THE UNIVERSITY OF MANITOBA

TREATMENT OF SINGULARITIES IN THE FINITE - ELEMENT METHOD

by

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CHAPTER ONE

INTRODUCTION

In problems involving the solution of partial differential equations, a major difficulty, for all numerical methods of approximation, is the effect of singularities. Either the process may not converge at all or else it may converge to something quite different from the actual solution.

Much attention has always been directed to suitable amendments of standard techniques to make their use near a singular point more feasible. These modifications often use, in one way or another, some information about the analytical behaviour of the field at these special points.

This work will deal with scalar or vector potentials from which the scalar part may be extracted. The main attention will be given to Laplace's and Helmholtz's equation. Both homogeneous and inhomogeneous media will be considered.

In Chapter 2, an analysis of the near field around singular points will be presented. Further, different methods for handling singularities will be reviewed.

In Chapter 3, a method of solution using a standard variational scheme with extra trial functions will be described. These extra functions are chosen to approximate the asymptotic behaviour of the field near the singular point. Although giving some good results in the examples considered, the method suffers from a lack of algorithmic generality, and a likelihood of slow convergence due to non-local considerations and significant inter-element incompatibility.

Chapter 4 will present another approach to the problem, using an integral formulation and the concept of the Green's function. In the case of a Dirichlet problem with Laplace's equation, a variational technique will be used, since a positive-definite and self-adjoint operator will be found.

This integral formulation will lead to the consideration of another way to construct asymptotic functions. These new trial functions presented in Chapter 5 are derived from the equivalent charge distribution at the boundary near the singularity. This approach is valid for all kinds of corners, boundary conditions and media. It represents a completely algorithmic way to solve singularity problems, and can be easily incorporated into a standard general finite-element program.

CHAPTER TWO

BEHAVIOUR OF SCALAR POTENTIALS NEAR CORNERS

As long as a region, where a partial differential equation has to be solved, does not contain any singular point, most of the existing approximation methods converge very well towards the exact solution. Good accuracy is obtained with relatively small computer time. But many difficulties arise with singular problems.

These singularities can occur through the differential equation itself. The presence of a source, for instance, gives the following equation

$$L[u] = \delta(x - a) \tag{2.1}$$

where L is a given partial differential operator. A singular point occurs at $x = a$. They also come from the boundary conditions. An abrupt change in direction, the corner problem, leads to infinite values of the derivative of the field. Boundary values may change discontinuously. Also, the form of the boundary condition may change from one kind to another with an incompatibility at the common point.

For a long time, it has been known that singular points cause the standard methods to converge to something different from the exact solution [13; 23]. However, this can be sufficiently close in some cases, but the convergence will be very slow. A significant example will be given in Chapter 5.

For many years, a considerable amount of work was done to describe, as precisely as possible, what was actually occuring at a singular point.

Attention was first oriented towards specific problems in electromagnetics and scattering theory [2; 5; 10; 18]. Much work was done concerning expansions of the solution near corners or jumps in boundary values. First, in the case of Laplace's equation, then more generally for all kinds of elliptic operators, different boundary conditions were considered. These results were then used to improve the existing approximation methods for singularities. This was applied for waveguides, heat flows around corners, etc.

2.1 Laplace's Equation with Homogeneous Dirichlet or Neumann Boundary Conditions

The expansion of the solution of Laplace's equation

$$\nabla^2 \Phi = 0 \qquad (2.2)$$

near a corner, Figure 2.1, is obtained by separation of variables r and Θ in the cylindrical co-ordinate system, with the origin at the corner 0.

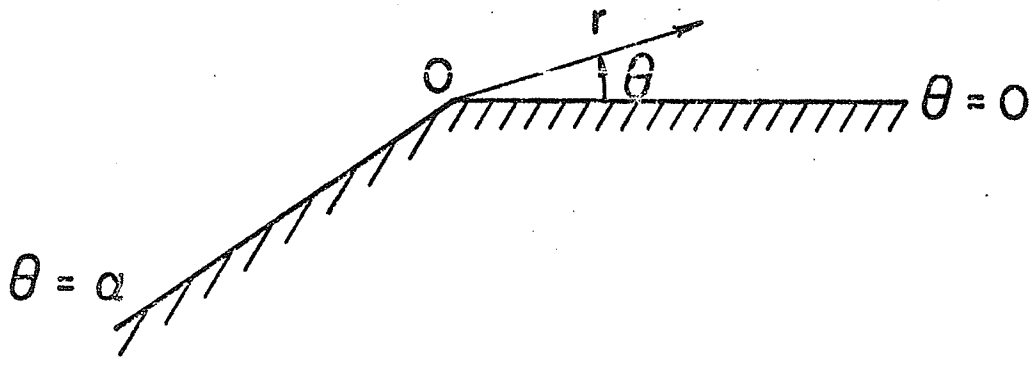

Figure 2.1 Re-entrant corner of angle α

Equation (2.2) becomes

$$r^2 \frac{\partial^2 \Phi}{\partial r^2} + r \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial \Theta^2} = 0 \tag{2.3}$$

and the solution is assumed to be

$$\Phi(r,\Theta) = R(r)\ \Psi\ (\Theta) \tag{2.4}$$

whose functions R and Ψ are solutions of

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - s^2 R = 0 \tag{2.5}$$

$$\frac{d^2 \Psi}{d\Theta^2} + s^2 \Psi = 0 \tag{2.6}$$

The general solution can be written as

$$\Phi = \sum_{S=-\infty}^{-\infty} (a_S r^S \sin S\Theta + b_S r^S \cos S\Theta) \tag{2.7}$$

where the summation extends over all real s.

When the boundary conditions on the corner are defined, the values of s will be restricted. For instance, when

$$\Phi = 0 \tag{2.8}$$

at $\Theta = 0$ and $\Theta = \alpha$, (2.7) reduces to

$$\Phi = \sum_{k=0}^{+\infty} a_k r^{k\frac{\pi}{\alpha}} \sin k\frac{\pi}{\alpha}\Theta \tag{2.9}$$

with k being integer. On the other hand, if (2.8) is valid at $\Theta = 0$ and

$$\left.\frac{\partial \Phi}{\partial n}\right|_S = 0 \tag{2.10}$$

at $\Theta = \alpha$, we get

$$\Phi = \sum_{k=0}^{+\infty} a_k r^{(k+\frac{1}{2})\frac{\pi}{\alpha}} \sin(k+\tfrac{1}{2})\tfrac{\pi}{\alpha}\Theta \qquad (2.11)$$

For the condition (2.10), valid at $\Theta = 0$ and $\Theta = \alpha$, we have

$$\Phi = \sum_{k=0}^{+\infty} b_k r^{k\frac{\pi}{\alpha}} \cos k\tfrac{\pi}{\alpha}\Theta \qquad (2.12)$$

In order to avoid an infinite value of Φ at $r = 0$, only positive values of the exponent are taken in formulas (2.9), (2.11) and (2.12), except in the case of a source at the origin in which case the field can be infinite. The singularity appears in these formulas when the derivative is taken with respect to r. Some terms will have negative exponent when $\alpha > \pi$ which is the case of a re-entrant corner. For formula (2.7), an expression can be found for $\frac{\partial \phi}{\partial n}$ along the edges of the corner, giving the charge distribution. This is presented in Appendix B.

The same derivation can be applied to the solution of the Helmholtz equation

$$\nabla^2\Phi + \lambda\,\Phi = 0 \qquad (2.13)$$

which reduces to

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{S^2}{r^2} - \lambda\right) R = 0 \qquad (2.14)$$

$$\frac{d^2\Psi}{d\Theta^2} + S^2\Psi = 0 \qquad (2.15)$$

The first equation is the Bessel equation and the general solution will be

$$\Phi = \sum_{S=0}^{\infty} J_S(\sqrt{\lambda}r)[a_S \cos S\Theta + b_S \sin S\Theta] \qquad (2.16)$$

where the terms with negative S have been removed in order to have a finite field at the origin.

It is easy to verify that for $\lambda \to 0$, the solution of Laplace's equation (1.7) is found, since [9, p. 202]

$$\lim J_s(\sqrt{\lambda}r) = r^s \quad \text{for} \lambda \to 0 \tag{2.17}$$

which causes (2.16) to become (2.7).

2.2 Inhomogeneous Boundary Conditions [8; 7, pp. 301 - 310]

Let us consider the boundary conditions

$$\Phi = g(r) \tag{2.18}$$

and

$$\left.\frac{\partial \Phi}{\partial \hat{n}}\right|_S = h(r) \tag{2.19}$$

where, by restricting the functions g and h to polynomial expressions

$$g(r) = \sum_{n=0}^{\infty} g_n r^{n+\beta} \tag{2.20}$$

and

$$h(r) = \sum_{n=0}^{\infty} h_n r^{n+\gamma} \tag{2.21}$$

on the sides of the corner. β and γ are real numbers, not necessarily positive integers.

Each combination of boundary conditions is the superposition of of the complete homogeneous case and solutions of one or more of the four following cases, or their reversals.

1. $\Phi(\Theta = 0) = g_n r^{n+\beta}$; $\phi(\Theta = \alpha) = 0$ (2.22)
2. $\Phi(\Theta = 0) = g_n r^{n+\beta}$; $\frac{\partial \phi}{\partial n}(\Theta = \alpha) = 0$ (2.23)
3. $\frac{\partial \Phi}{\partial n}(\Theta = 0) = h_n r^{n+\gamma}$; $\phi(\Theta = \alpha) = 0$ (2.24)
4. $\frac{\partial \Phi}{\partial n}(\Theta = 0) = h_n r^{n+\gamma}$; $\frac{\partial \phi}{\partial n}(\Theta = \alpha) = 0$ (2.25)

The validity of this superposition comes from the fact that the sum of two solutions with given boundary conditions is a solution for the sum of the boundary conditions.

The differential equation

$$\nabla^2 \Phi + f(r,\theta)\ \Phi = 0 \tag{2.26}$$

is considered. It generalizes the cases of Laplace's and Helmholtz's equations. We consider the case where $f(r,\theta)$ can be expanded in the following form

$$f(r,\theta) = \sum_{k=0}^{\infty} f_k(\theta)\ r^k \tag{2.27}$$

and a solution of the form

$$\Phi = \sum_{j=0}^{\infty} a_j(\theta)\ r^{w+j}\ , \tag{2.28}$$

a generalization of (2.7), is sought. w is a real number, not necessarily integer, to cater for the fractional exponent that will be seen to be required when singularities are present.

Substituting (2.28) into (2.26) gives

$$\ddot{a}_m(\theta) + (\alpha + m)^2\ a_m(\theta) = 0 \text{ for } m = 0,1 \tag{2.29}$$

$$\ddot{a}_m(\theta) + (\alpha + m)^2\ a_m(\theta) + \sum_{j=0}^{m-2} f_{m-2-j}\ a_j(\theta) \text{ for } m \geq 2 \tag{2.30}$$

The boundary conditions are then applied to this system of equations. This gives the following results, corresponding to the previous four cases.

1. $w = n + \beta$; $a_0 = -g_n \dfrac{\sin(n+\beta)(\theta-\alpha)}{\sin(n+\beta)\alpha}$ (2.31)

and the other coefficients are solved from (2.29) and (2.30),

knowing that

$$a_j(0) = a_j(\alpha) = 0 \tag{2.32}$$

2. $w = n + \beta$; $a_0 = g_n \dfrac{\cos(n+\beta)(\theta-\alpha)}{\cos(n+\beta)\alpha}$ (2.33)

3. $w = n + \gamma + 1$; $a_0 = h_n \dfrac{\sin(n+\gamma+1)(\alpha-\theta)}{(n+\gamma+1)\sin(n+\gamma+1)\alpha}$ (2.34)

4. $w = n + \gamma + 1$; $a_0 = h_n \dfrac{\cos(n+\gamma+1)(\theta-\alpha)}{(n+\gamma+1)\sin(n+\gamma+1)\alpha}$ (2.35)

We are now able to find the asymptotic series near any corner for Neumann and Dirichlet boundary conditions. In the case of mixed boundary conditions or for any other elliptic operators, a generalization of these results is necessary. This will be done in the next two sections.

2.3 Developments of Solutions at Corners Using Conformal Transformation [14; 15; 24; 26]

For mixed boundary conditions and all kinds of elliptic equations, a more general approach must be used to find the asymptotic series near the singularity. This can be done by a conformal transformation technique which will transform the original problem into an upper half-plane. For this new region, Green's functions are available and the problem can be formulated in terms of two-dimensional integrals in the complex plane. An iterative procedure, used to solve them, permits one to obtain the asymptotic series term by term.

The expansion obtained for the solution is valid not only in the real plane, but also in a complex domain containing the real plane.

An advantage of this is that these developments can be differentiated formally to obtain asymptotic expansions for the derivatives also. It is thus possible to obtain information about the behaviour of the derivatives at the corner.

This technique was mainly developed by Lewy [16], Lehman [14] and Carter [5] for Dirichlet problems and by Wigley [26] for mixed boundary value problems.

The main result of their work was that there exists an asymptotic expansion to the solution in all these cases in the form of a polynomial with the following arguments

$$z,\ \bar{z},\ z^{1/\alpha},\ \bar{z}^{1/\alpha},\ \log z,\ \log \bar{z}$$

where $z = x+jy$ and, $\bar{z} = x - jy$ and α is the interior angle at the corner. The existence of these expansions, as well as the completeness of the series with respect to the solution, are shown. This method, however, is very slow and not very algorithmic. A better way to solve these problems will be presented in Chapter 5.

2.4 Inhomogeneous Regions

The asymptotic expansions can be extended to more general cases where different media meet at the corner. For instance, in Figure (2.2), two different dielectrics exist near the corner, with characteristics ϵ_1, μ_1 and ϵ_2, μ_2 respectively.

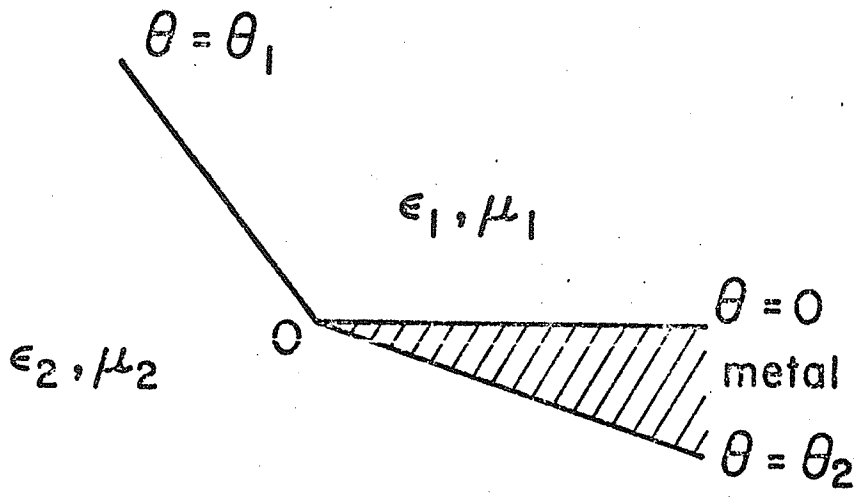

Figure 2.2 Different dielectric media near a corner.

These cases were thoroughly investigated by Meixner [18] and Minor [20]. They handle general electromagnetic problems, and so assume an expansion in terms of r and Θ for all the components of the electric and magnetic field. Here, however, only electrostatic problems are considered and so, only the scalar potential $\Phi(r,\Theta)$ must be expanded. The corresponding series for $\Phi(r,\Theta)$ is

$$\Phi(r,\Theta) = \sum_{n=0}^{\infty} a_n(\Theta)\, r^{t+n} \tag{2.36}$$

where t is a real number depending on ε_1, ε_2, Θ_1 and Θ_2. Using then the boundary condition at $\Theta = 0$ and $\Theta = \Theta_2$ and the continuity requirements of the flux on $\Theta = \Theta_1$, the following relationship is obtained.

$$\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = \frac{\sin t\Theta_2}{\sin t(2\Theta_1 - \Theta_2)} \tag{2.37}$$

with $0 \leq t < 1$.

This equation can be plotted in charts giving the values of t in function of the parameter [18]. A computer program could also easily generate the solutions of this equation. When t is known, the singular part of the field is found, since only the first term of (2.36) where n = 0 has an infinite derivative. From the form of (2.36), the equivalent charge distribution can be found along $\Theta = 0$, $\Theta = \Theta_1$ and $\Theta = \Theta_2$. The basic singularity will then be r^{t-1} for the charge distribution since the normal derivative $\frac{1}{r}\frac{\partial \phi}{\partial n}$ has to be taken.

The same method could be extended to other kinds of boundary conditions: inhomogeneous Dirichlet and Neumann, and mixed.

2.5 Treatment of Singularities by Numerical Methods

Difficulties encountered

Edges cause singularities of the solution derivatives, as was seen in the previous sections. Most of the methods, in their basic form, use polynomial approximations in an explicit or implicit way. The difficulties encountered in the convergence of these methods come from the fact that a polynomial, no matter how high its order, cannot approximate the field near the singular point as accurately as desired. The set of polynomial functions is not "complete" for the kind of solution to approximate [17].

An easy way to prove this incompleteness is found in the theory of functions of a complex variable [12]. It is known that if f(z) is analytic in the domain D, z_0 being in D, and R is the radius of the

largest circle with center z_0 whose interior is in D, then there is a power series

$$p_n(z) = \sum_0^\infty C_n (z - z_0)^n \tag{2.38}$$

which converges to f(z) for

$$|z - z_0| \leq R \tag{2.39}$$

In the case of functions of two variables, the power series in z becomes a polynomial in x and y.

If an isolated singularity occurs at z_0, a Laurent series

$$g_n(z) = \sum_{n=0}^\infty \left[b_n \frac{1}{(z - z_0)^n} + a_n (z - z_0)^n \right] \tag{2.40}$$

converges uniformly, within a ring, centered at z_0, but not in the interior of the smaller circle of the ring [12].

The limit circles of the ring are chosen such that all the singularities are either inside the inner circle or outside the outer one.

Furthermore, if the point z_0 is a branch point, i.e. the function is not single-valued, as for $\sqrt{z - z_0}$ and log $(z - z_0)$, there is no convergent series of type (2.38) or (2.40) possible. In these cases, a polynomial will not be able to approximate the function as closely as desired.

Modification of the region by conformal transformation

When the shape of the region is not too complicated and there is no change of dielectric constant, a conformal transformation can be used in order to get a new singularity-free region where a standard method of solution can be used without difficulty. In fact, the main problem is to find the conformal transformation, and this

approach has therefore been limited to simple regions.

Modifications of the finite-difference method

The specific problem of the finite-difference method near a singular point is the propagation of the local error to the whole region by the actual process of the iterative method. Much work was done on the expansion of the "area of infection", and its consequence on the convergence of the method [13].

The first attempt to modify the finite difference technique was the refinement of the mesh size near the singularity. This could better approximate the rapid variation of the field, but the amount of computation time was considerable and, for very large systems, the iterative method did not always converge very easily. This is a crude approach and not necessarily useful.

Another approach was to use a higher order approximation near the singular point by taking polynomials of order 2 or 3 as interpolation functions. This gave some improvement in the accuracy but the singularity was still not, of course, approximated correctly [6].

A better modification is to use the development of the solution near the singularity as an interpolation function. This gives very good results as shown by Motz [21] and Wigley [27]. It was also used very successfully by Fox and Sankar [8] for discontinuous boundary conditions. But, on the other hand, their numerical process suffered from the lack of flexibility of the finite-difference scheme, i.e. very large systems, no guaranteed convergence for the asymptotic functions, etc.

The expansion was also used in a different way by Wigley [27],

Whiting [25] and Milne [19]. They subtracted the singular part of the solution near the corner and solved separately the regular and singular parts. If S is the singular part, then

$$\Psi = \Phi - S \tag{2.44}$$

is a regular solution over the whole region. The original differential equation

$$L\,\Phi = f \tag{2.42}$$

becomes

$$L\,\Psi = f - LS \tag{2.43}$$

and the boundary conditions are changed accordingly. Ψ is then found without difficulty when S can be expanded asymptoticaly near the singularity. The main difficulty of this approach is to find the coefficients of the S part. With an elaborate derivation using a conformal transformation, it is possible to find these coefficients in some special cases ($\frac{\pi}{\alpha}$ integer) and an iterative approximation is necessary in the other ones [27]. This requires a considerable extra computer time, making this technique impractical for general problems.

Modifications of the variational methods

When variational techniques are considered with a Rayleigh-Ritz type of approximation, two kinds of refinements can be used. First, a finite element scheme with a higher number of small elements but a relatively low order of approximation on each element can be used. A second approach is to consider a small number of large elements with a higher order approximation in each of them.

The first attempt seems to converge very slowly near the singularity. The number of elements has to be very large and the computer time becomes unreasonably large [11].

Using the other approach, some work was done with higher-order polynomial approximations [3, 4], or other types of polynomials, like the Chebyshev or Legendre polynomials [17, 22, 20].

In both approaches, the matrices involved become very big. Their condition numbers become very large, growing almost exponentially with the order, and the solution of the linear system is difficult to find with good accuracy.

The reason for the increasing condition number can be understood with the aid of an intuitive analogy with the matrix eigenvalue problem

$$A\Phi = \lambda\Phi \tag{2.42}$$

solved by a finite-difference method where $\lambda = (k_c h)^2$ with k_c being the wavenumber and h being the mesh spacing. Clearly, λ goes to zero when the matrix A becomes larger and h tends to zero. On the other hand, the Laplace's equation gives

$$A\Phi = \underline{b} \tag{2.43}$$

with $\underline{b}$ depending on the boundary conditions. When A increases, and that the size of the elements decreases, the boundary node points become a smaller fraction of the total number of nodes. The boundary points are considered here because they are the only ones for which (2.42) differs from (2.43). Both A's become the same and so

$$|A| \to 0 \tag{2.44}$$

The matrix A is then singular. Increasing the number of elements causes

the matrix to become more ill-conditioned. This also applies to finite elements.

It is sometimes possible, by using proper combinations of the trial functions to keep the condition number very low, for instance by taking the eigenfunctions of the linear matrix. This, however, appears some times to give numerical instabilities in the computation of the eigenfunctions.

Some work was also done by Babuska [1] involving refinement of the finite elements in ways that eliminate the problems of increasing condition numbers and keep the rate of convergence reasonable. The method however was limited to natural boundary conditions, especially Neumann conditions.

In the next chapter, a new approach will be used. The terms of the asymptotic development of the solution around the singular point will be added to the standard polynomial set of trial functions and so make it complete with respect to the solution near the singularity.

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CHAPTER THREE

ASYMPTOTIC FUNCTIONS IN THE RAYLEIGH - RITZ APPROXIMATION TECHNIQUE

In this chapter, a method is presented to handle problems with singularities. A variational approach with a linear Rayleigh - Ritz approximation technique is used. Moreover, the finite element approach is applied to improve the accuracy. Standard methods use polynomial trial functions in each element, which are incomplete for singular problems. Here, some extra trial functions are added to the polynomial set in all the elements surrounding the singularity. These extra functions are terms of the asymptotic expansion of the solution near this point. Results are presented for the Laplace and the Helmholtz equations. The results are discussed and compared with those obtained by standard methods.

3.1 Polynomial Approximation with Laplace's Equation [9, 13, 1, 2]

The standard method using polynomial trial functions is first reviewed briefly. The modification of it near the singular point will then be presented in the next section.

Using variational techniques, the solution of Laplace's equation in the region R is found as the function minimizing the following functional

$$F[u] = \iint_R (\nabla u)^2 dxdy + \int_C [\sigma(s)u^2 - 2h(s)u]\, ds \qquad (3.1)$$

where u is an approximate trial function whose first derivatives are

square - integrable. C is the part of the boundary where the mixed condition

$$\frac{\partial u}{\partial n}\Big|_s + \sigma(s)u = h(s) \tag{3.2}$$

holds. The Neumann case is included in (3.2) when $\sigma(s)$ is zero. This condition (3.2) can be proved to be natural for the functional (3.1), i.e. it is satisfied naturally when F [u] is minimized. On the other hand, a Dirichlet condition

$$u(s) = g(s) \tag{3.3}$$

is principal and the trial function u to be used must be constrained to satisfy (3.3).

Rayleigh-Ritz Procedure

A set of functions

$$f_1, f_2, \ldots\ldots\ldots\ldots f_m$$

is chosen. The solution is assumed to have the form

$$u(x,y) = \sum_{i=1}^{M} c_i f_i \tag{3.4}$$

instead of using the c_i as the variational parameters, it is often convenient to take the values of u at M node points. This makes the Dirichet boundary conditions and the interelement continuity more easy to satisfy.

The solution is then written as

$$u(x,y) = \sum_{i=1}^{M} u_i \alpha_i \tag{3.5}$$

where the u_i are the values of u(x,y) at the M node points and $\alpha_i(x,y)$ are appropriate interpolation polynomials [10].

The functional F [u] will be minimized with respect to these new variational parameters u_i. This gives a set of M simultaneous linear equations on the unknowns u_i. The nodes at the Dirichlet boundaries are given the required values and at the interelement boundaries the nodes on two different elements are assigned the same value. This eliminates some of the equations and the matrix equation

$$S \underline{u} = \underline{b} \tag{3.6}$$

is finally obtained, where $\underline{u}$ is the vector of the variational parameters u_i.

Finite - Elements [10, 11, 14]

In order to increase the accuracy, a finite element division of the region is made and a particular combination of functions is chosen in each element. We will use the triangle in this work as the basic element.

The functional is then the sum of the contribution of each element.

A condition generally required is the continuity of the functions along the inter-element boundaries. This is easily satisfied with polynomials, as shown above, by assigning the same value of the trial function to common nodes.

It can also be proved that the continuity of normal derivatives, and more generally of the fluxes, is a natural condition of the chosen functional. The solution tends to approach the condition

$$\varepsilon_1 \frac{\partial u}{\partial n}\Big|_{S_1} = \varepsilon_2 \frac{\partial u}{\partial n}\Big|_{S_2} \tag{3.7}$$

along the interelement boundary when the dielectric constants are ε_1 in one element and ε_2 in the other one, as the polynomial order increases.

The use of polynomials as trial functions is thus certainly the most convenient regarding the handling of principal boundary conditions and inter-element continuity requirements. It also significantly simplifies programming effort, computer time and reduces storage requirements.

Mathematical treatment

Using the form (3.5) for the assumed solution and taking M node points on each element we can write

$$u = \underline{u}^T \underline{\alpha} \tag{3.8}$$

where $\underline{u}^T$ and $\underline{\alpha}$ are vectors of elements u_i and $\alpha_i(x,y)$ respectively. For the interpolation polynomials, we choose functions which have the value 1 at the node point i and 0 at the other ones [10,14]. These polynomials can be found uniquely and so the representation (3.5) is equivalent to the form (3.4). The M node points correspond to an order N of the polynomials $\alpha_i(x,y)$ and in order to avoid any preferred orientation, we take [3]

$$M = \tfrac{1}{2} (N + 1) (N + 2) \tag{3.9}$$

The node points will be chosen in a regular pattern such as to have (N + 1) ones on each side. Examples are given for M = 2, 3, 4 in Figure 3.1.

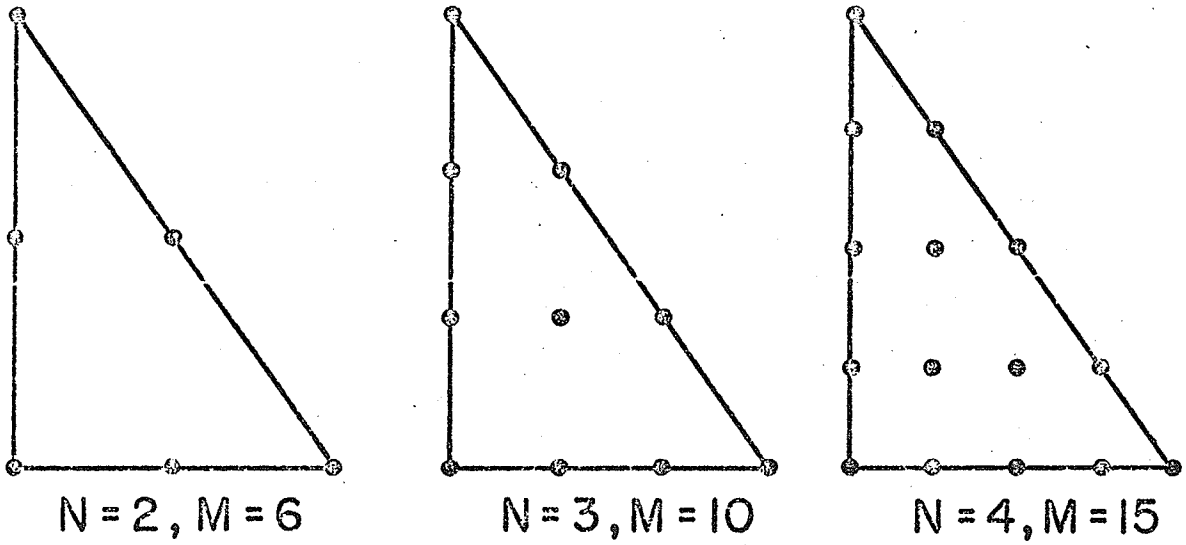

Figure 3.1 Node-points arrangements in triangular elements.

The polynomial is then defined uniquely inside the element and on the boundary.

The gradient of (3.8) is

$$\nabla u = \underline{u}^T \underline{\nabla\alpha} \tag{3.10}$$

and so

$$\nabla u . \nabla u = \underline{u}^T [\underline{\nabla\alpha} (\underline{\nabla\alpha})^T] \underline{u} \tag{3.11}$$

and if we call D the square matrix whose elements are

$$d_{ij} = \iint_R \nabla\alpha_i \nabla\alpha_j dxdy \tag{3.12}$$

we obtain for the first term of (3.1)

$$\iint_C (\nabla u)^2 \, dxdy = \underline{u}^T D \underline{u} \tag{3.13}$$

The second term becomes

$$\int_C [\sigma(s)\, u^2 - 2h(s)u]ds = \underline{u}^T E \underline{u} - 2\underline{u}^T \underline{p} \tag{3.14}$$

where the elements of the matrix E and the vector $\underline{p}$ are

$$e_{ij} = \int_C \sigma(s)\, \alpha_i \alpha_j \, ds \tag{3.15}$$

$$p_i = \int_C h(s) \alpha_i \, ds \tag{3.16}$$

When F is differentiated with respect to the parameters u_i of $\underline{u}$, and the derivatives are made equal to zero, we get a set of simultaneous linear equations (3.6) with

$$S = 2(D+E) \tag{3.17}$$

$$\underline{b} = 2\underline{p} \tag{3.18}$$

The matrices D and E are easily computed for polynomials. The integration is done analytically or numerically (usually by Gaussian quadrature)over each triangular element.

If several elements exist, a summation of all the contributions is performed, the variables corresponding to inter-element boundary nodes being common to adjacent triangles.

The solution of the linear set gives the vector $\underline{u}$ and the field can then easily be computed for each element.

3.2 Use of Asymptotic Functions

In Chapter 1, the asymptotic expansion of the solution near a

singular point was presented. Some of the terms of these series were singular, and therefore a polynomial of standard form could never approximate the solution accurately. In this section, we will outline a modification of the standard variational method where some terms of the asymptotic expansion will be added to the set of polynomial trial functions. This new series of trial functions becomes complete and the solution can, theoretically, be approximated as closely as wanted.

Mathematical Treatment

The solution is assumed to have the form

$$u = \sum_{i=1}^{N} u_i \alpha_i(x,y) + \sum_{j=1}^{Q} a_j t_j(x,y) \qquad (3.19)$$

where the t_i (x, y) are the first terms of the asymptotic expansion near the singularity.

In each element, a certain number Q of extra node points could be chosen, and the values of u at these Q extra points could be taken as the new variational parameters. However, it is easier to consider the coefficients a_i as the extra variational parameters since this does not require new non-polynomial interpolation function α_i (x, y) which are difficult to find.

The main difficulty which arises with these extra functions is related to the continuity requirements along inter-element boundaries. Actually, in this case, it is not sufficient to match the field at a number of points to ensure continuity on the whole boundary. The

elements will in general be non-compatible.

In order to avoid this difficulty, a special treatment of these asymptotic functions will be necessary. First the division in elements will be done in a star-like pattern around the singularity, as shown in Figure 3.2.

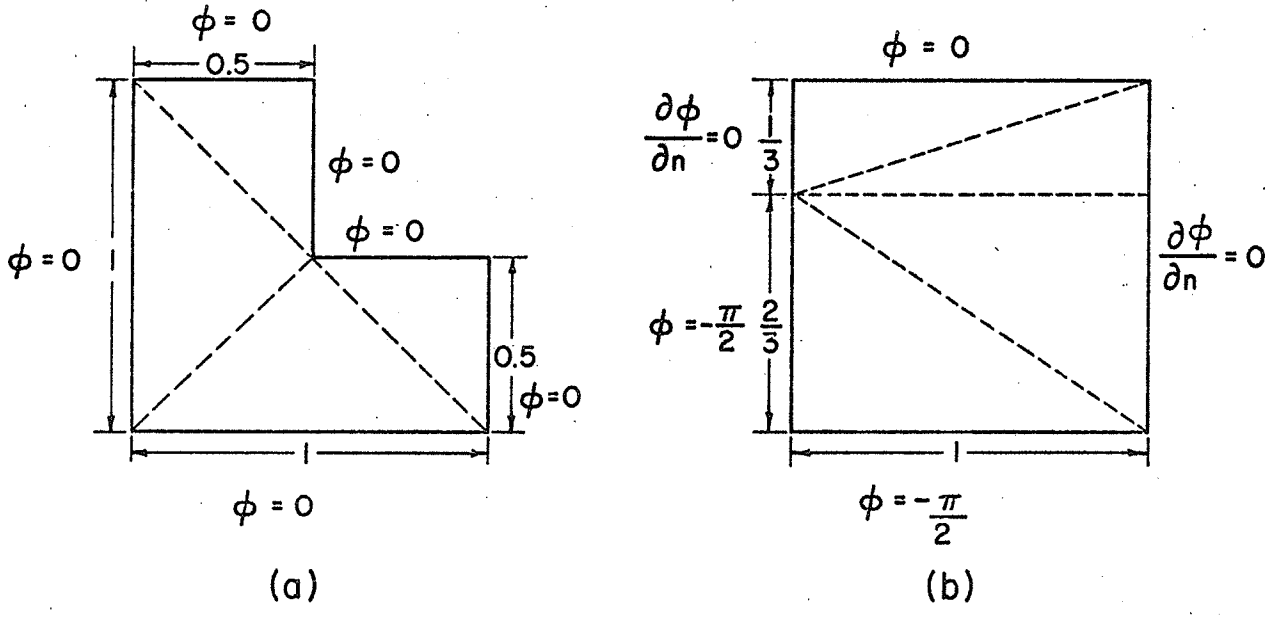

Figure 3.2 Star like division in elements around a singular point, corner for a), change of boundary conditions for b)

Figure 3.2b) is one half of the strip transmission line, solved as an example in the next section.

The asymptotic functions t_i (x, y) are then defined over the whole region instead of over each element separately. In fact, this means that an overlapping of elements exists, first the small triangles on which the polynomial functions are defined separately and then a

larger element, which is here the whole region, where the t_i (x, y) are defined. The coefficients a_i are the same over the whole domain and so, no discontinuity exists at inter-element boundaries. This then gives a complete compatibility and square-integrability between elements.

The mathematical formulation will be modified slightly. We can write

$$u = \underline{u}^T \underline{\alpha} + \underline{a}^T \underline{t} \tag{3.20}$$

where $\underline{u}$ and $\underline{\alpha}$ are defined as in (3.8) and $\underline{a}$ and $\underline{t}$ are vectors with elements a_i and t_i (x, y). By the same procedure as with polynomials, we obtain

$$\iint_R (\nabla u)^2 dxdy = \underline{u}^T D \underline{u} + 2\underline{u}^T G \underline{a} + \underline{a}^T \underline{t} \tag{3.21}$$

and

$$\int_C [\sigma(s)u^2 - 2h(s)u]ds = \underline{u}^T E \underline{u} + 2\underline{u}^T K \underline{a} + \underline{a}^T M \underline{a} - 2\underline{u}^T \underline{p} - 2\underline{a}^T \underline{q} \tag{3.22}$$

where the elements of A, D, E and $\underline{p}$ are defined as previously and the matrixes G, H, K, M and the vector $\underline{q}$ as

$$g_{ij} = \iint_R \nabla \alpha_i \cdot \nabla t_j \, dxdy \tag{3.23}$$

$$h_{ij} = \iint_R \nabla t_i \cdot \nabla t_j dxdy \tag{3.24}$$

$$k_{ij} = \int_C \alpha_i \, t_j \, \sigma(s)ds \tag{3.25}$$

$$m_{ij} = \int_C t_i \, t_j \, \sigma(s)ds \tag{3.26}$$

$$q_i = \int_C t_i h(s)ds \tag{3.27}$$

From (3.21) and (3.22), the matrix S and the vector $\underline{b}$ of (3.6) are easily computed. The computations involved in the evaluation of the integrals (3.23) to (3.27) are done by a numerical procedure, since no analytical method can easily be applied. A Gaussian quadrature formula, for instance, gives very good results in this case [14]. Actually the integrand is always separable in r and θ. The factor in r will be very easily integrated analytically and the Gaussian formula will then be used for the one dimensional integral in θ.

3.3 Numerical Results

The problem chosen, as an example, is the solution of Laplace's equation for a strip transmission line, as shown in figure 3.3.

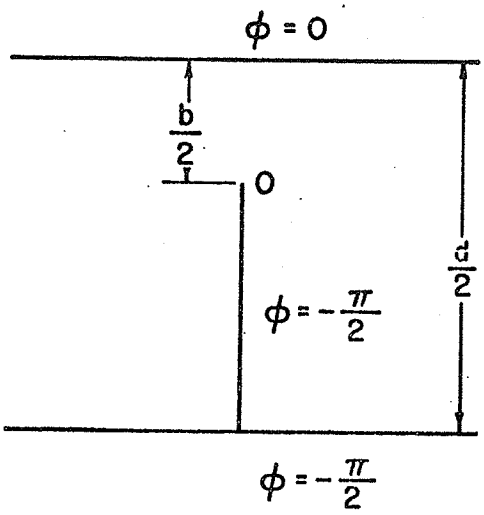

Figure 3.3 Strip transmission line

By reason of symmetry, and supposing the field to be undisturbed at a distance of the diaphragm, the problem reduces to the one shown on

Figure 3.4.

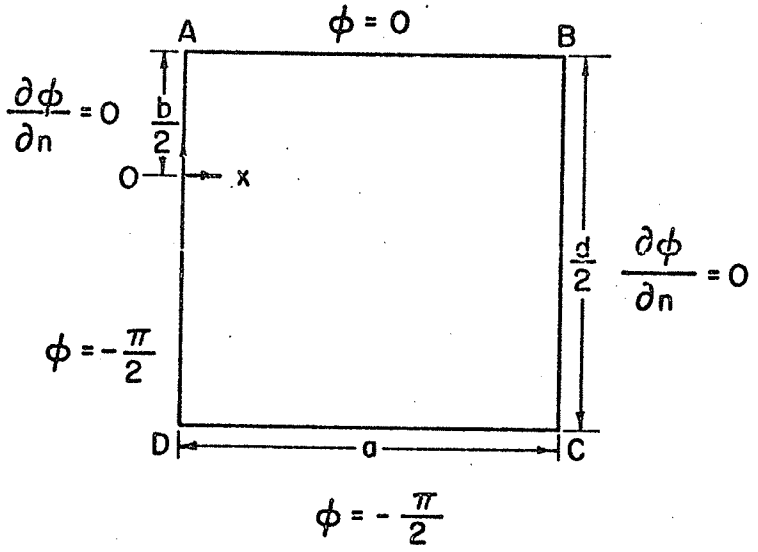

Figure 3.4 Simplified strip transmission line problem

The Neumann boundary condition along BC, although approximate, is valid within an error of 10^{-6} at each point. This was checked using the exact solution of this problem obtained through a conformal transformation of the region.

At the singular point O, where a corner of angle 2π appears, the asymptotic expansion, described in the second chapter (2.9), is

$$\Phi = -\frac{\pi}{2} + \sum_{k=1}^{\infty} a_k \, r^{k/2} \sin(\tfrac{1}{2}k\theta) \qquad (3.28)$$

A division into element will be done as in Figure 3.2(b)to avoid any non-compatibility between elements which would introduce another uncertainty in the results.

The experiments performed are presented in Table 2.1. The errors

were computed using the exact solution found by a conformal transformation

$$u = \mathrm{Re}\ \{\ \sin^{-1}\ \{\frac{\sin\ (y+jx)\pi/d}{\sin(\pi b/2d)}\}\} \qquad (3.29)$$

by taking the absolute value of the difference. They were integrated over the region indicated. The asymptotic terms used were $r^{1/2}\sin(\frac{1}{2}\theta)$ and $r^{3/2}\sin(3/2\theta)$, the case k = 2 in (3.28) is redundant with the polynomial term y.

Table 3.1

Errors for the solution of the strip transmission line problem with parameters $b = \frac{2}{3}$, a = 1, and d = 2

Order of Polynomial	Number of Asymptotic functions used	Number of independent variational parameters	Normalized error in the whole region with respect to the field values	Normalized error in a small neighborhood of the point 0. (20% of the whole region)
2	0	15	0.0893	0.0932
3	0	28	0.0652	0.0784
2	1	16	0.0153	0.0113
2	2	17	0.0115	0.0091
3	1	29	0.0111	0.0087
3	2	30	0.0092	0.0061

Although, by the positive-definiteness of the operator, the addition of

extra trial function could not worsen the solution, the choice of the asymptotic functions give a very important improvement, especially near the singularity. Actually, with the singular terms the errors are smaller near the point 0 than in the whole region. The reverse is found when only polynomials are used. This leads to the idea of a compromise between a good solution near the singularity and over the whole region. It may also be noted that the extra work involved in using one asymptotic function is much smaller than for a third order polynomial instead of a second order. This can easily be seen by the number of free variables of the whole system in both cases. One reduces, for instance, the error from 0.0653 to 0.0115 by using 17 free variables instead of 28. This will reduce the storage requirements (fewer variables) and the computer time (smaller matrices).

Although good results are obtained, the major disadvantage of this type of approximation, is the fact that the extra singular functions do not tend to zero when the distance from the singular point increases. In fact all the asymptotic terms contain r with a positive exponent.* This means that for large regions, the far-field will have an important influence on the coefficients of these terms and will force them to vanish to meet the finite boundary conditions, when the region becomes very large. If on the other hand, the domain of definition of these terms is limited to a small neighborhood of the singular point, a severe problem of incompatibility may occur between elements in this domain and elements outside. This will not enable us to use the method as a general algorithmic procedure for singular

* This difficulty could perhaps be solved by multiplying the singular term by a factor e^{-pr} for instance.

problems.

3.4 Helmholtz Equation and the Eigenvalue Problem

For the Helmholtz equation

$$\nabla^2 u + \lambda u = 0 \tag{3.30}$$

the corresponding functional is

$$F[u] = \iint_R (\nabla u)^2 \, dxdy - \lambda \iint_R u^2 dxdy \tag{3.31}$$

The Rayleigh-Ritz method can also be used here to get the minimum of F [u]. The set of simultaneous linear equations is [10, 11]

$$S\underline{u} + \lambda R\underline{u} = 0 \tag{3.32}$$

which defines an eigenvalue problem. The finite element scheme will be treated by the same way as for Laplace's equation.

Mathematical treatment

A form like (3.20) will be assumed for u. By the same method as for Laplace's equation, the first term of (3.31) leads to equation (3.21). The second term gives

$$\iint_R u^2 dxdy = \underline{u}^T W \underline{u} + \underline{a}^T V \underline{a} + 2\underline{u}^T Y \underline{a} \tag{3.33}$$

where W, V, and Y are matrices whose elements are

$$w_{ij} = \iint_R \alpha_i \alpha_j \, dxdy \tag{3.34}$$

$$v_{ij} = \iint_R t_i t_j \, dxdy \tag{3.35}$$

$$y_{ij} = \iint_R \alpha_i t_j \, dxdy \tag{3.36}$$

A particular difficulty with Helmholtz equation is the fact that the eigen values λ are part of the argument of the asymptotic functions and so the eigenvalue matrix problem becomes non linear. To avoid this difficulty, an expansion of the asymptotic functions, in this case the Bessel functions, is taken around the origin. The development of $J_s(\sqrt{\lambda}r)$ around $\sqrt{\lambda}r=0$ is [4]

$$J_s(\sqrt{\lambda}r) = \frac{(\sqrt{\lambda}r/2)^s}{\Pi s} \{1 - \frac{(\sqrt{\lambda}r/2)^2}{s+1} + \frac{(\sqrt{\lambda}r/2)^4}{2(s+1)(s+2)} - \cdots\} \tag{3.37}$$

The use of the basic expansion functions r^{s+2n} comes to the same approximation as considering a static field, i.e. Laplace's equation, in a small neighborhood of the singular point, instead of the eigenfunction solution of equation (3.30). These functions were also used by Van Bladel [12] in the same case.

We will use the functions

$$t_i = r^{s+2i} (a_s \cos(s\theta) + b_s \sin(s\theta)) \tag{3.38}$$

as the asymptotic trial functions t_i.

3.5 Numerical Results

The example chosen is the L-shaped membrane of Figure 3.2 a) with a homogeneous Dirichlet boundary condition on the whole contour. The division into elements is done in the star-like pattern shown on the figure. Around the corner of angle $\frac{3\pi}{2}$, the development is

$$\sum_{k=0}^{\infty} a_k J_{k2/3}(\sqrt{\lambda} r) \sin(k2/3\theta)$$

as presented in the first chapter.

The trial functions are then

$$r^{k2/3+2n} \sin(k2/3\theta)$$

The value of λ_1 will be computed in a number of different experiments and compared with results obtained by the standard methods (Table 3.2).

As a reference, a very precise solution obtained by Fox, Henrici and Moler [6] can be used. This value is accurate up to the sixth decimal place. They obtained

$$\lambda_1 = 9.639724$$

Their method was based on an analytical development in the whole region and the evaluation of upper and lower bounds for the eigenvalue, in an iterative process.

Table 3.2

First eigenvalue of the L shaped membrane

Polynomial order	Number of asymptotic functions	Number of free variables	First eigenvalue	Percentage error with [12] as reference
2	0	36	11.86987	23.2%
3	0	55	10.65606	10.5%
2	1	37	9.91287	2.83%
2	2	38	9.80292	1.70%
3	1	56	9.69538	0.57%
3	2	57	9.65311	0.14%
"Exact" [12]	-	-	9.639724	—

The usefulness of the extra singular terms is clearly emphasized by this table. This improvement is achieved at a relatively low cost. Going from 10.5% error (polynomial of order 3 without singular term) to 0.14% (use of singular terms) only requires two extra free variables. But the same disadvantages appear in this example as with Laplace's equation. The asymptotic form can only be used in a small neighborhood of the singular point. The functions increase when the distance from the singular point increases. If the region is too large, the far field will cause the coefficients of the singular terms to decrease, in order to meet the finite boundary conditions far from the origin. On the other hand, a division in elements separating the neighborhood of the singularity from the rest of the region will

cause severe incompatibility at the inter-element boundaries which will reduce, in some cases, the rate of convergence considerably, It is to be noted that although at first sight it would be better to use low order polynomials and more asymptotic terms, the disproportionality between the number of extra terms and the polynomial order will make the Dirichlet condition more difficult to constrain, and so limit the accuracy of the approximating solution.

CONCLUSION

The method consisting in using asymptotic functions in the standard variational process was shown to give good results for small regions around a singular point. The difficulties arising from the fact that these functions do not tend to zero when the distance from the origin increases, do not recommend this approach as a general algorithmic procedure for the singular field problems. In spite of this criticism, however, the results are a vast improvement and are obtained with relative ease analytically and at negligible extra computational cost. The extra computational work and storage will be low since only a few extra free variables will be necessary. The linear system, not much larger, will not require much more computer time to be solved. An amelioration could perhaps be achieved by using an attenuation factor multiplying the singular term.

We will see in the next chapter how the integral method presents a more general approach to cater for singularity problem.

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CHAPTER FOUR

GREEN'S FUNCTIONS AND INTEGRAL EQUATIONS

A brief review of the concept of the Green's function is presented in this chapter. It is used to elaborate two different integral formulations of the solution of the partial differential equation. One of these integral operators is shown to be positive definite and self-adjoint in the case of a Dirichlet boundary condition and so, a variational technique combined with a Rayleigh-Ritz type of approximation can be applied. This is used to solve some particular problems with singularities and a comparison with differential techniques is presented in each case.

4.1 Green's Functions

Let us consider the solution of a partial differential equation with operator L

$$L\,\Phi(\underline{x}) = f(\underline{x}) \tag{4.1}$$

By definition [6], the Green's function is the solution of

$$LG(\underline{x};\underline{x}_o) = \delta(\underline{x}-\underline{x}_o) \tag{4.2}$$

where δ is the delta function. G represents the effect of a point source located at $\underline{x}_o$.

In free space, with L being the Laplacian operator, we have

$$G = k\ 1/r \tag{4.3}$$

where r is the distance between the points $\underline{x}_o$ and $\underline{x}$, (three dimensional space)

$$r^2 = (x_o-x)^2+(y_o-y)^2+(z_o-z)^2 \tag{4.4}$$

and k being a constant of proportionality. When only a two-dimensional space is considered, we have, with log meaning the natural logarithm,

$$G = k \log|r| \tag{4.5}$$

For the Helmholtz operator

$$L\Phi = \nabla^2\Phi + \lambda\Phi \tag{4.6}$$

The corresponding free space Green's function is

$$G = e^{j\sqrt{\lambda}r}/r \tag{4.7}$$

for a three-dimensional space and

$$G = j\pi H_o^1(\sqrt{\lambda}r) \tag{4.8}$$

in a two-dimensional space, where H_o^1 is the first Hankel function.

4.2 Integral Formulation

Let us multiply (4.1) by G and (4.2) by Φ and subtract. We get

$$-G\ L\Phi + \Phi LG = -fG + \delta\Phi \tag{4.9}$$

This equation, integrated over the whole space, gives

$$\int\int_R [\Phi(LG) - G(L\Phi)]\,dr = \Phi(\underline{x}_o) - \int\int_R fG\,dr \tag{4.10}$$

which is the basic integral equation for the solution of (4.1).

In the particular case of Laplace's equation, where $L = -\nabla^2$ and $f=0$ we get, using the Green's identity,

$$-\int_S \left(\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n} \right) ds = \Phi(\underline{x}_o) \qquad (4.11)$$

where S is the surface enclosing the region R. This equation (4.11) is also valid for the Helmholtz operator since the terms in λ appearing at the left-hand side will cancel. In this case, equation (4.9) becomes

$$-G(\nabla^2\Phi + \lambda\Phi) + \Phi(\nabla^2 G + \lambda G) = \delta\Phi$$

which clearly reduces to (4.11).

Nothing was implied in the choice of G except that it satisfies equation (4.2). The free space Green's function can thus be used on (4.11), which will make this equation completely general for all kinds of boundary conditions.

Usually [6], the integral formulations used do not consider the first term on the left hand side of (4.11). The integral representation is then

$$\int_S \frac{\partial \Phi}{\partial n} G \, ds = \Phi(\underline{x}_o) \qquad (4.12)$$

where $\frac{\partial \Phi}{\partial n}$ is supposed to be the charge on the boundary S. This implies, however, the condition

$$\int_S \Phi \frac{\partial G}{\partial n} \, ds = 0 \qquad (4.13)$$

This condition is satisfied when G is chosen to be the solution of (4.2) with homogeneous Neumann boundary condition on the surface S of the region [6,1]. A similar formula could be found if G satisfies the homogeneous Dirichlet condition in a Dirichlet problem. However

this simplified formulation like (4.12) can only be used if the Green's function corresponding to the problem is known. This was, for instance, applied, by Silvester for the solution of a microstrip problem [10]. But for general boundary shapes, these functions are very hard to find, and usually this represents as much work as the actual solution of the problem.

An alternative way to construct the solution of (4.1) in terms of an integral equation is to use the well known single-layer logarithmic potential [5] given by

$$\Phi(\underline{x}_o) = \int_C \sigma(s)\, G(s;\underline{x}_o)\, ds \qquad (4.14)$$

where G is the free-space Green's function but $\sigma(s)$ is not yet $\frac{\partial \Phi}{\partial n}\big|_s$. It is a fictious charge around the boundary such as to satisfy the required conditions on S. This charge will take into account the influence of the outer space on the inner region. This can be clearly shown by a simple example: the condenser made by two infinite parallel plates, a section of which is given in figure (4.1).

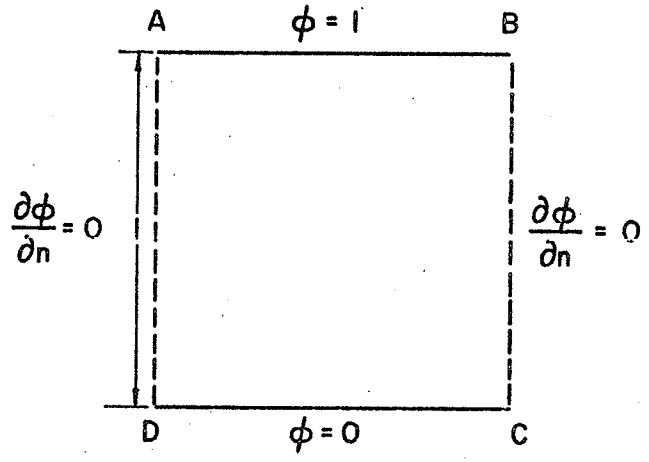

Figure 4.1 Section of a two infinite parallel plates condenser.

The exact solution is

$$\Phi = y \tag{4.15}$$

when formula (4.11) is applied, the charge to consider along the boundary is exactly $\frac{\partial \Phi}{\partial n}$ which is zero on AD and BC. However, if (4.14) has to be applied, a charge σ must be assumed on the whole contour. This fictitious charge along AD and BC will take into account the infinity of the plates at both sides. A solution of (4.14) with $\mathcal{F}= 0$ on AD and BC would solve a different non-physical problem: the two infinite plates with charges only on a small portion AB and DC. The fictitious charge $\sigma(s)$ will be such as to give the right boundary condition . For a Dirichlet conditions (4.14) should be used. For a Neumann, we use

$$\frac{\partial \Phi}{\partial n}\Big|_s = \int_S \sigma(s) \frac{\partial G}{\partial n}\Big|_s ds + \pi \sigma(s)$$

which corresponds to (4.14) where the normal derivative is taken on both sides [12; 13].

The two formulations, although equivalent in theory, will not be so when a method of solution must be applied. In the next section, it will be seen that equation (4.14), by the fact that the charge cannot be related to the field, will not be found to be positive-definite operator and so do not permit the use of a variational technique. On the other hand, the first formula (4.11) in the case of a Dirichlet boundary problem will be positive definite and self-adjoint and so a variational technique may be used effectively.

4.3 Variational Solution of the Integral Equations for a Dirichlet Boundary Problem

The theory will be given here for Laplace's equation. An extension to other equations, like Poisson's or Helmholtz's equation, is easily achieved by using the corresponding Green's function.

The integral equations (4.11) and (4.14) are often solved by some kind of point-matching or projection method [9, 10, 11]. Also some time-consuming numerical iterative approximations are used [7, 12, 5, 13]. The solution here will be obtained through a variational approach by writing the problem in the form of an operator acting on an unknown function to produce a given result.

Equation (4.11) can be written as

$$K\left[\frac{\partial \Phi}{\partial n}\right] = \Phi(s) + \int_S \Phi(s) \frac{\partial G}{\partial n} \, ds \tag{4.16}$$

where G is the free-space Green's function. The operator K is defined as

$$K\left[\frac{\partial \Phi}{\partial n}\right] = \int_S \frac{\partial \Phi}{\partial n} G \, ds \tag{4.17}$$

It is shown in Appendix A that this operator is self-adjoint and positive-definite. It is therefore possible to define a functional

$$F[\sigma] = \langle L\sigma, \sigma \rangle - 2\langle \sigma, f(\Phi) \rangle \tag{4.18}$$

where

$$f(\Phi) = \Phi(s) + \int_S \Phi(s) \frac{\partial G}{\partial n} \, ds \tag{4.19}$$

is known for a Dirichlet problem where $\Phi(s)$ is specified on the boundary. The brackets of (4.18) stand for a surface integral over S. The functional $F[\sigma]$ will then have a minimum value for the solution of the corresponding equation.

It is to be noted that a positive definite and self-adjoint operator was obtained only when equation (4.11) was written as (4.16), the unknown being the charge σ . In a problem with Neumann or mixed boundary conditions the same equation (4.11) cannot be written in terms of a positive-definite operator. Another formulation of the problem should be necessary in these cases, for instance, a variational approach based on the differential equation.

It is also to be noted that the second integral formulation (single layer logarithmic potential) of equation (4.14) was not found to be positive-definite and so will not be used with the variational technique.

Rayleigh-Ritz Method

A Rayleigh-Ritz method will be used to find the σ minimizing the functional $F(\sigma)$ in (4.18). A form of the charge like

$$\sigma(\xi) = \sum_{i=1}^{N} a_i f_i(\xi) \tag{4.20}$$

will be assumed, where the $f_i(\xi)$ form a set of appropriate trial functions. When no singularity occurs, a polynomial type is used for $f_i(\xi)$, like

$$f_i(\xi) = \xi^k \tag{4.21}$$

However, in the neighborhood of a singular point, special functions are added. For instance, for a corner of angle $\alpha > \pi$, a term like

$$f_i(\xi) = \xi^{\pi/\alpha - 1} \tag{4.22}$$

must be considered. In appendix B, a general method is presented to find these special terms.

When using (4.20) in (4.18) and minimizing with respect to the a_i, we get a matrix equation.

$$\underline{A}\underline{a} = \underline{b}$$

with A and $\underline{b}$ having the following elements

$$a_{ij} = \int_{S_2} f_i(s_2)\left[\int_{S_1} f_j(s_1)G(s_1;s_2)ds_1\right]ds_2 \tag{4.23}$$

$$b_i = \int_{S_2} f_i(s_2)\left[\Phi(s_2) + \int_{S_1}\Phi(s_1)\frac{\partial G}{\partial n}\,ds_1\right]ds_2 \tag{4.24}$$

S_1 and S_2 are the same surfaces or lines. The indices refer only to the integration variables.

4.4 Numerical Treatment

The evaluation of the integrals in (4.23) and (4.24) must require special attention since the integrand is singular. Actually, either $G(s_1, s_2)$, singular at $s_1 = s_2$ or $f_i(s)$, singular at the corner as for example, (4.22), will make the use of standard quadrature formulas very difficult.

One way to avoid these problems is to choose two different paths of integration S_1 and S_2 very close together. The function G is never infinite. This approach, however, is only a crude approximation, as the charge is supposed to be the same on both curves. In simple problems, without singularity, the errors introduced are very small. But when the shape of the region is complicated and the charge varies quickly, this method seems to be difficult to apply with great accuracy.

Another way to solve the problem is to perform the integrations on the boundary itself. A numerical technique, like the Gaussian quadrature formulas with suitable weighting factors, give some good results[8]In these cases, the singular part of the integrand becomes the weighting function.

General Gaussian quadrature formulas are given in [8], [2], and [4]. Actually in [4] a program able to handle almost all kinds of weighting functions is presented.

In our case, however, the Green's function has a simple form and an analytical formula can be found. The basic integrals to solve are

$$\int_a^b \xi^i \log|(\xi - \xi_o)| \, d\xi \tag{4.25}$$

with

$$i > -1 \tag{4.26}$$

The complete analytical solution is presented in Appendix C. The main advantage of this kind of approach to the singularity problem is the fact that the approximating charge is defined in a subspace of the original space and contains one variable less than

the field, which makes the approximation easier to perform. Moreover, the function involving the singular term $\sigma_s(s)$

$$\int_S \sigma_s(s)\ G\ ds$$

will only affect a small neighborhood of the region around the singularity since its value decreases when the distance from the corner becomes larger. The procedure used to approximate the singularity is thus only local.

It is sometimes convenient to use the concept of finite elements for the approximation of the charge distribution along the boundary. The boundary is divided into a small number of segments and a form like (4.20) is assumed over each segment. This will cause more flexibility by allowing lower order polynomial trial functions and will enable us to consider arbitrary curved boundaries. The integration should be performed piecewise over each linear segment. On the other hand, as for all finite element schemes, the final matrix will be a band matrix with most of the off-diagonal terms being zero. This will reduce the storage requirements and make the solution of the system much faster.

4.5 Examples and Numerical Results

Some problems involving Laplace's equation with and without singular point will be solved and the results compared with standard differential techniques.

Example I

Although an integral method is completely equivalent to a

differential one for the exact solution, the question can be asked if this is also true for the approximate solution. A simple problem, for which the exact solution is known, is taken as an example. It does not contain any singular point, so the comparison is not biased by the effect of a singularity.

Let us consider the problem of Figure 4.2.

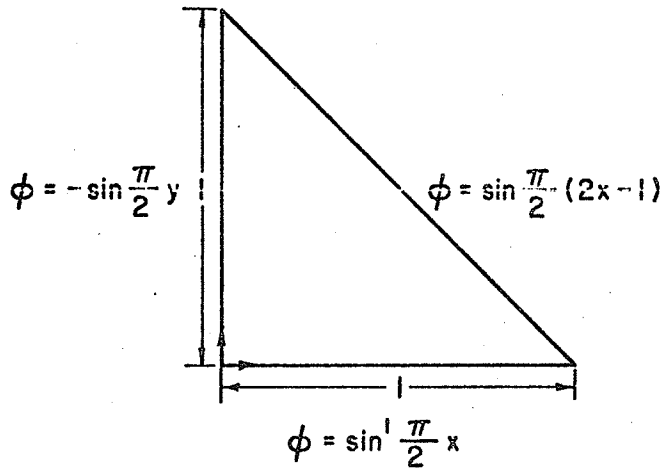

Figure 4.2 Laplace's equation in a region without a singularity.

The exact solution is

$$\Phi = \sin\frac{\pi}{2}(x - y) \tag{4.27}$$

With a differential approach, combined with a variational technique, a certain number of points will be chosen on the boundary and the interior of the region. On the other hand, with the integral method, all the variational parameters will be at the boundary. In both cases, only polynomial trial functions are used, since nothing requires here the presence of asymptotic terms. The errors are computed and normalized using (4.27) and integrated over the region.

The results are presented on Table 4.1

Table 4.1

Normalized errors with differential and integral method for example I

Type of Solution	Polynomial Order	Data Points at the Boundary	Total Number of Data Points	Normalized Error Over the Region
differential	2 (field)	6	6	0.32512
differential	3 (field)	9	10	0.10231
differential	4 (field)	12	15	0.05001
differential	5 (field)	15	21	0.00782
integral	2 (charge)	9	9	0.08325
integral	3 (charge)	12	12	0.04725
integral	4 (charge)	15	15	0.00563

We clearly see that for the same number of data points at the boundary, but less parameters in total, the results of the differential and the integral methods are of competitive accuracy. The methods appear thus to be equivalent also for the approximate solution.

By looking at the total number of data points, the integral method is much more preferable. Actually, this comes from the fact that the Green's function used in the integral formulation contains in itself some information about the solution of the system. This explains the fact that fewer data points will be needed. Only the boundary nodes will be necessary.

Example II

A complete integral approach will be used to solve the following example involving a singular point O in the form of a jump in Dirichlet boundary conditions (Figure 4.3).

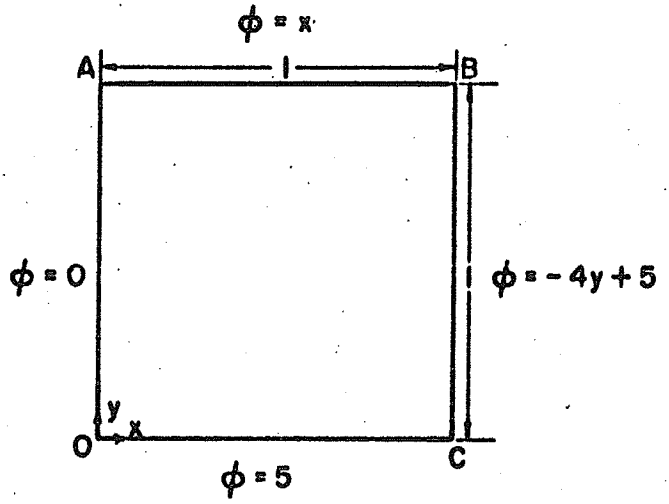

Figure 4.3 Laplace's equation with discontinuity in boundary condition

Around the point O, the normal derivative will be very large and infinite at the corner. The field behaves as

$$5\ \frac{(\pi/2)-\theta}{\pi/2}$$

with

$$\theta = \arctan y/x$$

in the neighborhood of the corner. The normal derivative along OA and OC is $\frac{1}{y}$ and $-\frac{1}{x}$ respectively. These two terms will be added to the polynomial set of trial functions for the charge. Moreover the boundary will be divided here in four segments, the four sides of the

square.

No exact solution is available here but the boundary conditions could be checked.Although this is not a sufficient criterion for the usual differential method since the field can satisfy the boundary conditions without satisfying the differential equation inside, it is enough to evaluate the integral equation accurately. Actually, although the field could not be exact inside, the differential operator is always satisfied.

The normalized errors integrated over the boundary are shown in Table 4.2.

Table 4.2

Normalized errors on the boundary

for example II

Type of Method	Polynomial Order	Free Variables	Number of Elements	Singular Term	Normalized Error
differential	2	13	4	no	0.0923
differential	3	25	4	no	0.0602
integral	2	12	4	no	0.0514
integral	2	13	4	yes	0.0042
integral	3	17	4	yes	0.0009

This shows the real importance of the singular term for the solution of this problem. By using polynomials, either by a differential or an integral method, the error will not tend to zero, no matter how high the order of the polynomial is. The ordinary polynomial is incomplete for the solution to be found. Actually, the asymptotic solution near

the singular point can be written as

$$\Phi = \text{arc tan } (y/x)$$

and we have [3] for $x \neq 0$

$$\text{arc tan } (y/x) = \frac{yx}{x^2+y^2} \{1+ \frac{2}{3}(\frac{y^2}{x^2+y^2})+\frac{8}{5}(\frac{y^2}{x^2+y^2})^2+\ldots\}$$

which will give a polynomial in x and y with negative powers when the normal derivative will be taken. The usual polynomial used (with positive exponents) is thus incomplete and will never be able to give the right solution. However, with the use of a singular term, the errors go quickly to zero as it can be seen from table 3.2. The singular part of the field is taken into account and the remaining one, which is regular, will be approximated quite easily by the remaining polynomial terms of the charge.

Example III

In the following problem of Figure 4.4, a jump in boundary conditions occurs at point 0. The charge is assumed to be polynomial everywhere except on OA and OD where a term $\frac{1}{y}$ is added. The basic error criterion will also be the boundary conditions as for the previous example.

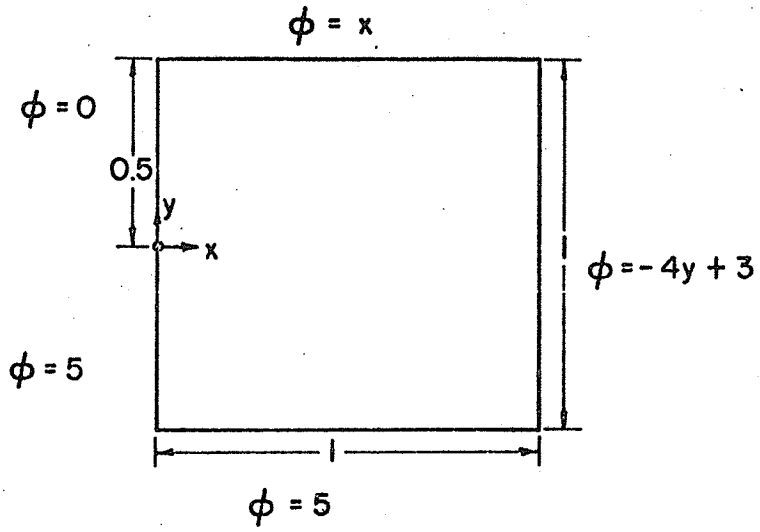

Figure 4.4. Laplace's equation with a jump in Dirichlet boundary condition.

The results are presented in Table 4.3

Table 4.3

Normalized errors on the boundary

for example III

Type of Method	Polynomial Order	Free Variables	Number of Elements	Singular Term	Normalized Error
differential	2	16	4	no	0.1257
differential	3	28	4	no	0.0921
integral	2	15	5	no	0.0853
integral	3	20	5	no	0.0628
integral	2	16	5	yes	0.0094
integral	3	21	5	yes	0.0015

As for the previous example, the importance of the asymptotic term is clearly emphasized. If the integral method is used without this term, the results are similar to the ones obtained with the differential method. The error does not converge to zero, since the polynomial is not complete. On the other hand, the convergence is very fast with the use of the singular term, at a relatively low cost, since only one more free variable reduces the error by an order of magnitude.

CONCLUSION

In this chapter, a new method to handle the integral formulation of the solution of the Dirichlet problem for Laplace's equation, using the variational techniques, was presented. It was shown that for singularity-free problems, the accuracy of the results was equivalent to previously used differential methods. However, in the case of singularities, the function to approximate is one dimension less than the field and so the use of asymptotic functions is much easier to handle. The approximation of the singularity was only local since the singular function concerns only the local singular charge near the corner.

For other types of boundary conditions than the Dirichlet one, no positive-definite operator was found. The convergence of a variational method in these cases is thus not guaranteed, since the functional can have a stationary point at the solution instead of a true minimum.

The finite element concept was used to divide the boundaries in segments, improving so the flexibility of the method with complicated boundary shapes.

Moreover, on the contrary of most of the existing methods, this approach can handle many singularities in the same region, since each of them is approximated only locally.

The major drawback here is the fact that an elementary solution, like the Green's function, must exist in the whole region, in order to apply a pure integral method. With an inhomogeneous region, for instance, or with an anisotropic medium, this general Green's function does not exist and another method must be used. This will be discussed in the next chapter.

In other cases, like for Helmholtz equation, the asymptotic functions for the charge are functions of the eigenvalue λ, which is unknown. The eigenvalue problem becomes non-linear. Here also, a special treatment must be found, which will be discussed later.

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CHAPTER FIVE

TRIAL FUNCTIONS GENERATED BY CORNER SOURCE DISTRIBUTION

The pure integral method is limited to homogeneous, isotropic regions where an elementary solution of the partial differential equation, i.e. the Green's function, can be found over the whole domain. Moreover it appears that its variational solution can only be formulated with a Dirichlet boundary problem, since only in that case was a positive-definite and self-adjoint operator found. For the Helmholtz equation,where a non-linear eigenvalue problem must be solved, the pure integral method cannot be applied very easily. In all these cases, a different approach must be used. The basic idea of the new method presented here is to use the integral formulation of the problem to find new singular trial functions to approximate the field near the singularity. These new functions are added to the polynomial set used in the standard Rayleigh-Ritz variational scheme, as it was done in Chapter 3. Here, however, these functions are only local, ameliorating therefore the problems encountered with the asymptotic functions used previously, i.e. incompatibility and far field effect.

5.1 Construction of the Asymptotic Functions

If the Green's function G can be defined as an elementary solution of Laplace's equation in a certain region, the integral formulation

used in Chapter 4, (4.11) may be used

$$\Phi(x,y) = \int_S \frac{\partial \Phi}{\partial n} G \, ds - \int_S \Phi \frac{\partial G}{\partial n} ds \qquad (5.1)$$

where G is the free space Green's function, $-\log|r|$. $\Phi(x,y)$ is a solution of Laplace's equation, for any source distribution

$$\sigma(s) = \frac{\partial \Phi}{\partial n}\Big|_s$$

or any field at the boundary, $\Phi(s)$.

Without any singularity, $\sigma(s)$ and $\Phi(s)$ would be analytical and $\Phi(x,y)$ would be regular. However in the occurrence of a singularity like a corner, for instance, the charge distribution will have a singular part that can be extracted as shown in Appendix B. The field at the boundary also has a singular term in its asymptotic expansion. The effect on the field of these singular terms can be found by (5.1). We have a new singular function $\Phi_s(x,y)$ representing the asymptotic behaviour, at least the singular part of it, near the corner. This function $\Phi_s(x,y)$ has one great advantage. It represents only a local effect since only the local singular charge or boundary field is considered. In Chapter 3, an asymptotic expansion was presented taking into account all the charges along the corner, but the functions increased with distance from the singularity. On the other hand, here, when the integration in (5.1) is taken over the boundary in the neighborhood of the singularity, the function decreases in a logarithmic fashion away from the corner.

The equation given in (5.1) is completely general. For specific boundary conditions, however, a simplified formulation could be used. With, for instance, a Dirichlet boundary condition, the second term

at the right-hand side of (5.1) is a fixed regular function near the corner and its use in the Rayleigh-Ritz process is not necessary since polynomial trial functions are already present. Only the first term will be used with the corresponding singular charge at the corner. For Neumann boundary conditions near the singular point, only the second term will be used since the first one will have a regular behaviour for a fixed charge distribution along S. Only in case of mixed boundary conditions must the two terms be applied but the condition will relate them together. For instance, for the condition

$$\frac{\partial \Phi}{\partial n}\Big|_S + \tau(s)\Phi(s) = h(s)$$

(5.1) becomes

$$\Phi(x,y) = \int \frac{\partial \Phi}{\partial n} \left[G + \frac{1}{\tau(s)} \frac{\partial G}{\partial n}\right] ds + \int \frac{h(s)}{\tau(s)} \frac{\partial G}{\partial n} ds$$

where the normal derivative of the field is the unknown functions. The expression will then be used as the extra trial function.

5.2 Use of the Singular Functions in a Finite Element Scheme

Since the extra functions are only needed in a small neighborhood of the singularity, a division into elements such as to have the corner enclosed in a small element, is certainly the most convenient finite element scheme. Examples of such a division are given in Figure 5.1 for different types of singularity. The small shaded region enclosing the corner is the domain of definition of the extra singular function. This area could be found to have an optimum size for finite orders of polynomials, although the results present an improvement for all reasonable choice. In the limit, however, for polynomials of infinite order, the choice is arbitrary.

The rest of the region is divided into elements as for a singularity free problem and polynomials are defined in each element. Polynomials are also defined in the shaded region with perhaps a division into triangular elements also there. So an overlapping of the domains of definition can exist. An example is given in Figure 5.2 where the shaded region is divided into four elements. Each polynomial set of trial functions will extend over one triangle only

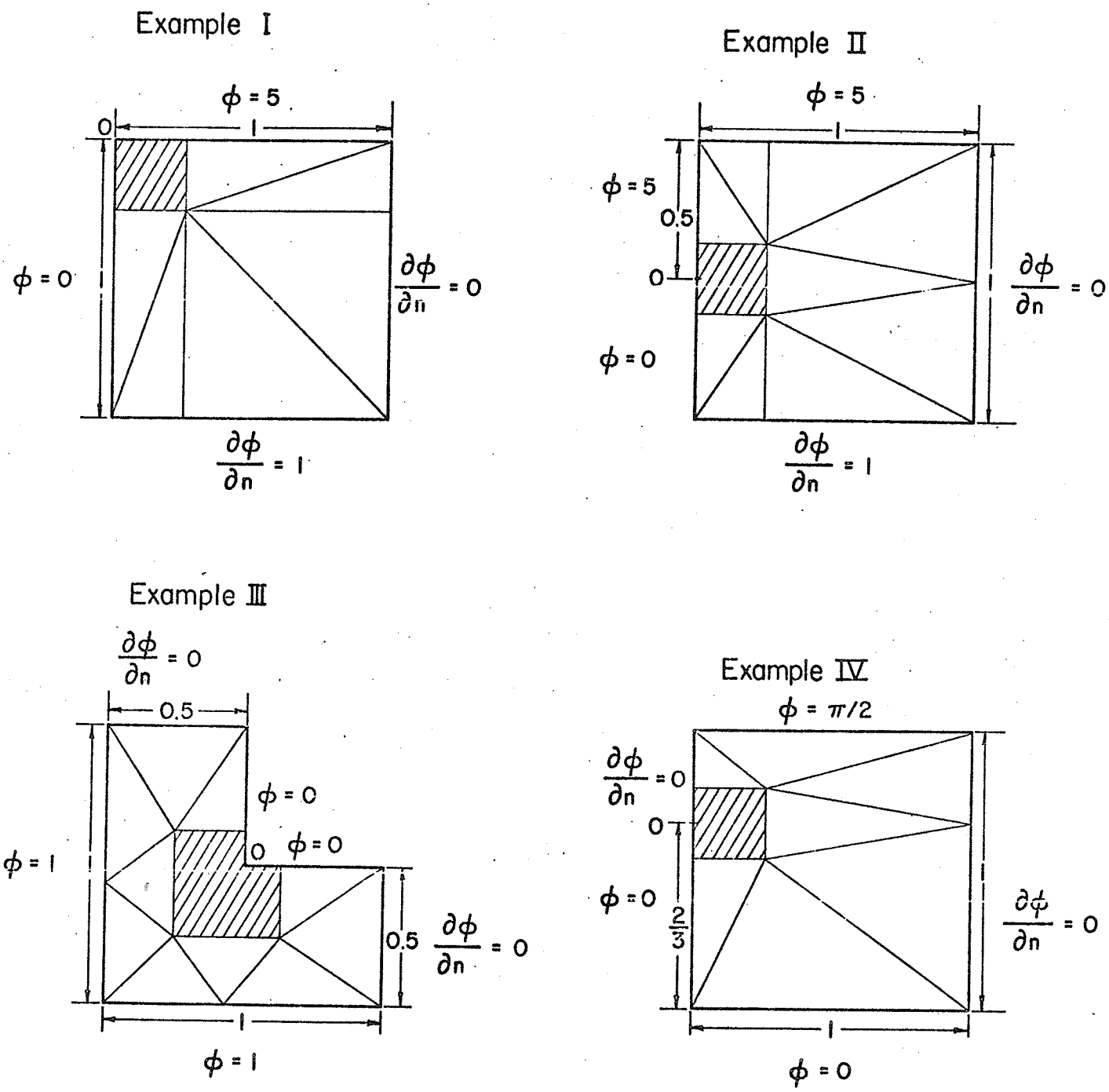

Figure 5.1 Examples of division in elements

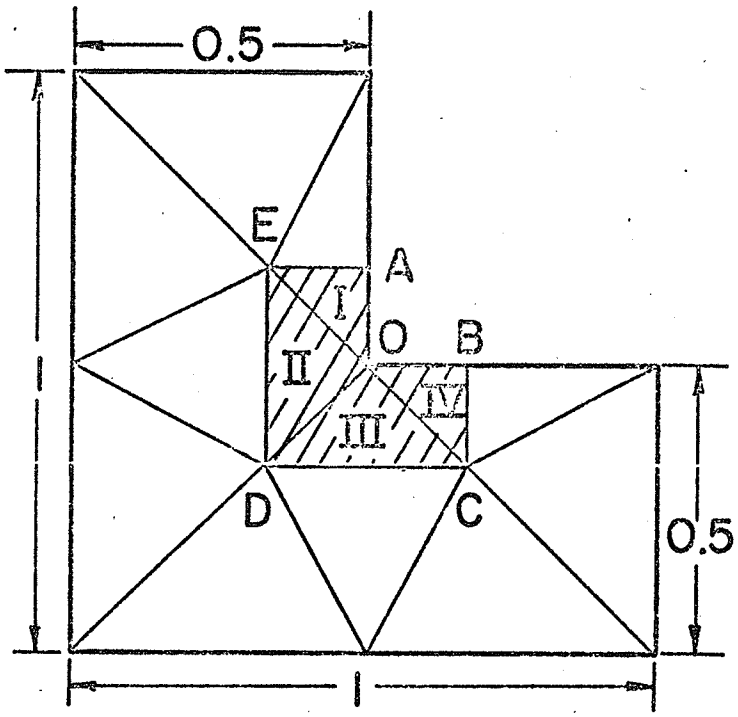

Figure 5.2 L-shaped region divided in elements

but the singular functions

$$\Phi = \int_S \sigma(s)\ G(s;x,y)ds$$

will be defined in the whole shaded area. S is the side of the corner, AOB. If node point potentials are used, M nodes will be chosen in each triangle (for a polynomial of M terms) and Q extra nodes in the shaded area. The interpolation formula will then be written as

$$u = \sum_{i=1}^{M+Q} \alpha_i(x,y)\ u_i + \sum_{i=1}^{M+Q} \beta_i(x,y)\ u_i \qquad (5.2)$$

where the $\alpha_i(x,y)$ are polynomial and the $\beta_i(x,y)$ functions of (5.1).

The presence of a higher order approximation in one region will cause some incompatibility along the interface. However the effect of this incompatibility will be slight since the values of the higher-order function decrease with distance from the corner. If the boundary of the shaded region is not chosen too close to the singular point,

the value of the extra function is very small along that boundary. Moreover, in the limit, the interface requirement will be satisfied, i.e. continuity of the field and its normal derivative. In fact, the behaviour of the singular function, at this interface, is regular since the only singularity occurs at the corner. So when the order of the polynomial increases inside and outside of the shaded region, the incompatibility becomes less and less significant. This can be intuitively* shown as follows: Since the asymptotic function is regular at the interface, it can be represented by a polynomial series with perhaps an infinite number of terms. It can thus be approximated by a polynomial of order P to any given accuracy. If a polynomial of order N is used in each element, the compatibility will be complete when N is equal to P. Letting P and consequently N go to infinity the asymptotic function is perfectly represented and the interface conditions are completely satisfied. However, for finite orders of polynomial, the slight incompatibility will not cause much difficulty. Actually, incompatible elements have already been used to solve partial differential equations [5]. They are used with confidence if we can be sure that the interface conditions are satisfied in some sort of limiting case. The method is valid for homogeneous and isotopic media with Laplace's equation but more emphasis will be given here for the cases where other methods of approximation, like the pure integral method, cannot be used. First the Helmholtz eigenvalue equation will be solved in a particular case, then the method of solution of Laplace's equation in inhomogeneous and anisotropic media will be described.

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* This is not an actual proof, although the proof could be easily constructed.

5.3 Helmholtz Equation

The difficulty of solving the Helmholtz problem is the fact that when the Green's function

$$\pi\, jH_o^1\,(\sqrt{\lambda}\, r)$$

is used in equation (5.1), the eigenvalue λ is part of the argument of the Hankel function and the eigenproblem becomes non-linear in λ. However, near the singularity, we have (Reference,Chapter 2, [12]).

$$\pi\, jH_o^1(\sqrt{\lambda}r) = \log|r| \quad \text{for} \quad \sqrt{\lambda}r \to 0 \qquad (5.3$$

and so, in a small neighborhood about the origin, we can use the static approximation of the Green's function, which is the elementary solution for Laplace's equation. The method is applied here to the L-shaped membrane already solved in Chapter 3 using another asymptotic expansion. The region is divided into elements like in Figure 5.2 and in the shaded region, a solution of the form given in (5.1) is used. The charge distribution σ along OA and OB can be found using the results presented in Appendix B. The re-entrant angle is here $\alpha = 3/2\pi$ and the singular term of the charge is $r^{-1/3}$. The Green's function used in (5.1) is

$$G = -\log|r| \qquad (5.4)$$

The final eigenvalue λ is computed and compared with the results of Moler [7] Fix [6], Beaubien and Wexler [2] and Muilwyk [3] for the same case. They all used a finite difference scheme with [6] and [7] employing asymptotic interpolation functions. The value obtained by Fox, Henrici and Moler [8] is taken as a reference, as in Chapter 3.

The results are summarized in Table 5.1. It can be seen that the results are similar to or better than the ones obtained by the very time-consuming finite difference approaches. They can also be compared with the method described in Chapter 3. The improvement is very significant.

Table 5.1

Eigenvalue λ_1 of the L-shaped membrane

Type of Solution	Polynomial Order	Free Variables	Use of Asymptotic Functions	Eigenvalue λ_1	Percentage Error with [8] as Reference
finite element	2	39	no	10.52312	9.15%
finite element	3	80	no	9.95136	3.23%
finite element	1	14	one function	10.31535	7.02%
finite element	2	40	one function	9.65012	0.108%
finite element	2	41	two functions	9.64612	0.065%
finite element	2	42	three functions	9.64455	0.051%
finite element	3	80	one function	9.64281	0.032%
finite element	3	81	two functions	9.64182	0.021%
finite element (Table 3.2,p.37)	3	56	one function	9.69538	0.570%
finite element (Table 3.2, p.37)	3	57	two functions	9.65311	0.140%
finite difference [6]	-	-	yes	9.64238	0.027%
finite difference [7]	-	-	yes	9.64393	0.043%
finite difference [2]	-	-	no	9.65345	0.142%
finite difference [3]	-	-	no	9.64613	0.065%
"Exact" [8]	-	-	-	9.639724	-

Moreover, the effect of the singular term is very important. Although this term is only a static approximation near the singularity, it takes into account the singular contribution of the field at this point. The remaining part of the field is regular and can be approximated by the usual polynomial series. Some experiments were also made with more asymptotic terms, which are singular with respect to higher derivatives. The result shows a large improvement when the number of free variables is considered. However, the computation involved in the integration of these asymptotic functions is much longer than with the polynomial terms and a compromise is to be found between time (number of asymptotic functions) and storage (number of free variables). When only a polynomial of order 1 is used, or when too many singular terms are present in comparison with the number of polynomial terms, the results are a little bit less accurate. The reason is that the interface condition is more difficult to satisfy in these cases. It is interesting to note that the result presented by Moler [7] was claimed to be very accurate (up to six figures) since the iterative process converged very well to this value of λ . He, however, considered only the convergence of the discrete system.

5.4 Inhomogeneous and Anisotropic Regions

For the solution of Laplace's equation in a region filled with an inhomogeneous or anisotropic medium, the complete integral method is not applicable since the Green's function cannot be found for the whole region. However, in some cases, a good approximation can be obtained by considering (5.1) valid in a small neighborhood of the singular point.

In a region, for instance, with a slowly varying dielectric constant, the value at the corner could be used in the region immediately surrounding the irregularity. In other cases, like the one of Figure 5.3, a different equation (5.1) could be used in each region with the corresponding Green's function. The singular parts of the field and the charge for this case have been derived in Chapter 2.

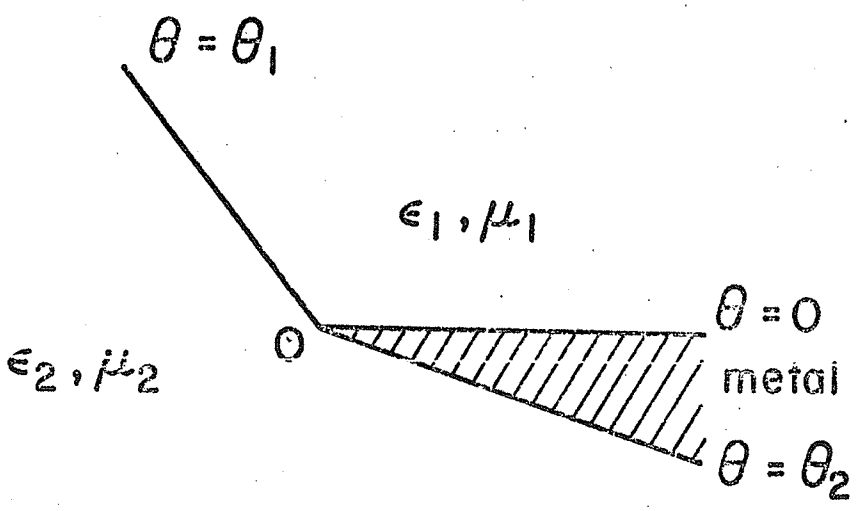

Figure 5.3 Different dielectric media near a corner.

For more complicated problems, such as an anisotropic region, special Green's functions are derived from the properties of the region themselves since they represent the potential of a point source. For non-constant anisotropic properties, the Green's function must be used as an approximation near the corner, as for inhomogeneous regions.

5.5 Other Kinds of Singularity Problems

The method presented here can also be applied to Poisson's equation. For a source located at $\underline{x}_o$, a term like $\log |x-x_o|$ will be used as a trial function in addition to the usual polynomial ones. It represents the effect at point $\underline{x}$ of the source, and is actually the Green's function. The same remarks hold here as for equation (5.1) about the domain of validity of the Green's function.

Only corner singularities were investigated in this work. However the method can also be extended to curved re-entrant parts. These non-convex regions, although not mathematically singular, often cause slow convergence of the approximation process. In some cases, the use of an asymptotic function could be useful. For instance for the temperature distribution in a finned nuclear fuel sheath, investigated by Richards and Wexler [4] (Figure 5.4),

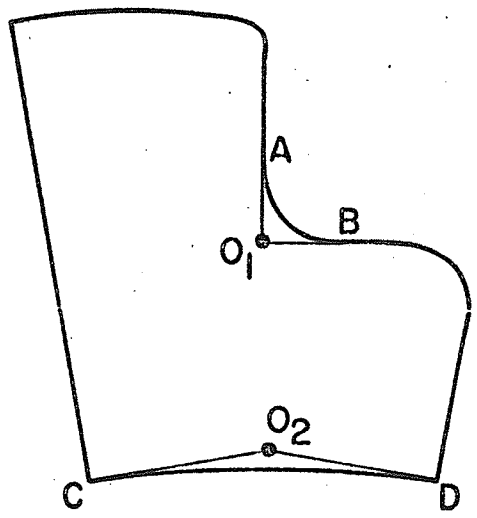

Figure 5.4 Finned nuclear fuel sheath

difficulties to converge were encountered around the re-entrant part AB. Asymptotic corner functions could be used about 0_1. Although no trouble was detected near CD, provision could be made around 0_2 as well, in an automated program. Actually, these functions would be the more efficient if the re-entrant part is very sharp. In cases like for CD, their contribution would be small, but they will never worsen the solution since the variational process used is positive-definite and that each trial function will either make the solution better or leave it unchanged.

These asymptotic functions do not here represent the singular part of the field since no mathematical singularity occurs but they introduce a biased combination of polynomial terms. Some of these terms would otherwise not have been used until the order N became large. Here, they are introduced at an early stage, thus possibly permitting N to remain low.

CONCLUSION

A general method, catering for singularity problems, was presented in this chapter. By using the asymptotic expansion of the charge distribution along the boundary near the singular point, new extra trial functions are constructed to approximate the singular behaviour of the field near the corner. These functions have only a local effect, they will thus not influence the far field and the method can be considered as a general algorithmic approach to the singularity problem. It can handle all kinds of corners, boundary conditions and different

media. Also Poisson's equation can be solved for singular sources. The technique also permits some improvements in cases where no real singularities appear but where the presence of, for instance, curved re-entrant parts, makes the approximation scheme converge more slowly.

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CHAPTER SIX

C O N C L U S I O N

The amount of previous work involved in partial differential equation problems for regions with singularities points out the real importance of a simple algorithmic solution for these applications. Actually, all the existing standard methods of approximation that do not explicitly cater for the problem, suffer major drawbacks as a consequence of the singularity.

Refinements introduced in recent years were essentially based on reduced mesh sizes, increased number of elements or order of polynomial approximation near the singular point. Although giving some improvements, these methods increased very much the computer effort and, some times, the numerical instability of the system. Other methods were based on the use of asymptotic functions as interpolation formulas. The improvement obtained was very significant but they suffered from all the lack of flexibility of the finite difference approach.

The first attempt presented here to solve these kinds of problems was the use of the analytical expansion of the solution near the singular point as a source of new trial functions for the variational technique. This method gave considerably improved results in the neighborhood of the singularity, but a main disadvantage was the fact that these asymptotic functions increased with the distance from the corner. So, for large regions, many difficulties arose either by the presence of severe inter-element incompatibilities or by the important effect of the far field

on the whole solution. This prevented to consider this attempt as a completely algorithmic approximation technique for these problems. The method would always need human intervention to decide the number of asymptotic functions to use, and to keep the incompatibilities within acceptable limits.

The second technique presented was based on a completely different approach: the integral equation method. Using an elementary solution of the Laplace's equation, known as the Green's function, the problem was formulated as an integral over the boundary. All the unknowns of the problem were then concentrated along the periphery of the region, and the number of dimensions of the system to solve was reduced by one. For the exact solution, this method is completely equivalent to the usual differential one. However, it was found by experiment that this is also true for the approximate solution. For the same number of data points on the boundary, but less in total, the integral approach presented the same accuracy as the differential method. Handling fewer independent variables made the use of asymptotic expansions of the source much easier and more efficient than the previous method. In this approach, instead of trying to approximate the differential equation and the boundary conditions at the same time, only the latter conditions had to be satisfied since, by definition of the Green's function, the trial functions are solutions of the differential equation. For Laplace's equation under Dirichlet boundary conditions, the integral operator obtained by this approach was proved to be self-adjoint and positive-definite. A variational technique was thus used with confidence. For other boundary conditions,

however, and for inhomogeneous or anisotropic regions where an elementary solution was difficult to find, this technique was not proved to be valid.

Although not generally applicable, as indicated above, the integral formulation leads to a new way to find asymptotic functions for the field near a singularity. The final method presented here consisted in using the charge distribution along the boundary near the singularity, to construct an asymptotic expansion of the field near this point. The singular part of this expansion was then added to the standard polynomial set of trial functions employed in the usual variational scheme. This method is quite general since it can handle inhomogeneous regions and the three kinds of boundary conditions, which was not the case with the pure integral method. Also the Helmholtz eigenvalue problem where the integral method resulted in a nonlinear eigenvalue problem was solved by approximating the singular solution near the corner as a static field. The method is also adapted to point source singularities in Poisson's equation. It may also improve the usual polynomial approximation in the case of re-entrant curved boundary where convergence in some times seem to be slower. This appears to be possible by expanding the field around fictious corners made by the tangents to the curved boundary. These extra functions would introduce a biased combination of the polynomial terms, allowing thus a lower order.

The main advantage of the trial functions used was their local aspect. Only the local singular charge was considered, which was not the case for

the asymptotic expansion used in the first method, where all the charges along the edges were taken into account. This permits the handling of large regions without the important effect of the far field and the slow convergence due to severe inter-element incompatibility. Regions with several singularities could also be considered.

The technique was accommodated into a finite-element scheme where the extra functions were only defined in the elements located around the singular point. The interface condition was satisfied in the limit when the order of the polynomial went to infinity. For finite orders, the incompatibility was slight since the asymptotic functions were only local. The finite-element procedure has here the same advantages of time and storage savings as with usual convex regions where only polynomials are used. Actually, by using the asymptotic functions, the singular part is taken into account and the remaining problem is regular. Block sparse matrices are constructed. This will give a more efficient computer storage and make the solution of the system much faster. It is to be noted that the computations involved in the extra asymptotic terms is much higher than for polynomials, since these terms contain integrals that must be solved numerically. However, they appear only in a few rows and columns of the matrices and their use saves so many free variables that much time is regained when the matrix equation must be solved.

The method could be generalized to three-dimensional problems for which an asymptotic expansion is known. The number of these solutions is very limited, however, and more work in this field would be very valuable.

Due to its algorithmic generality, this method could easily be incorporated into a general automated finite element program dealing with a broad variety of regions, equations and boundary conditions.

APPENDIX A

POSITIVE-DEFINITENESS AND SELF-ADJOINTNESS OF THE INTEGRAL OPERATOR

The integral operator

$$L\sigma = \int_S \sigma(s)\, G(s;\underline{x})ds$$

is proved to be self-adjoint and positive-definite

1. Self-adjoint

We have to prove that

$$\langle L\sigma, \tau\rangle = \langle L\tau, \sigma\rangle$$

where the brackets stand for a line integral along S. This reduces to

$$\int_{S_1}\int_{S_2} \sigma(s_1)\tau(s_2)\, G(s_1;s_2)ds_2ds_1$$

$$= \int_{S_1}\int_{S_2} \tau(s_1)\, \sigma(s_2)\, G(s_2;s_1)ds_2ds_1$$

which is true since S_1 and S_2 represent the same line and $G(s_2;s_1)$ is symmetric.

2. Positive-definite

We have to show that

$$\langle L\sigma, \sigma\rangle > 0$$

for all $\sigma \neq 0$. We will use the relationship between σ and Φ saying that for each $\sigma(s)$, a function Φ can be found such that

$$\nabla^2\Phi = 0 \quad \text{inside } S,$$

$$\sigma(s) = \frac{\partial\Phi}{\partial n}\Big|_s$$

and

$$\int_S \sigma(s)\, G(s;\underline{x})ds = \Phi(\underline{x}) + \int_S \Phi(s)\, \frac{\partial G}{\partial n}\, ds$$

we so get

$$<L\sigma,\sigma> = \int_{S_1}\int_{S_2} \sigma(s_1)\sigma(s_2)\, G(s_1;s_2)ds_1ds_2$$

$$= \int_{S_1} \sigma(s_1)\Phi(s_1)ds_1 + \int_{S_1}\int_{S_2} \sigma(s_1)\Phi(s_2)\frac{\partial G}{\partial n}ds_1ds_2$$

The first term is always positive since

$$\int_S \frac{\partial \Phi}{\partial n}\, \Phi ds = \int_V (\nabla^2\Phi)\Phi dv + \int_V (\nabla\Phi)^2 dv \quad > 0$$

The second term can be transformed by the Cauchy integral formula of a complex function, analytic in the region inside S and on S,

$$\int_S \frac{f(z)}{(z-z_o)}\, dz = f(z_o)\, 2\pi j$$

here, we have for Laplace's equation

$$G = -\log |r|$$

where r is the distance between z and z_o and so,

$$f(z) = \frac{\partial G}{\partial n}\, \Phi(z)\, (z-z_o)$$

which is analytic everywhere, including the point z_o. Actually at $z = z_o$, we have

$$\lim f(z) = \Phi(z_o)\, 2\pi$$

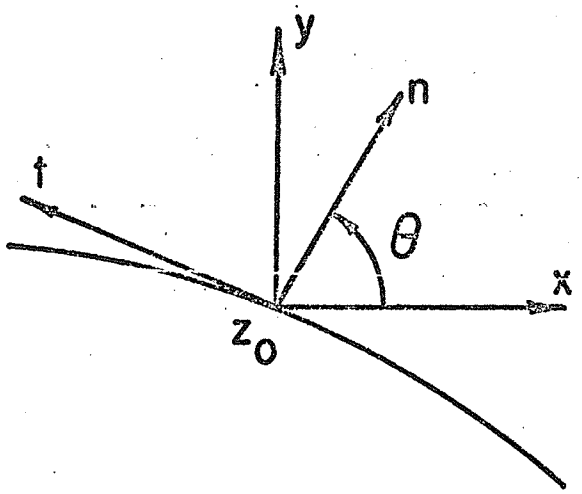
This can be proved by the following derivation.

If n is the normal derivative along S

$$n = x \cos\theta - j\, y \sin\theta$$

$$z - z_o = -r \sin\theta + j\, r \cos\theta$$

when z approaches z_o along t.



$$\frac{\partial G}{\partial n} = -\cos\theta \ \frac{\partial}{\partial x} \log r \ + j \sin\theta \ \frac{\partial}{\partial y} \log r$$

$$= \frac{1}{r^2} \left(-\cos\theta \ (x-x_o) + j \sin\theta \ (y-y_o) \right)$$

We so obtain

$$\lim_{z \to z_o} (z-z_o) \ \frac{\partial G}{\partial n} = -j \cos\theta \ + j \sin\theta \ = -j$$

and the second term is equivalent to the first one. This proof is only valid when the integral equation can be written as

$$\int_S G \ \frac{\partial \Phi}{\partial n} \, ds = \Phi(x_o) + \int_S \Phi \frac{\partial G}{\partial n} \, ds$$

for a Dirichlet problem. In case of Neumann and Dirichlet boundary condition, $\frac{\partial \Phi}{\partial n}$ and Φ are unknown on part of the boundary, and so the same proof does not apply. However, the integral operator is the inverse of a positive-definite differential operator and should be expected to have the same property. No proof could be found here.

APPENDIX B

ASYMPTOTIC FUNCTIONS FOR THE CHARGE DISTRIBUTION NEAR CORNERS.

When the boundary does not present any singularities, the charge can be approximated by a regular polynomial

$$\sigma = \sum_{i=1}^{N} a_i \xi^i$$

However, if a corner singularity occurs, or a discontinuity in boundary values, special approximation functions must be used.

Let us consider a corner of angle α. The general expression for the field near this corner can be found easily as it was presented in Chapter 2. The normal derivative along the edges of the corner is

$$\frac{\partial \Phi}{\partial n} = \frac{1}{r} \frac{\partial \Phi}{\partial \theta}$$

in cylindrical coordinates. For instance, in the case of homogeneous Neumann or Dirichlet boundary conditions with Laplace's equation, the asymptotic expansion of the field is

$$\Phi = \sum_{i=0}^{\infty} \left(a_i \, r^{i\pi/\alpha} \cos(i\theta\pi/\alpha) + b_i \, r^{i\pi/\alpha} \sin(i\theta\pi/\alpha) \right)$$

The normal derivative is then

$$\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \frac{1}{r} \sum_{i=1}^{\infty} \left(-a_i \, r^{i\pi/\alpha} i\frac{\pi}{\alpha} \sin(i\frac{\pi}{\alpha}\theta) + b_i \, r^{i\pi/\alpha} i\frac{\pi}{\alpha} \cos(i\frac{\pi}{\alpha}\theta) \right)$$

On $\theta = 0$

$$\frac{\partial \Phi}{\partial n} = \frac{\pi}{\alpha} \sum_{i=1}^{\infty} b_i \, r^{(\pi/\alpha i)-1} \, i$$

and on $\theta = \alpha$

$$\frac{\partial \Phi}{\partial n} = \frac{\pi}{\alpha} \sum_{i=1}^{\infty} b_i r^{(\pi/\alpha i)-1} \, i \, (-1)^i$$

The two expressions have a singular term if $\alpha > \pi$:

$$\frac{\pi}{\alpha} \, b_1 r^{(\pi/\alpha)-1}$$

which is infinite at r = 0.

The total charge, however, is finite on each side of the corner, since

$$\int_0^a r^{\pi/\alpha-1}\, dr = \frac{\alpha}{\pi}\, a^{\pi/\alpha}$$

where a is the length of the side. The trial function to use in this case is $\xi^{(\pi/\alpha)-1}$.

For other types of boundary conditions, an expansion of the field can be found and the charge can be derived by a similar method. It is to be noted that a conformal transformation using Schwarz's formula gives the same results. This approach, however, requires more complicated algebraic manipulations.

It is to be noted that in case of several singularities this method treats each singularity only locally. This is valid since the interaction between singularities is a regular function which can be approximated by the polynomial terms. An example of this is given by Acton (see Bibliography) who uses a form like$(c^2-x^2)^{-0.5}$ for the finite strip of length 2c with the origin in the middle. This is exactly the exact solution for that problem. On the other hand, the method presented here would use $(c-x)^{-\frac{1}{2}}$ near the point x=c, and $(c+x)^{-\frac{1}{2}}$ near x=-c. These functions are the singular parts of the exact one near each end point.

APPENDIX C

ANALYTICAL SOLUTION OF THE INTEGRAL EQUATION

The integrals involved in the computation of the matrix A of (4.23) are

$$\int_{S_1}\int_{S_2}\phi_i(s_1)\phi_j(s_2)G(s_1;s_2)ds_1ds_2$$

In the case of Laplace's equation, we have

$$G=-\log|r|$$

r being the distance between point s_1 and s_2 on the curve S. The curve will be approximated by straight segments C_m and only the integrals of type

$$\int_{C_{1m}}\int_{C_{2m}}\phi_i(s_1)\phi_j(s_2)G(s_1;s_2)ds_1ds_2$$

will be singular. On the other hand, we have

$$\phi_i = \xi^l \quad ; \quad \phi_j = \xi^k$$

where l and k can sometimes be non-integer. We end with

$$\int_a^b \int_a^b \xi_1^l \ \xi_2^k \log|\xi_1-\xi_2| \ d\xi_1 d\xi_2$$

where a and b are the limits of integration for the segment C_m.

We can write

$$\int_a^b \xi_2^k \log|\xi_2-\xi_1|d\xi_2 = \int_a^{\xi_1}\xi_2^k \log(\xi_1-\xi_2)d\xi_2 + \int_{\xi_2}^b \xi_2^k \log(\xi_2-\xi_1)d\xi_2$$

Each term is then put into the form

$$\gamma \int_0^1 (\alpha x+\beta)^k \log(h(1-x))dx$$

by a change of variables.

$$(1-x)h = \xi_1 - \xi_2$$

For the first term, we have

$$h = \xi_1 - a, \qquad \gamma = h, \qquad \alpha = h, \qquad \beta = \alpha$$

and for the second, $h = \xi_1 - b$, $\gamma = h = -\alpha$, $\beta = b$.

We get for each term

$$\gamma \int_0^1 (\alpha x+\beta)^k \log h \, dx + \gamma \int_0^1 (\alpha x+\beta)^k \log(1-x)dx$$

And we can expand $(\alpha x + \beta)$ in terms of powers of x. This will be a closed form if k is integer, and a convergent infinite series for k real, of which a few terms will be sufficient to get a good accuracy.

The final answer will have the form

$$M_1 \log(b-\xi_1) + M_2 \log(\xi_1 - a) + \sum_{r=0}^{k} \sum_{s=0}^{r} N_r \, f_r(\xi_1)$$

with

$$f_r(\xi_1) = (\xi_1 - b)^{r+1} b^{k-r} - (\xi_1 - a)^{r+1} a^{k-r}$$

and M_1 M_2, N_2 are constant where value can easily be computed.

To get this result, we used the standard integral [8]

$$\int_0^1 x^p \log(1-x)dx = -\frac{1}{p+1} \sum_{s=0}^{p} \frac{1}{s+1}$$

The outer integral will then be

$$\int_a^b \int_{\xi_1}^1 [M_1 \log(b-\xi_1) + M_2 \log(\xi_1 - a) + \sum_{r=0}^{k} \sum_{s=0}^{r} N_r \, f_r(\xi_1)] d\xi_1$$

which can be solved by the same method and using also the formula given in [8].

These computations give then a general formula for the elements of A which can be easily automated in the computer program. The evaluation of each element a_{ij} will be very fast.

For the contribution of different segments C_m, a Gaussian quadrature formula with weight 1 is used, since no singularity appears.

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