B$ such that $X \notin$ csp. $\overset{\infty}{\underset{i=1}{\cup}} M_i$. From a corollary to the Hahn-Banach theorem there exists an $f \in B^*$ such that $f(X) = \|X\|$ and $f \equiv 0$ on csp. $\overset{\infty}{\underset{i=1}{\cup}} M_i$. However, this would mean that $\lim_n. f(\sum_{i=1}^{n} P_i(X)) \not\to f(X)$ which contradicts the fact that (M_i, P_i) is a weak decomposition.

(ii) Given $X \in B$ we show the sequence $(\sum_{i=1}^{n} P_i(X))$ is bounded. For each n, $\sum_{i=1}^{n} \widehat{P_i}(X)$ is a continuous linear functional on B^*, where $\widehat{P}(x)$ is the image of $P(x)$ under the canonical map from B into B^{**}. Now, for $f \in B^*$, $\lim_n. \sum_{i=1}^{n} \widehat{P_i}(X)[f] = \lim_n. \sum_{i=1}^{n} f(P_i(X)) = f(X)$. Hence the sequence $(\sum_{i=1}^{n} \widehat{P_i}(X))$ is point-wise bounded on B^*, and so, by the Uniform Boundedness Principle, $\sup_n. \|\sum_{i=1}^{n} \widehat{P_i}(X)\| = \sup_n. \|\sum_{i=1}^{n} P_i(X)\| < \infty$.

In view of (i) and (ii) above and prop. 1 we need only show that the P_i are continuous to get the required result.

Since, by (ii), $(\sum_{i=1}^{n} P_i(X))$ is bounded for each $X \in B$ we have that $\| \ \|' : B \to R$, where $\|X\|' = \sup_n. \|\sum_{i=1}^{n} P_i(X)\|$, is a well-defined functional on B. It is easily verified that $\| \ \|'$ is a norm on B.

(iii) B is complete in the norm $\| \ \|'$. Let (X_j) be a Cauchy sequence in $B' = (B, \| \ \|')$. Then, as in Theorem 1, we get, for each i, the existence of an $X^{(i)} \in M_i$ such that $P_i(X_j) \to X^{(i)}$ (strongly). Now, since $\sum_{i=1}^{k} P_i(X_m - X_n) \to X_m - X_n$ weakly, we get, by Fatou's lemma,* that $\|X_m - X_n\| \le \lim_k.\inf. \|\sum_{i=1}^{k} P_i(X_m - X_n)\| \le \sup_k. \|\sum_{i=1}^{k} P_i(X_m - X_n)\| = \|X_m - X_n\|'$. Thus (X_j) will also be a Cauchy sequence in $B = (B, \| \ \|)$. The completeness of B gives an $X \in B$ such that $X_j \to X$.

*Fatou's lemma states that if a sequence (X_n) in a Banach Space B approaches $X \in B$ weakly then $\|X\| \le \lim_n.\inf. \|X_n\|$. For a proof see [7].

We show $X = \sum_{i=1}^{\infty} X^{(i)}$ where convergence is in the weak sense. Consider an arbitrary $f \in B^*$ and let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that $|f(Y)| < \epsilon/3$ whenever $\|Y\| < \delta$. Now, since (x_j) is a Cauchy sequence in B' there exists an N_1 such that $\|\sum_{i=1}^{k} P_i(X_m - X_n)\| < \delta/2$ for $m, n \geq N_1$, $k = 1, 2, \ldots.$ Letting $n \to \infty$ we get $\|\sum_{i=1}^{k} P_i(X_m) - \sum_{i=1}^{k} X^{(i)}\| \leq \delta/2 < \delta$ for all k and for $m > N_1$. Thus $|f(\sum_{i=1}^{k} P_i(X_m)) - f(\sum_{i=1}^{k} X^{(i)})| < \epsilon/3$ for all k and for $m > N_1$. (1)

Since $(X_j) \to X$ strongly in $B = (B, \| \ \|)$ it will approach X weakly. Hence there exists N_2 such that $|f(X_m) - f(X)| < \epsilon/3$ for $m \geq N_2$. (2)

Let $M = \max. [N_1, N_2]$. Then for all k, $|f(\sum_{i=1}^{k} X^{(i)}) - f(X)| \leq |f(\sum_{i=1}^{k} X^{(i)}) - f(\sum_{i=1}^{k} P_i(X_M))| + |f(\sum_{i=1}^{k} P_i(X_M)) - f(X_M)| + |f(X_M) - f(X)| < 2\epsilon/3 + |f(\sum_{i=1}^{k} P_i(X_M)) - f(X_M)|$ by (1) and (2). $\sum_{i=1}^{k} P_i(X_M) \to X_M$ weakly. Thus there exists N such that $|f(\sum_{i=1}^{k} P_i(X_M)) - f(X_M)| < \epsilon/3$ for $k \geq N$. Thus $|f(\sum_{i=1}^{k} X^{(i)}) - f(X)| < \epsilon$ for $k \geq N$ and so $\sum_{i=1}^{k} X^{(i)} \to X$ weakly.

We now show that $(x_j) \to X$ in B'. Let $\epsilon > 0$ be given. Since (x_j) is a Cauchy sequence in B', there exists m_ϵ such that $\|X_m - X_n\|' < \epsilon$ for $m, n > m_\epsilon$. Thus for each k, $\|S_k(X_m - X_n)\| < \epsilon$. Letting $n \to \infty$ we get $\|S_k(X_m - X)\| \leq \epsilon$ for $m > m_\epsilon$. Since k was arbitrary it follows that $\|X_m - X\|' = \sup_k . \|S_k(X_m - X_n)\| \leq \epsilon$ for $m > m_\epsilon$. This establishes the completeness of B'.

If it can be shown that B' is isomorphic to B, the continuity of the P_i will follow as in Theorem 1. The isomorphism of B and B' under the identity mapping on B will follow if $\| \ \|$ and $\| \ \|'$ can be compared.

Now, $\sum_{i=1}^{k} P_i(X) \to X$ weakly for each $X \in B$. By Fatou's lemma this means that $\|X\| \leq \lim_k . \inf. \|\sum_{i=1}^{k} P_i(X)\| \leq \sup_k . \|\sum_{i=1}^{k} P_i(X)\| = \|X\|'$.

Section II

PROPOSITION 2.1

Let (M_i, P_i) be a Schauder decomposition of a Banach Space B. Denote by P_i^* the adjoint of P_i and denote the range of P_i^* by $R(P_i^*)$. Then $(R(P_i^*), P_i^*)$ is a Schauder decomposition for csp. $\overset{\infty}{\underset{i=1}{\cup}} R(P_i^*)$.

PROOF:

As is well known P_i^* will be a continuous linear mapping from B^* into itself. We show P_i^* is a projection from B^* onto $R(P_i^*)$. To this end consider an arbitrary $f \in R(P_i^*)$. By definition $f = g \cdot P_i$ for some $g \in B^*$. Again, by definition, $P_i^*(f) = (g \cdot P_i) \cdot P_i = g \cdot (P_i \cdot P_i) = g \cdot P_i$ since P_i is a projection.

We show $R(P_i^*) \cap R(P_j^*) = \{0\}$ for $i \neq j$. Suppose $f \in R(P_i^*) \cap R(P_j^*)$ and $i \neq j$. Then $f = g \cdot P_i = h \cdot P_j$ where $g, h \in B^*$. Given $x_k \in M_k$, $k \neq i$, we have $f(x_k) = g \cdot P_i\ (x_k) = 0$. On the other hand for $x_i \in M_i$ we have $f(x_i) = h.\ P_j(x_i) = 0.$. Hence $f(x_k) = 0$ where $x_k \in M_k$ for all k. Given $x \in B$, $x = \sum_{i=1}^{\infty} x_i$, $x_i \in M_i$, thus $f(x) = f(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{\infty} f(x_i) = 0$ by continuity; i.e. $f \equiv 0$ on B.

From the previous two paragraphs we get that $((R(P_i^*)), (P_i^*))$ is a biorthogonal sequence in B^*. To show that $((R(P_i^*)), (P_i^*))$ is a Schauder decomposition for csp. $\overset{\infty}{\underset{i=1}{\cup}} R(P_i^*)$ we need only show, in view of proposition 1.1 that $(\sum_{i=1}^{n} P_i^*)$ is point-wise bounded on csp. $\overset{\infty}{\underset{i=1}{\cup}} R(P_i^*)$.

Take an $F \in B^*$. Then for $x \in B$ we have $\sum_{i=1}^{n} P_i^*(F)[x] = \sum_{i=1}^{n} F[P_i(x)] \to F(x)$ by continuity. Thus $(\sum_{i=1}^{n} P_i^*(F)[x])$ is point-wise bounded on B and so by the Uniform Boundedness Principle there exists M such that $\|\sum_{i=1}^{n} P_i^*(F)\| \leq M$ for all n. Since F was arbitrary, $(\sum_{i=1}^{n} P_i^*)$ is point-wise bounded.

THEOREM 2.1

Let (M_i, P_i) be a Schauder decomposition of a Banach Space B. Then, the following conditions are equivalent:

(i) $(R(P_i^*), P_i^*)$ is a Schauder decomposition of B^*.

(ii) $\lim_n \|T_n^*(f)\| = 0$ for each $f \in B^*$

(iii) (M_i, P_i) is shrinking.

(iv) csp. $\bigcup_{i=1}^{\infty} R(P_i^*) = B^*$.

PROOF: Proposition 1 gives immediately the equivalance of (i) and (iv). The equivalence of (i) and (ii) follows from observing that $T_n^* =$ Identity $- \sum_{i=1}^{n} P_i^*$ and thus $\lim_n \|T_n^*(f)\| = 0$ iff $\lim_n \|f - \sum_{i=1}^{n} P_i^*(f)\| = 0$; i.e. iff $\sum_{i=1}^{\infty} P_i^*(f) = f$ for all $f \in B^*$. The proof will be completed by showing the equivalence of (ii) and (iii). For each $f \in B^*$, $\|f\|_n = \sup. \{|f(x)| : X = T_n(X), \|X\| \le 1\} \le \sup. \{|f(T_n[X])| : \|X\| \le 1\} = \|T_n^*(f)\|$. Thus (ii) implies (iii). Since (M_i, P_i) is a Schauder decomposition, there exists, by Theorem 1.1, a K such that $\|S_n\| \le K$ for all n. Hence $\|T_n\| \le K + 1$ for all n. Thus for each n and for each $f \in B^*$, $\|T_n^*(f)\| = \sup. \{|f(T_n)(X)| : \|X\| \le 1\} \le \sup. \{\|f\|_n \|T_n(X)\| : \|X\| \le 1\} \le \|f\|_n \|T_n\| \le \|f\|_n (K+1)$ and consequently (iii) implies (ii).

LEMMA 2.1

Let (M_i) be a Schauder decomposition of a Banach Space B and let (F_i) be a sequence of subspaces of B^* satisfying:

(a) (F_i) is a Schauder decomposition for csp. $\bigcup_{i=1}^{\infty} F_i = \Gamma$

(b) given $f_i \in F_i$ and arbitrary $X_k \in M_k$, $k \neq i$, $f_i(X_k) = 0$.

Then, if (X_i), $X_i \in M_i$ is a sequence for which $\sup. \|U_n\| < \infty$ then, there exists $F \in B^{**}$ such that $\hat{U}_n(\gamma) \to F(\gamma)$ for $\gamma \in \Gamma$, where $U_n = \sum_{i=1}^{n} X_i$.

PROOF:

We can assume without loss of generality that $\|X_n\| \le 1$ for all n. Clearly, given $f_i \in F_i$ we have $f_i(U_n) = f_i(X_i)$ for all $n \geq i$. We show that for every $\gamma \in \text{csp.} \bigcup_{i=1}^{\infty} F_i$, $\lim_n \gamma(U_n)$ exists. Now γ can be expressed uniquely in the form $\sum_{i=1}^{\infty} f_i$ where $f_i \in F_i$. There will exist $N \ni \cdot \|\gamma - \sum_{i=1}^{N} f_i\| < \epsilon/2$. For any $n > N$ we have, by the foregoing, that $|\gamma(U_n) - \sum_{i=1}^{N} f_i(X_i)| = |[\gamma - \sum_{i=1}^{N} f_i](U_n)| < \epsilon/2$. Thus for $p, q > N$, $|\gamma(U_p) - \gamma(U_q)| \leq |\gamma(U_p) - \sum_{i=1}^{N} f_i(X_i)| + |\sum_{i=1}^{N} f_i(X_i) - \gamma(U_q)| < \epsilon$. This means that $(\gamma(U_n))$ is a Cauchy sequence and hence a convergent sequence of real numbers.

Since $\lim. \hat{U}_n / \Gamma(\gamma) = \lim_n \gamma(U_n)$ exists there is, by a corollary to the Uniform Boundedness Principle, a $\phi \in \Gamma^*$ such that $\lim. \hat{U}_n/_\Gamma(\gamma) = \phi(\gamma)$ for all $\gamma \in \Gamma$. By the Hahn-Banach Theorem there is a $F \in B^{**}$ such that $\|F\| = \|\phi\|$ and F is an extension of ϕ. Clearly, $F(\gamma) = \lim_n \hat{U}_n(\gamma)$ for all $\gamma \in \Gamma$.

THEOREM 2.2

Let (M_i, P_i) be a shrinking decomposition for a Banach Space B and let $(R(P_i^*))$ be the corresponding decomposition for B^*. Then (M_i, P_i) is a monotone decomposition iff. $(R(P_i^*), P_i^*)$ is a monotone decomposition.

PROOF:

Suppose (M_i, P_i) is monotone. To show $(R(P_i^*), P_i^*)$ is monotone it is sufficient to show $\|\sum_{i=1}^{n} P_i^*\| \leq 1$ for all n. Consider then arbitrary $f \in B^*$ with $\|f\| = 1$. We must show that $\|\sum_{i=1}^{n} P_i^*(f)\| \leq 1$. Consider $X \in B$, with $\|X\| = 1$. Then $|\sum_{i=1}^{n} P_i^*(f)(X)| = |f(\sum_{i=1}^{n} P_i(X))| \leq \|f\| \, \|\sum_{i=1}^{n} P_i(X)\| \leq \|\sum_{i=1}^{n} P_i(X)\| \leq 1$ since (M_i, P_i) is monotone.

Suppose now that $(R(P_i^*), P_i^*)$ is monotone and let $X \in B$, $\|X\|=1$ be given. To show that $\|\sum_{i=1}^{n} P_i(X)\| \leq 1$, and hence $\|\sum_{i=1}^{n} P_i\| \leq 1$, it will be sufficient to show that $|f(\sum_{i=1}^{n} P_i(X))| \leq 1$ for arbitrary $f \in B^*$ with $\|f\| = 1$. However, $|f(\sum_{i=1}^{n} P_i(X))| = |\sum_{i=1}^{n} f(P_i)(X)| =$

$|\sum_{i=1}^{n} f \cdot P_i(X)| = |\sum_{i=1}^{n} P_i^*(f)\ [X]| \le \|\sum_{i=1}^{n} P_i^*(f)\| \le \|\sum_{i=1}^{n} P_i^*\| \le 1$ since $(R\ (P_i^*),\ P_i^*)$ is monotone.

DEFINITION 2.1

If L_o and L_1 are subspaces of a linear space L then $T : L \to L_o$ is said to be a projection of L onto L_o along L_1 if $T\ [T\ (X)] = T\ (X)$ for all $X \in L_o$ and the kernel of T is L_1.

DEFINITION 2.2

If L_o is a closed linear manifold in a Banach Space B then $L_o^{\perp}$ is the set of all elements of B^* whose kernel contains L_o.

LEMMA 2.2

If i is the identity mapping into a Banach Space N of a closed linear subspace N_o then i^*, the adjoint of i, determines a linear isometry i^1 between N_o^* and $N^*/N_o^{\perp}$.

PROOF:

i^* is a continuous linear mapping from N^* into N_o^* under which two elements have the same image iff. they agree on N_o. Given $f \in N_o^*$, there exists, by the Hahn-Banach Theorem, a norm-preserving extension F of f to all of N and clearly $i^*(F) = f$. Thus i^* is onto and $N^*/N_o^{\perp}$ is isomorphic to N_o^* under the mapping $i^1 : N^*/N_o^{\perp} \to N_o^*$ where $i^1(N_o^{\perp} + F) = i^*(F)$. We show i^1 is an isometry. Consider $f \in N_o^*$ such that $f = i^1(N_o^{\perp} + F)$. We must show $\|f\| = \|N_o^{\perp} + F\|$. Given $\epsilon > 0$ there exists a $g \in N_o^{\perp}$ such that $\|F + g\| < \|N_o^{\perp} + F\| + \epsilon$. Since $F + g$ is an extension of f we get $\|f\| \le \|F + g\| < \|N_o^{\perp} + F\| + \epsilon$ and so $\|f\| \le \|N_o^{\perp} + F\|$.
On the other hand there exists, by the Hahn-Banach Theorem, a norm-preserving extension $\bar{f}$ of f to all of N. $\bar{f} = F + g$ for some $g \in N_o^{\perp}$ and so $\|f\| = \|F + g\| \ge \|N_o^{\perp} + F\|$.

LEMMA 2.3

The conjugate space of any Banach Space is w^* – sequentially complete. For a proof see Day.*

THEOREM 2.3

Let (b_i) be a monotone basis for a Banach Space B and let (β_i) be the corresponding biorthogonal sequence in B^*, and let Γ be the closed linear space spanned by the β_i. Then (i), (ii) and (iii) below are equivalent and imply (iv).

(i) (b_i) is a boundedly complete basis.

(ii) For each F in B^{**} the series $\sum_{i=1}^{\infty} F(\beta_i)\, b_i$ converges to a point Y_F of B with $\|Y_F\| \le \|F\|$.

(iii) There is a projection T of B^{**} onto Q(B) along $\Gamma^{\perp}$ of norm 1 where Q is the canonical map from B to B^{**}.

(iv) B is isometric to an adjoint space (Γ^*).

PROOF:

To show (i) implies (ii). We show that given $F \in B^{**}$, $\sup_n \|\sum_{i=1}^{n} F(\beta_i)\, b_i\| < \infty$. By the Uniform Boundedness Principle this will be the case if $\sup_n \|\sum_{i=1}^{n} F(\beta_i)\, \hat{b}_i(f)\| < \infty$ for every $f \in B^*$. This in turn will be true if $\sup_n \|\sum_{i=1}^{n} \beta_i\, \hat{b}_i\,(f)\| = \sup_n \|\sum_{i=1}^{n} \beta_i\, f\,(b_i)\| < \infty$. Again, by Uniform Boundedness Principle, this will follow if, for $X \in B$, $\sup_n \|\sum_{i=1}^{n} \beta_i(X)\, f\,(b_i)\| < \infty$. Finally this will be so if $\sup_n \|\sum_{i=1}^{n} \beta_i(X)\, b_i\| < \infty$ which it is since (b_i) is a basis for B. Thus $\sup_n \|\sum_{i=1}^{n} F(\beta_i)\, b_i\| < \infty$ and the bounded completedness of (b_i) implies there is a $Y_F \in B \ni Y_F = \sum_{i=1}^{\infty} F(\beta_i)\, b_i$. We show $\|Y_F\| \le \|F\|$. This will be so if we can show that $|f\,(Y_F)| \le \|F\|$ for all $f \in B^*$ with $\|f\| = 1$. Since $f(Y_F) = \sum_{i=1}^{\infty} F(\beta_i)\, f\,(b_i)$ we get $|f(Y_F)| \le \lim_n \sup |\sum_{i=1}^{n} F\,(\beta_i)\, f\,(b_i)|$. Now $|\sum_{i=1}^{n} F(\beta_i)\, f\,(b_i)| \le \|F\|\, \|\sum_{i=1}^{n} \beta_i\, f\,(b_i)\|$. Again, for any $X \in B$ such that $\|X\| = 1$, $|\sum_{i=1}^{n} \beta_i(X)\, f\,(b_i)| = |f\,(\sum_{i=1}^{n} \beta_i(X)\, b_i)| \le \|f\|\, \|\sum_{i=1}^{n} \beta_i(X)\, b_i\| \le \|\sum_{i=1}^{n} \beta_i\,(X)\, b_i\| \le 1$ since (b_i) is a monotone basis. Thus $\|\sum_{i=1}^{n} \beta_i\, f\,(b_i)\| \le 1$. This means and so $|\sum_{i=1}^{n} F(\beta_i)\, f\,(b_i)| \le \|F\|$ for each n.

*[1] p. 69

$|f(Y_F)| \le \lim._n \sup. |\sum_{i=1}^{n} F(\beta_i)\, b_i| \le \|F\|$.

To show (ii) implies (iii). Let $T(F) = \hat{Y}_F$; then T is linear, $\|T(F)\| = \|\hat{Y}_F\| = \|Y_F\| \le \|F\|$ (i.e. T of norm 1), and given $F \in Q(B)$, $Q(Y_F) = F$ (i.e. T is onto Q(B)). Also, since $\hat{Y}_F(\beta_k) = \sum_{i=1}^{\infty} F(\beta_i)\, \hat{b}_k\, [\beta_k] = \sum_{i=1}^{\infty} F(\beta_i)\, \beta_k\, (b_i) = F(\beta_k)$ [(b_i) and (β_i) being a biorthogonal system] we get $F - T(F)$ vanishing on each β_k whence $F - T(F) \in \Gamma^{\perp}$. If $g \in \Gamma^{\perp}$ then $T(g) = \hat{Y}_g = \sum_{i=1}^{\infty} g(\beta_i)\, b_i = 0$. Hence $T(F - T(F)) = 0$ for all F ; i.e. $T(F) - T^2(F) = 0$ or $T(F) = T^2(F)$ whence $T = T^2$. Finally, if $T(F) = 0$, $F(\beta_k) = 0$ for all k i.e. $F \in \Gamma^{\perp}$. Thus T is a projection of norm 1 of B^{**} onto Q(B) along $\Gamma^{\perp}$.

To show (iii) implies (i). Let (a_i) be a sequence of real numbers such that if $X_n = \sum_{i=1}^{n} a_i\, b_i$ then $\sup_n \|X_n\| < \infty$, say $\|X_n\| \le 1$ for all n. Then $\beta_i\,(X_n) = a_i$ for all $n \ge i$ since $((b_i), (\beta_i))$ is a biorthogonal system. We show that for every $\gamma \in \Gamma$ $(\gamma\,(X_n))$ is a Cauchy, and hence a convergent sequence of real numbers. For let $\epsilon > 0$ be given. Then, since $\gamma = \sum_{i=1}^{\infty} \beta_i\, \gamma\,(b_i)$ there exists N such that $\|\gamma - \sum_{i=1}^{N} \beta_i\, \gamma(b_i)\| < \epsilon/2$. For any $n > N$ we have, by the foregoing, that $|\gamma(X_n) - \sum_{i=1}^{N} a_i\, \gamma\,(b_i)| = |[\gamma - \sum_{i=1}^{N} \beta_i\, \gamma(b_i)]\,(X_n)| \le \|\gamma - \sum_{i=1}^{N} \beta_i\, \gamma(b_i)\|\, \|X_n\| < \epsilon/2$. Thus for p, $q > N$ $|\gamma\,(X_p) - \gamma\,(X_q)| \le |\gamma\,(X_p) - \sum_{i=1}^{N} a_i\, \gamma(b_i)| + |\sum_{i=1}^{N} a_i\, \gamma(b_i) - \gamma(X_q)| < \epsilon$. This means that for every $\gamma \in \Gamma$ the sequence of functionals $\hat{X}_n/\Gamma \in \Gamma^*$ are such that $\lim_n.\, \hat{X}_n/\Gamma\,(\gamma)$ exists. By Lemma 2.3 there is a ϕ in Γ^* such that $\lim_n.\, \hat{X}_n/\Gamma\,(\gamma) = \phi(\gamma)$. Further, by Fatou's Lemma, $\|\phi\| \le \lim_n. \inf. \|\hat{X}_n/\Gamma\| \le \lim._n \inf. \|\hat{X}_n\| = \lim_n. \inf. \|X_n\| \le 1$. By the Hahn-Banach Theorem there is a $F \in B^{**}$ so that $\|F\| = \|\phi\|$ and F is an extension of ϕ. Let $Y = Q^{-1}[T(F)]$. Since $T^2 = T$ we have that $F - T(F) \in \Gamma^{\perp}$ whence $[F - T(F)]\,(\gamma) = 0$ i.e. $F\,(\gamma) = T\,(F)\,(\gamma)$.

$\gamma(Y) = \gamma(Q^{-1}[T(F)]) = T(F)(\gamma) = F(\gamma) = \phi(\gamma)$. In particular $\beta_i(Y) = \phi(\beta_i) = \lim_n \beta_i(X_n) = a_i$. Hence the expansion of Y in the basis (b_i) is

$\sum_{i=1}^{\infty} a_i b_i$.

This completes the proof that (i), (ii) and (iii) are equivalent. We now show that (iii) implies (iv). B is isometric to Q(B). From (iii) we get that $i' : B^{**}/\Gamma^{\perp} \to Q(B)$ where $i'(\Gamma^{\perp} + F) = T(F)$, is an isomorphism. Since T is of norm 1, $\|T(F)\| \le \|\gamma^{\perp} + F\|$ for every $\gamma^{\perp} \in \Gamma^{\perp}$ and so $\|T(F)\| \le \inf_{\gamma^{\perp} \in \Gamma^{\perp}} \|\gamma^{\perp} + F\| = \|\Gamma^{\perp} + F\|$. On the other hand, T being the identity on Q(B), $T(F) \in \Gamma^{\perp} + F$ and so $\|\Gamma^{\perp} + F\| = \|T(F)\|$. Thus Q(B) is isometric to $\frac{B^{**}}{\Gamma^{\perp}}$. From Lemma 2.2, however, $\frac{B^{**}}{\Gamma^{\perp}}$ is isometric to Γ^* and so B is isometric to Γ^*.

DEFINITION 2.3

Let L be a subspace of a Banach Space B and let Γ be a subset of B^*. Then Γ is said to be total over L if $f(X) = 0$ for all f in Γ implies that $X = 0$, where $X \in L$.

LEMMA 2.4

Let (M_i) be a Schauder decomposition of a Banach Space B and suppose that (F_i) is a sequence of subspaces of B^* satisfying:

(a) (F_i) is a Schauder decomposition of $\Gamma = \text{csp.} \bigcup_{i=1}^{\infty} F_i$.

(b) Given $f_i \in F_i$ and arbitrary $X_k \in M_k$, $k \ne i$, $f_i(X_k) = 0$.

(c) F_i is total over M_i. Then, if there exists a projection T of B^{**} onto Q(B) along $\Gamma^{\perp}$, (M_i) is boundedly complete.

PROOF:

Let (X_i), $X_i \in M_i$, be a sequence such that if $z_n = \sum_{i=1}^{n} X_i$, then $\sup_n \|z_n\| < \infty$, say $\|z_n\| \le 1$ for all n. By (b) we get that if $f_i \in F_i$ then $f_i(z_n) = f_i(X_i)$ for all $n \ge i$. Then for every γ in Γ, $\lim_n \gamma(z_n)$ exists. Since, by (a), $\gamma = \sum_{i=1}^{\infty} f_i$ where $f_i \in F_i$, there exists N such that $\|\gamma - \sum_{i=1}^{N} f_i\| < \epsilon/2$. For any $n > N$ we have, by the foregoing, that $|\gamma(z_n) - \sum_{i=1}^{N} f_i(z_n)| = |[\gamma - \sum_{i=1}^{N} f_i](z_n)| < \epsilon/2$.

For p, q, $> N$ we get $|\gamma(z_p) - \gamma(z_q)| < \epsilon$ as in Theorem 2.3, whence $(\gamma(z_n))$ is a Cauchy and hence a convergent sequence of real numbers.

Thus for every $\gamma \in \Gamma$ the sequence of functionals $\hat{z}_n/\Gamma \in \Gamma^*$ are such that $\lim_n \hat{z}_n/\Gamma(\gamma)$ exists. By Lemma 2.3 there is a $\phi \in \Gamma^*$ such that $\lim_n \hat{z}_n/\Gamma(\gamma) = \phi(\gamma)$. By the Hahn-Banach theorem there is a $F \in B^{**}$ so that $\|F\| = \|\phi\|$ and F is an extension of ϕ. Let $Y = Q^{-1}[T(F)]$.
Since $T^2 = T$ we have that $T[F - T(F)] = T(F) - T(F) = 0$, *, and so $[F - T(F)](\gamma) = 0$ i.e. $F(\gamma) = T(F)(\gamma)$. Then $\gamma(Y) = \gamma(Q^{-1}(T(F))) = T(F)(\gamma) = F(\gamma) = \phi(\gamma) = \lim_n \gamma(z_n)$.

Let the expansion of Y w.r.t. (M_i) be $\sum_{i=1}^{\infty} Y_i$. By (b) $f_i(Y) = f_i(Y_i)$ for all $f_i \in F_i$. However, by the previous paragraph since $f_i \in \Gamma$, $f_i(Y) = \lim_n f_i(z_n) = f_i(X_i)$. Thus $f_i(Y_i) = f_i(X_i)$ for all $f_i \in F_i$ i.e. $f_i(X_i - Y_i) = 0$ for $f_i \in F_i$. Since F_i is total over M_i we get $X_i = Y_i$. Hence $Y = \sum_{i=1}^{\infty} X_i$. Therefore, (M_i) is boundedly complete.

THEOREM 2.4

Let (M_i, P_i) be a shrinking decomposition for a Banach Space B and let $(R(P_i^*), P_i^*)$ be the corresponding decomposition for B^*. Then $(R(P_i^*), P_i^*)$ is a boundedly complete decomposition of B^*.

PROOF:

Let $Q M_i$ be the image of M_i under the canonical mapping of B into B^{**}. We show that $(Q M_i)$ satisfies the properties of the previous Lemma w.r.t. the Schauder decomposition $(R(P_i^*))$ of B^*. Firstly, since (M_i) is a Schauder decomposition of B and since Q(B) is isomorphic to B under the mapping Q, $(Q M_i)$ will be a Schauder decomposition for $Q(B) = \text{csp.} \bigcup_{i=1}^{\infty} Q M_i$. Again, let $\chi_i \in Q M_i$ and consider arbitrary $f_k \in R(P_k^*)$, $k \neq i$. Then $f_k = g_k \cdot P_k$ where $g_k \in M_k^*$ and so $\hat{X}_i(f_k) = \hat{X}_i(g_k \circ P_k) = g_k \cdot P_k(X_i) = g_k(P_k(X_i)) = g_k(0) = 0$. Finally, suppose that

* i.e. $F - T(F) \in \Gamma^{\perp}$

$f_i \in R(P_i^*)$ is such that $\hat{X}_i(f_i) = 0$ for $\hat{X}_i \in Q\,M_i$.

There exists a $g_i \in B^*$ with $f_i = g_i \cdot P_i$. Now, $0 = \hat{X}_i\ (f_i) = \hat{X}_i\ (g_i \cdot P_i) = g_i\ (P_i\ (X_i)\ ) = g_i(X_i)$ for all $X_i \in M_i$. This means that $g_i \equiv 0$ on M_i whence $f_i = g_i \cdot P_i \equiv 0$.

In view of Lemma 2.4 and the preceeding paragraph we need only show the existence of a projection. T of B^{***} onto $Q'(B^*)$ along $\Gamma^\perp$ where $\Gamma = Q(B)$ †. We show $Q'\ Q^*$ is such a projection. If $f \in Q'(B^*)$ then $Q^*(f) = Q'^{-1}\ (f)$ and so $Q'Q^*$ is a mapping of B^{***} onto $Q'(B^*)$ and idempotent. If $f \in \Gamma^\perp$ then f vanishes on $\Gamma = Q(B)$ and so $Q^*(f) \equiv 0$ whence $Q'Q^*(f) \equiv 0$. Again, if $Q'Q^*(f) = 0$ then $Q^*(f)$ must be the zero functional on B which means that f must vanish on $Q(B) = \Gamma$ i.e. $f \in \Gamma^\perp$. Thus the kernel of $Q'Q^*$ is $\Gamma^\perp$.

<u>LEMMA 2.5</u>

If (X_i) is a sequence in a Banach Space B which converges weakly to X, then X is the limit of a sequence of finite linear combinations of the X_i i.e. $X \in \text{csp.} \bigcup_{i=1}^{\infty} X_i$.

<u>PROOF</u>:

Suppose not. Then there exists a continuous linear functional f such that $f(X) = ||X||$ and $f(Y) = 0$ for all $Y \in \text{csp.}\bigcup_{i=1}^{\infty} X_i$. Then $f(X_n) \to 0 \neq f(X)$.

<u>EBERLEIN'S THEOREM 2.5</u>

A Banach Space B is reflexive iff. its unit sphere is weakly compact.* For a proof see [2]

<u>THEOREM 2.6</u>

A Banach Space B with a Schauder decomposition (M_i, P_i) is reflexive iff. each M_i is reflexive and the decomposition is both shrinking and boundedly complete.

* i.e. every infinite sequence of points in the unit sphere has a weakly convergent subsequence.

† Q' is the canonical map from B^* into B^{***}.

PROOF:

Necessity. Suppose B is reflexive. This means that each of the M_i is reflexive. Firstly, since (M_i, P_i) is a Schauder decomposition M_i will be closed and hence a Banach Space for each i. In view of Eberlein's Theorem we need only show that the unit sphere in M_i is weakly compact. To this end let (X_i) be a sequence in M_i such that $\sup_i \|X_i\| \leq 1$. Since B is reflexive, we have, by Eberlein's theorem, that the unit ball is weak sequentially compact and so there exists an $X \in B$ and a subsequence (X_{i_k}) such that $X_{i_k} \to X$ weakly in B. This means that X belongs to the weak closure of M_i and hence to $\overline{M_i} = M_i$ the weak closure and the strong closure of a linear manifold in a normed linear space being the same.

(i) To show (M_i, P_i) is boundedly complete. Suppose (X_i) is a sequence in B such that $X_i \in M_i$ for all i and $\sup_n \|U_n\| < \infty$ where $U_n = \sum_{i=1}^{n} X_i$. Since $\sup_n \|U_n\| < \infty$ and B is reflexive there exists X in B and a subsequence (U_{n_k}) of (U_n) such that $U_{n_k} \to X$ weakly in B. By the previous Lemma there exists a sequence $Y_n \to X$ strongly where each Y_n is a finite linear combination of the U_i and hence of the X_i. If it can be shown that $P_i(X) = X_i$ for all i we will be done. If $X_i = 0$ then the strong convergence of (Y_n) to X gives us that $\lim_n P_i(Y_n) = X_i = 0 = P_i(X)$. Suppose then $X_i \neq 0$. This means that $X_i \notin$ csp. $\bigcup_{j=1, j\neq i}^{\infty} M_j = A$. Hence there exists a $g \in B^*$ such that $g(X_i) = \|X_i\|$ and $g(a) = 0$ for all $a \in A$. (i)

Thus $\lim_k g(U_{nk}) = g(X_i) = g(X) = g(P_i(X))$. . Furthermore, for each n there exists a real number a_n such that $P_i(Y_n) = a_n X_i$ whence $P_i(X) = b_i X_i$ * by the strong convergence of (Y_n) to X. Thus $g(P_i(X)) = b_i\, g(X_i)$.

* where b_i is a real number.

. This together with 1 yields $g(X_i) = g(P_i(X)) = b_i\ g(X_i)$ and so $b_i = 1$ since $g(X_i) \neq 0$. Thus $P_i(X) = X_i$.

(ii) To show (M_i, P_i) is shrinking. Suppose not. Then there exists g in B^*, an $\epsilon > 0$, and a subsequence $m(n)$ of the positive integers such that $\|g\|_{m(n)} \geq \epsilon$ for all n. Choose X_n so that

$$|g\ (X_n)\ | \geq \epsilon/2 \text{ where } X_n = T_{m(n)}\ [X_n] \text{ and } \|X_n\| \leq 1 \qquad (2)$$

By Eberlein's Theorem the reflexivity of B implies the existence of an $X \in B$ and a subsequence $(X_{n(i)})$ of (X_n) such that $X_{n(i)} \to X$ weakly. We show X must $= 0$. . For suppose that $P_k(X) \neq 0$ for some k. Then $P_k(X) \notin$ csp. $\overset{\infty}{\underset{j=1\ j\neq k}{U}} M_j = A$. . This means that there exists an $h \in B^*$ such that $h(P_k(X)\ ) = \|P_k(X)\|$ and $g(a) = 0$ for all $a \in A$. Choose t so that $m(n(t)) > k$. Then for all $i > t$ we have $\lim_i$. $h(X_{n(i)}) = \lim_i.\ h[Tm_{n(i)}\ (X_{n(i)})] = 0 = h(\sum_{i=1}^{\infty} P_i(X)) = h\ (P_k(X))$ which is a contradiction. Thus $X = 0$ and so $\lim_i.\ g(X_{n(i)}) = g(X) = g(0) = 0$ which contradicts (2).

<u>SUFFICIENCY</u>:

By Eberlein's Theorem it will be sufficient to show that the unit ball is weak sequentially compact. To this end let (Y_n) be a sequence in B such that $\|Y_n\| \leq 1$ for each n. P being continuous for each i we get that $(P_i(Y_n)\ )$ is bounded in M_i for each i. This means, by the reflexivity of M_i, that there exists $X_1 \in M_1$ and a subsequence $(Y_{n,1})$ of (Y_n) such that $(P_1(Y_{n,1})\ ) \to X_1$ weakly. Again, by the reflexivity of M_2 there exists $X_2 \in M_2$ and a subsequence $(Y_{n,2})$ of $(Y_{n,1})$ such that $(P_2)(Y_{n,2}) \to Y_2$. By induction we get a sequence of subsequences of (Y_n), $[(Y_{n,j})_{n=1}^{\infty}]_{j=1}^{\infty}$ each a subsequence of the previous, such that $P_j(Y_{n,j}) \to X_j \in M_j$ weakly. Consider now the sequence $(Y_{i,i})_{i=1}^{\infty}$. Since $(Y_{i,i})$, $i \geq j$, is a subsequence of $(Y_{n,j})$

we get $P_j(Y_{i,i}) \to X_j$ weakly for all j. Consider a fixed $f \in B^*$.

Then $|f(\sum_{i=1}^{n} P_j(Y_{i,i}))| \le \|f\| \, \|\sum_{j=1}^{n} P_j(Y_{i,i})\| \le \|f\| \, \|S_n\| \, \|Y_{i,i}\| \le k\|f\|$

for all i where k is an upper bound of $(\|S_n\|)$ whose existence is assured by theorem 1.1. Letting $i \to \infty$ we get $|f(\sum_{j=1}^{n} X_j)| \le k\|f\|$. Since the R.H.S. is independent of n we get that the sequence of functions $(\sum_{i=1}^{n} \hat{X}_i)$ is point-wise bounded on B^* and hence by the Uniform Boundedness Principle $\sup_n . \|\sum_{i=1}^{n} X_i\| < \infty$. The assumption of bounded completeness gives us the existence of a $Y \in B$ such that $\sum_{i=1}^{\infty} X_i = Y$ and $P_j(Y) = X_j$. The proof will be completed if it can be shown that $(Y_{i,i}) \to Y$ weakly. To this end consider an arbitrary $f \in B^*$ and let $\epsilon > 0$ be given. Let $K = k + 1$. Then $\|T_n\| \le K$ for all n. Choose $\mu = \epsilon/4K$ if $Y = 0$ or $= \min. (\epsilon/4K, \epsilon/4K\|Y\|)$ if $Y \ne 0$. Since the decomposition is shrinking there exists N such that $\|f\|_N < \mu$. Now $|f(Y - Y_{i,i})| \le |f(S_N(Y - Y_{i,i})| + |f(T_N(Y))| + |f(T_N(Y_{i,i}))|$. Since $P_j(Y_{i,i}) \to X_j = P_j(Y)$ weakly there exists M such that $|f(S_N(Y - Y_{i,i}))| < \epsilon/2$ for $i \ge M$. Now $\|T_N \frac{(Y_{i,i})}{K}\| \le \|T_N\|(1/K)\|Y_{i,i}\| \le K(1/K) = 1$. Thus $|f(T_N(Y_{i,i}))| \le \|f\|_N \, K \, \|T_N \frac{(Y_{i,i})}{K}\| \le \mu K \le \epsilon/4$. If $Y = 0$ we are done. If not, then $\|T_N \frac{(Y)}{K\|Y\|}\| \le \|T_N\|\left(1/K_{\|Y\|}\right)\|Y\| \le K\left(1/K_{\|Y\|}\right)\|Y\| = 1$ whence $|f(T_N(Y))| \le \|f\|_N \, K \, \|Y\| \, \|T_N \frac{(Y)}{K\|Y\|}\| \le \mu K \|Y\| < \epsilon/4$.

Section III

DEFINITION 3.1

A real valued function $(X_s, s \in S)$ on a set S is called unconditionally summable to $r \in R$ iff., where Σ is the collection of all finite subsets of S, to each $\epsilon > 0$ there exists a $\sigma \in \Sigma$ such that $\left|\sum_{s \in S} X_s - r\right| < \epsilon$ for all $\rho \in \Sigma$, $\rho \geq \sigma$.

LEMMA 3.1

$\sum_{s \in S} X_s$ is unconditionally convergent to some $r \in R$ iff. there exists K such that $\sum_{s \in \sigma} |X_s| \leq K$ for all $\sigma \in \Sigma$. In this case only countably many X_s are different from zero.

DEFINITION 3.2

(a) $m(S)$ denotes the set of all bounded real valued functions X on S with $\|X\| = \sup_{s \in S} |X(s)|$

(b) $C_o(S)$ denotes the closed linear subspace of all those X in $m(S)$ such that for each $\epsilon > 0$ $\{s : |X(s)| > \epsilon\}$ is finite.

(c) $\ell'(S)$ denotes the set of all real valued functions unconditionally summable on S with norm defined by $\|X\| = \sum_{s \in S} |X_s|$

As in the case of $S = \omega$, where ω is the set of + integers, $\ell'(S)$ is linearly isometric with $C_o(S)^*$ under the mapping $T(Y) = f$, if for every $X \in C_o(S)$, $f(X) = \sum_{s \in S} Y(s)\, X(s)$. Again, $m(S)$ is linearly isometric with $\ell'(S)^*$ under the mapping $T(Y) = f$, if for every $X \in \ell'(S)$, $f(X) = \sum_{s \in S} Y(s)\, X(s)$.

DEFINITION 3.3

Let $A(S)$ be the set of all subsets of S. Then a set function ϕ on $A(S)$ is said to be finitely additive if whenever $E_1, E_2 \in A(S)$ and $E_1 \cap E_2 = 0$ then $\phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2)$. ϕ is said to be of bounded variation if $V\phi(S) = \sup.\{\sum_{i=1}^{n} |\phi(E_i)| : E_i \cap E_j = 0 \; i \neq j\}$ is finite.

Let $B V(S)$ be the set of all finitely additive functions of bounded variation on $A(S)$. $B V(S)$ is a linear space under the following definitions of addition and scaler multiplication: $[\phi + \psi](E) = \phi(E) + \psi(E)$ and $[\lambda\phi](E) = \lambda[\phi(E)]$. Norming $B V(S)$ by $\|\phi\| = V\phi(S)$, $B V(S)$ becomes a normed linear space.

LEMMA 3.2

$B V(S)$ is linearly isometric to $m(S)^*$.

PROOF:

Given $F \in m(S)^*$ define the set function ϕ by $\phi(E) = F(\chi_E)$ where χ_E is the characteristic function of E. If $E_1, E_2 \in A(S)$ are disjoint then $\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2}$ whence $\phi(E_1 \cup E_2) = F(\chi_{E_1} + \chi_{E_2}) = F(\chi_{E_1}) + F(\chi_{E_2}) = \phi(E_1) + \phi(E_2)$. Now, let $E_1, E_2, \ldots, E_n \in A(S)$ be pair-wise disjoint. Then $\sum_{i=1}^{n} |\phi(E_i)| = \sum_{i=1}^{n} (\text{sgn}\, F(\chi_{E_i}))\, F(\chi_{E_i}) = F(\sum_{i=1}^{n} (\text{sgn}\, F(\chi_{E_i}))\, (\chi_{E_i})) \leq \|F\|$ since $\sum_{i=1}^{n} (\text{sgn}\, F(\chi_{E_i}))\, \chi_{E_i}$ is an element of $m(S)$ of norm ≤ 1, the E_i being pair-wise disjoint. Thus ϕ is of bounded variation and $V\phi \leq \|F\|$. (i)

Define $T : m(S)^* \to B V(S)$ by $T(F) = \phi$ where ϕ is defined as above. T is clearly linear. We wish to show it is 1 - 1 and onto. To this end consider $\phi \in B V(S)$. We show that it comes from one and only one F under the mapping T. If there is to be a $F \in m(S)^*$ such that $T(F) = \phi$ then $F(\chi_E)$ must $= \phi(E)$ for any $E \in A(S)$. F will then be determined on the linear space generated by the characteristic functions (denoted sp. $\{\chi_E, E \in A(S)\}$ by $F(\sum_{i=1}^{n} c_i \chi_{E_i}) = \sum_{i=1}^{n} c_i\, \phi(E_i)$. We show that F as defined on sp. $\{\chi_E, E \in A(S)\}$ is bounded. For suppose $X \in$ sp. $\{\chi_E, E \in A(S)\}$

with $\|X\| \le 1$. Then, X can be expressed as $\sum_{i=1}^{n} c_i \chi_{E_i}$ where the E_i are pairwise disjoint and $\max_i |c_i| \le 1$. Now, $[|F(X)| = |F(\sum_{i=1}^{n} c_i \chi_{E_i})| = |\sum_{i=1}^{n} c_i \phi(E_i)| \le \max_i |c_i| \sum_{i=1}^{n} | \phi(E_i)| \equiv \sum_{i=1}^{n} |\phi(E_i)| \le V\phi(S)]$ (2)

The Hahn-Banach Theorem now implies that F has a norm-preserving extension to all of $m(S)$. However $m(S) \le \text{csp.}\{\chi_E, E \in A(S)\}$ an so F is uniquely determined by ϕ. From (2) and the fact that $\|F\|$ was preserved under the extension we have $\|F\| \le V\phi(S)$. This result together with (1) gives $\|F\| = V\phi(S)$ and so T is a linear isometry. |

REMARK:

Let ϕ be a finitely additive set function on $A(S)$ which is of bounded variation. Then the total variation function of ϕ, denoted by $V\phi$ and defined on $E \in A(S)$ by $V\phi(E) = \sup.\{\sum_{i=1}^{n} |\phi(E_i)|, E_i \in A(E)$ and $E_i \cap E_j = 0\ i \ne j\}$, is also finitely additive and of bounded variation.

PHILLIP'S LEMMA 3.3

Let (T_n) be a sequence of elements in $c_o(S)***$ which is w*-convergent to zero. Then $\lim. \|Q^*(T_n)\| = 0$ where Q^* is the adjoint of Q, Q being the canonical map from $c_o(S)$ to $c_o(S)**$.

PROOF:

We begin by finding an expression for $\|Q^*(T_n)\|$ using the linear isometries obtained earlier in this section.

Let i_1 be the linear isometry, previously obtained, from $\ell'(S)$ to $c_o(S)^*$. Suppose i is the linear isometry, previously obtained, from $m(S)$ to $\ell'(S)^*$. Define $I : \ell'(S)^* \to c_o(S)^{**}$ by $I(f) = g$ where $g(X) = f(i_1^{-1}(X))$ for all $X \in c_o(S)^*$. It is readily seen that I is a linear isometry. Furthermore $i_2 = i^{-1}I^{-1}$ will be a linear isometry from $c_o(S)^{**}$ to $m(S)$ such that $i_2(Q(X)) = j(X)$ for all $X \in c_o(S)$ where j is the natural embedding of $c_o(S)$ into $m(S)$. For consider arbitrary $Y = (Y_s) \in \ell'(S)$. Then $\{I^{-1}Q(X)\}(Y) = [Q(X)](i_1(Y)) = [i_1(Y)](X) = \sum_{s \in S} X_s Y_s = [i\,j(X)](Y)$ i.e. $I^{-1}Q = i\,j$ whence $i^{-1}I^{-1}Q = j$.

Let i_3 be the linear isometry, previously obtained, from $BV(S)$ to $m(S)^*$. In a manner similar to that in which we obtained i_2 we get a linear isometry i_4 from $c_o(S)^{***}$ to $BV(S)$. Let $\phi_n = i_4(T_n)$ $n = 1, 2, \ldots$. To determine $Q^*(T_n)$ we must analyse the action of T_n on $Q(c_o(S))$. Consider then an $X \in Q(c_o(S))$. Let $Y = i_2(X)$. Then Y satisfies the property that $\{s : |Y(s)| > \epsilon\}$ is finite for every $\epsilon > 0$ (whence $Y(s) = 0$ for all but a countable number of s). Any such Y can be represented in the form $Y = \sum_{j=1}^{\infty} Y_{s_j} \delta_{s_j}$ where $\delta_{s_j}(s) = \begin{matrix} 0, & s \neq s_j \\ 1, & s = s_j \end{matrix}$ and $Y_{s_j} = Y(s_j)$, convergence being in the sense of $m(S)$. Hence $T_n(X) = \{i_3(\phi_n)\}(Y) = \{i_3(\phi_n)\}(\sum_{j=1}^{\infty} Y_{s_j} \delta_{s_j}) = \sum_{j=1}^{\infty} Y_{s_j} \{i_3(\phi_n)\}(\delta_{s_j}) = \sum_{j=1}^{\infty} Y_{s_j} \phi_n(\{s_j\})$ by the continuity of $i_3(\phi_n)$. Now, ϕ_n of bounded variation implies $\sum_{s \in \sigma} |\phi_n(\{s\})| \leq V\phi_n$ where $\sigma \in \Sigma$. By Lemma 3.1 we get that the function ψ_n defined by $\psi_n(s) = \phi_n(\{s\})$ belongs to $\ell'(S)$. Thus for $X \in c_o(S)$ we have $\{i_1(\psi_n)\}(X) = \sum_{k=1}^{\infty} \psi_n(s_k) X_{s_k} = \sum_{k=1}^{\infty} \phi_n(\{s_k\}) X_{s_k} = 1 = T_n(Q(X))$. Thus $Q^*(T_n) = i_1(\psi_n)$ whence $\|Q^*(T_n)\| = \|i_1(\psi_n)\| = \|\psi_n\| = \sum_{s \in S} |\psi_n(s)| = \sum_{s \in S} |\phi_n(\{s\})|$.

If we can now show that $\lim_n \sum_{s \in S} |\phi_n(\{s\})| = 0$ we will be done. (Before proceeding we note that (T_n) w^*-convergent to zero implies that for $E \subseteq S$ $\lim_n \phi_n(E) = \lim_n T_n(i_2^{-1}(\chi_E)) = 0$

For suppose that $\lim_n \sum |\phi_n(\{s\})|$ doesn't equal zero. Then there is an $\epsilon > 0$ and a subsequence (ϕ_n') of (ϕ_n) such that $\sum_{s \in S} |\phi_n'(\{s\})| \geq \epsilon$ for all n'.

We will ultimately construct a subsequence (ρ_i) of (ϕ_n') and an element X of $m(S)$ so that the sequence $\{i_4^{-1}(\rho_i)\}(i_2^{-1}(X))$ oscillates from above $\frac{7\epsilon}{10}$ to below $\frac{-7\epsilon}{10}$. Thus the assumption that $\lim_n \sum |\phi_n(\{s\})| \neq 0$ will have contradicted the fact that (T_n) is w^*-convergent to zero.

Let $\theta_1 = \phi_1'$. Then there is a $\sigma_1 \in \Sigma$ such that $\sum_{s\in\sigma_1} |\theta_1(\{s\})| > \sum_{s\in S} |\theta_1(\{s\})| - \frac{\epsilon}{10}$. Now $\lim_n. \sum_{s\in\sigma_1} \phi_n(\{s\}) = \lim_n. \phi_n(\sigma_1) = 0$ (from the note above) implies that $\lim_n. \sum_{s\in\sigma_1} |\phi_n(\{s\})| = 0$. Thus there exists an n_1 such that $\sum_{s\in\sigma} |\phi_{n_1}(\{s\})| < \frac{\epsilon}{20}$. Let $\theta_2 = \phi_{n_1}$. Then, because of the meaning of $\sum_{s\in S-\sigma_1} |\theta_2(\{s\})|$ there is a $\sigma_2 \in \Sigma$, $\sigma_2 \subseteq S - \sigma_1$, such that $|\sum_{s\in S-\sigma_2} |\theta_2(\{s\})| - \sum_{s\in\sigma_2}^{1} |\theta_2(\{s\})| < \frac{\epsilon}{20}$ whence $|\sum_{s\in\sigma_2} |\theta_2(\{s\})| - \sum_{s\in S} |\theta_2(\{s\})|| < \frac{\epsilon}{10}$ i.e. $\sum_{s\in\sigma_2} |\theta_2(\{s\}) > \sum_{s\in S} |\theta_2(\{s\})| - \frac{\epsilon}{10}$. By induction we get a sequence of disjoint finite subsets (σ_k) and a subsequence (θ_k) of (ϕ_n') such that

$$\sum_{s\in\sigma_k} |\theta_k(\{s\})| > \sum_{s\in S} |\theta_k(\{s\})| - \frac{\epsilon}{10} \qquad (1)$$

We now define subsequences of (θ_k) and (σ_k) satisfying further desirable properties. Let $\tau_1 = \sigma_1$ and $\rho_1 = \theta_1$. Divide the sequence $\sigma_2, \dots, \sigma_k, \dots$ into more than $\frac{10\, V\rho_1(S)}{\epsilon}$ disjoint infinite subsequences. Then one of these subsequences $\sigma_{11}, \dots, \sigma_{1n}, \dots$ is such that $V_{\rho_1}(\bigcup_{j\in} \sigma_{ij}) < \frac{\epsilon}{10}$. This follows from the fact that $V\,\theta_k$ is finitely additive and of bounded variation since θ_k is. Let $\tau_2 = \sigma_{11}$, let k_2 be the place of σ_{11} in the sequence (σ_k), and let $\rho_2 = \theta_{k_2}$. By induction, this process determines a subsequence (τ_i) of (σ_k) and a subsequence (ρ_i) of (θ_k) such that

$$\sum_{s\in\tau_i} |\rho_i(\{s\})| > \sum_{s\in S} |\rho_i(\{s\})| - \frac{\epsilon}{10}$$

and

$$V_{\rho_i}(\bigcup_{j>i} \tau_j) < \frac{\epsilon}{10} \qquad (2)$$

Define $X \in m(S)$ by $X(s) = 0$ if $s \notin \bigcup_{i\in} \tau_i$ and $X(s) = (-1)^i$ sgn. $\rho_i(s)$ if $s \in \tau_i$. Then, letting F_i be the element of $m(S)^*$ corresponding to ρ_i, under the correspondence established in Lemma 3.2 we have $F_i(X) = F_i(X\cdot\chi_{\bigcup_{j<i}\tau_j}) + F_i(X\cdot\chi_{\tau_i}) + F_i(X\cdot\chi_{\bigcup_{j>i}\tau_j}) + F_i(X\cdot\chi_{S-\bigcup_{j=1}^{\infty}\tau_j}) = \sum_{j<i}\sum_{s\in\tau_j} X_s F_i(\delta_s) + \sum_{s\in\tau_i} X_s F_i(\delta_s) + F_i(X\cdot\chi_{\bigcup_{j>i}\tau_j}) = \sum_{j<i}\sum_{s\in\tau_j} (-1)^j \text{sgn.}\rho_j(\{s\}) [\rho_i(\{s\})] + (-1)^i \sum_{s\in\tau_i} |\rho_i(\{s\})| + F_i(X\cdot\chi_{\bigcup_{j>i}\tau_j})$. Now, $|\sum_{j<i}\sum_{s\in\tau_j} (-1)^j$ sgn. $\rho_j(\{s\})[\rho_i(\{s\})]| \le \sum_{j<i}\sum_{s\in\tau_j} |\rho_i(\{s\})| \le \sum_{s\notin\tau_i} |\rho_i(\{s\})| < \frac{\epsilon}{10}$

by (2). Hence $|F_i(X) - (-1)^i \sum_{s\in\tau_i} |\rho_i(\{s\})|\ | \le |\sum_{j<i}\sum_{s\in\tau_j} (-1)^j \operatorname{sgn}.\rho_j(\{s\})$ $[\rho_i(\{s\})]| + |\ F_i(X\cdot \chi_{\bigcup_{j>i}\tau_j})| < \frac{\epsilon}{10} + |F_i(X\cdot\chi_{\bigcup_{j>i}\tau_j})|$. Any element Y of m(S) which vanishes outside $\bigcup_{j>i}\tau_j$ is the limit of an increasing sequence of elements of the form $\sum_{k=1}^{n} c_k \chi_{E_k}$ where E_k is contained in $\bigcup_{j>i}\tau_j$. Suppose $\|Y\| \le 1$. Then, since $|F_i(\sum_{i=1}^{n} (k\ \chi_{E_k})| = |\sum_{k=1}^{n} (c_k\rho_i(E_k)| \le [\max_k. |c_k|] \sum_{k=1}^{n}|\rho_i\ (E_k)| \le$ $V\rho_i(\bigcup_{j>i}\tau_j)$ we have $|F_i(Y)| \le V_{\rho_i}(\bigcup_{j>i}\tau_j)$. In particular $|F_i(X\cdot\chi_{\bigcup_{j>i}\tau_j})| \le$ $V\rho_i(\bigcup_{j>i}\tau_j)$ which is less than $\epsilon/10$ by (2). This gives us that $|F_i(X) - (-1)^i$ $\sum_{s\in\tau_i} |\rho_i(\{s\})|\ | < \frac{2\epsilon}{10}$. This means, in view of (1) and (2) that for i even $F_i(X) > \frac{7\epsilon}{10}$ and for i odd $F_i(X) < -\frac{7\epsilon}{10}$ whence $[i_4^{-1}(\rho_i)]\ (i_2^{-1}(X))$ oscillates from above $\frac{7\epsilon}{10}$ to below $-\frac{7\epsilon}{10}$. |

COROLLARY 1

Let T be a continuous linear operator from m(S) into a normed linear space B and let $b_s = T(\delta_s)$ (where δ_s is that element of m(S) defined by $\delta_s(s') = 1$ if $s' = s$ and $\delta_s(s') = 0$ if $s' \neq s$) . Then for each sequence (ξ_n) in B* such that $w^*\text{-}\lim_n.\ \xi_n = 0$ it follows that $\lim_n. \sum_{s\in S} |\xi_n\ (b_s)| = 0$.

PROOF:

If (ξ_n) is w*- convergent to zero then $(T^*(\xi_n))$ is w*-convergent to zero since, for $X \in m(S)$, $T^*[\xi_n\ (X)] = \xi_n[T(X)]$. The result will follow from Phillip's Lemma if we can show that $\|Q^*[T^*(\xi_n)]\| =$ $\sum_{s\in S} |\xi_n\ (b_s)|$. This however is trivial since the s^{th} coordinate of $Q^*[T^*\ (\xi_n)] \in \ell'(S)$ is equal to $Q^*[T^*\ \{\xi_n(\delta_s)\}] =$ $T^*[\xi_n\{Q\ (\delta_s)\}] = T^*[\xi_n\ (\delta_s)[= \xi_n[T\ (\delta_s)] = \xi_n\ (b_s)$. |

COROLLARY 2

In $\ell'(S)$ weak and norm convergence of a sequence to an element are equivalent.

PROOF:

Clearly, norm convergence implies weak convergence. Suppose then that $(f_n) \to f$ weakly. Let Q_1 be the canonical map from $\ell'(S)$ to its second conjugate. Then $Q_1(f_n - f)$ is w *-convergent to zero. However Q^*Q_1 is the identity and so $\|f_n - f\|$ tends to zero by Phillip's Lemma. |

COROLLARY 3

$\ell'(S)$ is weak sequentially complete.

PROOF:

Suppose (f_n) is a sequence in $\ell'(S)$ such that $\lim_n . X(f_n)$ exists for every $X \in \ell'(S)^*$. Then, if Q is the canonical map from $\ell'(S)$ to $\ell'(S)^{**}$ the sequence $(Q(f_n))$ is such that $\lim_n . Q[f_n(X)]$ exists for every $X \in \ell'(S)^*$. Let F be the element of $\ell'(S)^{**}$ defined by $F(X) = \lim_n . Q[f_n(X)]$. Then $Q(f_n) - F$ is w *-convergent to zero. By Phillip's Lemma $\|Q^*[Q(f_n) - F]\| = \|f_n - Q^*(F)\| \to 0$, whence $Q^*(F)$ is the required weak limit of (f_n). |

DEFINITION 3.4

The series $\sum_{i=1}^{\infty} X_i$ is said to be unordered convergent to X if, letting Σ be the system of finite subsets σ of ω, directed by $\supseteq$,

$$\lim_{\sigma \in \Sigma} . \sum_{i \in \sigma} X_i = X.$$

DEFINITION 3.5

The series $\sum_{i=1}^{\infty} X_i$ is subseries convergent if for every increasing sequence (n_i) of integers the series $\sum_{i=1}^{\infty} X_{n_i}$ is convergent.

DEFINITION 3.6

The series $\sum_{i=1}^{\infty} X_i$ is bounded multiplier convergent if for each bounded real sequence (a_i) the series $\sum_{i=1}^{\infty} a_i X_i$ is convergent.

REMARK 1

These forms of convergence are all equivalent to absolute convergence in the case of the real numbers.

DEFINITION 3.7

A series $\sum_{i=1}^{\infty} X_i$ in a normed linear space N is said to be unordered Cauchy if, given $\epsilon > 0$, there is a $\sigma \in \Sigma$ such that for all $\rho \in \Sigma$, $\rho \cap \sigma = 0$, $\|\sum_{i \in \rho} X_i\| < \epsilon$.

REMARK 2

In a Banach Space a series is norm unordered convergent iff. it is norm unordered Cauchy.

DEFINITION 3.8

A series $\sum_{i=1}^{\infty} X_i$ in a normed linear space is said to be weakly unordered Cauchy if for $f \in B^*$ the series of real numbers $\sum_{i=1}^{\infty} f(X_i)$ is unordered Cauchy.

LEMMA 3.4

A series of real numbers is unordered Cauchy iff. $\sup_{\sigma \in \Sigma} |\sum_{i \in \sigma} X_i| < \infty$.

LEMMA 3.5

If $\sum_{i=1}^{\infty} X_i$ is a series in a Banach Space B then $\sup_{\sigma \in \Sigma} \|\sum_{i \in \sigma} X_i\| = K < \infty$ iff. $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy.

PROOF:

Suppose $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy. Then given any $f \in B^*$ and $\epsilon > 0$ there is a $\sigma \in \Sigma$ such that for any $\rho \in \Sigma$, $\rho \cap \sigma = 0$, $|\sum_{i \in \rho} f(X_i)| < \epsilon$. Since there are only a finite number of subsets of σ there is a M such that $|\sum_{i \in r} f(X_i)| < M$ for $r \subseteq \sigma$. Now for any $\mu \in \Sigma$, $\mu = \mu_1 + \mu_2$ where $\mu_1 \subseteq \sigma$ and $\mu_2 \cap \sigma = 0$. Thus $|\sum_{i \in \mu} f(X_i)| \leq |\sum_{i \in \mu_1} f(X_i)| + |\sum_{i \in \mu_2} f(X_i)| < M + \epsilon$. By the Uniform Boundedness Principle this means that $\sup_{\sigma \in \Sigma} \|\sum_{i \in \sigma} X_i\| < \infty$.

Suppose now that $K = \sup_{\sigma \in \Sigma} \|\sum_{i \in \sigma} X_i\| < \infty$. Then for any $f \in B^*$

and $\sigma \in \Sigma$, $\left|\sum_{i\in\sigma} f(X_i)\right| = \left|f(\sum_{i\in\sigma} X_i)\right| \le \|f\| \, \|\sum_{i\in\sigma} X_i\| \le \|f\| \, K$. Thus $\sup_{\sigma\in\Sigma} \left|\sum_{i\in\sigma} f(X_i)\right| < \infty$ and so by Lemma 1 the series of real numbers $\sum_{i=1}^{\infty} f(X_i)$ is unordered Cauchy i.e. $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy. |

LEMMA 3.6

If $\sum_{i=1}^{\infty} X_i$ is a series in a Banach Space B then $\sup_{\sigma\in\Sigma} \|\sum_{i\in\sigma} X_i\| = K < \infty$ iff. $K' = \sup_{\sigma\in\Sigma} \{\|\sum_{i\in\sigma} a_i X_i\| : |a_i| \le 1\} < \infty$. Furthermore $K \le K' \le 2K$.

PROOF:

If $K' < \infty$, then taking $a_i = 1$ for all i we get $K \le K'$. Suppose that $K < \infty$. Given $\sigma \in \Sigma$ consider the sum $\sum_{i\in\sigma} Y_i$ where $Y_i = X_i$, $-X_i$ or 0. There exists $\sigma_1, \sigma_2 \in \Sigma$ such that $\sigma = \sigma_1 \cup \sigma_2$ and $\sum_{i\in\sigma} Y_i = \sum_{i\in\sigma_1} X_i - \sum_{i\in\sigma_2} X_i$, whence $\|\sum_{i\in\sigma} Y_i\| \le \|\sum_{i\in\sigma_1} X_i\| + \|\sum_{i\in\sigma_2} X_i\| \le 2K$. This proves that $\sup_{\sigma\in\Sigma} \{\|\sum_{i\in\sigma} a_i X_i\| : a_i = 1, -1, \text{or } 0\} \le 2K$. Consider an $f \in B^*$ with $\|f\| = 1$. Define $a_i = \text{sgn. } f(X_i)$. Then $\sum_{i=1}^{n} |f(X_i)| = |f(\sum_{i=1}^{n} a_i X_i)| \le \|f\| \, \|\sum_{i=1}^{n} a_i X_i\| \le 2K$. Thus $\sum_{i=1}^{\infty} |f(X_i)| \le 2K$. Now, let (a_i) be any real number sequence with $|a_i| \le 1$ for all i and let $\hat{X}_i$ be the image of X_i under the canonical mapping. Then for any $f \in B^*$ of norm 1 and any $\sigma \in \Sigma$ we have $\left|\sum_{i\in\sigma} a_i \hat{X}_i(f)\right| = \left|\sum_{i\in\sigma} a_i f(X_i)\right| \le \sup_i |a_i| \sum_{i\in\sigma} |f(X_i)| \le 2K$. Therefore $\sup_{\sigma\in\Sigma} \{\|\sum_{i\in\sigma} a_i X_i\| : |a_i| \le 1\} = \sup_{\sigma\in\Sigma} \{\|\sum_{i\in\sigma} a_i \hat{X}_i\| : |a_i| \le 1\} \le 2K$.

LEMMA 3.7

If B is a Banach Space and if $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy, then F, defined by $F(f) = (f(X_i))$ is a bounded linear operator from B^* into $\ell'(\omega)$.

PROOF:

Since $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy there exists, by Lemma 2, a K such that $\|\sum_{i\in\sigma} X_i\| \le K$ for all $\sigma \in \Sigma$. By Lemma 3,

$\|\sum_{i\in\sigma} a_i X_i\| \leq 2K$ for $|a_i| \leq 1$ and $\sigma \in \Sigma$. Given $f \in B^*$ define $a_i =$ sgn $f(X_i)$; then $\sum_{i=1}^{n} |f(X_i)| = |f(\sum_{i=1}^{n} a_i X_i)| \leq \|f\| \, \|\sum_{i=1}^{n} a_i X_i\| \leq 2K \|f\|$. Since the R.H.S. of the inequality is independent of n, $\sum_{i=1}^{\infty} |f(X_i)| \leq 2K \|f\|$. Thus F is well defined, clearly linear, and $\|F\| \leq 2K$. |

LEMMA 3.8

If, in a Banach Space B, $\sum_{i=1}^{\infty} X_i$ is weakly subseries convergent, then it is weakly bounded-multiplier convergent and T, defined for each $a = \{a_i\}$ in $m(\omega)$ by $T(a) = \sum_{i=1}^{\infty} a_i X_i$ (convergence considered in the weak sense) is a bounded linear operator from $m(\omega)$ into B. Furthermore, letting Q be the canonical map from $c_0(\omega)$ into $c_0(\omega)^{**}$, the F of Lemma 4 is $Q^* T^*$.

PROOF:

$\sum_{i=1}^{\infty} X_i$ weakly subseries convergent means that for every $f \in B^*$ the series of real numbers $\sum_{i=1}^{\infty} f(X_i)$ is subseries convergent. Thus (by Remark 1) $\sum_{i=1}^{\infty} f(X_i)$ will be unordered convergent and so (by Remark 2) unordered Cauchy. $\sum_{i=1}^{\infty} f(X_i)$ unordered Cauchy for every $f \in B^*$ means that $\sum_{i=1}^{\infty} X_i$ is weakly unordered Cauchy. By Lemma 2 there exists K such that $\|\sum_{i\in\sigma} X_i\| \leq K$ for all $\sigma \in \Sigma$. Again, by Lemma 3, $\|\sum_{i\in\sigma} a_i X_i\| \leq 2K$ for all $\sigma \in \Sigma$ and $|a_i| \leq 1$. Consider now $(a_i) \in m(\omega)$ where $a_i = 1, -1$, or 0. Thus will be shown that $\sum_{i=1}^{\infty} a_i X_i$ is weakly convergent to an element of B. For let (i_j) be the increasing subsequence of (i) such that $a_{i_j} = 1$ for all j and (i_k) the increasing subsequence such that $a_{i_k} = -1$ for all k. Then $\sum_{j=1}^{\infty} X_{i_j} - \sum_{k=1}^{\infty} X_{i_k}$ is a weakly subseries convergent series. As above, this means that $\sum_{j=1}^{\infty} X_{i_j} - \sum_{k=1}^{\infty} X_{i_k}$ will be weakly unordered convergent. Since $\sum_{i=1}^{\infty} a_i X_i$ is simply a rearrangement of $\sum_{j=1}^{\infty} X_{i_j} - \sum_{k=1}^{\infty} X_{i_k}$ the above assertion is established.

Thus, $\sum_{i=1}^{\infty} b_i X_i$, where (b_i) belongs to the smallest linear manifold M_c containing "characteristic" functions, will be a weakly convergent series in B.

The mapping $T : M_c \to B$ defined by $T[(a_i)] = \sum_{i=1}^{\infty} a_i X_i$, which is clearly linear, is also bounded. For let $N = \max_i (|a_i|)$. Then $\|T[(a_i)]\| = \|N\sum_{i=1}^{\infty} \frac{a_i X_i}{N}\| = N \|\sum_{i=1}^{\infty} \frac{a_i X_i}{N}\| \le N \lim_n \inf (\|\sum_{i=1}^{\infty} \frac{a_i X_i}{N}\|) \le N\, 2\, K = 2K \|(a_i)\|$.

Since M_c is dense in $m(\omega)$ there is a unique extension $\overline{T}$ of T to all of $m(\omega)$ and $\|\overline{T}\| = \|T\|$. If it can be shown that for $a = (a_i) \in m(\omega)$ the series $\sum_{i=1}^{\infty} a_i X_i$ is weakly convergent and $\overline{T}(a) = \sum_{i=1}^{\infty} a_i X_i$ the first assertion of the Lemma will be proved.

Let $a = (a_i)$ be given. Since M_c is dense in $m(\omega)$ there is a sequence $(a_i^{(n)})_{n=1}^{\infty}$ of elements in $M_c \to a$ as a limit. By the continuity of $\overline{T}$, $\overline{T}[(a_i^{(n)})] \to \overline{T}(a)$. We show that $w\text{-}\lim_m \sum_{i=1}^{m} a_i X_i = \overline{T}(a)$. Let $f \in B^*$ and $\epsilon > 0$ be given. Since $(a_i^{(n)}) \to (a_i)$ there exists a N_1 such that

$$\|(a_1^{(n)}, a_2^{(n)}, \cdots, a_m^{(n)}) - (a_1, a_2, \cdots, a_m)\| < \frac{\epsilon}{3\|f\|\,\|T\|} \quad (1)$$

for all m, and $n \ge N_1$. Again, since $\overline{T}(a_i^{(n)}) \to \overline{T}(a)$ there exists a N_2 such that $n \ge N_2 \Rightarrow$

$$\|f[\overline{T}(a_i^{(n)})] - f[\overline{T}(a)]\| < \frac{\epsilon}{3}\|. \quad (2)$$

Let $N = \max. (N_1, N_2)$. Then, since $\sum_{i=1}^{\infty} a_i^{(N)} X_i$ is weakly convergent and $\overline{T}[(a_i^{(N)})] = \sum_{i=1}^{\infty} a_i^{(N)} X_i$ there exists a M such that

$$\|f(\sum_{i=1}^{m} a_i^{(N)} X_i) - f(\overline{T}[(a_i)^{(N)}])\| < \frac{\epsilon}{3} \text{ for } m \ge M, \quad (3)$$

Thus $\|f(\sum_{i=1}^{m} a_i X_i) - f(\overline{T}(a))\| = \|f(\sum_{i=1}^{m} a_i X_i) - f(\sum_{i=1}^{m} a_i^{(N)} X_i) + f(\sum_{i=1}^{m} a_i^{(N)} X_i) - f(\overline{T}[(a_i^{(N)})]) + f(\overline{T}[(a_i^{(N)})]) - f(\overline{T}[(a_i)]) \le \|f(\sum_{i=1}^{m} a_i X_i - \sum_{i=1}^{m} a_i^{(N)} X_i)\| + \frac{2\epsilon}{3}$ for $m \ge M$ (by 2 and 3). This last expression is $\le \|f\|\|T[(a_1, a_2, \ldots, a_m) - (a_1^{(N)}, \ldots, a_m^{(N)})]\| + \frac{2\epsilon}{3} \le \|f\|\,\|T\|\,\|(a_1, \ldots, a_m) - (a_1^{(N)}, \ldots, a_m^{(N)})\| + \frac{2\epsilon}{3} < \epsilon$ by (1). This establishes the claim of the first assertion. The second assertion is trivial. |

ORLICZ-PETTIS THEOREM 3.1

If in the weak topology of a Banach Space B the series $\sum_{i=1}^{\infty} X_i$ is subseries convergent, then it is subseries convergent in the norm topology of B.

PROOF:

Let L be the smallest closed linear subset of B containing the X_i. By a corollary to the Hahn-Banach Theorem, there exists, for each i, a $g_i \in L^*$ with $\|g_i\| = 1$ and $g_i(X_i) = \|X_i\|$. The separability of L implies the existence of a subsequence (g_{i_j}) of (g_i) and an $h \in L^*$ such that $h = w^*\text{-}\lim_j g_{i_j}$. $\sum_{i=1}^{\infty} h(X_i)$ is convergent and so $\lim_i h(X_i) = 0$. Letting $f_j = g_{i_j} - h$ we have (1) $w^*\text{-}\lim_j f_j = 0$ and (2) $\lim_j [f_j(X_{i_j}) - \|X_{i_j}\|] = \lim_j [g_{i_j}(X_{i_j}) - h(X_{i_j}) - \|X_{i_j}\|] = \lim_j [-h(X_{i_j})] = 0$.

Now by Lemma 5, $T : m(\omega) \rightarrow L$ defined by $T[(a_i)] = \sum_{i=1}^{\infty} a_i X_i$ is a bounded linear operator with $T(\delta_i) = X_i$. By corollary 1 of Phillip's Lemma and (1) we get that $\lim_j \sum_{i=1}^{\infty} |f_j(X_i)| = 0$ and so, since $|f_j(X_{i_j})| \leq \sum_{i=1}^{\infty} |f_j(X_i)|$, $\lim_j |f_j(X_{i_j})| \rightarrow 0$. Together with (2) this means that $\lim_j \|X_{i_j}\| = 0$.

This shows that if $\sum_{i=1}^{\infty} X_i$ is weakly subseries convergent then some subseries have elements tending in norm to zero.

Suppose now that $\sum_{k=1}^{\infty} X_k$ is a subseries of $\sum_{i=1}^{\infty} X_i$ which is not norm convergent. Then, by the completeness of B, it will not be Cauchy. Hence there will exist an $\epsilon > 0$ and subsequences (n_j) and (m_j) of (k) such that $n_j < m_j < n_{j+1}$ and $\|\sum_{k=n_j}^{m_{j-1}} X_k\| \geq \epsilon$ for all j. Let $y_j = \sum_{k=n_j}^{m_{j-1}} X_k$. Then $\sum_{j=1}^{\infty} Y_j$ is weakly subseries convergent, since every subseries will, in effect, be a subseries of $\sum_{i=1}^{\infty} X_i$. But $\sum_{j=1}^{\infty} Y_j$ can have no subseries with terms norm convergent to zero. This contradiction proves the theorem. |

Section IV

DEFINITION 4.1

A decomposition (M_i, P_i) is said to be an unconditional decomposition iff. for each X the series $\sum_{i=1}^{\infty} P_i \ (X)$ is unordered convergent to X.

DEFINITION 4.2

A real valued function $f(t)$ of the real variable t is said to be convex if the inequality $f(\frac{t_1 + t_2}{2}) \le \frac{1}{2} [f(t_1) + f(t_2)]$ is satisfied for all values of t_1 and t_2.

PROPOSITION 4.1

A continuous, even, convex function $f(t)$ is a non-decreasing function of $|t|$.

PROOF:

It will be sufficient to show that $f(t)$ is non-decreasing for $t \ge 0$ since f is even. Given any $t \ne 0$, $f(t) \ge f(0)$ since $f(0) = f(\frac{t + (-t)}{2}) \le \frac{1}{2}[f(t) + f(-t)] = f(t)$. Suppose now there exists $t_1, t_2 > 0$ such that $0 < t_1 < t_2$ and $f(t_2) < f(t_1)$. Since $f(0) \le f(t_2) < f(t_1)$ there exists, by the continuity of f, a t_3 such that $0 \le t_3 < t_1$ and $f(t_3) = f(t_2)$. Let $\alpha_1 = \frac{t_3 + t_2}{2}$. Then, by the convexity of f, $f(\alpha_1) \le \frac{1}{2}[f(t_2) + f(t_3)] = f(t_2)$. Clearly $\alpha_1 \ne t_1$. Assume, without loss of generality, that $\alpha_1 < t_1$. Let $\beta_1 = t_2$. Then $f(\frac{\alpha_1 + \beta_1}{2}) \le \frac{1}{2}[f(\alpha_1) + f(\beta_1)] \le f(t_2)$. This means $\frac{\alpha_1 + \beta_1}{2} \ne t_1$. If $\frac{\alpha_1 + \beta_1}{2} < t_1$ define $\alpha_2 = \frac{\alpha_1 + \beta_1}{2}$ and $\beta_2 = t_2$. If $\frac{\alpha_1 + \beta_1}{2} > t_1$ then define $\alpha_2 = \alpha_1$ and $\beta_2 = \frac{\alpha_1 + \beta_1}{2}$. Continuing in this manner, we get two sequences (α_n) and (β_n) such that $f(\alpha_n) \le f(t_2)$, $f(\beta_n) \le f(t_2)$, (α_n) monotonic increasing with $\alpha_n \le t_1$ for all n, and (β_n) monotonic decreasing with $\beta_n \geq t_1$ for all n. Let $\alpha = \lim_n. \alpha_n$ and $\beta = \lim_n. \beta_n$. Then $\alpha \leq t_1 \leq \beta$. We have at least one of α, β equal to t_1 since $\lim_n. \frac{\alpha_n + \beta_n}{2} = \frac{\alpha + \beta}{2}$. Assume, without loss of generality, that $(\alpha_n) \to t_1$. Then, by the continuity of f, $\lim_n. f(\alpha_n) = f(t_1)$. However, $f(\alpha_n) \leq f(t_2) < f(t_1)$ for all n. |

If Σ is the set of all finite subsets ν of ω, the set of positive integers, define $U_\nu(X) = \sum_{i \in \nu} P_i(X)$ and $U_0(X) = 0$ for all X. Then define $V_\nu = i - U_\nu$ and $W_\nu = i - 2U_\nu = V_\nu - U_\nu$.

THEOREM 4.1

Let (M_i, P_i) be an unconditional Schauder decomposition for a Banach Space B. Let $\|X\|' = \sup_\nu. \{\|U_\nu(X)\| : \nu \in \Sigma\}$; let $\|X\|'' = \sup_{\nu \in \Sigma}. \|W_\nu(X)\|$. Let B' and B'' be the spaces obtained by renorming B with $\| \|'$ and $\| \|''$.

(i) B' and B'' are Banach Spaces isomorphic to B.

(ii) Setting $\lambda =$ the symmetric difference of μ and ν, $W_\mu W_\nu = W_\lambda$.

(iii) $\|U_\mu\|' \leq 1$, $\|W_\mu\|'' \leq 1$, and $\|U_\mu\|'' \leq 1$ for all $\mu \in \Sigma$. If μ and ν are disjoint and nonempty, if $X = U_\mu(X) \neq 0$ and $Y = U_\nu(Y) \neq 0$, then $\|X + tY\|''$ is a non-decreasing function of $|t|$.

PROOF:

We show first that $\| \|'$ and $\| \|''$ are defined. For, given $X \in B$, $\{U_\nu(X) : \nu \in \Sigma\}$ is a family of continuous linear functionals in B^{**}. Given any $f \in B^*$ the series $\sum_{i=1}^{\infty} f(P_i(X))$ (1)

is unconditionally convergent to f(X) since $\sum_{i=1}^{\infty} P_i(X)$ is unconditionally convergent to X. Since (1) is an unconditionally convergent series of real numbers it will be absolutely convergent. This means there exists a M such that $|f(U_\nu(X))| \leq M$ for all $\nu \in \Sigma$. Thus, the family $\{U_\nu(X) : \nu \in \Sigma\}$

is point-wise bounded on B^* and so, by the Uniform Boundedness Principle there exists a positive real number N such that $\|U_\nu(X)\| \leq N$ for $\nu \in \Sigma$. This in turn means that the family $\{U_\nu : \nu \in \Sigma\}$ of continuous linear functionals on B is point-wise bounded, whence, by the Uniform Boundedness Principle, there is a positive real number P such that $\|U_\nu\| \leq p$ for all $\nu \in \Sigma$. Thus $\| \|'$ is well defined. Now $\|W_\nu\| = \|i - 2U_\nu\| \leq 2\|U_\nu\| + 1 \leq 2P + 1$ for $\nu \in \Sigma$, and so $\| \|''$ is defined.

Needless to say $\| \|'$ and $\| \|''$ are norms. The proof of the isomorphism of B and B' is the same as that of Theorem 1.1. Also, using the methods of Theorem 1.1 and the fact that $U_\mu \; U_\nu = U_{\mu\cap\nu}$ we get $\|U_\mu\|' \leq 1$ for all $\mu \in \Sigma$. Whence there exists a K such that $\|U_\nu\| \leq K$ and $\|V_\nu\| \leq K + 1$ for all $\nu \in \Sigma$. Let $U_n(X) = \sum_{i=1}^{n} P_i(X)$ and $W_n(X) = X - 2\sum_{i=1}^{n} P_i(X)$. Clearly $\lim_n W_n(X) = -X$ and so $\|X\|'' = \sup_{\nu\in\Sigma} \|W_\nu(X)\| \geq \sup_n \|W_n(X)\| \geq \|X\|$ for all X. On the other hand $\|W_\nu(X)\| = \|[i - 2U_\nu](X)\| \leq \|i - 2U_\nu\| \, \|X\| \leq (2K + 1)\|X\|$ for all $\nu \in \Sigma$ and $X \in B$ whence $\|X\|'' = \sup_{\nu\in\Sigma} \|W_\nu(X)\| \leq [2K + 1]\|X\|$ for all X. Thus B and B'' are isomorphic.

Now $W_\mu W_\nu = [i - 2U_\mu][i - 2U_\nu] = i - 2U_\mu - 2U_\nu + 4U_{\mu\cap\nu} = i - 2U_{\mu\cap\nu} - 2U_{\mu-\nu} - 2U_{\mu\cap\nu} - 2U_{\nu-\mu} + 4U_{\mu\cap\nu} = i - 2U_{\mu-\nu} - 2U_{\nu-\mu} = i - 2U_{[(\mu\cup\nu) - (\mu\cap\nu)]} = i - 2U_\lambda = W_\lambda$ where λ is the symmetric difference of μ and ν. This yields $\|W_\mu\|'' \leq 1$ for all $\mu \in \Sigma$. For consider an $X \in B$ such that $\|X\|'' \leq 1$. Then for fixed ν, $\|W_\nu W_\mu(X)\| = \|W_\lambda(X)\| \leq 1$ whence $\sup_{\nu\in\Sigma} \|W_\nu W_\mu(X)\| \leq 1$ i.e. $\|W_\mu\|'' \leq 1$.

We show that $\|U_\mu\|'' \leq 1$. Consider $X \in B$ such that $\|X\|'' \leq 1$. Since $W_\mu(X) = i(X) - 2U_\mu(X)$ we get $U_\mu(X) = \dfrac{i(X) - W_\mu(X)}{2}$. For fixed ν, $\|W_\nu U_\mu(X)\| = \left\|W_\nu\left[\dfrac{i(X) - W_\mu(X)}{2}\right]\right\| = \frac{1}{2}\| W_\nu(X) - W_\lambda(X)\|$ where λ is the symmetric difference of μ and ν. Thus $\|W_\nu U_\mu(X)\| \leq \frac{1}{2}\left[\|W_\nu(X)\| + \|W_\lambda(X)\|\right] \leq 1$. Since ν was arbitrary we get

$\|U_\mu(X)\|'' \leq 1$ and thus $\|U_\mu\|'' \leq 1$.

We now prove the last assertion of (iii). Clearly $f(t) = \|X+tY\|''$ is a continuous real valued function of the real variable t. Again, $\|X + \left[\frac{t_1 + t_2}{2}\right] Y\|'' = \frac{1}{2}\|2X + t_1Y + t_2Y\|'' \leq \frac{1}{2}\left[\|X + t_1Y\|'' + \|X + t_2Y\|''\right]$ and so f is convex. Thus, in view of proposition 1 we need only show that $f(t)$ is even. To show this, it will be sufficient to show that the range of each W_ν, for all ν, on $X + tY$ is the same as that on $X - tY$. Consider an arbitrary $\lambda \in \Sigma$. Then $W_\lambda(X + tY) = W_\lambda[U_\mu(X) + t\,U_\nu(Y)] =$ $(i - 2U_\lambda)(U_\mu(X) + t\,U_\nu(Y)) = U_\mu(X) + t\,U_\nu(Y) - 2U_{\lambda\cap\mu}(X) - 2t\,U_{\lambda\cap\nu}(Y) =$ $U_{\mu\cap\lambda}(X) + U_{\mu-\lambda}(X) + t\,U_{\lambda\cap\nu}(Y) + t\,U_{\nu-\lambda}(Y) - 2U_{\lambda\cap\mu}(X) - 2t\,U_{\lambda\cap\nu}(Y) =$ $U_{\mu-\lambda}(X) + t\,U_{\nu-\lambda}(Y) - U_{\lambda\cap\mu}(X) - t\,U_{\lambda\cap\nu}(Y) = (1)$. We show $W_\sigma(X - tY) = (1)$ where $\sigma = (\lambda \cap \mu) \cup (\nu - \lambda)$. For $W_\sigma(X - tY) =$ $\left[i - 2U_{[(\lambda\cap\mu)\cup(\nu-\lambda)]}\right]\{U_\mu(X) - t\,U_\nu(Y)\} = U_\mu(X) - t\,U_\nu(Y) -$

$2U_{[(\lambda\cap\mu)\cup(\nu-\lambda)]\cap\mu}(X) + 2t\,U_{[(\lambda\cap\mu)\cup(\nu-\lambda)]\cap\nu}(Y) =$

$U_\mu(X) - t\,U_\nu(Y) - 2U_{\lambda\cap\mu}(X) + 2t\,U_{\nu-\lambda}(Y)$ since $\mu \cap \nu = 0$. This expression may be written as $U_{\mu\cap\lambda}(X) + U_{\mu-\lambda}(X) - tU_{\lambda\cap\nu}(Y) - tU_{\nu-\lambda}(Y) - 2U_{\lambda\cap\mu}(X) +$ $2tU_{\nu-\lambda}(Y) = U_{\mu-\lambda}(X) + t\,U_{\nu-\lambda}(Y) - U_{\lambda\cap\mu}(X) - t\,U_{\nu\cap\lambda}(Y) = (1)$.

COROLLARY 1

If $\mu, \nu \in \Sigma$ and $\mu \supseteq \nu$ then $\|U_\mu(X)\|' \leq \|U_\nu(X)\|'$ for all X.

This property may be called the unconditional monotony of the decomposition (M_i, P_i). Thus any B-space with an unconditional Schauder decomposition can be renormed isomorphically so that the decomposition is unconditionally monotone. The proof is the same as the one for an ordinary decomposition.

THEOREM 4.2

If (M_i, P_i) is an unconditional decomposition then (i) implies (ii) implies (iii): (i) B is weak sequentially complete.

(ii) There is no subspace of B isomorphic to $c_0(\omega)$

(iii) (M_i, P_i) is boundedly complete.

PROOF:

To show (i) implies (ii). If B is weak sequentially complete then every dosed linear subspace of B is also weak sequentially complete. For suppose M is a closed subspace of B and (X_n) a weak Cauchy sequence in M and therefore $(f(X_n))$ is Cauchy for each $f \in M^*$, so $\lim. f(X_n)$ exists for every $f \in M^*$. Then given any $g \in B^*$ $\lim_n. g(X_n) = \lim_n. g/M (X_n)$ exists since $g/M \in M^*$. So (X_n) is weak Cauchy in B. By the weak sequential completeness of B there exists an $X \in B$ such that $(X_n) \to X$ weakly in B. The weak closure of a linear manifold being the same as the norm closure, as a result of the Hahn-Banach Theorem, we get $X \in \bar{M}$. Now, given $f \in M^*$, there exists by the Hahn-Banach Theorem a functional $F \in B^*$ which is an extension of f to all of B. Thus $\lim_n. f(X_n) = \lim_n. F(X_n) = F(X) = f(X)$ and so M is weak sequentially complete. Now, $C_o(\omega)$ is not weak sequentially complete and hence no isomorphic image of $C_o(\omega)$ can appear in B.

To show (ii) implies (iii). Without loss of generality, it may be assumed that the decomposition and the norm in B have the properties expressed in Theorem 1 (iii) for $\| \|''$. Suppose there is a sequence (X_i), $X_i \in M_i$, such that $\sup_n. \|\sum_{i=1}^{n} X_i\| < \infty$ and $\sum_{i=1}^{\infty} X_i$ does not converge. Then it can be assumed that $\|\sum_{i \in \mu} X_i\| < 1$.* Then $\sum_{i=1}^{\infty} \frac{X_i}{M}$ will not converge and $\|\sum_{i=1}^{n} \frac{X_i}{M}\| \le 1$** for all n. Again, for $\mu \in \Sigma$ let $N = \max.\{i : i \in \mu\}$. Then by the unconditional monotony of the decomposition $\|\sum_{i \in \mu} X_i\| \le \|\sum_{i=1}^{N} X_i\| \le 1$.

Since $\sum_{i=1}^{\infty} X_i$ is not convergent, the completeness of B implies that $\left(\sum_{i=1}^{n} X_i\right)$ is not Cauchy. Thus there exists sequences of integers $(n_k), (m_k)$ and a positive number d such that $n_k < m_k < n_{k+1}$ for every k in ω and, setting z_k equal to the sum of all X_i with $n_k \le i < m_k$, $\|z_k\| \ge d$ for all k.

* for all $\mu \in \Sigma$.
** where $M = \sup_n. \|\sum_{i=1}^{n} X_i\|$.

It will suffice to show that (z_k) is a basis for a subspace of B isomorphic to $C_o(\omega)$. Take $\mu \in \Sigma$ and real numbers t_k, $k \in \mu$; by (iii) of Theorem 1, $\|\sum_{k\in\mu} t_k z_k\|$ is an even, non-decreasing function of each $|t_k|$, $k \in \mu$. Hence $\|\sum_{k\in\mu} t_k z_k\| = \|\sum_{k\in\mu} |t_k| z_k\| \le [\sup_{k\in\mu} \{t_k\}] \|\sum_{k\in\mu} z_k\| \le \sup_{k\in\mu} |t_k|$ for if $\nu = \bigcup_{k\in\mu} \{i : n_k \le i < m_k\}$ then $\|\sum_{k\in\mu} z_k\| = \|\sum_{i\in\nu} X_i\| \le 1$. But also, by Theorem 1 (iii), $\|\sum_{k\in\mu} t_k z_k\| \ge \sup_{k\in\mu} \|t_k z_k\|$ by the unconditional monotony of the basis. Now, $\sup_{k\in\mu} \|t_k z_k\| = \sup_{k\in\mu} |t_k| \|z_k\| \ge d \sup_{k\in\mu} |t_k|$. Therefore, if T carries each finite linear combination X of the basis vectors in $C_o(\omega)$ to the same combination of the z_k, we have $\|X\| \ge \|T(X)\| \ge d\|X\|$. Since the set of all finite linear combinations of the basis vectors in $C_o(\omega)$ is dense in $C_o(\omega)$ there is a unique extension $\overline{T}_1$ of T_1 to all of $C_o(\omega)$. By Lemma 1 below, $\overline{T}$ will be an isomorphism of $C_o(\omega)$ into B. |

LEMMA 4.1

Let T be a linear mapping of a linear manifold M, dense in a normed linear space N, into a Banach Space B. Suppose there exists positive real numbers K_1, K_2 such that $K_1\|X\| \le \|T(X)\| \le K_2\|X\|$. Then the unique continuous extension $\overline{T}$ of T to all of N will be an isomorphism of N into B.

PROOF:

For any $X \in N$ there is a sequence (X_n) in M such that $\lim_n X_n = X$. For each n, $K_1\|X_n\| \le \|T(X_n)\| \le K_2 \|X_n\|$. By the continuity of $\overline{T}$ and the fact that $\overline{T}$ is an extension of T_1 we get $K_1\|X\| \le \|\overline{T}(X)\| \le K_2 \|X\|$. This immediately shows $\overline{T}$ is one-one. For if $X, Y \in M$ are such that $\overline{T}(X) = \overline{T}(Y)$ then $K_1\|X - Y\| \le \|\overline{T}(X - Y)\| \le \|\overline{T}(X) - \overline{T}(Y)\| = 0$ i.e. $X = Y$. |

THEOREM 4.3

Suppose (M_i, P_i) is an unconditional boundedly complete decomposition of a Banach Space B. Suppose further that, for each i, M_i is weak sequentially complete and satisfies the property that every weak convergent sequence is norm convergent. Then B is weak sequentially complete.

PROOF:

Consider a weak Cauchy sequence in B, that is a sequence (X_n) such that $\lim_n . f(X_n)$ exists for every f in B^*. By a corollary to the Uniform Boundedness Principle* there is a K such that $\|X_n\| \leq K$ for all n. Let $f \in M_i^*$ be given. By the Hahn-Banach Theorem there exists a norm-preserving extension F, of f, to all of B. Clearly, $f(P_i(X)) = F(P_i(X))$ for all $X \in B$. Thus $\lim_n . f(P_i(X_n)) = \lim_n . F(P_i(X_n)) =$ exists since $F \circ P_i$ is a continuous linear functional on B. By the weak sequential completeness of M_i there exists an $X^{(i)} \in M_i$ such that $(P_i(X_n)) \to X^{(i)}$ weakly. By the further condition on M_i, $(P_i(X_n)) \to X^{(i)}$ in the norm. For fixed m consider the sequence $(\sum_{i=1}^{m} P_i(X_n))$. By the foregoing, $\sum_{i=1}^{m} X^{(i)}$ is the weak limit (in fact the norm limit) of this sequence and so by Fatou's Lemma $\|\sum_{i=1}^{m} X^{(i)}\| \leq \lim._n \inf. \|\sum_{i=1}^{m} P_i(X_n)\| \leq \lim._n \inf. \|X_n\| \leq K$ (the second inequality following from the monotony of the decomposition). Since m was arbitrary $\sup_m . \|\sum_{i=1}^{m} X^{(i)}\| < \infty$ and so, by bounded completeness, there is an $X \in B$ such that $X = \sum_{i=1}^{\infty} X^{(i)}$ with $P_i(X) = X^{(i)}$.

Setting $Y_n = X_n - X$ we need now only show that (Y_n) tends weakly to zero. If it does not, then there exists a f of norm 1 in B^*, an $\epsilon > 0$, and a sequence (n_m) such that $f(Y_{n_m}) > \epsilon$ for all m. Let $z_m = Y_{n_m}$. Then since $P_i(X_n) \to X^{(i)}$ in norm for each i we have $\lim_m . \|U_\mu(z_m)\| = 0$ for each $\mu \in \Sigma$. Now take $\eta = \frac{\epsilon}{6}$ and let $z_{n_1} = z_1$. Since $z_{n_1} = \sum_{i=1}^{\infty} P_i(z_{n_1})$ there is an m_1 such that $\|V_{m_1}(z_{n_1})\| < \eta$. The fact that $\lim_m .$

* any sequence (X_n) in a Banach Space B for which $\sup_n . (f(X_n)) < \infty$ for each $f \in B^*$ is norm bounded.

$\|U_{m_1}(z_m)\| = 0$ implies there is an $n_2 > n_1$ such that $\|U_{m_1}(z_n)\| < \eta$ for $n \geq n_2$. Again, since $z_{n_2} = \sum_{i=1}^{\infty} P_i(z_{n_2})$ there is an $m_2 > m_1$ such that $\|V_{m_2}(z_{n_2})\| < \eta$. Continuing in this manner we get increasing sequences (n_k) and (m_k) such that $\|V_{m_k}(z_{n_k})\| < \eta$ and $\|U_{m_k}(z_n)\| < \eta$ if $n \geq n_{k+1}$. Let $W_k = z_{n_k} - U_{m_{k-1}}(z_{n_k}) - V_{m_k}(z_{n_k})$. Then $\|W_k - z_{n_k}\| < 2\eta$ and and $f(W_k) = f(z_{n_k}) - f(U_{m_{k-1}})(z_{n_k}) - f(V_{n_k})(z_{m_k}) > \epsilon - 2\eta$. If δ_k is that element of $\ell'(\omega)$ whose k^{th} term is 1 and whose remaining terms are 0 and if $\tau = \sum_{k\in\rho} t_k \delta_k$, where $\rho \in \Sigma$, set $T(\tau) = \sum_{k\in\rho} t_k W_k$; then $\|T(\tau)\| = \|\sum_{k\in\rho} t_k W_k\| \leq [\sup_{k\in\rho}\|W_k\|] \sum_{k\in\rho} |t_k| = \|\tau\|_{\ell'} [\sup_{k\in\rho}\|W_k\|] < 2(K+\eta)\|\tau\|_{\ell'}$, since $\|W_k\| = \|z_{n_k} - U_{m_{k-1}}(z_{n_1}) - V_{m_k}(z_{n_k})\| \leq \|z_{n_k}\| + \|U_{m_{k-1}}(z_{n_k})\| + \|V_{m_k}(z_{n_k})\| \leq \|z_{n_k}\| + 2\eta = \|X_{n_k} - X\| + 2\eta \leq \|X_{n_k}\| + \|X\| + 2\eta \leq 2K + 2\eta$ since from the expansion for X we can show $\|X\| \leq K$. But, by Theorem 1 (iii) $\|\sum_{k\in\rho} t_k W_k\| = \|\sum_{k\in\rho} |t_k| W_k\| \geq |f(\sum_{k\in\rho} |t_k| W_k)| = \sum_{k\in\rho} |t_k| f(W_k) \geq (\epsilon - 2\eta)\sum_{k\in\rho} |t_k| = (\epsilon - 2\eta)\|\tau\|_{\ell'}$. Now if $T'(\tau) = \sum_{k\in\rho} t_k z_{n_k}$ then $\|T'(\tau) - T(\tau)\| \leq \|\sum_{k\in\rho} t_k(W_k - z_{n_k})\| \leq \sup_{k\in\rho} \|W_k - z_{n_k}\| \|\tau\|_{\ell'} < 2\eta\|\tau\|_{\ell'}$ (i.e. $\|T'(\tau)\| < \|T(\tau)\| + 2\eta\|\tau\|_{\ell'}$ and $\|T'(\tau)\| > \|T(\tau)\| - 2\eta\|\tau\|_{\ell'}$) and so $(2K + 4\eta)\|\tau\|_{\ell'} \geq \|T'(\tau)\| \geq (\epsilon - 4\eta)\|\tau\|_{\ell'}$. By Lemma 1, T' can be extended to a isomorphism $\overline{T}'$ of $\ell'(\omega)$ with a subspace of csp. (z_{nk}). If $B_1 = \overline{T}'(\ell'(\omega))$, then B_1 is weak sequentially complete since $\ell'(\omega)$ is (corollary 3 of Phillip's Lemma). Because $\lim_k f(z_{n_k}) = \lim_n f(Y_n)$ exists for all f in B^*, (z_{n_k}) converges weakly to some element z of B_1. For arbitrary $f \in B^*$, $f \cdot P_i$ is also in B^*, so $f \cdot P_i(z) = \lim_k f \cdot P_i(z_{n_k}) = \lim_n f \cdot P_i(Y_n) = \lim_n f \cdot P_i(X_n - X) = f(\lim P_i(X_n) - P_i(X)) = f(X^{(i)} - X^{(i)}) = 0$. Since $f \cdot P_i(z) = 0$ for all $f \in B^*$ it follows that $P_i(z) = 0$ and hence $z = 0$ i.e. 0 is the weak limit of the Y_n after all. This contradiction shows that B is weak sequentially complete. |

REMARK 1

Since every finite dimensional space is both weak sequentially complete and satisfies the property that every weak convergent sequence is norm convergent, we get that every Banach space with a boundedly complete unconditional basis is weak sequentially complete. Hence (i), (ii) and (iii), of Theorem 4.2 are equivalent for Banach Spaces with unconditional bases.

REMARK 2

From Theorem 4.2 it is seen that bounded completeness of the decomposition and weak sequential completeness of the coordinate spaces are necessary conditions for the weak sequential completeness of B. It is unknown whether they are sufficient. However, the property that in each coordinate space every weak convergent sequence is norm convergent is not necessary as is shown in Example 2 below.

EXAMPLE 1

The purpose of this example is to show that Theorem 4.3 is indeed a partial extension of the corresponding theorem for unconditional bases. Consider the set $\ell^{(p)}(\ell'(S))$, $p \geq 1$, consisting of all sequences (t_i) with t_i in $\ell'(S)$ such that $\sum_{i=1}^{\infty} \|t_i\|_{\ell'}{}^p < \infty$. With component-wise definitions of addition and scalar multiplication and with norm $\|(t_i)\| = (\sum_{i=1}^{\infty} \|t_i\|_{\ell'}{}^p)^{1/p}$, $\ell^{(p)}(\ell'(S))$ is a Banach Space the proof reducing to a consideration of the norm in $\ell'(\omega)$ in much the same way as the proof for $\ell^p(\omega)$ reduces to a consideration of the absolute value for the real numbers. Define M_i as the set of all elements of $\ell^{(p)}(\ell'(S))$ all of whose terms, except possibly the i^{th} are zero. It is readily seen that (M_i) is an unconditional boundedly complete decomposition of $\ell^{(p)}(\ell'(S))$ since we are, in effect, dealing with absolutely convergent series of real numbers. Again, (M_i) will be a Schauder decomposition of $\ell^{(p)}(\ell'(S))$ since each of the M_i is isomorphic to $\ell'(S)$ and hence closed. By corollaries 2 and 3 of Phillip's Lemma, $\ell'(S)$ is both weak sequentially complete and satisfies the property that every weak convergent sequence is norm convergent. Thus, by the above theorem, $\ell^{(p)}(\ell'(S))$ will be weak sequentially complete.

EXAMPLE 2

Let $\ell^{(2)}(\ell^2(S))$ be the set of all sequences in $\ell^2(S)$ such that the corresponding sequence of norms is square summable. Defining addition, scalar multiplication, and the norm in a manner analogous to example 1 we find that $\ell^{(2)}(\ell^2(S))$ is a Banach Space. If $X = (t_i)$, $y = (r_i) \in \ell^{(2)}(\ell^2(S))$ then $(,) : \ell^{(2)}(\ell^2(S)) \to R$ defined by $(X,y) = \sum_{i=1}^{\infty} (t_i, r_i)$, where the expression inside is the usual inner product in $\ell^2(S)$, is seen to satisfy the requirements of an inner product w.r.t. the norm defined in $\ell^{(2)}(\ell^2(S))$. Thus $\ell^{(2)}(\ell^2(S))$ is a Hilbert Space. Consequently it will be reflexive and hence weak sequentially complete.

Defining M_i as the set of all elements in $\ell^{(2)}(\ell^2(S))$ all of whose terms, except possibly the i^{th}, are zero, we get that (M_i) is an unconditional Schauder decomposition of $\ell^{(2)}(\ell^2(S))$ with each of the M_i isomorphic to $\ell^2(S)$. However, not every weak convergent sequence in $\ell^2(S)$ is norm convergent (for example consider any infinite sequence (X_{s_i}) with $X_{s_i} = 1$ at s_i and 0 elsewhere and with $X_{s_i} \neq X_{s_j}$, $i \neq j$).* Thus the third property of the previous theorem is not necessary.

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* Given $F \in \ell^2(S)^*$ let (y_s) be the element of $\ell^2(S)$ corresponding to f in the usual representation. Then $\lim_i F(X_{s_i}) = \lim_i y_{s_i} = 0$. However (X_{s_i}) is not norm convergent to zero.

THEOREM 4.4

Let (b_i) be an unconditional basis for B; then the basis is shrinking iff. there is no subspace of B isomorphic to $\ell'(\omega)$.

PROOF:

If necessary B can be renormed isomorphically to have the property of Theorem 1 (iii). Suppose now that (b_i) is not shrinking. Then there exists $\epsilon > 0$, $f \in B^*$, indices m_k increasing indefinitely and points Y_k in B such that $\|f\| = \|Y_k\| = 1$, $f(Y_k) > \epsilon$, $V_{m_k}(Y_k) = Y_k$ and $V_{m_{k+1}}(Y_k) = 0$. For $\mu \in \Sigma$ we have, by Theorem 1, (iii) that $\|\sum_{k\in\mu} t_k Y_k\| = \|\sum_{k\in\mu} |t_k| Y_k\| > |-f(\sum_{k\in\mu} |t_k| Y_k)| = \sum_{k\in\mu} |t_k| f(Y_k) \geq \epsilon \sum_{k\in\mu} |t_k|$. Also $\|\sum_{k\in\mu} t_k Y_k\| \leq \sum_{k\in\mu} |t_k| \|Y_k\| = \sum_{k\in\mu} |t_k|$. It follows, from Lemma 1, that $T[(t_k)] = \sum_{k\in\mu} t_k Y_k$ determines an isomorphism of $\ell'(\omega)$ with a subset of csp. $\bigcup_{k=1}^{\infty} Y_k$.

If (b_i) is a shrinking basis, (β_i) is a basis for B^* by Theorem 2.1 and so B^* is separable. Suppose there is a subspace B' of B isomorphic with $\ell'(\omega)$ By Lemma 2.2 $m(\omega) = \ell'(\omega)^*$ is isometric to $\frac{B^*}{B_1^{\perp}}$ This means that $\frac{B^*}{B_1^{\perp}}$ is inseparable since $m(\omega)$ is. But $f : B^* \to \frac{B^*}{B_1^{\perp}}$ defined by $f(g) = B_1^{\perp} + g$ is continuous since $\|B_1^{\perp} + g\| \leq \|g\|$. Thus, since the continuous image of a separable space is separable, $\frac{B^*}{B_1^{\perp}}$ will be separable. This contradiction establishes the theorem.

LEMMA 4.2

If $\ell'(S)$ is a homomorphic image of a Banach Space B then it is isomorphic to a subspace of B.

PROOF:

Suppose T is a bounded linear operator from a Banach Space B onto $\ell'(S)$ with kernel L. Then, by the Open Mapping Theorem, $\ell'(S)$ is isomorphic to $\frac{B}{L}$. This means that $I : \frac{B}{L} \to \ell'(S)$ defined by $I(L + b) = T(b)$ is such that there exists K_1, K_2 with $K_1 \|L + b\| \leq \|T(b)\| \leq K_2\|L + b\|$ for all $b \in B$. For each basis vector δ_s in $\ell'(S)$ there will be a b_s in B such that $T(b_s) = \delta_s$ and $\|b_s\| \leq 2/K_1$ since b_s can be chosen so that $\|b_s\| - \|L + b_s\| < \frac{1}{K_1}$. If $\tau = \sum_{k\in\sigma} t_k \delta_k$, where $\sigma \in \Sigma$, set $U(\tau) = \sum_{k\in\sigma} t_k b_k$. Then $\|U(\tau)\| = \|\sum_{k\in\sigma} t_k b_k\| \leq \sum_{k\in\sigma} |t_k| \|b_k\| \leq \|\tau\|_{\ell'} 2/K_1$.

Again, $\|U(\tau)\| \geq \|L + \sum_{k\in\sigma} t_k b_k\| \geq \frac{1}{K_2} \| T(\sum_{k\in\sigma} t_k b_k)\| = \frac{1}{K_2} \|\sum_{k\in\sigma} t_k \delta_k\| \geq \frac{\|\tau\|\ell'}{K_2}$. By Lemma 1, the unique continuous extension of U to all of $\ell'(S)$ will be the desired isomorphism.

LEMMA 4.3

If the conjugate B^* of a Banach Space B is separable, then so is B.

PROOF:

If B^* is separable, then there is a countable set dense in the unit sphere of B^*. Let (f_n) be a countable set which is everywhere dense on the set $\{f:\|f\|= 1\}$ in B^*. Choose $X_n \in B$ so that $\|X_n\| = 1$ and $|f_n (X_n)| \geq \frac{3}{4}$ which is possible since $1 = \|f_n\| = \sup_{\|X\|=1} |f_n(X)|$. Let M be the closed linear manifold in B generated by the sequence (X_n). Suppose $M \neq B$ and $X_o \in B - M$ Then by a corollary to the Hahn-Banach Theorem we get that there exists $f \in B^*$ such that $\|f\| = 1$, $f(X_o) = \|X_o\|$ and $f(X) = 0$ if $X \in M$. Then $f(X_n) = 0$ if $n = 1, 2, \ldots$ and $\frac{3}{4} \leq |f_n(X_n)| \leq |f_n (X_n) - f(X_n)| + |f (X_n)|$ whence $\frac{3}{4} \leq \|f_n - f\| \|X_n\| = \|f_n - f\|$. This contradicts the fact that $\|f_n - f\|$ can be made as small as we please by suitable choice of n. Hence $M = B$. It then follows that linear combinations formed from (X_n) with rational scalar coefficients constitute a countable set everywhere dense in B, so that B is separable.

THEOREM 4.5

Let B be a Banach Space with an unconditional basis. Then the following conditions are equivalent.

(i) B is reflexive.

(ii) B is weak sequentially complete and contains no subspace isomorphic to $\ell'(\omega)$.

(iii) B contains no subspace isomorphic to $c_0(\omega)$ or $\ell'(\omega)$

(iv) Neither B nor B^* contains a subspace isomorphic to $\ell'(\omega)$.

(v) B^{**} is separable.

PROOF:

By the previous three theorems, (see Remark 1), and the fact that a space with a basis is reflexive iff. the basis is shrinking and boundedly complete, we get the equivalence of the first three conditions.

(i) implies (v) since $B^{**} = Q(B)$ and B, having a basis, is separable.

To show (v) implies (iv). Firstly, B^* cannot have a subspace isomorphic to $\ell'(\omega)$ since if it did, then, as in the previous theorem, B^{**} would have a factor space isomorphic to $\ell'(\omega)^*$ and hence could not be separable. Again, B cannot have a subspace isomorphic to $\ell'(\omega)$ since that would mean that B^* would be inseparable. However, in view of Lemma 3 this cannot be the case.

To show (iv) implies (iii). Suppose T is an isomorphism of $c_0(\omega)$ into B. Then T^* will be a mapping of B^* into $c_0(\omega)^* = \ell'(\omega)$. Given $g \in c_0(\omega)^*$, the functional $f = g\,T^{-1}$ defined on $T(c_0(\omega))$ can be extended to a continuous functional on all of B. Then $T^*\phi = \phi\,T = g$, so T^* is an onto mapping. By Lemma 2, $\ell'(\omega)$ will be isomorphic to a subspace of B^*, contradicting (iv).

THEOREM 4.6

If (b_i) is an unconditional basis for B and if (β_i) is the corresponding sequence in B* then the following conditions are equivalent.

(i) (b_i) is a shrinking basis for B.

(ii) No subspace of B is isomorphic to $\ell'(\omega)$.

(iii) (β_i) is a basis for B*.

(iv) (β_i) is a boundedly complete basis for B*.

(v) B* is separable.

(vi) (β_i) is an unconditional basis for B*.

(vii) B* is weak sequentially complete.

PROOF:

The equivalence of (i), (ii) and (iii) has been established before and (iv) implies (iii) trivially and (i) implies (iv) follows from Theorem 2.4. (iii) implies (v) since any space with a base is separable, whereas (v) implies (ii) since B*, being separable, cannot have a factor space isomorphic to $\ell'(\omega)^*$. (vi) implies (v) and (iv) implies (v), since any space with a basis is separable, while by Remark 1 (iv) and (vi) together imply (vii).

To show (iii) $\Rightarrow$(vi). Renorm if necessary so that the basis (b_i) in B satisfies the properties of Theorem 1 (iii). We show that $\|V_\mu^*\| \le 2$ for all $\mu \in \Sigma$. This will follow if for every $f \in B^*$, $\|f\| = 1$, we have $\|V_\mu^*(f)\| \le 2$. This in turn will be so if for every $X \in B$, $\|X\| = 1$, we have $|V_\mu^*(f)[X]| \le 2$. But $|V_\mu^*(f)[X]| = |f(V_\mu(X))| \le \|V_\mu(X)\| \le 2$.

Now, given $\beta \in B^*$ and $\epsilon > 0$ there is a m such that $\|V_n^*(\beta)\| < \epsilon/2$ for $n \ge m$. However, if $\mu \in \Sigma$ and μ contains all integers $\le m$ we have $V_m V_\mu = V_\mu$ so that $V_\mu^* V_m^* = V_\mu^*$. Thus $\|V_\mu^*(\beta)\| = \|V_\mu^* V_m^*(\beta)\| \le$

$\|V_\mu{}^*\| \, \|V_m{}^*(\beta)\| < 2 \, \epsilon/2 = \epsilon$ so $\lim_{\mu \in \Sigma} \|V_\mu{}^* (\beta)\| = 0$ i.e. (vi) holds.

To show (vii) implies (iii). Renorming B to satisfy the properties of 1 (iii) we can get, as in (iii) ⇒ (vi), $\|U_\mu{}^*\| \le 1$.

Thus, for $\mu \in \Sigma$, $\|\sum_{i \in \mu} \beta(b_i)\, \beta_i\| \le \|\beta\|$ for each $\beta \in B$. Let $\sum_{k=1}^{\infty} \beta(b_{i_k})\, \beta_{i_k}$ be any subseries of $\sum_{i=1}^{\infty} \beta(b_i)\, \beta_i$. Then given any $f \in B^{**}$, $|\sum_{k \in \mu} \beta(b_{i_k})\, f(\beta_{i_k})| = |f(\sum_{k \in \mu} \beta(b_{i_k})\, \beta_{i_k})| \le \|f\| \, \|\sum_{k \in \mu} \beta(b_{i_k})\, \beta_{i_k}\| \le \|f\| \, \|\beta\|$. Thus the series $\sum_{k=1}^{\infty} \beta(b_{i_k})\, f(\beta_{i_k})$ of real numbers has all finite sums bounded. By Lemma 3.4 it will therefore converge. Since f was arbitrary in B^{**}, the weak sequential completeness of B^* ensures the existence of $\gamma \in B^*$ so that $\sum_{k=1}^{\infty} \beta(b_{i_k})\, \beta_{i_k}$ converges weakly to γ.

This shows that $\sum_{i=1}^{\infty} \beta(b_i)\, \beta_i$ is weak subseries convergent. By the Orlicz-Pettis Theorem it will then be norm convergent. Now, for any $X \in B$, $\sum_{i=1}^{\infty} \beta(b_i)\, \beta_i(X) = \beta(X)$ i.e. $\sum_{i=1}^{\infty} \beta(b_i)\, \beta_i$ is convergent to β. Hence it must be norm convergent to β. Thus (vii) ⇒ (iii), the uniqueness of the expansion being assured by the fact that (b_i) and (β_i) form a biorthogonal system.

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