A STUDY OF THE EIGENVALUES OF ELECTRICAL NETWORKS

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ABSTRACT

In this thesis expressions for the real and imaginary parts of the eigenvalues of an RLC network are derived in terms of the matrices used in the state-variable formulation. Consequences of the real or purely imaginary property of the eigenvalues are also derived and topological interpretations given. Finally, a topological scheme is presented whereby it is possible to obtain transformation matrices, by inspecting network topology, which can be applied to eliminate the zero eigenvalues.

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CHAPTER I

INTRODUCTION

The unforced response of electrical networks is characterised by the eigenvalues of the A-matrix of the state equation [I] . Thus if the eigenvalues are the set of distinct complex numbers

$$\lambda_i = \alpha_i + j\beta_i \qquad \ldots(I.I)$$

then the zero input state response can be written (for a single state variable)

$$x_j(t) = \sum_i K_i e^{\lambda_i t} \qquad \ldots(I.2)$$

where the K_i are constants (possibly complex) which depend on the initial conditions. The eigenvalues λ_i can be

(a) zero, in which case the corresponding term is a constant,

(b) real, and necessarily negative for passive physical networks, in which case the corresponding term is a decaying exponential,

(c) pure imaginary, in which case the corresponding term is an undamped oscillatory one, and

(d) complex, with necessarily negative real part, in which case the corresponding term is a damped oscillatory one.

Zero Eigenvalues

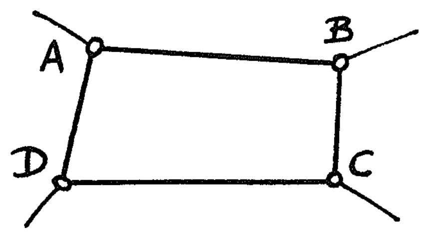
The eigenvalues can be determined in several ways. If the non-zero eigenvalues are desired, the roots of the appropriate loop impedance or node admittance determinant equated to zero suffices.

These methods of calculating the eigenvalues do not give the number of the zero eigenvalues. However, much information concerning the number of zero eigenvalues has been obtained. Thus Bryant [I] has shown that if $s^{l}|Y(s)|$ has r zeroes at the origin (and no more), then the A-matrix must have r zero eigenvalues. Here $|Y(s)|$ is the determinant of the nodal admittance matrix, and l is the number of inductors in the network. Hakimi and Kuo [2] further elaborate on the rank of the Bashkow-Bryant A-matrix and give a network characterisation of the interdependence of the state variables.

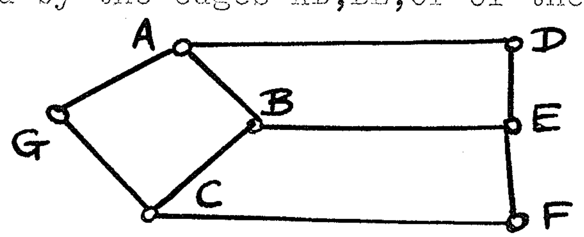
In the computation of the response of networks by the state variable technique, unnecessary complexity is introduced by the presence of these zero eigenvalues. Thus their elimination is a desirable thing. Parkin [3] discusses elimination of the zero eigenvalues of the transition matrix but he uses a nodal state variable technique rather than the conventional one. Martens and Guerra [II] use topological techniques to eliminate the zero eigenvalues. In this thesis a topological technique for the elimination of the zero eigenvalues of the A-matrix to reduce circuit computation complexity is presented.

Graph Theoretical Concepts.

It is the purpose of this thesis to study the eigenvalues of electrical networks using topological techniques. Some of the topological concepts employed will be reviewed. Seshu and Reed [4] is a comprehensive text on the subject with special reference to electrical applications. The concepts of a node,and of an edge joining a pair of nodes,are self explanatory. The concept of a loop is illustrated in the diagram below: edges AB,BC,CD,AD constitute a loop



The dual concept of the loop is the cut-set and this is a minimal set of edges,whose removal results in the decompo -sition of the graph into two disconnected sets of nodes, each set of nodes being connected by itself. This is illustrated by the edges AD,BE,CF of the graph below



Topological relationships are conveniently represented by means of matrices which are of various kinds such as incidence matrix,cut-set matrix,circuit matrix. Ore [5] is a good reference on pure graph theory.

The entries of these matrices are 0 or I if the edges are not oriented, and 0,I or -I if they are oriented. The concept of a tree is one of the most important in graph theory, and intimately connects the concepts of loop and cut-set. Defined in an intuitive manner, a tree is a connected set of edges of a graph, such that ~~no~~ it includes all the nodes of the graph but such that no subset of edges of the tree forms a loop. RLC networks without sources, which are the subject of this thesis, have the feature that every edge of the graph involved has the property of being either resistive, inductive or capacitive. Bryant found that the Normal Tree ---- which contains the maximum number of capacitances and the minimum number of inductances-----plays a central role in the state-variable analysis of electrical networks.

State Variable Characterisation

It is necessary to choose the voltages and currents which are to be state variables. Not all capacitance voltages are independently specifiable when there are capacitance loops in the network; not all inductance currents are independently specifiable when there are inductance cut-sets.

The set of state variables is given by

Set S (of state variables)

= S (of all C voltages)

- Subset_C(where order of Subset_C=number of C loops)

+ S (of all L currents)

-Subset_L(where order of Subset_L=number of L cut-sets)

...(I.3)

Thus to find the state equations,eliminate all nonstate variables i.e. all resistance voltages and currents,and those C voltages and L currents corresponding to certain members of C loops and L cut-sets.

Network Topology

Suppose there are n nodes,b branches and l fundamental loops. Let V be the voltage matrix, and I the current matrix and let them be partitioned into link elements and tree branch elements

$$V = \begin{bmatrix} V_l \\ V_t \end{bmatrix} \qquad I = \begin{bmatrix} I_l \\ I_t \end{bmatrix} \qquad \text{...(I.4)}$$

The state equations for an RLC network are now derived. The matrix equations which describe the network are

KVL:

$$\underset{(l\times b)(b\times I)}{\begin{bmatrix} B \cdot V \end{bmatrix}} = \underset{(l\times l)\;(l\times(n-1))}{\begin{bmatrix} U & B_f \end{bmatrix}} \begin{bmatrix} V_l \\ V_t \end{bmatrix} = 0 \qquad \text{...(I.5)}$$

KCL:

$$\underset{(n-1)\times b}{Q}\ \underset{b\times 1}{I} = \begin{bmatrix} \underset{(n-1)\times l}{Q_f} & \underset{(n-1)\times(n-1)}{U} \end{bmatrix} \begin{bmatrix} I_l \\ I_t \end{bmatrix} = 0 \qquad \ldots(\mathrm{I.6})$$

The element relations are

$$V_l = Z_l I_l \quad \text{and} \quad I_t = Y_t V_t \qquad \ldots(\mathrm{I.7})$$

where V, I, B_f, Q_f, Z_l, Y_t and U stand for voltage, current, fundamental circuit matrix, fundamental cut-set matrix, link impedance, twig admittance and unit matrix, respectively. By orthogonality

$$B_f = -Q_f^T \qquad \ldots(\mathrm{I.8})$$

Combining KVL, KCL in a single equation

$$\begin{bmatrix} Z_l & B_f \\ Q_f & Y_t \end{bmatrix} \begin{bmatrix} I_l \\ V_t \end{bmatrix} = 0 \qquad \ldots(\mathrm{I.9})$$

If we choose a normal tree and partition accordingly, we get

$$\begin{bmatrix} \frac{1}{p}S_1 & 0 & 0 & B_{SC} & 0 & 0 \\ 0 & R_2 & 0 & B_{RC} & B_{RG} & 0 \\ 0 & 0 & L_3 & B_{LC} & B_{LG} & B_{L\Gamma} \\ Q_{CS} & Q_{CR} & Q_{CL} & C_4 & 0 & 0 \\ 0 & Q_{GR} & Q_{GL} & 0 & G_5 & 0 \\ 0 & 0 & Q_{\Gamma L} & 0 & 0 & \frac{1}{p}\Gamma_6 \end{bmatrix} \begin{bmatrix} I_S \\ I_R \\ I_L \\ V_C \\ V_G \\ V_\Gamma \end{bmatrix} = 0 \qquad \ldots(\mathrm{I.10})$$

where p is the differential operator d/dt, and where the subscript denotes the element type. Here S, R and L denote chord elastances, resistances and inductances, respectively; C, G and Γ denote tree branch capacitances, conductances and reciprocal inductances, respectively; Q together with subscripts stands for the appropriate submatrices of the fundamental cut-set matrix partitioned according to element type; and similarly for B.

In the topological partitioning of B_f as

$$B_f = \begin{bmatrix} B_{SC} & 0 & 0 \\ B_{RC} & B_{RG} & 0 \\ B_{LC} & B_{LG} & B_{L\Gamma} \end{bmatrix} \qquad \ldots(\mathrm{I.11})$$

the B_f matrix expresses the topological relation between the links and the tree branches. Thus B_{SC} expresses the the topological relation between S links and C twigs, etc.

The coefficient matrix of (I.IO) has been called by Martens [6] the primitive hybrid matrix of the network. If the nonstate variables I_S, I_R, V_G and V are eliminated from (I.IO), the following matrix is obtained:

$$\begin{bmatrix} p\mathcal{L} + H_{II} & H_{I2} \\ -H_{I2}^T & p\mathcal{C} + H_{22} \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} = 0 \qquad \ldots (I.I2)$$

where

$$\left.\begin{aligned}
\mathcal{L} &= L_3 + B_{L\Gamma} L_6 B_{L\Gamma}^T \\
L_6 &= \Gamma_6^{-I} \\
\mathcal{C} &= C_4 + Q_{CS} C_I Q_{CS}^T \\
C_I &= S_I^{-I} \\
H_{II} &= Q_{GL}^T G^{-I} Q_{GL} \\
G &= G_5 + Q_{CR} G_2 Q_{CR}^T \\
G_2 &= R_2^{-I} \\
H_{22} &= Q_{CR} R^{-I} Q_{CR}^T \\
R &= R_2 + B_{RG} R_5 B_{RG}^T \\
R_5 &= G_5^{-I} \\
H_{I2} &= -Q_{CL}^T + Q_{GL}^T G^{-I} Q_{GR} G_2 Q_{CR}^T
\end{aligned}\right\} \quad \ldots (1.13)$$

It is noted that matrices $\mathcal{L}$ and $\mathcal{C}$ are symmetric and positive definite; H_{II} and H_{22} are symmetric and positive semidefinite or positive definite.

Martens [6] calls the coefficient matrix of (I.I2) the <u>near primitive hybrid matrix</u> and it can be written in the form of the matrix binomial

$$\mathcal{H} = p\Lambda + H \qquad \text{...(I.I4)}$$

where

$$\Lambda = \begin{bmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{C} \end{bmatrix} \qquad \text{...(I.I5)}$$

and

$$H = \begin{bmatrix} H_{II} & H_{I2} \\ -H_{I2}^{T} & H_{22} \end{bmatrix} \qquad \text{...(I.I6)}$$

If x denotes the state vector, then the free network is described by

$$(p\Lambda + H)x = 0 \qquad \text{...(I.I7)}$$

or

$$px = -\Lambda^{-I}Hx \qquad \text{..}$$

which, when compared with the unforced state equation,

$$px = Ax \qquad \text{...(I.I8)}$$

yields the result that the A matrix is given by

$$A = -\Lambda^{-I}H \qquad \text{...(I.I9)}$$

<u>Non-zero Eigenvalues</u>

Having discussed the significance of the zero eigenvalues, outlined the graph theoretical concepts, and introduced the relevant network topology we now discuss non-zero eigenvalues. The eigenvalues are the natural frequencies, and together with a knowledge of the initial conditions permits a determination of the response of a network or system to any forcing functions. The equations describing the linear systems with x(t) being the state vector, u(t) the input vector, y(t) the output vector, are (cf. L.Zadeh and C.A.Desoer [7])

$$px = Ax + Bu \qquad \ldots(I.20)$$

$$y = Cx + Du \qquad \ldots(I.21)$$

With u(t)=0, the zero-input state response is

$$x(t) = \exp(At)\, x(0) \qquad \ldots(I.22)$$

and the zero-input output response is

$$y(t) = C \exp(At)\, x(0) \qquad \ldots(I.23)$$

To evaluate exp(At) is the major task, and this is done through a knowledge of the eigenvalues of A, and an application of the Cayley-Hamilton theorem (cf. E.Bodewig [8])

Taking the initial state to be zero i.e.

$$x(t=0)=0 \qquad \ldots(I.24)$$

the zero-state state response is given by

$$x(t)=\int_0^t \exp(A(t-\tau))\, B\, u(\tau)d\tau \qquad \ldots(I.25)$$

and the zero-state output response is given by

$$y(t)=\int_0^t C \exp(A(t-\tau))\, B\, u(\tau)d\tau + Du(t) \qquad \ldots(I.26)$$

Both these latter two integrals contain the matrix exponential function $\exp(A(t-\tau))$.

Martens [6] has obtained some bounds on the eigenvalues, in terms of the network elements and topology, by making an application of the theorems of Bendixson, Pick, Wittmeyer, Wegner and Gerschgorin [8]. Thus if an eigenvalue is $\lambda = \alpha + j\beta$, then

$$|\beta| \leq \frac{\cot(\pi/\,2n)}{\min\,(\,\min(C_4),\min(L_3)\,)} \qquad \ldots(I.27)$$

where min(F) stands for the minimum entry of the matrix F and n is the order of the A-matrix.

The real part α is bounded above and below by

$$\min(\,m_I, m_2) \leq -\alpha \leq \max(\,M_I, M_2) \ldots(I.28)$$

where the constants m_I, m_2, M_I, M_2 are determined from the element values and an inspection of network topology.

Some further expressions are obtained in the next chapter for the real part α and the imaginary part β of an eigenvalue λ, and conditions are obtained for the eigenvalue to be purely real i.e. $\beta=0$, and for it to be purely imaginary i.e. $\alpha=0$.

CHAPTER II

Expressions for the real and imaginary parts of the eigenvalues and topological consequences of their reality or pure imaginariness

The eigenvalues or natural frequencies of a network are the eigenvalues of the A matrix. But since

$$A = -\Lambda^{-1} H \qquad \ldots(2.I)$$

$$pI - A = pI + \Lambda^{-1} H$$

$$\Lambda(pI - A) = p\Lambda + H$$

and taking determinants

$$\det \Lambda \det(pI - A) = \det(p\Lambda + H) \qquad \ldots(2.2)$$

Consequently the eigenvalues of A which are the zeroes of the characteristic polynomial, are also the zeroes of the determinant of the near-primitive hybrid matrix,

$$\mathcal{H} = p\Lambda + H \qquad \ldots(2.3)$$

The symmetric and skew symmetric parts of H are denoted by H_s and H_k where

$$H_s = \begin{bmatrix} H_{II} & 0 \\ 0 & H_{22} \end{bmatrix} = H_s^T; \quad H_k = \begin{bmatrix} 0 & H_{I2} \\ -H_{I2}^T & 0 \end{bmatrix} = -H_k^T \qquad \ldots(2.4)$$

$$H = H_s + H_k \qquad \ldots(2.5)$$

That H_s is indeed symmetric follows from the fact that

$$H_{II} = Q_{GL}^T G^{-I} Q_{GL}$$
$$H_{22} = Q_{CR} R^{-I} Q_{CR}^T \qquad \ldots(2.6)$$

are both symmetric matrices. That H_k is indeed skew symmetric follows upon application of the transpose operation to H_k and noting its relation to H_k; it is obvious that

$$H_k = -H_k^T$$

With these observations, we proceed in the next section to find expressions for the eigenvalues and conditions under which they are real or pure imaginary.

Derivation of Expressions for the real and imaginary parts of the eigenvalue $\lambda = \alpha + j\beta$.

The eigenvalue $\lambda = \alpha + j\beta$, and the eigenvector x are related by

$$Ax = \lambda x \qquad \ldots(2.7)$$

Using equation (2.I) for A in terms of Λ and H, we obtain

$$Hx = -\lambda \Lambda x \qquad \ldots(2.8)$$

Premultiply by $\bar{x}^T$ to obtain

$$-\lambda = \frac{\bar{x}^T H x}{\bar{x}^T \Lambda x} \qquad \ldots(2.9)$$

Consider the denominator of this expression for λ viz. $\bar{x}^T \Lambda x$, for reality or otherwise. By taking its complex conjugate we get

$$\begin{aligned} \overline{\bar{x}^T \Lambda x} &= x^T \bar{\Lambda} \bar{x} \\ &= x^T \Lambda \bar{x}, \end{aligned}$$

because Λ is real,

$$= (x^T \Lambda \bar{x})^T,$$

because the preceding quantity is a single number, that is a 1x1 matrix, and therefore since Λ is symmetric

$$\overline{\bar{x}^T \Lambda x} = \bar{x}^T \Lambda x, \qquad \ldots(2.10)$$

which assures us that the denominator expression in the formula for $-\lambda$ in equation (2.9) is a real number.

Consider now the numerator expression for $-\lambda$ in formula (2.9). Using the equation (2.5) which says

$$H = H_s + H_k$$

we have

$$\bar{x}^T H x = \bar{x}^T H_s x + \bar{x}^T H_k x \qquad \ldots(2.11)$$

We now test each component part in the right hand side of equation (2.11) for reality or otherwise by taking the complex conjugate.

Thus

$$\overline{\bar{x}^T H_s x} = x^T \bar{H}_s \bar{x}$$
$$\equiv x^T H_s \bar{x},$$

because H_s is real,

$$= (x^T H_s \bar{x})^T,$$
$$= \bar{x}^T H_s x$$

because H_s is symmetric.

Thus

$$\overline{\bar{x}^T H_s x} = \bar{x}^T H_s x \qquad \ldots(2.12)$$

from which we conclude that $\bar{x}^T H_s x$ is real.

Taking the complex conjugate of $\bar{x}^T H_k x$ we get

$$\overline{\bar{x}^T H_k x} = x^T \bar{H}_k \bar{x}$$
$$\equiv x^T H_k \bar{x}$$

because H_k is real,

$$= (\ x^T H_k \bar{x}\ )^T$$
$$= \bar{x}^T H_k^T x \qquad \ldots(2.13)$$

But since H_k is skew symmetric

$$H_k^T = -H_k \qquad \ldots(2.14)$$

we conclude that

$$\overline{\bar{x}^T H_k x} = -\bar{x}^T H_k x \qquad \ldots(2.15)$$

so that we have proved that the second component in the right hand side of equation (2.11) is pure imaginary.

Thus from equation (2.9)

$$-\lambda = \frac{\bar{x}^T H\ x}{\bar{x}^T \Lambda\ x}$$
$$= \frac{\bar{x}^T H_s x + \bar{x}^T H_k x}{\bar{x}^T \Lambda\ x}$$
$$= \frac{\bar{x}^T H_s x}{\bar{x}^T \Lambda\ x} + \frac{\bar{x}^T H_k x}{\bar{x}^T \Lambda\ x}$$
$$= -\alpha \ - j\beta$$

Using (2.12) and (2.15) we see that

$$-\alpha = \frac{\bar{x}^T H_s x}{\bar{x}^T \Lambda\ x} \qquad \ldots(2.16)$$

and

$$-j\beta = \frac{\bar{x}^T H_k x}{\bar{x}^T \Lambda x} \qquad \text{...(2.I7)}$$

From the above and the fact that H_s is positive semidefinite we note that $\alpha \leq 0$, which is well known. Alternate expressions for α and β may be derived as follows. The eigenvector x is in general complex, so we can write

$$x = x_r + jx_i \qquad \text{...(2.I8)}$$

where x_r and x_i are real vectors.

Then

$$\bar{x}^T = x_r^T - jx_i^T \qquad \text{...(2.I9)}$$

We now expand $\bar{x}^T H_s x$ as follows

$$\begin{aligned} \bar{x}^T H_s x &= (x_r^T - jx_i^T) H_s (x_r + jx_i) \ \ldots \\ &= x_r^T H_s x_r + x_i^T H_s x_i \\ &\quad + j(x_r^T H_s x_i - x_i^T H_s x_r) \end{aligned} \qquad \text{...(2.20)}$$

Because of the symmetry of H_s the second term of equation (2.20) vanishes, which proves in another way that $\bar{x}^T H_s x$ is real and given by

$$\bar{x}^T H_s x = x_r^T H_s x_r + x_i^T H_s x_i \qquad \text{...(2.2I)}$$

We also expand $\bar{x}^T H_k x$ as follows

$$\begin{aligned} \bar{x}^T H_k x &= (x_r^T - jx_i^T)\ H_k (x_r + jx_i) \\ &= (x_r^T H_k x_r + x_i^T H_k x_i) \\ &\quad + j(\ x_r^T H_k x_i - x_i^T H_k x_r) \end{aligned} \qquad \text{...(2.22)}$$

Because of the skew symmetry of H_k the first term in equation (2.22) vanishes, which leaves us with

$$\bar{x}^T H_k x = j(x_r^T H_k x_i - x_i^T H_k x_r)$$

$$= 2j\, x_r^T H_k x_i \qquad \ldots(2.23)$$

This proves in another way, that $\bar{x}^T H_k x$ is imaginary.

Finally $\bar{x}^T \Lambda x$ is found to be

$$\bar{x}^T \Lambda x = x_r^T \Lambda x_r + x_i^T \Lambda x_i \qquad \ldots(2.24)$$

We now use these results in the expression for $-\lambda$ in the equation (2.9)

$$-\lambda = \frac{\bar{x}^T H x}{\bar{x}^T \Lambda x}$$

$$= \frac{\bar{x}^T H_s x}{\bar{x}^T \Lambda x} + \frac{\bar{x}^T H_k x}{\bar{x}^T \Lambda x}$$

$$= \frac{(x_r^T H_s x_r + x_i^T H_s x_i}{x_r^T \Lambda x_r + x_i^T \Lambda x_i} + 2j \frac{x_r^T H_k x_i}{x_r^T \Lambda x_r + x_i^T \Lambda x_i}$$

$$\ldots(2.25)$$

Thus $-\lambda$ has been separated into a real and an imaginary part in the equation (2.25).

Consequently we have

$$-\alpha = \frac{x_r^T H_s x_r + x_i^T H_s x_i}{x_r^T \Lambda x_r + x_i^T \Lambda x_i} \qquad \ldots(2.26)$$

$$-\beta = \frac{2 x_r^T H_k x_i}{x_r^T \Lambda x_r + x_i^T \Lambda x_i} \qquad \ldots(2.27)$$

Classification of the Eigenvalues.

The eigenvalues λ of the network may be

(I) zero, i.e. $\lambda=0$ or $\alpha=0$, $\beta=0$;

(2) real, so that $\alpha \neq 0$, $\beta=0$;

(3) imaginary, so that $\alpha=0$, $\beta \neq 0$;

(4) complex, so that $\alpha \neq 0$, $\beta \neq 0$.

Consequences of any of these eventualities will be derived, and topological interpretations of these consequences given.

Consider the consequence of the real part being zero i.e. $\alpha=0$. By virtue of equation (2.I6)

$$-\alpha = \frac{\bar{x}^T H_s x}{\bar{x}^T \Lambda x}$$

Hence we must have

$$\bar{x}^T H_s x = 0 \qquad \ldots(2.28)$$

We partition the vector x into subvectors x_I, x_2 where x_I corresponds to the inductive link currents and x_2 corresponds to the capacitive twig voltages,

$$x = \begin{bmatrix} x_I \\ x_2 \end{bmatrix} \qquad \ldots(2.29)$$

We then have

$$\bar{x}^T H_s x = \begin{bmatrix} \bar{x}_I^T & \bar{x}_2^T \end{bmatrix} \begin{bmatrix} H_{II} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} x_I \\ x_2 \end{bmatrix}$$

$$= \bar{x}_I^T H_{II} x_1 + \bar{x}_2^T H_{22} x_2$$

Thus because $\alpha=0$, and using its consequence, equation (2.28), we must have

$$\bar{x}_1^T H_{11} x_1 + \bar{x}_2^T H_{22} x_2 = 0 \qquad \ldots(2.30)$$

Now H_{11}, H_{22} are both positive semidefinite. Hence

$$\bar{x}_1^T H_{11} x_1 \geq 0 \qquad \ldots(2.31)$$

$$\bar{x}_2^T H_{22} x_2 \geq 0 \qquad \ldots(2.32)$$

Equations (2.31),(2.32) in conjunction with equation (2.30) yields the following two equations

$$\bar{x}_1^T H_{11} x_1 = 0 \qquad \ldots(2.33)$$

$$\bar{x}_2^T H_{22} x_2 = 0 \qquad \ldots(2.34)$$

Consider equation (2.33). Making use of the expression for H_{11} given in equation (2.6) we have

$$\bar{x}_1^T Q_{GL}^T G^{-1} Q_{GL} x_1 = 0 \qquad \ldots(2.35)$$

But since G is positive definite, so is G^{-1}. This fact enables us to conclude that

$$Q_{GL} x_1 = 0 \qquad \ldots(2.36)$$

Using the equation (2.34) and substituting the expression for H_{22} given in equation (2.6) , gives us

$$\bar{x}_2^T Q_{CR} R^{-1} Q_{CR}^T x_2 = 0 \qquad \ldots(2.37)$$

But R is positive definite, and therefore R^{-1} is also positive definite. Hence

$$Q_{CR}^T x_2 = 0 \qquad \ldots(2.38)$$

The equation (2.36) is the Kirchoff Current Law for the subnetwork described by Q_{GL}, whereas equation (2.38) is the Kirchoff Voltage Law for the subnetwork described by Q_{CR}^{T} or B_{RC}.

Let x_I have real and imaginary parts given by

$$x_I = x_{Ir} + j\, x_{Ii}$$

and similarly for x_2,

$$x_2 = x_{2r} + j\, x_{2i} \, .$$

Then equation (2.36) implies that

$$Q_{GL}x_{Ir} = 0, \; Q_{GL}x_{Ii} = 0 \, ,$$

and equation (2.38) implies that

$$Q_{CR}^{T}x_{2r} = 0, \; Q_{CR}^{T}x_{2i} = 0.$$

Using these equations together with (I.I3), the numerator of the expression for $-\beta$ reduces to

$$x_{2r}^{T}Q_{CL}x_{Ii} - x_{Ir}^{T}Q_{CL}^{T}x_{2i}$$

which gives a simplified expression for $-\beta$ for the case of a pure imaginary eigenvalue.

A further consequence of the pure imaginary nature of an eigenvalue (i.e. $\alpha = 0$, $\beta \neq 0$) will now be derived. If λ is an eigenvalue of the A matrix with eigenvector x, then

$$Ax = \lambda x \qquad \ldots(2.39)$$

But

$$A = -\Lambda^{-1}H.$$

Hence

Hence

$$-\Lambda^{-1}Hx = \lambda x$$

or

$$(\lambda\Lambda + H)x = 0 \qquad \ldots(2.40)$$

We now replace Λ, H, x by their components by using the compatible partitioning mentioned above, so that

$$\left\{\lambda\begin{bmatrix}\mathcal{L} & 0\\ 0 & -\mathcal{C}\end{bmatrix} + \begin{bmatrix}H_{II} & H_{I2}\\ -H_{I2}^{T} & H_{22}\end{bmatrix}\right\}\begin{bmatrix}x_I\\ x_2\end{bmatrix} = 0 \qquad \ldots(2.4I)$$

Multiplying out we get the following two matrix equations

$$\lambda\mathcal{L}x_I + H_{II}x_I + H_{I2}x_2 = 0 \qquad \ldots(2.42)$$
$$\lambda\mathcal{C}x_2 + H_{22}x_2 - H_{I2}^{T}x_I = 0 \qquad \ldots(2.43)$$

These two equations can be simplified as follows. Consider $H_{II}x_I$. The expression for H_{II} is

$$H_{II} = Q_{GL}^{T}G^{-I}Q_{GL}.$$

Then

$$H_{II}x_I = Q_{GL}^{T}G^{-I}Q_{GL}x_I \qquad \ldots(2.44)$$

But by equation (2.36), which is a consequence of $\alpha = 0$, we have

$$Q_{GL}x_I = 0.$$

Hence

$$H_{II}x_I = 0 \qquad \ldots(2.45)$$

Similarly consider $H_{22}x_2$. The expression for the H_{22} matrix is

$$H_{22} = Q_{CR}R^{-I}Q_{CR}^{T}$$

Then

$$H_{22}x_2 = Q_{CR}R^{-I}Q^T_{CR}x_2 \qquad \ldots(2.46)$$

But by equation (2.38) which is a consequence of the fact that $\alpha = 0$, we have

$$Q^T_{CR}x_2 = 0$$

Hence

$$H_{22}x_2 = 0 \qquad \ldots(2.47)$$

Using these equations viz. (2.45),(2.47) we have the following simplified forms of equations (2.42) and (2.43) for the case of a pure imaginary eigenvalue

$$\lambda \mathcal{L}\, x_I + H_{I2}x_2 = 0 \qquad \ldots(2.48)$$

$$\lambda \mathcal{C}\, x_2 - H^T_{I2}x_I = 0 \qquad \ldots(2.49)$$

Premultiply the equation (2.48) by x^T_I, and the equation (2.49) by x^T_2, and add

$$\lambda(x^T_I \mathcal{L}\, x_I + x^T_2 \mathcal{C}\, x_2) + (x^T_I H_{I2}x_2 - x^T_2 H^T_{I2}x_I) = 0 \qquad \ldots(2.50)$$

In equation (2.50) we observe the second bracket as contributing zero,

$$x^T_I H_{I2}x_2 - x^T_2 H^T_{I2}x_I = 0 \qquad \ldots(2.5I)$$

Hence equation (2.50) becomes

$$\lambda(x^T_I \mathcal{L}\, x_I + x^T_2 \mathcal{C}\, x_2) = 0 \qquad \ldots(2.52)$$

Since we have assumed $\lambda = 0 + j\beta \neq 0$, we finally get

$$x^T_I \mathcal{L}\, x_I + x^T_2 \mathcal{C}\, x_2 = 0 \qquad \ldots(2.53)$$

Next some consequences of the reality of the eigenvalue λ of a network will be derived, i.e. $\alpha \neq 0$, $\beta = 0$

where $\lambda \doteq \alpha + j\beta$. By equation (2.17)

$$-j\beta = (\bar{x}^T H_k x)/(\bar{x}^T \Lambda x)$$

If $\beta = 0$, then

$$\bar{x}^T H_k x = 0 \qquad \ldots(2.54)$$

But

$$\bar{x}^T H_k x = \begin{bmatrix} \bar{x}_I^T & \bar{x}_2^T \end{bmatrix} \begin{bmatrix} 0 & H_{I2} \\ -H_{I2}^T & 0 \end{bmatrix} \begin{bmatrix} x_I \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{x}_I^T & \bar{x}_2^T \end{bmatrix} \begin{bmatrix} H_{I2} x_2 \\ -H_{I2}^T x_I \end{bmatrix}$$

$$= \bar{x}_I^T H_{I2} x_2 - \bar{x}_2^T H_{I2}^T x_I$$

Hence from equation (2.54) we have as a consequence of $\beta = 0$, the equation

$$\bar{x}_I^T H_{I2} x_2 - \bar{x}_2^T H_{I2}^T x_I = 0 \qquad \ldots(2.55)$$

If as before we consider the real and imaginary parts of x_I as x_{Ir} and x_{Ii} respectively, and similarly for x_2, then the equation (2.55) simplifies to

$$x_{Ir}^T H_{I2} x_{2i} - x_{2r}^T H_{I2}^T x_{Ii} = 0 \qquad \ldots(2.55a)$$

The equation (2.55) will now be written in terms of topological relationships and of element values.

We use the following expression for H_{I2} derived in Chapter I, viz.,

$$H_{I2} = -Q_{CL}^T + Q_{GL}^T G^{-I} Q_{GR} G_2 Q_{CR}^T$$

to obtain the following consequence of the reality of the eigenvalue λ :

$$-\bar{x}_1^T Q_{CL}^T x_2 + \bar{x}_1^T Q_{GL}^T G^{-1} Q_{GR} G_2 Q_{CR}^T x_2 + \bar{x}_2^T Q_{CL} x_1$$
$$-\bar{x}_2^T Q_{CR} G_2^T Q_{GR}^T (G^{-1})^T Q_{GL} x_1 = 0 \qquad \ldots(2.56)$$

In terms of the real and imaginary parts of the vectors x_1, x_2 the equation (2.56) becomes

$$-x_{1r}^T Q_{CL}^T x_{2i} + x_{1r}^T Q_{GL}^T G^{-1} Q_{GR} G_2 Q_{CR}^T x_{2i}$$
$$+x_{2r}^T Q_{CL} x_{1i} - x_{2r}^T Q_{CR} G_2^T Q_{GR}^T (G^{-1})^T Q_{GL} x_{1i} = 0 \qquad \ldots(2.56a)$$

We next derive the consequences of an eigenvalue λ of a network being zero, so that $\alpha = 0$, $\beta = 0$. Substituting the consequences of α alone being zero, i.e.

$$Q_{GL} x_1 = 0 ,$$
$$Q_{CR}^T x_2 = 0,$$

into the equation (2.56) which is the consequence of $\beta = 0$, we get

$$\bar{x}_1^T Q_{CL}^T x_2 = \bar{x}_2^T Q_{CL} x_1 = \overline{(\bar{x}_1^T Q_{CL}^T x_2)}^T \qquad \ldots(2.57)$$

Hence both $\bar{x}_1^T Q_{CL}^T x_2$, $\bar{x}_2^T Q_{CL} x_1$ are real numbers.

Now

$$\bar{x}_1^T Q_{CL}^T x_2 = (x_{1r}^T - j x_{1i}^T)\, Q_{CL}^T (x_{2r} + j x_{2i})$$
$$= (x_{1r}^T Q_{CL}^T x_{2r} + x_{1i}^T Q_{CL}^T x_{2i})$$
$$+ j (x_{1r}^T Q_{CL}^T x_{2i} - x_{1i}^T Q_{CL}^T x_{2r})$$

But since $\bar{x}_1^T Q_{CL}^T x_2$ is real, therefore

$$\bar{x}_1^T Q_{CL}^T x_2 = x_{1r}^T Q_{CL}^T x_{2r} + x_{1i}^T Q_{CL}^T x_{2i} \qquad \ldots(2.58)$$

and

$$x_{1r}^T Q_{CL}^T x_{2i} = x_{1i}^T Q_{CL}^T x_{2r} \qquad \ldots(2.59)$$

Complex Eigenvalues.

Finally, the expressions for α, β the real and imaginary parts of the eigenvalue $\lambda = \alpha + j\beta$ will be stated in terms of the state vector x (partitioned into the inductive link currents and the capacitive twig voltages) and matrices of element values a and of topological relationships.

Thus

$$-\alpha = \frac{\bar{x}^T H_s x}{\bar{x}^T \Lambda x}$$

$$= \frac{\begin{bmatrix} \bar{I}_L^T & \bar{V}_C^T \end{bmatrix} \begin{bmatrix} H_{II} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix}}{\begin{bmatrix} \bar{I}_L^T & \bar{V}_C^T \end{bmatrix} \begin{bmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{C} \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix}}$$

$$= \frac{\bar{I}_L^T H_{II} I_L + \bar{V}_C^T H_{22} V_C}{\bar{I}_L^T \mathcal{L}^I I_L + \bar{V}_C^T \mathcal{C} V_C}$$

Substituting for $H_{II}, H_{22}, \mathcal{L}, \mathcal{C}$ the matrices of element values and of topological relationships obtained earlier, we get finally

$$-\alpha = \frac{\bar{I}_L^T Q_{GL}^T G^{-I} Q_{GL} I_L + \bar{V}_C^T Q_{CR} R^{-I} Q_{CR}^T V_C}{\bar{I}_L^T (L_3 + B_{L\Gamma} L_6 B_{L\Gamma}^T) I_L + \bar{V}_C^T (C_4 + Q_{CS} C_I Q_{CS}^T) V_C} \qquad \ldots(2.60)$$

In the above I_L are the inductive link currents, and V_C the capacitive twig voltages.

For the imaginary part of the eigenvalue we have

$$-j\beta = \frac{\bar{x}^T H_k x}{\bar{x}^T \Lambda x}$$

But

$$\bar{x}^T = [\bar{I}_L^T \quad \bar{V}_C^T]$$

$$H_k = \begin{bmatrix} 0 & H_{12} \\ -H_{12}^T & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \mathscr{A} & 0 \\ 0 & \mathscr{B} \end{bmatrix}$$

Then

$$-j\beta = \frac{\bar{I}_L^T H_{12} V_C - \bar{V}_C^T H_{12}^T I_L}{\bar{I}_L^T \mathscr{A}\, I_L + \bar{V}_C^T \mathscr{B}\, V_C}$$

Substituting for H_{12}, $\mathscr{A}$, $\mathscr{B}$ in terms of matrices of element values and of topological relationships we get

$$-j\beta = \frac{\begin{aligned}&\bar{I}_L^T(-Q_{CL}^T + Q_{GL}^T G^{-1} Q_{GR} G_2 Q_{CR}^T) V_C \\ &- \bar{V}_C^T(-Q_{CL} + Q_{CR} G_2^T Q_{GR}^T (G^{-1})^T Q_{GL}) I_L\end{aligned}}{\begin{aligned}&\bar{I}_L^T(L_3 + B_{L\Gamma} L_6 B_{L\Gamma}^T) I_L \\ &+ \bar{V}_C^T(C_4 + Q_{CS} C_1 Q_{CS}^T) V_C\end{aligned}}$$

...(2.61)

In terms of the real and imaginary parts of I_L, V_C

$$-\frac{\beta}{2} = \frac{\begin{aligned}&I_{Lr}^T(-Q_{CL}^T + Q_{GL}^T G^{-1} Q_{GR} G_2 Q_{CR}^T) V_{Ci} \\ &- \bar{V}_{Cr}^T(-Q_{CL} + Q_{CR} G_2^T Q_{GR}^T (G^{-1})^T Q_{GL}) I_{Li}\end{aligned}}{\begin{aligned}&I_{Lr}^T(L_3 + B_{L\Gamma} L_6 B_{L\Gamma}^T) I_{Lr} + I_{Li}^T(L_3 + B_{L\Gamma} L_6 B_{L\Gamma}^T) I_{Li} \\ &+ V_{Cr}^T(C_4 + Q_{CS} C_1 Q_{CS}^T) V_{Cr} + V_{Ci}^T(C_4 + Q_{CS} C_1 Q_{CS}^T) V_{Ci}\end{aligned}}$$

CHAPTER III

Topological elimination of the zero eigenvalues of the A-matrix

Introduction. The state equation for an unforced linear system is

$$px = Ax \qquad \ldots(3.1)$$

where p is the differential operator d/dt. If the A matrix contains zero eigenvalues, the computation of the state vector is beset with difficulty (see for example Varga [9]). The Bashkow-Bryant A-matrix which is used in the state equation description of an RLC network includes zero eigenvalues if there are L-loops and C-cut-sets. In this case the order of the A matrix is greater than its rank, and thus unnecessarily large, increasing the computational complexity.

Some work has been done on this problem. Thus Hakimi and Kuo [2] have given for-mulas for the order of A and the number of zero eigenvalues. Parkin [3] discusses the elimination of the zero eigenvalues of the transition matrix but uses a nodal state variable technique.

In this chapter, a direct topological method to eliminate the zero eigenvalues is described.

The method is in the form of topological rules for obtaining transformation matrices which reduce Λ and H ($A=-\Lambda^{-I}H$) to $\hat{\Lambda}$ and $\hat{H}$ ($\hat{A}=-\hat{\Lambda}^{-I}\hat{H}$) where the zero eigenvalues have been eliminated.

The number of zero eigenvalues, n_o , is given by the number of independent L-loops plus the number of independent C-cut-sets, [4,6] .Alternatively in terms of the submatrices of equation (I.IO)

$$
\begin{aligned}
n_o = {} & \text{number of rows of } [B_{LC} \quad B_{LG}] \\
& + \text{number of rows of } [Q_{CR} \quad Q_{CL}] \\
& - \text{rank of } [B_{LC} \quad B_{LG}] \\
& - \text{rank of } [Q_{CR} \quad Q_{CL}] \qquad \ldots(3.2)
\end{aligned}
$$

This suggests that in order to eliminate the zero eigenvalues, elementary row operations could be applied to reduce the dependent rows of the above two matrices.

Elementary Transformation of Topological Matrices.

In order to obtain the elementary transformation to eliminate the zero eigenvalues we consider two subnetworks: an L-subnetwork and a C-subnetwork. The L-subnetwork is obtained by open-circuiting (removing) all non-inductive elements plus the inductive twigs. From equation (I.IO) it is then clear that this subnetwork is described by the cut-set matrix

$$\begin{bmatrix} Q_{CL} \\ Q_{GL} \\ Q_{\Gamma L} \end{bmatrix}$$

The inductors in the L subnetwork may be partitioned onto links (denoted by subscript l) and twigs (denoted by subscript t) by choosing a tree or a forest. Then the columns corresponding to the twigs are linearly independent [4] and the remaining link columns are dependent and may be expressed as linear combinations of the twig columns. Therefore, there exists a matrix

$$T_I = \begin{bmatrix} U & M \\ 0 & U \end{bmatrix} \qquad \ldots(3.3)$$

where U is the unit matrix, such that

$$\begin{bmatrix} Q_{CL} \\ Q_{GL} \\ Q_{\Gamma L} \end{bmatrix} T_I^T = \begin{bmatrix} Q_{CL_l} & Q_{CL_t} \\ Q_{GL_l} & Q_{GL_t} \\ Q_{\Gamma L_l} & Q_{\Gamma L_t} \end{bmatrix} \begin{bmatrix} U & 0 \\ M^T & U \end{bmatrix} = \begin{bmatrix} 0 & Q_{CL_t} \\ 0 & Q_{GL_t} \\ 0 & Q_{\Gamma L_t} \end{bmatrix} \qquad \ldots(3.4)$$

or, in terms of the circuit submatrices

$$T_I \begin{bmatrix} B_{LC} & B_{LG} & B_{L\Gamma} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ B_{L_tC} & B_{L_tG} & B_{L_t\Gamma} \end{bmatrix} \qquad \ldots(3.5)$$

The C-subnetwork is obtained by short-circuiting all non-capacitive elements and, in addition, open circuiting all C-links of the original network. This subnetwork is then described by the circuit matrix

$$\begin{bmatrix} B_{SC} \\ B_{RC} \\ B_{LC} \end{bmatrix}$$

If we choose a tree in this subnetwork, the capacitors may be partitioned into links and twigs. The twig columns are linear combinations of the linearly independent link columns. Thus there exists a matrix

$$T_2 = \begin{bmatrix} U & 0 \\ N & U \end{bmatrix} \qquad \ldots(3.6)$$

such that

$$\begin{bmatrix} B_{SC} \\ B_{RC} \\ B_{LC} \end{bmatrix} T_2^T = \begin{bmatrix} B_{SC_l} & B_{SC_t} \\ B_{RC_l} & B_{RC_t} \\ B_{LC_l} & B_{LC_t} \end{bmatrix} \begin{bmatrix} U & N^T \\ 0 & U \end{bmatrix} = \begin{bmatrix} B_{SC_l} & 0 \\ B_{RC_l} & 0 \\ B_{LC_l} & 0 \end{bmatrix} \qquad \ldots(3.7)$$

and in terms of the corresponding cut-set matrix

$$T_2 \begin{bmatrix} Q_{CS} & Q_{CR} & Q_{CL} \end{bmatrix} = \begin{bmatrix} Q_{C_lS} & Q_{C_lR} & Q_{C_lL} \\ 0 & 0 & 0 \end{bmatrix} \qquad \ldots(3.8)$$

<u>Elimination of Zero Eigenvalues.</u>

We now apply the transformations T_I and T_2 to the equation (I.IO), and make a change of variables in order to apply the corresponding operations to the columns. We partition I_L and V_C and define new variables as follows.

$$\begin{bmatrix} I_{L_1} \\ I_{L_t} \end{bmatrix} = \begin{bmatrix} U & 0 \\ M^T & U \end{bmatrix} \begin{bmatrix} I_{L_1} \\ J_{L_t} \end{bmatrix} \qquad \ldots(3.9)$$

$$\begin{bmatrix} V_{C_1} \\ V_{C_t} \end{bmatrix} = \begin{bmatrix} U & N^T \\ 0 & U \end{bmatrix} \begin{bmatrix} E_{C_1} \\ V_{C_t} \end{bmatrix} \qquad \ldots(3.10)$$

If we eliminate I_S and V from equation (1.10), apply the transformations T_1 and T_2, and make the above change of variables, then we obtain

$$\begin{bmatrix} R_2 & 0 & 0 & B_{RC_1} & 0 & B_{RG} \\ 0 & p\tilde{L}_{11} & p\tilde{L}_{1t} & 0 & 0 & 0 \\ 0 & p\tilde{L}_{t1} & p\tilde{L}_{tt} & B_{L_tC_1} & 0 & B_{L_tG} \\ Q_{C_1R} & 0 & Q_{CL_t} & p\tilde{C}_{11} & p\tilde{C}_{1t} & 0 \\ 0 & 0 & 0 & p\tilde{C}_{t1} & p\tilde{C}_{tt} & 0 \\ Q_{GR} & 0 & Q_{GL_t} & 0 & 0 & G_5 \end{bmatrix} \begin{bmatrix} I_R \\ I_{L_1} \\ J_{L_t} \\ E_{C_1} \\ V_{C_t} \\ V_G \end{bmatrix} = 0 \qquad \ldots(3.11)$$

We note in passing that $\tilde{L}_{11}$ and $\tilde{C}_{tt}$ are positive definite since they are nonsingular congruence transformations of positive definite matrices.

Next we eliminate the variables I_R, V_G, I_{L_1} and V_{C_t} from equation (3.II) and as a result obtain the reduced equation

$$\begin{bmatrix} p\hat{L} + \hat{H}_{II} & \hat{H}_{I2} \\ -\hat{H}_{I2}^T & p\hat{C} + \hat{H}_{22} \end{bmatrix} \begin{bmatrix} J_{L_t} \\ E_{C_1} \end{bmatrix} = 0 \qquad \ldots(3.I2)$$

or more compactly,

$$(p\hat{\Lambda} + \hat{H})\hat{x} \doteq 0 \qquad \ldots(3.I3)$$

At this point we note from equation (3.II) that

$$p\tilde{L}_{11} I_{L_1} + p\tilde{L}_{1t} J_{L_t} = 0 \qquad \ldots(3.I4)$$

and therefore

$$I_{L_1} = -\tilde{L}_{11}^{-I}\tilde{L}_{1t} J_{L_t} + K_I \qquad \ldots(3.I5)$$

and similarly

$$V_{C_t} = -\tilde{C}_{tt}\tilde{C}_{t1} E_{C_1} + K_2 \qquad \ldots(3.I6)$$

where the constant vectors K_I, K_2 are determined by the initial state of the network.

If we further define transformations

$$F_I = \begin{bmatrix} U & 0 \\ -\tilde{L}_{t1}\tilde{L}_{11}^{-I} & U \end{bmatrix} \qquad \ldots(3.I7)$$

$$F_2 = \begin{bmatrix} U & -\tilde{C}_{1t}\tilde{C}_{tt}^{-1} \\ 0 & U \end{bmatrix} \qquad \ldots(3.18)$$

and

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \qquad \ldots(3.19)$$

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \qquad \ldots(3.20)$$

then

$$\det(\, F\ T\ (\ p\Lambda + H)\ T^T F^T) = \det(p\Lambda\ + H) \qquad \ldots(3.21)$$

However,

$$FT(p\Lambda\ + H\ )\ T^T F^T = \begin{bmatrix} p\tilde{L}_{11} & 0 & 0 \\ 0 & p\hat{\Lambda} + \hat{H} & 0 \\ 0 & 0 & p\tilde{C}_{tt} \end{bmatrix} \qquad \ldots(3.22)$$

and therefore

$$\det(pI-A) = \frac{\det(p\Lambda\ + H\ )}{\det \Lambda}$$

$$= \frac{\det\tilde{L}_{11}\det\tilde{C}_{tt}}{\det \Lambda}\ p^{n_8}\ \det(p\hat{\Lambda}\ + \hat{H}) \qquad \ldots(3.23)$$

Thus the zero eigenvalues have clearly been eliminated to yield the reduced matrix binomial equation

$$(p\hat{\Lambda} + \hat{H}) \hat{x} = 0 \qquad \ldots(3.24)$$

or equivalently the reduced A-matrix

$$\hat{A} = -\hat{\Lambda}^{-I} \hat{H} \qquad \ldots(3.25)$$

The order of $\hat{A}$ is of course equal to the order of A minus the number of zero eigenvalues.

Example.

To illustrate the method for the elimination of the zero eigenvalues, we consider the network of Fig.I. This network contains one L-loop and one C-cut-set, therefore there are two zero eigenvalues. A normal tree is chosen with the state vector

$$x = \begin{bmatrix} i_{L_c} \\ i_{L_a} \\ i_{L_b} \\ v_{C_a} \\ v_{C_b} \\ v_C \end{bmatrix}$$

The relevant submatrices of Λ and H are

$$\mathcal{L} = \begin{bmatrix} L_c + L & 0 & -L \\ 0 & L_a & 0 \\ -L & 0 & L_b + L \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} C_a + C_c & 0 & 0 \\ C_c & C_b + C_c & 0 \\ 0 & 0 & C \end{bmatrix}$$

$$H_{11} = \begin{bmatrix} R_a + R_b & -R_b & -R_a \\ -R_b & R_b + R_c & -R_c \\ -R_a & -R_c & R_c + R_a \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

and H_{22} is a 3x3 zero matrix.

In the L-subnetwork, L_c may be a link and L_a and L_b twigs. In the C-subnetwork, C may be a twig and C_a and C_b links.

The transformations, T_I of equation (3.3) and T_2 of equation (3.6) are then

$$T_I = \begin{bmatrix} I & I & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & -I & I \end{bmatrix}$$

and

$$\tilde{\mathcal{L}} = T_I \, \mathcal{L} \, T_I^T = \begin{bmatrix} L_a+L_b+L_c & L_a & L_b \\ L_a & L_a & 0 \\ L_b & 0 & L_b + L \end{bmatrix}$$

$$\tilde{\mathcal{C}} = T_2 \, \mathcal{C} \, T_2^T = \begin{bmatrix} C_a + C_c & C_c & C_a \\ C_c & C_b+C_c & -C_b \\ C_a & -C_b & C_a+C_b+C \end{bmatrix}$$

The transformations, F_I of equation (3.I7) and F_2 of equation (3.I8) become

$$F_I = \begin{bmatrix} I & 0 & 0 \\ \dfrac{-La}{L_a+L_b+L_c} & I & 0 \\ \dfrac{-L_b}{L_a+L_b+L_c} & 0 & I \end{bmatrix}$$

$$F_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \dfrac{-C_a}{C_a+C_b+C} & \dfrac{C_b}{C_a+C_b+C} & I \end{bmatrix}$$

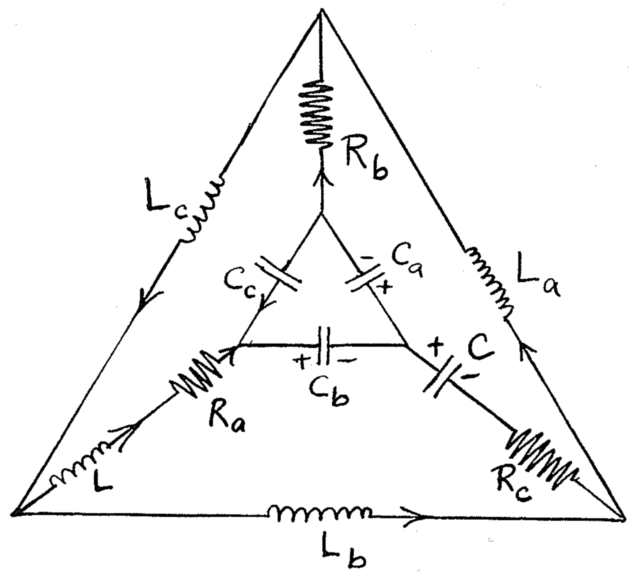
Finally, as in (3.22) we obtain

$$\hat{L} = \begin{bmatrix} \dfrac{L_a(L_b+L_c)}{L_a+L_b+L_c} & \dfrac{-L_aL_b}{L_a+L_b+L_c} \\ \dfrac{-L_aL_b}{L_a+L_b+L_c} & L+\dfrac{L_b(L_a+L_c)}{L_a+L_b+L_c} \end{bmatrix}$$

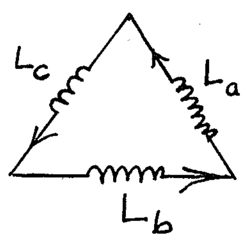
$$\hat{C} = \begin{bmatrix} \dfrac{C_a(C_b+C)}{C_a+C_b+C}+C_c & \dfrac{C_aC_b}{C_a+C_b+C}+C_c \\ \dfrac{C_aC_b}{C_a+C_b+C}+C_c & \dfrac{C_b(C_a+C)}{C_a+C_b+C}+C_c \end{bmatrix}$$

and

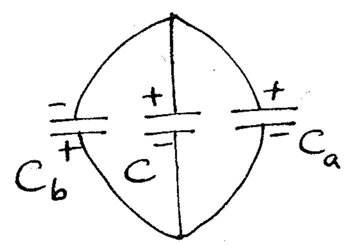
$$H = \begin{bmatrix} R_b+R_c & -R_c & -I & 0 \\ -R_c & R_a+R_c & 0 & -I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} -i_{L_c}+i_{L_a} \\ -i_{L_c}+i_{L_b} \\ v_{C_a}-v_C \\ v_{C_b}+v_C \end{bmatrix}$$



Network



L - subnetwork



C - subnetwork

Fig. I

Conclusion

By inspecting the topology of the network it is possible to obtain transformation matrices which can be applied to eliminate the zero eigenvalues of the A-matrix of the RLC network. The dimension of the A-matrix is consequently reduced by the number of these zero eigenvalues. In a network with many L-loops and C-cut-sets, this reduction in the dimension of the A matrix is correspondingly large, which is of significance in the numerical computations. This reduction is useful both for the determination of all the eigenvalues and for the calculation of the network response by the numerical integration of the state equations.

CHAPTER IV

Conclusion and Recommendation for future work.

The eigenvalues of a network, i.e. the eigenvalues of the A matrix, in the state variable description

$$\dot{x} = Ax \qquad \ldots(4.I)$$

of the unforced system, are important in characterising the transient response of the system. For RLC networks they may lie only in the closed left half complex λ plane. These eigenvalues are also needed in evaluating the response when external sources u are introduced into the network which is then described by the state equation

$$\dot{x} = Ax + Bu \qquad \ldots(4.2)$$

where B is also a matrix like A, but not necessarily of the same order, nor necessarily square. This is because the orders of the state vector x and of the forcing vector u are usually unequal.

It should be clearly pointed out that the networks under study are RLC networks. Thus for a network containing negative resistances or time-varying elements, the matrices H_{II} and H_{22} are not positive definite or semidefinite and the bound on $-\alpha$ given by equation (I.28)

$$\min(m_I, m_2) \leq -\alpha \leq \max(M_I, M_2)$$

is no longer applicable. Such networks may be decomposed by the Murata method [IO].

After describing the significance of the eigenvalues in characterising the network response, the first chapter continued with a description of the topological techniques as used in state variable studies of networks. It was shown that the calculation of the network response involves evaluation of matrix functions, which is complicated by the existence of zero eigenvalues. Thus if the A-matrix has zero eigenvalues, it is singular i.e. det A = 0, and the inverse

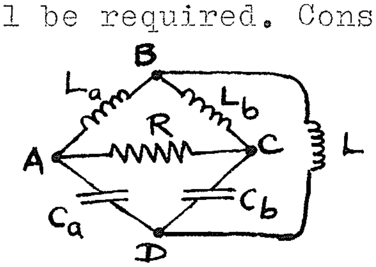
$$A^{-I} = (\text{Adjugate } A)/\det A$$

is undefined. Thus elimination of the zero eigenvalues is desirable. Next all the relevant equations which describe network topology were derived. These equations are used in later chapters, and demonstrate the merit of the topological technique. The chapter closed with some results on non-zero eigenvalues, such as bounds on their real and imaginary parts.

In the second chapter some further expressions for the real part α and the imaginary part β of a typical eigenvalue λ, were given explicitly in terms of the partitioned state variable x (partitioned into inductive link currents and capacitive twig voltages) and the matrices of element values and of topological relationships. Consequences of the reality of an eigenvalue, its pure imaginariness, and its being zero were derived and some topological interpretations were given.

In the third chapter a topological scheme for the elimination of zero eigenvalues of the A-matrix was described. The method is in the form of topological rules which enable one to obtain, by inspection of the graph of the network, elementary transformation matrices which reduce Λ and H to $\hat{\Lambda}$ and $\hat{H}$, where the zero eigenvalues of $A = -\Lambda^{-1}H$ have been eliminated in $\hat{A} = -\hat{\Lambda}^{-1}\hat{H}$.

The conditions under which zero eigenvalues occur, their number, and their elimination have been studied. It would be similarly of interest to determine like information for pure imaginary and for real eigenvalues. Their elimination would then enable us to determine the complex eigenvalues, and the reduction in order involved would imply immense reduction in numerical effort. It is quite possible that novel graph theoretical techniques will be required. Consider the network below



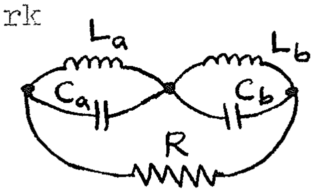
for which $L_aC_a=L_bC_b$ and which has two pure imaginary and two complex eigenvalues. The eigenequation is

$$p^4RC_aC_b(L_aL_b+LL_a+LL_b)+p^3(C_a+C_b)(L_aL_b+LL_a+LL_b)$$
$$+p^2R(C_a(L+L_a)+C_b(L+L_b))+p(L_a+L_b)+R=0.$$

Here we have a network with two pure imaginary eigenvalues, and a topology and an exact equation relating its element values, necessary to assure us of the pair of pure imaginary eigenvalues. An attempt was made to relate this exact equation, $L_aC_a=L_bC_b$, to the topological matrices and graph theoretic entities already in general use in electrical network theory. Under the condition $L_aC_a=L_bC_b$, it was observed that the above quartic eigenequation factors into two eigen-equations, firstly,

$$d_I(p)=p^2RL_aC_a+p(L_a+L_b)+R=0$$

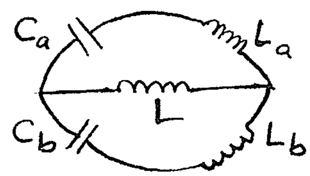
which is the eigenequation for the network below, which is obtainable by short circuiting nodes B, D of the original network



and secondly

$$d_{II}(p)=p^2(L_aC_a+LC_a+LC_b)+1=0$$

which is the eigenequation for the resistanceless network below, which is obtainable by open cicuiting the branch AC in the original network.



This shows that with certain topology, and certain exact relationships between element values, we can obtain a resistanceless subnetwork which will possess the pure imaginary eigenvalues. This was in fact observed by the author for other networks with different topology and a set of equations relating element values. More research is needed to arrive at general conditions for pure imaginary eigenvalues, and for purely real ones.

A STUDY OF THE EIGENVALUES OF
ELECTRICAL NETWORKS

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