THE UNIVERSITY OF MANITOBA

CATEGORICAL COMBINATORICS

A Category-theoretic Approach to Combinatorial Analysis and Enumeration

by

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ABSTRACT

A *finitary category* is defined in this work as a skeletally small category $\underline{C}$ such that the set of morphisms between any two $\underline{C}$-objects is finite.

Many combinatorial problems can be viewed as relating to quantities involving appropriate finitary categories. In this work general techniques for analyzing such problems are developed and applied to various typical examples.

The chief tool in such an analysis is the notion of *S-representability*. We call a set-valued functor S-representable if it is the disjoint union of representable functors. With this idea, various other categorical concepts are generalized. For example, we derive a concept of an *S-product*, a good illustration of which is given by the collection of subdirect products in the cartesian product of two finite sets. It is an S-product in the category of finite sets and surjective maps.

When such an S-product is present, it is possible to associate with the finitary category a commutative ring within which many combinatorial calculations can be carried out with facility. A concept of *S-adjointness* allows one to establish homomorphisms between such rings which also frequently encapsulate a great deal of combinatorial information. For example,

using such techniques, we are able to prove the following result:

Let the function Q_n on the positive integers be defined as the function whose value at the integer k is the "n-th difference of 0^k"; that is,

$$Q_n(k) = \Delta^n 0^k = n!S(k,n),$$

where $S(k,n)$ is the designated Stirling number of the second kind. Then, Q_n is equal to a unique polynomial with integral coefficients in the functions Q_p for primes p less than or equal to n.

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CHAPTER ONE

INTRODUCTION

This chapter will provide some examples and (hopefully) some motivation for the theory to be developed in the following chapters.

A "multiplication" of two finite algebras (in the sense of universal algebra) of the same type is defined as the formal sum of their subdirect products (loosely speaking). The problem is then posed as to whether this multiplication is associative. That this is so is proved in two different ways. En route, a number of interesting combinatorial relations are derived, which become the focus of later developments.

Two specific cases are then studied: first, that of finite algebras without any operations (i.e., finite sets) and then, finite Boolean algebras (which are dual, in the sense of category theory, to finite sets). Several classical results involving binomial coefficients and Stirling numbers are proved, plus a few new ones. Perhaps the most satisfying of the new results is the following:

For integers $n, k \geq 1$, let $S(n,k)$ denote a Stirling number of the second kind and $S(-,k)$ denote the corresponding function on the positive integers. Then,

$S(-,k)$ is expressible as a unique polynomial in the functions $S(-,p_1)$, $S(-,p_2)$, ..., $S(-,p_r)$, where the p_i are the primes less than or equal to k.
(For example, $S(-,4) = \frac{1}{6} S(-,2)^2 - S(-,3) - \frac{1}{6} S(-,2)$.)

The chapter concludes with some notational remarks that anticipate the category-theoretic viewpoint which will dominate the rest of this dissertation.

1. A Multiplication Derived from Subdirect Products:

We use the term "algebra" in the sense of universal algebra; i.e., a set with operations. Two algebras are of the same type if, loosely speaking, they have the "same" operations. (In that case, it is possible to speak of an algebra homomorphism from one to the other.)

Now let $\underline{C}$ be a class of finite algebras of the same type, which is closed under the formation of subalgebras and of the direct product of two algebras. Thus, $\underline{C}$ could consist of all finite groups, all finite p-groups, finite lattices, finite distributive lattices, etc.

Let C be a skeletal set for $\underline{C}$; that is, C contains exactly one member from each isomorphism class of algebras in $\underline{C}$. Form the free Z-module Z(C) on the elements of C; loosely speaking, Z(C) consists of all formal, finite linear combinations of the members of C with integer coefficients. Thus, if we write $C = \{A_i : i \in I\}$, where I is some suitable indexing set, then an arbitrary element $\underline{v}$ of Z(C) can be written as:

$$\underline{v} = \sum_{i \in I} v(i) A_i$$

where the v(i) are integers only a finite number of which are non-zero.

If B_1 and B_2 are algebras in $\underline{C}$, then $B_1 \times B_2$ denotes their direct product; while $\pi_1 : B_1 \times B_2 \to B_1$ and $\pi_2 : B_1 \times B_2 \to B_2$ denote the natural projections of the direct product onto its first and second factors, respectively. A subalgebra B of $B_1 \times B_2$ is <u>subdirect</u> in $B_1 \times B_2$ if $\pi_1(B) = B_1$ and $\pi_2(B) = B_2$. In other words, B is subdirect in $B_1 \times B_2$ if the restrictions of π_1 and π_2 to B are surjective. Define a multiplication in Z(C) as follows:

For A_i and A_j in C, set

$$(1.1) \quad A_i \circ A_j = \sum_k r(i,j;k) A_k,$$

where r(i,j;k) is the number of subalgebras B which are subdirect in $A_i \times A_j$ and isomorphic to A_k. (Thus, r(i,j;k) is the number of subdirect algebras in the isomorphism class defined by A_k.)

We now pose the following problem: Is the multiplication defined by equation (1.1) associative? We shall prove this in two ways. For our first proof we require a few definitions:

If B_1 and B_2 are two algebras in $\underline{C}$, let $\underline{D}[B_1, B_2]$ denote the family of all surjective homomorphisms from B_1 to B_2. For each A_i in C define a corresponding Z-linear map $\underline{d}_i$: Z(C) $\rightarrow$ Z on basis elements by means of the formula:

$$(1.2) \quad \underline{d}_i(A_j) = \#\underline{D}[A_i, A_j].$$

(In this work, the symbol "#" shall stand for "the number of elements in" a given finite set.)

Proposition (1.3): Let B_1, B_2, and B_3 be arbitrary algebras in $\underline{C}$, and let W be the family of all algebra homomorphisms $h: B_1 \to B_2 \times B_3$ such that $h(B_1)$, the image of B_1 under h, is subdirect in $B_2 \times B_3$. Then, there is a natural one-one correspondence between W and $\underline{D}[B_1, B_2] \times \underline{D}[B_1, B_3]$.

Proof: Given a pair of surjective homomorphisms $(f, g) \varepsilon \underline{D}[B_1, B_2] \times \underline{D}[B_1, B_3]$, one derives a corresponding homomorphism $\langle f, g\rangle: B_1 \to B_2 \times B_3$ by the prescription $\langle f, g\rangle(x) = (f(x), g(x))$. Since $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$, and f and g are surjective, it follows that the image of $\langle f, g\rangle$ is subdirect in $B_2 \times B_3$, and hence $\langle f, g\rangle \varepsilon W$. Conversely, if $h: B_1 \to B_2 \times B_3$ has a subdirect image, then it yields a pair $(\pi_1 \circ h, \pi_2 \circ h)$ of surjective homomorphisms. QED

The above proposition can also be stated as follows:

Let $\{B_x : x \in X\}$ be the family of all subdirect algebras in $B_2 \times B_3$ (suitably indexed). Then,

$$(1.4) \qquad \underline{D}[B_1, B_2] \times \underline{D}[B_1, B_3] \cong \bigcup_{(x)} \underline{D}[B_1, B_x].$$

The union on the right hand side of (1.4) is of course disjoint. This immediately gives us the following proposition:

<u>Proposition (1.5):</u> For each index $i \in I$, the map $\underline{d}_i$ defined by (1.2) preserves multiplication. That is,

$$(1.6) \qquad \underline{d}_i(A_j)\underline{d}_i(A_k) = \underline{d}_i(A_j \cdot A_k)$$

for all A_j, $A_k \in C$.

<u>Proof:</u> Note that the left hand side of (1.6) is simply $\#(\underline{D}[A_i, A_j] \times \underline{D}[A_i, A_k])$; while the right hand side is

$$\underline{d}_i(\sum_m r(j,k;m) \cdot A_m) = \sum_m r(j,k;m)(\#\underline{D}\ [A_i, A_m]),$$

which clearly counts the number of elements in the (disjoint) union $\bigcup_{(x)} \underline{D}[A_i, B_x]$, where the B_x vary over the subdirect algebras of $A_j \times A_k$. (Simply replace each B_x by its isomorphic copy in C.) But by proposition (1.3) these two counts must be equal. QED

We used the awkward phrase "preserves multiplication" in the above instead of the preferable description "$\underline{d}_i$ is a ring homomorphism" since we do not yet know that Z(C) is a ring under the multiplication (1.1); Z(C) will be a ring if this multiplication is associative.

Since the multiplication in Z is associative, it follows that for any elements $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$ ε Z(C),

$$\underline{d}_i((\underline{v}_1 \circ \underline{v}_2) \circ \underline{v}_3) = \underline{d}_i(\underline{v}_1 \circ (\underline{v}_2 \circ \underline{v}_3))$$

for all indices i in I. We shall show that there are "enough" of these homomorphisms $\underline{d}_i$ so that this fact implies that

$$((\underline{v}_1 \circ \underline{v}_2) \circ \underline{v}_3) = (\underline{v}_1 \circ (\underline{v}_2 \circ \underline{v}_3)),$$

and thus complete our first proof of the associativity of (1.1).

Towards this end, consider Z^C, the space of <u>all</u> functions $\underline{w} : C \to Z$. It is a ring under pointwise operations. That is, if $\underline{w}_1$, $\underline{w}_2$ ε Z^C, then the functions $\underline{w}_1 + \underline{w}_2$ and $\underline{w}_1 \circ \underline{w}_2$ are defined ("pointwise") by the equations:

$$(\underline{w}_1 + \underline{w}_2)(A_i) = \underline{w}_1(A_i) + \underline{w}_2(A_i),$$

and $$(\underline{w}_1 \circ \underline{w}_2)(A_i) = \underline{w}_1(A_i) \circ \underline{w}_2(A_i),$$

for all A_i ε C. Now define the Z-linear map $\underline{d}$: $Z(C) \to Z^C$ on basis elements by letting $\underline{d}(A_i)$ be the function whose value at A_j, $\underline{d}(A_i)(A_j)$, is simply $\underline{d}_j(A_i)$, and, of course, extend $\underline{d}$ to the rest of Z(C) by linearity.

Thus, by definition, we have that

$$\underline{d}(A_i)(A_j) = \underline{d}_j(A_i) = \#\underline{D}[A_j, A_i];$$

consequently, if $\underline{v} = \sum_i v(i)A_i$ is an arbitrary element of $Z(C)$, then $\underline{d}(\underline{v}) = \sum_i v(i)\underline{d}(A_i)$, the function in Z^C whose value at a point $A_j \in C$ is given by the equation

$$(1.7) \quad \underline{d}(\underline{v})(A_j) = \sum_i v(i)(\#\underline{D}[A_j, A_i]).$$

(Because of the form of the equation (1.7), in which summation occurs over the right hand argument in the expression $\#\underline{D}[A_j, A_i]$, we shall refer to the process of forming $\underline{d}$ from its "component" homomorphisms $\underline{d}_j$ as "linearization on the right".)

It is clear that the Z-linear map $\underline{d}\colon Z(C) \to Z^C$ is also "multiplication-preserving", since the multiplication in Z^C is defined pointwise, and each "component" $\underline{d}_j$ of $\underline{d}$ preserves multiplication.

We now aim to show that $\underline{d}$ is faithful (i.e., injective); from which fact the associativity of (1.1) is immediately deducible.

In order to show that $\underline{d}$ is faithful, it is necessary and sufficient to show that the image of the basis elements of $Z(C)$, namely $\{\underline{d}(A_i) : i \in I\}$, forms a linearly independent set in Z^C. As a first step towards this result, define the relation ">" on the set C by:

(1.8) $\qquad A_i > A_j$ if $\underline{D}[A_i, A_j] \neq \emptyset$.

It is immediately obvious that the relation > is both transitive and reflexive. A little thought shows that it is also anti-symmetric. (If there exist surjective algebra homomorphisms $f: A_i \to A_j$ and $g: A_j \to A_i$, then since the underlying sets are finite, both f and g must be bijective. But an algebra homomorphism which is a set isomorphism is also an algebra isomorphism. Consequently, A_i and A_j are isomorphic as algebras; which by the definition of C means that $A_i = A_j$.)

In other words, > is a partial order on C. Also note that $\underline{d}(A_i)(A_j) = \underline{d}_j(A_i) = 0$ unless $A_j > A_i$; while $\underline{d}(A_i)(A_i) \neq 0$. (That is, $\underline{d}_j(A_i) \neq 0$ if and only if $A_j > A_i$.)

<u>Proposition (1.9):</u> The Z-linear map $\underline{d} : Z(C) \to Z^C$ is faithful.

<u>Proof:</u> As stated, we must show that the set $\{\underline{d}(A_i) : i \in I\}$ is linearly independent. For infinite sets, linear independence means that any finite subset is linearly independent. Thus, let $\Omega = \{A_{i(1)}, A_{i(2)}, \ldots, A_{i(n)}\}$ be an

arbitrary finite subset of C, and suppose that:

$$r_1\underline{d}(A_{i(1)}) + r_2\underline{d}(A_{i(2)}) + \ldots + r_n\underline{d}(A_{i(n)}) = 0.$$

We claim that $r_1 = r_2 = \ldots = r_n = 0$, and therefore Ω is linearly independent. For suppose otherwise, that at least one of the r_i is non-zero: We may assume Ω has been indexed in "non-decreasing" order; i.e., if $A_{i(s)} < A_{i(t)}$, then $s \leq t$. This means that the function $\underline{d}(A_{i(s)}) \in Z^C$ takes on zero values at points in the sequence Ω after $A_{i(s)}$.

Let r_s be the first non-zero term in the series $(r_1, r_2, \ldots, r_n)$; then,

$$[r_1\underline{d}(A_{i(1)}) + r_2\underline{d}(A_{i(2)}) + \ldots + r_n\underline{d}(A_{i(n)})](A_{i(s)})$$
$$= r_s\underline{d}(A_{i(s)})(A_{i(s)}) = 0.$$

But, $\underline{d}(A_{i(s)})(A_{i(s)}) \neq 0$, and therefore $r_s = 0$, a contradiction. QED

Corollary (1.10): The multiplication (1.1) on Z(C) is associative.

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This completes our first proof of the associativity of the multiplication (1.1). Under this multiplication, Z(C) is a ring. Indeed, it is clearly a commutative ring.

Before we begin the development of the second proof, it will be helpful to develop some notation. Let us use $Z\langle C, \underline{D}\rangle$ to denote the Z-module Z(C) equipped with the multiplication (1.1), and with the (faithful) ring homomorphism $\underline{d}$: $Z(C) \to Z^C$.

We now construct a second such structure. Begin by defining a second multiplication on Z(C) via the prescription:

$$(1.11) \qquad A_i \circ A_j = A_k,$$

where A_k is the unique algebra in C isomorphic to $A_i \times A_j$.

As before, we extend (1.11) to all of Z(C) by linearity. This multiplication is clearly associative, and under it Z(C) is again a commutative ring.

It is also possible to find a natural ring homomorphism from Z(C) (equipped with the multiplication (1.11)) to Z^C :

For B_1 and B_2 in $\underline{C}$, let $\underline{C}[B_1, B_2]$ denote the family of <u>all</u> algebra homomorphisms from B_1 to B_2. For each index $i \in I$, define the Z-linear map $\underline{c}_i$: $Z(C) \to Z$ on basis elements by $\underline{c}_i(A_j) = \#\underline{C}[A_i, A_j]$.

<u>Proposition (1.12):</u> For all $i \in I$, the map $\underline{c}_i : Z(C) \to Z^C$ is a ring homomorphism, assuming Z(C) is equipped with the multiplication (1.11).

<u>Proof:</u> The proof follows immediately from the fact that for all B, B_1, $B_2 \in \underline{C}$, the sets $\underline{C}[B, B_1] \times \underline{C}[B, B_2]$ and

$\underline{C}[B, B_1 \times B_2]$ are in a natural one-one correspondence, and therefore their cardinalities are equal. QED

We now proceed to define as before, from the collection of all the ring homomorphisms $\underline{c}_i$ the ring homomorphism $\underline{c} : Z(C) \to Z^C$. To be quite explicit, $\underline{c}(A_i)$ is the function on C whose value at A_j is given by:

$$\underline{c}(A_i)(A_j) = \#\underline{C}[A_j, A_i] = \underline{c}_j(A_i).$$

(Alternatively, we say that c is obtained by "linearization on the right" of the expression $\#\underline{C}[A_j, A_i]$.)

Following the same notation introduced above, let $Z<C, \underline{C}>$ denote the structure consisting of Z(C) equipped with the multiplication (1.11) and the ring homomorphism $\underline{c}$. We shall now show that the two structures $Z<C, \underline{D}>$ and $Z<C, \underline{C}>$ are isomorphic; that is, there is a Z-linear isomorphism $\underline{t}$: $Z(C) \to Z(C)$ which is both an isomorphism of the multiplications involved, and also commutes with the ring homomorphisms $\underline{c}$ and $\underline{d}$.

We define $\underline{t}$ on basis elements $A_i \in C$ by the formula:

$$(1.13) \quad \underline{t}(A_i) = \sum_j t(i,j)A_j,$$

where t(i,j) is the number of subalgebras of A_i isomorphic to A_j.

Proposition (1.14): The following diagram commutes for all $i \in I$:

$$\begin{tikzcd}
Z(C) \arrow[d, "\underline{t}"'] \arrow[dr, "\underline{c}_i"] & \\
Z(C) \arrow[r, "\underline{d}_i"'] & Z
\end{tikzcd}$$

Proof: It suffices to verify this for the basis elements $A_j \in C$. Thus, we must show

$$\underline{c}_i(A_j) = \underline{d}_i(\underline{t}(A_j)) = \sum_k t(j,k)\underline{d}_i(A_k),$$

or, (referring to the definitions of $\underline{c}_i$ and $\underline{d}_i$) that

$$(*) \qquad \#\underline{C}[A_i,\ A_j] = \sum_k t(j,k)(\#\underline{D}[A_i,\ A_k]).$$

Let $\{B_x : x \in X\}$ be the (suitably indexed) family of all subalgebras of A_k. Since every algebra homomorphism from A_i to A_j corresponds to a unique surjective homomorphism from A_i to a subalgebra of A_j, it is clear that the sets $\underline{C}[A_i,\ A_j]$ and $\bigcup_{(x)} \underline{D}[A_i,\ B_x]$ are in a one-one correspondence.

Furthermore, from the definition of the coefficients $t(j,k)$, it is clear that the right hand side of (*) is equal to $\#\bigcup_{(x)} \underline{D}[A_i, B_x]$, which from what we have just said must be equal to the left hand side of (*). QED

We shall now show that $\underline{t}$ is a homomorphism of the multiplications involved. Again, this needs to be verified only on basis elements. Write the multiplication (1.11) in the form: $A_i \circ A_j = A_{p(i,j)}$, where $p(i,j)$ is the unique index in I such that $A_i \times A_j$ is isomorphic to $A_{p(i,j)}$. Thus, we want to prove:

$$(1.15) \quad \underline{t}(A_{p(i,j)}) = \underline{t}(A_i) \circ \underline{t}(A_j),$$

where the multiplication on the right hand side of (1.15) is that given by (1.1). The proof hinges on the following simple lemma:

Lemma (1.16): Given algebras $A, B \in \underline{C}$, set:

$\{A_x : x \in X\}$ to be the family of subalgebras of A,

$\{B_y : y \in Y\}$ to be the family of subalgebras of B,

(both sets appropriately indexed). For each index $i \in I$, let:

$U_i(A \times B)$ be the family of subalgebras of $A \times B$ isomorphic to A_i,

$W_i(A \times B)$ be the family of subdirect subalgebras of $A \times B$ isomorphic to A_i.

Then, for all $i \in I$,

$$U_i(A \times B) = \bigcup_{(x,y)} W_i(A_x \times B_y).$$

Proof: Clearly, $U_i(A \times B) \supseteq \bigcup_{(x,y)} W_i(A_x \times B_y)$. On the other hand, if A' is an arbitrary subalgebra of $A \times B$ isomorphic to A_i, then A' is subdirect in $A_x \times B_y$, where $A_x = \pi_1(A')$ and $B_y = \pi_2(A')$. (Here π_1 and π_2 are the natural projections of $A \times B$ onto its first and second factors, respectively.) Consequently,

$$U_i(A \times B) \subseteq \bigcup_{(x,y)} W_i(A_x \times B_y). \quad \text{QED}$$

Proposition (1.17): The map $\underline{t}$ is a multiplication preserving map, $\underline{t} : Z\langle C, \underline{C}\rangle \to Z\langle C, \underline{D}\rangle$.

Proof: Expanding both sides of (1.15) we get:

$$\sum_m t(p(i,j),m)A_m = \sum_{k,n} t(i,k)t(j,n)(\sum_m A_m)$$

$$= \sum_m (\sum_{k,n} t(i,k)t(j,n)r(k,n;m))A_m.$$

Equating coefficients, we see that the required result is equivalent to the equation:

$$(1.18) \quad t(p(i,j),m) = \sum_{k,n} t(i,k)t(j,n)r(k,n;m),$$

for all $m \in I$. The left hand side of (1.18) is clearly the number of elements in $U_m(A_i \times A_j)$. On the other hand, a term of the form $t(i,k)t(j,n)r(k,n;m)$ counts the number of subalgebras (isomorphic to A_m) of $A_i \times A_j$ which are subdirect in $A_x \times B_y$, where $A_x \leqq A_i$, $B_y \leqq A_j$, with $A_x \cong A_k$ and $B_y \cong A_n$. This sum is equal (via lemma 1.16) to the number of elements in $U_m(A_i \times A_j)$. QED

It remains only to show that $\underline{t}$ is in fact an <u>isomorphism</u>. It suffices to show that $\underline{t}$ is invertible as a Z-linear map $\underline{t} : Z(C) \to Z(C)$; if it is invertible, then $\underline{t}^{-1}$ also preserves the multiplications involved. Towards this result, we define another partial order on C:

Set $A_i \leqq A_j$ if A_i is isomorphic to a subalgebra of A_j. Thus, $A_i \leqq A_j$ if and only if there exists an injective algebra homomorphism $A_i \to A_j$. The following are some easily verified assertions:

(1.19) a) $\leqq$ is a partial order on C.

b) The set $(A_i)^-$ defined by $(A_i)^- = \{A_j \in C : A_j \leqq A_i\}$, (the principal order ideal generated by A_i) is finite for all $A_i \in C$.

c) $t(i,j) \neq 0$ if and only if $A_j \leqq A_i$; thus, we may write $\underline{t}(A_i) = \sum \{t(i,j)A_j : (A_j \leqq A_i)\}$.

d) $t(i,i) = 1$ for all $i \in I$.

Because of (c) and (d) above, we can write:

$$\underline{t} = \underline{1} + \underline{u} \ ,$$

where $\underline{1}$ is the identity mapping on Z(C), and $\underline{u}$ is defined by:

$$(1.20) \qquad \underline{u}(A_i) = \sum_{A_j < A_i} t(i,j)\, A_j$$

where of course "$A_j < A_i$" means that $A_j \leq A_i$ but $A_j \neq A_i$.

Call a linear endomorphism $\underline{s}$: $Z(C) \to Z(C)$ locally nilpotent if for all $w \in Z(C)$ there exists a positive integer n such that $\underline{s}^n(w) = 0$. Clearly, for $\underline{s}$ to be locally nilpotent it suffices that $\underline{s}$ satisfy this condition on basis elements.

<u>Proposition (1.21):</u> The Z-linear endomorphism $\underline{u}$ defined by (1.20) is locally nilpotent.

<u>Proof:</u> Define the <u>support</u> of an element $w = \sum_i w(i)A_i$, denoted by supp(w), by:

$$\mathrm{supp}(w) = \{A_i \in C : w(i) \neq 0\}.$$

Thus, for example, the support of $\underline{t}(A_i)$ for any i is the order ideal $(A_i)^-$, and $\mathrm{supp}(\underline{u}(A_i)) = (A_i)^- - \{A_i\}$. Of course for any $w \in Z(C)$, supp(w) is a finite subset of C.

Now note that if A_i is a maximal element (under $\leq$) of supp(w), then $A_i \notin \mathrm{supp}(\underline{u}(w))$. Let $(\mathrm{supp}(w))^-$ denote the order ideal generated by by the set supp(w).

Since $(\mathrm{supp}(w))^-$ is the union of the prinipal order ideals $(A_i)^-$ for A_i in supp(w) (ie, a finite union of finite sets), it is clear that $(\mathrm{supp}(w))^-$ is also finite. But now if $w \neq 0$ (and hence $\mathrm{supp}(w) \neq \emptyset$), we have a proper containment

$$(\mathrm{supp}(w))^- \supset (\mathrm{supp}(\underline{u}(w)))^-,$$

since the maximal elements of supp(w) are not in $\mathrm{supp}(\underline{u}(w))$.

Consequently, the sequence:

$$(\mathrm{supp}(\underline{w}))^- \supset (\mathrm{supp}(\underline{u}(\underline{w})))^- \supset (\mathrm{supp}(\underline{u}^2(\underline{w})))^- \supset \ldots$$

is strictly decreasing until we hit a point at which $\underline{u}^n(w) = 0$, but since the sets involved are finite, this must occur after a finite number of steps. QED

<u>Proposition (1.22):</u> If $\underline{s}$ is a locally nilpotent linear mapping $Z(C) \rightarrow Z(C)$, then $\underline{1} + \underline{s}$ is invertible.

<u>Proof:</u> The inverse of $\underline{s}$ is given by the formula:

$$(\underline{1} + \underline{s})^{-1} = \underline{1} - \underline{s} + \underline{s}^2 - \underline{s}^3 + - \ldots.$$

Of course the sequence on the right is infinite, but it "converges" in the sense that when applied to any element w of Z(C) it yields only a finite number of non-zero terms. That it is the inverse of $(\underline{1} + \underline{s})$ is easily verified by applying $(\underline{1} + \underline{s})$ to the right hand side. QED

The above proposition shows that $\underline{t}$ is invertible, and thus our second demonstration of the associativity of the multiplication (1.1) is complete:

Since the multiplication in $Z\langle C, \underline{C}\rangle$ is associative, and since $Z\langle C, \underline{D}\rangle$ is isomorphic to $Z\langle C, \underline{C}\rangle$, it follows that the multiplication in $Z\langle C, \underline{D}\rangle$ is also associative.

***

Needless to say, the byproducts of our investigations are of greater interest than their putative object, the associativity of (1.1).

The results can be briefly summed up in the following commutative triangle of ring homomorphisms:

$$
\begin{tikzcd}
Z(C) \arrow[dr, "\underline{c}"] \arrow[dd, "\underline{t}"'] & \\
 & Z^C \\
Z(C) \arrow[ur, "\underline{d}"'] &
\end{tikzcd}
\tag{1.23}
$$

together with the fact that $\underline{t}$ is invertible.

Before we go on to apply these concepts to some special cases, there is one more formula that we wish to derive. Towards this objective, let us suppose that $\underline{t}^{-1}$ is given by the formula:

$$\underline{t}^{-1}(A_i) = \sum_j w(i,j)A_j$$

Then, it is not hard to derive the "inversion" of formula (1.18):

$$(1.24) \quad r(i,j;n) = \sum_{k,m} w(i,k)w(j,m)t(p(k,m),n).$$

The proof follows the same pattern as that of (1.18): expand the equation $\underline{t}^{-1}(A_i \circ A_j) = \underline{t}^{-1}(A_i) \circ \underline{t}^{-1}(A_j)$ (keeping in mind that the multiplication on the left of the equal sign is in $Z\langle C, \underline{D}\rangle$, while that on the right is in $Z\langle C, \underline{C}\rangle$), and then equate coefficients.

We shall now see what these results look like in one particular (and interesting) case.

2. Finite Sets:

We now turn our attention to the simplest possible class of finite algebras: those without any operations at all; i.e., finite sets without any additional structure.

Denote the class of all finite sets by $\underline{\mathbb{N}}$. A skeletal set for $\underline{\mathbb{N}}$ is given by:

$$N = \{A_1, A_2, \ldots, A_n, \ldots\},$$

where A_n is a set with n elements, say $A_n = \{1,2,\ldots,n\}$. (In line with our attitude that these are finite algebras, we exclude the empty set from consideration.)

In order to be able to interpret the results of the previous section in this context, we introduce some notation:

Let $q(i,j;k)$ = number of subdirect subsets of $A_i \times A_j$ of cardinality k. (This is perhaps more easily conceptualized as the number of i-by-j (0,1)-matrices which have a 1 in every row and column, and which contain precisely k 1's altogether as entries.) Then the multiplication (1.1) takes the form:

$$(2.1) \quad A_i \circ A_j = \sum_k q(i,j;k) A_k .$$

Denote the family of all surjective maps from a finite set A to a finite set B by $\underline{Q}[A, B]$, and set $Q(i,j)$ equal to $\#\underline{Q}[A_i, A_j]$; thus $Q(i,j)$ is the number of surjective maps from an i-set to a j-set. The quantity $Q(i,j)$ can be expressed in terms of more traditional combinatorial expressions as follows:

The number of ways of partitioning an i-set into j blocks is given by the Stirling number of the second kind $S(i,j)$. Alternatively, $S(i,j)$ is the number of distinct quotient sets of an i-set which are of cardinality j. Since every surjective map from an i-set to a j-set can be decomposed uniquely as the natural projection onto a quotient set (of cardinality j) followed by a set isomorphism, it is easily deduced that $Q(i,j) = j!S(i,j)$.

With this in mind, we know that the Z-linear map $\underline{q}_r\colon Z(N) \to Z$ defined by $\underline{q}_r(A_j) = Q(r,j) = \#\underline{Q}[A_r, A_j]$ is a ring homomorphism when $Z(N)$ is given the multiplication (2.1) . Consequently, applying $\underline{q}_r$ to both sides of (2.1) at once yields the following multiplicative identity for the quantities $Q(i,j)$:

$$\text{(2.2)}\quad Q(r,i)Q(r,j) = \sum_k q(i,j;k)Q(r,k).$$

A little fiddling with this (using the equation $Q(i,j) = j!S(i,j)$) then yields the corresponding multiplicative formula for the Stirling numbers of the second kind:

$$(2.3)\quad S(r,i)S(r,j) = \sum_k \{(q(i,j;k)k!)/(i!j!)\}S(r,k).$$

The following (easily verified) results are also worth mentioning:

a) $\underline{q}_i(A_j) = Q(i,j) = 0$ unless $i \geq j$. (The natural order on the integers corresponds to the order ">" of the previous section.)

b) $\underline{q}_i(A_i) = Q(i,i) = i!$

c) A_1 is the identity of the ring $Z\langle N, \underline{Q}\rangle$; thus in the sequel we shall generally write 1 instead of A_1.

d) $q(i,j;k) \neq 0$ if and only if $\max\{i,j\} \leq k \leq ij$;
$q(i,j;ij) = 1$.

e) If $i \geq j$, then $q(i,j;i) = Q(i,j)$. (<u>Proof:</u> Apply $\underline{q}_i$ to the equation $A_i \circ A_j = \sum_k q(i,j;k)A_k$ to get:

$$i!Q(i,j) = q(i,j;i)i!.$$

Alternatively, one may note that the only subsets of cardinality i that are subdirect in $A_i \times A_j$ are the "graphs" of surjective maps from A_i to A_j.)

Now let $\underline{N}[A, B]$ denote the family of <u>all</u> maps from A to B (A and B both finite). The multiplication derived from the direct product is of course simply:

$$(2.4) \qquad A_i \circ A_j = A_{ij}.$$

For each index i, we get the Z-linear map $\underline{n}_i$ defined by:

$$(2.5) \qquad \underline{n}_i(A_j) \;=\; \#\underline{N}[A_i, A_j] \;=\; j^i.$$

Thus $\underline{n}$: $Z(N) \to Z^N$ is the Z-linear map whose i-th component is $\underline{n}_i$; essentially, $\underline{n}(A_j)$ is the function (identifying A_i with the integer i) $i \mapsto j^i$.

This defines the ring $Z\langle N, \underline{N}\rangle$, which also has A_1 as its identity. The fact that $\underline{n}_k$ is a ring homomorphism for each index k reduces to the trivial equation $(ij)^k = i^k j^k$. The ring homomorphism $\underline{t}$: $Z\langle N, \underline{N}\rangle \to Z\langle N, \underline{Q}\rangle$, à la (1.13), is given by:

$$(2.6) \qquad \underline{t}(A_i) \;=\; \sum_{i \geq 1} C(i,j)A_j,$$

where C(i,j) is the number of combinations of j things out of i, or equally, the number of subsets of A_i of cardinality j .

(Note that writing 1 for A_1 and hence i for iA_1, this sum is $i + C(i,2)A_2 + \ldots + C(i,i-1)A_{i-1} + A_i$.)

Our basic commutative triangle means that $(\underline{q}_i \circ \underline{t})(A_k) = \underline{n}_i(A_k)$, which expanded becomes:

$$(2.7) \qquad \sum_{j\geq 1} C(k,j)Q(i,j) \quad = \quad \sum_{j\geq 1} C(k,j)j!S(i,j) = k^i,$$

a well-known identity involving the Stirling numbers $S(i,j)$ (cf. [R1], p.34).

Now using the fact that $\underline{t}$ is a ring homomorphism, and in particular equation (1.18), we get the interesting identity:

$$(2.8) \qquad C(ij,m) \quad = \sum_{k,r\geq 1} q(k,r;m)C(i,k)C(j,r),$$

since in this context the index $p(i,j)$ is simply ij. We can, with a little extra work, specialize (2.8) to the case $i = 2$ and obtain the following identity:

$$(2.9) \qquad C(2j,m) \quad = \sum_{r\geq(m/2)} C(r,m-r)2^{2r-m}C(j,r).$$

The proof of (2.9) follows from the following lemma:

<u>Lemma (2.10):</u>

a) $q(2,r;m) = C(r,m-r)2^{2r-m} \qquad \text{if } r < m \leq 2r;$

b) $q(2,r;r) = 2^r - 2$ (which is also $Q(r,2)$).

<u>Proof:</u> Interpret $q(2,r;m)$ as the number of 2-by-r

(0,1)-matrices such that every row and column contains a 1, and with precisely m entries equal to 1. In such a matrix there must be precisely (m - r) columns of the form $\binom{1}{1}$, while the remaining r - (m - r) = 2r - m columns must be of the form $\binom{1}{0}$ or $\binom{0}{1}$. Thus there are C(r,m-r) ways of choosing the $\binom{1}{1}$ columns, and 2^{2r-m} of choosing the remaining columns; and this gives us (a).

The above argument breaks down for the case m = r, when there are no columns of the form $\binom{1}{1}$. As before, there are 2r ways of choosing the columns, but two of these (namely $\begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0 & \cdots & 0\\ 1 & 1 & \cdots & 1\end{pmatrix}$) do not satisfy the conditions laid down. Hence, $q(2,r;r) = 2^r - 2$.
QED

Proof of (2.9):

Using (2.8), we can write:

$$C(2j,m) = \sum_{\substack{1\leq k\leq 2\\ 1\leq r\leq j}} q(k,r;m)C(2,k)C(j,r)$$

$$= \sum_{r\geq 1} 2q(1,r;m)C(j,r) + \sum_{r\geq 1} q(2,r;m)C(j,r).$$

Now, in the first place, q(1,r;m) = 0 unless r = m, in which case q(1,m;m) = 1; secondly, q(2,r;m) = 0 unless $r \leq m \leq 2r$; i.e., unless $(m/2) \leq r \leq m$. Finally substituting in the above via lemma (2.10) and simplifying slightly, we get (2.9). QED

Let us now turn again to the ring homomorphism $\underline{t}$; the inverse of a map of the form $\underline{t}(A_i) = \sum_{j \geq 1} C(i,j)A_j$ is well-known (cf. [R 2], p.44), and is given by:

$$(2.11) \quad \underline{t}^{-1}(A_i) = \sum_{j \geq 1} (-1)^{i-j} C(i,j) A_j$$

$$= \sum_{0 \leq k < i} (-1)^k C(i,k) A_{i-k}.$$

From our commutative triangle, we have that $(\underline{n}_i \circ \underline{t}^{-1})(A_k) = \underline{q}_i(A_k)$, which expanded gives us the identity:

$$(2.12) \quad \sum_{j \geq 1} (-1)^{k-j} C(k,j) k^j = Q(i,k) = k!S(i,k),$$

a standard way of defining $S(i,k)$ (cf. [Li], p.39). (Of course it is merely the inversion of (2.7).)

We can now apply equation (1.24) to derive a formula for the "structural constants" $q(i,j;n)$ of the ring $Z\langle N, \underline{Q}\rangle$. Noting that in this context $w(i,k) = (-1)^{i-k} C(i,k)$ and $p(k,m) = km$, we get:

$$(2.13) \quad q(i,j;n) = \sum_{\substack{k \geq 1 \\ m \geq 1}} (-1)^{(i+j)-(k+m)} C(i,k) C(j,m) C(km,n).$$

This is actually a fairly efficient formula, since the terms on the right hand side are non-zero only when $i \geq k$ and $j \geq m$ but $km \geq n$. Using the fact that $Q(i,j)$

$= j!S(i,j) = q(j,i;i)$ when $i \geq j$ leads via (2.13) to another formula for the Stirling numbers $S(i,j)$:

$$(2.14) \quad Q(i,j) = j!S(i,j)$$

$$= \sum_{k,m\geq 1} (-1)^{(i+j)-(k+m)} C(i,k)C(j,m)C(km,i),$$

when $i \geq j$.

Before proceeding with our investigations, we need to introduce another tool:

Let $X = \{x_i : i \in J\}$ be a family (finite or infinite) of "indeterminates". Then the ring $Z[X]$ of polynomials (with integer coefficients) in the commuting indeterminates x_i can be described as the free Z-module $Z(F[X])$, where $F[X]$ is the free commutative semigroup (with identity) on the set X, and $Z(F[X])$ has the multiplication derived from the multiplication in $F[X]$. The basis elements in $F[X]$ of the free Z-module $Z(F[X]) = Z[X]$ are of course simply the <u>monomials</u> in $Z[X]$. (In particular, the identity 1 in $Z[X]$ is considered a monomial of degree zero.)

As is well-known, the ring $Z[X]$ has the important "universal mapping property" that any map $\bar{f} : X \to R$, where R is any commutative ring with identity, extends to a unique (identity preserving) ring homomorphism

$f: Z[X] \to R$.

Usually if X is a singleton set $X = \{x\}$, then $Z[X]$ is denoted by $Z[x]$; similarly, if X is finite and is equal to $\{x_1, x_2, \ldots, x_n\}$, then $Z[X]$ is often denoted by $Z[x_1, x_2, \ldots, x_n]$.

Now if R is a commutative ring with identity and $A = \{a_i : i \in J\}$ is a family of (not necessarily distinct) elements of R also indexed by J, then a necessary and sufficient condition that R be generated by A (as a ring with identity) is that the ring homomorphism $f: Z[X] \to R$ determined by the map $x_i \mapsto a_i$ be surjective. If, on the other hand, f is injective, then the elements a_i are said to be algebraically independent. (Thus, $\{a_i\}$ forms an algebraically independent set if and only if there exists no polynomial relation connecting the a_i.)

If the set $\{a_i\}$ generates R and is also algebraically independent, then we shall call $\{a_i\}$ an algebraic basis for R. Under those conditions R is clearly isomorphic to $Z[X]$.

The ring $Z\langle N, \underline{N}\rangle$ provides an illustration of a ring with an algebraic basis:

For the rest of this section, $X = \{x_p : p \text{ a prime}\}$ will denote a family of indeterminates indexed by the rational primes ($p = 2, 3, 5, \ldots$). Define the ring homomorphism $\phi^S: Z[X] \to Z\langle N, \underline{N}\rangle$ by $x_p \mapsto A_p$. Since under multiplication the basis elements A_n of $Z\langle N, \underline{N}\rangle$ form a semigroup with identity isomorphic to the multiplicative semigroup of positive integers, and since that semigroup is freely generated by the prime numbers, it is easy to see that ϕ^S is an isomorphism. Hence, the set $\Phi = \{A_p : p \text{ a prime}\}$ forms an algebraic basis for $Z\langle N, \underline{N}\rangle$.

A more interesting question is: could it be that Φ forms an algebraic basis for $Z\langle N, \underline{Q}\rangle$? This question will be answered (eventually) in the affirmative. But first we will show that Φ generates $Z\langle N, \underline{Q}\rangle$ as a ring:

<u>Proposition (2.15):</u> For all $n \geq 2$, A_n is equal, in $Z\langle N, \underline{Q}\rangle$, to a polynomial without a constant term in the elements of Φ corresponding to the primes $\leq n$.

<u>Proof:</u> We proceed by induction on n. For $n = 2$, there is nothing to show since A_2 is an element of Φ. Suppose now that the proposition holds for all k such that $2 \leq k < n$. If n is prime, again there is nothing to show. Hence suppose n is composite, and write $n = pm$, where p is the smallest prime divisor of n. Then using the multiplication in $Z\langle N, \underline{Q}\rangle$ we may write:

$$A_p \circ A_m = A_n + \sum_{m \leq k \leq n-1} q(p,m;k)A_k$$

and therefore we have:

$$A_n = A_p \circ A_m - \sum_{m \leq k \leq n-1} q(p,m;k)A_k$$

By the induction hypothesis, A_m and each A_k is equal to a polynomial in the elements of Φ indexed by the primes less than n, and without a constant term. It follows by substitution that the same holds for A_n. QED

Notice that in the proof we have given an inductive procedure for constructing a sequence of polynomials SD_2, SD_3, SD_4,..., SD_n,... in Z[X] with the property that the substitution of x_p by A_p yields A_n as the value for SD_n. There is another point worth making about these polynomials:

Let m be a positive integer; we define the corresponding monomial g_m in Z[X] by taking the prime decomposition of m, and replacing each prime factor p in it by the indeterminate x_p. We set $g_1 = 1$. Note that $g_k g_m = g_{km}$. Indeed, every monomial in Z[X] is equal to g_m for some integer m, and the semigroup they form is isomorphic to the multiplicative semigroup of the positive integers. Thus, any polynomial f in Z[X] can be uniquely expressed in the form:

$$f = \sum_{k \geq 1} r(k) g_k .$$

Let us then write:

$$(2.16) \qquad SD_n = \sum_{k\geq 1} SD(n,k)g_k.$$

We can now note the following description of what SD_n looks like:

Proposition (2.17):

a) $SD(n,1) = 0$ for all $n \geq 2$,

b) for a prime p, $SD_p = x_p$,

c) $SD_n = \sum_{2\leq k\leq n} SD(n,k)g_k$;

that is, $SD(n,k) = 0$ for $k > n$;

d) $SD(n,n) = 1$.

Proof: (a) is merely a restatement of the fact that the polynomials SD_n have no constant term, and (b) is obvious. To show (c) and (d), just add them to the induction hypothesis in the proof of (2.15), and note that these properties are preserved by the construction employed. QED

For the sake of completeness, we set $SD_1 = 1$, so that $SD(1,1) = 1$, but $SD(1,n) = 0$ for $n \geq 2$.

In order to show that Φ forms an algebraic basis for $Z\langle N, \underline{Q}\rangle$, we must now make a short digression into ring theory:

A commutative ring R is <u>Noetherian</u> if it satisfies the ascending chain condition on ideals; or, to put it in another way, if there are no infinite strictly increasing sequences of ideals in R. As is well-known, Z is Noetherian.

The Hilbert Basis Theorem (viz. [F], p.16) asserts that if R is Noetherian, then so is the ring of polynomials over R in a finite number of indeterminates. The following results explore some of the properties of Noetherian rings of interest to us:

<u>Lemma (2.18):</u> Let R be a ring, and $\phi: R \to R$ a surjective endomorphism of R. Then, if $\phi^{-1}(0) \neq 0$, the sequence of ideals:

$$\{0\} \quad \phi^{-1}(0) \subset \phi^{-2}(0) \subset \ldots \subset \phi^{-r}(0) \subset \ldots,$$

where $\phi^{-r}(0) = \mathrm{Ker}(\phi^{r}) = \{a \in R : \phi^{r}(a) = 0\}$, is <u>strictly increasing</u> (i.e., the containments are all proper).

<u>Proof:</u> It is clear that this sequence of ideals is at least increasing. We shall show that the containment $\phi^{-(r-1)}(0) \subset \phi^{-r}(0)$ (where $\phi^{0}(0) = 0$) is proper for all $r \geq 1$ by induction. The case $r = 1$ is a proper inclusion by assumption.

Now suppose that $\phi^{-(r-1)}(0) \subset \phi^{-r}(0)$ is a proper inclusion, and choose an element b in the set difference $\phi^{-r}(0) - \phi^{-(r-1)}(0)$. Since ϕ is surjective,

$\phi^{-1}(b)$ is non-empty. Choose c in $\phi^{-1}(b)$. Then c is in $\phi^{-(r+1)}(0)$, but $c \notin \phi^{-r}(0)$; for $c \in \phi^{-r}(0)$ implies that $\phi(c) = b \in \phi^{-(r-1)}(0)$, a contradiction. Thus, $\phi^{-r}(0) \subset \phi^{-(r+1)}(0)$ is a proper inclusion. QED

<u>Corollary (2.19):</u> If R is a Noetherian ring, and $\phi: R \to R$ is a surjective ring endomorphism, then ϕ is an isomorphism.

The following gives a nice application of these ideas:

<u>Proposition (2.20):</u> Let R be the ring of polynomials over Z in a finite number of indeterminates; say,

$$R = Z[x_1, x_2, \ldots, x_n].$$

Suppose $\{f_1, f_2, \ldots, f_n\}$ is a set of elements of R which generate R as a ring. Then $\{f_1, f_2, \ldots, f_n\}$ is an algebraically independent set.

<u>Proof:</u> The map $x_i \mapsto f_i$ determines a ring homomorphism $\phi: Z[x_1, x_2, \ldots, x_n] \to Z[x_1, x_2, \ldots, x_n]$, ("replace x_i by f_i").

Since the f_i generate R as a ring, the map ϕ is surjective. But, by the Hilbert Basis Theorem, $Z[x_1, x_2, \ldots, x_n]$ is Noetherian, and therefore ϕ must be an isomorphism. QED

Of course the above proposition also applies to any ring isomorphic to a polynomial ring; that is, if a ring has an algebraic basis consisting of n elements, then any other n elements that generate the ring must also form an algebraic basis.

Now let us return to the consideration of the ring $Z\langle N, \underline{Q}\rangle$. Let $\phi^Q\colon Z[X] \to Z\langle N, \underline{Q}\rangle$ denote the ring homomorphism defined by the mapping $x_p \mapsto A_p$. For example, earlier we showed that $\phi^Q(SD_n) = A_n$, and that therefore ϕ^Q is surjective and Φ generates $Z\langle N, \underline{Q}\rangle$ as a ring.

It would be nice to apply the above results immediately to the ring $Z\langle N, \underline{Q}\rangle$, which we know is isomorphic to $Z[X]$, since it is isomorphic to $Z\langle N, \underline{N}\rangle$. Unfortunately, since $Z[X]$ is a polynomial ring in an infinite number of indeterminates, it is not Noetherian. We can, however, get around this problem in the following manner:

For each prime p, let Φ_p denote the family of elements $A_{p'}$ in Φ for which $p' \leq p$. Furthermore, let $Z^{(p)}\langle N, \underline{N}\rangle$ be the subring of $Z\langle N, \underline{N}\rangle$ generated by Φ_p, and $Z^{(p)}\langle N, \underline{Q}\rangle$ the subring of $Z\langle N, \underline{Q}\rangle$ generated by Φ_p. It is clear that Φ_p is an algebraic basis for $Z^{(p)}\langle N, \underline{N}\rangle$. We shall now show that the same is true for Φ_p in $Z^{(p)}\langle N, \underline{Q}\rangle$:

Proposition (2.21): For each prime p, the set Φ_p forms an algebraic basis for $Z^{(p)}\langle N, \underline{Q}\rangle$.

Proof: Consider the restriction of the ring homomorphism $\underline{t}$: $Z\langle N, \underline{N}\rangle \to Z\langle N, \underline{Q}\rangle$ to the subring $Z^{(p)}\langle N, \underline{N}\rangle$. The image of an element $A_{p'}$ in Φ_p under $\underline{t}$ is a linear combination of elements A_k for $k \leq p$. By proposition (2.15), these elements lie in $Z^{(p)}\langle N, \underline{Q}\rangle$. Hence, $\underline{t}$ defines a corresponding homomorphism $\underline{t}_p$ from $Z^{(p)}\langle N, \underline{N}\rangle$ to $Z^{(p)}\langle N, \underline{Q}\rangle$. (Basically, it is the restriction of $\underline{t}$ to $Z^{(p)}\langle N, \underline{N}\rangle$, except that we are also restricting the codomain of the map.) Similarly, it is easy to see that the image of $Z^{(p)}\langle N, \underline{Q}\rangle$ under $\underline{t}^{-1}$ is contained in $Z^{(p)}\langle N, \underline{N}\rangle$. Thus it defines a corresponding homomorphism from $Z^{(p)}\langle N, \underline{Q}\rangle$ to $Z^{(p)}\langle N, \underline{N}\rangle$ which we denote by $\underline{t}_p^{-1}$. It is clear that $\underline{t}_p$ and $\underline{t}_p^{-1}$ are inverses of each other. Consequently, $Z^{(p)}\langle N, \underline{Q}\rangle$ also has an algebraic basis consisting of the same number elements as there are in Φ_p; but since by definition Φ_p generates $Z^{(p)}\langle N, \underline{Q}\rangle$, it must also be an algebraic basis for $Z^{(p)}\langle N, \underline{Q}\rangle$. QED

From the above proposition we also see that Φ_p is an algebraically independent set in $Z\langle N, \underline{Q}\rangle$ for all primes p. But since every finite set of the elements $A_{p'}$, p' a prime, is contained in Φ_p for sufficiently large p, it follows that every finite subset of Φ is algebraically independent, and therefore that Φ is an algebraic basis for $Z\langle N, \underline{Q}\rangle$.

We state this fact as a proposition, together with some immediate corollaries:

Proposition (2.22): Let $X = \{x_p : p \text{ a prime}\}$ be a family of indeterminates in a one-one correspondence with the rational primes. Then the ring homomorphism

$$\phi^Q: Z[X] \to Z\langle N, \underline{Q}\rangle$$

defined by the map $x_p \mapsto A_p$ is a ring isomorphism.

That is, the set $\Phi = \{A_p : p \text{ a prime}\}$ forms an algebraic basis for $Z\langle N, \underline{Q}\rangle$.

Corollary (2.23): Every element of $Z\langle N, \underline{Q}\rangle$ is expressible as a unique polynomial with integer coefficients in the elements A_p, p prime. In particular, SD_n is the unique polynomial with "value" A_n in $Z\langle N, \underline{Q}\rangle$ under the substitution $x_p \mapsto A_p$ (i.e., the unique polynomial such that $\phi^Q(SD_n) = A_n$).

For the sake of illustration, we give SD_4 and SD_6 explicitly:

$$SD_4 = x_2^2 - 4x_3 - 2x_2$$

$$SD_6 = x_2x_3 - 6x_5 - 12x_2^2 + 42x_3 + 24x_2$$

Thus, in $Z\langle N, \underline{Q}\rangle$:

$$A_4 = A_2^2 - 4A_3 - 2A_2$$

$$A_6 = A_2 \circ A_3 - 6A_5 - 12A_2^2 + 42A_3 + 24A_2$$

Now, as noted earlier, $\underline{q}$ is a faithful ring homomorphism $Z\langle N, \underline{Q}\rangle \rightarrow Z^N$; that is, an isomorphism with its image in Z^N. Under this isomorphism, A_n is essentially mapped into the function $Q(-,n)$ (i.e., $\underline{q}(A_n)$ is the function $A_k \mapsto \#\underline{Q}[A_k, A_n] = Q(k,n)$). Thus, we at once have the following result:

<u>Corollary (2.24):</u> For all $n \geq 1$, the function $Q(-,n)$ is expressible as a unique polynomial with integer coefficients (namely, SD_n) in the functions $Q(-,p)$ for primes $p \leq n$.

(For example:

$$Q(-,4) = Q(-,2)^2 - 4Q(-,3) - 2Q(-,2).)$$

<u>Corollary (2.25):</u> For all $n \geq 1$, the "n-th Stirling function" $S(-,n)$ (defined as the mapping $k \mapsto S(k,n)$) is expressible as a unique polynomial (with rational coefficients) in the Stirling functions $S(-,p)$ for primes $p \leq n$.

To arrive at the polynomials of corollary (2.25), one makes use of the fact that $n!S(-,n) = Q(-,n)$.

***

Knowing the existence of the polynomials SD_n, one would like to have at hand some convenient way of calculating them. The inductive procedure described in proposition (2.15) is quite unwieldy in that it requires a knowledge of the structural constants $q(i,j;k)$. The following describes a more efficient and elegant procedure for finding the polynomials SD_n:

According to our results above, Z[X] is isomorphic to $Z\langle N, \underline{Q}\rangle$ under ϕ^Q, with the polynomial SD_n corresponding to A_n; and Z[X] is also isomorphic to $Z\langle N, \underline{N}\rangle$ under ϕ^S, but with A_n now corresponding to the monomial g_n.

Now look at the ring isomorphism $\underline{t}: Z\langle N, \underline{N}\rangle \to Z\langle N, \underline{Q}\rangle$. Since it is a ring isomorphism, the elements:

$$\underline{t}(A_n) = \sum_{k\geq 1} C(n,k)A_k$$

have the property in $Z\langle N, \underline{Q}\rangle$ that $\underline{t}(A_m) \circ \underline{t}(A_n) = \underline{t}(A_{mn})$. Thus, if one transfers this result (via ϕ^Q)

to Z[X] by defining the polynomial sd_n by:

$$(2.26) \qquad sd_n = \sum_{k \geq 1} C(n,k) SD_k,$$

then one immediately has the result that:

$$(2.27) \qquad sd_m sd_n = sd_{mn}.$$

Define the quantities sd(n,k) by means of the equation:

$$sd_n = \sum_k sd(n,k) g_k.$$

Then from (2.26) one deduces that $sd_1 = 1$, while for $n \geq 2$ we have:

$$(2.28) \qquad sd(n,k) = \sum_j C(n,j) SD(j,k),$$

in which the summation only takes place over $k \leq j \leq n$, since $SD(j,k) = 0$ if $j < k$. Thus, it is also true that $sd(n,k) = 0$ if $n < k$. Additionally, one can also easily deduce that $sd(n,n) = SD(n,n) = 1$, while $sd(n,1) = n$. (Recall that $SD_1 = 1$ by definition, while SD_n contains no constant term; so that (2.26) can be written as :

$$sd_n = n + \sum_{k \geq 2} C(n,k) SD_k.)$$

On the other hand, if we interpret equation (2.27) in terms of the coefficients sd(n,k) (using the fact that $g_j g_k = g_{jk}$), we get:

$$(2.29) \quad sd(mn,r) = \sum_{\substack{j,k \\ jk=r}} sd(m,j)sd(n,k).$$

Finally, we note that the relation (2.26) can be inverted to obtain:

$$(2.30) \qquad SD_n = \sum_{k\geq 1} (-1)^{n-k} C(n,k) sd_k.$$

In terms of the quantities SD(n,k) and sd(n,k), this becomes for $n \geq 2$:

$$(2.31) \quad SD(n,j) = \sum_{k\leq j\leq n} (-1)^{n-k} C(n,k) sd(k,j).$$

These relations form the basis of a simple inductive procedure for jointly computing the polynomials SD_n and sd_n:

Suppose that SD_k and sd_k are known for $1 \leq k \leq n-1$. There are then two possibilities for n: either n is prime; or n is composite and hence we can write $n = rs$, where r and s are proper factors of n.

If n is prime, then $SD_n = x_n$, and we can use (2.26) to compute sd_n. On the other hand if n is not prime, then we can compute sd_n as $sd_n = sd_r sd_s$, and then use (2.30) to calculate SD_n.

Let us illustrate the process:

$$SD_2 = x_2 , \quad SD_3 = x_3;$$

thus $sd_2 = SD_2 + 2SD_1 = x_2 + 2,$

and $sd_3 = SD_3 + 3SD_2 + 3SD_1 = x_3 + 3x_2 + 3.$

$$sd_4 = sd_2^{\,2} = (x_2 + 2)^2 = x_2^{\,2} + 4x_2 + 4;$$

thus $SD_4 = sd_4 - 4sd_3 + 6sd_2 - 4$

$\quad = x_2^{\,2} - 4x_3 - 2x_2.$

$$SD_5 = x_5;$$

thus $sd_5 = SD_5 + 5SD_4 + 10SD_3 + 10SD_2 + 5$

$\quad = x_5 + 5x_2^{\,2} - 10x_3 + 5.$

$$sd_6 = sd_2 sd_3 = (x_2 + 2)(x_3 + 3x_2 + 3)$$

$$= x_2x_3 + 3x_2^{\,2} + 2x_3 + 9\ x_2 + 6;$$

thus $SD_6 = sd_6 - 6sd_5 + 15sd_4 - 20sd_3 + 15sd_2 - 6$

$\quad = x_2x_3 - 6x_5 - 12x_2^{\,2} + 42x_3 + 24x_2,$

and so on.

The procedure described above for recursively defining the polynomials SD_n and sd_n can be made the basis of a simple computer program for calculating the coefficients SD(n,k) and sd(n,k). Partial results from one such program are displayed in tables I(a) and I(b).

Table I(a), SD(n,k):

n/k	2	3	4	5	6	7	8	9	10	11	12
4	-2	-4	1								
5	0	0	0	1							
6	24	42	-12	-6	1						
7	0	0	0	0	0	1					
8	-548	-952	272	112	-28	-8	1				
9	3150	5466	-1557	-630	174	36	-9	1			
10	-11500	-19940	5650	2270	-700	-120	50	-10	1		
11	0	0	0	0	0	0	0	0	0	1	
12	316032	547716	-154392	-61908	20860	3168	-1812	440	-66	-12	1

Note: $SD_1 = 1$, $SD_2 = x_2$, $SD_3 = x_3$.

Table I(b), sd(n,k):

n/k	2	3	4	5	6	7	8	9	10	11	12
2	1										
3	3	1									
4	4	0	1								
5	0	-10	5	1							
6	9	2	3	0	1						
7	119	189	-49	-21	7	1					
8	12	0	6	0	0	0	1				
9	18	6	9	0	6	0	0	1			
10	5	-20	10	2	-10	0	5	0	1		
11	-33187	-57541	16181	6490	-2288	-330	220	-55	11	1	
12	24	4	15	0	4	0	3	0	0	0	1

Note: sd(n,1) = n for all n.

3. Boolean algebras:

It should be apparent to the reader that there is a distinct category-theoretic flavour to our work. In the general theory developed in section 1 of this chapter, we were concerned in fact with categories of finite algebras; indeed, a good deal of the theory simply involved the act of counting morphisms within a given category. In section 2 we particularized matters to the category of finite sets and mappings (which we shall henceforth denote by $\underline{N}$), and to the subcategory of finite sets and surjective mappings (which we shall denote by $\underline{Q}$). It is our intention in this section to reinforce this impression.

We will apply the ideas of section 1 to the category of finite boolean algebras and boolean algebra homomorphisms (which we shall henceforth denote by $\underline{BA}$) together with the subcategory of finite boolean algebras and surjective boolean algebra homomorphisms (which in this section will be denoted by $\underline{D}$). Our method of attack, however, will consist in utilizing the fact that $\underline{BA}$ is "dual" (in the sense of category theory) to $\underline{N}$ to transport our considerations back to the consideration of sets and mappings. In doing so, we will be able to show that there is indeed a dual aspect to our ideas which has not yet become apparent.

Our first order of business is to describe (succinctly) the duality between BA and N:

As an algebra, a boolean algebra has five operations: the two binary operations of meet and join (set $\vee$ = join and $\wedge$ = meet), one unary operation of complementation (denoted by ') and two nullary operations that give the distinguished elements 0 and 1 of the boolean algebra. Of course a boolean algebra homomorphism must preserve all five operations.

We have a (contravariant) functor PW: $\underline{N} \to \underline{BA}$ ("power set functor") which can be defined as follows:

Given any finite set A, we have the finite boolean algebra PW(A) of all subsets of A, in which the meet operation is given by set intersection, the join operation by set union, complementation by set complementation in A, and of which the 0 and 1 are given by the empty set Ø and A itself, respectively. If f: $A \to B$ is a mapping between finite sets, then the corresponding boolean algebra homomorphism $PW(f) = f^*: PW(B) \to PW(A)$ is defined by letting $f^*(Y) = f^{-1}(Y)$ = pre-image of Y under f, for all $Y \in PW(B)$.

The functor PW is in fact a (dual) equivalence of categories, but its "inverse" (so to speak) is not quite as easy to describe:

Any boolean algebra is a poset if one defines $b \leq c$ to mean $b \wedge c = b$ (or equivalently, $b \vee c = c$). An element x in B is an <u>atom</u> if it covers 0; that is, if $0 < x$ ($0 \leq x$ and $0 \neq x$), and there is no element between 0 and x. If B is finite, it is clear that every element of B besides 0 "contains" an atom; that is, for all $b \in B$, $b \neq 0$, there is an atom x such that $x \leq b$. For $B \in$ ob <u>BA</u>, let M(B) denote the family of atoms in B. (Note: For any category <u>C</u>, we shall denote the object class of <u>C</u> by ob <u>C</u>, and the morphism class of <u>C</u> by mor <u>C</u>.) For $b \in B$, let $M_b(B)$ denote the set of atoms in B which b contains. (If $b = 0$, then $M_b(B)$ is the empty set.) As is well-known, b is completely determined in B by the set $M_b(B)$. To be more precise, every element b in B is uniquely expressible as a join of atoms, namely the atoms in $M_b(B)$. Thus, for elements $b, b' \in B$, $b = b'$ if and only $M_b(B) = M_{b'}(B)$. (Note: To be quite correct, one must adopt the convention that the join of an empty set of elements from B is 0.)

We wish now to show that M extends to a contravariant functor <u>BA</u> $\rightarrow$ <u>N</u>. For this we need a few more results, which happen to apply to a wider setting than that of boolean algebras. Thus, suppose that L_1 and L_2 are finite lattices, and suppose that $f:L_1 \rightarrow L_2$

is a mapping which preserves meets and 1's. (Since L_1 and L_2 are finite, each has both a 0 and 1, given respectively by the meet and the join of the elements of each.) We then have the following simple lemma:

Lemma (3.1): Let L_1, L_2 and f be as described above. Then, for each $b \in L_2$ there is a unique element $f^\wedge(b) \in L_1$ such that, for all $c \in L_1$, we have the relation:

$$f(c) \geq b \text{ if and only if } c \geq f^\wedge(b).$$

The map $f^\wedge: L_2 \to L_1$ so defined satisfies:

a) $f^\wedge(0) = 0$;

b) $f^\wedge$ is join preserving.

Proof: Set $U_f(b) = \{d \in L_1 : f(d) \geq b\}$. Then $U_f(b)$ is non-empty since it contains $1 \in L_1$. It is easy to see that $U_f(b)$ is closed under meets, and hence it contains a least element, namely the meet of all the elements in $U_f(b)$; this we set equal to $f^\wedge(b)$. Then it satisfies the given relation by its very definition.
Statement (a) is also immediate.

For $c \in L_1$, let $(c)^+ = \{d \in L_1 : d \geq c\}$; i.e., $(c)^+$ is the "order co-ideal in L_1 generated by c". It is easy to show that $(c_1)^+ \cap (c_2)^+ = (c_1 \vee c_2)^+$. On the other hand, we have essentially shown that $U_f(b) = (f^\wedge(b))^+$, and it is easy to prove that $U_f(b_1 \vee b_2) = U_f(b_1) \cap U_f(b_2)$. Thus (b) follows immediately. QED

(As an aside, it is interesting to note that the above has a categorical interpretation. Any poset may be interpreted as being a category in its own right, with an "arrow" from b to c if $b \leq c$. Then, meets are categorical products, while joins are coproducts. An order-preserving map is a functor. The relation between f and $f^\wedge$ given in the above lemma simply asserts that $f^\wedge$ is a left adjoint, and the fact that $f^\wedge$ preserves joins can then be viewed as an instance of the fact that left adjoints always preserve coproducts.)

Proposition (3.2): Let $f: B_1 \to B_2$ be a homomorphism of finite boolean algebras. Then if x is an atom in B_2, $f^\wedge(x)$ is an atom in B_1.

Proof: First note that $f^\wedge(x) \neq 0$ (since the smallest element y satisfying $f(y) \geq 0$ is zero itself). We can then express $f^\wedge(x)$ as a join of atoms, say:

$$f^\wedge(x) = y_1 \vee y_2 \vee \cdots \vee y_r, \quad y_i \in M(B_1).$$

But, by definition of $f^\wedge$, this implies that

$$x \leq f(y_1) \vee \ldots \vee f(y_r).$$

(We also used the fact that f is join preserving.)
But since x is an atom, this means that for some i we have that $x \leq f(y_i)$, and thus $f^\wedge(x) \leq y_i$. But since y_i

is an atom, and $f^\wedge(x) \neq 0$, we must have $f^\wedge(x) = y_i$.
QED

Now, given a homomorphism $f: B_1 \to B_2$, we define $M(f): M(B_2) \to M(B_1)$ by setting $M(f)(x) = f^\wedge(x)$ for all $x \in M(B_2)$. It is easy to show that M so defined is indeed a (contravariant) functor.

The verification that the pair (PW, M) defines an eqivalence between the categories $\underline{BA}$ and $\underline{N}$ now rests on noting that for all A in ob $\underline{N}$, we have a natural isomorphism between A and M(PW(A)) (the atoms in PW(A) are simply the singleton subsets of A), and similarly, for B in ob $\underline{BA}$, a natural isomorphism between PW(M(B)) and B (under the one-one correspondence between subsets of M(B) and elements of B). We omit the details, which are elementary.

This dual equivalence shows that every finite boolean algebra "looks like" the power set of a finite set, and every boolean algebra homomorphism "looks like" a map of the form $f^*: PW(A) \to PW(B)$ for some function $f: B \to A$ between finite sets. In particular, if $N = \{A_0, A_1, \ldots, A_k, \ldots\}$ is a skeletal set for $\underline{N}$, with A_k being a k-element set, then $\{B_0, B_1, \ldots, B_k, \ldots\}$, where $B_k = PW(A_k)$, is a skeletal set for $\underline{BA}$. (Note that in this context we do include the empty set as an object of $\underline{N}$, unlike the previous section.) Since PW is a dual equivalence, it establishes a one-one correspondence between $\underline{BA}[B_i, B_k]$ and $\underline{N}[A_k, A_i]$; in particular $\#\underline{BA}[B_i, B_k] = \#\underline{N}[A_k, A_i] = i^k$. Also, under PW the product in $\underline{BA}$ corresponds to the coproduct in $\underline{N}$, which is given by the disjoint union operation on sets, and which we shall denote by "\/".

Now, it is possible to "linearize on the left" of $\underline{N}$ in the same way that we have "linearized on the right" previously. That is, we again form the free Z-module Z(N), but this time define Z-linear maps $\underline{n}_i^*: Z(N) \to Z$ on basis elements by:

$$\underline{n}_i^*(A_k) = \#\underline{N}[A_k, A_i]. \tag{3.3}$$

As before, the collection of all such maps defines a Z-linear map $\underline{n}^*: Z(N) \to Z^N$. Similarly, we turn Z(N) into a ring by using the coproduct (i.e., disjoint union) in the category $\underline{N}$; thus, multiplying A_j and A_k yields the unique element of N isomorphic to $A_j \vee A_k$, namely A_{j+k}. Then it is easy to see that the maps $\underline{n}_i^*$ (and hence $\underline{n}^*$) are ring homomorphisms. The resultant structure (consisting of Z(N), the multipllication derived from the coproduct, and the ring homomorphism $\underline{n}^*$) we shall denote by $Z\langle N, \underline{N}^*\rangle$. This structure, however, because of the duality outlined above, is seen to be completely isomorphic to the right linearization $Z\langle \underline{BA}, \underline{BA}\rangle$ of $\underline{BA}$.

Now, from a combinatorial viewpoint, the subdirect products of two boolean algebras are much more interesting than their direct product. As is generally true, the family of subdirect products of a pair of boolean algebras is connected with the category $\underline{D}$ of boolean algebras and surjective boolean algebra homomorphisms. At this point, we can again make use of duality. It is not difficult to show that a homomorphism $PW(f)\colon PW(B) \to PW(A)$ is surjective if and only if the map $f\colon A \to B$ is injective; and similarly, $PW(f)$ is injective if amd only if f is surjective. Consequently, one sees that under our dual equivalence the subcategory of $\underline{N}$ corresponding to $\underline{D}$ is that of finite sets and injective maps, which we shall denote by $\underline{P}$. Thus, $\underline{P}[A,B]$ will denote the family of all injective maps from A to B. The reason for this notation is that we have:

$$(3.4) \qquad \#\underline{P}[A_k, A_n] = P(n,k),$$

where $P(n,k) = n(n-1)\ldots(n-k+1)$ is a standard symbol for the number of "r-permutations of n objects" (cf. [R1], p.2). (Note, however, the reversal of order of the indices k and n.)

Given this duality, we naturally ask what in the category $\underline{P}$ corresponds to subdirect products in $\underline{D}$. The answer to this question is not hard to give. A subdirect product of two boolean algebras is a subalgebra of the direct product with the property that the natural projections, restricted to it, are surjective. The dual notion for two sets A and B, then, is a quotient of the disjoint union $A \vee B$ such that the composition of the natural injections $\iota_1 : A \to A \vee B$ and $\iota_2 : B \to A \vee B$ with the natural map of $A \vee B$ onto the quotient, remains injective.

One constructs such a quotient in the following manner: Let X and Y be subsets of A and B respectively with the same cardinality, and let $g: X \to Y$ be a one-one correspondence. Let $A \vee_g B$ be the quotient space of $A \vee B$ obtained by identifying each point x in X with the corresponding point g(x) in Y. If $h: A \vee B \to A \vee_g B$ is the natural surjection, then it is clear that the maps $h \circ \iota_1$ and $h \circ \iota_2$ (which in the sequel we shall denote by ι_1^g and ι_2^g respectively) are injective. Any quotient of $A \vee B$ with this property is in fact determined by such a one-one correspondence between subsets of A and B.

Now let W(A,B) denote the family of all such one-one correspondences $g: X \to Y$ between respective subsets of A and B. Then the set $\{A \vee_g B : g \in W(A,B)\}$ of all such "amalgamations" of A and B is universal with respect to injective maps in the same way as the set of subdirect algebras of a product of two algebras is universal with respect to surjective algebra homomorphisms (except that all "arrows" are reversed). To be precise, if $f_1: A \to C$ and $f_2: B \to C$ are both injective maps, then there is a unique g in W(A,B) and a unique <u>injective</u> map $f: A \vee_g B \to C$ such that $f \circ \iota_1^g = f_1$ and $f \circ \iota_2^g = f_2$. This can be seen by noting that the overlap of the images of A and B in C under f_1 and f_2 respectively, defines a one-one correspondence g between the subsets X and Y (of A and B respectively) which correspond to this overlap. We use the family of such amalgamations to define a multiplication in Z(N) by writing:

$$(3.5) \qquad A_j \circ A_k = \sum_r p(j,k;r)A_r,$$

where p(j,k;r) is simply the number of amalgamations of A and B isomorphic to A_r (i.e., of cardinality r).

Through duality, it is not hard to show that there is a natural one-one correspondence between amalgamations of A_j and A_k and subdirect products of B_j and B_k. Thus, if we replace A_j, A_k, and A_r in (3.5) by B_j, B_k, and B_r respectively, we have the multiplication derived from subdirect products in the right linearization $Z\langle BA, \underline{D}\rangle$ of $\underline{D}$. Of course the ring homomorphisms $\underline{d}_i$: $Z(BA) \to Z$ defined by $\underline{d}_i(B_j) = \#\underline{D}[B_i, B_j]$ correspond entirely to the Z-linear maps $\underline{p}_i^*$: $Z(N) \to Z$ defined by $\underline{p}_i^*(A_j) = \#\underline{P}[A_j, A_i] = P(i,j)$; consequently, the maps $\underline{p}_i^*$ (together with the map $\underline{p}^*$ from $Z(N)$ to Z^N defined in terms of the $\underline{p}_i^*$) are ring homomorphisms (assuming $Z(N)$ is given the multiplication (3.5)). Of course this can be proved directly. This entire "right linearization" of $\underline{P}$ is (following our usual convention) denoted by $Z\langle N, \underline{P}^*\rangle$. Of course, it is entirely isomorphic to the right linearization $Z\langle BA, \underline{D}\rangle$ of $\underline{D}$.

Continuing to use duality, since subalgebras of B_j correspond to quotient sets of A_j, it is not hard to see that the ring homomorphism $Z\langle BA, \underline{BA}\rangle \to Z\langle BA, \underline{D}\rangle$ becomes the ring homomorphism $\underline{t}$: $Z\langle N, \underline{N}^*\rangle \to Z\langle N, \underline{P}^*\rangle$ defined by:

$$\underline{t}(A_j) = \sum_k S(j,k)A_k, \tag{3.6}$$

where $S(j,k)$ is the indicated Stirling number of the second kind, since $S(j,k)$ counts the number of quotient sets of a j-set which are of cardinality k. We then have the commutative diagram:

$$(3.7)\qquad \begin{tikzcd}
Z\langle N, \underline{N}^{*}\rangle \arrow[dd, "\underline{t}"'] \arrow[dr, "\underline{n}_i^{*}"] & \\
& Z \\
Z\langle N, \underline{P}^{*}\rangle \arrow[ur, "\underline{p}_i^{*}"'] &
\end{tikzcd}$$

for each index i; and expanding $\underline{p}_i^{*} \circ \underline{t}(Aj) = \underline{n}_i^{*}(Aj)$ then yields the identity:

$$(3.8)\qquad \sum_k S(j,k)P(i,k) = i^j,$$

which, since $P(i,k) = C(i,k)k!$, is the same as (2.7). (But note the considerable difference in interpretation of the same identity under the two contexts.)

Let us investigate what other results of combinatorial interest can be obtained from this line of thought. First of all, let us compute the structural constants p(i,j;k). To do so, consider the number of ways it is possible to form amalgamations of A_i and A_j by identifying an r-subset of A_i with an r-subset of A_j:

Since there are C(i,r) ways of picking such a subset of A_i, C(j,r) of picking an r-subset of A_i, and r! ways of identifying the two subsets, this number is C(i,r)C(j,r)r!. The resulting amalgamation, however, has cardinality i + j - r. Consequently, we have that

$$p(i,j;i+j-r) = C(i,r)C(j,r)r!,$$

whence the substitutions $k = i + j - r$, $r = i + j - k$, give us the formula:

$$(3.9) \quad p(i,j;k) = C(i,i+j-k)C(j,i+j-k)(i+j-k)!.$$

Now applying $\underline{p}_i^*$ to both sides of (3.5) yields the following multiplicative formula for the quantities $P(i,j)$:

$$(3.10) \qquad P(i,j)P(i,k) = \sum_r p(j,k;r)P(i,r).$$

This identity is known (cf. [R2], p.15), usually in the form of a corresponding multiplicative identity for the binomial coefficients $C(i,j)$, which can be obtained from (3.10) by using the identity $P(i,j) = C(i,j)j!$.

The fact that $\underline{t}$ is a ring homomorphism allows us to apply equation (1.18); in this context $p(i,j) = i + j$, and the formula then gives the interesting identity:

$$(3.11) \quad S(i+j,m) = \sum_{k,n} p(k,n;m)S(i,k)S(j,n).$$

Now, it is easy to see that in $Z\langle N, \underline{N}^*\rangle$ A_0 is the identity 1, while $A_1{}^n = A_n$. From this we see that $Z\langle N, \underline{N}^*\rangle$ is isomorphic to the polynomial ring $Z[x]$ under the homomorphism defined by mapping the indeterminate x to A_1. Under this isomorphism, A_n

corresponds to the monomial x^n, and $\underline{n}_k^*$ to evaluation at $x = k$.

It is also not hard to see that A_1 generates $Z\langle N, \underline{P}^*\rangle$:

Note that $A_1 \circ A_n = A_{n+1} + nA_n$, so that in $Z\langle N, \underline{P}^*\rangle$ $A_{n+1} = A_n \circ (A_1 - n)$. By induction it is then clear that:

$$A_n = A_1 \circ (A_1-1) \circ (A_1-2) \circ \ldots \circ (A_1-n+1).$$

Thus it follows that $Z\langle N, \underline{P}^*\rangle$ is also isomorphic to $Z[x]$ under the map determined by sending x to A_1 and that under this isomorphism A_n corresponds to the polynomial $(x)_n = x(x-1)(x-2)\ldots(x-n+1)$. (Note that therefore equation (3.5) can also be interpreted as a formula for multiplying the "falling factorials" $(x)_n$.)

Finally, let us look again at the map $\underline{t}$. We know that it is invertible; we can, with a little work, give an explicit description of $\underline{t}^{-1}$. Define the map $\underline{s}$: $Z\langle N, \underline{P}^*\rangle \to Z\langle N, \underline{N}^*\rangle$ on basis elements by setting

$$\underline{s}(A_n) = A_1 \circ (A_1-1) \circ (A_1-2) \circ \ldots \circ (A_n-n+1),$$

where the multiplication on the right hand side of the above equation is in $Z\langle N, \underline{N}^*\rangle$ (<u>not</u> in $Z\langle N, \underline{P}^*\rangle$). Thus, we have that:

$$(3.12) \qquad \underline{s}(A_n) = \sum_k s(n,k)A_k,$$

where $s(n,k)$ above is the designated Stirling number of the first kind (which by definition (viz. [R1], p.33) is the coefficient of x^k in the expansion of $(x)_n = x(x-1)\ldots(x-n+1))$.

Now it is clear that $\underline{t}(A_1) = A_1$ (since a one element set has no quotients other than itself), and of course $\underline{t}(A_0) = \underline{t}(1) = 1$. Since $\underline{t}$ is a ring homomorphism, we can calculate the composition $(\underline{t} \circ \underline{s})(A_n)$ as follows:

$$\begin{aligned} \underline{t}(\underline{s}(A_n)) &= \underline{t}(A_1 \circ (A_1-1) \circ \ldots \circ (A_1-n+1)) \\ &= \underline{t}(A_1) \circ (\underline{t}(A_1)-1) \circ \ldots \circ (\underline{t}(A_1)-n+1) \\ &= A_1 \circ (A_1-1) \circ \ldots \circ (A_1-n+1) \end{aligned}$$

in which last expression the multiplication is now in $Z\langle N, \underline{P}^*\rangle$, and in that ring the expression is equal to A_n. Thus, $(\underline{t} \circ \underline{s})(A_n) = A_n$ for all basis elements A_n, whence $\underline{s} = \underline{t}^{-1}$. We have thereby proved the well-known result (viz. [R1], p.34) that the Stirling numbers of the first and second kind are related to each other as the coefficients of inverse transformations.

4. Further Remarks:

It should now be clear that the proper setting for the techniques used in this chapter lies in category theory. Let us isolate the particular class of categories which can be studied by these methods:

One of the things we have been doing is simply counting; and in particular, we have been counting the family of morphisms from one object of the category in question to another. In order to do this, we require that this family be finite.

A second thing we have assumed is that it is possible to form a set which picks one representative out of each isomorphism class of objects in the category. In order to be able to do this we must assume that the category in question is <u>skeletally small</u>. This concept can be explicated as follows:

If $\underline{C}$ is a category, then a subcategory $\underline{D}$ is a <u>skeleton</u> for $\underline{C}$ if every object in ob $\underline{C}$ is isomorphic to precisely one object in ob $\underline{D}$. A category is <u>small</u> if its object class is a set. (Then its morphism class also forms a set.) Thus, a category is skeletally small if it has a small skeleton. If $\underline{C}$ is skeletally small, we shall call the object class of a skeleton of

$\underline{C}$ a skeletal set for $\underline{C}$.

Definition (4.1): Call a category $\underline{C}$ finitary if it satisfies the following two conditions:

a) $\underline{C}$ is skeletally small,

b) for all objects A, $B \in$ ob $\underline{C}$, the set $\underline{C}[A, B]$ (of all morphisms in $\underline{C}$ from A to B) is finite.

Given a finitary category $\underline{C}$, one can already construct its "right linearization". That is, choose a skeletal set $C = \{A_i : i \in I\}$ for $\underline{C}$ and form the free Z-module $Z(C)$, and then define the Z-linear map $\underline{c}$: $Z(C) \to Z^C$ (as we already have done in our earlier examples) as the map whose "i-th component" is defined on basis elements by:

$$\underline{c}_i(A_j) = \#\underline{C}[A_i, A_j].$$

Thus, $\underline{c}(A_j)$ is essentially the function $\#\underline{C}[-,A_j]$.

Clearly, since there is a natural bijection between any two skeltal sets for $\underline{C}$, the choice of skeletal set is immaterial. We shall adopt the notation $Z\langle C, \underline{C}\rangle$ to denote the right linearization of a category $\underline{C}$ using the skeletal set C. Generally, we use one or two roman letters, underlined, for the category involved, and (unless it becomes to unwieldy) the same roman letters, but lower case and underlined, to denote the corresponding Z-linear map $Z(C) \to Z^C$. In fact, this homomorphism will itself be frequently referred to as the "right linearization" of the category. Its "components" will be distinguished by suitable sub-indices.

Of course the "left linearization" of a finitary category can be defined analogously. It should be clear, however, that the left linearization of a finitary category $\underline{C}$ is identical to the right linearization of the dual (or "opposite") category, which we denote by $\underline{C}^*$. ($\underline{C}^*$ has the same objects as $\underline{C}$, and the same morphisms but with their "directions" reversed; i.e., $\underline{C}^*[A, B] = \underline{C}[B, A]$, and the order of compositions is reversed.) For this reason, we shall denote the left linearization of $\underline{C}$ by $Z\langle C, \underline{C}^*\rangle$.

Of course, there is nothing very interesting in such linearizations without the development of some further structure. This will be done in the next chapter. There is, however, a question we can ask, and give a partial answer to, now. The question is: when is the homomorphism $\underline{c}$: $Z(C) \to Z^C$ (of the right linearization $Z\langle C, \underline{C}\rangle$ of a finitary category) faithful? We shall show (in basically the same way we did in the case of a category of algebras and surjective algebra homomorphisms) that if mor $\underline{C}$ consists only of epimorphisms, then $\underline{c}$ is indeed faithful.

Lemma (4.2): Let $\underline{C}$ be a category, and suppose $A \varepsilon$ ob $\underline{C}$ satisfies the condition that $\underline{C}[A, A]$ is finite. If e in $\underline{C}[A, A]$ is an epimorphism, then e is an isomorphism.

Proof: Since $\underline{C}[A, A]$ is finite, there must be a term e^r in the sequence e, $e \circ e = e^2$, $e^3, \ldots$ which is equal to a succeeding term $e^{r+s} = e^s \circ e^r$; that is,

$$1_A \circ e^r = e^s \circ e^r,$$

where 1_A is the identity on A. But the composition of epimorphisms is again an epimorphism, and since by definition epimorphisms are right cancellable, the above equation implies:

$$e^s = 1_A, \text{ and } e^{-1} = e^{s-1}. \quad \text{QED}$$

Corollary (4.3): Let $\underline{C}$ be a finitary category, and A, B in ob $\underline{C}$. Suppose $e_1: A \to B$ and $e_2: B \to A$ are both epimorphisms. Then they are both isomorphisms.

Proof: By the above lemma, both $e_1 \circ e_2$ and $e_2 \circ e_1$ are isomorphisms. Clearly $e_1 \circ (e_2 \circ e_1)^{-1}$ is a right inverse, and $(e_1 \circ e_2)^{-1} \circ e_1$ a left inverse, for e_2. By a standard argument, these inverses must be equal and hence e_2 is invertible. The proof that e_1 is invertible proceeds similarly. QED

Corollary (4.4): Let $\underline{C}$ be a finitary category all of whose morphisms are epimorphisms, and suppose that $C = \{A_i : i \in I\}$ a skeletal set for $\underline{C}$. Define the relation $>$ on C by:

$$A_i > A_j \quad \text{if} \quad \underline{C}[A_i, A_j] \neq \emptyset.$$

Then $>$ is a partial order on C.

Proof: Transitivity and reflexivity are immediate, while anti-symmetry follows from (4.3). QED

We can now apply the same arguments used in the proof of proposition (1.9) to show that:

Proposition (4.5): Let $\underline{C}$ be a finitary category all of whose morphisms are epimorphisms, and let C be a skeletal set for $\underline{C}$. Then the homomorphism $\underline{c}$: $Z(C) \to Z^C$ associated with the right linearization of $\underline{C}$ is faithful.

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Chapter II

Elements of a Theory of Categorical Combinatorics

1. Introductory remarks:

In this chapter we outline the fundamentals of a category-theoretic approach to combinatorial problems. We assume on the part of the reader knowledge of the basics of category theory, including familiarity with the concepts of functor, natural transformation, limit, and adjointness (viz., for example, [HS] or [P]).

If $\underline{C}$ is a category, then (as we have already said) $\underline{C}[A, B]$ represents the set of $\underline{C}$-morphisms with domain A and codomain B. If we have two categories $\underline{C}$ and $\underline{D}$, then $\langle\underline{C}, \underline{D}\rangle$ will denote the "quasi-category" whose objects are (covariant) functors from $\underline{C}$ to $\underline{D}$, and whose morphisms are natural transformations between such functors. (Depending on the set-theoretic foundations that one adopts, $\langle\underline{C}, \underline{D}\rangle$ may not be a category because of set-theoretic niceties, or it may be a category in a higher order "universe". In this work, we will generally ignore these difficulties.) Thus, if F and G are functors $\underline{C} \to \underline{D}$, then $\langle\underline{C}, \underline{D}\rangle[F, G]$ is the class of all natural transformations from F to G. If α is an element of $\langle \underline{C}, \underline{D} \rangle[F, G]$, we also write $\alpha: F \to G$, and for A in ob $\underline{C}$, $\alpha_A: F(A) \to G(A)$ denotes the corresponding element of mor $\underline{D}$.

A contravariant functor from $\underline{C}$ to $\underline{D}$ can be equally well conceived as a covariant functor from $\underline{C}^*$ to $\underline{D}$, or from $\underline{C}$ to $\underline{D}^*$. We shall generally adopt the former viewpoint, and thus denote the quasi-category of contravariant functors from $\underline{C}$ to $\underline{D}$ (and natural transformations) by $\langle \underline{C}^*, \underline{D} \rangle$.

In this work, $\underline{S}$ will denote the category of sets and mappings. Set-valued functors are of special importance in category theory. If $\underline{C}$ is a category and $A \in$ ob $\underline{C}$, then $\underline{C}[-, A]$ will generally be used to stand for the (contravariant) functor which assigns to each $B \in$ ob $\underline{C}$ the set $\underline{C}[B, A]$, and to a $\underline{C}$-morphism $f: B_1 \to B_2$ the mapping:
$\underline{C}[f, A] = f^*: \underline{C}[B_2, A] \to \underline{C}[B_1, A]$, defined by:
$f^*(g) = g \circ f$ for g in $\underline{C}[B_2, A]$.

Similarly, we let $\underline{C}[A,-]$ denote the (covariant) set-valued functor which assigns to each $B \in$ ob $\underline{C}$ the set $\underline{C}[A, B]$, and to the $\underline{C}$-morphism $f: B_1 \to B_2$ the mapping $\underline{C}[A, f] = f_*: \underline{C}[A, B_1] \to \underline{C}[A, B_2]$ defined by:
$f_*(g) = f \circ g$ for $g \in \underline{C}[A, B_1]$.

(Frequently we shall denote the image of a morphism

f under a covariant functor by f_*, and under a contravariant functor by f^*, as long as the context makes it clear what functor is intended. Thus, we always have $(f_1 \circ f_2)_* = f_{1*} \circ f_{2*}$, and $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$.)

The single most important result for the study of set-valued functors is Yoneda's Lemma (viz. [HS], pp. 221-230), which we state in the following form:

Proposition (1.1):

(a) Let $\underline{C}$ be a category, $A \in$ ob $\underline{C}$, and $K: \underline{C} \to \underline{S}$ any contravariant set-valued functor. Then the family of natural transformations $\alpha: \underline{C}[-, A] \to K$ is in a one-one correspondence with the elements of the set $K(A)$.

To be precise, each element $x \in K(A)$ determines a natural transformation $\alpha^x: \underline{C}[-, A] \to K$ under which the map $\alpha^x{}_B: \underline{C}[B, A] \to K(B)$ is defined by the prescription:

$$\alpha^x{}_B(g) = (K(g))(x) \quad \text{for all } g \in \underline{C}[B, A],$$

and every natural transformation $\underline{C}[-, A] \to K$ is equal to α^x for some $x \in K(A)$.

(b) Similarly, if $K: \underline{C} \to \underline{S}$ is any covariant set-valued functor, then the family of natural transformations $\beta: \underline{C}[A, -] \to K$ is in a one-one correspondence with the elements of the set $K(A)$.

That is, each element $x \in K(A)$ determines a natural transformation β^x: $\underline{C}[A, -] \to K$ under which the map $\beta^x{}_B$: $\underline{C}[A, B] \to K(B)$ is defined by the prescription:

$$\beta^x{}_B(g) = (K(g))(x) \quad \text{for all } g \in \underline{C}[A, B].$$

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By Yoneda's lemma, the natural transformations from $\underline{C}[-, A_1]$ to $\underline{C}[-, A_2]$ are all of the form α^f for f in $\underline{C}[A_1, A_2]$, and it is easily shown that $\alpha^f \circ \alpha^g$ is equal to $\alpha^{f \circ g}$. Similarly, natural tranformations from $\underline{C}[A_1, -]$ to $\underline{C}[A_2, -]$ are of the form β^f for f in $\underline{C}[A_2, A_1]$, and $\beta^f \circ \beta^g = \beta^{f \circ g}$. Thus, the rule that sends A to the functor $\underline{C}[-, A]$, and a morphism f to the natural transformation α^f, defines a functor (indeed, an imbedding) of $\underline{C}$ into the quasi-category $\langle \underline{C}^*, \underline{S} \rangle$, called the Yoneda emebedding. Similarly, the assignment $A \mapsto \underline{C}[A, -]$, $f \mapsto \beta^f$ is contravariant, and hence can be regarded as an embeddding of $\underline{C}^*$ into $\langle \underline{C}, \underline{S} \rangle$.

Another important point about the category $\langle \underline{C}, \underline{S} \rangle$ is that limits and colimits in it can be defined "pointwise" in terms of limits and colimits of sets. For our purposes, the most important instances of this are those of the product and coproduct of two set-valued functors. The product in $\underline{S}$ is given by the cartesian product of sets, while the coproduct is given by the disjoint union of sets. Thus if K_1 and K_2 are two set-valued functors (either both covariant or both contravariant), then $K_1 \times K_2$ denotes the product of the two, defined point-wise by:

$$(K_1 \times K_2)(A) = K_1(A) \times K_2(A) \text{ for } A \in \text{ob } \underline{C},$$

and $(K_1 \times K_2)(f) = K_1(f) \times K_2(f)$ is the mapping from $K_1(A) \times K_2(A)$ to $K_1(B) \times K_2(B)$ for any $\underline{C}$-morphism $f\colon A \to B$. (Assuming K_1 and K_2 are covariant. Of course the arrows are reversed if they are contravariant.)

Similarly, the coproduct (or disjoint union) of the two will be denoted by $K_1 \vee K_2$, defined pointwise by:

$$(K_1 \vee K_2)(A) = K_1(A) \vee K_2(A) \quad \text{for } A \in \text{ob } \underline{C},$$

$$(K_1 \vee K_2)(f) = K_1(f) \vee K_2(f) \quad \text{for } f \in \text{mor } \underline{C}.$$

The cartesian product and disjoint union of an arbitrary family $\{K_i : i \in I\}$ of set-valued functors is defined similarly.

Another useful concept is that of a <u>subfunctor</u> of a set-valued functor:

If K is a set-valued functor (covariant or contravariant) on the category $\underline{C}$, a subfunctor of K is a second set-valued functor (of the same "variance" as K) such that:

a) $L(B) \subseteq K(B)$ for all $B \in \text{ob } \underline{C}$,

b) the inclusion maps $\sigma_B\colon L(B) \to K(B)$, as B varies over ob $\underline{C}$, define a natural transformation $\sigma\colon L \to K$.

It is easy to see that if K is a covariant (respectively, contravariant) set-valued functor on $\underline{C}$, and L is any function which assigns to each $A \varepsilon$ ob $\underline{C}$ a subset $L(A)$ of $K(A)$, then L extends to a (unique) subfunctor of K if it satifies the condition:

(1.2) $\quad (K(f))(L(A)) \subseteq L(B)$ for all $\underline{C}$-morphisms $f: A \to B$ (respectively, $f: B \to A$).

Subfunctors can almost be treated like subsets of a set. For example, if L_1 and L_2 are subfunctors of K, then we also have subfunctors $L_1 \cup L_2$ and $L_1 \cap L_2$ such that for all $A \varepsilon$ ob $\underline{C}$:

$(L_1 \cup L_2)(A) = (L_1(A)) \cup (L_2(A))$,

and $(L_1 \cap L_2)(A) = (L_1(A)) \cap (L_2(A))$.

Continuing with this analogy with sets, we note that if K_1 and K_2 are two set-valued functors (of the same variance) on $\underline{C}$, and $\alpha: K_1 \to K_2$ is a natural transformation, then α determines a subfunctor $im(\alpha)$ of K_2 by setting $im(\alpha)(A)$ equal to the image of $K_1(A)$ under α_A for all $A \varepsilon$ ob $\underline{C}$. Proceeding pointwise, we see that α factors uniquely as $\alpha = \sigma \circ \tilde{\alpha}$, where $\sigma: im(\alpha) \to K_2$ is the inclusion natural transformation and $\tilde{\alpha}: K_1 \to im(\alpha)$ is a surjective transformation (i.e., $\tilde{\alpha}_A$ is surjective for all A in ob $\underline{C}$).

2. Representability and S-representability:

A contravariant set-valued functor $K: \underline{C} \to \underline{S}$ is said to be <u>representable</u> if there exists an object A in ob $\underline{C}$ such that K is naturally equivalent to $\underline{C}[-, A]$. We say that A represents K. From the Yoneda lemma it is clear that an object representing K is unique up to isomorphism in $\underline{C}$.

Similarly, a covariant functor $K: \underline{C} \to \underline{S}$ is representable if there is an $A \in$ ob $\underline{C}$ such that K is naturally equivalent to $\underline{C}[A, -]$. Again, a representing object is unique up to isomorphism in $\underline{C}$.

Most categorical concepts can be expressed in terms of representability. For example, the product of two objects A_1 and A_2 in ob $\underline{C}$ exists if and only if the functor $\underline{C}[-, A_1] \times \underline{C}[-, A_2]$ is representable in $\underline{C}$. The object $A_1 \pi A_2$ (if it exists) which represents $\underline{C}[-, A_1] \times \underline{C}[-, A_2]$ is the product. One regains the natural projections usually associated with the product by choosing a fixed natural equivalence

$$\alpha: \underline{C}[-, A_1 \pi A_2] \to \underline{C}[-, A_1] \times \underline{C}[-, A_2];$$

then by Yoneda's lemma, α is equal to the natural transformation α^x for a unique $x = (\pi_1, \pi_2)$ in $\underline{C}[A_1 \pi A_2, A_1] \times \underline{C}[A_1 \pi A_2, A_2]$. Thus, since α^x is an equivalence, for any $B \in$ ob $\underline{C}$ and (f_1, f_2) in

$\underline{C}[B, A_1] \times \underline{C}[B, A_2]$ there is a unique $f \in \underline{C}[B, A_1 \pi A_2]$ such that $\alpha^x{}_B(f) = (\pi_1 \circ f, \pi_2 \circ f) = (f_1, f_2)$, which is the usual definition of a product in category theory.

Similarly, the coproduct of A_1 and A_2 exists if and only if $\underline{C}[A_1, -] \times \underline{C}[A_2, -]$ is representable, and the coproduct $A_1 \mu A_2$ is a representing object for that functor.

In the following we describe a few other categorical concepts from the viewpoint of representability:

a) Sub-objects: A $\underline{C}$-morphism $f: A_1 \to A_2$ is a monomorphism if and only if α^f is an injective natural transformation from $\underline{C}[-, A_1]$ to $\underline{C}[-, A_2]$. Dually, f is an epimorphism if β^f is an injective natural transformation $\underline{C}[A_2, -] \to \underline{C}[A_1, -]$.

Usually a sub-object of an object A is defined as an equivalence class of monomorphisms with codomain A. Note, however, that if f is a monomorphism, then the subfunctor $\mathrm{im}(\alpha^f)$ of $\underline{C}[-, A_2]$ is naturally equivalent to $\underline{C}[-, A_1]$, and is therefore representable, that if two monomorphisms f and g with codomain A_2 represent the same subobject of A_2, then $\mathrm{im}(\alpha^f) = \mathrm{im}(\alpha^g)$, and that if a subfunctor of $\underline{C}[-, A_2]$ is representable, then a representation of it determines a corresponding monomorphism with codomain A_2. The upshot of all this is that there is a one-one correspondence between representable subfunctors of $\underline{C}[-, A_2]$ and subobjects of A_2.

In the same way, there is a one-one correspondence between representable subfunctors of $\underline{C}[A_1, -]$ and quotient objects of A_1.

b) Image and co-image: We can also use these ideas to arrive at the concept of the "image" of a $\underline{C}$-morphism. If $f\colon A_1 \to A_2$ is a morphism, we get the natural transformation $\alpha = \alpha^f\colon \underline{C}[-, A_1] \to \underline{C}[-, A_2]$, which in turn defines the subfunctor $\mathrm{im}(\alpha)$ of $\underline{C}[-, A_2]$. If $\mathrm{im}(\alpha)$ is representable, then it corresponds to a unique subobject of A_2; let $\mathrm{Im}(f) \in \mathrm{ob}\ \underline{C}$ be a representing object for $\mathrm{im}(\alpha)$. (Of course $\mathrm{Im}(f)$ is unique up to isomorphism in $\underline{C}$.) Now, we have the unique factorization of α as:

$$\underline{C}[-, A_1] \xrightarrow{\tilde{\alpha}} \mathrm{im}(\alpha) \xrightarrow{\sigma} \underline{C}[-, A_2],$$

where σ is the inclusion natural transformation. Using the isomorphism between $\mathrm{im}(\alpha)$ and $\underline{C}[-, \mathrm{Im}(f)]$, we can then perform a substitution to get a factorization of α of the form

$$\underline{C}[-, A_1] \xrightarrow{\gamma} \underline{C}[-, \mathrm{Im}(f)] \xrightarrow{\eta} \underline{C}[-, A_2],$$

in which we must have that $\gamma = \alpha^g$ where g is a morphism from A_1 to Im(f), $\eta = \alpha^h$ where h is a morphism from Im(f) to A_2, and $f = h \circ g$. We can call the object Im(f) (or, more precisely, the subobject it represents via the monomorphism h) the image of f. It is of course unique up to isomorphism; the factorization of f as $f = h \circ g$ is also essentially unique, and we shall call it the image factorization of f. Note that $\alpha^g = \gamma$ is surjective in the sense that γ_A is a surjective map for all $A \in$ ob $\underline{C}$. Thus, it must be an epimorphism in $\langle \underline{C}^*, \underline{S} \rangle$, and, since the Yoneda emebedding is indeed an embedding, this implies that g is an epimorphism in $\underline{C}$.

Of course, all this is immediately dualizable. The morphism $f: A_1 \to A_2$ also determines the natural transformation $\beta = \beta^f: \underline{C}[A_2, -] \to \underline{C}[A_1, -]$. If im($\beta$) is representable, denote a representing object for it by Coim(f); we then derive an (essentially unique) factorization of f as $f = g \circ h$, where h is an epimorphism and g a monomorphism such that α^g is a surjective natural transformation. Thus, call Coim(f) (or the quotient object it represents via h) the coimage of f, and the factorization $f = g \circ h$ the coimage factorization of f.

These ideas may be regarded as generalizations of the factorization of set mappings into the composition of a surjective and an injective map. Note that in general, these two factorizations (if they exist at all) are distinct. It can, however, be shown that if $\underline{C}$ is balanced, then the two factorizations must coincide. (A morphism is called a bimorphism if it is both an epimorphism an a monomorphism; a category is balanced if the only bimorphisms in $\underline{C}$ are isomorphisms.)

c) Limits and colimits: In general, the existence of a particular limit or colimit can be reduced to the question of the representability of an appropriate set-valued functor. We have already seen this in the case of products and coproducts. We further illustrate this in the notions of equalizer and coequalizer:

If f and g are $\underline{C}$-morphisms $A_1 \to A_2$, then define the subfunctor $K = K(f,g)$ of $\underline{C}[-, A_1]$ by defining:

$K(B) = \{h \in \underline{C}[B, A_1] : f \circ h = g \circ h\}$

for all B in ob $\underline{C}$. If K is representable, then a representing object for K (or more precisely, the subobject it represents) is called the equalizer of f and g.

Similarly, if the subfunctor $L = L(f,g)$ of $\underline{C}[A_2, -]$ defined by:

$L(B) = \{h \in \underline{C}[A_2, B] : h \circ f = h \circ g\}$

for all B in ob $\underline{C}$, is representable, then the quotient object of A_2 that it determines is the <u>coequalizer</u> of f and g.

d) <u>Adjointness:</u> As one more example, look at the concept of adjointness. We shall discuss this at much greater length in section 4 of this chapter. At this point, we simply note that a (covariant) functor $F: \underline{C} \to \underline{D}$ has a <u>left adjoint</u> if and only if the set-valued functor $\underline{D}[B, F(-)]$ is representable for all B in ob $\underline{D}$. (Note that $\underline{D}[B, F(-)]$ is a set-valued functor on $\underline{C}$, and hence when we say representable we mean representable in $\underline{C}$.)

Similarly, it has a right adjoint if and only if $D[F(-), B]$ is representable (in $\underline{C}$) for all $B \in$ ob $\underline{D}$.

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For the purpose of this dissertation, the concept of representability is not quite general enough. We extend it via the following definition:

<u>Definition (2.1):</u> A (contravariant) set-valued functor K on the category $\underline{C}$ is <u>S-representable</u> if there exists an indexed family $\{A_i : i \in I\}$ of objects $A_i \in$ ob $\underline{C}$ such that K is naturally equivalent to the disjoint union $\bigvee_i \underline{C}[-, A_i]$ of the set-valued functors $\underline{C}[-, A_i]$.

Similarly, a (covariant) set-valued functor K on $\underline{C}$

is S-representable if there exists an indexed family $\{A_i : i \in I\}$ of objects such that K is naturally equivalent to the disjoint union $\bigvee_i \underline{C}[A_i, -]$.

The disjoint union $\bigvee_i X_i$ of an indexed family $\{X_i : i \in I\}$ of sets can be unambiguously defined as the set of all pairs (i,x), with $i \in I$ and $x \in X_i$. Then, the natural injection $\iota_k: X_k \to \bigvee_i X_i$ (for $k \in I$) is simply the mapping $x \mapsto (k,x)$.

It is usual (when there is no danger of confusion) to identify X_k with its image in $\bigvee_i X_i$ under the map ι_k. We shall usually do so here, and call X_k the "k-th component" of the disjoint union $\bigvee_i X_i$. We can transport these concepts to the case of a disjoint union of set-valued functors $\bigvee_i K_i$ on $\underline{C}$, and (for $j \in I$) call the functor K_j the j-th component of the disjoint union. Of course, working pointwise, we have the injective natural transformation $\iota_j: K_j \to \bigvee_i K_i$, under which it is possible to identify K_j with the corresponding subfunctor of $\bigvee_i K_i$.

The important thing about a disjoint union $\bigvee_i X_i$ is that maps from it may be defined "component-wise". In particular, let us consider the case of two indexed families of sets $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$, together with their corresponding disjoint unions $\bigvee_i X_i$ and $\bigvee_j Y_j$:

Let $f: I \to J$ be a map between the index sets, and suppose additionally we have a function F which assigns to each index $i \in I$ a mappping $F_i: X_i \to Y_{f(i)}$. Then the pair (f,F) defines a corresponding map $\vee(f,F): \vee_i X_i \to \vee_j Y_j$ via the formula:

$$(\vee(f,F))(i,x) = (f(i),F_i(x)).$$

Essentially, we simply "glue" together the maps F_i.

Now, the above discussion applies (pointwise) to disjoint unions of set-valued functors:

Suppose $\{K_i : i \in I\}$ and $\{L_j : j \in J\}$ are two indexed families of set-valued functors on $\underline{C}$ (all of the same variance), $f: I \to J$ is a mapping between the indexing sets, and F a function which assigns to each $i \in I$ a natural transformation $F_i: K_i \to L_{f(i)}$. For each $B \in \text{ob}\ \underline{C}$, let $F_{(B,i)}$ denote the mapping $K_i(B) \to L_{f(i)}(B)$ that F_i assigns to B, and let F_B be the function $i \mapsto F_{(B,i)}$. Then the pair (f,F) defines a natural transformation $\vee(f,F): \vee_i K_i \to \vee_j L_j$ which is defined by the recipe $(\vee(f,F))_B = \vee(f,F_B)$.

In particular, suppose $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are indexed collections of $\underline{C}$-objects, $f: I \to J$ a mapping, and $F: I \to \text{mor}\ \underline{C}$, $i \mapsto F(i) = F_i$, is a function such that $F_i \in \underline{C}[A_i, B_{f(i)}]$ for all i in I. Then each F_i determines the corresponding natural transformation $\underline{C}[-, A_i] \to \underline{C}[-, B_{f(i)}]$; and therefore the collection of these natural transformations determines (as above) a natural transformation $\bigvee_i \underline{C}[-, A_i] \to \bigvee_j \underline{C}[-, B_j]$ which we shall henceforth denote by $\alpha^{(f,F)}$.

A similar definition applies to disjoint unions of covariant functors of the form $\bigvee_i \underline{C}[A_i, -]$ and $\bigvee_j \underline{C}[B_j, -]$ except that we require that F_i be an element of $\underline{C}[B_{f(i)}, A_i]$ in order that the corresponding natural transformation go in the correct direction, from $\underline{C}[A_i, -]$ to $\underline{C}[B_{f(i)}, -]$. Thus, a pair (F,f) in which f is a mapping $I \to J$ and $F: I \to \text{mor } \underline{C}$ is a function such that $F(i) = F_i \in \underline{C}[B_{f(i)}, A_i]$ defines by the above process a natural transformation from $\bigvee_i \underline{C}[A_i, -]$ to $\bigvee_j \underline{C}[B_j, -]$ which we shall from now on denote by $\beta^{(F,f)}$.

<u>Lemma (2.2):</u> Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two indexed collections of objects in ob $\underline{C}$, and suppose that $\alpha: \bigvee_i \underline{C}[-, A_i] \to \bigvee_j \underline{C}[-, B_j]$ is a natural transformation. Then there is a unique mapping $f: I \to J$ and a unique function $F: I \to \text{mor } \underline{C}$ satisfying $F_i \in \underline{C}[A_i, B_{f(i)}]$ such that $\alpha = \alpha^{(f,F)}$.

Similarly, if $\beta: \bigvee_i \underline{C}[A_i, -] \to \bigvee_j \underline{C}[B_j, -]$ is a natural transformation, then there is a unique mapping $f: I \to J$ and function $F: I \to \text{mor } \underline{C}$ satisfying $F_i \in \underline{C}[B_{f(i)}, A_i]$ such that $\beta = \beta^{(F,f)}$.

Proof: The "restriction" of α to the component $\underline{C}[-,A_i]$

of the disjoint union yields a natural transformation $\underline{C}[-, A_i] \to \bigvee_j \underline{C}[-, B_j]$ for each $i \in I$. By Yoneda's lemma, this transformation is given by a unique element of $\bigvee_j \underline{C}[A_i, B_j]$. Therefore let F_i be this element, which must lie in a unique component $\underline{C}[A_i, B_{f(i)}]$ of the disjoint union $\bigvee_j \underline{C}[A_i, B_j]$. This defines the pair (f,F), and it is clear that $\alpha = \alpha^{(f,F)}$.

A similar proof applies to the natural transformation β. QED

Proposition (2.3): Suppose $K: \underline{C} \to \underline{S}$ is a contravariant functor which is S-representable. Then, its representation as a disjoint union of functors of the form $\underline{C}[-, A]$ is essentially unique.

To be precise, if K is naturally equivalent to both $\bigvee_i \underline{C}[-, A_i]$ and $\bigvee_j \underline{C}[-, B_j]$, where $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ are given indexed families of objects from ob $\underline{C}$, then there exists a bijection $f: I \to J$ such that, for all $i \in I$, A_i is is isomorphic in $\underline{C}$ to $B_{f(i)}$.

Proof: If both disjoint unions are naturally equivalent to K, then there is a natural equivalence $\alpha: \bigvee_i \underline{C}[-, A_i] \to \bigvee_j \underline{C}[-, B_j]$. By lemma (2.3), there are unique pairs (f,F) and (g,G) such that $\alpha = \alpha^{(f,F)}$ and $\alpha^{-1} = \alpha^{(g,G)}$. It is easy to show, then, that $g = f^{-1}$ and $G_{f(i)} = F_i^{-1}$. QED

Of course, the dual of proposition (2.3) also holds. That is, the representation of a covariant set-valued functor on $\underline{C}$ by a disjoint union of the form $\bigvee_i \underline{C}[A_i, -]$ is essentially unique.

Call an S-representable functor <u>finitary</u> if in its representation as a disjoint union $\bigvee_i \underline{C}[-, A_i]$ (or, in the case of a covariant functor, as a disjoint union $\bigvee_i \underline{C}[A_i, -]$), the indexing set I is finite.

***

We now apply the notion of S-representability to arrive at the following generalization of the notions of product and coproduct:

<u>Definition (2.4):</u> The pair (A_1, A_2) of objects in ob $\underline{C}$ will be said to <u>have an S-product</u> in $\underline{C}$ if the set-valued functor $\underline{C}[-, A_1] \times \underline{C}[-, A_2]$ is S-representable.

If so, let $A_1 \pi A_2 = \{(A_1 \pi A_2)(x) : x \in \mathrm{dom}(A_1 \pi A_2)\}$ be the (unique up to equivalence) indexed collection of objects that represents the functor, and call it the <u>S-product</u> of A and B. Note that $\mathrm{dom}(A_1 \pi A_2)$ is the indexing set for the S-product.

More generally, we say an n-tuple $(A_1, A_2, \ldots, A_n)$ of $\underline{C}$-objects has an S-product in $\underline{C}$ if the functor $\underline{C}[-, A_1] \times \underline{C}[-, A_2] \times \ldots \times \underline{C}[-, A_n]$ is S-representable in $\underline{C}$. The indexed set of objects representing this functor (assuming it is S-representable) will be denoted by $A_1 \pi A_2 \pi \ldots \pi A_n$ and will be called the S-product of the n-tuple. Of course, $A_1 \pi \ldots \pi A_n$ represents an indexing which assigns to each x in a set $\mathrm{dom}(A_1 \pi \ldots \pi A_n)$ a $\underline{C}$-object $(A_1 \pi \ldots \pi A_n)(x)$.

(Clearly the definition can be extended to apply to an infinite number of factors; in this work, however, we shall be concerned with only a finite number.)

Definition (2.5): Say that a pair (A_1, A_2) of $\underline{C}$-objects has an S-coproduct if the set-valued functor $\underline{C}[A_1, -] \times \underline{C}[A_2, -]$ is S-representable in $\underline{C}$. If so, we write:

$$A_1 \mu A_2 = \{(A_1 \mu A_2)(x) : x \in \mathrm{dom}(A_1 \mu A_2)\}$$

for the indexed collection of objects that represents $\underline{C}[A_1, -] \times \underline{C}[A_2, -]$, and call it the S-coproduct of A_1 and A_2.

Similar notation will apply for the S-coproduct of an n-tuple $(A_1, A_2, \ldots, A_n)$ of $\underline{C}$-objects.

***

Suppose that a pair (A_1, A_2) of $\underline{C}$-objects has an S-product $A_1 \pi A_2$. Then choose a fixed natural transformation α from $\bigvee_x \underline{C}[-, (A_1 \pi A_2)(x)]$ to $\underline{C}[-, A_1] \times \underline{C}[-, A_2]$. Then α is determined on each component $\underline{C}[-, (A_1 \pi A_2)(x)]$ of the disjoint union by a pair $(\pi_1{}^x, \pi_2{}^x)$ in the morphism set $\underline{C}[(A_1 \pi A_2)(x), A_1] \times \underline{C}[(A_1 \pi A_2)(x), A_2]$. For any object $B \in \text{ob } \underline{C}$, the restriction of α_B to the component $\underline{C}[B, (A_1 \pi A_2)(x)]$ of the disjoint union $\bigvee_x \underline{C}[B, (A_1 \pi A_2)(x)]$ is given therefore by the map $f \mapsto (\pi_1{}^x \circ f, \pi_2{}^x \circ f)$. Thus, since α is a natural equivalence, the S-product has the following property:

For all $B \in \text{ob } \underline{C}$, and every pair (f_1, f_2) of $\underline{C}$-morphisms $f_1 \colon B \to A_1$ and $f_2 \colon B \to A_2$, there is a unique $x \in \text{dom}(A_1 \pi A_2)$ and morphism f in $\underline{C}[B, (A_1 \pi A_2)(x)]$ such that $\pi_1{}^x \circ f = f_1$ and $\pi_2{}^x \circ f = f_2$.

Conversely, given an indexed family $\{(A_1 \pi A_2)(x)\colon \ x \in \text{dom}(A_1 \pi A_2)\}$ of $\underline{C}$-objects equipped with a family of "natural projections" $(\pi_1{}^x, \pi_2{}^x)$ in $\underline{C}[(A_1 \pi A_2)(x), A_1] \times \underline{C}[(A_1 \pi A_2)(x), A_2]$ having the above property, one can conclude that the family $A_1 \pi A_2$ is the S-product of the pair (A_1, A_2).

Similar remarks apply to the S-coproduct of a pair (A_1, A_2), which can be equipped with a family of "natural injections" $(\iota_1{}^x, \iota_2{}^x)$ from A_1 to $(A_1 \mu A_2)(x)$ and from A_2 to $(A_1 \mu A_2)(x)$ respectively.

We shall say that a category $\underline{C}$ has S-products if the S-product of any pair of $\underline{C}$-objects exists in $\underline{C}$, and similarly that it has S-coproducts if the S-coproduct of any pair exists.

***

Examples:

a) Any product can be regarded as a finitary S-product, any coproduct as a finitary S-coproduct.

b) If $\underline{A}$ is an appropriate class of finite algebras of the same type, as in chapter I, then the category $\underline{C}$ of algebras in $\underline{A}$ and surjective algebra homomorphisms has finitary S-products. As we saw, the S-product of a pair (A_1, A_2) in $\underline{C}$ is provided by the family of subdirect algebras of $A \times B$.

In particular, the category $\underline{Q}$ of finite sets and surjective maps has finitary S-products.

The same considerations in fact apply to non-finite algebras, but then the S-product will not in general be finitary.

c) As we also saw in chapter I, the category $\underline{P}$ of finite sets and injective maps has a finitary

S-coproduct, the S-coproduct of a pair (A_1, A_2) being given by the family of all amalgamations $A_1 \vee_g A_2$.

Similar considerations apply to the category of all sets and injective maps, but then if either A_1 or A_2 is infinite, the S-coproduct will not be finitary.

d) It is worth noting that the S-product (and likewise the S-coproduct) of a pair (A_1, A_2) may exist but be <u>empty</u>. That is, we may have $\mathrm{dom}(A_1 \pi A_2) = \emptyset$. This simply means that for all $B \in \mathrm{ob}\, \underline{C}$, at least one of the sets $\underline{C}[B, A_1]$ or $\underline{C}[B, A_2]$ is empty.

For an example of this situation, let us consider posets. A poset $(P, \leq)$ can be viewed as a category by considering the relation $a \leq b$ as indicating the existence of a single morphism $a \to b$ (which we identify with the pair (a,b)), and that otherwise $P[a, b] = \emptyset$. Thus, $\mathrm{ob}\,(P, \leq) = P$, while $\mathrm{mor}\,(P, \leq)$ consists of all pairs (a,b) such that $a \leq b$; i.e., it is simply the graph of the relation $\leq$. Composition is provided by the transitivity of $\leq$.

Under this interpretation, the product of a pair (a,b) of elements in P is given by their meet $a \wedge b$ (if it exists), and the coproduct by their join $a \vee b$.

A poset P is a <u>partial meet semi-lattice</u> if when elements a and b of P have a lower bound, they have a greatest lower bound $a \wedge b$. Similarly, it is a <u>partial join semi-lattice</u> if when a and b have an upper bound they also have a least upper bound $a \vee b$ in P.

From our point of view, a partial meet semi-lattice P has finitary S-products (when viewed as a category), with $a \pi b = \{a \wedge b\}$ when a and b have a lower bound, and $a \pi b = \emptyset$ otherwise. Similarly, a partial join semi-lattice, viewed as a category, has finitary S-coproducts.

e) For one more example, we look at finite graphs. Following what now seems standard terminology, the term "graph" here means an undirected graph without loops or multiple edges. We must also distinguish between different candidates for the title of a morphism from one graph to another.

If G_1 and G_2 are graphs with vertex sets $V(G_1)$ and $V(G_2)$, respectively, then call a map $f: V(G_1) \to V(G2)$ adjacency-preserving if, when x_1 and x_2 are adjacent vertices in G_1, then $f(x_1)$ and $f(x_2)$ are adjacent in G_2. Note that this means that two adjacent points in G_1 cannot be mapped into a single point of G_2, since a vertex is not considered to be adjacent to itself.

Another type of map between graphs is what we shall call a simplicial map: A graph can be considered as a one dimensional simplicial complex, with its vertices as its 0-simplexes, and its edges as its 1-simplexes. A simplicial map is then one which is "simplex-preserving"; i.e., the image of a simplex must be a simplex. In this case, one is allowed to map two adjacent points into a single point.

Both of these candidates are possible choices for the role of a morphism between graphs. Let G then denote the category of finite graphs and adjacency-preserving maps, and GS the category of finite graphs and simplicial maps. Clearly G is a subcategory of GS.

Both categories have (distinct) products. In G the product of two graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set, with a pair (x_1,y_1) being adjacent to (x_2,y_2) if and only if x_1 is adjacent to y_1 in G_1, and x_2 is adjacent to y_2 in G_2. In GS, however, one has $V(G_1) \times V(G_2)$ as the vertex set of the product, but in addition to the adjacencies already given, (x,y_1) is adjacent to (x,y_2) if y_1 and y_2 are adjacent in G_2, and (x_1,y) is adjacent to (x_2,y) if x_1 and x_2 are adjacent in G_2.

In GS the product of two *connected* graphs is again connected; in G, however, the product of two connected graphs may well be disconnected. For example, the product of the graph  with itself (in G) is the graph , which has two components.

There is another way of looking at this situation. Let CG be the full subcategory of G generated by conncted graphs. Then CG does not have a product, but it does have an S-product, under which the S-product of two connected graphs is the family of the connected components of their product in $\underline{G}$.

***

It is clear that S-products, like products, are <u>commutative</u> in the sense that $A_1 \pi A_2$ is naturally isomorphic to $A_2 \pi A_1$. This follows from the fact that the functors $\underline{C}[-, A_1] \times \underline{C}[-, A_2]$ and $\underline{C}[-, A_2] \times \underline{C}[-, A_1]$ are naturally equivalent. Another "product-like" property is that if the S-product of any two objects exists in $\underline{C}$, then the S-product of any finite n-tuple of objects also exists in $\underline{C}$. This fact we shall now prove:

<u>Lemma (2.7):</u> Suppose that $\underline{C}$ has S-products (i.e., the S-product of any pair of objects exists). If K_1 and K_2 are (contravariant) S-representable set-valued functors, then $K_1 \times K_2$ is again S-representable.

<u>Proof:</u> Suppose K_1 and K_2 are naturally equivalent to $\bigvee_i \underline{C}[-, A_i]$ and $\bigvee_j \underline{C}[-, A_j]$ respectively. Among set-valued functors as among sets, the cartesian product distributes over the disjoint union. Thus it follows that $K_1 \times K_2$ is naturally equivalent to the disjoint union $\bigvee_{(i,j)} \underline{C}[-, A_i] \times \underline{C}[-, A_j]$. But by the hypothesis each component of this disjoint union is S-representable, and it is clear that the disjoint union of S-representable functors is again

S-representable. QED

<u>Proposition (2.8):</u> Suppose the category $\underline{C}$ has S-products. Then, for all positive integers n and all n-tuples $(A_1, A_2, \ldots, A_n)$ of $\underline{C}$-objects, the S-product of $(A_1, A_2, \ldots, A_n)$ also exists. If $\underline{C}$ has finitary S-products, then $A_1 \pi A_2 \pi \ldots \pi A_n$ is also finitary.

<u>Proof:</u> The proof is by induction on n. The case n = 2 is given. Suppose then that $n \geq 2$, and that the S-product of n factors always exists. Let $(A_1, A_2, \ldots, A_{n+1})$ be any (n+1)-tuple of $\underline{C}$-objects. Then it is clear that the cartesian product

$\underline{C}[-, A_1] \times \underline{C}[-, A_2] \times \ldots \times \underline{C}[-, A_{n+1}]$

of functors is naturally equivalent to

$(\underline{C}[-, A_1] \times \underline{C}[-, A_2] \times \ldots \times \underline{C}[-, A_n]) \times \underline{C}[-, A_{n+1}]$.

By the induction hypothesis, the functor in the round brackets above is S-representable, $\underline{C}[-, A_{n+1}]$ is trivially S-representable, and therefore by lemma (2.7) their cartesian product is S-representable.

In case that $\underline{C}$ has finitary S-products, just add "finitariness" to the induction hypothesis, and note that it is preserved by the proof. QED

Of course the analogous theorem for S-coproducts is also true.

One way of looking at our results so far is to distinguish the "full sub-quasi-categories" of $\langle \underline{C}^*, \underline{S}\rangle$ and of $\langle \underline{C}, \underline{S}\rangle$ generated by representable functors and by S-representable functors. Let $R\langle \underline{C}, \underline{S}\rangle$ denote the category of representable covariant functors and natural transformations, and $SR\langle \underline{C}, \underline{S}\rangle$ the category of S-representable (covariant) functors and natural transformations. Both are full subcategories of $\langle \underline{C}, \underline{S}\rangle$, and recall that $\langle \underline{C}, \underline{S}\rangle$ has products given pointwise by the cartesian product in $\underline{S}$. Then the situation for products and coproducts is as follows:

$\underline{C}$ has products if and only if $R\langle \underline{C}^*, \underline{S}\rangle$ is closed under the formation of products in $\langle \underline{C}^*, \underline{S}\rangle$, and it has coproducts if and only if $R\langle \underline{C}, \underline{S}\rangle$ is closed under the formation of products in $\langle \underline{C}, \underline{S}\rangle$.

The situation for S-products and S-coproducts is similar:

If $SR\langle \underline{C}^*, \underline{S}\rangle$ is closed under the product in $\langle \underline{C}^*, \underline{S}\rangle$, then $\underline{C}$ has S-products (and conversely); while if $SR\langle \underline{C}, \underline{S}\rangle$ is closed under the product in $\langle \underline{C}, \underline{S}\rangle$, then $\underline{C}$ has S-coproducts. It is for this reason that general results about products and coproducts also hold for S-products and S-coproducts (when properly interpreted); what looks like an S-product in $\underline{C}$ is simply a product in $SR\langle \underline{C}^*, \underline{S}\rangle$, while what looks like an S-coproduct is simply a product in $SR\langle \underline{C}, \underline{S}\rangle$.

Let us now turn to the case in which we are especially interested:

Let $\underline{C}$ be a finitary category with skeletal set $C = \{A_i : i \in I\}$, and suppose that $\underline{C}$ has finitary S-products. We then define a multiplication in the right linearization $Z\langle C, \underline{C}\rangle$ of $\underline{C}$ by writing:

$$(2.9) \qquad A_i \circ A_j = \sum_k r(i,j;k) A_k$$

where $r(i,j;k)$ is the number of elements x in $\mathrm{dom}(A_i \; \pi \; A_j)$ such that $(A_i \; \pi \; A_j)(x)$ is isomorphic to A_k, and extending to all of $Z\langle C, \underline{C}\rangle$ by linearity.

There is another convenient way of writing the above multiplication. For this, we introduce what we shall call the "angle bracket convention":

For any $A \in \mathrm{ob}\, \underline{C}$, let $\langle A\rangle$ denote the unique element A_i in the skeletal set C such that A is isomorphic to A_i. Then, we may alternatively write the above multiplication in the form:

$$(2.10) \qquad A_i \circ A_j = \sum_x \langle (A_i \; \pi \; A_j)(x)\rangle,$$

where x varies over the set $\mathrm{dom}(A_i \; \pi \; A_j)$.

Proposition (2.11): The multiplication law (2.9) is commutative and associative, and hence makes Z(C) into a commutative ring.

Proof: The proof consists in showing that both $(A_i \circ A_j) \circ A_k$ and $A_i \circ (A_j \circ A_k)$ are equal to the sum

$$\sum_x \langle (A_i \pi A_j \pi A_k)(x) \rangle,$$

where x varies over the set $\mathrm{dom}(A_i \ \pi \ A_j \ \pi \ A_k)$. This is done by noting that the S-product $A_i \ \pi \ A_j \ \pi \ A_k$ can be arrived at either by forming the indexed set

$\{((A_i \ \pi \ A_j)(x) \ \pi \ A_k)(y):$

$x \in \mathrm{dom}(A_i \ \pi \ A_j),\ y \in \mathrm{dom}((A_i \ \pi \ A_j)(x) \ \pi \ A_k)\}$

which corresponds to the expression $(A_i \circ A_j) \circ A_k$, or else by forming the indexed set

$\{(A_i \ \pi \ (A_j \ \pi \ A_k)(x))(y):$

$x \in \mathrm{dom}(A_j \ \pi \ A_k),\ y \in \mathrm{dom}(A_i \ \pi \ (A_j \ \pi \ A_k))(x)\}$

which corresponds to the expression $A_i \circ (A_j \circ A_k)$.

For example,

$$\sum_{(x,y)} \langle ((A_i \ \pi \ A_j)(x) \ \pi \ A_k)(y) \rangle$$

$$= \sum_x (\sum_y \langle ((A_i \ \pi \ A_j)(x) \ \pi \ A_k)(y) \rangle$$

$$\text{(by (2.10))} \quad = \sum_x (\langle (A_i \ \pi \ A_j)(x) \rangle \circ A_k$$

$$= (\sum_x \langle (A_i \ \pi \ A_j)(x) \rangle) \circ A_k$$

$$= (A_i \circ A_j) \circ A_k.$$

But by the uniqueness of S-representability, we must have the equivalence of these two ways of arriving at $A_i \pi A_j \pi A_k$, and hence the equality of the corresponding sums. QED

Let us call the multiplication (2.9) on Z(C) the <u>multiplication derived from the S-product</u> in $\underline{C}$. It is obvious from the manner in which the S-product is defined that the Z-linear maps $\underline{c}_i$ are ring homomorphisms from Z(C) to Z when Z(C) is given the multiplication derived from the S-product. Thus, the map $\underline{c}$: $Z(C) \to Z^C$ is also a ring homomorphism.

Of course by duality the same concepts and results apply to the left linearization $Z\langle C, \underline{C}^*\rangle$ of a finitary category $\underline{C}$. If $\underline{C}$ has finitary S-coproducts, then we define the <u>multiplication derived from the S-coproduct</u> by:

$$(2.12) \qquad A_i \circ A_j = \sum_x \langle (A_i \mu A_j)(x)\rangle,$$

where x varies over the set $\mathrm{dom}(A_i \mu A_j)$.

Under this multiplication, $Z\langle C, \underline{C}^*\rangle$ is a commutative ring and $\underline{c}^*$: $Z(C) \to Z^C$ is a ring homomorphism.

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Before we leave this section, note that other limits and colimits besides products and coproducts can be generalized via the notion of S-representability, to give us the concepts of S-limits and S-colimits. The key to such a generalization is the fact (which we noted earlier) that the existence of a particular limit or colimit can be reduced to the question of the representability of a corresponding set-valued functor. To generalize, then, we just replace the condition of representability with the condition of S-representability.

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3. Factorizations:

In the first chapter, we showed that if $\underline{A}$ is a class of finite algebras closed under the formation of direct products and sub-algebras, then both the category $\underline{C}$ of $\underline{A}$-algebras and algebra homomorphisms, and the category $\underline{D}$ of $\underline{A}$-algebras and surjective algebra homomorphisms are finitary categories with finitary S-products. (Of course the S-product in $\underline{C}$ is a product.) Thus, if C is a skeletal set for $\underline{C}$ (and hence a skeletal set for $\underline{D}$), then both $Z\langle C, \underline{C}\rangle$ and $Z\langle C, \underline{D}\rangle$ are equipped with multiplications (derived from their respective S-products) making them into commutative rings. But in addition to this, we also have a ring homomorphism (indeed, an isomorphism) $\underline{t}\colon Z\langle C, \underline{C}\rangle \to Z\langle C, \underline{D}\rangle$ which commutes with the ring homomorphisms $\underline{c}$ and $\underline{d}$. In this section we shall give a category-theoretic account of this phenomenon.

Fundamental to the definition of $\underline{t}$ was the fact that every algebra homomorphism $f\colon A \to B$ has the (essentially unique) factorization:

$$A \xrightarrow{e} B_1 \xrightarrow{m} B, \quad f = m \circ e,$$

where e is a surjective homomorphism and m is injective.

In the previous section, we gave a brief account of what we called "image factorizations" and (dually) "co-image factorizations". These are examples of so-called $(\underline{M},\underline{E})$-factorizations (viz. [HS]), which we now define:

Let $\underline{M}$ be a class of $\underline{C}$-monomorphisms closed under composition with isomorphisms, and $\underline{E}$ a class of $\underline{C}$-epimorphisms also closed under composition with isomorphisms. We shall say that a C-morphism $f: A_1 \to A_2$ has an $(\underline{M},\underline{E})$-factorization if it is possible to "factor" f as a composition $f = m \circ e$ with $m \in \underline{M}$ and $e \in \underline{E}$. Say that f is uniquely $(\underline{M},\underline{E})$-factorizable if f has an $(\underline{M},\underline{E})$-factorization, and any two $(\underline{M},\underline{E})$-factorizations of f, $f = m \circ e$ and $f = m' \circ e'$, are equivalent in the sense that, supposing the factorizations are given by the diagrams:

$$A_1 \xrightarrow{e} B \xrightarrow{m} A_2$$

and $$A_1 \xrightarrow{e'} B' \xrightarrow{m'} A_2,$$

then there is an isomorphism $i: B \to B'$ such that the diagram:

$$\begin{tikzcd} & B \arrow[dr, "m"] \arrow[dd, "i"] & \\ A_1 \arrow[ur, "e"] \arrow[dr, "e'"'] & & A_2 \\ & B' \arrow[ur, "m'"'] & \end{tikzcd}$$

commutes. If every morphism in mor $\underline{C}$ is uniquely $(\underline{M},\underline{E})$-factorizable we shall say that $\underline{C}$ is $(\underline{M},\underline{E})$-factorizable.

We have already seen some important examples of such factorizability:

In the first place, we have the family of examples given by (injective, surjective)-factorizations of homomorphisms of $\underline{A}$-algebras as in chapter I. More generally, if $\underline{C}$ is a category all of whose morphisms admit image factorizations (as defined in section 2 above), then $\underline{C}$ is $(\underline{M},\underline{E})$-factorizable, where $\underline{M}$ is the class of all $\underline{C}$-monomorphisms, and $\underline{E}$ is the class of epimorphisms e such that the natural transformation α^e is surjective. Similarly, if $\underline{C}$ is a category all of whose morphisms admit co-image factorizations, then $\underline{C}$ is $(\underline{M},\underline{E})$-factorizable where $\underline{E}$ is the class of all $\underline{C}$-epimorphisms and $\underline{M}$ is the class of monomorphisms m such that the natural transformation β^m is surjective.

As general as the concept of ($\underline{M}$,$\underline{E}$)-factorizations is, it is not quite general enough for our purposes. Consider the following example:

Let $\underline{PN}$ be the category of finite sets and <u>partial functions</u>. A partial function f: A $\to$ B is a function which may be defined only on a subset of A. The subset of A on which f is defined we shall call the <u>domain of definition</u> of f, and denote by Def(f). (It must be distinguished from the <u>domain</u> of f, which, in category theory, is the set A.) For example, between any two sets A and B we have the "empty" partial function, whose domain of definition is the empty set $\emptyset \subseteq A$.

Note that $\underline{N}$, the category of finite sets and mappings, is a subcategory of $\underline{PN}$. Now consider the class $\underline{E}$ of $\underline{PN}$-morphisms e: $A_1 \to A_2$ such that e is a set isomorphism of Def(e) with A_2. (Essentially, e is the "inverse" of an injective map $A_2 \to A_1$.) Note that the elements of $\underline{E}$ are all are all $\underline{PN}$-epimorphisms. Then it is not hard to see that $\underline{PN}$ is ($\underline{N}$,$\underline{E}$)-factorizable in the sense that every partial function f: $A_1 \to A_2$ admits a factorization of the form $f = f_1 \circ e$ where $f_1 \in \text{mor}\ \underline{N}$ and $e \in \underline{E}$; simply choose e to be the "inverse" of the inclusion of B = Def(f) $\to A_1$, and f_1: B $\to$ A to be the restriction of f to B. And it is not hard to verify that this factorization is unique if one uses the same definition of uniqueness as that used for

$(\underline{E},\underline{M})$-factorizations. But, mor $\underline{N}$ is <u>not</u> a class of monomorphisms in $\underline{PN}$.

The above example suggests the need for a wider definition of factorizability. In fact, we shall provide two definitions, which will be dual to each other.

<u>Definition (3.1):</u> A <u>right factorization</u> of the category $\underline{C}$ is a pair $(\underline{M},\underline{D})$ such that:

a) $\underline{D}$ is a subcategory of $\underline{C}$ with same object class as $\underline{C}$ and containing all $\underline{C}$-isomorphisms,

b) $\underline{M}$ is a class of $\underline{C}$-monomorphisms closed under composition with $\underline{C}$-isomorphisms,

c) every $\underline{C}$-morphism f has a unique factorization of the form $f = m \circ f_1$, where $m \in \underline{M}$ and $f_1 \in$ mor $\underline{D}$. "Uniqueness" in this context means, as before, that if $f = m' \circ f_1'$ is a second such factorization, then there is an isomorphism which makes the following diagram commute:

$$\begin{tikzcd} & B_1 \arrow[dr, "m"] \arrow[dd, "i"] & \\ A \arrow[ur, "f"] \arrow[dr, "f_1'"'] & & B \\ & B_1' \arrow[ur, "m'"'] & \end{tikzcd}$$

Definition (3.2): A left factorization of the category $\underline{C}$ is a pair $(\underline{D},\underline{E})$ such that:

a) $\underline{D}$ is a subcategory of $\underline{C}$ with the same object class as $\underline{C}$, and containing all $\underline{C}$-isomorphisms,

b) $\underline{E}$ is a class of $\underline{C}$-epimorphisms closed under composition with $\underline{C}$-isomorphisms,

c) every $\underline{C}$-morphism has a unique (as above) factorization $f = f_1 \circ e$, where $f_1 \in \text{mor } \underline{D}$ and $e \in \underline{E}$.

If $(\underline{M},\underline{D})$ is a right factorization of $\underline{C}$, we shall also say that $\underline{D}$ is a right factor of $\underline{C}$, and call $\underline{M}$ its associated class of monomorphisms. Similarly, if $(\underline{D},\underline{E})$ is a left factorization of $\underline{C}$, we shall call $\underline{D}$ a left factor of $\underline{C}$, $\underline{E}$ being its associated class of epimorphisms.

We give a few examples of factorizations:

a) Suppose $\underline{C}$ is a category such that every f in mor $\underline{C}$ has an image factorization. If we let $\underline{M}$ be the class of all $\underline{C}$-monomorphisms and $\underline{E}$ be the class of all epimorphisms e such that α^e is surjective, then (as we have already noted) $\underline{C}$ is $(\underline{E},\underline{M})$-factorizable. Additionally, however, both $\underline{E}$ and $\underline{M}$ are closed under composition. Thus, if we define subcategories $\underline{CE}$ and $\underline{CM}$ of $\underline{C}$ by setting ob $\underline{CE}$ = ob $\underline{CM}$ = ob $\underline{C}$ and mor CE = $\underline{E}$, mor $\underline{CM}$ = $\underline{M}$, it is clear that $\underline{CE}$ is a right factor of $\underline{C}$

(with $\underline{M}$ as its associated class of monomorphisms) and $\underline{CM}$ is a left factor of $\underline{C}$ (with $\underline{E}$ as its associated class of epimorphisms).

Similar remarks apply if every $\underline{C}$-morphism has a co-image factorization.

b) As we saw in the example preceding our definitions, $\underline{N}$ is a left factor of $\underline{PN}$.

c) Consider again the categories $\underline{G}$ (finite graphs and adjacency-preserving maps) and $\underline{GS}$ (finite graphs and simplicial maps). We have noted that $\underline{G}$ is a subcategory of $\underline{GS}$; in fact, it is a left factor of $\underline{GS}$.

To see this, notice that a $\underline{GS}$-morphism $f\colon G_1 \to G_2$ is a $\underline{G}$-morphism if and only if, for all $y \in V(G_2)$, the subset $f^{-1}(y)$ is "totally disconnected" in G_1 (i.e., no two points in $f^{-1}(y)$ are adjacent in G_1). Now let $\underline{E}$ consist of all morphisms $e\colon G_1 \to G_2$ in mor $\underline{GS}$ which satisfy:

i) e is surjective as a function between vertex sets,

ii) for all $y \in V(G_2)$ the subgraph of G_1 generated by $f^{-1}(y)$ is connected.

(If G is a graph and Y a subset of its vertex set, then the subgraph of G generated by Y, which is denoted by $G(Y)$, is the graph with Y as its vertex set and having two points in it adjacent if they are adjacent in G.)

Let $f\colon G_1 \to G_2$ be any $\underline{GS}$-morphism. Define a "quotient" graph $\bar{G}_1$ of G_1 by identifying to a point each connected component of $G_1(f^{-1}(y))$, for each y in $V(G_2)$, with two such points being adjacent in $\bar{G}_1$ if the corresponding connected subgraphs of G_1 have at least one edge between them. Then the natural projection $e\colon G_1 \to \bar{G}_1$ is an element of $\underline{E}$, while f factors (uniquely) through $\bar{G}_1$ as $f = f_1 \circ e$ with $f_1 \in \text{mor } \underline{G}$. It is not hard to show that this factorization is unique (in the sense of our definition of factorizations), and that therefore $(\underline{G},\underline{E})$ is a left factorization of $\underline{GS}$.

It is also interesting to notice that $\underline{E}$ is <u>not</u> closed under composition, unlike our other examples.

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Suppose $(\underline{M},\underline{D})$ is a right factorization of the category $\underline{C}$. Call a subobject of an object $A \in \text{ob } \underline{C}$ an $\underline{M}$-subobject of A if it can be represented by a monomorphism from $\underline{M}$. (Then, since $\underline{M}$ is closed under composition with $\underline{C}$-isomorphisms, it can only be represented by monomorphisms from $\underline{M}$.) We shall call the factorization $(\underline{M},\underline{D})$ <u>locally small</u> if the class of $\underline{M}$-subobjects of A, for any A in ob $\underline{C}$, form a set.

Further, call $(\underline{M},\underline{D})$ *finitary* if for all $\underline{C}$-objects A the class of $\underline{M}$-subobjects of A form a finite set.

Similarly, if $(\underline{D},\underline{E})$ is a left factorization of $\underline{C}$, then a $\underline{E}$-quotient object of $A \in \text{ob } \underline{C}$ is a quotient object of A which can be represented by an element of $\underline{E}$; $(\underline{D},\underline{E})$ is locally small if the class of $\underline{E}$-quotient objects of any $A \in \text{ob } \underline{C}$ form a set and is finitary if that class forms a finite set.

With this terminology established, we can state and prove the following proposition:

Proposition (3.3): Suppose $(\underline{M},\underline{D})$ is a locally small right factorization of $\underline{C}$. Then for all $A \in \text{ob } \underline{C}$, the set-valued functor $\underline{C}[-, A]$, restricted to $\underline{D}$, is S-representable in $\underline{D}$.

Specifically, if $\{A_x \xrightarrow{m^x} A : x \in X\}$ is a family of representatives of the $\underline{M}$-subobjects of A, then the restriction of $\underline{C}[-, A]$ to $\underline{D}$ is naturally equivalent to $\bigvee_x \underline{D}[-, A_x]$.

Proof: The family of morphisms m^x define (component-wise) a natural transformation

$$\alpha: \bigvee_x \underline{D}[-, A_x] \to \underline{C}[-, A].$$

The definition of a right factorization asserts that for all $B \in \text{ob } \underline{C}$, α_B is a set isomorphism. QED

Of course, the dual of the above proposition also holds. That is, if $(\underline{D},\underline{E})$ is a locally small left factorization of $\underline{C}$, then for all A in ob $\underline{C}$ the set-valued functor $\underline{C}[A, -]$, restricted to $\underline{D}$, is S-representable in $\underline{D}$. That is, if $\{A \xrightarrow{e^x} A_x : x \in X\}$ is a family of representatives of the distinct $\underline{E}$-quotient objects of A, then the restriction of $\underline{C}[A, -]$ to $\underline{D}$ is naturally equivalent to $\bigvee_x \underline{D}[A_x, -]$.

We now look at what these concepts so far entail for finitary categories and their linearizations. Thus, let $\underline{C}$ be a finitary category with skeletal set $C = \{A_i : i \in I\}$. Clearly, if $\underline{D}$ is a right or left factor of $\underline{C}$, then $\underline{D}$ is also finitary, and C is also a skeletal set for $\underline{D}$.

Let us then suppose that $\underline{D}$ is a finitary right factor of $\underline{C}$, with associated class of monomorphisms $\underline{M}$. Corresponding to the factorization $(\underline{M},\underline{D})$ of $\underline{C}$ define a Z-linear map $Z(C) \to Z(C)$ as follows:

For each $A_i \in C$, let $\{A_x : x \in X(i)\}$ be a family of distinct representatives for the $\underline{M}$-subobjects of A_i. By assumption, this set is finite for all i. Thus, using the angle bracket convention, we write:

$$\underline{m}(A_i) = \sum_x \langle A_x \rangle, \tag{3.4}$$

where the summation is over the elements x in X(i).

Of course it follows at once that the above can also be written as:

$$(3.5) \qquad \underline{m}(A_i) = \sum_j m(i,j)A_j$$

where m(i,j) is the number of distinct $\underline{M}$-subobjects of A_i representable by A_j.

It is almost immediate from our definitions, that the following diagram commutes for all $k \in I$:

$$(3.6) \qquad \begin{tikzcd} Z(C) \arrow[dr, "\underline{c}_k"] \arrow[dd, "\underline{m}"'] & \\ & Z \\ Z(C) \arrow[ur, "\underline{d}_k"'] & \end{tikzcd}$$

since $\underline{c}_k$ and $\underline{d}_k$ simply count morphisms (that is, $\underline{c}_k(A_i) = \#\underline{C}[A_k, A_i]$ and $\underline{d}_k(A_j) = \#\underline{D}[A_k, A_j]$).

Dually, if $(\underline{D},\underline{E})$ is a finitary left factorization of $\underline{C}$, then it determines a a corresponding Z-linear map $\underline{e}\colon Z\langle C, \underline{C}^*\rangle \to Z\langle C, \underline{D}^*\rangle$ defined on basis elements by:

$$(3.7) \qquad \underline{e}(A_i) = \sum_j e(i,j)A_j,$$

where $e(i,j)$ is the number of distinct $\underline{E}$-quotient objects of A_i representable by A_j. Clearly, the following diagram commutes for all $k \in I$:

$$\begin{tikzcd}
Z(C) \arrow[dd, "\underline{e}"'] \arrow[dr, "\underline{c}_k^*"] & \\
& Z \\
Z(C) \arrow[ur, "\underline{d}_k^*"'] &
\end{tikzcd} \tag{3.6}$$

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It is our intention at this point to turn our attention to the question of the relationship between factorizations and S-products. Before going on to our principal result on this matter, we need some further properties of factorizations:

<u>Proposition (3.9):</u> Suppose that $(\underline{M},\underline{D})$ is a right factorization of the category $\underline{C}$. Then the following statements are true:

a) All $\underline{C}$-isomorphisms are elements of $\underline{M}$; in particular, all identities are in $\underline{M}$.

b) If $m \in \underline{M}$ and $f \in \operatorname{mor} \underline{D}$ are such that $m \circ f \in \operatorname{mor} \underline{D}$, then m is an isomorphism.

c) If $f \in$ mor $\underline{D}$ and $g \in$ mor $\underline{C}$ are such that $g \circ f \in$ mor $\underline{D}$, then $g \in$ mor $\underline{D}$.

Proof: (a) Let $1_A\colon A \to A$ be an identity in $\underline{C}$. Then 1_A admits the factorization $1_A = m \circ f$, where $m \in \underline{M}$ and $f \in$ mor $\underline{D}$. Say the diagram looks like:

$$A \underset{f}{\overset{m}{\rightleftarrows}} B$$

Then, $m \circ (f \circ m) = (m \circ f) \circ m = m = m \circ 1_B$. But since m is a monomorphism (i.e., left cancellable), this implies that $f \circ m = 1_B$. Thus, f and m are inverses of one another. Since $\underline{M}$ is closed under composition with isomorphisms, it follows that $f \circ m = 1_A$ is an element of $\underline{M}$. Since all identities are in $\underline{M}$, and $\underline{M}$ is closed under composition with isomorphisms, $\underline{M}$ also contains all isomorphisms.

b) Let $f' = m \circ f$. Then $f' = m \circ f$ is an $(\underline{M},\underline{D})$-factorization of f'. But since by assumption f' is an element of mor $\underline{D}$, and all identities are in $\underline{M}$, it follows that $f' = 1_B \circ f'$ is a second such factorization of f', where 1_B is the identity on the codomain of f. By the uniqueness of such factorizations, there must be a $\underline{C}$-isomorphism i such that the following diagram commutes:

$$\begin{tikzcd}
 & B_1 \arrow[dd, "i"] \arrow[dr, "m"] & \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & & B \\
 & B \arrow[ur, "1_B"'] &
\end{tikzcd}$$

But then we see that $m = 1_B \circ i = i$ is an isomorphism.

c) The morphism g has an $(\underline{M},\underline{D})$-factorization $g = m \circ g_1$ ($m \in \underline{M}$, $g_1 \in \text{mor } \underline{D}$). But then since $g \circ f = m \circ (g_1 \circ f)$, with $m \in \underline{M}$ and $g_1 \circ f \in \text{mor } \underline{D}$, by (a) m must be an isomorphism. And since $\underline{D}$ contains all $\underline{C}$-isomorphisms, it follows that $m \circ g_1 = g$ is an element of mor $\underline{D}$. QED

<u>Proposition (3.10):</u> Let $\underline{C}$ be a category which has S-products. Suppose that $(\underline{M},\underline{D})$ is a locally small right factorization of $\underline{C}$. Then $\underline{D}$ also has S-products. Specifically, the relationship is as follows:

For objects A_1 and A_2, let
$A_1 \pi A_2 = \{(A_1 \pi A_2)(x) : x \in \text{dom}(A_1 \pi A_2)\}$
denote the S-product of A_1 and A_2 in the category $\underline{C}$. For each $x \in \text{dom}(A_1 \pi A_2)$, let π_1^x and π_2^x be the "natural projections" from $(A_1 \pi A_2)(x)$ to A_1 and A_2,

respectively. For each $x \in \mathrm{dom}(A_1 \pi A_2)$, also let:

$$\{(A_1 \pi_D A_2)(x,y) \xrightarrow{m^y} (A_1 \pi A_2)(x) : y \in Y(x)\}$$

be the family of representatives of the distinct $\underline{M}$-subobjects of $(A_1 \pi A_2)(x)$ which satisfy the following additional condition:

both $\pi_1{}^x \circ m^y$ and $\pi_2{}^x \circ m^y$ are elements of mor $\underline{D}$.

Then, the indexed set

$$A_1 \pi_D A_2 = \{(A_1 \pi_D A_2)(x,y) : (x,y) \in \bigvee_x Y(x)\}$$

is the S-product of A_1 and A_2 in the category $\underline{D}$.

<u>Proof:</u> We show that, for any object B and any pair of $\underline{D}$-morphism $(f_1,f_2) \in \underline{D}[B, A_1] \times \underline{D}[B, A_2]$, there is a unique (x,y) and a unique $\underline{D}$-morphism $f: B \to (A_1 \pi_D A_2)(x,y)$ such that $(\pi_1{}^x \circ m^y) \circ f = f_1$ and $(\pi_2{}^x \circ m^y) \circ f = f_2$. This will establish a natural equivalence between $\underline{D}[-, A_1] \times \underline{D}[-, A_2]$ and $\bigvee_{(x,y)} \underline{D}[-, (A_1 \pi_D A_2)(x,y)]$.

First, since f_1 and f_2 are $\underline{C}$-morphisms (as well as $\underline{D}$-morphisms), there is a unique $x \in \mathrm{dom}(A_1 \pi A_2)$ and a unique morphism $\bar{f}: B \to (A_1 \pi A_2)(x)$ such that $\pi_1{}^x \circ \bar{f} = f_1$ and $\pi_2{}^x \circ \bar{f} = f_2$. Furthermore, $\bar{f}$ has a unique factorization as $\bar{f} = m \circ f$, where $f \in$ mor $\underline{D}$ and $m: H \to (A_1 \pi A_2)(x)$ is a representative of a unique $\underline{M}$-subobject of $(A_1 \pi A_2)(x)$.

We must show that m can be chosen as one of the

m^y's described in the proposition. That is, we must show that both $\pi_1{}^x \circ m$ and $\pi_2{}^x \circ m$ are elements of mor $\underline{D}$. But, by proposition (3.9a), since $(\pi_1{}^x \circ m) \circ f = f_1$, $(\pi_2{}^x \circ m) \circ f = f_2$, and f, are all elements of mor $\underline{D}$, it follows that $\pi_1{}^x \circ m$ and $\pi_2{}^x \circ m$ are also elements of mor $\underline{D}$. Thus there is a unique $y \in Y(x)$ such that $\bar{f} = m^y \circ f$, and therefore such that $\pi_1{}^x \circ m^y \circ f = f_1$ and $\pi_2{}^x \circ m^y \circ f = f_2$. QED

From the manner in which the "inherited" S-product is constructed, it is clear that if $\underline{C}$ has finitary S-products, and $(\underline{M},\underline{D})$ is a finitary right factorization of $\underline{C}$, then the S-product in $\underline{D}$ is also finitary.

The dual of proposition (3.10) can be stated as follows:

If $(\underline{D},\underline{E})$ is a locally small left factorization of $\underline{C}$, and $\underline{C}$ has S-coproducts, then $\underline{D}$ also has S-coproducts. Specifically, if

$A_1 \mu A_2 = \{(A_1 \mu A_2)(x) : x \in \mathrm{dom}(A_1 \mu A_2)\}$

is the S-coproduct of A_1 and A_2 in $\underline{C}$, then their S-coproduct in $\underline{D}$ can be constructed as follows: For each $x \in \mathrm{dom}(A_1 \mu A_2)$ let

$$\{(A_1 \mu A_2)(x) \xrightarrow{e^y} (A_1 \mu_D A_2)(x,y) : y \in Y(x)\}$$

be a family of representatives of the distinct $\underline{E}$-quotient objects of $(A_1 \mu A_2)(x)$ with the property that the compositions $e^y \circ \iota_1{}^x$ and $e^y \circ \iota_2{}^x$ are both elements of mor $\underline{D}$. (Here, of course, $\iota_1{}^x$ and $\iota_2{}^x$ are the "natural injections" from A_1 and A_2 respectively to $(A_1 \mu A_2)(x)$ associated with the S-coproduct.) Then we can form the S-coproduct in $\underline{D}$ as the indexed family of objects

$$A_1 \mu_D A_2 = \{(A_1 \mu_D A_2(x,y) : (x,y) \in \bigvee_x Y(x)\}.$$

We have already seen this phenomenon of the inheritance of S-products and S-coproducts at work in the case of subdirect products and amalgamations of sets. Let us look at one more example of interest, in this case an S-coproduct inherited from a coproduct:

Example: Consider the category $\underline{G}$ of finite graphs and adjacency-preserving maps. Call a $\underline{G}$-morphism $f: G_1 \to G_2$ adjacency-reflecting if whenever $f(x_1)$ and $f(x_2)$ are adjacent in G_2, then x_1 and x_2 are also adjacent in G_1. It is not hard to verify that finite graphs and adjacency-reflecting maps form a subcategory of $\underline{G}$ which we shall denote by $\underline{GR}$.

Now let $\underline{B}$ be the class of adjacency-preserving maps between finite graphs which are bijective as maps between vertex sets. It is not hard to see that $\underline{B}$ consists entirely of bimorphisms; i.e., each element of $\underline{B}$ is both a monomorphism and an epimorphism in $\underline{G}$. (In fact, $\underline{B}$ is precisely the class of all bimorphisms in

G.) A typical example of such a bimorphism is given by graphs G_1 and G_2 such that $V(G_1) = V(G_2)$ but $E(G_1) \subsetneq E(G_2)$; then the identity mapping on the vertex set defines such a bimorphism $G_1 \to G_2$.

One can verify that (GR,B) is a left factorization of G. Now, G has a coproduct, namely the disjoint union operation (defined in the obvious way) on graphs. Thus, by the above result, we can assert that GR has an S-coproduct (at least). To describe it with some clarity we introduce yet another notion:

Given two graphs G_1 and G_2, suppose that $\rho \subseteq V(G_1) \times V(G_2)$ is a relation between their respective vertex sets. We define the new graph $G_1 \vee_\rho G_2$ as the disjoint union $G_1 \vee G_2$ given additional edges making x and y adjacent for each pair $(x,y) \in \rho$. Clearly the natural map $e: G_1 \vee G_2 \to G_1 \vee_\rho G_2$ is a B-morphism, and thus determines a B-quotient object of $G_1 \vee G_2$; furthermore, if ι_1 and ι_2 are the natural injections from G_1 and G_2 respectively into $G_1 \vee G_2$, then $e \circ \iota_1$ and $e \circ \iota_2$ are elements of mor GR. In fact, one verifies that these are essentially the only B-quotient objects of $G_1 \vee G_2$ with this property. Hence, by our results above, we see that for finite graphs G_1 and G_2, the indexed set $\{G_1 \vee_\rho G_2 : \rho \subseteq V(G_1) \times V(G_2)\}$ is the S-coproduct of G_1 and G_2 in the category GR .

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A short note on the "smallness" conditions in the statement of (2.10):

If $\underline{C}$ is "locally small" (or well-powered, as it is more commonly termed: the class of subobjects of any object form a set), then any right factorization of $\underline{C}$ is *a fortiori* locally small. Similarly, if $\underline{C}$ is co-well-powered (quotient objects of an object are a set), then any left factorization of $\underline{C}$ is locally small. These set-theoretical complications do not arise with finitary categories, as it is easy to prove that any finitary (indeed, any skeletally small) category is both well-powered and co-well-powered.

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We now turn to the following result:

Proposition (3.11): Let $\underline{C}$ be a finitary category with finitary S-products, and $(\underline{M},\underline{D})$ be a finitary right factorization of $\underline{C}$. (Thus $\underline{D}$ also has finitary S-products.)

Consider the right linearizations $Z\langle C, \underline{C}\rangle$ and $Z\langle C, \underline{D}\rangle$ (where C is a skeletal set for $\underline{C}$ and hence for $\underline{D}$) as rings under the multiplications derived from the respective S-products.

Then, if $\underline{m}$: $Z\langle C, \underline{C}\rangle \to Z\langle C, \underline{D}\rangle$ is the Z-linear map corresponding to the given factorization, $\underline{m}$ is a ring homomorphism.

Proof: Set $C = \{A_i : i \in I\}$. We must show the equality, for all $i,j \in I$, of $\underline{m}(A_i \circ A_j)$ (where the multiplication is in $Z\langle C, \underline{C}\rangle$) and $\underline{m}(A_i) \circ \underline{m}(A_j)$ (where the multiplication is in $Z\langle C, \underline{D}\rangle$). Let π denote the S-product in $\underline{C}$, and π_D the S-product in $\underline{D}$.

Consider $\underline{m}(A_i \circ A_j)$ first. Using the angle bracket convention, we write:

$$A_i \circ A_j = \sum_x \langle (A_i \pi A_j)(x) \rangle,$$

and we can also write $\underline{m}(\langle A_i \pi A_j(x)\rangle)$ as:

$$\underline{m}(\langle A_i \pi A_j(x)\rangle = \sum_y \langle A_{(x,y)}\rangle,$$

y varying over the set $Y(x)$,

where $\{A_{(x,y)} \xrightarrow{m^{(x,y)}} (A_i \pi A_j)(x) : y \in Y(x)\}$ is for each x a family of representatives of the distinct $\underline{M}$-subobjects of $(A_i \pi A_j)(x)$. Putting this together, we can write:

$$\underline{m}(A_i \circ A_j) = \sum_{(x,y)} \langle A_{(x,y)}\rangle.$$

Now let $\{B_u \xrightarrow{m^u} A_i : u \in Y(i)\}$ and $\{B_v \xrightarrow{m^v} A_j : v \in Y(j)\}$

be families of representatives of the distinct $\underline{M}$-subobjects of A_i and A_j respectively. Then, by reasoning similar to the above, we can express

$\underline{m}(A_i) \circ \underline{m}(A_j)$ as in the equation:

$$\underline{m}(A_i) \circ \underline{m}(A_j) = \sum_{(u,v,z)} \langle (B_u \, \pi_D \, B_v)(z) \rangle,$$

where in the sum u and v vary over the index sets Y(i) and Y(j), while z varies over $\mathrm{dom}(B_u \, \pi_D \, B_v)$.

Finally, consider the set-valued functor $\underline{C}[-,A_i] \times \underline{C}[-,A_j]$ restricted to $\underline{D}$. By proposition (2.7), it is S-representable in $\underline{D}$. There are two ways of arriving at particular S-representations (in $\underline{D}$) of this functor:

In the first place, we may use the S-product in $\underline{C}$ to give it as the disjoint union $\bigvee_x \underline{C}[-, (A_i \, \pi \, A_j)(x)]$ (restricted to $\underline{D}$), and then use (3.3) to express this as the disjoint union $\bigvee_{(x,y)} \underline{D}[-, A_{(x,y)}]$.

Alternatively, we may use (3.3) first to express this functor as the cartesian product $(\bigvee_u \underline{D}[-, B_u]) \times (\bigvee_v \underline{D}[-, B_v])$, which is by distributivity equivalent to $\bigvee_{(u,v)} (\underline{D}[-, B_u] \times \underline{D}[-, B_v])$, and then use the S-product in $\underline{D}$ to express this as the disjoint union $\bigvee_{(u,v,z)} \underline{D}[-, (B_u \, \pi_D \, B_v)(z)]$. But by the uniqueness of S-representability, the two representations are equivalent; i.e., there is a one-one correspondence between the objects $A_{(x,y)}$ and $(B_u \, \pi_D \, B_v)(z)$ under which corresponding objects are isomorphic. But since these are precisely the objects that go into the two different expressions in question, we conclude that they are equal. QED

The dual of the above proposition can be stated as follows:

Let $\underline{C}$ be a finitary category with finitary S-products, and let $(\underline{D},\underline{E})$ be a finitary left factorization of $\underline{C}$. Consider the left linearizations $Z\langle C, \underline{C}^*\rangle$ and $Z\langle \underline{C}, \underline{D}^*\rangle$ of $\underline{C}$ and $\underline{D}$ (where of course C is a skeletal set for $\underline{C}$ and hence for $\underline{D}$) as rings under the multiplications derived from their respective S-products. The Z-linear map $\underline{e}$: $Z\langle C, C^*\rangle \to Z\langle C, D^*\rangle$ corresponding to the given factorization is a ring homomorphism.

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There is one further question relating to all this that we should ask:

Under what circumstances is the map $\underline{m}$ (or $\underline{e}$), defined as above, invertible?

In giving conditions under which this is so, we shall essentially follow the same techniques used in the previous chapter. In other words, to show that $\underline{m}$ is invertible (in a particular case), we show that $\underline{m}$ can be written as $\underline{m} = \underline{1} + \underline{u}$, where $\underline{1}$ is the identity linear transformation, and $\underline{u}$ is locally nilpotent.

Let $\underline{C}$ be a finitary category, and $(\underline{M},\underline{D})$ a finitary right factorization of $\underline{C}$; suppose $C = \{A_i : i \in I\}$ is a skeletal set for $\underline{C}$, and let $\underline{m}$: $Z\langle C, \underline{C}\rangle \to Z\langle C, \underline{D}\rangle$ be the linear transformation corresponding to the factorization.

The class $\underline{M}$ of monomorphisms is not necessarily closed under composition. It does, however, generate such a class $\underline{M}'$ of monomorphisms; $\underline{M}'$ simply consists of monomorphisms m which can be expressed as a composition $m_1 \circ m_2 \circ \ldots \circ m_r$ (for some r) of monomorphisms m_i in $\underline{M}$. We shall call $\underline{M}'$ the <u>derived class</u> of $\underline{M}$. By an "$\underline{M}$-derived subobject" of a $\underline{C}$-object A we mean a subobject representable by a monomorphism in $\underline{M}'$ (i.e., an $\underline{M}'$-subobject of A).

Now define a relation "$\leqq$" on the skeletal set C as follows:

(3.12) $A_i \leqq A_j$ if there exists $m: A_i \to A_j$, $m \in \underline{M}'$.

<u>Proposition (3.13):</u> The relation $\leqq$ on C defined by (3.12) is a partial order.

<u>Proof:</u> By proposition (3.9), $\underline{M}$ (and therefore $\underline{M}'$) contains all identities. Thus $A_i \leqq A_i$. From the fact that $\underline{C}'$ is closed under composition, it follows at once that $\leqq$ is transitive. Finally, since $\underline{C}$ is finitary, the dual of corollary (4.3) in chapter I holds; that is, if m_1: $A_i \to A_j$ and m_2: $A_j \to A_i$ are both

monomorphisms, then they are both isomorphisms. It follows that $\leq$ is anti-symmetric. QED

As before, let us write $(A_i)^-$ for the principal order ideal $(A_i)^- = \{A_j : A_j \leq A_i\}$. More generally, if W is any subset of C, let $(W)^-$ denote the order ideal generated by W; i.e., $(W)^- = \{ A_j : A_j \leq A_i$ for some A_i in W$\}$. Also write $A_i < A_j$ to mean that $A_i \leq A_j$ but $A_i \neq A_j$.

In the equation $\underline{m}(A_i) = \sum_j m(i,j)A$, it is clear by the definition of $\underline{m}$ that $m(i,j) = 0$ unless $A_j \leq A_i$; while $m(i,i) = 1$ since (by the dual of lemma (4.2) in chapter I) A_i is a representative of an $\underline{M}$-subobject of itself precisely once. Consequently, we can write the defining equation of $\underline{m}$ in the form:

$$(3.14) \qquad \underline{m}(A_i) = A_i + \sum_{A_j < A_i} m(i,j)A_j.$$

In other words, $\underline{m} = \underline{1} + \underline{u}$, where $\underline{1}$ is the identity linear transformation on Z(C) and $\underline{u}$ is defined by:

$$(3.15) \qquad \underline{u}(A_i) = \sum_{A_j < A_i} m(i,j)A_j.$$

Now, it is easily seen that for an element $\underline{w}$ of Z(C), if we denote the support of $\underline{w}$ by supp($\underline{w}$), then:

$$(\mathrm{supp}(\underline{w}))^- \supset (\mathrm{supp}(\underline{u}(\underline{w})))^-,$$

and that the containment is proper if $\underline{w} \neq 0$. Consequently, if $(A_i)^-$ is *finite* for all $i \in I$, we can conclude (as we did in chapter I) that for all i there is a corresponding positive integer n such that $\underline{u}^n(A_i) = 0$, and that therefore $\underline{u}$ is locally nilpotent and $\underline{m} = \underline{1} + \underline{u}$ is invertible.

It is not hard to show that the finiteness of the order ideal $(A_i)^-$ for all indices i is equivalent to the condition that any object A in ob C have only finitely many $\underline{M}$-derived subobjects. Thus we may state these conclusions in the form of the following proposition:

Proposition (3.16): Let $\underline{m}$: $Z\langle C, \underline{C}\rangle \to Z\langle C, \underline{D}\rangle$ be the Z-linear map corresponding to the finitary right factorization $(\underline{M},\underline{D})$ of the finitary category $\underline{C}$.

If each $\underline{C}$-object A has only finitely many $\underline{M}$-derived subobjects, then $\underline{m}$ is invertible.

In most of the cases we shall deal with, the family of all subobjects of a $\underline{C}$-object will be itself finite, and therefore the conditions of this proposition will be met *a fortiori*. The dual of (3.16) can be phrased as follows:

If $\underline{e}$: $Z\langle C, \underline{C}^*\rangle \to Z\langle C, \underline{D}^*\rangle$ is the Z-linear map corresponding to the finitary left factorization $(\underline{D},\underline{E})$

of the finitary category $\underline{C}$, and if the $\underline{E}$-derived quotient objects of any $\underline{C}$-object form a finite set, then $\underline{e}$ is invertible.

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Before we go on to the topics of the next section, we note one application of this invertibility result:

When we form the right linearization $Z\langle C, \underline{C}\rangle$ of a finitary category $\underline{C}$, we would frequently like to know whether or not the "linearizing" homomorphism $\underline{c}: Z(C) \to Z^C$ is faithful. In the first chapter we showed that if mor $\underline{C}$ consisted only of epimorphisms then this was indeed the case. Using proposition (3.16) we can extend this result somewhat further. For suppose $(\underline{M},\underline{D})$ is a right factorization of the finitary category $\underline{C}$ satisfying the conditions of (3.16), and suppose further that mor $\underline{D}$ consists only of epimorphisms. Then the fact that $\underline{d}: Z(C) \to Z^C$ is then faithful, coupled with the invertibility of $\underline{m}$ and the fact that $\underline{c} = \underline{d} \circ \underline{m}$, allows one to say that $\underline{c}$ must also be faithful.

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4. Connections and S-adjointness:

Essentially, a "connection" is a supplementary class of "arrows" going from the objects of one category to the objects of another category. The concept was introduced in [P] as a way of describing the notion of adjointness. We shall find it a convenient tool for generalizing our results still further, as well as a means of introducing a concept of "S-adjointness".

Definition (4.1): Let $\underline{C}$ and $\underline{D}$ be categories. A connection $\underline{W}$ from $\underline{C}$ to $\underline{D}$ is given by the following data:

a) a function which assigns to each pair (A,B) of objects in ob $\underline{C} \times$ ob $\underline{D}$ a set $\underline{W}[A, B]$, whose elements will be called $\underline{W}$-morphisms (with domain A and codomain B);

b) for A_1, $A \in$ ob $\underline{C}$ and B_1, $B \in$ ob $\underline{D}$, both a right composition law which assigns to a $\underline{W}$-morphism $v: A \to B$ and a $\underline{C}$-morphism $f: A_1 \to A$ their "composition" $v \circ f$ in $\underline{W}[A_1, B]$, and also a left composition law which assigns to a $\underline{W}$-morphism $v: A \to B$ and a $\underline{D}$-morphism $g: B \to B_1$ their composition $g \circ v$ in $\underline{W}[A, B_1]$;

c) the stipulation that composition with $\underline{W}$-morphisms is associative and behaves correctly under composition with identities.

Specifically, (c) in the above definition means that:

(i) if $v: A \to B$ is a $\underline{W}$-morphism, then

$$v \circ 1_A = v = 1_B \circ v,$$

and (ii) if we have a diagram of the form:

$$A_2 \xrightarrow{f'} A_1 \xrightarrow{f} A \xrightarrow{v} B \xrightarrow{g} B_1 \xrightarrow{g'} B_2$$

in which v is a $\underline{W}$-morphism, f and f' are $\underline{C}$-morphisms, and g and g' are $\underline{D}$-morphisms, then:

$$(v \circ f) \circ f' = v \circ (f \circ f'),$$

$$(g' \circ g) \circ v = g' \circ (g \circ v),$$

$$\text{and} \quad (g \circ v) \circ f = g \circ (v \circ f).$$

We shall write $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ to indicate that $\underline{W}$ is connection from $\underline{C}$ to $\underline{D}$. The following are a few examples:

a) Suppose $\underline{B}$ and $\underline{D}$ are subcategories of a category $\underline{C}$. Then we have the connection $\underline{W} = \underline{W}(\underline{B}, \underline{D})$ defined by setting $\underline{W}[A, B] = \underline{C}[A, B]$ for all pairs (A,B) in ob $\underline{B}$ × ob $\underline{D}$, with the composition laws inherited from $\underline{C}$.

b) There is a "trivial" connection between any two categories. For any pair (A, B) in ob $\underline{C}$ × ob $\underline{D}$, simply set $\underline{W}[A, B]$ equal to the empty set.

c) Let $\underline{C}$ be the category of groups and group homomorphisms, $\underline{S}$ the category of sets and mappings. Define a connection $\underline{W} = \underline{W}(\underline{S}, \underline{C})$ by letting, for any set X and any group G, $\underline{W}[X, G]$ be the family of all maps from X to (the underlying set of) G, together with the obvious composition of functions.

d) For an example with more of a combinatorial flavour, take the categories $\underline{N}$ (finite sets and mappings) and $\underline{GR}$ (finite graphs and adjacency-reflecting maps). Define $\underline{W} = \underline{W}(\underline{N}, \underline{GR})$ by setting, for any finite set A and finite graph G, $\underline{W}[A, G]$ equal to the family of all maps f from A to V(G) such that, for any x and y in A, f(x) and g(y) are <u>not</u> adjacent in G, together with the obvious composition laws. It is not hard to verify that $\underline{W}$, so defined, is indeed a connection.

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Just as in a category $\underline{C}$ where one can form the set-valued functors $\underline{C}[- ,A]$ and $\underline{C}[A, -]$, so when one is given a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ one can also form:

a) for A in ob $\underline{C}$, the covariant set-valued functor $\underline{W}[A, -]: \underline{D} \to \underline{S}$ under which an object B in ob $\underline{B}$ is mapped to the set $\underline{W}[A, B]$, and a morphism $f: B_1 \to B_2$ is mapped to the function $\underline{W}[A, f] = f_*: \underline{W}[A, B_1] \to \underline{W}[A, -]$

defined by $f_*(v) = f \circ v$ for all v in $\underline{W}[A, B_1]$;

b) for B in ob $\underline{D}$, the contravariant set-valued functor $\underline{W}[-, B]: \underline{C} \to \underline{S}$, under which an object A in ob $\underline{C}$ is mapped to the set $\underline{W}[A, B]$, and a $\underline{C}$-morphism $f: A_1 \to A_2$ to the function $\underline{W}[f, B] = f^*: \underline{W}[A_2, B] \to \underline{W}[A_1, B]$ defined by setting $f^*(v) = v \circ f$ for all v in $\underline{W}[A_2, B]$.

Additionally, if $g: A_1 \to A_2$ is a $\underline{C}$-morphism, then it defines a natural transformation

$$\beta^g: \underline{W}[A_2, -] \to \underline{W}[A_1, -]$$

under which the map (for any B in ob $\underline{D}$)

$$\beta^g{}_B: \underline{W}[A_2, B] \to \underline{W}[A_1, B]$$

is defined by the prescription $\beta^g{}_B(v) = v \circ g$ (for v in $\underline{W}[A_2, B]$).

In a similar manner, a $\underline{D}$-morphism $f: B_1 \to B_2$ defines a corresponding natural transformation α^f from $\underline{W}[-, B_1]$ to $\underline{W}[-, B_2]$.

Following the pattern established in the previous section, one easily verifies that the rule which assigns to each $\underline{D}$-object B the contravariant set-valued functor $\underline{W}[-, B]$ and to a $\underline{D}$-morphism $g: B_1 \to B_2$ the natural transformation α^g can be considered as a (covariant) functor from $\underline{D}$ to the quasi-category $\langle \underline{C}^*, \underline{S} \rangle$. On the other hand, the rule assigning to a $\underline{C}$-object A the set-valued functor $\underline{W}[A, -]$ and to a $\underline{C}$-morphism $g: A_1 \to A_2$ the natural transformation β^g is a (contravariant) functor from $\underline{C}$ to the quasi-category $\langle \underline{D}, \underline{S} \rangle$. This is of course completely analogous to the Yoneda embeddings discussed earlier (except that we do not generally have embeddings from connections, lacking the resources of Yoneda's lemma).

Call a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ finitary if the set $\underline{W}[A, B]$ is finite for all pairs (A,B) in ob $\underline{C} \times$ ob $\underline{D}$.

Let us now consider the situation in which we are given finitary categories $\underline{C}$ and $\underline{D}$, with respective skeletal sets $C = \{A_i : i \in I\}$ and $D = \{B_j : j \in J\}$, and also a finitary connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$. We can then define Z-linear maps $\underline{w}$: $Z(D) \to Z^C$ and $\underline{w}^*$: $Z(C) \to Z^D$ as follows:

For each i in I, define $\underline{w}_i$: $Z(D) \to Z$ on basis elements by:

$$(4.2) \qquad \underline{w}_i(B_j) = \#\underline{W}[A_i, B_j].$$

Then, the family of all such maps defines a corresponding Z-linear map $\underline{w}$: $Z(D) \to Z^C$ under which $\underline{w}(B_j)$ is the function $A_i \mapsto (\underline{w}(B_j))(A_i) = \underline{w}_i(B_j)$. Call $\underline{w}$ the right linearization of $\underline{W}$.

Similarly, if we define, for each j in J, the Z-linear map $\underline{w}_j^*$: $Z(C) \to Z$ on basis elements by:

$$(4.3) \qquad \underline{w}_j^*(A_i) = \#\underline{W}[A_i, B_j],$$

then the family of these maps defines (component-wise) a Z-linear map $\underline{w}^*: Z(C) \to Z^D$, which we shall call the <u>left linearization</u> of $\underline{W}$.

The notions of the left and right linearizations of a finitary connection are clearly dual to one another. (Indeed, it is not hard to see that a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ gives rise to a dual or "opposite" connection $\underline{W}^* = \underline{W}^*(\underline{D}^*, \underline{C}^*)$ obtained by "reversing arrows", and that the left linearization of $\underline{W}$ is the right linearization of $\underline{W}^*$.)

Now suppose $\underline{C}$ and $\underline{D}$ are arbitrary categories. As we have seen, if $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ is a connection, then each $\underline{D}$-object B determines a corresponding (contravariant) set-valued functor $\underline{W}[-, B]$ on $\underline{C}$. Thus we can ask whether or not $\underline{W}[-, B]$ is representable, or S-representable, in $\underline{C}$. We adopt the following terminology:

Call the connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ <u>realizable on the right</u> if the set-valued functor $\underline{W}[-, B]$ is representable in $\underline{C}$ for all B in ob $\underline{D}$. Call it <u>S-realizable on the right</u> if $\underline{W}[-, B]$ is S-representable in $\underline{C}$ for all B in ob $\underline{D}$.

Similarly, $\underline{W}$ is <u>realizable on the left</u> if the the functor $\underline{W}[A, -]$ is representable in $\underline{D}$ for all $\underline{C}$-objects A, and <u>S-realizable on the left</u> if $\underline{W}[A, -]$ is S-representable in $\underline{D}$ for all $\underline{C}$-objects A.

If we are given a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ which is realizable on the right, then it determines a corresponding functor $F: \underline{D} \to \underline{C}$ in the following manner:

For each B in ob $\underline{D}$, let $F(B)$ be a $\underline{C}$-object which represents $\underline{W}[-, B]$, and also let θ^B be a fixed natural equivalence from $\underline{C}[-, F(B)]$ to $\underline{W}[-, B]$. By Yoneda's lemma, θ^B is given by a unique element v_B in $\underline{W}[F(B), B]$, and the set isomorphism from $\underline{C}[A, F(B)]$ to $\underline{W}[A, B]$ provided by θ^B is given by the assignment $f \mapsto v_B \circ f$.

It is not hard to see that the function $B \mapsto F(B)$ extends to a functor F from $\underline{D}$ to $\underline{C}$:

If $g: B_1 \to B_2$ is a $\underline{D}$-morphism, then it determines a corresponding natural transformation α^g from $\underline{W}[-, B_1]$ to $\underline{W}[-, B_2]$; using the natural equivalences θ_{B_1} and θ_{B_2}, one then transfers this to a corresponding natural transformation from $\underline{C}[-,F(B_1)]$ to $\underline{C}[-,F(B_2)]$, which by Yoneda's lemma is given by a unique $\underline{C}$-morphism $F(g)$. That the function so defined is indeed a functor is a matter of routine verification. It is also not hard to show that the functor F is unique up to natural equivalence. We shall call it the <u>right realization</u> of the connection $\underline{W}$.

Conversely, if $F: \underline{D} \to \underline{C}$ is a (covariant) functor, it defines a corresponding connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ obtained by setting $\underline{W}[A, B] = \underline{C}[A, F(B)]$ for all pairs

(A,B) in ob $\underline{C} \times$ ob $\underline{D}$. The composition on the right (i.e., for a morphism $f: A_1 \to A$) is simply given by the usual composition in $\underline{C}$; while the composition on the left (for morphisms $v: A \to F(B)$ in $\underline{W}[A, B]$ and $h: B \to B_1$) is given by the function $(h, v) \mapsto F(h) \circ v$. One verifies without difficulty that $\underline{W}$ so defined is indeed a connection; and because of the way it is defined, we shall denote it by $\underline{C}[-, F(-)]$. It clearly has F as a right realization.

The same reasoning can be carried out on the left. Thus if $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ is a connection which is realizable on the left, then there is a (covariant!) functor $G: \underline{C} \to \underline{D}$ (unique up to natural equivalence) such that $\underline{W}[A, -]$ is naturally equivalent to $\underline{D}[G(A), -]$ for all $\underline{C}$-objects A. We shall call it the <u>left realization of</u> W. Conversely, if $G: \underline{C} \to \underline{D}$ is a covariant functor it defines a corresponding connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ which is denoted by $\underline{D}[G(-), -]$, and which has G as a left realization.

The relationship of these concepts to the notion of adjointness is straightforward:

If $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ is a connection which is realizable on both the right and left, and $G: \underline{C} \to \underline{D}$ and $F: \underline{D} \to \underline{C}$ are the left and right realizations respectively of $\underline{W}$, then (G, F) forms an <u>adjoint pair</u> of functors. One also says that G is a _left adjoint_ of F, and that F is a _right adjoint_ of G. (In many applications of these ideas, one starts with one half of the pair, say F, and then forms the connection $\underline{C}[-, F(-)]$ which, if it is realizable on the left, then gives rise to the left adjoint of F.)

We now proceed to extend these concepts to the case in which a given connection is S-realizable on the left or right. Let us first consider a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ which is S-realizable on the right:

For each $\underline{D}$-object B, let $F(B) = \{F_x(B) : x \in X(B)\}$ be a suitably indexed family of $\underline{C}$-objects which represent $\underline{W}[-, B]$ in $\underline{C}$, and let θ^B be a corresponding natural equivalence from $\bigvee_x \underline{C}[-, F_x(B)]$ to $\underline{W}[-, B]$. Then, by Yoneda's lemma, θ^B is given on each component $\underline{C}[-, F_x(B)]$ by a unique $\underline{W}$-morphism v_{B_x} in $\underline{W}[F_x(B), B]$, and the natural transformation θ_B is thus defined by the family $\{v_{B_x} : x \in X(B)\}$ of these $\underline{W}$-morphisms.

Now, if $g: B_1 \to B_2$ is a $\underline{D}$-morphism, then it defines a corresponding natural transformation α_g from $\underline{W}[-, B_1]$ to $\underline{W}[-, B_2]$, which in turn can be transferred via the the natural equivalences θ^{B_1} and θ^{B_2} to a natural transformation from $\bigvee_x \underline{C}[-, F_x(B_1)]$ to $\bigvee_y \underline{C}[-, F_y(B_2)]$, where the disjoint unions are taken over the indexing sets $X(B_1)$ and $X(B_2)$, respectively. By lemma (2.2), however, such a natural transformation is given by a unique mapping $X(g) = g_*$ from $X(B_1)$ to $X(B_2)$ together with a unique function $F(g)$ from $X(B_1)$ to mor $\underline{C}$, $x \mapsto F(g)_x$, such that $F(g)_x \in \underline{C}[F_x(B_1), F_{g*(x)}(B_2)]$.

This pair of functions (X, F), assigning as it does to any $\underline{D}$-object B the family $\{F_x(B) : x \in X(B)\}$ of $\underline{C}$-objects, and to a $\underline{D}$-morphism g the pair (X(g), F(g)), looks very much like a functor. What must be done now is to describe the category which is the "codomain" of this (putative) functor. We shall do so by constructing, given a category $\underline{C}$, a new category which we shall denote by $S(\underline{C})$. Loosely speaking, $S(\underline{C})$ is the category of "indexed collections" of objects in $\underline{C}$. The formal definition is as follows:

An $S(\underline{C})$-object consists of a pair (X, A) in which X is a set and A is a function from X to ob $\underline{C}$, $x \mapsto A(x)$. We can call such a pair an <u>indexing</u> in $\underline{C}$. A morphism $(X, A) \to (Y, B)$ between two such objects consists of a pair (r, R), in which $r: X \to Y$ is a mapping and $R: X \to$ mor $\underline{C}$, $x \mapsto R_x$, is a function such that R_x is an element of $\underline{C}[A(x), B(f(x))]$ for all x in X.

We must still define the composition law for the morphisms of $S(\underline{C})$. If we have such a morphism (r, R) from the indexing (X^1, A^1) to the indexing (X^2, A^2), and (t, T) is a morphism from (X^2, A^2) to (X^3, A^3), we note that for x in X^1 we get the $\underline{C}$-morphisms $R_x: A^1(x) \to A^2(r(x))$ and $T_{r(x)}: A^2(r(x)) \to A^3(t(r(x)))$.

The mappings more or less "tell" where the $\underline{C}$-morphisms are to go, and in forming the composition of the pairs, we basically follow their "directions". Thus, under the composition, we want to have an arrow from $A(x)$ to $A((t \circ r)(x))$, which is provided by the composition $T_{r(x)} \circ R_x$. We put this idea in more precise terms as follows:

Define the function $T * r: X^1 \to \text{mor } \underline{C}$ by setting $(T * r)_x = T_{r(x)}$ (an element of the morphism set $\underline{C}[A^2(r(x)), A^3((t \circ r)(x))]$). Now define the composition $(t, T) \circ (r, R)$ by means of the prescription:

$$(t, T) \circ (r, R) = (t \circ r, (T * r) \circ R),$$

where $((T * r) \circ R)_x = (T * r)_x \circ R_x$ for all x in X^1.

The verification that $S(\underline{C})$ so defined forms a category is relatively straightforward. Also straightforward (but tedious) is the verification that the function (X, F) defined earlier in terms of the connection $\underline{W}$ and its "right realizability", is indeed a functor from $\underline{D}$ to $S(\underline{C})$.

We have now seen that a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ which is S-realizable on the right, determines a functor $(X,F): \underline{D} \to S(\underline{C})$, which we shall call the <u>right S-realization</u> of $\underline{W}$. It is unique up to natural equivalence.

In the sequel, we shall frequently refer to a functor from $\underline{D}$ to $S(\underline{C})$ as an "S-functor" from $\underline{D}$ to $\underline{C}$.

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Before we go on to dualize the above result (which process, unfortunately, is not quite straightforward), we shall briefly examine the category $S(\underline{C})$.

In the first place, note that $S(\underline{C})$ has a "disjoint union" operation inherited from $\underline{S}$. Given $S(\underline{C})$-objects (X, A) and (Y, B), we define $(X, A) \vee (Y, B)$ as the pair $(X \vee Y, A \upsilon B)$, where $X \vee Y$ is the ordinary disjoint union of sets, and $A \upsilon B$ is the function $X \vee Y \to \text{mor } \underline{C}$ defined by:

$$(A \upsilon B)(x) = \begin{cases} A(x) \text{ if } x \in X, \\ B(x) \text{ if } x \in Y, \end{cases}$$

where we identify X and Y with their natural images in $X \vee Y$. The disjoint union of any family of $S(\underline{C})$-objects is defined similarly.

The disjoint union operation in $S(\underline{C})$ can be shown to be the coproduct in $S(\underline{C})$. The natural injections of (X, A) and (Y, B) into $(X, A) \vee (Y, B)$ are given, respectively, by the pairs (ι_1, E^1) and (ι_2, E^2), where ι_1 and ι_2 are the natural injections of X and Y respectively into $X \vee Y$, and E^1_x is the identity on $A(x)$ for all x in X, and E^2_x is the identity on $B(x)$ for all x in Y. We leave the details to the reader.

The category $\underline{C}$ may itself be regarded as a subcategory of $S(\underline{C})$. Simply consider a $\underline{C}$-object A as being indexed by the singleton set $\{A\}$ consisting of A itself. Under this convention, an indexing (X, A) may be regarded as the disjoint union $\vee_x A(x)$, where x varies over the "base" set X. The fact that $\vee$ is a coproduct in $S(\underline{C})$ leads to the conclusion that the morphism set $S(\underline{C})[(X, A), (Y, B)]$ is naturally equivalent to the cartesian product

$$\times_x S(\underline{C})[A(x), (Y, B)], \text{ where x varies over X.}$$

This is not surprising if one recalls that the cartesian product of an indexed family $\{V_x : x \in X\}$ is simply the family of all functions f from X to the union of the sets V_x such that $f(x) \in V_x$. Applying this to the present situation, and using the fact that an $S(\underline{C})$-morphism from the $\underline{C}$-object A(x) to (Y, B) is simply a rule that selects an element r(x) in Y together with an element R_x in $\underline{C}[A(x), B(r(x))]$, we just recover our original definition of an $S(\underline{C})$-morphism.

On the other hand, it is easy to see that, for a $\underline{C}$-object A and an $S(\underline{C})$-object (Y, B), the morphism set $S(\underline{C})[A, (Y, B)] = S(\underline{C})[A, \vee_y B(y)]$ is naturally equivalent to the disjoint union $\vee_y \underline{C}[A, B(y)]$ (the disjoint unions taken over the set Y). This can be seen by noting that $\vee_y \underline{C}[A, B(y)]$ consists of the family of pairs (y, R_y) with y in Y and R_y in $\underline{C}[A, B(y)]$. But such a pair determines trivially an $S(\underline{C})$-morphism from A to (Y, B), which (since A is being taken as indexed by a singleton set) simply consists of selecting a y in Y and a $\underline{C}$-morphism R_y in $\underline{C}[A, B(y)]$, and vice-versa.

A more important point to notice is the relationship between S-products in $\underline{C}$, and products in $S(\underline{C})$. They are, in fact, essentially identical. In the first place, if $S(\underline{C})$ has products, then the product of two $\underline{C}$-objects exists in $S(\underline{C})$, since we take $\underline{C}$ as a subcategory of $S(\underline{C})$. This product is an indexing of $\underline{C}$-objects, which it is easily verified conforms to the definition of an S-product in $\underline{C}$. Conversely, if $\underline{C}$ has S-products, then the S-product of two $\underline{C}$-objects A and B, $A \pi B$, is essentially a pair $(\mathrm{dom}(A \pi B)), A \pi B)$ in which $A \pi B$ is a function from $\mathrm{dom}(A \pi B)$ to $\underline{C}$; i.e., an element of $S(\underline{C})$. This extends to a product of two $S(\underline{C})$-objects (X, A) and (Y, B) by "distributing" over the index sets. We simply define the product $(X, A) \pi (Y, B)$ to be the disjoint union

$$\bigvee_{(x,y)} (\mathrm{dom}(A(x) \pi B(y)), A(x) \pi B(y)).$$

The reader will verify the requisite universal properties of this definition, utilizing the fact that the cartesian product $X \times Y$ of X and Y is a product in $\underline{S}$, plus the universal properties of the S-product.

Note that, by the very definition of the product in $S(\underline{C})$ obtained from the S-product in $\underline{C}$, it distributes over the disjoint union in $S(\underline{C})$.

Finally, notice that lemma (2.2) can now be viewed as stating that natural transformations from the set-valued functor $\bigvee_x \underline{C}[-, A(x)]$ to $\bigvee_y \underline{C}[-, B(y)]$, where the disjoint unions are taken over the sets X and Y respectively, are in a one-one correspondence with $S(\underline{C})$-morphisms from (X, A) to (Y, B). Indeed, the functor $\bigvee_x \underline{C}[-, A(x)]$ can be regarded as the restriction of the functor $S(\underline{C})[-, (X, A)]$ to the subcategory $\underline{C}$.

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We leave off the discussion of $S(\underline{C})$ for the moment to turn to the problem of the dualization of the above concepts.

This process, as we said, is not quite straightforward. The problem is discernible in the second half of lemma (2.2):

According to the lemma, a natural transformation from the disjoint union $\bigvee_x \underline{D}[A(x), -]$ to the disjoint union $\bigvee_y \underline{D}[B(y), -]$ (with respective index sets X and Y) is given by a pair (F, f) in which $f: Y \to X$ is a mapping and $F: Y \to \text{mor } \underline{D}$, $y \mapsto F_y$, is a function such that F_y is in $\underline{D}[A(f(y)), B(y)]$. The problem is that the function f goes in the "opposite direction" to that of the morphisms F_y.

Thus, if we are given a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ which is S-realizable on the left, and we apply the same reasoning used above in the case of right "S-realizability", we derive a pair (F, Y) which assigns to each $\underline{C}$-object A a pair (F(A), Y(A)) in which Y(A) is a set and F(A) a function $Y(A) \to \text{mor}\ \underline{D}$, $x \mapsto F(A)_x$, and to each $\underline{C}$-morphism $f\colon A_1 \to A_2$ a pair (F(f), Y(f)) in which $Y(f) = f^*\colon Y(A_2) \to Y(A_1)$ is a mapping and F(f) is a function from $Y(A_2)$ to mor $\underline{D}$ such that $F(f)_x$ is an element of $\underline{D}[F_{f^*(x)}(A_1), F_x(A_2)]$ for all x in $Y(A_2)$. Of course, (F, Y) has the property that, for every $\underline{C}$-object A, $\underline{W}[A, -]$ is naturally equivalent to $\bigvee_x \underline{D}[F_x(A), -]$.

If we look upon this pair as a functor, it appears to be contravariant on the base sets, but covariant (more or less) as far as $\underline{D}$-morphisms are concerned. One way of solving this problem is by creating a second "indexing" category, which we shall denote by $S^*(\underline{D})$, to be the codomain of this (putative) functor:

The idea is to regard $\underline{D}$ as being indexed by $\underline{S}^*$, the category dual to $\underline{S}$. (We will not, however, find it necessary to bring $\underline{S}^*$ explicitly into the picture.) An object of $S^*(\underline{D})$ is a pair (A, Y) in which Y is a set and A: $Y \to \text{ob } \underline{D}$ $(x \mapsto A(x))$ is a function. (This is the same as in $S(\underline{D})$; the change in the order of the pair is for somewhat greater convenience in the writing of the composition of two morphisms.)

An $S^*(\underline{D})$-morphism $(A_1, Y_1) \to (A_2, Y_2)$ consists of a pair (R,r) in which $r: Y_2 \to Y_1$ is a mapping, and R is a function from Y_2 to mor $\underline{D}$ $(x \mapsto R_x)$ such that R_x is an element of $\underline{D}[A_1(r(x)), A_2(x)]$ for all x in Y_2. (Think of r as being an $\underline{S}^*$-morphism, and hence going from Y_1 to Y_2.)

If we have two such morphisms, (R,r) from (A_1, Y_1) to (A_2, Y_2), and (T,t) from (A_2, Y_2) to (A_3, Y_3), then define the composition $(T, t) \circ (R, r)$ by:

$$(T, t) \circ (R, r) = (T \circ (R * t), r \circ t),$$

where $(T \circ (R * t))_x = T_x \circ R_{t(x)}$. Also note the inversion in the order of composition of the set mappings (which of course is consistent with regarding them as morphisms in $\underline{S}^*$). Again, one can verify that $S^*(\underline{D})$ is a category. Also, we have the following result:

If $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ is a connection which is S-realizable on the left, then there is a functor $(F, Y): \underline{C} \to S^*(\underline{D})$ such that the set-valued functor $\underline{W}[A, -]$ (for all $\underline{C}$-objects A) is naturally equivalent to $\bigvee_x \underline{D}[F_x(A), -]$. The functor (F, Y) is unique up to natural equivalence, and we shall call it the <u>left S-realization</u> of the connection $\underline{W}$.

Again, we shall speak of a functor from $\underline{C}$ to $S^*(\underline{D})$ as an "S-functor" from $\underline{C}$ to $\underline{D}$; but to distinguish it from functors $\underline{C} \to S(\underline{D})$, we shall dub it a "semi-contravariant" S-functor.

The category $S^*(\underline{D})$ deserves a few remarks (the proofs of which, however, will be left to the reader). In the first place, $S^*(\underline{D})$ also inherits a disjoint union from $\underline{S}$ (or rather, $\underline{S}^*$). The disjoint union in this case, however, is a <u>product</u> rather than a coproduct (consistent with the fact that the coproduct in $\underline{S}$ is the product in $\underline{S}^*$). In order to avoid collision with the notation under which $\vee$ denotes the coproduct in $\underline{S}$ (as well as some other categories), we shall use the symbol $\wedge$ to denote the disjoint union operation in $S^*(\underline{D})$. Thus, for $S^*(\underline{D})$-objects (A, X) and (B, Y), their disjoint union $(A, X) \wedge (B, Y)$ is (still, as in $S(\underline{D})$) the pair $(A \cup B, X \vee Y)$.

The following additional statements concerning $S^*(\underline{D})$ are not very hard to establish:

If $S^*(\underline{D})$ has a coproduct, then $\underline{D}$ has a corresponding S-coproduct. Conversely, if $\underline{D}$ has an S-coproduct, it extends to a coproduct in $S^*(\underline{D})$. If μ is such a coproduct in $S^*(\underline{D})$, it distributes over the disjoint union $\wedge$ in $S^*(\underline{D})$.

Any $S^*(\underline{D})$-object (A, X) can be regarded as the disjoint union $/\backslash_x\ A(x)$ (x varying over X). In line with the fact that $/\backslash$ is a product in $S^*(\underline{D})$, the morphism set $S^*(\underline{D})[(B, Y), (A, X)]$ is naturally equivalent to the cartesian product

$$\mathop{X}_x S^*(\underline{D})[(B, Y), A(x)],$$ where x varies over X.

Again, $\underline{D}$ can be regarded as a subcategory of $S^*(\underline{D})$. Adopting this convention, we find that the morphism set $S^*(\underline{D})[(B, Y), A]$, for A in ob $\underline{D}$, is naturally equivalent to the $\backslash/_y\ \underline{D}[B(y), A]$ (y in Y).

Natural transformations from $\backslash/_x\ \underline{D}[A(x), -]$ to $\backslash/_y\ \underline{D}[B(y), -]$ (with respective index sets X and Y) are in a one-one correspondence with $S^*(\underline{D})$-morphisms from (A, X) to (B, Y). Indeed, the set-valued functor $\backslash/_x\ \underline{D}[A(x), -]$ may be regarded as the restriction to $\underline{D}$ of the functor $S^*(\underline{D})[(A, X), -]$.

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If (X, F) is an S-functor from $\underline{D}$ to $\underline{C}$ (i.e., a functor from $\underline{D}$ to $S(\underline{C})$), then X is simply a functor from $\underline{D}$ to $\underline{S}$, which we can call the "set-theoretic part" of (X, F). Similarly, in a semi-contravariant S-functor (F, Y) from $\underline{C}$ to $\underline{D}$, Y is simply a contravariant set-valued functor, the "set-theoretic" part of (F, Y). In either case, if the set-theoretic part takes on only finite sets as values, we shall call the S-functor in question <u>finitary</u>.

Now let $\underline{C}$ and $\underline{D}$ be finitary categories with skeletal sets $C = \{A_i : i \in I\}$ and $D = \{B_j : j \in J\}$ respectively. Suppose (X, F) is a finitary S-functor from $\underline{D}$ to $\underline{C}$. We then define the Z-linear map $\underline{r}: Z(D) \to Z(C)$ <u>derived from</u> (X, F) on basis elements by:

$$(4.4) \qquad \underline{r}(B_j) = \sum_{x \in X(B_j)} \langle F_x(B_j) \rangle .$$

Here we are using the angle-bracket convention. Clearly this can also be written as:

$$(4.5) \qquad \underline{r}(B_j) = \sum_i r(j,i) A_i ,$$

where $r(j,i)$ is the number of elements x in $X(B_j)$ such that $F_x(B_j)$ is isomorphic to A_i.

The Z-linear map derived from a semi-contravariant S-functor is defined in the same way. The following proposition is virtually immediate from our definitions:

<u>Proposition (4.6):</u> Suppose $\underline{C}$ and $\underline{D}$ are finitary categories with skeletal sets $C = \{A_i : i \in I\}$ and $D = \{B_j : j \in J\}$ respectively; and suppose $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ is a connection which has a finitary right realization $(X, F): \underline{D} \to S(\underline{C})$.

Let $\underline{w}$: $Z(D) \to Z^C$ be the right linearization of $\underline{W}$, and $\underline{r}$: $Z(D) \to Z(C)$ be the Z-linear map derived from (X, F). Then the following diagram commutes:

$$\begin{tikzcd} Z(D) \arrow[r, "\underline{w}"] \arrow[d, "\underline{r}"'] & Z^C \\ Z(C) \arrow[ur, "\underline{c}"'] & \end{tikzcd}$$

where of course $\underline{c}$ is the right linearization of $\underline{C}$.

We omit the proof, which is straightforward.

Of course the dual proposition holds, and may be stated as follows:

Suppose the situation is as above, except that now $\underline{W}$ has a finitary left realization (F, Y): $\underline{C} \to S^*(\underline{D})$, and let $\underline{s}$: $Z(C) \to Z(D)$ be the Z-linear map derived from (F, Y). Then the following diagram commutes:

$$\begin{tikzcd} Z(C) \arrow[r, "\underline{w}^*"] \arrow[d, "\underline{s}"'] & Z^D \\ Z(D) \arrow[ur, "\underline{d}^*"'] & \end{tikzcd}$$

* * *

Just as the simultaneous left and right realizability of a connection leads to the concept of adjointness, so we may speak of <u>S-adjointness</u> in the case of simultaneous left and right S-realizability. It turns out that there is not much of interest in such a general situation. What is of interest is the case in which one half of such an "S-adjoint" pair is a functor rather than an S-functor. Let us be more specific:

We shall say that the functor $G: \underline{C} \to \underline{D}$ has a <u>right S-adjoint</u> if G is the left realization of a connection $\underline{W} = \underline{W}(\underline{C}, \underline{D})$ (which we may take to be $\underline{D}[G(-), -]$), and $\underline{W}$ also has a right S-realization $(X, F): \underline{D} \to S(\underline{C})$. In such a case, we shall call (X, F) the right S-adjoint of G. Similarly, G will be said to have a <u>left S-adjoint</u> if $G: \underline{D} \to \underline{C}$ is the right realization of $\underline{W} = \underline{W}(\underline{C}, \underline{D})$, which also has a left S-realization $(F, Y): C \to S^{*}(\underline{D})$.

Let us consider the finitary case:

Thus, we suppose $\underline{C}$ and $\underline{D}$ are finitary categories having respective skeletal sets $C = \{A_i : i \in I\}$ and $D = \{B_j : j \in J\}$, and $G: \underline{D} \to \underline{C}$ is a functor with a finitary right S-adjoint $(X, F): \underline{C} \to S(\underline{D})$. That is, there is a connection $\underline{W} = \underline{W}(\underline{D}, \underline{C})$ (which may be taken to be $\underline{C}[G(-), -]$) that has G as its left realization and (X, F) as its right S-realization.

Note that G determines a corresponding function $g: D \to C$ defined by setting:

$$g(B_j) = \langle G(B_j)\rangle;$$

i.e., $g(B_j) = A_i$, where A_i is the unique element of C isomorphic to $G(B_j)$. For ease of notation, we transfer the function g to a function $g\colon J \to I$ between the index sets for D and C (and use g to denote both). Thus, we write:

$$g(B_j) = \langle G(B_j)\rangle = A_{g(j)}.$$

In this situation, we also have the Z-linear map $\underline{r}\colon Z(C) \to Z(D)$ derived from the S-functor (X, F).

Proposition (4.7): Let $\underline{C}$, $\underline{D}$; C, D; $G\colon \underline{D} \to \underline{C}$, $\underline{W} = \underline{W}(\underline{D}, \underline{C})$, $(X, F)\colon \underline{C} \to \underline{D}$; and $g\colon D \to C$ (as well as $J \to I$), and $\underline{r}\colon Z(C) \to Z(D)$ be as described above. Further, let $\underline{d}\colon Z(D) \to Z^D$ and $\underline{c}\colon Z(C) \to Z^D$ be the right linearizations of $\underline{C}$ and $\underline{D}$ respectively.

Then, the following diagram commutes for all indices j in J:

$$\begin{tikzcd}
Z(C) \arrow[d, "\underline{r}"'] \arrow[dr, "\underline{c}_{g(j)}"] & \\
Z(D) \arrow[r, "\underline{d}_j"'] & Z
\end{tikzcd}$$

Proof: We wish to show that for all j in J and i in I that the following equation holds:

$$\underline{d}_j(\underline{r}(A_i)) = \underline{c}_{g(j)}(A_i).$$

The right hand side is equal to $\#\underline{C}[G(B_j), A_i]$, while it is not hard to see that the left hand side is equal to

$\#(\setminus\!/_x\ \underline{D}[B_j,\ F_x(A_i)]$, the disjoint union taken over x in $X(A_i)$. But, by assumption, both of these are equal to $\#\underline{W}[B_j,\ A_i]$. QED

Of course, the dual of the above proposition also holds. Thus, if $G\colon \underline{D} \to \underline{C}$ has a finitary left S-adjoint $(F, Y)\colon \underline{C} \to S^*(\underline{D})$, then the following diagram commutes for all j in J:

$$\begin{tikzcd} Z(C) \arrow[d, "\underline{s}"'] \arrow[dr, "\underline{c}^*_{g(j)}"] & \\ Z(D) \arrow[r, "\underline{d}^*_j"'] & Z \end{tikzcd}$$

where $\underline{s}$ is the Z-homomorphism derived from (F, Y) and $g\colon C \to D$ and $J \to I$ is the function defined by G.

We have already seen examples of both a right S-adjoint and a left S-adjoint of a functor. These are given by a locally small factorizations (right and left, respectively).

For example, if $\underline{C}$ has a right factorization $(\underline{M},\underline{D})$, then we have the inherited connection $\underline{W} = \underline{W}(\underline{D},\ \underline{C})$ defined by setting $\underline{W}[B,\ A] = \underline{C}[B,\ A]$ for a $\underline{D}$-object B and a $\underline{C}$-object A (of course, ob $\underline{C}$ = ob $\underline{D}$). Then $\underline{W}$ clearly has as a left realization the <u>inclusion</u> functor $G\colon \underline{D} \to \underline{C}$. But, if the factorization is locally small, then the connection also is S-realizable on the right;

i.e., $W[-, A]$ is just the restriction of $\underline{C}[-, A]$ to $\underline{D}$, and as we showed, under these conditions $\underline{W}[-, A]$ is S-representable for all A in ob $\underline{C}$. Thus, we have a right realization (X, F): $\underline{C} \to S(\underline{D})$, under which the set $\{F_x(A) : x \in X(A)\}$ picks one representative for each distinct $\underline{M}$-subobject of A. Of course, if $\underline{C}$, $\underline{D}$, and $(\underline{M},\underline{D})$ are all finitary, then (X, F) is finitary, and the homomorphism derived from (X, F) is simply the homomorphism $\underline{t}$ corresponding to the factorization $(\underline{M},\underline{D})$. Similar remarks apply to a (locally small) left factorization $(\underline{D},\underline{E})$ of $\underline{C}$, which gives rise to a left S-adjoint of the inclusion functor.

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An important feature of adjointness is the well-known theorem which asserts that if a functor F is a right adjoint of another functor, then F preserves products (as well as other categorical limits). We shall prove an analogue of this in the case of S-adjointness; but first, in the way of preparation, let us briefly review the proof of the "classical" result:

Suppose then that $\underline{C}$ and $\underline{D}$ are categories with products π and $\tilde{\pi}$, respectively, and that $F: \underline{C} \to \underline{D}$ is the right adjoint of a functor $G: \underline{D} \to \underline{C}$. Essentially, we wish to prove that the set-valued functors $\underline{D}[-, F(A) \tilde{\pi} F(B)]$ and $\underline{D}[-, F(A \pi B)]$ are naturally equivalent for all $\underline{C}$-objects A and B. Then, by the uniqueness of representations, this yields the result that $F(A) \tilde{\pi} F(B)$ is isomorphic to $F(A \pi B)$.

Of course, by definition $\underline{D}[-, F(A) \tilde{\pi} F(B)]$ is naturally euivalent to $\underline{D}[-, F(A)] \times \underline{D}[-, F(B)]$. By adjointness, $\underline{D}[-, F(A \pi B)]$ is naturally equivalent to $\underline{C}[G(-), A \pi B]$. Also by adjointness, $D[-, F(A)] \times \underline{D}[- ,F(B)]$ is naturally equivalent to $\underline{C}[G(-), A] \times \underline{C}[G(-), B]$. But it is easy to prove that $\underline{C}[G(-), A] \times \underline{C}[G(-), B]$ and $\underline{C}[G(-), A \pi B]$ are naturally equivalent. Thus the result follows.

Of course, in the same way we have the result that left adjoints preserve coproducts.

We cannot extend this result to the case of a connection which is simultaneously both right and left S-realizable (as one might think). But we can extend it to the case in which a functor $G: \underline{D} \to \underline{C}$ has a right S-adjoint $(X, F): \underline{C} \to S(\underline{D})$.

To do so, we must return to our study of the category $S(\underline{D})$.

Suppose $(X, F): \underline{C} \to S(\underline{D})$ is a functor. Then it extends naturally to a functor $S(X, F): S(\underline{C}) \to S(\underline{D})$. The manner in which this is done is as follows:

An object (Y, A) in ob $S(\underline{C})$ can, as we have noted, be thought of as a disjoint union $\bigvee_y A(y)$, the disjoint union being over y in Y. Thus, we simply let

$S(X, F)(Y, A)$ be the disjoint union, in $S(\underline{D})$, $\bigvee_y (X(A(y)), F(A(y)))$. Similarly, given an $S(\underline{C})$-morphism $(r, R)\colon (Y_1, A_1) \to (Y_2, A_2)$, we take the images of the morphisms (as y varies over Y_1) $R_y\colon A_1(y) \to A_2(r(y))$ under the functor (X, F) and "glue" them together via the disjoint union in $S(\underline{D})$. (The details, though messy, are straightforward). One should note that, by the very definition of the extension $S(X, F)$, the extended functor preserves coproducts (i.e., the disjoint unions in $S(\underline{C})$ and $S(\underline{D})$).

Of course, this extension also works for a functor $G\colon \underline{D} \to \underline{C}$, which, since we regard $\underline{C}$ as a subcategory of $S(\underline{C})$, can also be looked upon as a functor from $\underline{D}$ to $S(\underline{C})$. We denote this extension simply by $S(G)$, from $S(\underline{D})$ to $S(\underline{C})$. Under it an $S(\underline{D})$-object (Y, B) is mapped to the pair $(Y, G(B))$, where $G(B)(y)$ is simply $G(B(y))$.

With this preparatory work out of the way, we may now state the following result:

Proposition (4.8): Suppose the functor $G\colon \underline{D} \to \underline{C}$ has a right S-adjoint $(X, F)\colon \underline{C} \to S(\underline{D})$. Then $S(X, F)$ is a right adjoint of the functor $S(G)$.

Proof: We shall merely sketch the proof.

For A in ob $\underline{D}$ and (V, B) in ob $S(\underline{C})$, the morphism set $S(\underline{C})[G(A), (V,B)]$ is, as we noted earlier, in a natural one-one correspondence with the (set) disjoint union $\bigvee_v \underline{C}[G(A), B(v)]$, where v varies over the set V. Since (X, F) is a right S-adjoint of G, this disjoint union is in a natural one-one correspondence with the (double) disjoint union $\bigvee_v \{\bigvee_x \underline{D}[A, F_x(B(y))]\}$, where the x varies, for each v, over X(B(v)). This last disjoint union, however, is identifiable with the morphism set $S(\underline{D})[A, S(X,F)(V, B)]$.

The above one-one correspondences are "natural", and imply that $S(\underline{C})[G(A), -]$ is naturally equivalent to $S(\underline{D})[A, S(X,F)(-)]$ for all $\underline{D}$-objects A.

On the other hand, using the fact that the disjoint union in $S(\underline{C})$ and $S(\underline{D})$ are their coproducts, and that consequently since an $S(\underline{D})$-object (Y, A) can be regarded as the disjoint union $\bigvee_y A(y)$, we have that $S(\underline{C})[S(G)(Y,A), -] = S(\underline{C})[(Y, G(A)), -]$ is naturally equivalent to the cartesian product

$$\mathop{X}_{y} S(\underline{C})[G(A(y)), -], \text{ y varying over Y,}$$

which in turn by our above remarks is naturally equivalent to:

$$\mathop{X}_{y} S(\underline{D})[A(y), S(X, F)(-)], \text{ y in Y}$$

which in turn is equivalent to $S(\underline{D})[(Y,A), S(X, F)(-)]$.

QED

We shall now briefly discuss the dualization of the preceding result. In a manner essentially the same as that used above, a functor $(F, Y)\colon \underline{C} \to S^*(\underline{D})$ can be extended (via the disjoint union $/\backslash$) to a functor $S^*(F, Y)\colon S^*(C) \to S^*(\underline{D})$. Similarly, a functor $G\colon \underline{D} \to \underline{C}$ extends to a functor $S^*(G)\colon S^*(\underline{D}) \to S^*(\underline{C})$. Then, as before, if (F, Y) is a left S-adjoint of G, then $S^*(F, Y)$ is a left adjoint of $S^*(G)$. Recall, however, that the disjoint union $/\backslash$ in $S^*(\underline{C})$ is a product, as it is in $S^*(\underline{D})$, and the functors $S^*(G)$ and $S^*(F, Y)$ are by their very definition product-preserving.

Now if $\underline{C}$ and $\underline{D}$ have S-products, then $S(\underline{C})$ and $S(\underline{D})$ have products (as we indicated above). Thus, if the functor $G\colon \underline{D} \to \underline{C}$ has a right S-adjoint $(X, F)\colon \underline{C} \to S(\underline{D})$ then $S(X, F)$ is a right adjoint of $S(G)$, and consequently it preserves products. Notice that since $S(G)$ is a left adjoint, it must preserve coproducts, but this represents no new information, since it preserves coproducts by definition. Similarly, if $(F, Y)\colon \underline{C} \to S^*(\underline{D})$ is a left S-adjoint of G, and $\underline{C}$ and $\underline{D}$ have S-coproducts, then $S^*(F, Y)$ preserves coproducts (but again we gain no new information about G).

Finally, let us apply these concepts to the finitary case:

Proposition (4.9): Suppose we are given finitary categories $\underline{C}$ and $\underline{D}$, both of which have finitary S-products. Also suppose the functor $G: \underline{D} \to \underline{C}$ has a finitary right S-adjoint $(X, F): \underline{C} \to S(\underline{D})$.

Let $C = \{A_i : i \in I\}$ and $D = \{B_j : j \in J\}$ be respective skeletal sets for $\underline{C}$ and $\underline{D}$, let $Z\langle C, \underline{C}\rangle$ and $Z\langle D, \underline{D}\rangle$ be given the multiplications derived from the respective S-products, and let $\underline{r}: Z(C) \to Z(D)$ be the Z-linear map derived from (X, F). Then $\underline{r}$ is a ring homomorphism.

Proof: The proof of this proposition merely requires a proper interpretation of the fact that $S(X, F)$ preserves products.

Let us denote by $N(\underline{C})$ the full subcategory of $S(\underline{C})$ generated by objects (Y, A) in which the set Y is finite. We shall extend our angle bracket convention by writing, for each $N(\underline{C})$-object (Y, A), the equation:

$$\langle Y, A\rangle = \sum_{y \in Y} \langle A(y)\rangle.$$

Now, it is easy to establish that, under this extension of the bracket notation, we have the following relations:

$$(4.10)\quad \begin{aligned} &(a)\ \langle (Y, A) \vee (V, B)\rangle = \langle (Y, A)\rangle + \langle (V, B)\rangle \\ &(b)\ \langle (Y, A)\ \pi\ (V, B)\rangle = \langle (Y, A)\rangle \circ \langle (V, B)\rangle \end{aligned}$$

in which π is the product in $S(\underline{C})$, and '$\circ$' is the multiplication derived from the S-product in $\underline{C}$. Also notice that if (Y, A) and (V, B) are isomorphic in $S(\underline{C})$, then $\langle (Y, A)\rangle = \langle (V, B)\rangle$. Of course similar remarks apply to $N(\underline{D})$. (Notice that it is the fact that the S-products are finitary that allows $N(\underline{C})$ and $N(\underline{D})$ to have products.)

Now we can also establish without difficulty that, for any S-functor $(X, F): \underline{C} \to S(\underline{D})$, we have the relation:

$$(4.11) \quad \langle S(X, F)(Y, A)\rangle_D = \underline{r}(\langle (Y, A)\rangle_C),$$

where $\underline{r}$ is the Z-linear map derived from (X, F), and we use the notation $\langle\rangle_D$ and $\langle\rangle_C$ to indicate that the elements so designated are in Z(D) and Z(C) respectively. In particular, we see that on basis elements A_i, we have that $\underline{r}(A_i) = \langle S(X, F)(A_i)\rangle = \langle (X, F)(A_i)\rangle$.

Thus, for A and B in ob C (and of course also considered as elements of ob $S(\underline{C})$), we have that $S(X, F)(A \pi B)$ is (since $S(X, F)$ is a right adjoint) isomorphic to $(S(X, F)(A)) \tilde{\pi} (S(X, F)(B))$, where '$\tilde{\pi}$' denotes the product in $S(\underline{D})$. Now, by the relations established above, the result follows without difficulty. QED

The dual result is, of course, that if $G: \underline{D} \to \underline{C}$ has a finitary left S-adjoint $(F, Y): \underline{C} \to \underline{D}$, and both $\underline{C}$ and

$\underline{D}$ have finitary S-products, then the map $\underline{r}$: $Z(C) \to Z(D)$ (derived from (F, Y)) is a ring homomorphism if $Z(C)$ and $Z(D)$ are given the multiplications derived from the respective S-products.

Example: Consider the category $\underline{N}$ of finite sets and mappings, together with the connection $\underline{W} = \underline{W}(\underline{N}, \underline{N})$ defined by letting $\underline{W}[A, B]$ be the set of all partial functions from A to B (A and B being finite sets), together with the obvious compositions.

Note that $\underline{W}$ is realizable on the right:

Let $\{p\}$ be a fixed singleton set. Then, for all finite sets B, $\underline{W}[-, B]$ is naturally equivalent to $\underline{N}[-, B \vee \{p\}]$. The correspondence between $\underline{W}[A, B]$ and $\underline{N}[A, B \vee \{p\}]$ is obtained as follows:

If $h: A \to B$ is a partial function, then it extends to a unique function $\tilde{h}: A \to B \vee \{p\}$ which maps all of the points outside the domain of definition of h into the point p. Conversely, if $\tilde{h}: A \to B \vee \{p\}$ is a mapping, it defines a corresponding partial function $h: A \to B$ defined on all points except those which were originally mapped onto p by $\tilde{h}$.

Of course the function $B \mapsto B \vee \{p\}$ extends to a functor $G: \underline{N} \to \underline{N}$ which is the right realization of $\underline{W}$. Notice that G does not preserve products. It cannot therefore have a left adjoint. It does, however, have a left S-adjoint, since $\underline{W}$ is S-realizable on the left:

For all finite sets A, $\underline{W}[A, -]$ is naturally equivalent to the disjoint union $\bigvee_x \underline{N}[A_x, -]$, where the A_x vary over the subsets of A. This is easily seen since a partial fuction on A corresponds to a unique function on a subset of A (namely, the domain of definition of the partial function), while a function defined on a subset of A clearly corresponds to a partial function on A.

Since a left S-realization is involved, the rule that assigns to a set A the (appropriately indexed) family of subsets of A extends to a functor $(F, X)\colon \underline{N} \to S^*(\underline{N})$. We can write the image of a set A under this functor as $(F, X)(A) = \bigwedge_x A_x$, the A_x varying over the subsets of A. (Of course, we are using the notation established earlier, under which $\bigwedge$ represents the disjoint union operation in $S^*(\underline{N})$, which happens to be the product in that category.) According to our results above, this functor (or rather its extension to a functor $S^*(F, X)$ defined on $S^*(\underline{N})$) preserves S-coproducts. Since the S-coproduct in $\underline{N}$ is in fact a coproduct (given by the disjoint union in $\underline{N}$, $\bigvee$), we expect (F, X) to preserve coproducts. Let us see how this looks:

We must have that, for a pair of sets A and B, $(F, X)(A \vee B) = \bigwedge_w D_w$, the D_w varying over the subsets of $A \vee B$, is naturally isomorphic with the indexing $(F, X)(A) \vee (F, X)(B) = (\wedge_x A_x) \vee (\wedge_y B_y)$.

Note that in the last equation above, '$\vee$' denotes the extension of the coproduct $\vee$ in $\underline{N}$ to $S^*(\underline{N})$, and in $S^*(\underline{N})$ the coproduct distributes over $\wedge$. Thus the last expression is equivalent to $\wedge_{(x,y)} (A_x \vee B_y)$. But now the equivalence we seek to verify is obvious, since it merely states that every subset of $A \vee B$ corresponds to a disjoint union $A_x \vee B_y$, where A_x is a subset of A and B_y is a subset of B.

Let us now look at the Z-linear map derived from (F, X). Let $N = \{A_0, A_1, \ldots, A_n, \ldots\}$ be a skeletal set for $\underline{N}$ in which A_n is an n-element set. The Z-linear map $\underline{r}$ derived from (F, X) is clearly given by the formula:

$$\underline{r}(A_n) = \sum_k C(n,k)A_k,$$

where of course $C(n,k)$ is the number of k-subsets of an n-set. As a ring, $Z\langle N, \underline{N}^*\rangle$ is (as we saw in chapter I) isomorphic to $Z[x]$. Thus if we transfer these considerations to $Z[x]$ via this isomorphism, we can write $\underline{r}$ as the map $Z[x] \to Z[x]$ defined by:

$$\underline{r}(x^n) = \sum_k C(n,k)x^k.$$

Now the map $g\colon N \to N$ defined by the functor G is clearly given by $g(A_i) = A_{i+1}$. The map $\underline{n}_i^*$, looked at from the point of view of Z[x], simply corresponds to evaluation at x = i. According to (the dual of) proposition (4.7), we should have that $\underline{n}_i^* \circ \underline{r}$ is equal to $\underline{n}^*_{g(i)} = \underline{n}^*_{i+1}$. In Z[x], this is equivalent to the equation:

$$(i + 1)^n = \sum_k C(n,k)i^k,$$

which is certainly true since $\underline{r}(x_n)$ is obviously $(x + 1)^n$. The dual of proposition (4.9) asserts that $\underline{r}$ must be a ring homomorphism, and again this is clearly the case.

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CHAPTER III

FURTHER RESULTS AND APPLICATIONS

In this chapter we shall slightly extend the theory developed in the previous chapters, and show their relationship to some established results.

In the first section, we look at categories whose objects can be regarded as being made up of "connected components". In section 2, we show how certain well known results concerning finite vector spaces may be derived by our methods. Finally in section 3 we make a short study of categories all of whose morphisms are epimorphisms (or, dually, monomorphisms). For example, we shall see that the "Möbius transform" can be regarded as the inverse of a Z-linear map corresponding to a factorization. More interesting, however, is the relationship between such categories and the poset of quotient objects of an object of the category.

1. Connectivity in categories:

In the preceding section, the category $S(\underline{C})$ was introduced primarily as a technical device in order to allow us to state certain results concerning S-functors and S-adjoints. It turns out, however, that it is not unusual for a category to be naturally equivalent to $S(\underline{C})$ for a suitable subcategory $\underline{C}$. The common denominator for such situations is that the category in question allow a notion of an object's being "connected" or not in some sense. In combinatorial applications (where "finitariness" usually reigns), the category of interest is frequently naturally equivalent to $N(\underline{D})$ for some suitable subcategory $\underline{D}$.

For example, one can verify without difficulty that the category $\underline{G}$ (finite graphs and adjacency-preserving maps) is naturally equivalent to $N(\underline{D})$, where $\underline{D}$ is the full subcategory of $\underline{G}$ generated by connected graphs.

Since in this section we shall be principally interested in categories of the form $N(\underline{D})$ rather than $S(\underline{D})$, the following easily verifiable results are in order:

a) A finitary functor (X, F): $\underline{D} \to S(\underline{C})$ can (and in this section, will) be regarded as a functor from $\underline{D}$ to $N(\underline{C})$. Such a functor extends to a functor $N(X, F)$ from $N(\underline{D})$ to $N(\underline{C})$; the functor $N(X, F)$ preserves coproducts (i.e., the disjoint union in $S(\underline{D})$ and $S(\underline{C})$).

In particular, if a connection has a finitary right

S-realization, it will be regarded as a functor taking its values in a category of the form $N(\underline{D})$.

b) The category $N(\underline{D})$ has products if and only if $\underline{C}$ has finitary S-products.

c) If the functor $G\colon \underline{C} \to \underline{D}$ has a finitary right S-adjoint $(X, F)\colon \underline{D} \to S(\underline{C})$, then $N(G)$ has $N(X, F)$ as a right adjoint.

Let us now turn to the question of "connectedness". The following definition represents one way of conceiving of this notion:

Definition (1.1): Let $\underline{C}$ be a category which has a coproduct μ. Then, a $\underline{C}$-object A will be said to be *connected* in $\underline{C}$ if, for any pair (B_1, B_2) of $\underline{C}$-objects, any $\underline{C}$-morphism $f\colon A \to B_1 \,\mu\, B_2$ factors uniquely through one (and only one) of the natural injections $\iota_1\colon B_1 \to B_1 \,\mu\, B_2$ and $\iota_2\colon B_1 \to B_1 \,\mu\, B_2$; that is, there is a unique morphism $\hat{f}$ in $\underline{C}[A, B_1] \cup \underline{C}[A, B_2]$ such that either $f = \iota_1 \circ \hat{f}$ or $f = \iota_2 \circ \hat{f}$ (but not both).

We can phrase the definition alternatively in the following manner:

A pair (B_1, B_2) of $\underline{C}$-objects, together with the natural injections ι_1 and ι_2 of B_1 and B_2 respectively into their coproduct $B_1 \,\mu\, B_2$ determines a natural transformation $\lambda = \lambda(B_1, B_2)$,

$$\lambda\colon \underline{C}[-, B_1] \vee \underline{C}[-, B_2] \to \underline{C}[-, B_1 \mu B_2],$$

simply defined on each component of the disjoint union as the natural transformation corresponding to ι_1 or ι_2.

Thus, A is connected in $\underline{C}$ if for all pairs (B_1, B_2) of $\underline{C}$-objects, the function λ_A from $\underline{C}[A, B_1] \vee \underline{C}[A, B_2]$ to $\underline{C}[A, B_1 \mu B_2]$ determined by the natural transformation $\lambda = \lambda(B_1, B_2)$ is a set isomorphism.

Now suppose that the class of connected objects in $\underline{C}$ is non-empty, and let $\underline{D}$ be the full subcategory of $\underline{C}$ generated by its connected objects. Call a $\underline{C}$-object A <u>componented</u> if it is isomorphic to a coproduct of a finite number of connected objects, and let $\underline{CO}$ be the full subcategory of $\underline{C}$ generated by componented objects. For the sake of simplicity, we shall also assume that $\underline{C}$ has an initial object A_0, and adopt the convention that A_0 is the coproduct of the empty family of objects from $\underline{D}$, so that A_0 is considered componented. (Of course, we also take the viewpoint that the coproduct of a singleton set of $\underline{C}$-objects is the single member of that set, so that $\underline{D}$ is a subcategory of $\underline{CO}$.)

Note that the definition of connectedness establishes a natural equivalence between $\underline{C}[A, B_1 \mu B_2]$ and $\underline{C}[A, B_1] \vee \underline{C}[A, B_2]$ for all $\underline{D}$-objects A and pairs (B_1, B_2) of $\underline{C}$-objects. One can

easily show by induction that, for any $\underline{D}$-object A and any n-tuple $(B_1, B_2, \ldots, B_n)$ of $\underline{C}$-objects, this extends to a natural equivalence between
$\underline{C}[A, B_1 \mu B_2 \mu \ldots \mu B_n]$ and $\underline{C}[A, B_1] \vee \ldots \vee \underline{C}[A, B_n]$, determined by the natural injections ι_j $(j = 1, \ldots, n)$ of each B_j into the coproduct.

<u>Proposition (1.2):</u> Let $\underline{C}$, $\underline{D}$, and $\underline{CO}$ be as above. The representation of a componented object B in ob $\underline{CO}$ as a coproduct is <u>unique</u> in the following sense:

If B is isomorphic to both $A_1 \mu A_2 \mu \ldots \mu A_m$ and $B_1 \mu B_2 \mu \ldots \mu B_n$, where the A_i and the B_j are connected objects, then m = n and there is a one-one correspondence $A_i \mapsto B_{g(i)}$ such that A_i and $B_{g(i)}$ are isomorphic in $\underline{C}$.

<u>Proof:</u> Let $\underline{W}$ be the inherited connection from $\underline{D}$ to CO; i.e., $\underline{W}[A, B] = \underline{C}[A, B]$ for A in ob $\underline{D}$ and B in ob $\underline{CO}$, with the obvious compositions. Note that for all $\underline{CO}$-objects B, $\underline{W}[-, B]$ is S-representable in $\underline{D}$ since, if B is isomorphic to $A_1 \mu A_2 \mu \ldots \mu A_m$, then $\underline{W}[-, B]$ is naturally equivalent to $\underline{D}[-, A_1] \vee \ldots \vee \underline{D}[-, A_m]$. Now, invoking the uniqueness of S-representability establishes the result. QED

We shall call the connected objects (unique in the above sense) which under the coproduct operation make up a componented object B, the <u>components</u> of B.

In the proof of the above proposition, it was established that the connection $\underline{W}$ is S-realizable on the right. Let (X, F) be its right S-realization. Since (X, F) is clearly finitary, we have (X, F) as a functor from $\underline{CO}$ to $N(\underline{D})$. On the other hand, $\underline{W}$ is certainly realizable on the left, its left realization being simply the inclusion functor of $\underline{D}$ in $\underline{CO}$; thus, (X, F) is a right S-adjoint.

<u>Proposition (1.3):</u> Let $\underline{C}$, $\underline{D}$, $\underline{CO}$, and $(X, F)\colon \underline{CO} \to N(\underline{D})$ be as above. Then (X, F) is an equivalence of $\underline{CO}$ with $N(\underline{D})$.

<u>Proof:</u> If (Y, A) is an $S(\underline{D})$-object, let $\mu(Y, A)$ denote the coproduct of the elements $A(y)$ as y varies over Y. This extends to a functor $\mu\colon N(\underline{D}) \to \underline{CO}$. (In fact, the functor μ so defined is the left realization of the connection $\underline{V} = \underline{V}(N(\underline{D}), \underline{CO})$ defined by letting $\underline{V}[(Y, A), B]$ be the family of functions T which assign to each y in Y a $\underline{C}$-morphism $T_y\colon A(y) \to B$, together with the obvious composition laws. So defined, $\underline{V}[(Y, A), B]$ is in fact the cartesian product of the morphism sets $\underline{C}[A(y), B]$, and since μ is a coproduct in $\underline{C}$, $\underline{V}[(Y, A), -]$ is naturally equivalent to $\underline{C}[\mu(Y, A), -]$.)

It is not hard to see that $((X, F) \circ \mu)(Y, A)$ is naturally isomorphic to (Y, A), since $\mu(Y, A)$ is the coproduct of connected objects, and (X, F) analyzes a componented object into its connected components. On the other hand, $(\mu \circ (X, F))(B)$ is clearly isomorphic to B. The required equivalence follows from these facts. QED

If in fact $\underline{C} = \underline{CO}$, we shall call $\underline{C}$ a <u>componented</u> category. That is, a componented category is a category $\underline{C}$ in which every object is isomorphic to a coproduct of connected objects. We have just seen that a componented category is equivalent to the category $N(\underline{D})$, where $\underline{D}$ is the full subcategory of $\underline{C}$ generated by connected objects.

(A note on the proof: We assumed that $\underline{C}$ had an initial object, which we regarded as being componented. This was necessary in order that the functor $\mu\colon N(\underline{D}) \to \underline{CO}$ could be defined on all of $N(\underline{D})$, since $N(\underline{D})$ contains the "empty indexing" as an object. This gets mapped by μ into the initial object of $\underline{CO}$. Of course, if $\underline{C}$ does not have an initial object, $\underline{CO}$ would still be equivalent to $N(\underline{D})$ minus the empty indexing.)

The simplest example of a componented category is $\underline{N}$ itself, the connected objects being the singleton sets. On the other hand, $N(\underline{D})$ is of course componented for any category $\underline{D}$, the connected objects being the objects indexed by singleton sets; i.e., essentially $\underline{D}$ itself.

The principal reason for introducing the notion of a componented category is to establish the relationship between the ideas developed in this thesis and those found in earlier work of W. Burnside and (especially) L. Lovasz. To do so succinctly, however, we require yet another concept:

Let $\underline{C}$ be a skeletally small category equipped with a commutative and associative operation ; that is, a functor $\oplus$: $\underline{C} \times \underline{C} \to \underline{C}$ such that the associated functors $((-)_1 \oplus (-)_2) \oplus (-)_3$ and $(-)_1 \oplus ((-)_2 \oplus (-)_3)$ from $\underline{C} \times \underline{C} \times \underline{C}$ to $\underline{C}$ (where the subscripts indicate which factor is being operated on by the operation) are naturally equivalent (associativity), and the functors $(-)_1 \oplus (-)_2 =$ and $(-)_2 \oplus (-)_1$ from $\underline{C} \times \underline{C}$ to $\underline{C}$ are also naturally equivalent (commutativity).

Then the <u>Grothendiek group</u> of the pair $(\underline{C}, \oplus)$ is defined as an (additive) abelian group $GG(\underline{C}, \oplus)$ equipped with a map b: ob $\underline{C} \to GG(\underline{C}, \oplus)$ such that:

a) $b(A_1) = b(A_2)$ if A_1 and A_2 are isomorphic in $\underline{C}$,

b) $b(A_1 \oplus A_2) = b(A_1) + b(A_2)$ for all $\underline{C}$-objects A_1 and A_2,

c) the function b: ob $\underline{C} \to GG(\underline{C}, \oplus)$ is universal with respect to properties (a) and (b) above. That is if b': ob $\underline{C} \to G$ (where G is an additive abelian group) is a function satifying (a) and (b) above, then there is a unique group homomorphism $h: GG(\underline{C},\oplus) \to G$ such that $b' = h \circ b$.

It is easy to show from the universal property (c) above that the Grothendiek group of a pair $(\underline{C},\oplus)$ is essentially unique. It is also true that given any pair $(\underline{C},\oplus)$ as described above, one can construct a Grothendiek group for it.

In many applications, in addition to the operation $\oplus$ as described above, $\underline{C}$ has a second commutative and associative operation $\otimes$: $\underline{C} \times \underline{C} \to \underline{C}$, which distributes over $\oplus$; that is, the functors $(-)_1 \otimes ((-)_2 \oplus (-)_3)$ and $((-)_1 \otimes (-)_2) \oplus ((-)_1 \otimes (-)_3)$ from $\underline{C} \times \underline{C} \times \underline{C}$ to $\underline{C}$ are naturally equivalent. In that case, if $GG(\underline{C},\oplus)$ is the Grothendiek group of the pair $(\underline{C},\oplus)$, then it is not hard to show that it can be equipped with a multiplication '$\circ$' making it into a commutative ring, and such that $b(A_1 \otimes A_2) = b(A_1) \circ b(A_2)$ for all $\underline{C}$-objects A_1 and A_2. When that is the case, we shall call the ring (equipped with the function b) the <u>Grothendiek ring</u> of the triple $(\underline{C},\oplus,\otimes)$, and denote it by $GR(\underline{C},\oplus,\otimes)$.

The reader will no doubt notice the similarity between the universal map b of a Grothendiek group $GG(\underline{C},\oplus)$ and the angle bracket function $\langle - \rangle$ from ob $S(\underline{D})$ to $Z(D)$ (where D is a skeletal set for $\underline{D}$)

introduced in the proof of proposition (4.9). In fact, we have the following proposition:

Proposition (1.4): Let $\underline{D}$ be a skeletally small category with skeletal set D. Consider the pair $(N(\underline{D}), \vee)$, where $\vee$ is the disjoint union operation (coproduct) on $N(\underline{D})$. Then the free Z-module Z(D), together with the bracket function <-> from ob $S(\underline{D})$ to Z(D) (as defined in the proof of (4.9)), is the Grothendiek group of $(N(\underline{D}), \vee)$.

Proof: We must show that the bracket function has the required universal property. Thus, let b: ob $N(\underline{D}) \to G$ be any function from ob $N(\underline{D})$ to an additive abelian group G satisfying properties (a) and (b) in the definition of a Grothendiek group.

As usual, we regard $\underline{D}$ as a subcategory of $N(\underline{D})$, and hence D as a subset of ob $N(\underline{D})$. Thus, b is defined on D. If there exists a map h: $Z(D) \to G$ such that $h \circ \langle - \rangle = b$, then we must have that $h(A_i) = b(A_i)$ for all A_i in D, since $\langle A_i \rangle = A_i$ by definition. But then, the assignment $A_i \mapsto b(A_i)$ already defines a unique map h: $Z(D) \to G$ since the A_i form a basis for Z(D). It is trivial to show that indeed $h \circ \langle - \rangle = b$. QED

It is not hard to see that equivalent categories have isomorphic Grothendiek groups. Thus, we have the following immediate corollary:

Proposition (1.5): Let $\underline{C}$ be a (skeletally small) componented category, $\underline{D}$ the full subcategory generated by the connected objects in $\underline{C}$, and (X, F): $\underline{C} \to S(\underline{D})$ the equivalence which assigns to each A in ob $\underline{C}$ the indexed family of its connected components. Let D be a skeletal set for $\underline{D}$. Then Z(D), equipped with the function b: ob $\underline{C} \to Z(D)$ defined by:

$$b(A) = \langle(X(A), F(A))\rangle, \tag{1.6}$$

is the Grothendiek group of the pair $(\underline{C}, \mu)$.

Clearly, by its equivalence with $N(\underline{D})$, a componented category $\underline{C}$ has a product π if and only if the subcategory $\underline{D}$ (generated by the connected objects) has an S-product. In that case, the product must then also distribute over the coproduct (since it does so in $S(\underline{D})$), and then the Grothendiek ring of the triple $(\underline{C}, \mu, \pi)$ is simply Z(D) equipped with the multiplication derived from the S-product in $\underline{D}$.

Now, let us continue with the situation as above (i.e., a componented category $\underline{C}$, a subcategory $\underline{D}$ of connected objects, etc.), but additionally suppose that

$\underline{C}$ is finitary. (Note that $\underline{C}$ is finitary if and only if $\underline{D}$ is finitary.) Also let $D = \{A_i : i \in I\}$ be a skeletal set for $\underline{D}$, and $\underline{d}$: $Z(D) \to Z^D$ be the usual right linearization. Then, clearly the composition $\underline{d}_i \circ b$, mapping ob $\underline{C}$ into Z^C, can be defined directly by the formula:

$$(1.7) \qquad (\underline{d}_i \circ b)(A) = \#\underline{C}[A_i, A],$$

for all A in ob $\underline{C}$ and A_i in D.

Note that if $\underline{d}$ is faithful (for example, if $\underline{D}$ has a finitary right factorization ($\underline{M}$, $\underline{D}'$) in which $\underline{D}'$ consists entirely of epimorphisms, and the class of subobjects of a $\underline{D}$-object is finite) then we have the following result:

If $\{A_i : i \in I\}$ is a set of representatives of the connected objects in $\underline{C}$, then two $\underline{C}$-objects B_1 and B_2 are isomorphic in $\underline{C}$ if and only if $\#\underline{C}[B_1, A_i]$ is equal to $\#\underline{C}[B_2, A_i]$ for all i in I.

With these remarks in mind, we can now turn to the examples we have in mind:

<u>The Burnside Ring:</u> If X is a set, let P(X) denote the group of all permutations of X; i.e., P(X) is the symmetric group on X.

Now let G be a finite group. By an action of G on a set X we mean a group homomorphism $\underline{a}$: G → P(X), g |→ $\underline{a}_g$. If $\underline{a}$ and $\underline{b}$ are actions of the group G on the sets X and Y respectively, then call a map f: X → Y an intertwining map from $\underline{a}$ to $\underline{b}$ if $f \circ \underline{a}_g = \underline{b}_g \circ f$ for all g in G (or equivalently, $f = \underline{b}_g \circ f \circ \underline{a}_g^{-1}$ for all g in G).

The class of actions of G on finite sets, together with intertwining maps as morphisms, forms a category which we shall denote by $\underline{B}$(G).

The category $\underline{B}$(G) inherits the operations of cartesian product and disjoint union from $\underline{N}$. The catesian product × and disjoint union \/ (as the product and coproduct in $\underline{N}$), extend to functors from $\underline{N} \times \underline{N}$ to $\underline{N}$, and thus $\underline{a} \times \underline{b}$ and $\underline{a}$ \/ $\underline{b}$ are simply defined by setting $(\underline{a} \times \underline{b})_g = \underline{a}_g \times \underline{b}_g$ and $(\underline{a} \vee \underline{b})_g = \underline{a}_g \vee \underline{b}_g$ for all g in G.

It is easy to show that × and \/ are the product and coproduct in $\underline{B}$(G), respectively, and of course × still distributes over \/. It is also clear that $\underline{B}$(G) is finitary. The Grothendiek group of the triple ($\underline{B}$(G), \/, ×) has been christened the Burnside ring of G by L. Solomon in [S], in honor of Burnside's work on the subject (viz. [B]).

An action $\underline{a}$ of G on a set X is called transitive if, for any x and y in X, there is an element g in G such that $\underline{a}_g(x) = y$. It can be shown that every action of G on a finite set X is isomorphic to a disjoint union of a finite number of transitive actions. More precisely, an action $\underline{a}$ of G on a set X determines a partition of X into <u>orbits</u> of the action (two elements x and y in X being in the same orbit if there exists a g in G such that $\underline{a}_g(x) = y$), and each orbit determines a corresponding transitive action of G; then $\underline{a}$ is naturally isomorphic to the disjoint union of these transitive actions. They are called the <u>transitive components</u> of $\underline{a}$. It is easy to see, in fact, that the transitive actions of G are the connected objects in $\underline{B}(G)$. Thus, $\underline{B}(G)$ is a componented category.

If H is a subgroup of G, let G/H denote the family $\{gH : g \in G\}$ of left cosets of H in G. Then there is a natural action $\underline{a}^H$ of G on G/H via left multiplication. This action is clearly transitive. In fact, it is not hard to prove that every transitive action $\underline{a}$ of G is isomorphic to an action $\underline{a}^H$ for some subgroup H of G. One can construct such an isomorphism as follows: Suppose $\underline{a}$ is a transitive action on the set X. Pick a point x in X, and let H be the subgroup of all g in G that <u>fix</u> x; i.e., H is defined as

$$H = \{g \in G : \underline{a}_g(x) = x\}.$$

We shall call H the <u>isotropy subgroup</u> of x under the action $\underline{a}$. Now define $f: X \to G/H$ by writing

$$f(y) = gH \text{ if } \underline{a}_g(x) = y.$$

It is not hard to show that f is well-defined by this prescription (i.e., independent of the choice of g in the above equation), and that f is in fact a $\underline{B}(G)$-isomorphism.

In the case of the action $\underline{a}^H$ of G on G/H, it is easy to see that the isotropy group of the coset H in G/H is simply H (now considered as a group rather than a point). More generally, the isotropy group of the coset gH is the conjugate gHg^{-1} of H. Thus, if H and K are conjugate in G, then $\underline{a}^H$ and $\underline{a}^K$ are isomorphic as actions of G. We can conclude, then, that there are only a finite number of transitive actions of G, up to isomorphism, and these are given by selecting one representative H out of each conjugacy class of subgroups of G, and taking the corresponding action $\underline{a}^H$.

The structure of the Burnside ring is now clear. Let $\underline{D}$ be the subcategory of $\underline{B}(G)$ generated by transitive (i.e., connected) actions. A skeletal set for $\underline{D}$ can then be chosen by first choosing a set $W = \{H_1, H_2, \ldots, H_r\}$ of subgroups H_i of G that selects precisely one representative out of each conjugacy class of subgroups of G, and then forming the skeletal set $D = \{\underline{a}^{(1)}, \underline{a}^{(2)}, \ldots, \underline{a}^{(r)}\}$, where $\underline{a}^{(i)}$ is the action of G on G/H_i. Then the Burnside ring can be taken as Z(D), together with the multiplication derived from the S-product in $\underline{D}$. (Note that $\underline{D}$ has only an S-product in general, since the product of two transitive actions may well not be transitive. In fact, it can be shown that the transitive components of the product $\underline{a}^H \times \underline{a}^K$ are in a one-one correspondence with the distinct double cosets HgK of H and K in G.) The universal map b: ob $\underline{B}(G) \to B(G)$ is thus defined by:

$$b(\underline{a}) = \sum_i r(i)\underline{a}^{(i)}$$

where r(i) is the number of transitive components of $\underline{a}$ isomorphic to $\underline{a}^{(i)}$.

All these results are essentially contained in [B]; Burnside also proved the following result:

Let W be as above. For any action $\underline{a}$ of G on a set X, define the function $m(\underline{a})\colon W \to Z$ by:

$$(m(\underline{a}))(H_i) = \#\{x \in X : \underline{a}_g(x) = x \text{ for all } g \text{ in } H_i\},$$

the number of points in X that H_i leaves fixed. (Burnside calls the quantity $(m(\underline{a}))(H_i)$ the <u>mark</u> of H in the action $\underline{a}$.) Then two actions $\underline{a}$ and $\underline{b}$ of G are isomorphic if and only if $m(\underline{a}) = m(\underline{b})$.

In order to see the connection between this result and the ideas we have developed in this section, it is only necessary to prove the following simple proposition:

Proposition (1.6): Let $\underline{a}$ be a transitive action of G on the set X, choose a point x in X, and let H be the isotropy subgroup of x under the action $\underline{a}$. Then, if $\underline{b}$ is any other action of G on a set Y, the intertwining maps from $\underline{a}$ to $\underline{b}$ are in a one-one correspondence with the points y in Y that are left fixed by H under the action $\underline{b}$.

Proof: In the first place, given an intertwining map f from $\underline{a}$ to $\underline{b}$, it is easy to see that f(x) is left fixed by H under $\underline{b}$, since for g in H we have that

$$\underline{b}_g(f(x)) = f(\underline{a}_g(x)) = f(x).$$

On the other hand, suppose y in Y is left fixed by H under $\underline{b}$. Then the assignment $x \mapsto y$ extends to a unique intertwining map f from X to Y defined by setting, for any x' in X,

$$f(x') = \underline{b}_g(y) \quad \text{if } x' = \underline{a}_g(x).$$

Again, it is easy to show that f is well-defined and is an intertwining map. QED

Thus, we can now see that for any action $\underline{a}$ of G, $(m(\underline{a}))(H_i)$ is simply the number of intertwining maps from $\underline{a}^{(i)}$ to $\underline{a}$; that is, $\#\underline{B}(g)[\underline{a}^{(i)}, \underline{a}]$. The function $m(\underline{a})$ is, for all practical purposes, $\underline{d}(b(\underline{a}))$, where of course $\underline{d}$ is the right linearization of $\underline{D}$.

One recovers Burnside's result by noting that mor $\underline{D}$ consists only of epimorphisms, (an intertwining map from one transitive action to another is necessarily surjective) and therefore $\underline{d}$ must be faithful.

Before we leave this example, we will look at one more representation of the Burnside ring:

Let L(G) be the lattice of subgroups of G, and form the free Z-module Z(L(G)). We can make Z(L(G)) into a commutative ring by using the meet operation, $\wedge$, in L(G) to define a multiplication on basis elements, and then extend to all of Z(L(G)) by linearity. Denote this ring by $Z\langle L(G), \wedge\rangle$.

For any action $\underline{a}$ of G on a set X, define the element $w(\underline{a})$ in $Z\langle L(G), \wedge\rangle$ by means of the formula:

$$(1.7) \qquad w(\underline{a}) = \sum_{x} I(\underline{a}, x),$$

where $I(\underline{a}, x)$ denotes the isotropy group of x under the action $\underline{a}$, and the summation is over all x in X. By definition, it is clear that $w(\underline{a} \vee \underline{b}) = w(\underline{a}) + w(\underline{b})$. On the other hand, it is easy to show that, for actions $\underline{a}$ and $\underline{b}$ of G on sets X and Y respectively,

$$I(\underline{a} \times \underline{b}, (x,y)) = I(\underline{a}, x) \wedge I(\underline{b}, y),$$

for all (x,y) in X × Y; whence it follows that $w(\underline{a} \times \underline{b}) = w(\underline{a}) \circ w(\underline{b})$ in $Z\langle L(G), \wedge\rangle$.

By the universal property of the map $b: \text{ob } \underline{B}(G) \to B(G)$, w must factor through b and determine a corresponding ring homomorphism w from B(G) to $Z\langle L(G), \wedge\rangle$. This homomorphism is faithful:

Under w, the basis element $\underline{a}^{(i)}$ of B(G) is mapped into $w(\underline{a}^{(i)})$, which it is easy to see is simply a multiple of the sum of the conjugates of H_i. But since H_i is not conjugate to H_j for $i \neq j$, these sums are clearly "non-overlapping" and hence linearly independent in Z(L(G)). It follows that w is faithful.

We shall have more to say about this representation of B(G) in a later section of this chapter. Now, however, we shall turn to another class of examples of componented categories.

Relational structures: For this example, we look at some of the ideas developed by L. Lovasz in [L].

If A is a finite set, let $\times^n(A)$ denote the cartesian product of A with itself n times. An element $\underline{x}$ in $\times^n(A)$ can be written as a n-tuple:

$$\underline{x} = (x(1), x(2), \ldots, x(n)),$$

or alternatively, we may regard $\underline{x}$ as a function from the set $A_n = \{1,2,\ldots,n\}$ to A, with x(i) denoting the value of $\underline{x}$ at the point i in A_n. Taking this latter viewpoint, we see that $\times^n(A)$ is simply the set $\underline{N}[A_n, A]$, and therefore the "operator" $\times^n$ can be identified with the (covariant) set-valued functor $\underline{N}[A_n, -]$ on $\underline{N}$. In particular, if f is a mapping from the set A to the set B, we have the corresponding mapping $\times^n(f)\colon \times^n(A) \to \times^n(B)$, under which an element $\underline{x}$ in $\times^n(A)$ simply goes to $f \circ \underline{x} = (f(x(1)),\ldots,f(x(n)))$.

An n-ary relation on the set A is simply any subset R of $\times^n(A)$. In the most general terms, a relational structure is a set A equipped with a family of relations (of varying "arities"); however, for the sake of simplicity we shall restrict ourselves here to sets equipped with a single n-ary relation. Thus, specifically, by an (n-ary) relational structure B we shall mean a pair $B = (V(B), R(B))$ in which $V = V(B)$ is a finite set and $R = R(B)$ is a subset of $\times^n(V)$. Given two n-ary relational structures B_1 and B_2, by a structure-preserving map from B_1 to B_2 we shall mean a map $f\colon V(B_1) \to V(B_2)$ such that $(\times^n(f))(R(B_1))$ is contained in $R(B_2)$. It is readily seen that n-ary relational structures and structure preserving maps form a category we shall denote by $\underline{L}(n)$. Clearly, $\underline{L}(n)$ is finitary.

The category $\underline{L}(n)$ inherits the operations of cartesian product and disjoint union from $\underline{N}$ in the following manner:

In the first place, given finite sets A and B, we see that $\times^n(A)$ and $\times^n(B)$ can be identified in a natural way with subsets of $\times^n(A \vee B)$; we simply identify an element $\underline{x}$ in $\times^n(A)$ with the element $i_1 \circ \underline{x} = (\times^n(\iota_1))(\underline{x})$ of $\times^n(A \vee B)$, where ι_1 is the natural injection of A into A $\vee$ B, and similarly identify an element $\underline{y}$ of $\times^n(B)$ with $\iota_2 \circ \underline{y}$. With this understanding, if R_1 and R_2 are n-ary relations on A and B respectively, then $R_1 \vee R_2$ (denoting the union of $(\times^n(\iota_1))(R_1)$ and $(\times^n(\iota_2))(R_1)$ in $\times^n(A \vee B)$) can be simply thought of as the union of R_1 and R_2.

Thus, the disjoint union of two $\underline{L}(n)$-objects B_1 and B_2, denoted by $B_1 \vee B_2$, is simply defined as the pair $(V(B_1) \vee V(B_2), R(B_1) \vee R(B_2))$. It is not hard to establish that $\vee$, so defined on $\underline{L}(n)$, is the coproduct in that category.

The cartesian product of two n-ary relational structures is defined in even a more natural manner; since $\times$ is the product in $\underline{N}$, it follows that $\times^n(A \times B) = \underline{N}[A_n, A \times B]$ is naturally isomorphic to $(\times^n(A)) \times (\times^n(B))$. Given $\underline{x}$ in $\times^n(A)$ and $\underline{y}$ in $\times^n(B)$, the product $\underline{x} \times \underline{y}$ is simply defined in the standard manner by the equation:

$$(\underline{x} \times \underline{y})(i) = (\underline{x}(i), \underline{y}(i)).$$

Thus, given n-ary relations R_1 and R_2 on A and B respectively, we define a new n-ary relation $R_1 \bullet R_2$ on $A \times B$ by:

$$R_1 \bullet R_2 = \{\underline{x} \times \underline{y} : \underline{x} \in R_1, \underline{y} \in R_2\}.$$

Then, the cartesian product of two relational structures B_1 and B_2, denoted by $B_1 \times B_2$, is defined as the pair $(V(B_1) \times V(B_1), R(B_1) \bullet R(B_2))$. It can be shown to be the product in the category $\underline{L}(n)$.

The cartesian product in $\underline{L}(n)$ distributes over the disjoint union, a property also inherited from $\underline{N}$. Consequently, we can form the Grothendiek ring of the triple ($\underline{L}(n)$, \/, ×); we shall denote it by L(n).

The category $\underline{L}(n)$ is componented. In it, an object B is connected if an only if it is not possible to divide V(B) into two non-empty subsets V_1 and V_2 such that R(B) is equal to the union of $R(B) \cap \times^n(V_1)$ and $R(B) \cap \times^n(V_2)$. If this is possible, then it is not hard to show that B is isomorphic to the disjoint union of the structures $(V_1, R(B) \cap \times^n(V_1))$ and $(V_2, R(B) \cap \times^n(V2))$, and then, by progressive refinement in this manner, one can show that every n-ary structure is isomorphic to a disjoint union of (a finite number of) connected structures.

Let $\underline{D}$ be the full subcategory of $\underline{L}(n)$ generated by the connected structures, and let $D = \{B_i : i \in I\}$ be a skeletal set for $\underline{D}$. One of Lovasz's principal results in [L] is the following:

Two n-ary relational structures B and B' are isomorphic in $\underline{L}(n)$ if and only if:

$$\#\underline{L}(n)[B_i, B] = \#\underline{L}(n)[B_i, B']$$

for all i in I.

Of course, from our vantage point we recognize that this result is equivalent to the faithfulness of the right linearization $\underline{d}$: $Z(D) \to Z^D$ of $\underline{D}$. And to show that $\underline{d}$ is faithful, it suffices to show that $\underline{D}$ has a finitary right factorization ($\underline{M}$, $\underline{D}'$) in which mor $\underline{D}'$ consists only of epimorphisms. (The invertability of the Z-homomorphism $\underline{m}$ corresponding to such a factorization follows easily from the fact that any $\underline{L}(n)$-object clearly has only a finite number of subobjects.)

In fact, it is not hard to show that one obtains such a factorization if one lets $\underline{M}$ be the class of all relation-preserving maps which (considered as maps in $\underline{N}$) are injective, and lets $\underline{D}'$ be the subcategory whose morphisms are surjective relation-preserving maps $f\colon B \to B'$ such that $(\times^n(f))(R(B)) = R(B')$.

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2. Finite vector spaces:

We begin by establishing some notation:

$F = GF(q)$ = the finite (Galois) field with q (a prime power) elements.

$\underline{W}$ = category of finite-dimensional F-vector spaces and F-linear transformations.

$C_q(i,j)$ = the number of j-dimensional subspaces in an i-dimensional F-vector space.

$P_q(i,j)$ = number of injective linear transformations from an j-dimensional F-vector space to an i-dimensional space.

$A_q(i)$ = number of F-linear automorphisms of an i-dimensional F-vector space.

$\underline{MW}$ = category of F-vector spaces and injective linear transformations; $\underline{M}$ = mor $\underline{MW}$.

$\underline{EW}$ = category of F-vector spaces and surjective linear transformations; $\underline{E}$ = mor $\underline{EW}$.

$W = \{V_0, V_1, V_2, \ldots, V_n, \ldots\}$ is a skeletal set for $\underline{W}$, with V_n being a n-dimensional space. Of course W is also a skeletal set for $\underline{MW}$ and $\underline{EW}$.

Define the polynomial $(x)_{(q,n)} \in Z[x]$ by means of the formula:

$$(x)_{(q,n)} = x(x-(q-1))(x-(q^2-1))\ldots(x-(q^{n-1}-1)).$$

Finally, define the quantity $(n!q)$ by the equations:

$$(n!q) = (q^n-1)(q^{n-1}-1)(q^{n-2}-1)\ldots(q-1),$$

$$(0!q) = 1.$$

The direct sum operation, $\oplus$, is both a product and coproduct in $\underline{W}$. Additionally, we have the "dual space" functor $\underline{W} \to \underline{W}$ which assigns to any F-vector space V the dual vector space V* (consisting of all linear maps from V to F, with pointwise operations). If $f\colon U \to V$ is a $\underline{W}$-morphism, then we have its dual $f^*\colon V^* \to U^*$ defined by $f^*(w) = w \circ f$ for all $w \in V^*$. The functor $()^*$ so defined is, as is well-known, a dual equivalence (i.e., a contravariant functor that is also an equivalence.) Specifically, $()^{*2}$ is naturally equivalent to the identity functor on $\underline{W}$. The functor $()^*$ of course preserves the direct sum; it can be regarded as mapping the product on $\underline{W}$ into the coproduct, or vice-versa. It also takes any injective linear map into a surjective linear map, and any surjective linear map into a injective linear map. Consequently, it also defines a dual equivalence between $\underline{MW}$ and $\underline{EW}$.

These facts allow us to establish some elementary facts about the quantities defined above. Define, temporarily, the quantities $P'_q(i,j)$ as the number of surjective linear maps from an i-dimensional to a j-dimensional space, and $C'_q(i,j)$ as the number of j-dimensional quotient spaces of an i-dimensional space. Of course, there is a one-one correspondence between subspaces $V' \subsetneq V$ of an i-dimensional space and its quotient spaces V/V'; under this correspondence, a j-dimensional subspace corresponds to an (i-j)-dimensional quotient space. Thus, we immediately deduce that $C_q(i,j) = C'_q(i,i-j)$. On the other hand, through the dual equivalence of <u>EW</u> and <u>MW</u> under $()^*$ it is easy to show that $P_q(i,j) = P'_q(i,j)$. Finally, since every injective linear map $f: V' \to V$ can be decomposed uniquely as an isomorphism of V' with its image $f(V')$ followed by an inclusion, and every surjective linear map $g: V \to V'$ can be decomposed uniquely as the natural projection of V onto the quotient $V/Ker(g)$ (where $Ker(g)$ is the kernel of g) followed by an isomorphism of the quotient with V', one can see that $P^q(i,j) = A_q(j)C_q(i,j)$ and $P'_q(i,j) = A_q(j)C'_q(i,j)$. Thus, we derive the following facts:

$C_q(i,j) = C_q(i,i-j)$ is both the number of j-dimensional subspaces of an i-dimensional space, and the number j-dimensional quotient space of an i-dimensional space,

$$P_q(i,j) = A_q(j)C_q(i,j) = \#\underline{MW}[V_j, V_i] = \#\underline{EW}[V_i, V_j].$$

Under the dual equivalence of $\underline{W}$ with itself, we see that the right and left linearizations of $\underline{W}$, $Z\langle W, \underline{W}\rangle$ and $Z\langle W, \underline{W}^*\rangle$ are not only isomorphic, but identical; indeed:

$$\underline{w}_i(V_j) = \underline{w}_i^*(V_j) = \#\underline{W}[V_i, V_j] = \#\underline{W}[V_j, V_i] = q^{ij},$$

since by choosing bases for the vector spaces we can establish a one one correspondence between linear maps $V_i \to V_j$ and i-by-j matrices with entries from F. The multiplication derived from the product (or coproduct) is easily seen to be given by

$$A_i \circ A_j = A_{i+j},$$

whence one concludes that $Z\langle W, \underline{W}\rangle$ is isomorphic to the polynomial ring $Z[x]$, with A_i corresponding to x^i, and the map $\underline{w}_i$: $Z\langle W,\underline{W}\rangle \to Z$ corresponding to evaluation at $x = q^i$.

Clearly $(\underline{M},\underline{EW})$ forms a finitary right factorization of $\underline{W}$, just as $(\underline{MW},\underline{E})$ forms a finitary left factorization. Thus, $\underline{EW}$ inherits a finitary S-product from $\underline{W}$, while $\underline{MW}$ inherits a finitary S-coproduct. Indeed, since the two categories are dually equivalent under $()^*$, the S-product in $\underline{EW}$ and the S-coproduct in $\underline{MW}$ correspond under $()^*$. In fact one easily verifies that the rings $Z\langle W, \underline{EW}\rangle$ and $Z\langle W, \underline{MW}\rangle$ are identical. Even the ring homomorphisms $Z\langle W, \underline{W}\rangle \to Z\langle W, \underline{EW}\rangle$ and $Z\langle W, \underline{W}^*\rangle \to Z\langle W, \underline{MW}^*\rangle$ derived from the respective factorizations are identical, being given in each case on basis elements by the formula:

$$(2.1) \qquad \underline{t}(A_i) = \sum_j C_q(i,j)A_j.$$

From the standard commutative triangle, we have that $\underline{w}_k = \underline{ew}_k \circ \underline{t}$, which when applied to A_i and expanded via (2.1) yields the identity:

$$(2.2) \qquad q^{ki} = \sum_j C_q(i,j)P_q(k,j).$$

(Of course $\underline{ew}_k(A_j) = \#\underline{EW}[A_k, A_j] = P_q(k,j)$.)

We now wish to exploit the multiplication in $Z\langle W, \underline{EW}\rangle$ derived from the S-product, or, equivalently, the multiplication in $Z\langle W, \underline{MW}^*\rangle$ derived from the S-coproduct. We take the latter viewpoint since it seems considerably easier to visualize what the S-coproduct looks like.

For $U, V \in \text{ob } \underline{W}$, let ι_1 and ι_2 denote the natural injections of U and V respectively into their direct sum $U \oplus V$, and π_1 and π_2 denote the natural projections of $U \oplus V$ onto U and V respectively. (As a set, $U \oplus V$ is simply the cartesian product $U \times V$.) Let $\Omega(U, V)$ denote the family of subspaces B of $U \oplus V$ such that $\iota_1 \circ \eta_B$ and $\iota_1 \circ \eta_B$ are both injective maps, where $\eta_B\colon U \oplus V \to (U \oplus V)/B$ is the natural map onto the indicated quotient space. By (the dual of) proposition (3.10) of chapter II, we know that the family of quotients spaces $\{(U \oplus V)/B : B \in \Omega(U,V)\}$ defines the S-coproduct in MW.

Proposition (2.3): Let B be any subspace of the direct sum $U \oplus V$. Then B is an element of $\Omega(U, V)$ if and only if B is the graph of a linear isomorphism from a subspace B_1 of U to a linear subspace B_2 of V.

Proof: First of all note that (identifying U and V with their images $U \oplus 0$ and $0 \oplus V$ under the natural injections ι_1 and ι_2, respectively) the kernels of $\iota_1 \circ \eta_B$ and $\iota_2 \circ \eta_B$, respectively, are the subspaces $U \oplus 0 \cap B$ and $0 \oplus V \cap B$, respectively. Thus, B is an element of $\Omega(U, V)$ if and only if both these intersections are zero.

Also note that the graph of a linear isomorphism from a subspace of U to a subspace of V is indeed a linear subspace of $U \oplus V$. It is also immediate that the intersections of a subspace with $U \oplus 0$ and $0 \oplus V$ are both zero. Thus, such a graph is an element of $\Omega(A, B)$.

Conversely, suppose B is an element of $\Omega(A, B)$. Let $B_1 = \pi_1(B)$ and $B_2 = \pi_2(B)$. Clearly, B is a subdirect subspace of $B_1 \oplus B_2$; i.e., if (u,v) is an

element of B, then $u \in B_1$ and $v \in B_2$. We claim, that for every element u in B_1, there is a unique element v in B_2 such that (u,v) is in B. First of all, if u = 0, then v must also be zero, since if we have (0,v) in B with $v \neq 0$, then v is a non-zero element of $B \cap 0 \oplus V$, a contradiction. Now, if (u,v_1) and (u,v_2) are two elements of B, then $(u,v_1) - (u,v_2) = (0, v_1 - v_2)$ is also in B, whence $v_1 - v_2 = 0$ and $v_1 = v_2$, as was to be shown. Thus, B is the graph of a function from B_1 to B_2, which it is easy to show is a linear map. But we can also apply the same reasoning on the right hand side of B to show that each v in B_2 determines a unique u in B_1. Thus it is the graph of a linear isomorphism.
QED

Now let $\Omega_k(V_i, V_j)$ be the subset of $\Omega(V_i, V_j)$ consisting of subspaces B of dimension k. Since each such B is the graph of a linear isomorphism from a subspace B_1 of V_i to a linear subspace B_2 of V_j, it is clear that B_1 and B_2 also have dimension k.
Consequently, we deduce that

$$\#\Omega_k(V_i, V_j) = C_q(i,k)C_q(j,k)A(k),$$

since there are $C_q(i,k)$ ways of choosing a k-dimensional subspace of V_i, $C_q(j,k)$ ways of choosing a k-dimensional subspace of V_j, and A(k) linear isomorphisms between two k-dimensional spaces.

Now, if we write the multiplication derived from the S-coproduct in $Z\langle W, \underline{MW}^*\rangle$ as follows:

$$(2.4) \quad V_i \circ V_j = \sum_m r_q(i,j;m)V_m,$$

then the above reasoning shows that $r_q(i,j;i+j-k)$ is equal to $C_q(i,k)C_q(j,k)A_q(k)$, since it is clear that the dimension of $(V_i \oplus V_j)/B$ is $i + j - k$ for B in $\Omega_k(V_i, V_j)$. Then, the substitutions $m = i + j - k$ and $k = i + j - m$ at once yield the following formula for $r_q(i,j;m)$:

$$(2.5) \quad r_q(i,j;m) = C_q(i,i+j-m)C_q(j,i+j-m)A_q(i+j-m).$$

Of course by duality, this is also the multiplication in $Z\langle W, EW\rangle$.

We can now establish a multiplicative identity for the quantities $P_q(i,k)$ by simply applying $\underline{ew}_k$ to equation (2.4):

$$(2.6) \quad P_q(k,i)P_q(k,j) = \sum_m r_q(i,j;m)P_q(k,m)$$

It is clear that V_0 is the identity of this ring; thus we shall generally denote it by 1. On the other hand, multiplication by V_1 is easy to write down, since V_1 has only two subspaces, namely 0 and itself. Thus, we deduce the equation:

$$V_1 \circ V_i = V_{i+1} + (q^i-1)V_i,$$

since the elements of $\Omega(V_1, V_i)$ are the zero subspace (which gives us the direct sum $V_i \oplus V_1 \cong V_{i+1}$), and the graphs of the linear injections of V_1 into V_i (which are in a one-one correspondence with the q^i-1 non-zero elements of V_i). From this equation we get:

$$V_{i+1} = V_i \circ (V_1 - (q_i-1)),$$

whence it is easy to show by induction that:

$$(2.7) \quad V_n = V_1 \circ (V_1-(q-1)) \circ \ldots \circ (V_1 -(q^{n-1}-1))$$

$$= (V_1)_{(q,n)},$$

where $(V_1)_{(q,n)}$ of course means the evaluation of the polynomial $(x)_{(q,n)}$ in $Z\langle W, \underline{EW}\rangle$ at $x = A_1$. By standard arguments, we can then conclude that the ring homomorphism from $Z[x]$ to $Z\langle W, \underline{EW}\rangle$ defined by the map $x \mapsto A_1$ is a ring isomorphism under which V_n corresponds to the polynomial $(x)_{(q,n)}$.

Now, since $\underline{ew}_n = \underline{mw}_n^*: Z(W) \to Z$ is a ring homomorphism when $Z(W)$ is given the multiplication of $Z\langle W, EW\rangle = Z\langle W, MW^*\rangle$, and $\underline{mw}_n^*(V_1) = \#\underline{MW}[V_1, V_n]$ $= q^n-1 = P_q(n,1)$, we get the following equations by applying this map to (2.7):

$$(2.8) \quad P_q(n,k) = \underline{mw}_n^*(A_k) = (q^n-1)_{(q,k)}$$

$$= (q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{k-1})$$

$$= q^{k(k-1)/2}(n!q)/(n-k!q),$$

where the last expression is obtained by factoring out of each bracket in the preceding line the highest power of q possible (we of course assume $n \geq k$), and collecting them in the factor $q^{k(k-1)/2}$.

Now, it is clear that $A_q(k) = P_q(k,k)$, and therefore we have:

$$(2.9) \quad A_q(k) = q^{k(k-1)/2}(k!q),$$

and, since $P_q(n,k) = C_q(n,k)A_q(k)$, we get:

$$(2.10) \quad C_q(n,k) = (n!q)/\{(k!q)((n-k)!q)\}.$$

Of course this is highly reminiscent of the standard equation for the binomial coefficient C(n,k). The quantity (n!q) was so defined to bring out this analogy. Notice how the equation $C_q(n,k) = C_q(n,n-k)$ is now displayed in the symmetry in n and n-k of the expression.

The fact that the multiplication in Z<W, $\underline{W}$> is given by $V_i \cdot V_j = V_{p(i,j)}$ where $p(i,j) = i+j$, allows us to appply equation (1.18) of chapter I:

$$t(p(i,j),m) = \sum_{k,n} t(i,k)t(j,n)r(k,n;m),$$

which in the present context becomes:

$$(2.11)\qquad C_q(i+j,m) = \sum_{k,n} C_q(i,k)C_q(j,n)r_q(k,n;m).$$

This last equation can be regarded as a generalization of the recursion formula for the quantity $C_q(i,m)$. To see this let j equal 1 in the above formula, and use the fact that $C_q(1,n) = 0$ except for $n = 0$ and $n = 1$, in which cases it has the value 1. Thus, (2.11) then becomes:

$$C_q(i+1,m) = \sum_k C_q(i,k)r_q(k,0;m) + \sum_k C_q(i,k)r_q(k,1;m).$$

Now, keeping in mind that the quantities $r_q(k,0;m)$ correspond to the multiplication of $V_0 = 1$ and V_k, we see that $r_q(k,0;m) = 0$ except when k equals m, and in that case, we have $r_q(m,0;m) = 1$. Similarly, the quantities $r_q(k,1;m)$ derive from the multiplication of V_k and V_1, but since $V_1 \circ V_k = V_{k+1} + (q^k-1)V_k$, we deduce that $r_q(k,1;m) = 0$ except for $k = m$ and $k = m-1$. In those cases, we have

$$r_q(m-1,1;m) = 1 \text{ and } r_q(m,1;m) = q^m-1.$$

Substituting these values in the above expression, we get the recursion:

$$(2.12)\qquad C_q(i+1,m) = C_q(i,m-1) + q^m C_q(i,m).$$

Finally, let us consider the inverse of the transformation t defined by (2.1). Of course, t is a ring homomorphism from Z<W, W> to Z<W, EW>. Let us then define the quantities $D_q(m,k)$ by means of the equation:

$$(2.12) \qquad \underline{t}^{-1}(V_m) = \sum_k D_q(m,k)V_k.$$

We wish to find a more explicit description of $\underline{t}^{-1}$. Thus, define the transformation $\underline{s}$: $Z(W) \to Z(W)$ on basis elements by means of the equation:

$$(2.13) \qquad \underline{s}(V_m) = (V_1-1) \circ (V_1-q) \circ \ldots \circ (V_1-q^m),$$

where the multiplication is that of $Z\langle W, \underline{W}\rangle$. Now, if one calculates $(\underline{t} \circ \underline{s})(V_m)$, using the fact that $\underline{t}$ is a ring homomorphism and that $\underline{t}(1) = 1$ and $\underline{t}(V_1) = V_1 + 1$, we get $(V_1)_{(q,m)}$ but with the multiplication now in $Z\langle W, \underline{EW}\rangle$, in which it is equal to V_m. Thus we conclude that $\underline{s}$ so defined is the inverse of $\underline{t}$, and, recalling the isomorphism of $Z\langle W, \underline{W}\rangle$ with $Z[x]$, we may assert that $D_q(m,k)$ is the coefficient of x_k in the expansion of the polynomial $(x-1)(x-q)\ldots(x-q^m)$.

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3. Epimorphic and monomorphic categories:

By an *epimorphic category* we simply mean one all of whose morphisms are epimorphisms. Similarly, call a category *monomorphic* if all of its morphisms are monomorphisms. (Following this line, we might as well call a category *bimorphic* if it is both epimorphic amd monomorphic; i.e., if all of its morphisms are bimorphisms.)

Epimorphic and monomorphic categories are of some interest, not only in their own right, but also because it is not unusual for them to appear as right or left factors of a given category. Consequently in this section we shall make a modest study of them, and then see what our results look like in a few applications.

Let $\underline{C}$ be an epimorphic category. Then it always has the following left factorization:

Take $\underline{D}$ to be the subcategory of $\underline{C}$ having the same object class as $\underline{C}$, but whose morphisms are $\underline{C}$-isomorphisms. We can call $\underline{D}$ the *isomorphism subcategory* of $\underline{C}$. Now, if we set $\underline{E}$ = mor $\underline{C}$, it is not hard to see that $(\underline{D},\underline{E})$ forms a left factorization of $\underline{C}$.

It is interesting to note that $\underline{D}$ has both S-products and S-coproducts (the two being basically the same in this case):

Proposition (3.1): Let $\underline{D}$ be a category all of whose morphisms are isomorphisms. Then $\underline{D}$ has both an S-product and an S-coproduct. Specifically, the S-product can be defined by:

$A_1 \;\pi\; A_2 = \emptyset$ if $\underline{D}[A_1, A_2] = \emptyset$,

$A_1 \;\pi\; A_2 = \{A_f : f \in \underline{D}[A_1, A_2]\}$ where we set $A_f = A_1$ for all $f \in \underline{D}[A_1, A_2]$.

(The S-coproduct can be defined in exactly the same way, except that it is more convenient to take $A_f = A_2$ for all $f \in \underline{D}[A_1, A_2]$.)

Proof: Clearly if $\underline{D}[A_1, A_2] = \emptyset$ (i.e., A_1 and A_2 are not isomorphic), then $\underline{D}[-, A_1] \times \underline{D}[-, A_2]$ is the empty functor on $\underline{D}$ since for all $A \in \mathrm{ob}\,\underline{D}$ at least one of $\underline{D}[A, A_1]$ or $\underline{D}[A, A_2]$ must be empty.

Now suppose that $\underline{D}[A_1, A_2]$ is non-empty; i.e., A_1 and A_2 are isomorphic. For each $f \in \underline{D}[A_1, A_2]$, define the "natural projections" $\pi_1^f \colon A_f = A_1 \to A_1$ and $\pi_2^f \colon A_1 \to A_2$ by setting π_1^f = identity on A_1, and $\pi_2^f = f$. Then the family of pairs (π_1^f, π_2^f) defines (component-wise) a natural transformation $\alpha \colon \bigvee_f \underline{D}[-, A_f] \to \underline{D}[-, A_1] \times \underline{D}[-, A_2]$.

It is not hard to see that α is a natural equivalence. Indeed, for (g_1, g_2) in the cartesian product $\underline{D}[A, A_1] \times \underline{D}[A, A_2]$ it is easy to see that $(g_1, g_2) = (\pi_1^f \circ g_1, \pi_2^f \circ g_1)$ where $f = g_2 \circ g_1^{-1}$, and that this is the unique $f \in \underline{D}[A_1, A_2]$ with this property.

The proof regarding S-coproducts is similar. QED

Now let us further assume that $\underline{C}$ is finitary, with a skeletal set $C = \{A_i : i \in I\}$, and that the left factorization $(\underline{D},\underline{E})$ is finitary. If $(\underline{D},\underline{E})$ is finitary, then any $\underline{C}$-object can have only a finite number of quotient objects in $\underline{C}$. Consequently, if $\underline{e}$: $Z(C) \to Z(C)$ is the Z-linear map corresponding to $(\underline{D},\underline{E})$, then by (the dual of) proposition (3.16) of chapter II, $\underline{e}$ must be invertible.

Let us define the quantity $B(i,j)$ as the cardinality of $\underline{C}[A_i, A_j]$, and $A(j)$ as the cardinality of $\underline{C}[A_j, A_j]$ (= $\underline{D}[A_j, A_j]$, the automorphism group of A_j in $\underline{C}$). We also have $\underline{e}$ defined by the equation

$$(3.2) \qquad \underline{e}(A_i) = \sum_j e(i,j)A_j,$$

where $e(i,j)$ is equal to the number of times that A_j represents a subobject of A_i in $\underline{C}$. We can easily see that we have the equation:

$$(3.3) \qquad B(i,j) = A(j)e(i,j).$$

Let us further define the quantities $d(i,j)$ by means of the formula:

$$(3.4) \qquad \underline{e}^{-1}(A_i) = \sum_j d(i,j)A_j .$$

Now using the fact that $\underline{c}_k^* = \underline{d}_k^* \circ \underline{e}$, and expanding this via (3.2), we derive the (rather trivial) identity:

$$\underline{c}_k^*(A_i) = \sum_j e(i,j)\underline{d}_k^*(A_j),$$

which can also be written as:

$$(3.5) \qquad B(i,k) = \sum_j e(i,j)\Delta(j,k),$$

where $\Delta(j,k)$ is the number of elements in $\underline{D}[A_j, A_k]$; that is, 0 if j is not equal to k, and A(k) if j = k. Thus, this essentially (3.3).

Of course precisely similar remarks apply to a category $\underline{C}$ which consists only of monomorphisms. It has an "obvious" right factorization ($\underline{M}$,$\underline{D}$) in which $\underline{D}$ is the isomorphism subcategory of $\underline{C}$ and $\underline{M}$ = mor $\underline{C}$. And naturally a bimorphic category has both factorizations.

A simple example of this phenomenon is given by any poset $\underline{C} = (P, \underline{\leq})$ regarded as a category. Recall that there is at most one morphism from a point x to a point y in $\underline{C}$, with $\underline{C}[x, y] = \{(x,y)\}$ if $x \leq y$, and $\underline{C}[x, y] = \emptyset$ otherwise. Clearly, all the morphisms in $\underline{C}$ are both monomorphisms and epimorphisms; i.e., $\underline{C}$ is bimorphic. Of course it is finitary, and since it forms its own skeleton, P is a skeletal set for $\underline{C}$.

Now, if $\underline{c}$: $Z(P) \to Z^P$ is the right linearization of $\underline{C}$, then $\underline{c}_x$: $Z(P) \to Z$ is defined by:

$$\underline{c}_x(y) = \#\underline{C}[x, y] = \begin{cases} 1 \text{ if } x \leq y, \\ 0 \text{ otherwise;} \end{cases}$$

i.e., $\underline{c}_x(y) = \zeta(x,y)$, where ζ: $P \times P \to Z$ is the zeta function of the partial order. Thus, $\underline{c}(y)$ is the function $\zeta(-,y)$. In a similar way, we see that the left linearization $\underline{c}^*$: $Z(P) \to Z^P$ maps an element x in P to the function $\zeta(x,-)$.

Now, we noted earlier that if $(P, \leq)$ is a meet semilattice, then $\underline{C}$ may be regarded as having an S-product. In that case it is clear that the multiplication in Z(P) derived from the S-product is clearly given on basis elements by:

$$(3.6) \qquad x \circ y = \begin{cases} x \wedge y, \text{ if the meet } x \wedge y \text{ exists,} \\ 0 \quad \text{otherwise.} \end{cases}$$

With the above multiplication, we shall call Z(P) the meet algebra of $(P, \leq)$, and denote it by $Z\langle P, \wedge \rangle$ (instead of using our usual notation). Of course, if $(P, \leq)$ is a join semilattice, then $\underline{C}$ may be regarded as having an S-coproduct, and we may define in an analogous manner the join algebra of $(P, \leq)$, which we shall denote by $Z\langle P, \vee \rangle$.

For $\underline{C} = (P, \leq)$, the isomorphism subcategory $\underline{D}$ of $\underline{C}$ is simply P considered as a discrete category (i.e., all morphisms are identities). The corresponding right factorization ($\leq$, $\underline{D}$) of $\underline{C}$ (letting $\leq$ stand for the family of all morphisms in $\underline{C}$) is finitary if and only if the the principal order ideal $(x)^{-}$ is finite for all x in P. In that case, let us denote the Z-linear map $Z(P) \to Z(P)$ corresponding to this factorization by T_ζ, which is defined by:

$$(3.7) \qquad T_\zeta(x) = \sum_{y \leq x} y = \sum_y \zeta(y,x)y.$$

It is the <u>zeta transform</u> of the poset (P, $\leq$). Of course it is invertible, and its inverse is the <u>Mobius transform</u>, which we shall denote by T_μ. Thus, we have

$$(3.8) \qquad T_\mu(x) = \sum_y \mu(y,x)y,$$

where μ: $P \times P \to Z$ so defined is the Mobius function on (P, $\leq$).

Of course similar remarks apply to the left factorization ($\underline{D}$, $\leq$) of $\underline{C}$ = (P, $\leq$).

In the case of posets regarded as categories, it would be nice to know what in general S-products, S-coproducts, and factorizations look like. The following proposition answers the question:

<u>Proposition (3.9):</u>

a) A poset $\underline{C}$ = (P, $\leq$), considered as a category, has an S-product if and only if the intersection of any two principal order ideals $(x)^-$ and $(y)^-$ in P can be expressed as an (internal) disjoint union of principal order ideals (i.e., a union of ideals $(x_i)^-$ such that $(x_i)^- \cap (x_j)^-$ is empty if $i \neq j$).

Similarly, $\underline{C}$ has an S-coproduct if and only if the intersection of any two principal order co-ideals $(x)^+$ and $(y)^+$ can be expressed as an internal disjoint union of principal order co-ideals.

b) Let $(P, \leq_1)$ be a second partial order on the set P, with $\leq_1$ weaker than $\leq$. Then $\underline{D} = (P, \leq_1)$ is a subcategory of $\underline{C}$. Denote the principal order ideal generated by an element x in $\underline{D}$ by $(x)_1^-$.

Then $\underline{D}$ is a right factor of $\underline{C}$ if and only if any principal order ideal $(x)^-$ in $\underline{C}$ is expressible as an internal disjoint union of ideals $(x_i)_1^-$ in $\underline{D}$.

Similarly, $\underline{D}$ is a left factor if any order co-ideal $(x)^+$ is expressible as an internal disjoint union of order co-ideals $(x_i)_1^+$.

We omit the proof, which just consists of checking definitions.

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We now shall introduce another general concept which we shall find useful. Thus, let $\underline{C}$ be an arbitrary category. For any $\underline{C}$-object A, define the right specialization of $\underline{C}$ at A, which we shall denote by (A, $\underline{C}$), as the full subcategory of $\underline{C}$ generated by $\underline{C}$-objects B such that $\underline{C}$[A, B] is non-empty. Thus, B is an (A, $\underline{C}$)-object if and only if there exists a $\underline{C}$-morphism from A to B. (Of course, it is perfectly possible that (A, $\underline{C}$) = $\underline{C}$.)

The following are some facts about such specializations:

a) The inclusion functor (A, $\underline{C}$) $\to$ $\underline{C}$ has a right S-adjoint, which is defined on $\underline{C}$-objects by the assignments B $\mapsto$ B if B is in ob (A, $\underline{C}$), and B $\mapsto$ $\emptyset$ (the empty indexing) if B is not in ob (A, $\underline{C}$). We shall call this S-functor the projection of $\underline{C}$ onto (A, $\underline{C}$). [Proof: Consider the "inherited" connection $\underline{W}$ = $\underline{W}$((A, $\underline{C}$), $\underline{C}$) defined by setting $\underline{W}$[B, B'] = $\underline{C}$[B, B'] for B in ob (A, $\underline{C}$) and B' in ob $\underline{C}$. It clearly has the inclusion functor as a left realization. On the other hand, if B' is not an element of ob (A, $\underline{C}$), then there cannot be a $\underline{C}$-morphism from B to B' (since the existence of one would imply that B' in ob (A, $\underline{C}$)). Thus, it is clear that the "projection" S-functor as described above is a right S-realization of $\underline{W}$.]

b) If $\underline{C}$ has an S-product, then (A, $\underline{C}$) also has an S-product. Indeed, the S-product of two (A, $\underline{C}$)-objects B_1 and B_2 is simply obtained from the S-product in $\underline{C}$, $A \pi B = \{(A \pi B)(x) : x \in \mathrm{dom}(A \pi B)\}$, by deleting from the indexing all objects $(A \pi B)(x)$ which are not in (A, $\underline{C}$).

c) If (A, $\underline{C}$) is a finitary category, and it has an S-product, then the S-product is finitary.

[Proof: If the S-product in (A, $\underline{C}$) of two objects B_1 and B_2 were assumed to have an infinite number of components, then $(A, \underline{C})[A, B_1] \times (A, \underline{C})[A, B_1]$ would be infinite, since by assumption $(A, \underline{C})[A, (B_1 \pi B_2)(x)]$ is non-empty for all indices x; a contradiction.]

Let us return to the case in which $\underline{C}$ is finitary and epimorphic. Choose a $\underline{C}$-object A and form the right specialization (A, $\underline{C}$).

Now, we form yet another category by adding some structure to (A, $\underline{C}$). Define the category A/$\underline{C}$ as follows:

The object class of A/$\underline{C}$ consists of pairs (f, B), where B is in ob (A, $\underline{C}$), and f is an (A, $\underline{C}$)-morphism from A to B. A morphism g from (f_1, B_1) to (f_2, B_2) is given by a morphism $g: B_1 \to B_2$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ B_1 \arrow[rr, "g"'] & & B_2 \end{tikzcd}$$

i.e., $g \circ f_1 = f_2$.

Note that since all the morphisms in $(A, \underline{C})$ are epi, there can be at most one $A/\underline{C}$-morphism from (f_1, B_1) to (f_2, B_2); for if $g_1 \circ f_1 = f_2 = g_2 \circ f_1$, we must have that $g_1 = g_2$ since f_1 is epi. Thus, $(A/\underline{C})[(f_1, B_1), (f_2, B_2)]$ is either a singleton set or is empty. We shall write $(f_1, B_1) \leqq (f_2, B_2)$ to mean that there exists an $A/\underline{C}$-morphism from (f_1, B_1) to (f_2, B_2).

Also notice that (f_1, B_1) and (f_2, B_2) are isomorphic in $A/\underline{C}$ if and only if they represent the same quotient object of A. Thus, if we let Quot(A) $= \{(f_i, A_i) : i \in J\}$ be a skeletal set for $A/\underline{C}$, then the elements of Quot(A) are in a one-one correspondence with the distinct quotient objects of A in $\underline{C}$, and the relation $(f_i, A_i) \leqq (f_j, A_j)$ simply means that the quotient object represented by (f_i, B_i) is "finer" than the quotient object represented by (f_j, B_j). (We use the term "finer" in analogy with the case in which A is a set, and hence the quotient objects may be identified with partitions of A; then the relation $\leqq$ can be identified with the relation of one partition being finer than another.) Thus, Quot(A), under $\leqq$, is naturally isomorphic with the poset of quotient objects of A in $\underline{C}$.

We have a natural connection $\underline{U}$ from $A/\underline{C}$ to $(A, \underline{C})$, given by setting

$$\underline{U}[(f, B), B'] = (A, \underline{C})[B, B'],$$

together with the obvious compositions. The connection has an obvious left realization, given by the "forgetful" functor from $A/\underline{C}$ to $(A, \underline{C})$, which on objects maps (f, B) to B. It is also S-realizable on the right:

To see this, for each $(A, \underline{C})$-object B let $(X, F)(B)$ be defined by setting $X(B) = (A, \underline{C})[A, B]$, and for each f in $X(B)$, let $F_f(B) = (f, B)$. The natural equivalence between $\underline{U}[-, B]$ and $\bigvee_f (A/\underline{C})[-, (f, B)]$ is then provided by mapping any g in $\underline{U}[(h, B'), B]$ into the unique element $g \circ h$ in $(A/\underline{C})[(h, B'), (g \circ h, B)]$. It follows that (X, F) extends to an S-functor from $(A, \underline{C})$ to $A/\underline{C}$, which is the right S-adjoint of the forgetful functor from $A/\underline{C}$ to $(A, \underline{C})$.

<u>Proposition (3.10):</u> If the category $(A, \underline{C})$ has an S-product, then $A/\underline{C}$ has a product, and therefore $(\mathrm{Quot}(A), \underline{\leq})$ has meets.

<u>Proof:</u> Looking at $A/\underline{C}$ as a quasi-ordered class, we wish to show that any two elements have a greatest lower bound. (Of course, the object $(1_A, A)$ is a lower

bound for any two elements of $A/\underline{C}$.)

Given any two $(A/\underline{C})$-objects (f_1, B_1) and (f_2, B_2), consider their S-product (in $(A, \underline{C})$):

$$B_1 \pi B_2 = \{(B_1 \pi B_2)(x) : x \in \mathrm{dom}(B_1 \pi B_2)\}.$$

The morphisms f_1 and f_2 determine a unique x and a unique $(A, \underline{C})$-morphism $f: A \to (B_1 \pi B_2)(x)$ such that $\pi_1{}^x \circ f = f_1$ and $\pi_2{}^x \circ f = f_2$. Thus, we have:

$$(f, (B_1 \pi B_2)(x)) \leqq (f_1, B_1)$$

$$\text{and } (f, (B_1 \pi B_2)(x)) \leqq (f_2, B_2).$$

We must now show that $(f, (B_1 \pi B_2)(x))$ is a greatest lower bound. Thus, suppose we have that $(g, B') \leqq (f_1, B_1)$ and $(g, B') \leqq (f_2, B_2)$. Then, we must have morphisms $h_1: B' \to B_1$ and $h_2: B' \to B_2$ such that $h_1 \circ g = f_1$ and $h_2 \circ g = f_2$.

In the S-product, there must be a unique y and a unique $(A, \underline{C})$-morphism $h: (B_1 \pi B_2)(y) \to B'$ such that $\pi_1{}^y \circ h = h_1$ and $\pi_2{}^y \circ h = h_2$. But then, we must also have $\pi_1{}^y \circ (h \circ g) = h_1 \circ g = f_1$ and $\pi_2{}^y \circ (h \circ g) = h_2 \circ g = f_2$, and by the uniqueness of x and f, we therefore have y = x and

$$(g, B') \leqq (f, (B_1 \pi B_2)(x))$$

$$\text{and } (g, B') \leqq (f, (B_1 \pi B_2)(x)).$$

QED

So, the situation is as follows:

If $\underline{C}$ is a finitary, epimorphic category with a skeletal set $C = \{A_i : i \in I\}$, we form the specialization $(A_i, \underline{C})$; it has a skeletal set $C_i = \{A_j : j \in I(i)\}$ which can be chosen to be a subset of C. Then the Z-linear map $\underline{p}$: $Z(C) \to Z(C_i)$ derived from the projection S-functor from $\underline{C}$ to $(A_i, \underline{C})$ is defined on basis elements simply by:

$$\underline{p}(A_j) = \begin{cases} A_j, & \text{if } A_j \in C_i \\ 0, & \text{otherwise.} \end{cases}$$

If $\underline{C}$ has a finitary S-product, then so does $(A_i, \underline{C})$, and if $Z(C)$ and $Z(C_i)$ are given the multiplications derived from the respective S-products, then $\underline{p}$ is a ring homomorphism (since the projection S-functor is a right S-adjoint of the inclusion functor). (Of course if $\underline{C}$ has an S-product, finitary or not, then $(A_i, \underline{C})$ inherits a finitary S-product.)

If we now form $A_i/\underline{C}$, then the S-functor (X, F) from $(A_i, \underline{C})$ to $A_i/\underline{C}$ (i.e., the right S-adjoint of the forgetful functor from $A_i/\underline{C}$ to $(A_i, \underline{C})$) is clearly finitary. The Z-linear map $\underline{w}$: $Z(C_i) \to Z(\mathrm{Quot}(A_i))$ can be described as follows:

For each A_j in C_i, let Ω_j be the family of all (f_k, B_k) in $\mathrm{Quot}(A_i)$ such that B_k is isomorphic to A_j. Of course the number of elements in Ω_j is simply the number of distinct quotient objects of A_i which can be represented by A_j. Let ω_j in $Z(\mathrm{Quot}(A_i))$ be the sum of the elements in Ω_j. Then, if $\Delta(j)$ is the cardinality of the automorphism group of A_j in $\underline{C}$, we have:

(3.11) $$\underline{w}(A_j) = \Delta(j)\omega_j.$$

This can be seen by noting that every pair (f, A_j), for f in $(A, \underline{C})[A_i, A_j]$ can be written as $(g \circ f_k, A_j)$, where (f_k, B_k) is a unique element of Ω_k, and g is a unique isomorphism from B_k to A_j.

Now, if $(A_i, \underline{C})$ has an S-product (which is necessarily finitary), then we have shown that the poset $(\mathrm{Quot}(A_i), \leq)$ has meets; indeed, the right linearization of $A_i/\underline{C}$ with the multiplication derived from the product in $A_i/\underline{C}$ is simply the meet algebra $Z\langle \mathrm{Quot}(Ai), \wedge \rangle$. And since (X, F) is a right S-adjoint, $\underline{w}$ is a ring homomorphism from $Z(C_i)$ (with the multiplication derived from its S-product) to $Z\langle \mathrm{Quot}(A_i), \wedge \rangle$. It is not difficult to see that the elements ω_j of $Z(\mathrm{Quot}(A_i))$ are linearly independent (since the sets Ω_j are, for different j, disjoint), and that therefore $\underline{w}$ is a faithful representation of $Z\langle C_i, (A_i, \underline{C})\rangle$.

We are now in a position better to understand the representation $w\colon B(G) \to Z\langle L(G), \wedge \rangle$ introduced in the first section. The Burnside ring $B(G)$ is the ring $Z\langle D, \underline{D}\rangle$ where $\underline{D}$ is the category of transitive actions of G. But if we take $\underline{a}$ to be the action of G on itself by right multiplication (i.e., the natural action of G on G/H where H is the identity subgroup of G), it is easy to show that $(\underline{a}, \underline{D}) = \underline{D}$. Furthermore, the quotient objects of $\underline{a}$ are in a one-one correspondence with the subgroups H of G (with the actions $\underline{a}^H$ forming a family of representatives of the distinct quotient actions of $\underline{a}$). In fact the poset L(G) is isomorphic to Quot($\underline{a}$) under this correspondence, and w is essentially the representation $\underline{w}$: $B(G) \rightarrow Z\langle Quot(\underline{a}), \wedge\rangle$ described above.

Continuing with the case in which $(A_i, \underline{C})$ has an S-product, we can use the facts established above to deduce some relationships between the structural constants defining the multiplication in $Z\langle C_i, (A_i, \underline{C})\rangle$ and the poset $(Quot(A_i), \underline{\leq})$. For elements A_j and A_k in C_i, we write:

$$A_j \circ A_k = \sum_s r(j,k;s) A_s,$$

where of course the multiplication is that derived from the S-product in $(A_i, \underline{C})$.

Now, for the elements ω_j and ω_k of $Z\langle Quot(A_i), \wedge\rangle$, we can write:

$$(3.12) \qquad \omega_j \circ \omega_k = \sum_s b(j,k;s) \omega_s.$$

Here, it is not hard to see that the quantity b(j,k;s) can be defined as the number of ways a given element of Ω_s can be expressed as a meet of an element of Ω_j with an element of Ω_k. But it is also easy to establish the relationship between the quantities $r(j,k;s)$ and $b(j,k;s)$; since $\underline{w}$ is a ring homomorphism (and faithful) we at once have:

$$\underline{w}(A_j) \circ \underline{w}(A_k) = \sum_s r(j,k;s)\underline{w}(A_s)$$

$$= (\Delta(j)\omega_j \circ \Delta(k)\omega_k = \sum_s r(j,k;s)\Delta(s)\omega_s$$

$$= \sum_s \Delta(j)\Delta(k)b(j,k;s)\omega_s,$$

from which we get the equation:

$$(3.13) \qquad b(j,k;s) = \{r(j,k;s)\Delta(s)\}/\{\Delta(j)\Delta(k)\}.$$

If we apply this to the category $\underline{Q}$ (finite sets and surjective maps), we get the following:

Let $q(i,j;k)$ be the number of subdirect k-subsets of the product of an i-set and a j-set. Then, if by a "k-partition" we mean a partition of a set into k blocks, the number of ways of forming a k-partition of a given finite set as the meet of an i-partition with a j-partition is given by the number:

$$b(i,j;k) = (q(i,j;k)k!)/(i!j!).$$

The reader will easily dualize the above results to arrive at the appropriate notion of left specialization and establish a similar relationship between a monomorphic category and the poset of subobjects of an object of the category.

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