DIFFRACTION BY A SLIT IN A THICK CONDUCTING SCREEN

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Doctor of Philosophy

by

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TO TERESA

AND

TO MY PARENTS

WITH DEEP GRATITUDE AND AFFECTION

ABSTRACT

The diffraction of a plane electromagnetic wave by a slit in a thick conducting screen is investigated using the Wiener-Hopf and the generalized scattering matrix techniques. For purposes of the analysis, the diffraction by two identical semi-infinite parallel plate waveguides forming a tandem slit configuration is treated first in order to determine the interaction between the open ends of the waveguides. This interaction is then utilized in solving for the thick slit geometry which is obtained by filling the parallel plate regions with a dielectric whose permittivity is allowed to approach infinity. For an E-polarized incident plane wave, the solution is expressed in ray-optical terms and the diffraction by each thick edge is viewed as that due to a thin edge centred at the middle of the thick edge and modified by an appropriate diffraction coefficient. The thick edge-edge interaction term, on the other hand, is also modified such that each thick edge is viewed by the other as the combination of an inhomogeneous line source as well as a line dipole. It is shown that, in general, the effective width of a slit becomes smaller with increasing screen thickness. The far field scattering properties of a thick slit thus obtained are employed to determine the scattering coefficients of symmetric and asymmetric waveguide diaphragms of finite thickness. This application, though best suited to the high frequency range far from modal cutoffs, is shown to lead to accurate results even in the dominant mode range.

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CHAPTER I

INTRODUCTION

Numerous microwave structures dealing with diffraction and waveguide propagation employ slits or slots of various sizes for coupling purposes. Although these structures are simple to construct in principle, they normally present serious mechanical difficulties unless the screen thickness is finite. Since the thickness also affects the electrical performance of these structures, it should be incorporated as a design parameter. As most of the available analysis applies to apertures in infinitesimally thin screens, any attempt to study this parameter should therefore be of interest to those engaged in design or application of such structures.

This thesis is concerned with the diffraction of a plane electromagnetic wave by a slit in a thick conducting screen. It also deals with the application of these results to finding the effect of finite thickness on reflection and transmission properties of asymmetric and symmetric waveguide diaphragms. Although the study is specifically confined to slits in free space or waveguides, some of the results may be also extended to deal with thick slots in free space or waveguides. The extension to the design of multicavity waveguide filters using thick diaphragms is another interesting application which is omitted here but reported elsewhere [1].

The diffraction of a plane electromagnetic wave by a slit (or strip) in a thin conducting screen has received wide attention due to its importance in microwave and optical instrumentation [2-6]. Considerable literature is available on the subject and is briefly reviewed here as background to the thick slit problem. It is well known that an exact solution for the thin slit can be found in terms of eigenfunction series of Mathieu functions [7,8], but its usefulness is limited to $ka < 10$ (where k is the wave number and a is the half aperture width) because of the difficulty in tabulating the Mathieu functions and poor convergence of the series. Power series solutions in ka, also restricted to small slit widths, have been reported by Groschwitz and Hönl [9], Hönl and Zimmer [10], Müller and Westpfahl [11], and Bouwkamp [12]. For large slit widths Clemmow [13] used the concept of edge currents, while Millar [14] presented an asymptotic solution of the integral equations by successive iteration. A Wiener-Hopf treatment of the integral equation approach was given by Levine [15].

In 1953, Keller [16] proposed a more promising approach called the geometrical theory of diffraction for a large class of diffraction problems. The proposed ray-optical method provides physical insight into the mechanism of diffraction since the geometrical parameters dictate the paths of the propagating rays. Another basic advantage of this method lies in its simplicity since it employs elementary (trigonometric) functions for the far field on a ray and the resulting asymptotic series is comparable in accuracy, though more rapidly convergent, than the infinite series resulting from the boundary value solution for large aperture dimensions. The theory, as proposed or modified, has been very successful in solving many problems in diffraction by apertures [2,17,18], by smooth objects [19,20], in antenna structures [21,22] and waveguides [23]. In the application to waveguides, special mention should be made to the work of Yee and Felsen [24-27] who proposed a novel approach to the scattering by discontinuities in a waveguide using the free space scattering properties of the scattering centres constituting the discontinuity. It was found that while the ray-optical technique is best suited to the high frequency (multimode) regime far from the modal cutoffs, it is capable of providing satisfactory results even in the range of propagation of only the dominant mode. More recent work on the problem of diffraction by a slit (or a strip) has been due to Khaskind and Vainshteyn [28], Popov [29], Yu and Rudduck [4], Tan [5] and Ufimtsev [6]. An assumption common to all these investigations is that the slit (or the strip) is infinitesimally thin. The difficulty in considering a finite screen thickness arises from the fact that no convenient extension of the conventional methods of approach can be readily found.

Very few investigations have dealt with the subject of slits or slots in thick screens. Nomura and Inawashiro [30] proposed a boundary value solution using Weber-Shafheitlin's integrals and Jacobi's polynomials for the problem of transmission of acoustic waves through a circular channel of a thick wall. Wilson and Soroka [31] obtained an approximate solution for acoustic incidence which coincided with the complicated exact solution of Nomura and Inawashiro [30] over a wide range of frequencies. Budach [32] presented an approximate solution for acoustic incidence on a thick slit of dimensions very small compared to the wavelength λ. More recently, Lehman, using the analytic properties of finite Fourier transforms, showed that the electromagnetic field distributions for a thick slit can be obtained by solving a single variable Fredholm equation of the second kind [33]. His numerical approach, though restricted to a symmetrical excitation (i.e. two plane waves illuminating the slit symmetrically at angles θ and $-\theta$), spans the Rayleigh to geometrical optics range. However, since the approach is numerical, there is a lack of physical insight into the mechanism of diffraction leading to a very limited application of the results obtained.

Since the ray-optical method has been successful in a large number of free space and waveguide problems, it presents a promising approach to the analysis of the thick slit problem. However, the conventional ray method is inconvenient to apply because of the four scattering centres and the large number of edge-edge interactions involved. Moreover, it is restricted to the case of sufficiently large screen thickness. Thus an appropriate diffraction coefficient, associated with an equivalent thin edge situated at the centre of the thick half plane, would be of great use in the elimination of these restrictions.

Hanson [34] apparently was the first to consider the thick half plane problem. Under the assumption that $kb \ll 1$ (where $2b$ is the thickness of the half plane) he obtained a solution correct to the zeroth order of kb. Such a solution bears strong similarity to that of diffraction by a semi-infinite parallel plate waveguide, which is also solvable by the Wiener-Hopf method. Sometime later Jones [35] formulated the thick half plane problem in terms of two equations of the Wiener-Hopf type and obtained an approximate solution under the assumption that $kb << 1$. Harden [36] has reported extensive near field experimental data on the field intensity and phase characteristics of the diffraction pattern of a thick half plane whose thickness is of the order of the wavelength. The most recent work on this topic is due to Lee and Mittra [37] who obtained a solution to the diffraction of a plane wave by a parallel plate waveguide, loaded with a dielectric recessed by an arbitrary distance δ from the aperture, in terms of a highly convergent Neumann series involving scattering matrices. The scattering matrices were obtained from the well-known solution to the problem of diffraction and radiation by a semi-infinite parallel plate waveguide [38]. The solution of the thick half plane problem for an incident H-polarized plane electromagnetic wave was obtained by letting $\delta \to 0$ and allowing the permittivity of the dielectric to approach infinity.

Since one of the novel applications of the thick slit solution is to the scattering by a thick waveguide diaphragm, it would be relevant to discuss some of the approaches that have been employed in the past to deal with this problem. Mumford [39] proposed an approximate formula deduced from experimental results to account for the finite thickness of waveguide irises. Akhiezer [40] extended Bethe's [41] theory for small holes to include the effect of the finite screen thickness. In investigating the properties of narrow slots, Oliner [42] viewed the thick slot as a composite structure consisting of a length of waveguide equal to the slot wall thickness and of cross-sectional dimensions equal to those of the slot. Cohn [43] using a conformal mapping technique suggested by Davy [44], obtained an approximate formula for capacitance between two infinite plates of finite thickness. The thickness of a capacitive iris was accounted for by considering a line section of certain length and the formulae obtained were applicable only for $2b/\ell >> 1$ (where $2b$ is the thickness and ℓ the width of the aperture). Garb et al. [45] made an integral equation formulation for the fields on the two sides of a thick slot connecting two cavities. Though their solution is applicable for any ratio of $2b$ to ℓ, it is valid only for both $2b$ and $\ell << \lambda$. Thus, most of these investigations are restricted to slit or slot dimensions small compared to the wavelength.

This thesis is concerned with a ray-optical solution of the thick slit problem for a plane electromagnetic wave incident at an oblique angle. The solution, expressed in terms of simple diffraction coefficients assigned to each thick half plane is shown to be applicable to a large number of related problems. For the purposes of the analysis, we first find the solution to the problem of diffraction by a tandem slit configuration. This forms the basis of Chapter II. The tandem slit problem is basic to the problem of a thick slit similar to the manner in which the solution of the parallel plate waveguide problem led to the solution of the thick half plane [37]. Numerical computations are performed for various values of slit width (ℓ) and slit separation

(2b) and the results compared with variational and experimental results of Alldredge [46]. It is shown that as the tandem slit separation increases, the first peak in the scattered cross-section decreases and moves towards values corresponding to larger slit widths.

In Chapter III we first obtain the solution to the thick half plane problem for an incident E-polarized electromagnetic plane wave. Diffraction patterns for various values of screen thickness (2b) and angle of incidence θ_o are obtained, and it is shown that the peak in the diffraction patterns moves gradually towards $\theta = -\theta_o$ with the increase of kb. In the latter part of Chapter III the results for the thick half plane and the tandem slit configuration are combined to form the solution to the thick slit problem. Computations of the diffraction patterns are performed for various values of slit width and screen thickness and compared with experimental data. It is found that the effective width of a slit decreases with increasing screen thickness.

In Chapter IV the solution for the far field diffraction by a thick slit is utilized to find the scattering coefficients of thick asymmetric and symmetric diaphragms. The results obtained are compared with experimental and/or available numerical data [47]. Chapter V summarizes the general conclusions of this thesis. Finally, most of the material in Chapters II to IV has already been reported elsewhere [1,48-51].

CHAPTER II

FORMULATION OF THE BOUNDARY VALUE PROBLEM OF A TANDEM SLIT

In this chapter, the diffraction of a plane electromagnetic wave by two slits in a tandem configuration (see Fig. 2.1) is investigated by the Wiener-Hopf technique. The solution of this problem is employed in the next chapter to solve for the thick slit by filling the space in between the parallel plate waveguides with a dielectric whose permittivity is allowed to approach infinity.

The diffraction of a plane electromagnetic wave by a slit in a conducting screen has received wide attention due to its importance in microwave and optical instrumentation [2-6]. Although an exact solution is available in terms of Mathieu functions for the thin screen case, the problem is of sufficient interest for testing asymptotic diffraction theories where edge-edge interaction is only known for large slit widths. However, when there are two slits in tandem, the problem becomes complicated since conventional methods cannot be readily extended. The geometrical theory of diffraction is inconvenient to apply because of the four scattering centres and large number of edge-edge interactions involved, unless a reduction in the number of scattering centres is achieved. Furthermore, the ray method is restricted to the case of large plate-plate separation and it has been shown [52] that, for the case of a single parallel plate waveguide, it yields results which are not in complete agreement with the asymptotic form of the exact solution. These difficulties may be overcome by using an

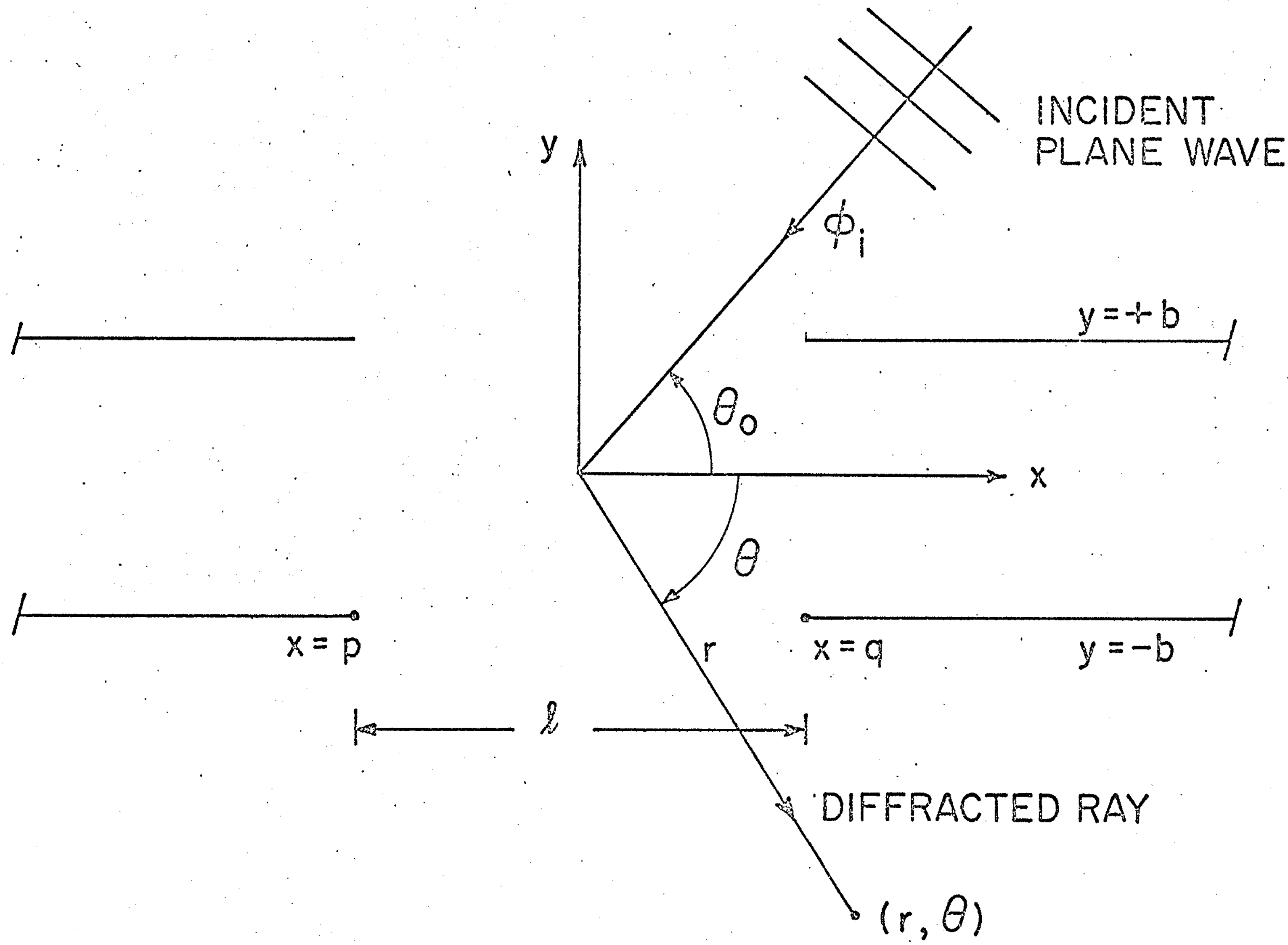


Figure 2.1 Tandem slit configuration.

equivalent edge concept and viewing the diffraction by each parallel plate waveguide as that due to a thin edge centred at the middle of the parallel plate waveguide modified by a multiplication factor, as shown later.

The problem has been solved in the past by Alldredge [46] using the variational principle. Though his analytical results are only valid for very small plate separation ($2kb < 0.985$, where k is the free space wave number and $2b$ is the plate separation), his experimental data for various values of slit width and plate separation is found to be very useful for comparing with our approximate solution.

Our approach is based on the boundary value method which leads to a Wiener-Hopf equation similar to that obtained by Jones [53] and Williams [54] for the complementary strip problem. This equation is solved in a manner similar to these authors and the results are used to extend the generalized scattering matrix solution of Lee and Mittra [37] to the thick slit problem. The final solution is obtained in the ray-optical form and expressed in terms of ray diffraction coefficients assigned to each parallel plate waveguide. Thus for small slit widths, we replace the Whittaker functions by Fresnel integrals and the resulting solution may be shown to be similar to that obtained by Yu and Rudduck [4] for the complementary strip problem. For large slit widths, the asymptotic evaluation of the Whittaker functions yields a solution similar to those of Keller [2] and Karp and Russek [3] for the case of a wide slit in a thin screen. This is shown to be possible using an equivalent edge concept where the diffraction by a parallel plate waveguide is attributed to a thin edge with an appropriate multiplication factor.

2.1 Derivation of the Integral Equation

Consider the parallel plate tandem slit configuration shown in Fig. 2.1. Let an E-polarized plane wave

$$\phi_i = \exp(-ikx\cos\theta_o - iky\sin\theta_o)e^{-i\omega t} \tag{2.1}$$

be incident on the plates. The time dependence $e^{-i\omega t}$ will henceforth be omitted. Let ϕ be the field scattered by the tandem slit and ϕ_t be the total field. Set

$$\phi_t = \begin{cases} \phi + e^{-ikx\cos\theta_o - iky\sin\theta_o} - e^{-ikx\cos\theta_o + ik(y-2b)\sin\theta_o} \; ; \; y \geq b \\ \\ \phi \; ; \; -b \leq y \leq b \;, \; -\infty < x < \infty \\ \qquad y \leq -b \;, \; -\infty < x < \infty \end{cases} \tag{2.2}$$

Let Φ be the Fourier transform of the scattered field ϕ with respect to the x direction. Thus

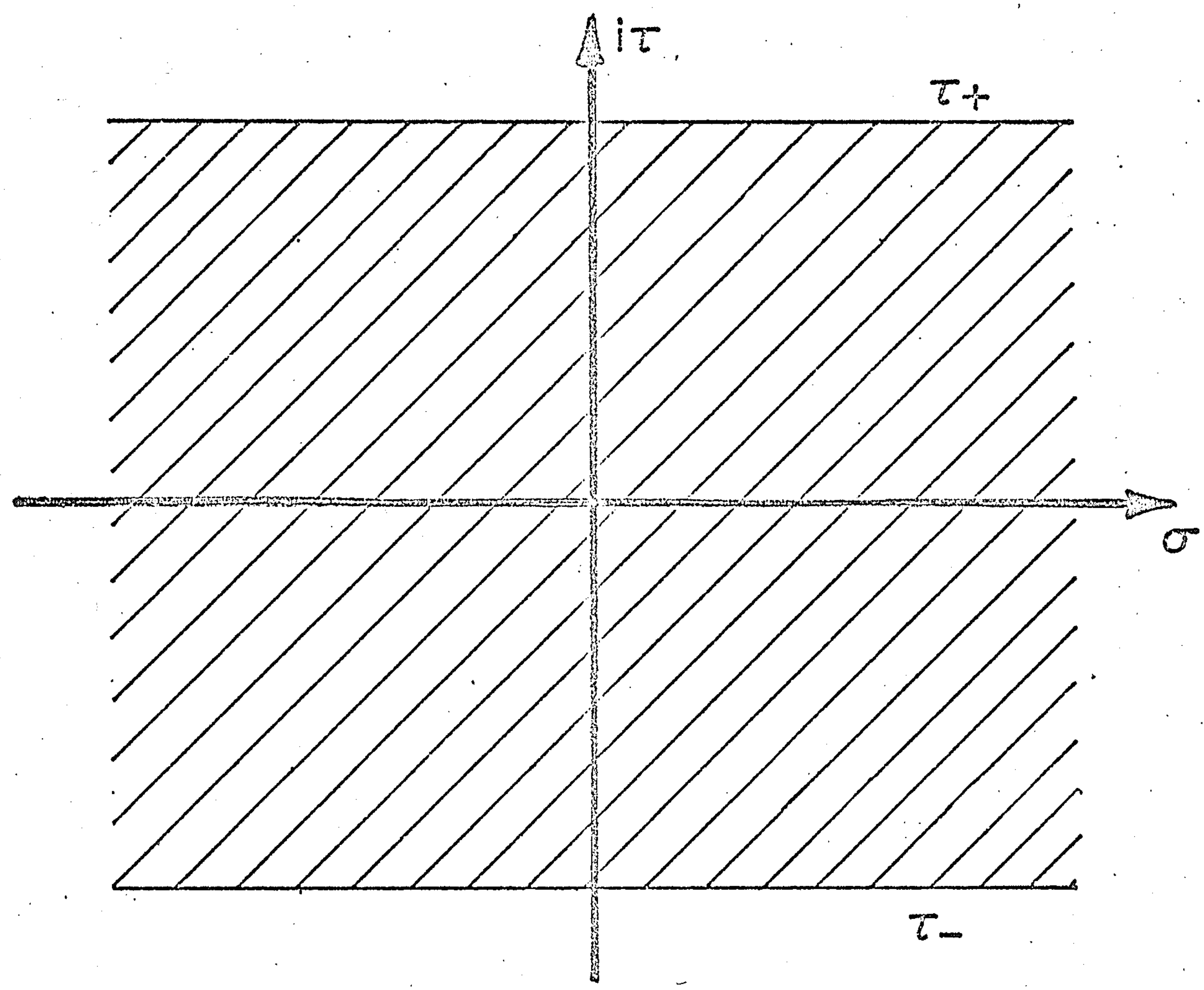
$$\Phi = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \phi \, e^{i\alpha x} \, dx \tag{2.3}$$

where α is a complex variable given by (see Fig. 2.2)

$$\alpha = \sigma + i\tau$$

The two-dimensional steady-state wave equation for ϕ can be written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0 \tag{2.4}$$



Complex α plane

Figure 2.2

Multiplying (2.4) by $(2\pi)^{-\frac{1}{2}} e^{i\alpha x}$ and integrating with respect to x from $-\infty$ to $+\infty$, we obtain

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{i\alpha x} dx + \frac{\partial^2 \Phi}{\partial y^2} + k^2 \Phi = 0 \tag{2.5}$$

which reduces to

$$\left[\frac{\partial \phi}{\partial x} e^{i\alpha x}\right]_{-\infty}^{\infty} - i\alpha\left[\phi \; e^{i\alpha x}\right]_{-\infty}^{\infty} + \frac{\partial^2 \Phi}{\partial y^2} - \gamma^2 \Phi = 0 \tag{2.6}$$

where

$$\gamma^2 = \alpha^2 - k^2 \tag{2.7}$$

and k is temporarily assumed to be complex i.e. $k = k_1 + ik_2$ $(k_1 > 0,\ k_2 > 0)$. Physically when $k_2 > 0$, ϕ behaves exponentially at infinity because of the finite damping—outgoing waves must decrease exponentially as we go to infinity. Since ϕ is an outgoing wave, contributions from the bracketed terms vanish. Thus

$$\frac{\partial^2 \Phi}{\partial y^2} - \gamma^2 \Phi = 0 \tag{2.8}$$

and the solution of this equation is of the form

$$\Phi = \begin{cases} Ae^{-\gamma y} & , \ y \geq b \qquad (2.9a) \\ De^{\gamma y} & , \ y \leq -b \qquad (2.9b) \\ Be^{\gamma y} + Ce^{-\gamma y} & , \ -b \leq y \leq b \qquad (2.9c) \end{cases}$$

In this solution there are branch points at $\alpha = \pm k$. The associated branch cuts have to be arranged in such a way that (2.9) represents a solution of (2.8) which can be inverted to give ϕ. This may be done by cutting the α-plane from $+k$ to $+\infty$ in the upper half plane and

$-k$ to $-\infty$ in the lower half plane and choosing the branch such that $\gamma \to \sigma$ if $\alpha = \sigma \to +\infty$. By analytic continuation we have $\gamma = -ik$ when $\alpha = 0$ and $\gamma \to |\sigma|$ when $\alpha = \sigma \to -\infty$ (Noble [38], p. 10). It is to be noted that (2.9a) and (2.9b) should also have terms of the type $A_1e^{\gamma y}$ and $D_1e^{-\gamma y}$, respectively. However, these terms vanish because $\Phi(\alpha,y)$ is bounded as $|y| \to \infty$ for all α in the strip $-k_2 < \tau < k_2$.

The boundary conditions for the various quantities are:

i) $\phi_t = 0$ on $y = \pm b$, $-\infty < x < p$, $q < x < \infty$

ii) ϕ_t and Φ_t are continuous on $y = +b$, $-\infty < x < \infty$

iii) ϕ and Φ are continuous on $y = -b$, $-\infty < x < \infty$

iv) $\frac{\partial \phi_t}{\partial y}$ $\left(\text{not } \frac{\partial \phi}{\partial y}\right)$ and $\frac{\partial \Phi_t}{\partial y}$ $\left(\text{not } \frac{\partial \Phi}{\partial y}\right)$ are continuous on $y = +b$, $p < x < q$

v) $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \Phi}{\partial y}$ are continuous on $y = -b$, $p < x < q$

vi) $\phi = O(r^{\frac{1}{2}})$ and $\frac{\partial \phi}{\partial y} = O(r^{-\frac{1}{2}})$ as $r \to 0$, where r is the distance from any of the edges [55].

The subscript "t" in the above conditions denotes total quantities.

Differentiating (2.9a) with respect to y, eliminating A and letting y tend to $(b + 0)$, we have (with $\Phi' = \frac{\partial \Phi}{\partial y}$, etc.)

$$\Phi'(b + 0) = -\gamma\, \Phi(b + 0) \tag{2.10}$$

which can be rewritten in the form

$$e^{i\alpha q}\Phi_+'(b + 0) + \Phi_1'(b + 0) + e^{i\alpha p}\Phi_-'(b + 0)$$
$$= -\gamma\left[e^{i\alpha q}\Phi_+(b + 0) + \Phi_1(b + 0) + e^{i\alpha p}\Phi_-(b + 0)\right] \tag{2.11}$$

where

$$\Phi_+ = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_q^\infty \phi \, e^{i\alpha(x-q)} \, dx \tag{2.12a}$$

$$\Phi_- = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^p \phi \, e^{i\alpha(x-p)} \, dx \tag{2.12b}$$

$$\Phi_1 = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_p^q \phi \, e^{i\alpha x} \, dx \tag{2.12c}$$

$$\Phi'_+ = \frac{\partial\Phi_+}{\partial y} \, , \; \Phi'_- = \frac{\partial\Phi_-}{\partial y} \, , \; \Phi'_1 = \frac{\partial\Phi_1}{\partial y} \tag{2.12d}$$

It is necessary to determine the behaviour of these transforms for large $|\xi|$. Since ϕ varies exponentially for large x as $\exp(-k_2|x|)$, it tends to zero as $|x| \to \infty$ and the transform Φ is therefore regular in the strip $-k_2 < \tau < k_2$ in the α-plane. Next consider $\Phi_-(\alpha)$ which is clearly regular in the lower half plane $\tau < k_2$. By a change of variable $z = x - p$, we have

$$\Phi_- = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^0 \phi(z + p) e^{i\alpha z} \, dz \tag{2.13}$$

Now the edge condition (vi, p. 14) gives $\phi(z + p) = 0(z^{\frac{1}{2}})$ as $z \to 0$, so that

$$\Phi_-(\alpha) = 0(|\alpha|^{-3/2}) \tag{2.14}$$

as $|\alpha| \to \infty$ in the lower half plane. Similarly $\Phi_+(\alpha)$ which is regular in the upper half plane $\tau > -k_2$, is $0(|\alpha|^{-3/2})$ as $|\alpha| \to \infty$

in the upper half plane. Also

$$e^{-i\alpha q}\Phi_1(\alpha) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_p^q \phi(x)e^{i\alpha(x-q)}\,dx = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{p-q}^{0} \phi(u+q)e^{i\alpha u}\,du \tag{2.15}$$

so that the left hand side is regular in the lower half plane and is $0(|\alpha|^{-3/2})$ as $|\alpha| \to \infty$ in the lower half plane. Similarly $e^{-i\alpha p}\Phi_1(\alpha)$ is regular in the upper half plane and is $0(|\alpha|^{-3/2})$ as $|\alpha| \to \infty$ in the upper half plane.

Differentiating (2.9b) with respect to y, eliminating D and letting y tend to $(-b - 0)$, we have

$$\Phi'(-b - 0) = \gamma\Phi(-b - 0) \tag{2.16}$$

which, using (2.12), can be rewritten in the form

$$e^{i\alpha q}\,\Phi_+'(-b - 0) + \Phi_1'(-b - 0) + e^{i\alpha p}\,\Phi_-'(-b - 0)$$
$$= \gamma\left[e^{i\alpha q}\,\Phi_+(-b - 0) + \Phi_1(-b - 0) + e^{i\alpha p}\,\Phi_-(-b - 0)\right] \tag{2.17}$$

Define

$$S_+'^{(o)} = \Phi_+'(b + 0) + \Phi_+'(-b - 0) \tag{2.18a}$$

$$S_+'^{(i)} = \Phi_+'(b - 0) + \Phi_+'(-b + 0) \tag{2.18b}$$

$$D_+'^{(o)} = \Phi_+'(b + 0) - \Phi_+'(-b - 0) \tag{2.18c}$$

$$D_+'^{(i)} = \Phi_+'(b - 0) - \Phi_+'(-b + 0) \tag{2.18d}$$

where the superscripts "o" and "i" refer to the outer and inner sides of the half planes, respectively. Adding and subtracting (2.11) and (2.17) and using (2.18), we obtain

$$S_+'^{(o)}e^{i\alpha q} + S_1'^{(o)} + S_-'^{(o)}e^{i\alpha p} = -\gamma\left[D_1^{(o)} + D_+^{(o)}e^{i\alpha q} + D_-^{(o)}e^{i\alpha p}\right] \tag{2.19}$$

and

$$D_+'^{(o)}e^{i\alpha q} + D_1'^{(o)} + D_-'^{(o)}e^{i\alpha p} = -\gamma\left[S_1^{(o)} + S_+^{(o)}e^{i\alpha q} + S_-^{(o)}e^{i\alpha p}\right] \tag{2.20}$$

where $S_1'^{(o)}$, $S_-'^{(o)}$, $D_1^{(o)}$, $D_+^{(o)}$ are analogous to the corresponding quantities in (2.18).

For eliminating the constants B and C, we proceed by setting $y = b - 0$ and $y = -b + 0$ in (2.9c). Using (2.12), we have

$$e^{i\alpha q}\Phi_+(b-0) + \Phi_1(b-0) + e^{i\alpha p}\Phi_-(b-0) = Be^{\gamma b} + Ce^{-\gamma b} \tag{2.21a}$$

$$e^{i\alpha q}\Phi_+(-b+0) + \Phi_1(-b+0) + e^{i\alpha p}\Phi_-(-b+0) = Be^{-\gamma b} + Ce^{\gamma b} \tag{2.21b}$$

Differentiating (2.9c) with respect to y and setting $y = b - 0$ and $y = -b + 0$, we have

$$e^{i\alpha q}\Phi_+'(b-0) + \Phi_1'(b-0) + e^{i\alpha p}\Phi_-'(b-0) = \gamma Be^{\gamma b} - \gamma Ce^{-\gamma b} \tag{2.22a}$$

$$e^{i\alpha q}\Phi_+'(-b+0) + \Phi_1'(-b+0) + e^{i\alpha p}\Phi_-'(-b+0) = \gamma Be^{-\gamma b} - \gamma Ce^{\gamma b} \tag{2.22b}$$

Eliminating B and C from (2.21), (2.22) and using (2.18), we have

$$e^{i\alpha q}\, S_+'^{(i)} + S_1'^{(i)} + e^{i\alpha p}\, S_-'^{(i)} = \gamma\coth\gamma b\left(D_1^{(i)} + D_+^{(i)}e^{i\alpha q} + D_-^{(i)}e^{i\alpha p}\right) \tag{2.23}$$

and

$$e^{i\alpha q}\, D_+'^{(i)} + D_1'^{(i)} + e^{i\alpha p}\, D_-'^{(i)} = \gamma\tanh\gamma b\left(S_1^{(i)} + S_+^{(i)}e^{i\alpha q} + S_-^{(i)}e^{i\alpha p}\right) \tag{2.24}$$

The condition of zero tangential electric field on the plates leads to

$$D_+ = D_- = S_+ = S_- = 0$$

where the superscripts have been omitted, since from the continuity of ϕ on $y = \pm b$, $S_+^{(o)} = S_+^{(i)} = S_+$. Similarly $D_1^{(o)}$, $D_1^{(i)}$ are written as D_1 and $S_1^{(o)}$, $S_1^{(i)}$ are written as S_1.

Now from (2.18a) and (2.18b), we have

$$S_1'^{(o)} - S_1'^{(i)} = \Phi_1'(b+0) + \Phi_1'(-b-0) - \Phi_1'(b-0) - \Phi_1'(-b+0)$$

where the second and fourth terms on the right hand side cancel out because of the boundary condition (v, p. 14). Thus we have

$$S_1'^{(o)} - S_1'^{(i)} = \Phi_1'(b+0) - \Phi_1'(b-0)$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_p^q (2ik \sin\theta_o) e^{-ikx \cos\theta_o - ikb \sin\theta_o} e^{i\alpha x} dx$$

$$= \frac{2k \sin\theta_o \, e^{-ikb \sin\theta_o}}{(2\pi)^{\frac{1}{2}}(\alpha - k\cos\theta_o)} \left[e^{i(\alpha - k\cos\theta_o)q} - e^{i(\alpha - k\cos\theta_o)p} \right] \qquad (2.25)$$

Subtracting (2.23) from (2.19), using (2.25) and recognizing that $D_1^{(o)} = D_1^{(i)} = D_1$, we have

$$e^{i\alpha q} \psi_{s_+} + A'G(\alpha) + e^{i\alpha p} \psi_{s_-} = -D_1/[L(\alpha)b] \qquad (2.26)$$

where

$$A' = 2k \sin\theta_o \, e^{-ikb \sin\theta_o} \qquad (2.27)$$

$$G(\alpha) = \frac{e^{i(\alpha - k\cos\theta_o)q} - e^{i(\alpha - k\cos\theta_o)p}}{(2\pi)^{\frac{1}{2}}(\alpha - k\cos\theta_o)} \qquad (2.28)$$