LINEAR PROGRAMMING TECHNIQUES IN PROBLEMS OF TRANSPORTATION

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PREFACE

The purpose of this preface is to give me an opportunity to express my gratitude to those who made this thesis possible. In chronological order, as well as in order of magnitude, my deep appreciation goes to Dr. Stephen George Triantis of the Department of Political Economy of the University of Toronto, without whose invaluable help my studies would not have been possible, Dr. Clarence L. Barber, Head of the Department of Economics, Professor Ralph F. Harris, Graduate Studies Adviser of same, and Professor P. S. Dhruvarajan, for their recommendations which made it possible for me to receive financial help for continuation of my studies, and to Dr. P. S. Dhruvarajan, Professor of Economics and supervisor of this thesis, whose help was instrumental in its preparation and extended beyond pure academic matters. To all the above, I am deeply grateful.

CONTENTS

CHAPTER I

INTRODUCTION

Linear programming methods are today an accepted and widely used technique of analysis in the field of economics. Generalization of such methods towards what T. C. Koopmans calls "Activity Analysis" has led to the realization of the connections that exist between linear programming and the rest of standard economic theory. Another generalization proceeded towards spatial models and yielded a series of complex interregional systems capable of practical applications to questions referring to multiregional economies. These latter models derive directly as generalizations of a most abstract scheme developed by T. C. Koopmans in [11], Ch. 3, and [12]. The mathematics of [11] is quite complicated while in [12] Koopmans makes an attempt at better communication with general economists. Up to now, however, very few attempts have been made at communication as regards the complex interregional systems developed out of Koopmans' general scheme, except in Isard [9].

The aim in this thesis is threefold: (a) to show how the complex interregional linear programming models can be derived from the simpler linear programming models and thus help in the understanding of these more complex systems, (b) to bring out, in the course of this integration, the economic interpretation of these models and relate them to standard notions of economic theory, and (c) to stress the formal similarity of these models by adopting a specific method of exposition that brings out that similarity and substantiates the contention that the abstract framework of Koopmans forms their basis. This helps those unacquainted with the literature on interregional linear programming carry over their knowledge of spaceless linear programming analysis in studying the literature.

Interregional linear programming considers the transportation sector explicitly, as was to be expected. From various linear programming models on this subject, I have chosen for consideration in this thesis a family of models that lend themselves to practical application and are at the same time quite general.

Chapter II of this thesis contains a simple spaceless linear programming model, an elementary exposition of the economic interpretation of the model and its method of solution which bring out the way in which the problem of economic choice is handled by the method. An attempt at relating the results of the technique to economic equilibrium is also made, and some differences from neo-classical analysis are pointed out. The "raison d'etre" of this chapter is twofold: to provide a simple basis for exposition and economic interpretation of linear programming and to set out a simple model (Model I) which constitutes the basis of a much more complicated model in Chapter V.

Chapter III examines the well known transportation problem in linear programming as well as a variation of it which yields to generalization towards the general model of Chapter IV. The dual to the transportation model is also examined and the connection of the model to spatial equilibrium and of the dual prices to equilibrium prices is examined.

Chapter IV proceeds to an account of the way in which the simple transportation model can be generalized and culminates in an interregional linear programming model by A. Hurter [7].

Chapter V generalizes the simple model of Chapter II to another interregional model by M. Harwitz [7].

Finally, Chapter VI is devoted to "formal comparison" of the models in Chapter IV and V to one another and to other models by W. Isard [9], B. Stevens [21] and L. Moses [16]. An evaluation and some conclusions end the Chapter.

I have attempted in this thesis to integrate the relevant literature by bringing its similarities to the fore. In some ways this has forced me to approach the models from a novel angle, and in some places I have run into unsettled questions. When I did, I attempted an explanation but I did not always succeed. I would very much hesitate to use the word "original" to indicate these efforts, since I believe that originality constitutes much more than what I have done. In footnotes, then, I have indicated when a passage in this thesis is a product of my efforts to understand the literature, by using the word "novel" - for lack of a more appropriate one.

CHAPTER II

LINEAR PROGRAMMING

The Chapter will be devoted to an elementary exposition of the technique of linear programming. As stated in the introduction, the general approach will consist of an attempt to interpret the mathematical features of this method in economic terms.

Section 2.0 sets out the mathematical structure of the general linear programming problem in a very elementary form. Pure mathematical questions connected with the method are not examined.

Section 2.1 gives a general economic interpretation of the linear programming problem. This procedure helps, by bringing out the general features of the method, to determine general situations where the method is applicable.

Section 2.2 is concerned with a specific economic example of the method and explains duality and efficiency in economic terms.

Finally, section 2.3 elaborates on some special characteristics of linear programming with regard to employment of resources.

2.0 THE MATHEMATICAL STRUCTURE OF LINEAR PROGRAMMING

Linear programming is basically a mathematical technique pertaining to the maximization or minimization of a linear function (called the "objective function") subject to a number of constraints in the form of linear inequalities or equalities. In this section I will only state the problem in its mathematical form and mention some mathematical questions connected with it, together with references for those interested in pursuing these matters further. In fact, the "raison d'etre" of this section is to provide a framework for the general economic interpretation in 2.1.

2.0.0. Maximization and Minimization: Dual Linear Programming Problems

A basic mathematical feature of the L.P.[1] technique is that of "duality". This means that to each maximization problem there corresponds a minimization problem and vice versa. The two problems are usually called "dual linear programming problems" or "dual linear programs". Generally, the problem one starts with is called the "primal", whether it is the maximization or the minimization problem. The corresponding problem is then called its "dual". In any case, the terms "primal" or "dual" should not be specifically associated with maximization or minimization: the primal problem may be either a maximization or a minimization problem. The dual will then be the opposite of the primal.

It should perhaps be noted that the dual programs are connected through their parameters. The parameters of the primal enter the dual, though in a rearranged form. This is made clearer in the next two subsections, and bears an important economic interpretation as will be seen later.[2]

2.0.1. The Primal: Mathematical Formulation

Let $f(x_1, x_2, \ldots x_n)$ be a linear function in n variables, which is to be maximized.

$f(x_1, \ldots x_n)$ can be written in any of the following alternative forms:

$$f(x_1 \ldots x_n) = p_1 x_1 + p_2 x_2 + \ldots p_n x_n \tag{1a}$$

$$f(x_1 \ldots x_n) = \sum_{i=1}^{n} p_i x_i \tag{1b}$$

$$f(x_1 \ldots x_n) = \left[p_1, p_2 \ldots p_n\right] \left[x_1, x_2 \ldots x_n\right]' \tag{1c}$$

$$f(x_1, \ldots x_n) = px \tag{1d}$$

In (1d) p is a row vector and x a column vector.

The linear inequality constraints may similarly be written in any of the following forms:

$$\begin{matrix} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leqslant b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leqslant b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leqslant b_m \end{matrix} \tag{2a}$$

or

$$\sum_{k=1}^{n} a_{ik} x_k \leqslant b_i \qquad (i = 1, \ldots m) \tag{2b}$$

or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} \leqslant \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix} \tag{2c}$$

or

$$Ax \leqslant b \qquad (2d)$$

In (2d), A is the matrix (a_{ik}) of (2c), x the column vector of x_i's and b the column vector of b_i's.

An additional constraint present in linear programming is that variables are not permitted to take up negative values. This is usually called the "non-negativity" condition, and is stated as follows:

$$x_i \geqslant 0 \quad (3a, b, c) \quad (i = 1, \ldots n)$$

or

$$x \geqslant 0 \quad (3d)$$

where x is the column vector of x_i's and 0 is the nxl null vector.

A typical maximization problem in L.P. is then,

maximize (1)

subject to

(2) and (3)

or, in the matrix notation,

$$\text{maximize } f(x_1, x_2 \ldots x_n) = px \qquad (1d)$$

$$\text{subject to} \quad Ax \leqslant b \qquad (2d)$$

$$\text{and} \quad x \geqslant 0 \qquad (3d)$$

2.0.2. <u>The Dual: Mathematical Formulation</u>

The dual corresponding to the problem in 2.0.1 is a minimization problem, stated as follows:

Minimize

$$g(y_1, y_2, \ldots y_m) = b_1y_1 + b_2y_2 + \ldots + b_my_m \qquad (4a)$$

or $$g(y_1, \dots y_m) = \sum_{i=1}^{m} b_i y_i \qquad (4b)$$

or $$g(y_1, \dots y_m) = [b_1, b_2 \dots b_m][y_1, y_2 \dots y_m]' \qquad (4c)$$

or $$g(y_1, \dots y_m) = b'y \qquad (4d)$$

Subject to

$$\begin{aligned} a_{11} y_1 + a_{21} y_2 + \dots + a_{m1} y_m &\geqslant p_1 \\ a_{12} y_1 + a_{22} y_2 + \dots + a_{m2} y_m &\geqslant p_2 \\ a_{1n} y_1 + a_{2n} y_2 + \dots + a_{mn} y_m &\geqslant p_n \end{aligned} \qquad (5a)$$

or

$$\sum_{k=1}^{m} a_{ki} y_k \geqslant p_i \qquad (i = 1 \dots n) \qquad (5b)$$

or

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \\ y_m \end{bmatrix} \geqslant \begin{bmatrix} p_1 \\ p_2 \\ \\ p_n \end{bmatrix} \qquad (5c)$$

or $$A' y \geqslant p' \qquad (5d)$$

and to the additional requirement that

$$y_i \geqslant 0 \qquad (i = 1, \dots m) \qquad (6\ a, b, c)$$

or

$$y \geqslant 0 \qquad (6d)$$

In (4d) b is the column vector of b_i's, as in (2d). In (5d) A is the matrix of a_{ik}, as in (2d), p is the row vector of p_j's, as in (1d). The transpose of a vector or matrix is denoted by a prime.

2.0.3. Formal Relations Between Dual Linear Programs

The relationship alleged in 2.0.0 between the primal and the dual problems now becomes apparent. Consider the two problems in their most compact notation:

PRIMAL		DUAL	
Maximize	px	Minimize	$b'y$
Subject to	$Ax \leqq b$	Subject to	$A'y \geqq p'$
and	$x \geqq 0$	and	$y \geqq 0$

First, the parameters involved in the two problems are the same.

Second, the symmetry in the transposition of parameters from one problem to the other is easily noticeable: (a) the vector p of the objective function in the primal has been transposed and transferred to the right-hand side of the constraints in the dual, (b) the vector b on the right-hand side of the constraints in the primal has been transposed and transferred to the objective function in the dual and, (c) the matrix A in the primal has been transposed in the dual.

Third, it should be noted that the number of constraints in the primal (including the non-negativity constraints) is the same as in the dual, i.e. $m+n$. If the non-negativity conditions are not taken into account, it turns out that the number of variables in the primal (n) becomes the number of constraints in the dual, and the number of constraints in the primal (m) becomes the number of variables in the dual.

An important property of the dual linear programs is that the maximum value of $f(x_1 \ldots. x_n)$ is equal to the minimum value of $g(y_1, \ldots y_m)$. Also, if a solution to the maximum (or minimum) problem exists, its dual also has a solution.[3]

These formal symmetries should not, however, give the impression that one of the dual programs is redundant; while it is true that $\max f\ (x_1, .. x_n) = \min g\ (y_1, .. y_m)$, the variables x_i and y_k bear distinct economic interpretations and are both useful in terms of understanding the basic notions in the theory of economic choice. This will become apparent in sections 2.2 and 2.3 below.

2.1 L.P. AND ECONOMIC CHOICE: A GENERAL INTERPRETATION

In (2c) of 2.0.1, each column of the mxn matrix may be generally interpreted to represent an economic "activity". The notion need not be restricted to any particular type of economic activity; it is quite general and may be used to represent an activity in production, in consumption, in transportation etc. This explains the variety of uses of the linear programming technique, from very specific and practical questions to quite general problems.

The (constant) numbers in each column of the matrix usually indicate what the respective activity needs to operate at a unit level; this is not a general interpretation, however, since sometimes the "results" (as contrasted to the "needs") of the activity are recorded in its column along with its "needs". Differentiation between the two is achieved through a sign convention (e.g. minus signs for the "needs" and plus signs for the "results").

A word about the term "unit level of activity" used in the previous paragraph is in order: this is defined with reference to one of the (possibly many) "results" of the activity by specifying a "unit" of it for the purpose, and then calculating what the activity "needs" to produce this "unit". Consider, for example, the activity of producing seed oil. Choose seed oil as the result of the activity in terms of which the unit level of activity is to be defined; choose an appropriate unit for the result, e.g., tons. The "needs" of the activity (i.e., some of the numbers, a_{ik} in the relevant column of A will now be calculated with reference to one ton of seed oil. Any other "results" of the activity besides seed oil will also be recorded along with the "needs" of the activity (only with a different sign).

Concentrating on the "needs" of each activity and turning to (2a) of 2.0.1, the meaning of the left-hand side of each inequality becomes clear: a_{11} is what must be used of "source" 1 by activity 1 in order that the latter operate at its unit level. Taking x_i as the level of operation of the particular activity i, it is easily seen that $a_{11}\ x_1$ represents the total "needs" of activity 1 for "source" 1 when the activity is operated at the level x_1. A similar interpretation holds for $a_{12}\, x_2$, except that it refers to activity 2. The left-hand side of the first equation therefore represents the sum total of "needs" of all activities from "source" 1, when they operate at levels $x_i (i=1,\ldots n)$ respectively. This interpretation carries over to any relation of system (2a).

To be sure, unless the "sources" are defined to be "primary resources", i.e., non-producible by any activity, the above interpretation is not so general. Suppose, for example, that "source" 3 is producible by one or more activities, say activities 2 and 4. Then, a_{32} and a_{34} are unit "results" of the relevant activities and the left-hand side of equation 3 of system (2a) of 2.0.1 must be separated in two parts: $a_{32}\, x_2 + a_{34}\, x_4$, which is the "contribution" of activities 2 and 4 to the availability of "source" 3; and the rest which is the sum total of "needs" of all activities from "source" 3. Obviously some kind of sign convention is required to distinguish a_{32} and a_{34} from all other a_{3k} $(k = 1, \ldots .\, n,\ k \neq 2\ \ k \neq 4)$.

The interpretation of the right-hand side of the equations of system(2a) of 2.0.1 is, of course, the availability limits of the "sources". Thus, b_3 is the available amount of "source" 3 (not taking into account what can be contributed by activities if "source" 3 is reproducible), and the same holds for every b_k $(k = 1, \ldots\, m)$. Feasibility of whatever is pursued is imposed by the restriction that the sum total of the "needs" of all activities from each "source" should not exceed the availability limits of the "source".

Turning to the objective function (1 of 2.0.1) the vector $p = [p_1, p_2, \dots p_n]$ represents the unit contribution of each activity to a quantitatively expressible objective. For example, p_2 represents the contribution of activity 2 to this aim, when the activity is operated at unit level. The purpose, of course, is to maximize the total contribution of the activities, subject to the constraints of "source" availability and to the additional constraint that the maximizing activity levels (x_i) must not be negative.

The non-triviality of this problem is assured when three conditions are satisfied: (a) the availability limits of at least two sources are finite, (b) the various activities compete between themselves for the use of limited "sources", and (c) no activity can be judged outright as technically superior to all others. An explanation follows.

First, suppose that the availability limits of all "sources" are infinite, i.e., sources are available in whatever amount needed. Since the objective function is linear (i.e., since the unit contribution of each activity to the aim remains constant irrespective of the level of operation of the activity) the value of the function varies directly with the levels of activities. But since the availability limits of all sources are infinite, any and all activities can be operated at any level. Hence, the value of the objective function can be made as large as desired,[4] and there is an infinite number of ways in which this can be done (e.g., by using any one activity, provided its contribution to the aim is positive). In this case, the economic problem is trivial or, rather, there exists no economic problem at all.[5]

As a second case, assume that the availability limits of all "sources" are infinite except one, say the fifth. In the system (2a) of 2.0.1 the only relevant constraint is then the fifth inequality namely,

$$a_{51} x_1 + a_{52} x_2 + \dots + a_{5n} x_n \leq b_5$$

together, of course, with the non-negativity conditions (3).

This problem is not formally trivial, but it is indeed very simple to solve. Consider the ratios:

$$\frac{p_1}{a_{51}}, \frac{p_2}{a_{52}}, \dots\dots\dots \frac{p_n}{a_{5n}}$$

The economic meaning of $\frac{p_i}{a_{5i}}$ is that it is the "return" of the ith activity per unit of the "source" used. It is the contribution of the ith activity to the aim, per unit of the "source" used by it. Obviously, the objective function will be maximized by using the activity with the highest "return per unit of source" at the maximum possible level.[6]

To illustrate case (b) above, consider the problem:

Maximize $p_1 x_1 + p_2 x_2 + p_3 x_3 = f(x)$

subject to

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \\ 0 & a_{42} & 0 \\ 0 & 0 & a_{53} \\ 0 & 0 & a_{63} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix}$$

In this example, the availability levels of "sources" are finite, but the activities do not compete for their use. In other words, what is needed by one activity is not needed by any other, and this simplifies the problem immensely. Assuming that all p's are positive (i.e., that all activities have something to contribute to the aim) the solution of the problem is obvious: for activity 1, check the ratios $\frac{b_1}{a_{11}}$ and $\frac{b_2}{a_{21}}$ and choose the smaller;[7] this will be the level that helps maximize f(x), subject to the constraints set by the availability limits b_1 and b_2. The same procedure is to be followed for each activity.

In this last case, the various activities do not compete for the same scarce "sources", neither does one produce something needed by another activity. Hence, they are independent in the sense that the level of operation of each does not depend on the levels of others.[8]

To show the need for condition (c), consider the following technology:

$$\begin{array}{l} \\ \text{Source 1} \\ \text{Source 2} \\ \text{Result} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Act. 1} & \text{Act. 2} & \text{Act. 3} \end{array} \\ \begin{bmatrix} -5 & -20 & -5 \\ -10 & -30 & -5 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

The result is the same in all activities (e.g. a certain commodity). Obviously, activity 3 is "technically" superior to the other two since it requires less of both "sources" to yield the same result. Choice is restricted from the start to activity 3.

To summarize the above, then, it can be said that the L.P. technique is suitable for tackling problems of choice between interdependent activities. Problems of choice with not obvious solutions arise in cases where there exist restrictions and where the elements subject to choice are interdependent, and, of course, where there exists more than one way of satisfying the exogenously given aim.[9]

It cannot be overstressed that a large category of allocation problems in economics are essentially problems of choice as described above. Linear programming can thus be a very useful technique in tackling these problems.

2.2 LINEAR PROGRAMMING: AN ECONOMIC EXAMPLE

In this section I shall make an attempt at presenting a simplified economic problem in linear programming form and at explaining the mathematical features of the problem in specific economic terms. Both the primal and the dual problems will be examined.

2.2.0 The Primal

2.2.0.0. The Setup and the Assumptions

Suppose a closed economy with the following characteristics:

(i) There are two "resources" or "primary factors of production". "Resources" are defined to be commodities available in nature and not reproducible by any activity, at least within the time-span considered in the problem.

(ii) The technology available to this economy specifies three "activities" of production. An "activity" is here defined as a combination of qualitatively defined commodities (in our case, resources) in fixed quantitative ratios, as inputs, to produce "final" commodities in fixed quantitative ratios to the inputs. A "final" commodity is one not available in nature but desired in itself (presumably by consumers, but we shall be abstracting from the demand side of the problem in this example).

To clear the above definition of an activity, consider the following example:

$$\begin{array}{ll} \text{Resource 1} & \\ \text{" \quad 2} & \\ \text{Final commodity 1} & \\ \text{Final commodity 2} & \end{array} \begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \end{bmatrix}$$

This activity uses a combination of 5 units of (qualitatively defined, that is, homogeneous) resource 1 and 6 units of (also homogeneous)[10] resource 2 to produce 1 and 0.5 units of final commodities 1 and 2. From the definition, the quantitative ratio of inputs ($\frac{5}{6}$ or $\frac{6}{5}$) is given and constant. This obviously means that <u>within this specific activity</u> there is no possibility of primary factor substitution. In other words, an activity as defined represents one point on the production isoquant of neo-classical economics.[11]

The requirement that the final commodities produced by the activity bear constant quantitative ratios to the inputs is another way of stating the assumption of constant returns to scale.

(iii) Each of the activities given by the existing technology can (not taking into account the restrictions imposed by the limited availability of the resources) be expanded or reduced to any level of operation.

This is the assumption of divisibility: it implies that given the activity

$$\begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \end{bmatrix}$$

of the previous example, any level of it is possible, e.g.,

$$0.1 \begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.6 \\ 0.1 \\ 0.05 \end{bmatrix}$$ is possible, and so is

$$10 \begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -50 \\ -60 \\ 10 \\ 5 \end{bmatrix}$$

As can be seen from the coefficients of inputs and outputs of the resultant vectors, this assumption means that resources as well as final commodities are perfectly divisible.

(iv) There is no interaction between the activities given by the technology. This means that the unit resource "needs" and the unit productive "results" of each activity are not affected by the fact that it is operated along with others. External economies and diseconomies, that is, are assumed away. In mathematical terms, given the activities

$$\begin{array}{cc} \text{Act. 1} & \text{Act. 2} \\ \begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \\ 0 \end{bmatrix} & \begin{bmatrix} -10 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \end{array}$$

one can perform operations like

$$2 \begin{bmatrix} -5 \\ -6 \\ 1 \\ 0.5 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -10 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -40 \\ -18 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

The resultant vector shows the composite needs and productive results of activities 1 and 2 when they are operated at levels 2 and 3 respectively.

(v) The prices of the final commodities are given and constant.

(vi) Intermediate commodities have been "netted out" (the relevant technique is in 4.2, below).

(vii) The purpose is to maximize the sales receipts.

2.2.0.1. <u>The Data</u>

(i) The quantities of resources available are as follows:

$$\begin{array}{l} \text{Resource 1} \\ \text{Resource 2} \end{array} \begin{bmatrix} 60 \\ 18 \end{bmatrix} \quad \text{measured in appropriate units}$$

(ii) Suppose that the technology is as follows:

$$\begin{array}{ll} & \begin{array}{ccc} \text{Act. 1} & \text{Act.2} & \text{Act. 3} \end{array} \\ \begin{array}{l} \text{Resource 1} \\ \text{Resource 2} \\ \text{Final Comm. 1} \\ \text{" \quad " \quad 2} \\ \text{" \quad " \quad 3} \end{array} & \begin{bmatrix} -10 & -24 & -21 \\ -4 & -6 & -3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

with resources measured in the same units as above, and final commodities measured in appropriate units

where each column represents one activity.

(iii) The prices of final commodities are

$$\begin{array}{l} \text{final commodity 1} \\ \text{" \quad " \quad 2} \\ \text{" \quad " \quad 3} \end{array} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

expressed in \$ per unit of commodity

Since the aim is to maximize sales receipts, the commodities can be taken out of the matrix by adopting as unit level for each activity that level which gives \$1 of revenue. The activities, as given in (ii), result in \$2, \$4, and \$3 per unit of operation, respectively. Division of the columns by 2, 2, and 3 respectively, gives what we want:

$$\begin{bmatrix} -5 & -6 & -7 \\ -2 & -1.5 & -1 \\ 0.5 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & \end{bmatrix}$$

It can be easily verified that each activity now gives \$1 of revenue when operated at this "new" unit level given by the matrix. We can thus dispose of the commodities and write the technological matrix as follows:[12]

$$\begin{array}{ll} \text{Resource 1} & \multirow{2}{*}{$\begin{bmatrix} 5 & 6 & 7 \\ 2 & 1.5 & 1 \end{bmatrix}$} \\ \text{Resource 2} & \end{array}$$

2.2.0.2. Formulation of the Problem: Model I

Denote by x_1, x_2, x_3 the activity levels (in dollar values) that maximize total receipts: since at the unit level each activity contributes \$1 to revenue, we seek to maximize

$$R = 1x_1 + 1x_2 + 1x_3$$

On the other hand, when the activities are operated at levels x_1, x_2, x_3, they absorb resources. We have:

total amount absorbed of resource 1 is $5x_1 + 6x_2 + 7x_3$

and the same as above of resource 2 is $2x_1 + 1.5x_2 + 1x_3$

and these amounts should not exceed the availability limits of resources as given by (i) of 2.2.0.1, i.e.,

$$5x_1 + 6x_2 + 7x_3 \leqslant 60$$
$$2x_1 + 1.5x_2 + 1x_3 \leqslant 18$$

As a last requirement, negative levels of operation of the activities have no economic meaning, i.e., we must have

$$x_1 \geqslant 0 \qquad x_2 \geqslant 0 \qquad x_3 \geqslant 0$$

To summarize, the problem, is:

$$\text{Maximize } R = \sum_i^3 x_i = [1 \quad 1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (1c)$$

subject to

$$\begin{bmatrix} 5 & 6 & 7 \\ 2 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leqslant \begin{bmatrix} 60 \\ 18 \end{bmatrix} \qquad (2c)$$

and to

$$x_i \geqslant 0 \qquad (i = 1,2,3) \qquad (3c)$$

This can be seen to be a standard linear programming problem, by comparison with what has been said in 2.0.1. Also, comparison with the examples set out in section 2.1 shows that the problem is neither trivial nor "easy" to solve, in the sense that (a) constraints exist (b) activities are interdependent in their use of limited resources and,

(c) no activity can be judged outright as technically superior to all the rest.

2.2.0.3. Slack Variables

It would be easier, for computational reasons, if the system (2c) of the previous section were one of equations rather than inequalities. This can easily be done by introducing additional variables (called "slack" variables) to the inequalities, as follows:

The first inequality of (2c) is

$$5x_1 + 6x_2 + 7x_3 \leqslant 60$$

by adding up a positive variable x_4 to the left-hand side

$$5x_1 + 6x_2 + 7x_3 + 1x_4 = 60$$

we have an equality. The economic meaning of the variable x_4 must be obvious; since the sum $5x_1 + 6x_2 + 7x_3$ represents the total amount of resource used by the three activities when they operate at the levels x_1, x_2 and x_3 respectively, the expression $1x_4$ represents the amount of the resource left unused. The amount used and that which remains idle must (by the definition of the unused amount) be equal to the amount available of the resource. The fact that an additional restriction of non-negativity is imposed on x_4 ($x_4 \geqslant 0$) ensures that the original inequality is satisfied when the corresponding equality is, and vice versa.

To elaborate on this last point, suppose that x_4 were permitted to take negative values. The result would be that x_4 could artificially augment the amount of the resource available, and the original inequality constraint would not bear a one-to-one correspondence with the derived equality constraint. For example, we could have $x_1 = 100$ $x_2 = 100$ $x_3 = 100$ and $x_4 = -1740$. This set of values would satisfy the equality constraint, but the values of x_1, x_2, x_3 would not satisfy the original inequality, which is the original meaningful constraint imposed by the data. Introduction of the slack variable x_4, that is, without the additional non-negativity restriction would <u>change</u> the problem, which was not intended.[13]

The same procedure can be followed with regard to the second inequality of 2c. We then get,

$$5x_1 + 6x_2 + 7x_3 + 1x_4 + 0x_5 = 60$$
$$2x_1 + 1.5x_2 + 1x_3 + 0x_4 + 1x_5 = 18$$

Obviously, we must use a separate slack variable for the second inequality, as there is no reason to impose the additional restriction that the amounts unused of each resource must be equal.

The system of (2c) and (3c) of 2.2.0.2 can then be written:

$$\begin{bmatrix} 5 & 6 & 7 & 1 & 0 \\ 2 & 1.5 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 60 \\ 18 \end{bmatrix} \qquad (2d)$$

$$x_i \geq 0 \qquad (i = 1, \ldots 5) \qquad (3d)$$

and it must be borne in mind that this system is equivalent to the system (2c),(3c) in the sense that a set of values of x_i that satisfy (2c), (3c) satisfy (2d), (3d) also, and vice versa. In economic terms, the nature of the constraints is not affected by the introduction of the slack (or surplus) variables.[14]

The usual assumption about the contribution of the slack variables to the objective function of the problem is that they do not contribute anything. The objective function (1c) is then written

$$R = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1, & x_2, & x_3, & x_4, & x_5 \end{bmatrix}' \qquad (1d)$$

which is the same function. We are thus assured that the system (1d), (2d), (3d) is equivalent to the original (1c), (2c), (3c).

As stated at the beginning of this section, the introduction of the slack variables was meant to simplify the problem from the computational point of view. For certain cases of theoretical analysis, however, slack variables acquire a quite different and important standing: it will be recalled that $1x_4$ represents the amount of resource 1 not used by any activity. The assumption is that the existence of this unused amount does not modify the objective function. This implies that the amount unused can be disposed of freely, that is, without requiring inputs for its disposal. In another, more theoretical, problem it might be desirable to investigate this situation without making the "free disposal" assumption. Naturally, in this case it is the system (1d) (2d) (3d) (with (1d) modified accordingly) which is the original problem.[15]

2.2.0.4 Feasible, Basic Feasible, and Optimum Solutions

Our aim in the problem of this section is to maximise the sales receipts subject to the constraints imposed by resource availability. A "solution" is defined as a set of values of the variables of the problem, x_1, x_2, x_3, x_4, x_5. E.g., a "solution" is $x_1 = 100$ $x_2 = 0$ $x_3 = 0$ $x_4 = 440$ $x_5 = -182$. In this "solution", the objective function takes the value 100. It will be observed, however, that though this set of values of the variables satisfies the constraints (2d), it violates the non-negativity conditions on x_4 and x_5. Hence, the first distinction with reference to solutions:

A "feasible" solution is a set of values of the variables (a set of activity levels) that satisfies the constraints (2d) and the non-negativity conditions (3d), if such a solution exists.[16]

The number of unknowns is usually greater than the number of equation constraints in a linear programming problem.[17] This means that, if there exists a solution, there will usually exist more than one solutions[18] to the system (2d). The same will usually hold for feasible solutions: if there exists one there will exist more than one.

In the specific problem of this section, for example, one can see that the sets of activity levels below constitute some of the feasible solutions (indeed, the number of feasible solutions is infinite).

Table 1: Some Feasible Solutions of (2d)

	Variable	x_1	x_2	x_3	x_4	x_5	f (x)
Solution	(a)	0	0	0	60	18	0
	(b)	9	0	0	15	0	9
	(c)	$7.\frac{1}{3}$	0	$3.\frac{1}{3}$	0	0	$10\frac{2}{3}$
	(d)	1	1	1	42	13.5	3
	(e)	2	1	5	9	7.5	8
	(f)	3	1	5	4	5.5	9

(continued)

	Variable	x_1	x_2	x_3	x_4	x_5	f(x)
Solution	(g)	4	1	5	15	3.5	10
	(h)	5	0	5	0	7	10
	(i)	4.5	0	0	37.5	9	4.5
	(j)	0	10	0	0	3	10
	(k)	0	0	$\frac{60}{7}$	0	$18-\frac{60}{7}$	8.000

The question immediately arises, then: how are we to proceed in finding, among the infinite number of feasible solutions available, one which maximizes the objective function?

A number of mathematical theorems eliminate the difficulty. These theorems assure us that in order to find an "optimum" solution (i.e., a feasible solution which maximizes the objective function) we need only examine the "basic feasible solutions". Since the number of basic feasible solutions is finite, the difficulty is surpassed.[19]

A "basic feasible solution" is a feasible solution in which the number of variables that can be different from zero is equal to the number of equations.[20] It can be seen that solutions (a), (b), (c), (j), (k) are basic feasible solutions, in Table 1.

An iterative procedure has been devised which proceeds from one basic feasible solution to another and, with the help of a choice criterion, determines an optimum solution (it should be borne in mind that this optimum solution is not necessarily unique; other basic - and non basic - feasible solutions may be optimum solutions, too).[21] The method is called the "simplex method". In accordance with the aim pursued in this thesis, I shall investigate only the economics of the simplex method in the next section.

2.2.0.5. Solution of the Problem: The Economics of the Simplex Method

From 2.2.4., our problem was:

To maximize sales receipts

$$R = f(x) = x_1 + x_2 + x_3 \qquad (1)$$

subject to the constraints

$$5x_1 + 6x_2 + 7x_3 + x_4 = 60 \qquad (2)$$

$$2x_1 + 1.5x_2 + 1x_3 + x_5 = 18 \qquad (3)$$

and $$x_1, x_2, x_3, x_4, x_5 \geqslant 0 \qquad (4)$$

From the discussion in the previous section we know that we need consider only two activities at a time.

An obvious choice is to produce nothing.[22] This will mean that the levels of activities 1, 2, 3, (x_1, x_2, x_3) are chosen to be zero. On the other hand, the total amounts of resources available will remain idle, i.e., the levels of activities 4 and 5 will be $x_4 = 60$ $x_5 = 18$. Total sales receipts are seen to be zero, since keeping resources idle does not contribute to sales receipts. This is solution (a) in Table 1 of 2.2.0.4.[23]

Since we are to proceed iteratively, we may find another (basic) feasible choice of activity levels, compare the sales receipts in the two choices and keep the choice which yields the greater receipts. Actually, however, the method proceeds in a slightly different way, by using a mathematical criterion which has a direct and very important interpretation for the theory of economic choice. I will proceed to state this criterion by example.

Let us isolate the activity with the highest unit contribution to sales receipts; since all activities have the same unit contribution, we will examine them in turn.

Consider activity 1. Its resource requirements per unit are 5 and 2 respectively and its unit contribution to receipts is $1. The basic economic question which arises is as follows: we can use 5 units of resource 1 and 2 of resource 2 <u>in two alternative ways</u>; either operate activities 4 and 5 at the levels 5 and 2 respectively, or operate activity 1 at the level of unity. If we make the first choice, we have receipts equal to zero. If, however, we choose to use these amounts of resources by operating activity 1 at the level of unity, we have receipts of $1. By sticking to the first choice, that is, we actually <u>lose</u> receipts that we could have had.[24]

The same procedure for activities 2 and 3 shows that by using each one of them at unit level we could increase our receipts by $1 respectively.

A little reflection shows that the question posed above lies at the heart of the allocation problem in economics. I will elaborate on this point in section 2.2.1.

Since there is no difference between the unit gains of activities 1, 2, 3, we can choose activity 1 to operate in the program. We could of course choose to operate activity 1 at a level that permits both activities 4 and 5 to operate at some positive level (as in solution (i) of Table 1 of 2.2.0.4). But this is to be avoided for two reasons: first, we will economize in computation time by not considering three-variable solutions (i.e., non-basic feasible solutions) and, second, it would pay to operate activity 1 at the highest possible level (since its unit contribution to the sales receipts is constant).

To find the highest possible level at which activity 1 can be operated we proceed as follows: we know that activity 1 operated at unit level is equivalent - in terms of resource use - to the sum of activities 4 at level five and 5 at level two. Since we are to consider two variable (i.e. basic feasible) solutions only, we seek to find the levels of activity 1 that will make it impossible for activities 4 and 5 to be operated at positive levels.

Now, in the first basic solution the activity levels were as follows: $x_4 = 60, x_5 = 18$. Since activity 1 at unit level is equivalent to $x_4 = 5$ and $x_5 = 2$, it is obvious that if we set $x_1 = 9$ we will have to have $x_5 = 0$.[25] Also, if we set $x_1 = 12$ we will have $x_4 = 0$. The choice is now between $x_1 = 9$ and $x_2 = 12$. A little reflection shows that the least of the two is the maximum possible level of x_1.[26] We have thus determined that $x_1 = 9$ must be substituted for x_5 in the next basic solution. The second basic solution, that is, contains $x_1 = 9$ and x_4 to be determined.

A look at the resource needs of activity 1 at the level 9 shows that only 15 units of resource 1 are left. Hence $x_4 = 15$. Our second basic feasible solution is, then, solution (b) of Table 1, i.e., $x_1 = 9$, $x_4 = 15$, $x_2 = x_3 = x_5 = 0$. The total revenue from activity 1 will be \$9 and that of activity 4 \$0. Hence the value of f(x) will be \$9.[27]

The question is whether this solution gives the maximum revenue possible. To answer it, we must go through the same procedure as with the first basic feasible solution.

Let us start with activity x_2: it needs 6 and 1.5 units of resources 1 and 2 respectively, and gives \$1 of revenue (all these, per unit level of operation). The same total amount of resources is needed by activities 1 and 4 if they are operated at levels $x_1 = 0.75$ and $x_4 = 2.25$. This can be easily checked:

$$0.75 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 2.25 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \end{bmatrix}$$, the unit requirements of activity 2.

We have, again, two alternatives: either use 6 units of resource 1 and 1.5 units of resource 2 by operating activities 1 and 4 at levels 0.75 and 2.25 respectively, or use those same amounts of resources by operating activity 2 at the level of unity. The receipts associated with the first choice are \$0.75 (since only activity 1 contributes to revenue), while those associated with the second choice are \$1. Between these two alternative uses of the same bundle of resources, it is the second which contributes more to the given purpose (i.e., to revenue maximisation). By sticking to the first, then, we lose \$0.75 - \$1 = \$0.25 per unit of non-operation of activity 2.

By the same reasoning we find that we can use 7 units of resource 1 and 1 unit of resource 2 in two alternative ways: either by operating activities 1 and 4 at the levels 0.5 and 4.5 respectively (with total revenue \$0.5) or by operating activity 3 at unit level. By sticking to the first alternative, we lose \$0.5 - \$1 = \$0.5 per unit of non-operation of activity 3.[28]

Finally, the same procedure shows that we can use 0 units of resource 1 and one unit of resource two either by operating activities 1 and 4 at levels 0.5 and -2.5 respectively, or by operating activity 5 at unit level. This alternative, however, will have to be discarded, since by sticking to the first alternative we gain \$05 - \$0 = \$0.5 per unit of non-operation of activity 5.[29]

To summarize the discussion, we have three alternatives:

(a) keep operating activities 1 and 4 at levels 9 and 15,

(b) substitute activity 2 in the program, to gain \$0.25 of revenue per unit of its operation, and

(c) substitute activity 3 in the program, to gain \$0.50 of revenue per unit.

Taken at face value (i.e. in terms of gains per unit of activity) it seems logical to proceed to alternative (c).[30] Following exactly the same procedure as when we moved from the first to the second basic solution, we find that activity 3 will be operated at the level $x_3 = 3\frac{1}{3}$ and activity 4 will be $x_4 = 0$. Activity 1 will be operated at level $x_1 = 7\frac{1}{3}$.[31] The full solution is, then, $x_1 = 7\frac{1}{3}$ $x_2 = 0$ $x_3 = 3\frac{1}{3}$ $x_4 = 0$ $x_1 = 0$ solution (c) in Table 1. The total revenue associated with this solution is \$10 $\frac{2}{3}$.

To check whether this is an optimal solution, we need only consider activity 2 as an alternative.[32] It is then found that the alternative is either activities 1 and 3 at levels 0.5 and 0.5 respectively, or activity

2 at unit level. Since both alternatives contribute the same revenue (\$1) we lose \$1 - \$1 = \$0 by non-operating activity 2. Hence, we have arrived at an optimum solution. That is, any other feasible set of activity levels will yield either \$10 $\frac{2}{3}$ of revenue or less.

The last statement of the preceding paragraph makes it clear that the optimum solution that we have found is not necessarily unique. Indeed, in this problem there are more than one optimum solutions.[33] This can be seen by the fact that the contributions to revenue by the two alternatives above are equal. This situation means that activity 2 can be substituted at unit level (or at any other possible level) for activity 1 at level 0.5 and activity 3 at level 0.5, and total revenue will not change. If, for example, we operate x_2 at the maximum possible level (in which case $x_3 = 0$) we get[34]

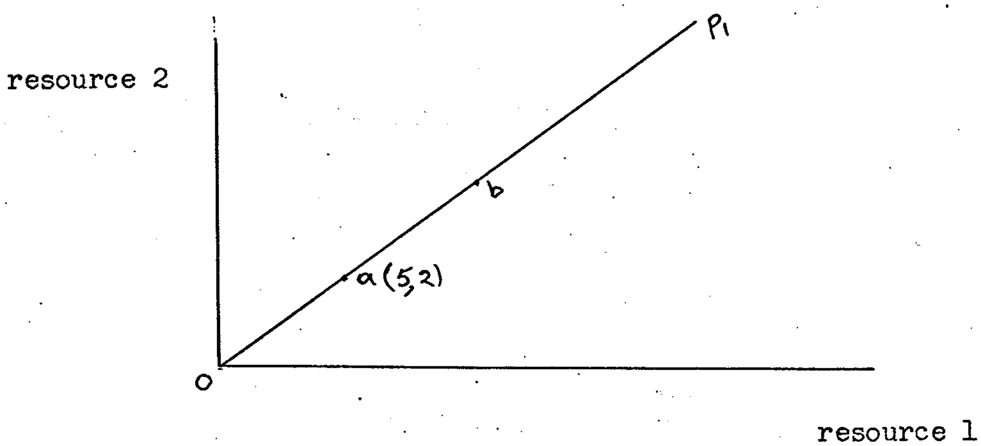
$x_1 = 4 \quad x_2 = 6\frac{2}{3} \quad x_3 = 0 \quad x_4 = 0 \quad x_5 = 0$ and total revenue of \$10 $\frac{2}{3}$.

Indeed it can be seen that if we start from the optimum solution $x_1 = 7\frac{1}{3} \quad x_2 = 0 \quad x_3 = 3\frac{1}{3} \quad x_4 = 0 \quad x_5 = 0$ and proceed substituting activity 2 for activities 1 and 3. in the ratio 1: 0.5, 0.5, we get an infinite number of optimum solutions, all yielding the same maximum revenue. This, of course, need not happen in all linear programming problems: most problems have a unique optimum solution. The specific example happens to have infinite optimum solutions.

2.2.0.6 Graphical Solution

The simple problem that we are working with allows a graphical solution, too. The graphical approach to the solution has its usefulness, though it cannot be a substitute to the method of section 2.2.0.5, since it brings out different aspects of the problem.

If we are to use two-dimensional diagrams we will have to measure resource quantities on the axes. Activities will then represent points in the positive quadrant, and all their possible levels of operation will be on straight lines passing through the origin. Consider, for example, activity 1:



Eig. 2.1

Point a on the line Op_1 shows the resource quantities needed by the activity at unit level of operation. Point b=(10, 4) shows what the activity needs in order to operate at level 2. Obviously, $Ob = 2Oa$ from the assumption of constant returns to scale. Any point on Op_1 between O and a represents the activity at a level smaller than unity. It will be clear that the needs of the activity at each level are represented by a unique point on Op_1 and vice versa.

In Fig. 2.2, I have graphed all three activities: points a, b, c, represent the activities 1, 2, 3 at unit level. The question arises about the meaning of the points on the line abc.

Note, to start with, that point a represents a combination of resources that yields \$1 of revenue, at the given final commodity prices. The same holds for points b and c.

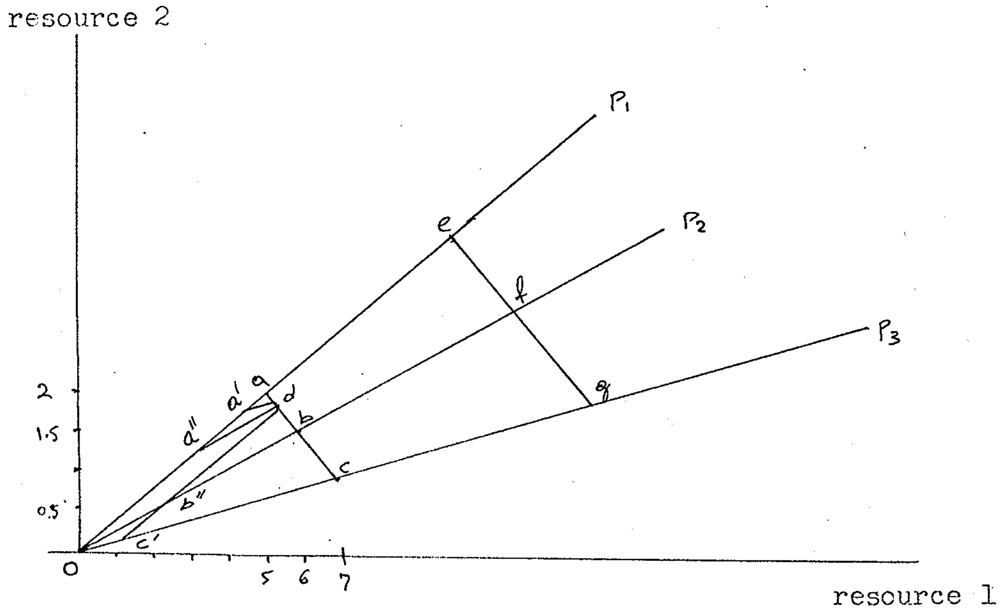


Fig. 2.2

Now consider point d on ab. It can be easily proved that (i) d represents the total factor quantities needed by activity p_1 at level a'' plus activity p_2 at level b'' (or, the quantities needed by p_1 at level a'' plus those needed by p_3 at level c''), and (ii) d represents combinations of activities that give a total of \$1 in revenue.[35] The same holds for every point on abc. It is then seen that abc is an iso-revenue curve, i.e., it represents activities or combinations of activities that result in \$1 of revenue. Note that the general shape of this line (whether it is a straight or broken line, whether its slope is negative or positive or mixed) depends on the commodity prices in relation to the technology.[36]

Now consider the line efg: it is parallel to abc and eO = 2aO, fO = 2bO, gO = 2cO. It can be easily seen that it is an iso-revenue curve for $2. Obviously, a whole family of such iso-revenue curves can be drawn, covering the area $p_1 0 \, p_3$.

Let us turn to the resource availabilities: resource 1 is available at a maximum of 60 units and resource 2 at a maximum of 18 units. This can be represented as point b in Fig. 2.3.

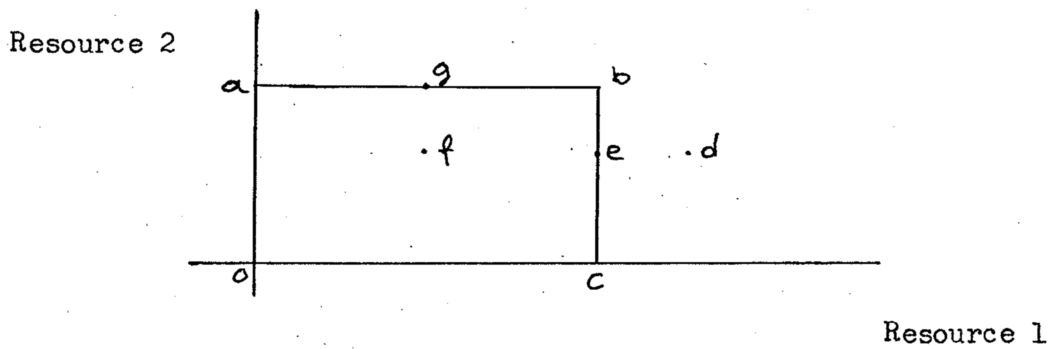


Fig. 2.3

Obviously, any point outside the rectangle Oabc is infeasible; any point inside is feasible, though it does not employ the resources fully,[37] and any point on the line ab (or bc), e.g., point g (point e) employs fully resource 2 (resource 1). Also, the only point that employs both resources fully is point b=(60, 18).

In Fig. 2.4 I have put the two diagrams together. It will be remembered that the problem was to maximise revenue subject to the constraints of resource availability and the non-negativity conditions

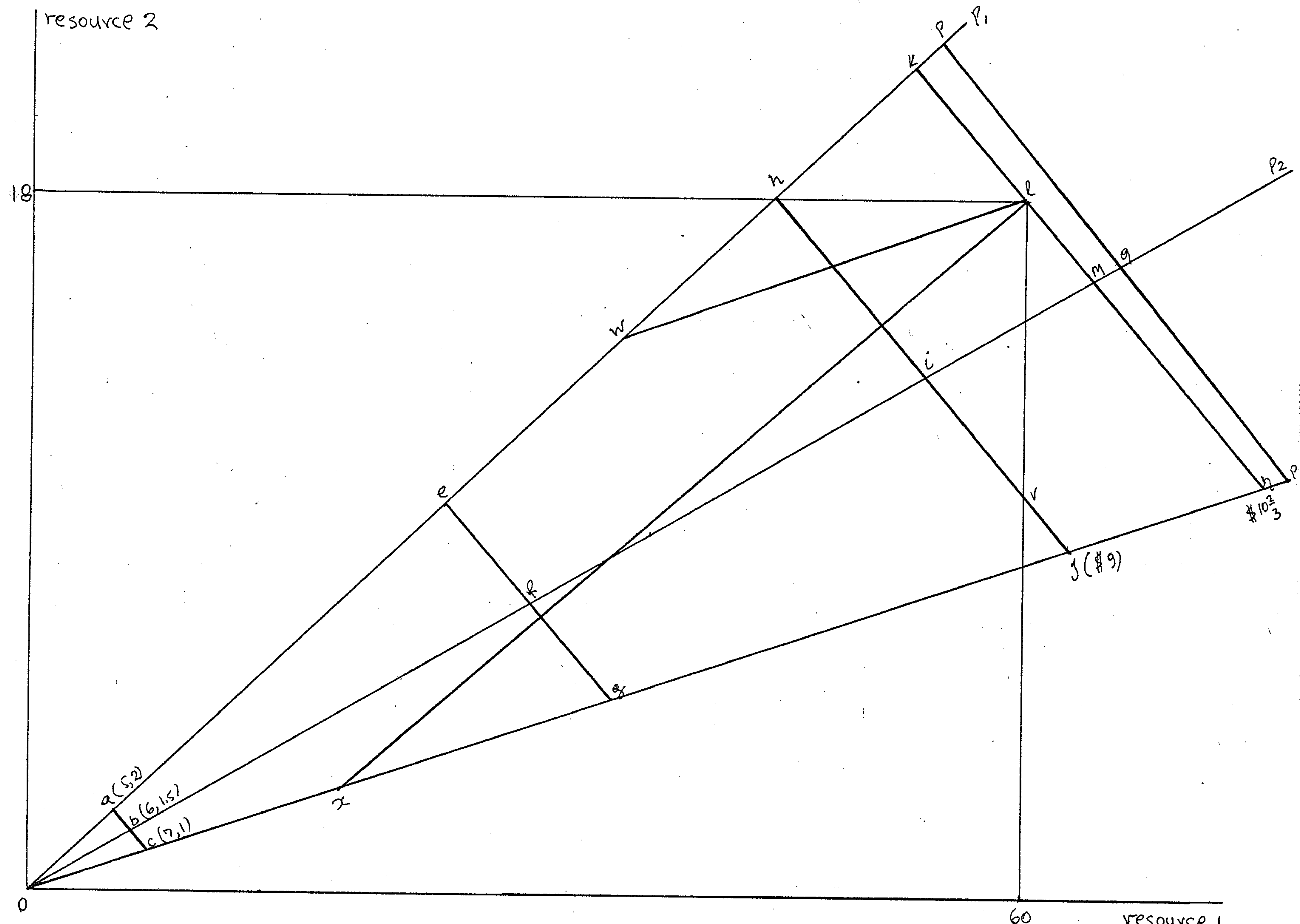

resource 2
18
p
P_1
k
h
l
P_2
M
g
w
i
e
v
h
p
$\$10\frac{2}{3}$
J ($9)
f
g
a (5,2)
b (6,1.5)
c (7,1)
x
0
60
resource 1

Fig 2.4

on the activity levels. Since we operate on the positive quadrant, non-negativity conditions are always satisfied; the other constraints are embodied in the figure. Note, finally, that the farther an iso-revenue curve is from the origin, the more revenue it shows.

Consider the iso-revenue curve efg. Since e0 = 5a0, efg shows activity combinations yielding \$5 of revenue. All combinations on efg are feasible. Obviously, we can do better than that, as in hij; this shows \$9 of revenue, and the part hiv is all feasible. Incidentally, our second basic solution in 2.2.0.1 is point h.

Now consider pqr. It yields more than \$10 $\frac{2}{3}$, but it contains no feasible point. Obviously, the line we seek is klmn, whose only feasible point is l, with revenue \$10 $\frac{2}{3}$. l can be reached by operating activity 1 at level W (7 $\frac{1}{3}$) and activity 3 at level x (3 $\frac{1}{3}$). Both resources are fully employed. (Indeed, l can be reached by a combination of activities 1 and 2 also, as well as by combinations of all activities, as explained in 2.2.0.5).

In this section, I have not exploited the particular advantages of the graphical approach. This will be done in section 2.3.

2.2.1 Economic Efficiency

2.2.1.0 Efficiency of Resource Allocation, and Resource Valuation: A Preview of the Dual

In checking the optimality of the third basic solution ($x_1 = 7\frac{1}{3}$ $x_3 = 3\frac{1}{3}$ $x_2 = x_4 = x_5 = 0$) of the previous subsection I asserted that we did not need to check the alternatives of operating activities 4 and 5.[38] It will, however, be instructive for quite another purpose to examine these alternatives.

First, it is to be remembered that $x_4 = 1$ (or $x_5 = 1$) means that activity 4 (or 5) is operated at unit level, that is, one unit of resource 1 (or 2) remains unemployed. It follows that in the third basic (and optimal) solution we have full employment[39] of both resources ($x_4 = x_5 = 0$).

Let us now check the alternatives of operating either activity 4 or 5: I start with activity 4.

It will be found that activity 4 operated at unit level is equivalent - from the point of view of resource use - to activities 1 and 3 operated at levels $-\frac{1}{9}$ and $\frac{2}{9}$ respectively.[40] We have

$$-\frac{1}{9}x_1 + \frac{2}{9}x_3 = x_4 \qquad (1)$$

The left-hand side combination of activities yields $-\frac{1}{9}(\$1) + \frac{2}{9}(\$1) = \$\frac{1}{9}$ of revenue, while the right-hand-side yields zero revenue. Hence, if we decide to operate activity 4 instead of activities 1 and 3, we <u>lose</u> $\$\frac{1}{9} - \$0 = \$\frac{1}{9}$ per unitary operation of activity 4. Alternatively, by non-operating activity 4, we gain $\$0 - \$\frac{1}{9} = \$ -\frac{1}{9}$ by unitary non-operation.

As far as activity 5 is concerned, we find[41] that it is equivalent to activities 1 and 3 operated at levels $\frac{7}{9}$ and $-\frac{5}{9}$ respectively. We have

$$\frac{7}{9}x_1 - \frac{5}{9}x_3 = x_5$$

The left-hand side yields $\$\frac{2}{9}$ of revenue. Hence, we lose $\$\frac{2}{9}$ per unitary operation of activity 5, or we gain $\$\frac{2}{9}$ by non-operation (per unit level).

The main question is about the economic meaning of the two numbers associated with the resources, namely, $\$\frac{1}{9}$ for resource 1 and $\$\frac{2}{9}$ for resource 2. What does it mean to say that we lose $\$\frac{1}{9}$ by unit operation of activity 4? It will be recalled that non-operation of activity 4 means full employment of the resource, while operation of activity 4 at positive level means unemployment of resource 1. Hence, we can say that $\$\frac{1}{9}$ is what we lose in revenue if we employ 59 units of resource 1 instead of 60 (recall that, in the optimal solution, resource 1 was fully employed). $\$\frac{1}{9}$, that is, is the marginal contribution of resource 1 to revenue, and can be called the marginal revenue productivity of the resource. The same interpretation holds for resource 2.

Interestingly, the implied analogy with standard neoclassical allocation theory proceeds further: multiplication of the quantities of factors employed (in the optimal solution) by their marginal revenue products (at the optimal solution) yields

$$60.\frac{1}{9} + 18.\frac{2}{9} = 10\frac{2}{3}$$

which is exactly the maximum revenue attainable.

The two "values" associated with the two resources are then seen to impute the total revenue to them, in conformity with the standard neoclassical theory of distribution.

2.2.1.1 Efficiency: L.P. and Neo-classical Economics

I shall now make an attempt at explanation of the notion of efficiency, and its relationship to resource valuation.

The simplest notion of efficiency refers to commodity bundles: given the technology and the quantities of resources available, a feasible commodity bundle is efficient if there is no other feasible bundle with a greater quantity of at least one commodity and with equal quantities of the rest of the commodities. Consider the case of two commodities. Given the technology and the resources available a standard production possibility area may help explain the notion of efficiency (Fig. 2.5).

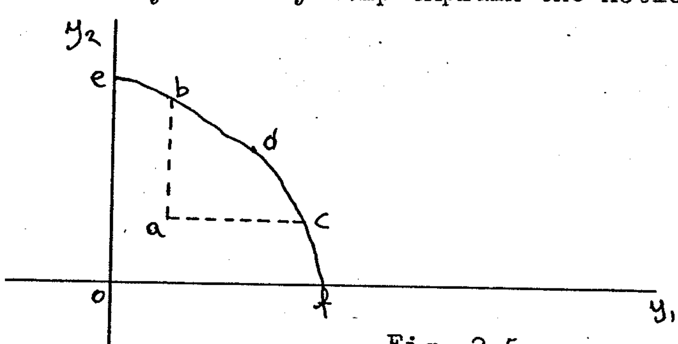


Fig. 2.5

According to our definition, bundle a is not efficient since there exists, for example, a feasible bundle b which has more of one commodity (y_2) and the same quantity of the other commodity (y_1). We could of course have picked point d or c for the comparison (indeed, any point in the area abc is superior to a).

Now let us examine point b. Clearly, there is no other feasible bundle with more units of one commodity and at least equal quantity of the other. Hence, b is an efficient bundle. The same will hold for every point on the production-possibility frontier, ebdcf.

The notion of an efficient point is thus seen to be a technical concept relating to production and allocation of resources. It does not, however, always refer to full employment of all resources, as the above example may imply.[42] This can be seen from the following "linear-programming-type" of example: Let

$$\begin{array}{c|c} & \Pi_1 \\ A & \\ B & \\ y_1 & \end{array}\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

represent the activity producing commodity y_1 by use of resources A, B, and

$$\begin{array}{c} \Pi_2 \\ \begin{array}{c} A \\ B \\ y_2 \end{array}\begin{bmatrix} -12.5 \\ -4 \\ 1 \end{bmatrix} \end{array}$$

the activity producing y_2. Also, let

$$\begin{array}{c} \Pi_0 \\ \begin{array}{c} A \\ B \end{array}\begin{bmatrix} -250 \\ -120 \end{bmatrix} \end{array}$$

represent the quantities of resources available and, finally, make the usual linear programming assumptions.[43] Given this technology and the quantities of resources, we can construct the production-possibility area as follows:

Full employment of factor A in the production of y_1 (notwithstanding the limits imposed by factor B) gives a maximum of 50 units of y_1. Full employment in the production of y_2 gives 20 units of y_2. Because of non-substitutability of resources and constant returns to scale, the marginal rate of transformation of y_1 to y_2 is a constant. Hence the production-possibility frontier for resource A is a straight line joining the two extreme combinations. The same reasoning applies for resource B. The combinations are shown in Fig. 2.6 below.

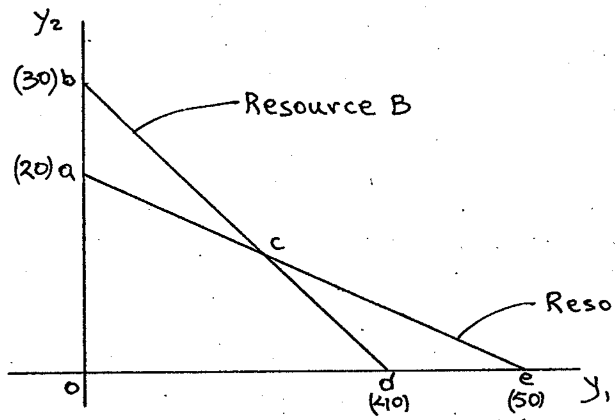


Fig. 2.6

Obviously, the set of feasible bundles from the point of view of both resources is the area Oacd. Moreover, on all bundles lying <u>within</u> (not on the North-east boundary of) the area Oacd need, for their production, fewer units of both resources than are available. Finally, all bundles lying on the segment ac (except bundle c) employ resource A fully, but not resource B. The reverse holds for bundles on the segment dc (except bundle c). To wit, only bundle c needs, for its production, to fully employ both resources.[44] By our definition, however, bundle c is not the only efficient point:

the locus of (the infinite number of) efficient points is the frontier acd. It is thus seen that the notions of efficiency of resource use and that of full employment of resources are not necessarily equivalent.[45] It is true, however, that an efficient bundle will employ at least one resource fully.[46]

Obviously, the notion of efficient commodity bundles (or equivalently, of efficient use of scarce resources) seems to be a technical one, disengaged from such economic considerations as demand, price, and profit. This is formally true, as can be seen from the relevant definition.

We can, however, define an efficient commodity bundle in relation to a set of prices for the commodities considered. Such a bundle will have to be, so to speak, "doubly" efficient, i.e., it will have to satisfy two requirements: first, it will have to be an efficient bundle in the sense defined above and, second, it will have to be efficient from the economic point of view. In other words, it will have to satisfy some kind of (usually) a monetary objective, besides being technically efficient.

In still other words, the notion of efficient commodity bundles (or, what is the same thing, the notion of efficient resource utilization) is closely related to the notion of the production function in standard economic theory.[47] Both notions represent <u>sets of efficient choices available</u> to an economic agent or agents (or, for that matter, to a planning authority). An efficient bundle **relative** to a set of prices, on the other hand, is an element of this set of choices that satisfies an additional re-

requirement. In standard economic theory such a bundle represents an optimal choice appropriate to the given set of prices, and is usually found by calculus methods. Consider, for example, the case (Fig. 2.7)

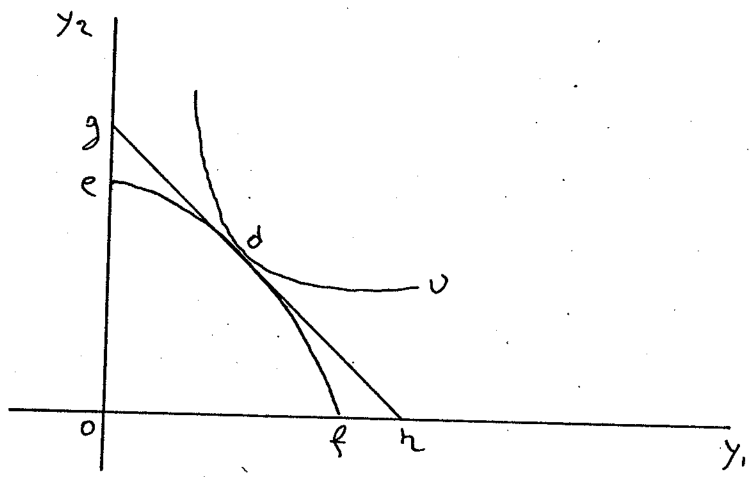


Fig. 2.7

where ef is the production possibility frontier of Fig. 2.5 and U is a horizontal section (isoquant) of the society's assumed utility surface. Point d is seen to satisfy two requirements: it is technically efficient (as it lies on ef) and it maximises a utility index. Point d can then be called efficient in relation to the set of prices given by the slope of the line gh.[48] No other efficient point is seen to satisfy this second requirement. And another set of prices (as implied in another utility surface) will produce another "efficient" point in this "double" sense.

Similarly, in the linear programming example, consider first the following set of constant prices: $p_1 = 5$ $p_2 = 4$. The total revenue function then becomes $R = p_1 y_1 + p_2 y_2 = 5y_1 + 4y_2$. The total revenue surface lies in the space of three dimensions, and is a plane through the origin (due to the assumption of constant prices). To find the efficient bundle relative to the given set of prices assume that the aim is revenue maximisation. Take horizontal sections of the revenue surface (iso-revenue lines) to get Fig. 2.8

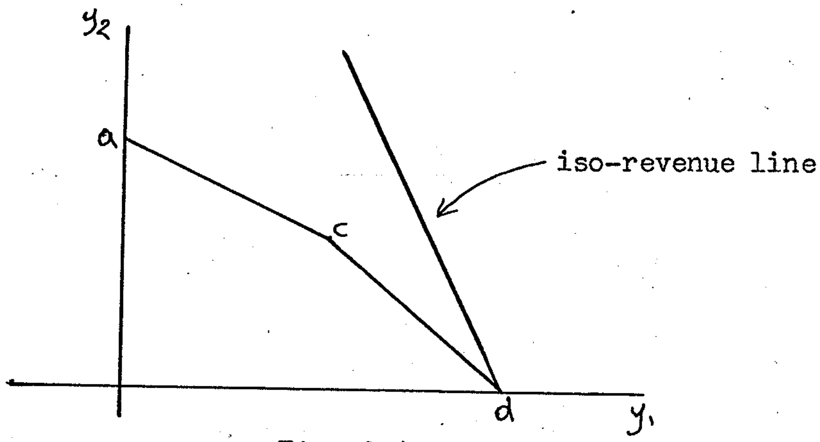


Fig. 2.8

where acd is the production-possibility frontier of Fig. 2.6. It turns out that bundle d (Od units of y_1 and zero units of y_2) is the efficient one, given this set of prices.[49] At another set of prices, say $p_1 = 2.5$ $p_2 = 4$, bundle c turns out to be efficient instead.

It will have become apparent in the above discussion that the simple linear programming example of this section and the slightly more complicated one that we have posed and solved in section 2.2.0 are formally equivalent. It is thus seen that linear programming is designed to solve specifically the problem of finding, among a number of efficient points, one that maximizes a given linear function for a given set of prices.[50]

In this sense, the technique is completely equivalent to the standard calculus techniques of economic theory (though, when it comes to specification of functions and formulation of constraints, L.P. turns out to be a much more general and deeper tool for economic analysis. Also, the results of the two analyses might differ considerably.)[51] Just as standard "continous" economic analysis depicts equilibrium points by solving problems of extrema of functions,[52] L.P. does precisely the same thing, in a setup of different specifications. This is sometimes supressed by the fact that L.P. (and, in its more general formulation, activity analysis) is a more general method of analysis, completely free of the institutional specifications that abound in standard neoclassical economic theory. As a result, it may look as if it bears no relationship to standard neoclassical economics. But while it is true that L.P. is more general and can be used in various institutional setups (for example in a centrally planned economy), it is also true that its categories can be given institutional specifications that bring it quite close to the perfectly competitive and welfare model of neoclassical economics.[53] The simple example of this section is a case in point: a number of assumptions and specifications can turn it into the standard profit-maximisation problem of the perfectly competitive firm in the short run.[54]

2.2.1.2 Efficient Choices and Shadow (Dual) Prices

It will be clear from the above discussion that the optimal solution $x_1 = 7\frac{1}{3}$ $x_3 = 3\frac{1}{3}$ $x_2 = x_4 = x_5 = 0$ to the problem of 2.2 is an efficient bundle at the given set of commodity prices. The maximum revenue associated with this bundle is \$10 $\frac{2}{3}$. The values associated with the two scarce resources were $\frac{1}{9}$ and $\frac{2}{9}$ respectively, and were seen to allocate the maximum total revenue attainable (at the given prices) to the scarce resources. These resource values are in effect the solution of the dual problem, and are usually called "shadow" or "accounting" prices. They can be used as guides for the efficient allocation of resources, in the sense that actual prices cannot exceed the shadow prices, if the (additional) units of the resources are to be employed.[55]

2.2.2 The Dual

2.2.2.1 A Graphical Solution

I now proceed to formulate the dual to the problem of 2.2.0. To recapitulate, the primal problem was:

$$\text{maximize } f(x) = [1, 1, 1] \; [x_1, x_2, x_3]' \tag{1a}$$

subject to

$$\begin{bmatrix} 5 & 6 & 7 \\ 2 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leqslant \begin{bmatrix} 60 \\ 18 \end{bmatrix} \tag{1b}$$

$$\text{and } x_i \geqslant 0 \qquad i = 1, .. 3 \tag{1c}$$

Formally, then, the dual is[56]

$$\text{minimize } g(u) = [60, 18] \; [u_1, u_2]' \tag{2a}$$

subject to

$$\begin{bmatrix} 5 & 2 \\ 6 & 1.5 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{2b}$$

$$\text{and } u_i \geqslant 0 \qquad i=1, 2. \tag{2c}$$

To see the economic meaning of this system, it may be easier to write it in the form:

$$\text{minimize } g(u) = 60\,u_1 + 18\,u_2 \qquad (2a)'$$

subject to

$$\begin{aligned} 5\,u_1 + 2\,u_2 &\geqslant 1 \\ 6\,u_1 + 1.5\,u_2 &\geqslant 1 \qquad (2b)' \\ 7\,u_1 + 1\,u_2 &\geqslant 1 \end{aligned}$$

$$u_1, u_2 \geqslant 0 \qquad (2c)'$$

Since the physical quantities 60 and 18 are multiplied and added into one homogeneous sum, u_1 and u_2 have to be some kind of prices for the resources. What is sought is a set of prices u_1, u_2 that will minimize the total value of resources $g(u)$.

In the first inequality of (2b)' the left-hand side sum is obviously the cost of activity 1 at unit level of operation, while the right-hand side is the revenue of the activity at unit level, given the prices of the outputs. The inequality stipulates that activity 1 is not permitted to create any profit. The same interpretation holds for the other two inequalities of (2b)'.

As for (2c)', the resource prices are not permitted to be negative. This is justified by the assumption of free disposal, mentioned in 2.2.0.3.

The computational procedure for the solution of the dual differs slightly from that of the solution of the primal. Since, however, the columns of the matrix in (2b) are purely fictitious activities, and their "levels" of operation are the prices of the resources, it would be rather confusing to go through the computations and try to explain the economics of the simplex method in this case. Instead, I will solve the problem by the graphical approach. But first some observations on the constraints and the objective function of the problem.

The requirement that no activity shall have its primary resource costs smaller than its revenue ensures complete allocations of the revenue to the scarce resources, which is compatible with traditional theory.

The objective function can be looked upon as the total "income" of the resources <u>when fully employed</u>. The economic explanation of minimization of this function proceeds as follows.

The actual quantities of resources employed (given the final commodity prices) depend on the technology and the resource prices. Given the technology, a set of high resource prices may be such that the resource costs of operating the activities exceed their revenues. In such a case, no activity will be operated and, of course, no resources will be employed. The <u>actual</u> income of resource holders will be zero. The objective function, however, will show a positive value. The unemployment of resources shows, on the other hand, that the value <u>imputed</u> to resources (by choice of the resource prices) is higher than the actual contribution they can make to revenue in circumstances of optimal allocation. If resource holders insist on these prices, misallocation (i.e. waste) of resources will occur since the commodity bundle chosen under these circumstances has zero quantities of all commodities and is obviously not efficient. The economy stays at the origin of Fig. 2.6 above. It is then seen that a lower set of resource prices are needed if the economy is to move to a more sensible resource allocation, i.e. to an efficient commodity bundle. When these prices are tried, it is found that (a) the economy operates on its production-possibility frontier (b) the actual returns to resource holders are equal to the maximum revenue attainable on the frontier and, (c) the value of the objective function is at its minimum, which is also equal to the maximum revenue attainable (as was already shown in 2.2.1.0).

I now proceed to a graphical presentation of the solution. In Fig. 2.8, the line AF represents all price combinations for which activity 1 breaks even.[57] Lines BE and CD represent the same things for activities 2 and 3 respectively. The line labelled $g(u) = 18$ has a slope of $-\frac{60}{18}$ and represents all sets of prices u_1, u_2 for which the objective function achieves the value 18.[58]

To start, choose the resource-price pair represented by poing G ($u_1 = 0.165$ $u_2 = 0.45$). These prices are feasible from the point of view of the constraints, as can be seen by substituting them into (2b)' and (2c)'. Graphically, this is ascertained by the fact that any pair of u_1, u_2 on the right of the "frontier" CJF satisfies all three inequalities of (2b)' and, of course, (2c)'.

As can be seen from (2b)', however, at these prices no activity breaks even. Hence no activity will be operated, and the full amount of resources available will be unemployed. Resources are "overpriced" and the result is waste. The "would-be" income of the resource-owners is \$18, but their

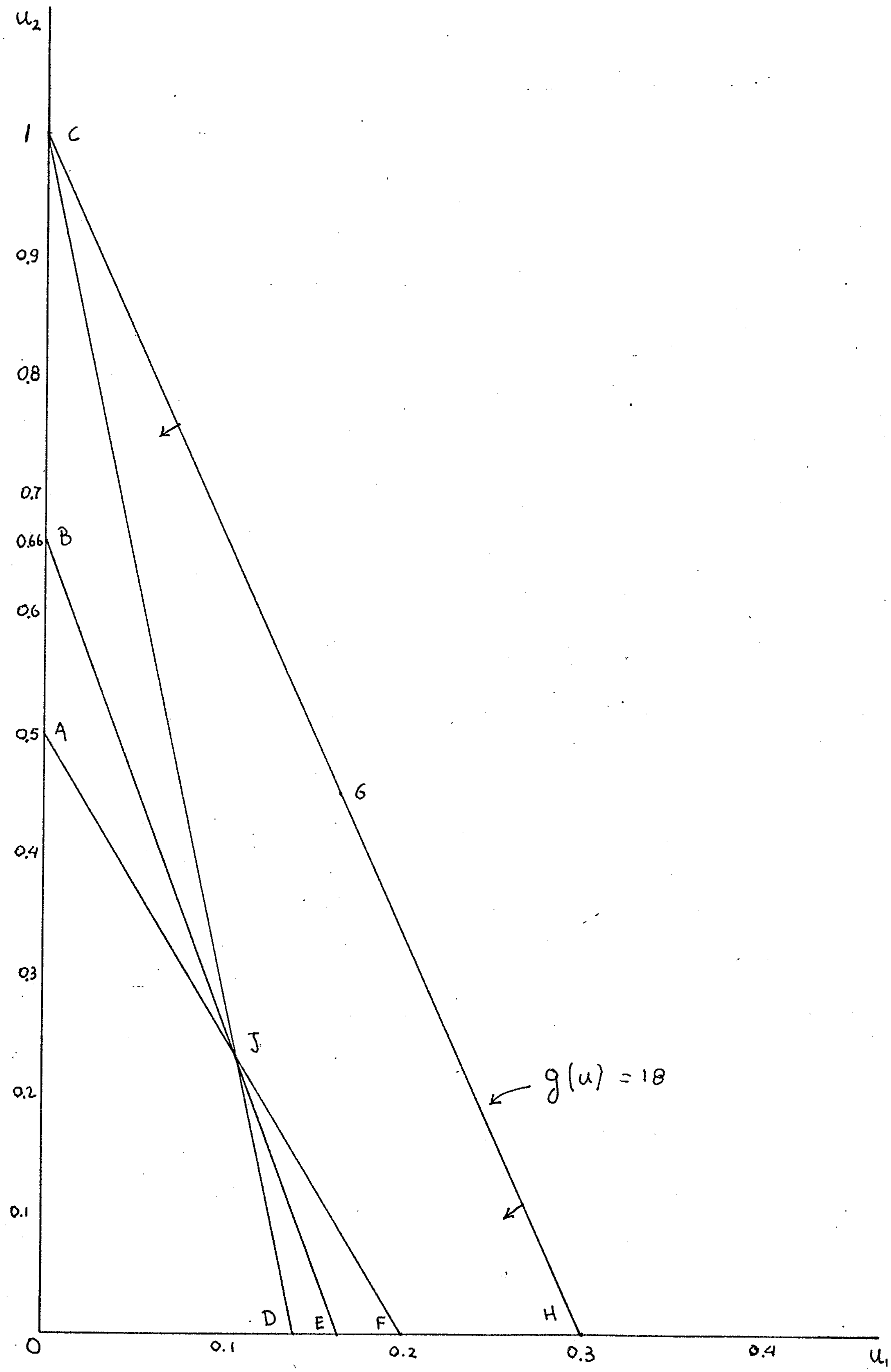


Fig 2.8

actual receipts are zero.

Obviously, what has been said of the price-pair G can be said in relation to any price pair lying to the right of the line CJF. On the other hand, any price-pair on the left of this same line is infeasible, as it violates at least one constraint. Consider, for example, point B ($u_1 = 0 \quad u_2 = 0.66$): it violates the third constraint of (2b)', i.e. makes activity 3 profitable.

Obviously, what remains are price combinations on the "frontier" CJF. Let us therefore consider point C ($u_1 = 0 \quad u_2 = 1$). At this price pair the "would-be" returns of resource owners are still $18. Moreover, activity 3 breaks even, and thus can be operated at the maximal possible level, which is $x_3 = \frac{60}{7}$ $\left(\frac{60}{7} < \frac{18}{1}\right)$. Resource 1 is thus fully employed, together with $\frac{60}{7}$ units of resource 2. Actual receipts of resource owners are $60.0 + \frac{60}{7} \cdot 1 = \$\frac{60}{7}$, which is exactly equal to the revenue attainable by this program ($x_1 = 0 \quad x_2 = 0 \quad x_3 = \frac{60}{7} \quad x_4 = 0 \quad x_5 = 18 - \frac{60}{7}$). That revenue, however, is not the maximum possible $\left(\frac{60}{7} < 10\frac{2}{3}\right)$. The resource prices are such that - given the commodity prices - the resulting allocation is not efficient in the "double" sense of section 2.2.1.1 (i.e., optimal) despite the fact that it lies on the frontier of the production-possibility area (it does because it employes one resource fully).

To wit, the situation stands as follows: given the technology and the resource quantities we have a production-possibility frontier, on which all points are usually efficient[59] in the technical sense. Now, given a set of commodity prices (perhaps, reflecting the relative desirability of commodities) there exist, among the efficient points of the frontier, one or more optimal points that maximise revenue. If we stipulate that no activity in an optimal program shall operate at a profit or loss, the set of resource prices compatible with the optimal points is unique. This unique set of resource prices is what we are trying to find. Any other set of resource prices will produce a program (a point in the production possibility area) which is either inefficient (like that given by the set of prices G) or it is technically efficient but not optimal in relation to the commodity prices.

Let us return to the problem of finding these resource prices. Obviously, on the line $g(u) = 18$ we have price sets that give us either no production at all (all points on line except point C) or some production which is not optimal. Moreover, (and this is in line with our aim of minimizing $g(u)$) there is ample room for moving the line $g(u)$ parallelly to the left (i.e. making its value less). In Fig. 2.9, then, let us try the line $g(u) = 12$.

On this line, all price combinations lying on the segment BK (except point K) are not feasible. On the other hand, all price combinations on the segment KF (except points K and F) produce a zero-production situation. What remains are price-sets K and F.

Consider, first, the price-pair K ($u_1 = \frac{1}{11}$ $u_2 = \frac{4}{11}$) where activity 3 breaks even. Obviously, we have the same situation as with the price pair C, except for "would-be" resource income. The latter is \$12 at K ($60 \cdot \frac{1}{11} + 18 \frac{4}{11}$). Activity 3 is operated at the level $\frac{60}{7}$ as at C, with actual resource income of $60 \cdot \frac{1}{11} + \frac{60}{7} \cdot \frac{4}{11} = \frac{660}{77} = \frac{60}{7}$. Activity revenue is again $\$\frac{60}{7}$.

We can then conclude that for all price pairs lying on CJ (except J) activity 3 will be operated at the level $\frac{60}{7}$, with actual resource employment 60 units of 1 and $\frac{60}{7}$ units of 2, and actual resource income of $\$\frac{60}{7}$. The only thing that differs on these points is the value of $g(u)$, which gets smaller as we move from C to J.

Now consider point F ($u_1 = 0.2$, $u_2 = 0$). At this price-pair, it is activity 1 which breaks even. It will be operated at the maximum possible level $x_1 = 9$ with full employment of resource 2 and employment of 45 units of resource 1. "Would be" resource income is \$12, while actual resource income is $45 \times 0.2 \quad 18 \times 0 = \$9$, equal to the revenue of the activity. Incidentally, this price pair (F) is better than K from the point of view of actual resource revenue.

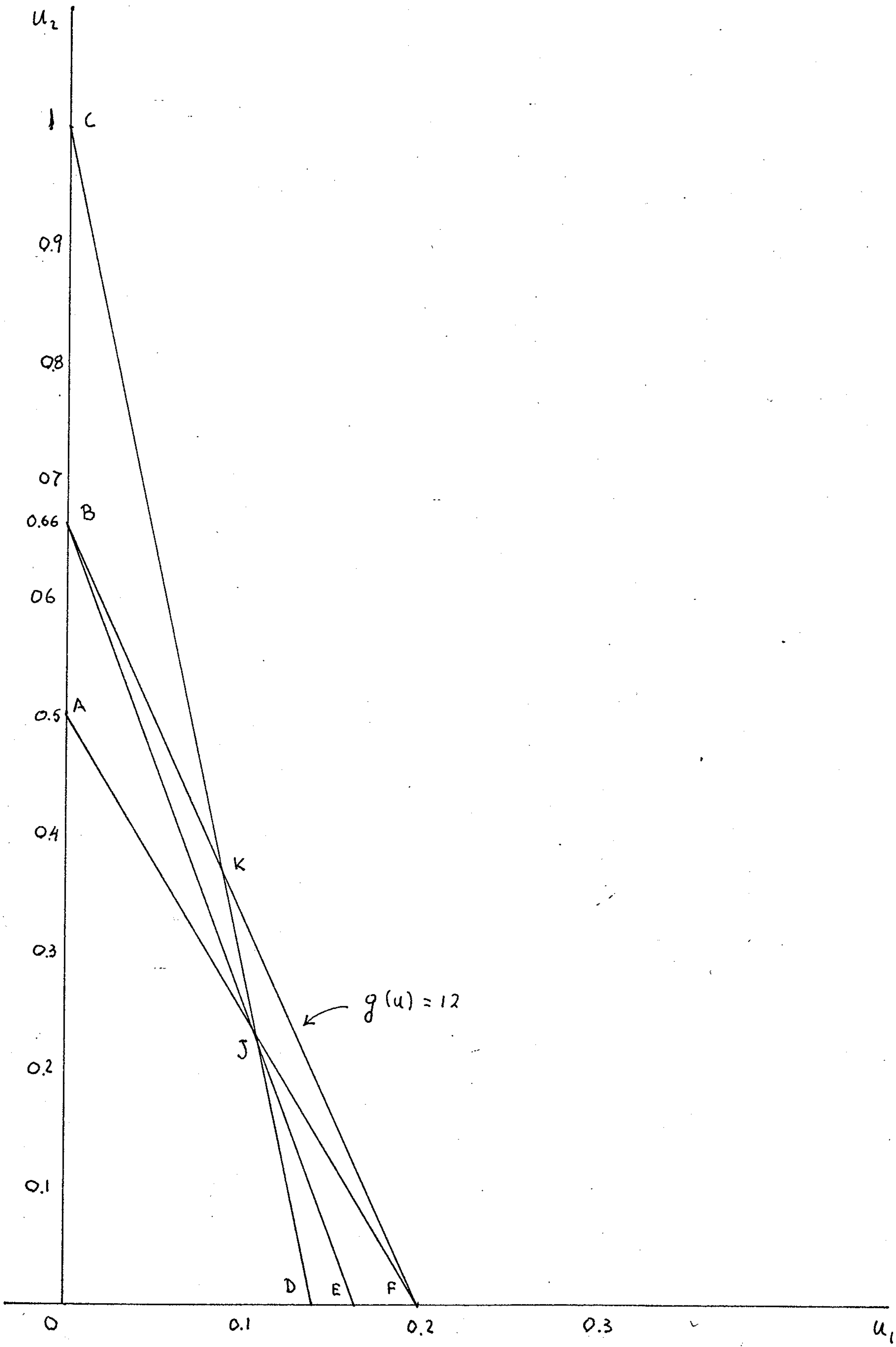


Fig 2.9

Thus, we have not yet reached the maximum revenue, neither have we reached the minimum $g(u)$, as the latter can be shifted to the left and still contain some feasible price-pair. Moreover, what we said about the relationship between price-pair C and all others on CJ (except J), obviously holds for the relationship between price-pair F and all others on FJ (except J). That is, for any price-pair on FJ (except J) the situation will be as for price-pair F (i.e., $x_1 = 9$ $x_2 = 0$ $x_3 = 0$ $x_4 = 15$ $x_5 = 0$, activity revenue = actual resource receipts = \$9) except for $g(u)$, which will be less.

It remains, then, to shift $g(u)$ to the left, to pass through J. This has been done in Fig. 2.10. First, it is to be observed that no other price pair on that line is feasible except J. ($u_1 = \frac{1}{9}$ $u_2 = \frac{2}{9}$). Total "would-be" returns are $60 \times \frac{1}{9} + 18 \times \frac{2}{9} = \frac{32}{3}$. Moreover, all activities break even! The problem emerges as to which activities to operate and at what levels, but we can postpone it for the next section. For the moment, we can pick activity 1 at level $7\frac{1}{3}$ and activity 2 at level $3\frac{1}{3}$. Their total resource needs are 60 and 18 respectively. Actual resource income is $60 \times \frac{1}{9} + 18 \times \frac{2}{9} = \frac{32}{3}$ equal to the "would-be" income and equal to the maximum revenue attainable in the optimal solutions to the primal problem. We have found two shadow prices that are compatible with the technology, the resource availability and the final commodity prices. We have reached the end of a rather long journey.

2.2.2.2 Dual Prices As Guides to Resource Allocation

I now take up the "quantity" problem which arose in the previous section. It will be remembered that at point J of Fig. 7 ($u_1 = \frac{1}{9}$ $u_2 = \frac{2}{9}$)

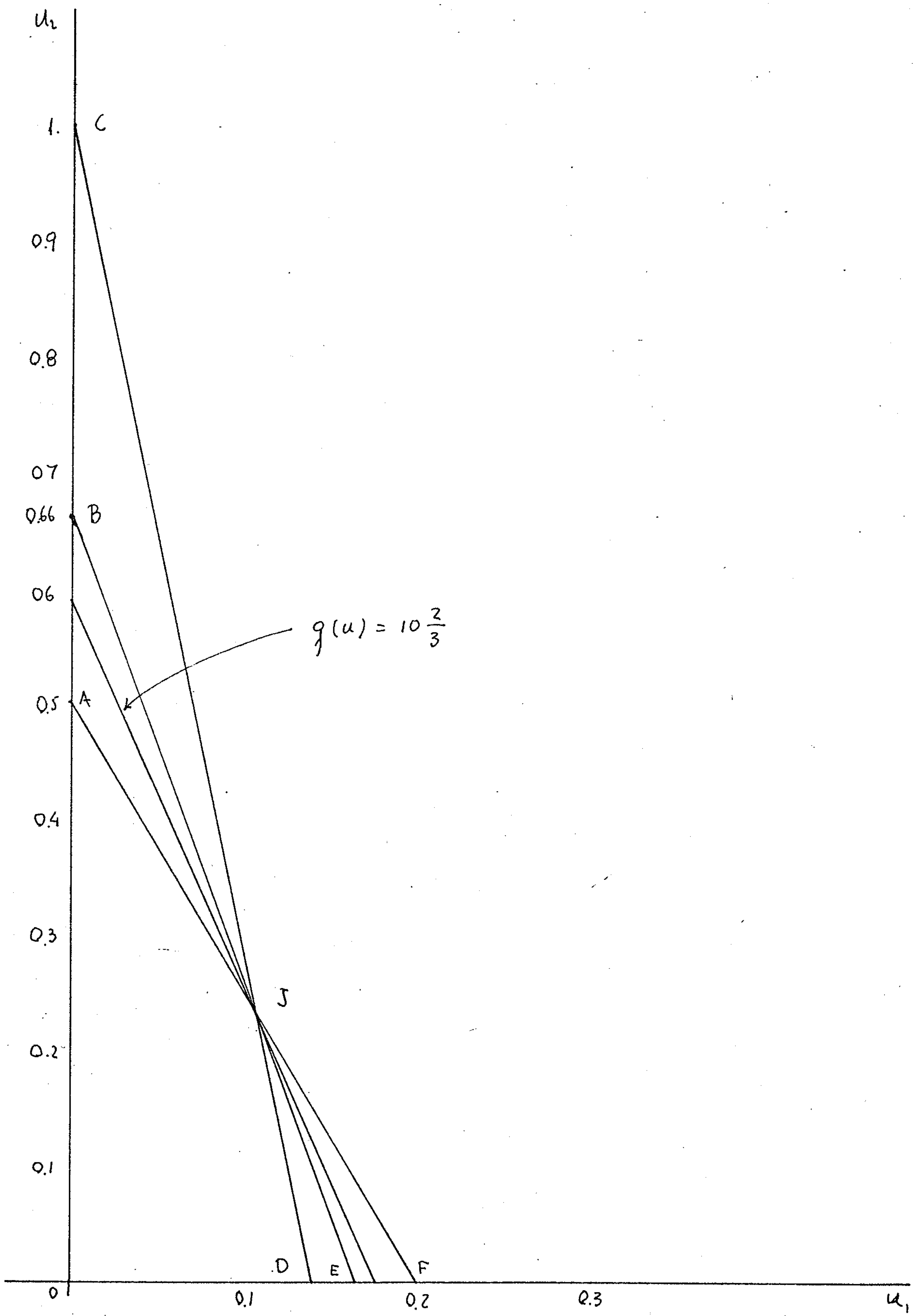

u_2
1.
C
0.9
0.8
0.7
0.66
B
0.6
$g(u) = 10\frac{2}{3}$
0.5
A
0.4
0.3
J
0.2
0.1
D
E
F
0
0.1
0.2
0.3
u_1

Fig 2.10

all activities broke even, so the problem arose of which activities to use at which level, in order to achieve the maximum revenue.

First, a word about the fact that, in our example, all three activities break even. This is really a rather special case, as will be seen in the next section, and corresponds to the case of an infinite number of optimal solutions to the primal. The usual case is that, at the optimizing dual prices, as many activities break even as there are resources,[60] i.e., the optimal solution to the primal is unique and contains only two activities at non-zero level.

As far as our "quantity" problem is concerned, however, it would remain if, say, only activities 1 and 3 broke even at point J $(u_1 = \frac{1}{9} \;\; u_2 = \frac{2}{9})$.

To illustrate, I will first examine the case for a centrally planned economy. The criterion that only those activities which break even be operated is not enough for specification of the levels of the activities, even when the dual resource-prices are known to the planning authority; a separate calculation is needed, with the following procedure:

Find out which activities break even (suppose that they are activity 1 and activity 2) at the given dual prices. In the primal, drop all other activities to get

$$\text{maximise} \quad x_1 + x_3 \qquad (1)$$

subject to

$$\begin{aligned} 5\,x_1 + 7\,x_3 &\leqslant 60 \\ 2\,x_1 + 1\,x_3 &\leqslant 18 \end{aligned} \qquad (2)$$

Drop all those constraints which correspond to resources with dual prices equal to zero (in our case, none).[61] Change the inequality signs into equation signs and solve[62] the system:

$$\begin{aligned} 5\,x_1 + 7\,x_3 &= 60 \\ 2\,x_1 + 1\,x_3 &= 18 \end{aligned} \qquad (2)'$$

to get $x_1 = 7\frac{1}{3}$ $x_3 = 3\frac{1}{3}$

and maximum revenue $x_1 + x_3 = 10\frac{2}{3}$

Now examine the case for a perfectly competitive economy with three industries, each firm in each industry facing given constant prices of resources ($u_1 = \frac{1}{9}$ $u_2 = \frac{7}{9}$) and of commodities ($p_1 = 2$ $p_2 = 3$ $p_3 = 4$) and each firm in each industry facing the same technology (as represented by the columns of the matrix (2c) of 2.2.0.2). Assume, as above, that only activities 1 and 3 break even at the given commodity and resource prices. Thus, only commodities 1 and 2 will be produced. However, the allocation of resources between the two industries (let alone among the firms in each industry) is indeterminate.[63] This is a common problem when constant-returns-to-scale production functions are assumed.[64] Hence, even in a competitive economy, a planning authority is needed to dictate the levels of operation of each industry that will make revenue attain its maximum. The same procedure will have to be followed as in the case of a centrally planned economy.

2.3 LINEAR PROGRAMMING: STRUCTURAL RELATIONSHIPS BETWEEN COMMODITY PRICES, RESOURCE QUANTITIES, AND TECHNOLOGY[65]

The example given in section 2.2. was rather special in two respects:

first, the relationship between technology and factor availability was such that full employment of <u>both</u> resources was possible within a relatively great range of commodity prices. Second, there were an infinite number of optimum solutions to the problem.

The aim in this section is to show (a) the effect that commodity prices can have on the optimal solution, given the technology and the resource availability, (b) The effect that commodity prices can have on the number of optimum solutions, (c) The effect that commodity prices can have on full employment of resources in the optimal solution, (d) The effect that the relationship between technology and resource availability can have on the employment of resources, irrespective of commodity prices. The approach in this section will be graphical only.

2.3.0 "Structural" Unemployment

This is case (d) of 2.3. Consider the same technology as in 2.2.0.2, with different factor availabilities

$$\begin{bmatrix} 5 & 6 & 7 \\ 2 & 1.5 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Resource 1} \\ \text{Resource 2} \end{array} \quad \begin{bmatrix} 40 \\ 18 \end{bmatrix}$$

and depict these activities in Fig. 2.11:

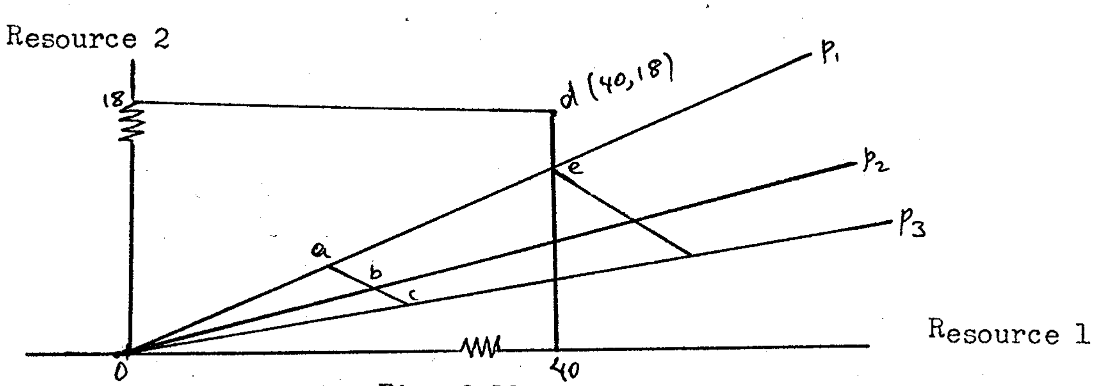


Fig. 2.11

I said in 2.2.0.6. that any point on the line abc represents a combination of activities, and that there is a family of such lines actually "covering" the area p_1 0 p_3 extending to infinity in the NE direction.[66] That is, every point in this area represents a combination of activities, and its co-ordinates show total quantities of resources employed by the respective combination(s). It follows that "bundles" of resources lying within this area are potentially[67] fully employable, that is, there exists a combination of activities that employs both resources fully.

Now consider the data on resource availability that yield point d (or, in fact, any point outside the area p_1 0 p_3); there exist two kinds of combinations of activities that can give this point: (i) a combination of p_1 and p_2 (or p_1 and p_3, or p_2 and p_3, or p_1, p_2 and p_3) with one activity operated at a negative level, and (ii) a combination of "actual" activities (p_1, p_2, p_3) and slack variables, with all "actual" activities operated at non-negative levels and the slack variables operated at positive levels.[68]

Combination (i) is, of course, to be rejected, since negative levels of activities are not admissible. As for combination (ii), the presence in it of a slack variable at positive level shows that both resource quantities shown by point d cannot be actually fully employed.

The above conclusion is insensitive to a change in commodity prices: such a change will only alter the shape of the line abc but will never make it extend outside the area p_1 0 p_3. It follows that at no set of commodity prices is full employment of resource 2 possible. We can call this case one of "structural" unemployment, as it occurs from the fact that the set of resource quantities available is not "balanced" with technology (i.e., it does not fall within the area p_1 0 p_3). Or we can say that resource 2 is a "free commodity", although this seems to me as too impersonal a terminology (suppose resource 2 were labour?)

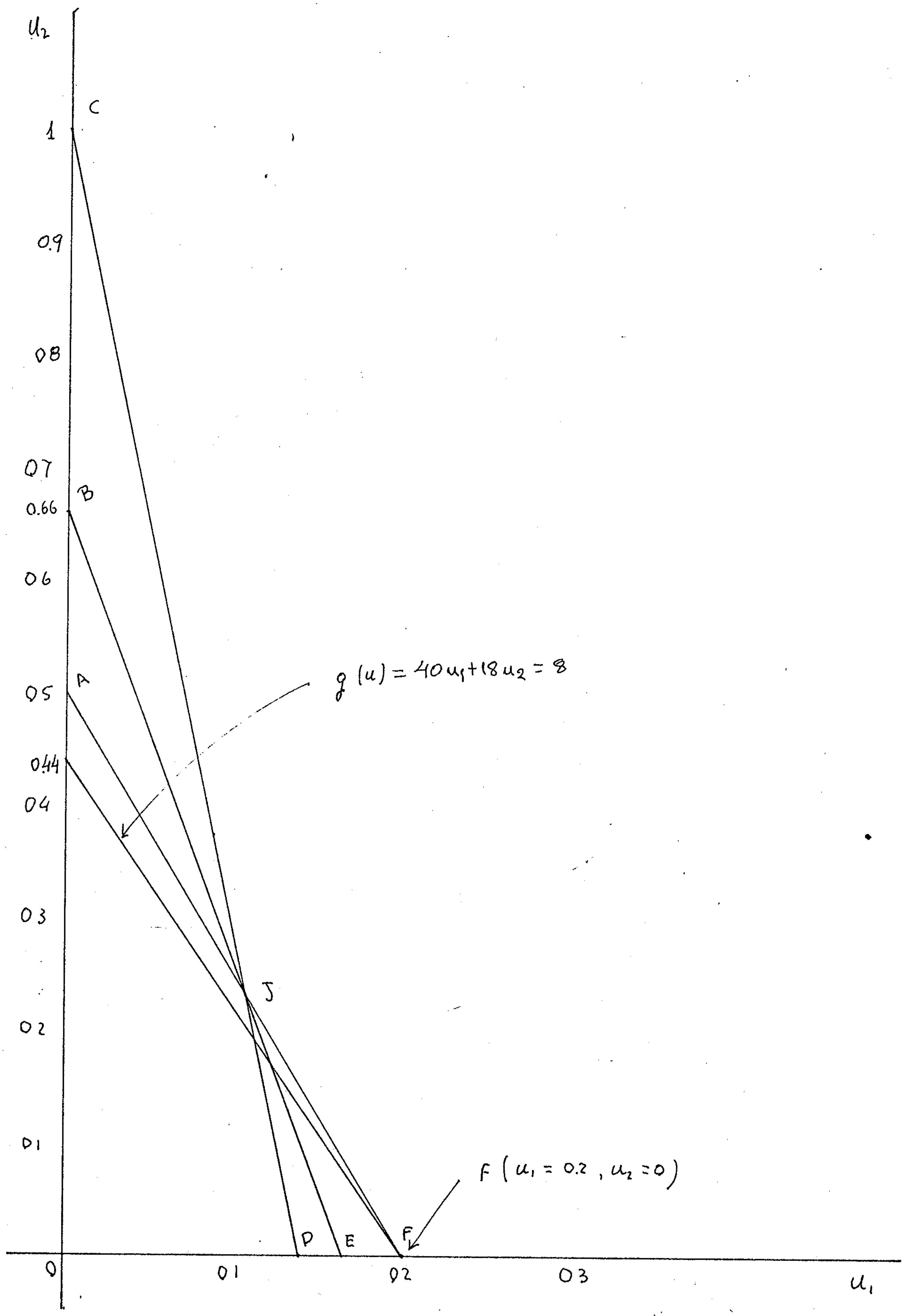
The solution to the primal is, of course, point e, with only activity 1 operated at the maximum possible level (8), with a maximum revenue of $8. The solution is unique.

In the dual, the slope of g(u) changes. From Fig. 2.12 it can be seen that the resource-price pair minimizing g(u) is $u_1 = 0.2$ $u_2 = 0$, with activity 1 operated. Hence, we see that when a resource comes out not fully employed in the final solution its dual price is zero (hence the terminology "free commodity").

This result of structural unemployment is rather uncommon in standard neoclassical economic theory, where the "well-behavedness" of production functions permits any degree of substitution of resources. In contrast, the assumption of fixed coefficients in linear programming permits substitution only in the indirect sense of combinations of activies, and this only within a certain range.[69]

2.3.1 Commodity Prices and the Optimal Solution (Cases a, b, c, of 2.3)

Consider the original data of the problem of 2.2 (given in 2.2.0.1) with a different set of prices:



u_2
C
1
0.9
0.8
0.7
B
0.66
0.6
A
0.5
0.44
0.4
$g(u) = 40u_1 + 18u_2 = 8$
0.3
J
0.2
0.1
$F\ (u_1 = 0.2,\ u_2 = 0)$
D
E
F
0
0.1
0.2
0.3
u_1

Fig 2.12

$$\begin{array}{l} \text{Res. 1} \\ \text{" 2} \\ \text{Comm. 1} \\ \text{" 2} \\ \text{" 3} \end{array} \begin{bmatrix} -10 & -24 & -21 \\ -4 & -6 & -3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \text{Resource 1 available} \\ \text{" 2 "} \\ \text{Price of commod. 1} \\ \text{" " " 2} \\ \text{" " " 3} \end{array} \begin{bmatrix} 60 \\ 18 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

Then, our "technology per dollar"[70] is as follows:

$$\begin{bmatrix} 10 & 24 & 7 \\ 4 & 6 & 1 \end{bmatrix}$$

and in Fig. 2.13 the optimal solution is point d with resource 2 not fully employed. This happens despite the fact that the resource availability point lies within the area $p_1 \, 0 \, p_3$ and is thus potentially employable. In this case, commodity prices are in such relation to the technology that they render activities 1 and 2 unacceptable. As can be seen from the \$1 iso-revenue "curve", point C lying on activity 3 is the only relevant one from the whole triangle, as it yields \$1 of revenue by using less resources than any other combination.[71]

So there seems to be one kind of unemployment arising simply from the strength of demand in relation to technology.

Fig. 2.14 shows the dual to this problem. The dual prices that minimize g(u) are at F, i.e., $u_1 = \frac{1}{7}$, $u_2 = 0$. The only activity operated is activity 3, at level $8\frac{4}{7}$.

It is thus seen that:

(a) Different sets of commodity prices yield different optimal solutions to the same problem ("same" in terms of technology and resource

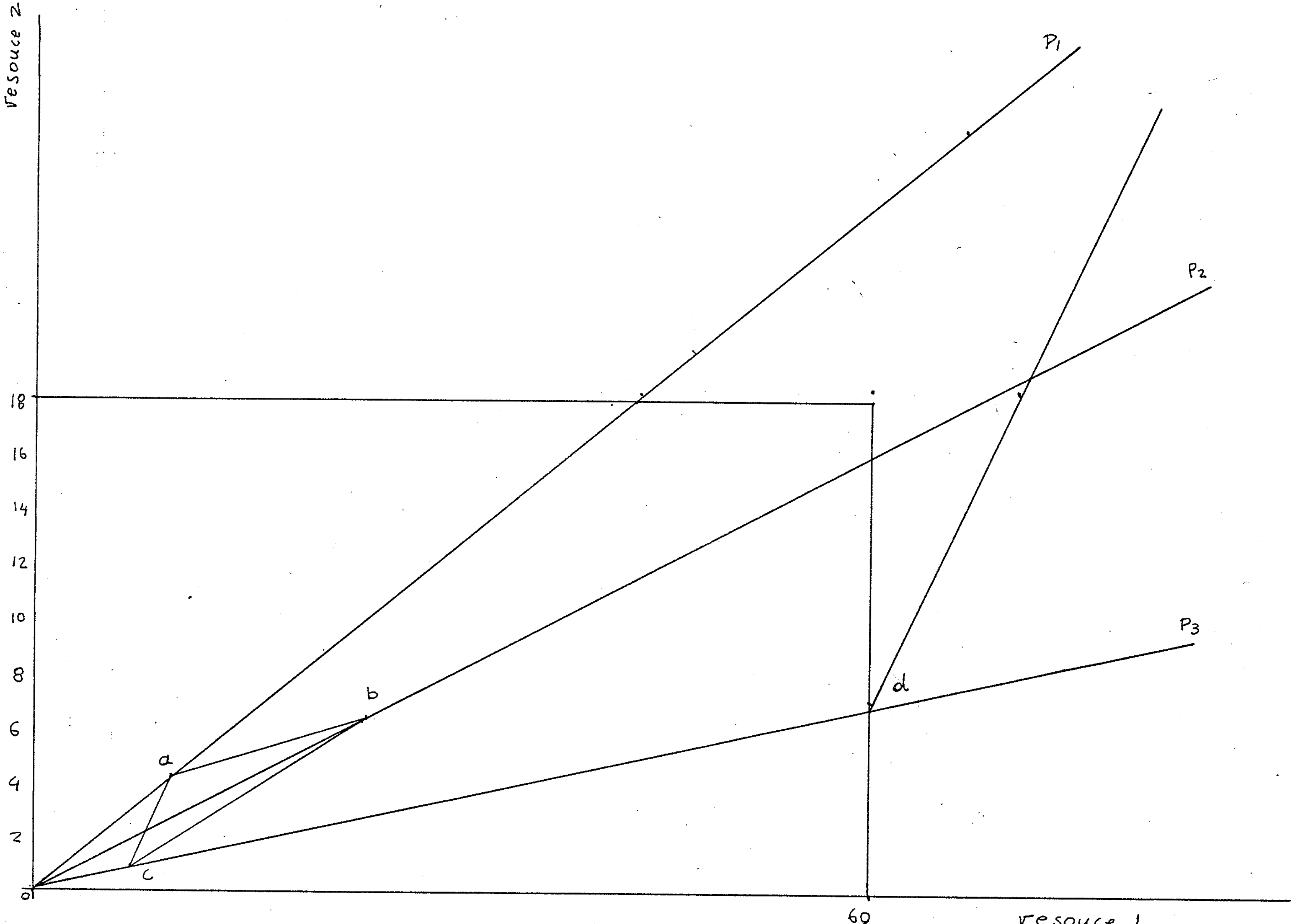
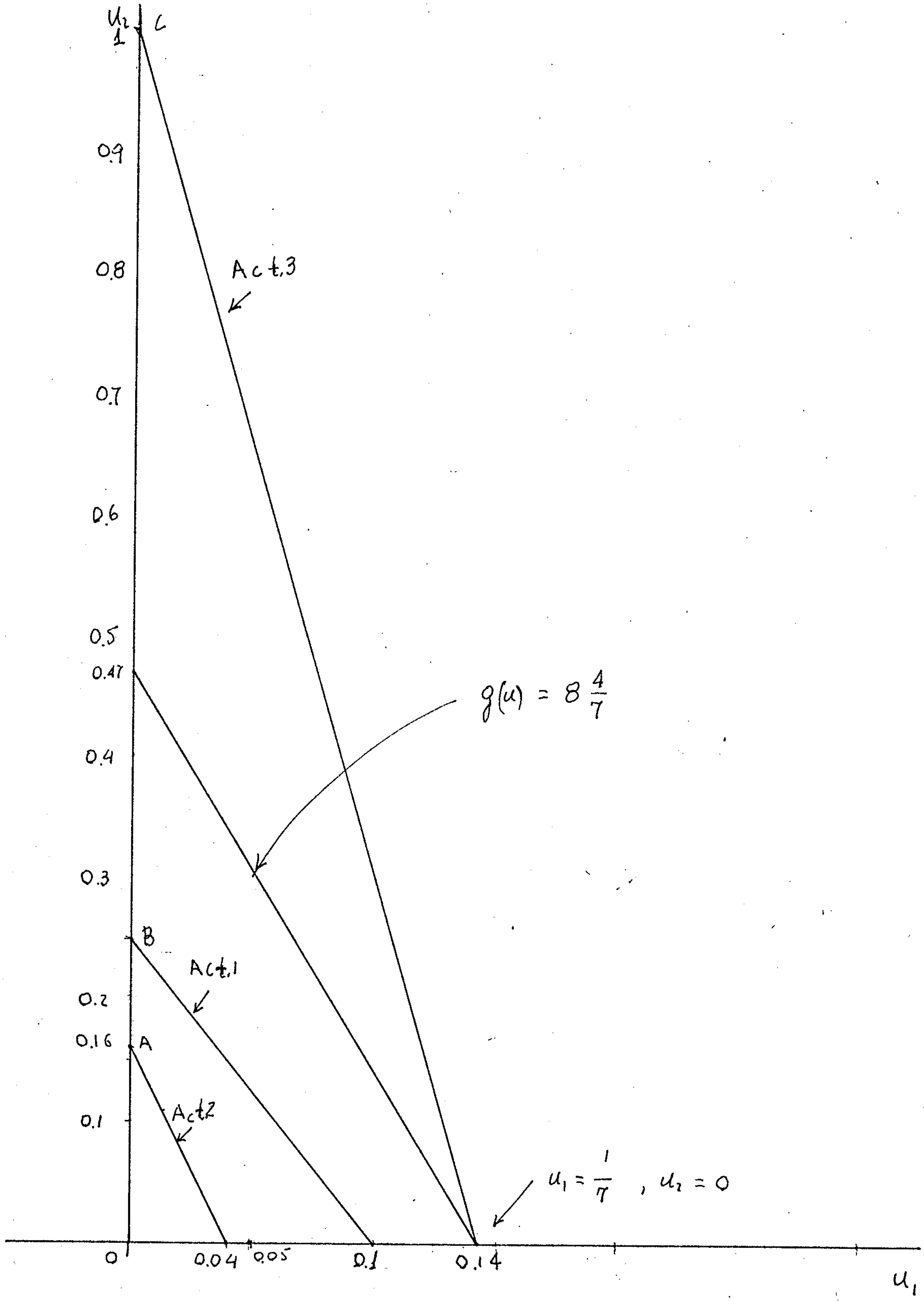


FIG 2.13

availability).

(b) The number of optimal solutions (one or infinite) depends on the set of prices.

(c) Full employment of resources in the optimal solution also depends on prices of commodities. For full employment of resources there have to be satisfied <u>two</u> conditions: first, the resource quantities available must be in balance with the existing technology (see 2.3.0) and, second, commodity prices must fall within certain ranges.



u_2
1
C
0.9
0.8
Act.3
0.7
0.6
0.5
0.47
$g(u) = 8\frac{4}{7}$
0.4
0.3
B
Act.1
0.2
0.16
A
Act2
0.1
$u_1 = \frac{1}{7}$, $u_2 = 0$
0
0.04
0.05
0.1
0.14
u_1

Fig 2.14

CHAPTER III

LINEAR PROGRAMMING MODELS WITH SPACIAL CONSIDERATIONS

This Chapter is devoted to an examination of the transportation model, an interpretation of its dual and a reexamination of the primal in the light of a non-linear interregional equilibrium model. Examination of these subjects paves the way to generalizations of the transportation model in Chapter IV.

3.0 THE TRANSPORTATION MODEL

This is a general linear programming model with a special and simpler mathematical structure being the only difference. This simpler structure allows the use of solution methods other than the simplex method. Since, however, this thesis is not concerned with computational problems except in so far they illustrate economic notions and since these aspects were taken up in Chapter II, this section will be devoted to some special characteristics of this model with reference to space.[72]

3.0.1 Model III: Total Exports Equal To Imports

The problem tackled by this transportation model is part of a more general problem of interregional equilibrium which will be examined briefly in the next section. With this generalization in view I will state the transportation problem in the most general terms possible.

Consider a closed economy divided into n regions[73] potentially producing and consuming[74] a single homogeneous commodity. Production costs may vary from region to region in both marginal and average terms. There is a maximum productive capacity for each region, which is fixed during the period of examination. Demand conditions may also vary between regions.[75] Minimum[76] transportation costs per unit of commodity from region to region are known and constant.

The questions arising in connection with this static partial equilibrium situation have to do with the possibilities of trade between regions, the amounts "exported" and "imported" and the after trade equilibrium prices in each region. For the purposes of this section assume that we are given part of the solution to this problem as follows:

(a) All regions participate in trade, i.e., either export or import.[77] We may then number the regions so that those from 1 to m ($< n$) are net exporters (later also called "origins") and those from $m+1$ to n are importers (also called destinations).

(b) The total amount of exports of each region is given, denoted by X_i ($i = 1, 2 \ldots m$).

(c) The total imports of each importing region are given, denoted by B_j ($j = m+1, \ldots n$).

(d) Minimum unit transportation costs are given, denoted by s_{ij}. ($i = 1, \ldots m$, $j = m+1, \ldots n$). The first subscript refers to the exporting and the second to the importing region.

The problem that remains in this case is the question of which origin will ship how much of the commodity to which destination. The criterion to be used in answering this question is obviously total transportation costs of the system. This is because comparative advantages of regions due to different conditions of supply and/or demand have already been taken into account in answering the question "which region is to export or import how much". Productive capacities have also been taken into account in connection with this latter question.

In linear programming form, the problem is, then,

$$\text{Minimize} \quad S = \sum_{i=1}^{m} \sum_{j=m+1}^{n} s_{ij}\, x_{ij} \qquad (3\text{-}1)$$

$$\text{Subject to} \quad \sum_{j=m+1}^{n} x_{ij} \leqslant X_i \quad (i = 1, \ldots m) \qquad (3\text{-}2)$$

$$\sum_{i=1}^{m} x_{ij} \geqslant B_j \quad (j = m+1, \ldots n) \qquad (3\text{-}3)$$

$$x_{ij} \geqslant 0 \quad \text{for all } i, j \qquad (3\text{-}4)$$

$$\text{and} \quad \sum_{i=1}^{m} X_i = \sum_{j=m+1}^{n} B_j \qquad (3\text{-}5)$$

Condition (3-5) implies that (3-2) and (3-3) are actually equality constraints. It is impossible, for example, for any origin not to ship all its exports X_i, since this will entail unsatisfied demand in at least one destination.[78] The same remark holds with respect to (3-3). Slack variables are obviously not necessary in this special case.

The unknowns in this problem are the individual shipments x_{ij}. The problem has $m(n-m)$ unknowns and n equality constraints. Actually, however, only n - 1 constraints are independent. That is, if a set of x_{ij} values satisfies all but one constraint in (3-2) and (3-3) with an equality sign, then it must satisfy that one constraint also. The reasoning depends on the fact that (3-5) must hold.[79] Say, for example, that a set of x_{ij} values satisfies all but the first equation constraint in (3-2)

$$\begin{aligned} \sum_j x_{2j} &= X_2 \\ \sum_j x_{3j} &= X_3 \\ &\vdots \\ \sum_j x_{mj} &= X_m \end{aligned} \qquad (3\text{-}2')$$

and all constraints in (3-3):

$$\begin{aligned} \sum_i x_{i,\,m+1} &= B_{m+1} \\ \sum_i x_{i,\,m+2} &= B_{m+2} \\ &\vdots \\ \sum_i x_{i,\,n} &= B_n \end{aligned} \qquad (3\text{-}3')$$

and, of course, the non-negativity conditions.

Adding up all equalities in (3-2') and in (3-3') we get

$$\sum_{i=2}^{m} \sum_{j=m+1}^{n} x_{ij} = \sum_{i=2}^{n} X_i$$

and $$\sum_{j=m+1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=m+1}^{n} B_j$$ respectively.

And subtraction of the first from the second sum yields

$$\sum_{j=m+1}^{n} x_{1j} = \sum_{j=m+1}^{n} B_j - \sum_{i=2}^{m} X_i$$

and the right-hand side of this last equation must be equal to X_1, by (3-5).

In economic terms, since the above set of x_{ij} values satisfies all the demands exactly (i.e., 3-3), it must exhaust all the available exports X_i, because total imports equal total exports. On the other hand, exports from regions 2 to m have been exhausted (the set of x_{ij} satisfies 3-2). Hence, the exports of origin 1 must also be exhausted.

The above remarks as to the number of effective constraints serve to indicate that a basic feasible (and hence the optimal) solution will contain n-1 variables. This is of use in the interpretation of the dual, below.

I will not take up the solution of a specific transportation problem, since the principle is here the same as in Chapter 2 (though a simpler solution method may be used). It seems worthwhile for the purposes of this thesis, however, to interpret the notion of "activity" in this model and state and interpret the dual. To simplify, I will assume three origins and four destinations. The primal is, then

$$\text{minimize } S = \sum_{i=1}^{3} \sum_{j=4}^{7} s_{ij} x_{ij} \qquad (3\text{-}6)$$

subject to

$$
\begin{array}{c}
\begin{array}{cccccccccccc}
x_{14} & x_{15} & x_{16} & x_{17} & x_{24} & x_{25} & x_{26} & x_{27} & x_{34} & x_{35} & x_{36} & x_{37}
\end{array} \\
\begin{bmatrix}
-1 & -1 & -1 & -1 & & & & & & & & \\
 & & & & -1 & -1 & -1 & -1 & & & & \\
 & & & & & & & & -1 & -1 & -1 & -1 \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{bmatrix} \\
\begin{array}{cccccccccccc}
s_{14} & s_{15} & s_{16} & s_{17} & s_{24} & s_{25} & s_{26} & s_{27} & s_{34} & s_{35} & s_{36} & s_{37}
\end{array}
\end{array}
.
\begin{bmatrix} x_{14} \\ x_{17} \\ x_{24} \\ x_{27} \\ x_{34} \\ \\ x_{37} \end{bmatrix}
\geqq
\begin{bmatrix} -X_1 \\ -X_2 \\ -X_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{bmatrix}. \qquad (3\text{-}7)
$$

$$x_{ij} \geqslant 0 \quad \text{for all } i, j \qquad (3\text{-}8)$$

and $$\sum_{i=1}^{3} X_i = \sum_{j=4}^{7} B_j \qquad (3\text{-}9)$$

The constraints in (3-7) are accompanied by the activity labels at the top and the relevant transportation costs at the bottom for expositional convenience. Looking down by columns, origin number one has four activities available, 14, 15, 16, and 17. Each of them "uses" one unit of the commodity available at this origin (hence the minus sign) and "produces" the same commodity at each destination, incurring the transportation cost shown at the bottom. The same holds with respect to the other two origins. The solution method will proceed by choosing (n-1) activities each time and calculating their total transportation cost. Comparison of activities outside the basis with those of the basis will proceed exactly along the lines of Chapter 2, till an optimal solution is reached.

The dual of this problem is as follows:

$$\text{Maximize } U = -\sum_{i=1}^{3} u_i X_i + \sum_{j=4}^{7} v_j B_j \qquad (3\text{-}10)$$

subject to

$$\left.\begin{array}{l}
\left.\begin{array}{l}
v_4 - u_1 \leqslant s_{14} \\
v_5 - u_1 \leqslant s_{15} \\
v_6 - u_1 \leqslant s_{16} \\
v_7 - u_1 \leqslant s_{17}
\end{array}\right] \text{(a)} \\
\left.\begin{array}{l}
v_4 - u_2 \leqslant s_{24} \\
v_5 - u_2 \leqslant s_{25} \\
v_6 - u_2 \leqslant s_{26} \\
v_7 - u_2 \leqslant s_{27}
\end{array}\right] \text{(b)} \\
\left.\begin{array}{l}
v_4 - u_3 \leqslant s_{34} \\
v_5 - u_3 \leqslant s_{35} \\
v_6 - u_3 \leqslant s_{36} \\
v_7 - u_3 \leqslant s_{37}
\end{array}\right] \text{(c)}
\end{array}\right. \qquad (3\text{-}11)$$

$$\text{and } v_j,\ u_i \geqslant 0 \quad \text{for all } i, j \qquad (3\text{-}12)$$

The optimal solution in the primal will include six activities, and of these six at least one will belong to each origin and at least one will transport the commodity to each destination (this is because of (3-5) above). By the symmetry of duality, at least one constraint from groups (a) (b) (c) will be satisfied with an equality, hence u_1, u_2, u_3 will appear without fail. Also, all v_j will appear since at least one shipment will be made to each destination.[80] The dual will then give six equations in seven unknowns, u_i and v_j, to be determined. The values to be determined will be the optimal ones.[81] The degree of freedom available in determining these values corresponds to the redundancy of one constraint in the primal.

In short, the dual does not determine absolute "values" but only "value" differences which should hold in equilibrium. The economic meaning of these "value" differences has intentionally been left ambiguous in this section. Economic interpretation is taken up in section 3.0.4.

3.0.2 Model III: $\sum_i X_i > \sum_j B_j$

As another variation of the transportation model consider the case where $\sum_i X_i > \sum_j B_j$: this, of course, could not have been given by the more general problem that was briefly referred to in section 3.0.1, the reason being that $\sum_i X_i = \sum_j B_j$ must hold if X_i is interpreted as exports and B_j as imports. Another interpretation is then needed here.

Consider a case where amounts X_i of a homogeneous commodity are available in $\bar{m}$ locations for distribution to $(n-\bar{m})$ consumption points. The consumption requirements of these points are B_j $(j = \bar{m}+1, \dots n)$ and the unit transportation costs are given. The question is to find a set of shipments x_{ij} which will minimize

$$S = \sum_{i=1}^{\bar{m}} \sum_{j=M+1}^{n} s_{ij} x_{ij} \qquad (3\text{-}1a)$$

subject to

$$\sum_{j=M+1}^{n} x_{ij} \leqslant X_i \quad (i = 1, \dots \bar{m}) \qquad (3\text{-}2a)$$

$$\sum_{i=1}^{\bar{m}} x_{ij} \geqslant B_j \quad (j = \bar{m}+1, \dots n) \qquad (3\text{-}3a)$$

$$x_{ij} \geqslant 0 \quad \text{for all } i, j \qquad (3\text{-}4a)$$

and
$$\sum_{i=1}^{m} X_i > \sum_{j=m+1}^{n} B_j \qquad (3\text{-}5a)$$

We can still write (3-3a) in this form, provided all $s_{ij} > 0$, since in this case the objective of cost minimization will see to it that no destination receives more than it requires. Constraints (3-3a) are thus in fact equality constraints.

Constraints (3-2a), however, are meaningful inequality constraints since (3-5a) holds. We are then in need of slack or disposal activities which will perform the task of transferring the excess of $\sum_i X_i$ over $\sum_j B_j$ to a fictitious destination at zero transportation costs. We need one such activity for each origin, and the relevant s_{ij} will be zero for such activities, though their coefficients will be the same as those of any other meaningful activity. To simplify the exposition, with three origins and three destinations and the slack variables added, the fictitious destination bearing the number 7, the problem is that of section 3.0.1, equations (3-6) to (3-9), with s_{17}, s_{27} and s_{37} equal to zero and B_7 equal to

$$B_7 = \sum_{i=1}^{3} X_i - \sum_{j=4}^{6} B_j$$

It is thus seen that the total amount of the slack is given before the solution of the problem. This permits equation (3-9) to remain after the introduction of the slack variables. Formally, then, a problem of the form (3-1a) to (3-5a) is transformed to one of the form (3-1) to (3-5), the differences being:

(a) If the problem (3-1a) to (3-5a) has m origins and (n-m) destinations, the problem (3-1) to (3-5) derived from it has m origins and (n+1 - m) destinations,

(b) While the problem (3-1a) to (3-5a) has m(n-m) unknowns and n constraints, the derived one has m (n+1 - m) unknowns and n+1 constraints. However, the derived problem has n meaningful constraints for reasons already noted in 3.0.1.

This form of the problem has not been given as much attention in the literature as the one in section 3.0.1. Note, however, that this form leads to generalizations that are quite close to full scale interregional models. The dual of this problem will be interpreted in section 3.0.4.

3.0.3 The General Economic Problem

In this section the general economic problem whose part can be solved by the transportation model of 3.0.1 is examined,[82] for the case of two regions. The supply and demand conditions in these regions can be represented in a back-to-back diagram as follows:

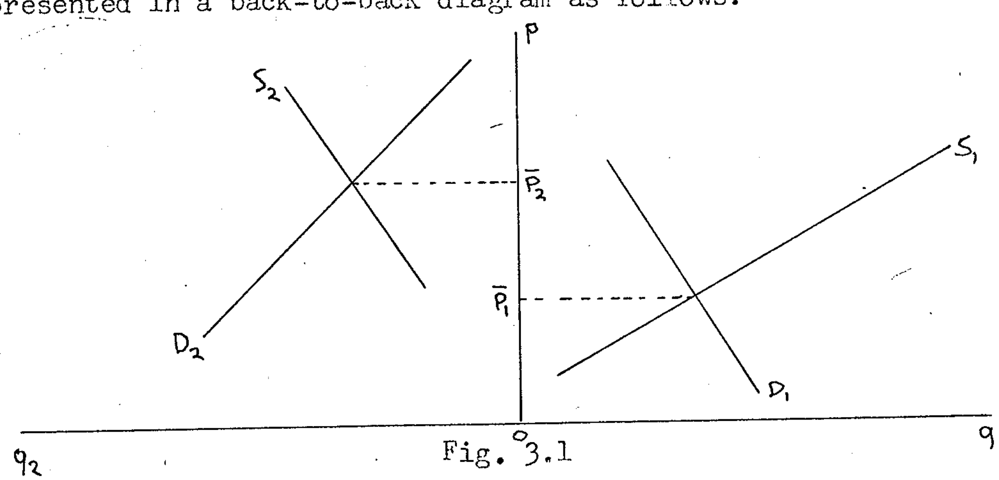


Fig. 3.1

The supply and demand functions are here represented by straight lines for purposes of diagrammatic convenience. The basic argument is not affected by this simplification. Both supply and demand in each region are functions of price.

In Figure 3.1, the pre-trade equilibrium prices in regions 1 and 2 are $\bar{p}_1$ and $\bar{p}_2$ respectively. The relevant questions in this partial equilibrium situation are: the possibility of trade, the direction of trade, its amount and the after-trade equilibrium prices and quantities.

The possibility and direction of trade obviously depend on the pre-trade price difference in relation to the given constant transportation costs. If $\bar{p}_1 - \bar{p}_2 > s_{21}$ where $\bar{p}_1$, $\bar{p}_2$ are pre-trade prices, region 2 will be an exporter. If $\bar{p}_2 - \bar{p}_1 > s_{12}$ region 1 will be an exporter. If the initial price difference is smaller than the relevant transportation cost ($\bar{p}_1 - \bar{p}_2 < s_{21}$ and $\bar{p}_2 - \bar{p}_1 < s_{12}$) no trade will take place.

The amount of trade (the X_i and B_j of the transportation problem) may be said to depend on elasticities of demand and supply in the two regions.

A solution of this problem[83] will yield the above as well as the quantities supplied locally (which are not considered in the transportation model). It will also yield equilibrium after-trade prices satisfying the following conditions:

$$p_1 - p_2 = s_{21} \quad \text{if region 2 exports}$$

and

$$p_2 - p_1 = s_{12} \quad \text{if region 1 exports}$$

The after-trade equilibrium prices yielded by this problem have the conventional meaning, since they are derived from equilibria of supply and demand, with one difference: transportation costs influence the price structure in that the price spread between regions in after-trade equilibrium cannot exceed them. Thus, given one after trade equilibrium price and the interregional pattern of trade (which region exports to which) all prices are "frozen" by the fact that they have to satisfy the equilibrium conditions above.[84]

In diagrammatic terms for the case of two regions, Fig. 3.2 depicts the excess curves

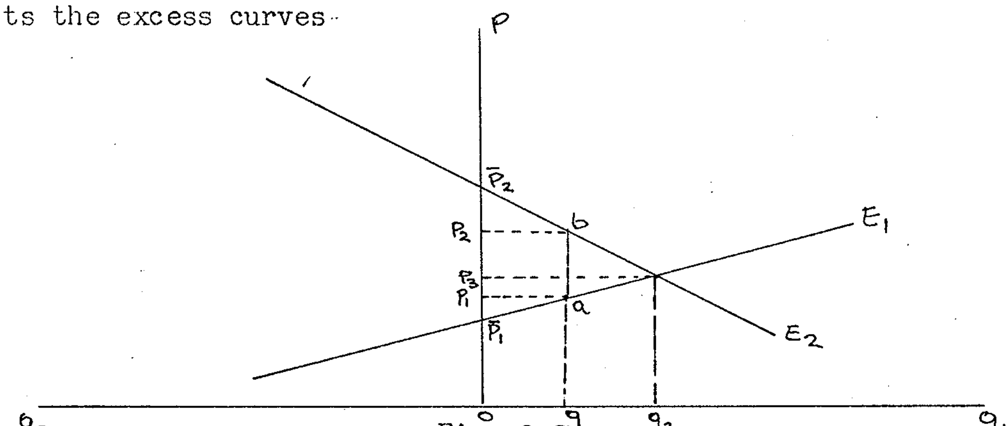


Fig. 3.2

of region 1 (E_1) and 2 (E_2), calculated as $S_i - D_i$ for all prices ($i = 1, 2$). $\bar{p}_1$ and $\bar{p}_2$ are the pre-trade equilibrium prices and p_3 would be the interregional equilibrium price (with region 1 exporting $X_1 = q_3$ and region 2 importing $B_2 = q_3$) if the transportation cost s_{12} were zero. On the other hand, if $s_{12} \gg \bar{p}_2 - \bar{p}_1$ there would be no trade. Finally, with $0 < s_{12} < \bar{p}_2 - \bar{p}_1$ say $s_{12} = ab$, the after-trade equilibrium prices will be p_1, p_2, with $p_2 - p_1 = s_{12}$ and $x_1 = q \equiv B_2$.

In his quoted article, P. Samuelson is concerned with a number of interesting questions that this thesis will not relate as they do not pertain directly to its content. The purpose of this section was simply to show the context in which the transportation model of 3.0.1 falls as far as general economic analysis is concerned, to help in the interpretation of the dual to the transportation problem, and to show the rationale by which extensions and generalizations of the transportation model have proceeded. These matters will be examined in Chapter 4.

3.0.4 Economic Interpretations of the Duals to the Transportation Models II and III

There are a number of interpretations of the dual values u_i and v_j in the transportation model. I consider all of them important and illuminating and I will take them up in turn.

One interpretation[85] of model II relates directly to the general problem of 3.0.3: the equilibrium conditions on after-trade prices in this problem are exactly the same as the constraints (3-11) with an equality sign, if u_i is interpreted as F.O.B. price at the exporting region i and v_j as delivered price at the importing region j. Since the dual to the transportation model has one degree of freedom, we need one price given from outside the model to determine the whole equilibrium price structure. If the price given from outside is taken from the solution of the general problem of 3.0.3, the prices derived from the dual will be identical to those found by the solution of the general problem. The transportation model II is thus seen to be embedded in the more general problem of 3.0.3 in the sense that, given the total amounts of exports and imports, one after-trade equilibrium price and the transportation costs, it determines (in the primal) the interregional trade structure which minimizes transport costs and which is identical to that yielded by the solution to the general problem,[86] and (in the dual) the interregional after-trade equilibrium price structure again identical to that yielded by the problem in 3.0.3.

Building up on this interpretation,[87] consider a large number of merchants buying the commodity from the (given) exporting regions and selling it to the importing regions. Their revenue would be, in total,

$$R = \sum_{j=m+1}^{n} v_j \; B_j \qquad (3\text{-}13)$$

and their cost, again in total,

$$C = \sum_{i=1}^{m} u_i \, X_i + \sum_{i=1}^{m} \sum_{j=m+1}^{n} s_{ij} \, x_{ij} \qquad (3\text{-}14)$$

In the aggregate, then, they will attempt to maximize:[88]

$$T = \Big(\sum_j v_j \, B_j - \sum_i u_i \, X_i\Big) - \sum_i \sum_j s_{ij} \, x_{ij} \quad (3\text{-}15)$$

The first part (in parentheses) of (3-15) is the quantity to be maximized in the dual, subject to the conditions of zero profit on all shipments actually made ($v_j - u_i \leqslant s_{ij}$). The second part is the value to be minimized in the primal subject to its availability and requirement constraints. It can then be said that the fictitious merchants attempt to maximize (3-15) subject to the constraints (3-2) to (3-5) and (3-11), (3-12). But maximization of (3-15) subject to these constraints is equivalent to separate minimization of (3-1) subject to (3-2) to (3-5), and maximization of (3-10) subject to (3-11) and (3-12).[89] The maximum value of T is zero, by the fundamental theorem of linear programming.[90] The merchants, that is, end up without even the normal profits at equilibrium. This is due to the fact that this is only a partial model, and the existence of these merchants to force the equilibrium in the competitive economy may be justified by reference to disequilibrium profits that they would make.[91]

In another interpreation,[92] one can disregard production costs and proceed to determine the dual values by arbitrarily setting one value u_i equal to zero and solving for the rest. Values so computed should be positive for a direct economic interpretation and this is assured by trial and error in setting various u_i equal to zero.[93] The computed values of u_i can be ranked in descending order of magnitude, and the interpretation would be that an origin with higher u_i than another is better located than the second, given the demands and transportation costs. Proximity to markets (given the demands and the transportation costs) is then reflected by the u_i. Transportation costs per unit do not reflect this characteristic because they are one factor in determining total transportation costs, the other being the interregional shipments found in the optimal solution of the primal.

When it comes to the v_j, however, this approach in interpreting the dual can only say that they correspond to the most economic distribution of output from the point of view of total transportation cost.

Lastly, one interpretation relates the dual variables to location rents.[94] This interpretation abstracts, like the previous one, from production and in this sense it removes the transportation model from its place as part of the general economic problem of 3.0.3. The given amounts of X_i at each location are here to be thought of as given by technology, e.g., warehouse capacity. The problem is then transformed into the following: A number m of warehouses having capacity X_i ($i = 1, .. m$) are full of a commodity to be distributed to a number n-m of consumption locations, each having a given requirement B_j ($j = m+1, ... n$). Given the unit transportation costs s_{ij}, the question is again to find the minimum-transportation-cost pattern of shipments. In this situation, the commodity available at the m warehouses may be assumed to have a given cost p, same for all locations. The dual constraints can then be written

$$(v_j + p) - (u_i + p) \leqslant s_{ij} \quad \begin{pmatrix} i = 1 \dots m \\ j = m+1, \dots n \end{pmatrix}$$

Calling $(u_i + p)$ the FOB <u>price</u> of the commodity at location i and $(v_j + p)$ the delivered price at j and assuming one consumption location and three warehouse locations we have, for the dual constraints:[95]

$$\begin{aligned} (v_5 + p) - (u_1 + p) &\leqslant s_{15} \\ (v_5 + p) - (u_2 + p) &\leqslant s_{25} \\ (v_5 + p) - (u_3 + p) &\leqslant s_{35} \end{aligned} \qquad (3\text{-}16)$$

Assuming $\sum_i X_i = \sum_j B_j$, all three constraints will be satisfied with equality. The solution to this problem, is, of course, trivial. The delivered <u>price</u> at the consumption location will be equal to p plus the highest unit transportation cost. Assuming $s_{15} > s_{25} > s_{35}$ the delivered price will be $p + s_{15}$. But it was also defined as $(v_5 + p)$. Substitution of $(p+s_{15})$ for (v_5+p) in the equality form of (3-16) yields

$$\left.\begin{aligned} u_1 &= s_{15} - s_{15} = 0 \\ u_2 &= s_{15} - s_{25} \\ u_3 &= s_{15} - s_{35} \end{aligned}\right) \qquad (3\text{-}17)$$

and the u_i can be directly interpreted as location rents. The warehouse manager at location 3, that is, can charge $p+u_3$ for his commodity, u_3 being his rent due to better location of his warehouse relative to the demand location. His rent is quantitatively equal to the difference between the highest transportation costs (s_{15}) and the transportation costs from his own (s_{35}). The warehouse at location 1 can dispose of its product only if it changes a FOB price equal to p, hence with $u_1 = 0$.

Now assume that $X_1 + X_2 + X_3 > B_5$ but also that $X_2 + X_3 < B_5$, so that all constraints in (3-16) are again satisfied with equality. If we then introduce a fourth warehouse with capacity X_4 such that

$$X_4 < B_5 - (X_2 + X_3)$$

and located at the point of consumption $(s_{45} = 0)$ we get

$$\left.\begin{aligned} u_1 &= s_{15} - s_{15} = 0 \\ u_2 &= s_{15} - s_{25} \\ u_3 &= s_{15} - s_{35} \\ u_4 &= s_{15} \end{aligned}\right) \qquad (3\text{-}18)$$

The warehouse at location 4, that is, can change $p+u_4$ for his commodity. But this will be equal to the delivered price, since

$$(v_5 + p) - (u_4 + p) = s_{45} = 0 \qquad (3\text{-}19)$$

Obviously, then, $v_5 = u_4$: the v_j's, that is, are the location rents that warehouses would earn if located at the points of consumption, and there is no conceptual difference between them and the u_j. Put in another way, the location rent v_5 is equal to the transportation cost of the first warehouse, since

$$(v_5 + p) = (s_{15} + p) \qquad (3\text{-}20)$$

and is "earned" by the warehouses at a rate discounted by the transportation cost:

$$\begin{aligned} u_4 &= s_{15} - s_{45} = v_5 \\ u_3 &= s_{15} - s_{35} = v_5 - s_{35} \\ u_2 &= s_{15} - s_{25} = v_5 - s_{25} \\ u_1 &= s_{15} - s_{15} = 0 \end{aligned} \qquad (3\text{-}21)$$

An explanation of the quantitative restrictions on the X_i imposed above is in order: $X_2 + X_3 < B_5$ was imposed so that warehouse 1 would remain in the "optimal" solution. If $X_2 + X_3 \geqslant B_5$, the given relationships between unit transport costs dictate satisfaction of B_5 solely from locations 2 and 3, in which case the rents change to

$$\begin{aligned} u_2 &= s_{25} - s_{25} = 0 \\ u_3 &= s_{25} - s_{35} \end{aligned} \qquad (3\text{-}22)$$

and $u_5 = s_{25}$

and the first constraint of (3-16) is satisfied with an inequality sign in the solution (that is, $x_{15} = 0$ in the solution of the primal).

Also, if the introduced X_4 is greater than the difference B_5 minus

$(X_2 + X_3)$, which was supplied by warehouse 1 before the introduction, the fourth warehouse will displace warehouse 1 (and possibly 2 and 3, but assume this away for the moment) and the structure of rents will be:

$$\begin{aligned} u_2 &= s_{25} - s_{25} = 0 \\ u_3 &= s_{25} - s_{35} \\ u_4 &= s_{25} = v_5 \end{aligned} \qquad (3\text{-}23)$$

It is obvious from (3-17) that, given the X_i and B_j, the rent structure depends on the transportation costs. Also, from (3-17) and (3-22), given the transportation costs, the rent structure depends on capacity in its relation to demand. Changes in capacity and/or demand that are large enough will change the rent structure (i.e. the solution to the problem). The example was given above as regards to changes in capacity. In a case pertaining to demand, if B_5 is changed to B'_5, where $B'_5 \leqslant X_2 + X_3$, the rent structure of (3-17) will be changed to that of (3-22).

The above examples, involving only one consumption location, are obviously simplified to such a degree that they render the solution to the transportation problem trivial. They do, however, make clear the notion of dual prices in terms of location rents. In a more complicated example the relationships between u_i, v_j, s_{ij} and capacities and demands would not be so simple, but the interpretation would carry over directly.

Finally, there is the formal "mathematical" interpretation, common to linear programming problems: u_i expresses the change (decrease) in total transportation costs that would result if the capacity of origin i were to be increased by one unit. Similarly, v_j can be interpreted exactly as u_i and, also, as the increase total in transportation costs that would result if the requirement B_j increased by one unit. This interpretation is, of course, in "marginal" terms, that is it does not necessarily apply for a "lump-sum" change. Also, it is conceptually more directly applicable to the problem in 3.0.2 since changes in capacity or demand would violate equality (3-5) of the problem in 3.0.1. This may be one reason why this explanation is not common in the literature on the transportation problem.

Needless to say, all four interpretations are useful. The first shows the interregional price structure to be determined by the dual, given one price. The second is of value in explaining locational advantage, though it faces some difficulties in explaining the v_j. The fourth is directly applicable as an investment criterion, as will be seen below. Finally, the third is to the writer of this thesis the most satisfactory one from the point of view of linking the transportation model to the general problem of 3.0.3. Stevens did not proceed to this linkage, however, and his concept of given cost at each warehouse may create some uneasiness since it is not linked to production. I shall attempt to bring forth this linkage in the next few paragraphs.

Consider two regions facing constant returns to scale in the production of one commodity, with the same variable costs per unit (but not necessarily with the same capacity in production). The back-to-back diagram in this case would be as in Fig. 3.3.

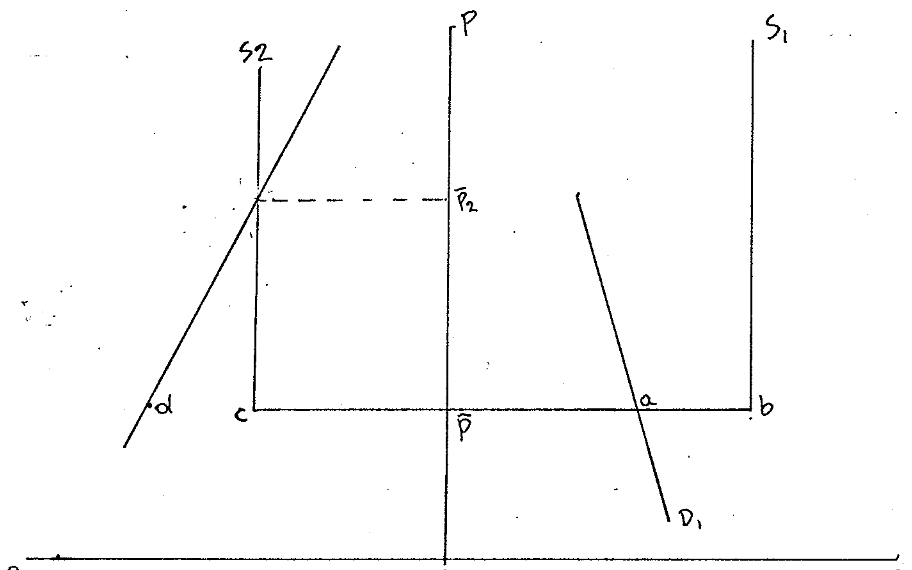


Fig. 3.3

Region 1 is seen to have an "excess capacity" of ab at the pre-trade equilibrium price $\bar{p}$. Region 2 is facing a strong demand in relation to its "capacity" and operates at full capacity at the price $\bar{p}_2$. The situation obviously involves the short-run. The capacity restrictions need not refer to capital stock; they may be due to any resource. The excess curves ($S_i - D_i$) are as in Fig. 3.4 below.

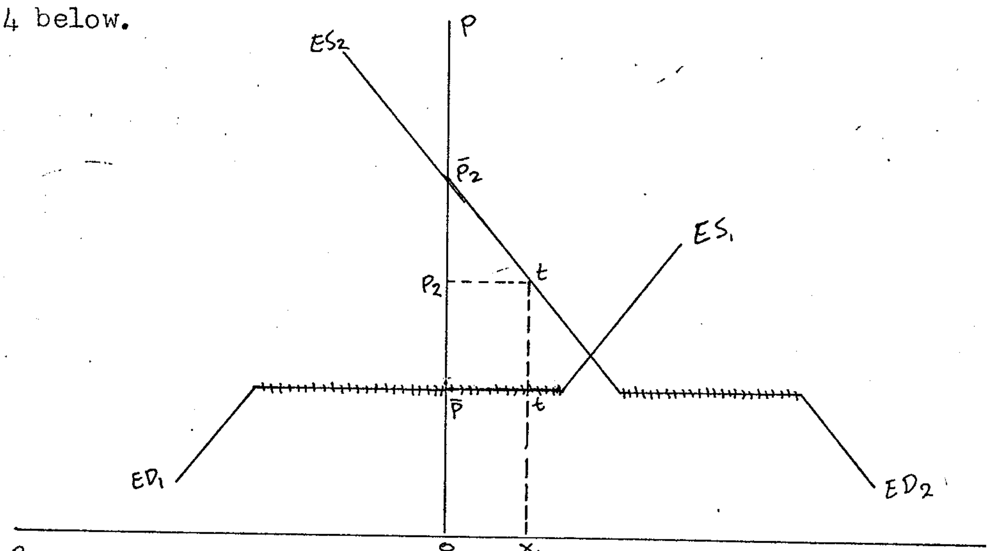


Fig. 3.4

The labelling is meant to show that an excess supply curve is actually an excess-demand curve at prices below the equilibrium.

If $s_{12} = tt$, the after-trade equilibrium price in region 2 will be

$$p_2 = \bar{p} + s_{12}$$

and imports B_2 will be equal to X_1. Producers in region 1 will be charging $\bar{p}$ in a world of perfect competition, and they will be exporting X_1. Their location rent, u_1, will then be equal to zero since they are the (single) farthest supplying origin to destination 2. On the other hand,

$$v_2 = s_{12} - s_{22} = s_{12}$$

In other words, Stevens' base price or cost, p, is $\bar{p}$ in our example. Thus, Stevens' interpretation is compatible with a general Samuelson-Cournot problem where technology is the same in all regions and exhibits constant returns to scale.

Actually, B. Stevens constructs a supply curve at the single destination which implies a constant base price p irrespective of output, and his demand curve is inelastic with respect to price. But he stops short of interpreting p as the common constant-per-unit variable costs in the general problem. The above explanation is deemed to justify the assumption of a "base price", as well as to make clear the reason why, in a linear technology, it is capacity combined with location that makes for the existence of rents (given the demand and transportation costs). For suppose that capacity in region 2 was not $\bar{p}c$but $\bar{p}d$: u_i and v_j would then be all zero. With the same unit variable costs and constant returns to scale, and given the demands and the transportation costs, it is only regional capacity constraints which give rise to trade, and hence earn rents. When the variable cost conditions are different we cannot ascribe the whole values of u_i and v_j to location. In fact, a Stevens approach seems inapplicable, since the base price is not unique but there is one distinct p_i for each origin which cannot be determined by the transportation model. In this case, location and scarcity rent are mixed in u_i.[96]

In fact, even in this last "linkage" case that I have been describing, things are not so clear: things were rather "engineered" so that the pre-trade and after-trade price in the exporting region is the same. This was so because exports of region 1 at price $\bar{p}$ were smaller than the excess capacity in region 1. Another type of diagram, equivalent to Fig. 3.4 may help (Fig. 5).

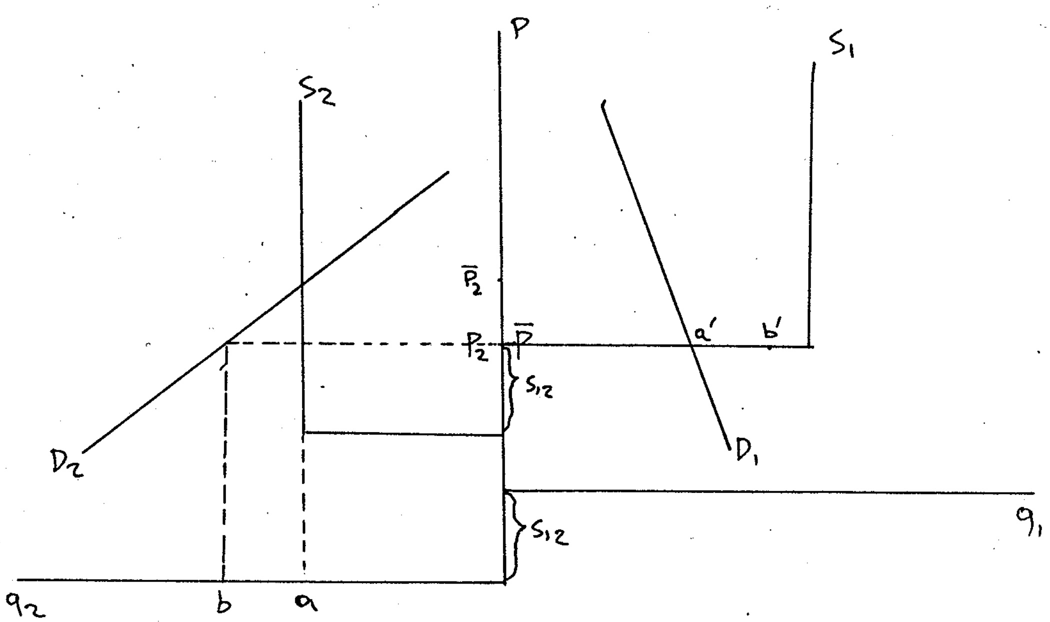


Fig. 3.5

In Fig. 3.5, the axes of region 1 have been displaced upwards by s_{12}, to show that exports from region 1 to 2 will cost the price charged at region 1 plus transport cost. When trade opens there is a meaningful price spread $\bar{p}_2 - \bar{p}$. Region 1 exports and 2 imports till this spread vanishes. This happens when the price $\bar{p}_2$ falls to p_2 (actually $\bar{p} + s_{12}$, due to axis transposition). Imports to region 2 (B_2) are ab, exports from 1 are a'b' and ab = a'b' = X_1 in Fig. 3.4.

The special characteristic in these figures is that exports of region 1 bring forth interregional price equilibrium before the excess capacity in the region is exhausted. If we consider Fig. 3.6, however (the difference is made by changing the capacity in 1, but could also have been brought about by reducing transport costs or by increasing demand) things are not as clear as in Fig. 3.5.

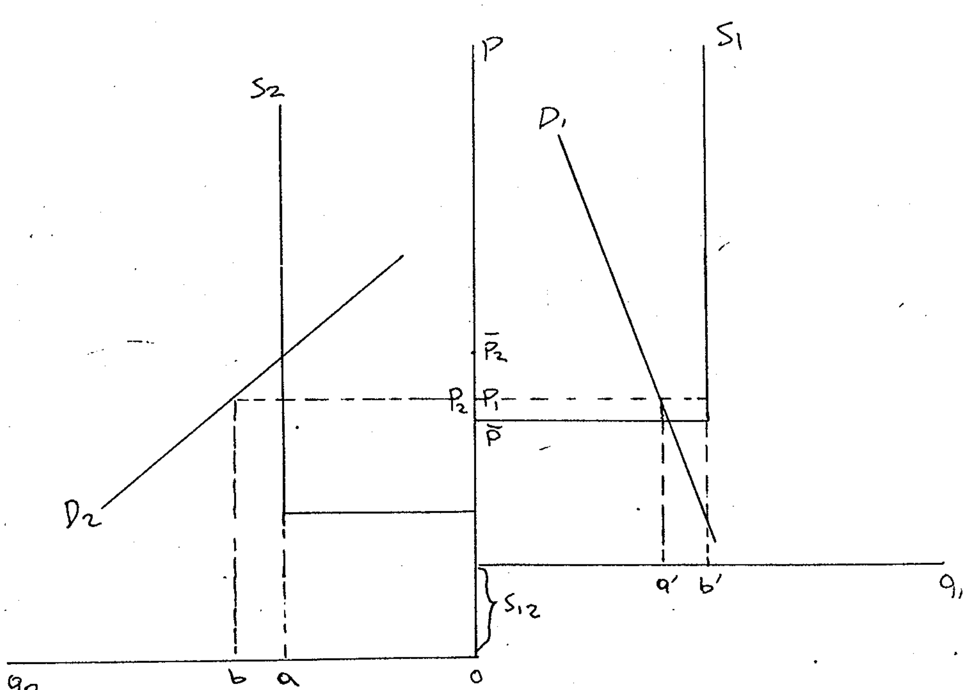


Fig. 3.6

Producers in region 1 change p_1 for their output in inter-regional equilibrium which is higher than $\bar{p}$. We still have, of course,

$$p_2 = p_1 + s_{12}$$

but the relationship of Stevens' base price to production costs is somehow lost: something has interfered, and I would hazard that it is relative scarcity of total capacity in relation to demand; the total rent $s_{12} = \bar{p}_2 - \bar{p}$ which was earned by location 2 in Figs. 3.4 and 3.5 is now increased to $s_{12} + \overrightarrow{\bar{p}\, p_2}$ and location 1 is also earning $p_1 - \bar{p}$. This analysis cannot be done through linear programming, however, as will be seen in section 4.0.

CHAPTER IV

SOME GENERALIZATIONS OF THE TRANSPORTATION MODEL (III)

In this Chapter I will attempt a presentation of two models by A. Hurter.[97] The first is a generalization of the transportation model in 3.02 to include production costs. The second is a full-fledged multi-commodity multi-region model utilizing techniques of general activity analysis. This chapter also includes an "intermediate" model (V) designed as a smooth step to Hurter's general model.

4.0 Model IV: Minimization of Production plus Transportation Costs

The applicability of the transportation model in 3.0.1 to questions of general interregional equilibrium is obviously limited by the assumptions of given supplies and demands. Such a model cannot consider questions of spatial allocation of production and the effects of transportation costs on such allocation. Moreover, it is a one-commodity model and in this sense it is partial. Sections 3.0.2 - 3.0.4 point towards generalizations of the transportation model to relax the assumption of given supplies at the origins. This is what the model of this section amounts to.

Consider, first, the model in 3.0.2 (III): one obvious "generalization" would be to drop the distinction between "origin" and "destination" regions and assign a supply and a demand at every region. As long as capacity in region i is not greater or equal to demand in region i for all i, there would exist deficit regions where $X_i - B_i < 0$, and surplus regions where $X_i - B_i > 0$. X_i obviously does not represent total exports of region i, neither does B_j represent total imports. Rather, when $X_i - B_i < 0$, this number represents the ~~total~~ imports of region i. When $X_i - B_i > 0$, this number represents the upper limit to exports (since $\sum X_i > \sum B_i$ the actual exports of a surplus region can be less than $X_i - B_i$).

The condition that $X_i - B_i < 0$ for some regions is necessary for the problem not to be trivial: if $X_i \geq B_i$ for all i and $s_{ii} = 0$ the solution is obviously that each region supplies its own demand and total transportation cost is zero.[98] For the possibility of trade to exist in this scheme, then, the above condition must hold.

Actually, given the assumption that intraregional transportation costs are zero ($s_{ii} = 0$), the above excursion is not really a generalization: for consider the possibilities for v regions;

$$\left.\begin{array}{lll} X_i - B_i > 0 & & i = 1 \; \ldots\ldots \; m \\ X_j - B_j < 0 & & j = m+1, \; \ldots\ldots \; n \\ X_k - B_k = 0 & & k = n+1 \; \ldots\ldots \; v \end{array}\right\} \quad (4\text{-}1)$$

each region must fall into one of these three categories, and the numbering was made with these categories in mind. Now define

$$\left.\begin{array}{lll} E_i \equiv X_i - B_i & & i = 1, \ldots m \\ I_j \equiv B_j - X_j & & j = m+1, \ldots n \end{array}\right\} \quad (4\text{-}2)$$

The above "generalization" is then shown to be trivial, in the sense that it introduces no new elements into the problem: since regions $m+1$ to v do not participate in trade, and since $s_{ii} = 0$ the problem is still

$$\text{Minimize} \quad \sum_{i=1}^{M} \sum_{j=M+1}^{n} s_{ij} \quad x_{ij} \tag{4-3}$$

subject to

$$\sum_{j=M+1}^{n} x_{ij} \leqslant E_i \quad (i = 1, \ldots M) \tag{4-4}$$

$$\sum_{i=1}^{n} x_{ij} \geqslant I_j \quad (j = m+1, \ldots n) \tag{4-5}$$

$$x_{ij} \geqslant 0 \text{ for all } i, j \tag{4-6}$$

and

$$\sum_i E_i > \sum_j I_j \tag{4-7}$$

and this is identical to the problem in 3.0.2 eq. (3-1a) to (3-5a) . This was intuitively obvious, given the assumptions.

A meaningful generalization is seen to involve production costs at each region. If the objective is defined as minimization of the sum of production and transportation costs we cannot be sure that a surplus region will not import. For suppose that $X_1 - B_1 > 0$ and that $c_1 > c_2 + s_{21}$ where c_i is the constant unit cost of production in region i; it is then quite possible that the region may import in the final solution (depending on the total surplus on the other regions and on the structure of transportation costs). Thus, we cannot maintain the definitions (4-2) any longer. Instead, the problem is,

$$\text{Minimize} \quad \sum_{i=1}^{v} \sum_{j=1}^{v} c_{ij} \quad x_{ij} \tag{4-8}$$

subject to

$$\sum_{j=1}^{v} x_{ij} \leqslant X_i \quad (i = 1, \ldots v) \tag{4-9}$$

$$\sum_{i=1}^{v} x_{ij} \geqslant B_j \qquad (j=1, .. v) \qquad (4\text{-}10)$$

$$x_{ij} \geqslant 0 \qquad \text{for all } i,j \qquad (4\text{-}11)$$

and

$$\sum_{i=1}^{v} X_i > \sum_{j=1}^{v} B_j \qquad (4\text{-}12)$$

where:

$c_{ij} = c_i + s_{ij}$ the sum of unit production cost in region i and the unit transportation cost from region i to region j

X_i = the maximum productive capacity in region i

B_j = the given demand in region j

In mathematical terms, the model (4-8) to (4-12) is seen to be the transportation model III (eqs. (3-1a) to (3-5a) with transportation costs equal to c_{ij}. The economic interpretation changes considerably, however: the model(4-8) to (4-12) incorporates production as well as transportation. The choice involved here is the combined one of where to produce and where to ship the commodity, and is to be made with the objective of minimization of total production and transportation costs.

That the implications of this model differ from those of 3.0.2 can be seen from an interpretation of the dual in the simple example which follows.[99]

Consider 2 regions with the following supply and demand relations:

$$D_1 = 100 - 4p_1 \qquad 0 \leqslant p_1 \leqslant 25 \qquad (4\text{-}13)$$

$$D_2 = 120 - 6p_2 \qquad 0 \leqslant p_2 \leqslant 20 \qquad (4\text{-}14)$$

$$S_1 \leqslant 100 \qquad p_1 \geqslant 5 \qquad (4\text{-}15)$$

$$S_1 = 0 \qquad p < 5$$

$$S_2 \leqslant 150 \qquad p \geqslant 15 \qquad (4\text{-}16)$$

$$S_2 = 0 \qquad p < 15$$

$$s_{12} = 5 \qquad s_{21} = 1 \qquad (4\text{-}17)$$

Where D_i, S_i, p_i represents demand, supply and price in region i and s_{ij} the cost of transportation from i to j. The supply functions in this example exhibit constant returns to scale in production, with capacity limits 100 and 150 units for regions 1 and 2 respectively. The pre-trade prices are (see Fig. 4.1, below)

$$\bar{p}_1 = 5 \qquad \bar{p}_2 = 15 \qquad (4\text{-}18)$$

with $D_1 = S_1 = 80$ and $D_2 = S_2 = 30$. (4-19)

Since $\bar{p}_1 - \bar{p}_2 < s_{21}$ and $\bar{p}_2 - \bar{p}_1 > s_{12}$ region 1 will be an exporter and region 2 an importer. The excess supply functions in the two regions are: (see Fig. 4.2, below)

$$ES_1 = 100 - (100 - 4p_1) = 4p_1 \qquad p_1 \geqslant 5 \quad (4\text{-}20)$$

$$ED_2 = 120 - 6p_2 \qquad p_2 \leqslant 15 \quad (4\text{-}21)$$

since, at after-trade equilibrium, we have

$$ES_1 = ED_2 \text{ and } p_2 - p_1 = s_{12} \qquad (4\text{-}22)$$

equating (4-20) to (4-21) and using (4-22) we find

$$p_1 = 9 \qquad p_2 = 14$$

$$D_1 = 64 \qquad D_2 = 36 \qquad (4\text{-}23)$$

$$S_1 = 100 \qquad S_2 = 0$$

Graphical representations of the pre-trade and after-trade solutions are shown in Figs. 4.1 and 4.2.

Consider now the linear programming model (4-8) to (4-12): the relevant data must be,

$$\begin{aligned} &D_1 = 64 && D_2 = 36 \\ &c_1 = 5 && c_2 = 15 \\ &s_{12} = 5 && s_{21} = 1 \\ &X_1 = 100 && X_2 = 150 \end{aligned} \qquad (4\text{-}24)$$

Note that the after-trade equilibrium quantities demanded are taken from (4-23). The rest is original data. Thus, the model (4-8) to (4-12) is going to determine total regional exports by itself, and not require them as data, as was the case in 3.0.1. Also, it differs fromthe model of section 3.0.2 in that the latter chooses the interregional shipments x_{ij} which minimize total transportation costs only.

What enables us to generalize from the model of 3.0.2 to the model (4-8) to (4-12) is obviously the assumption of constant returns to scale, coupled with an implicit assumption of perfectly elastic factor supplies at given factor prices in each region. These two assumptions make for the specific shape of the supply functions in Fig. 4.1. The concept of a maximum capacity can then be associated with an upper physical limit to the supply of one of the factors. More on this point will be said in relation to the full-fledged model.

Given the data in (4-24) the (trivial in this case) problem is

$$\text{Minimize } C = 5\,x_{11} + 10\,x_{12} + 16\,x_{21} + 15\,x_{22} \qquad (4\text{-}25)$$

Subject to

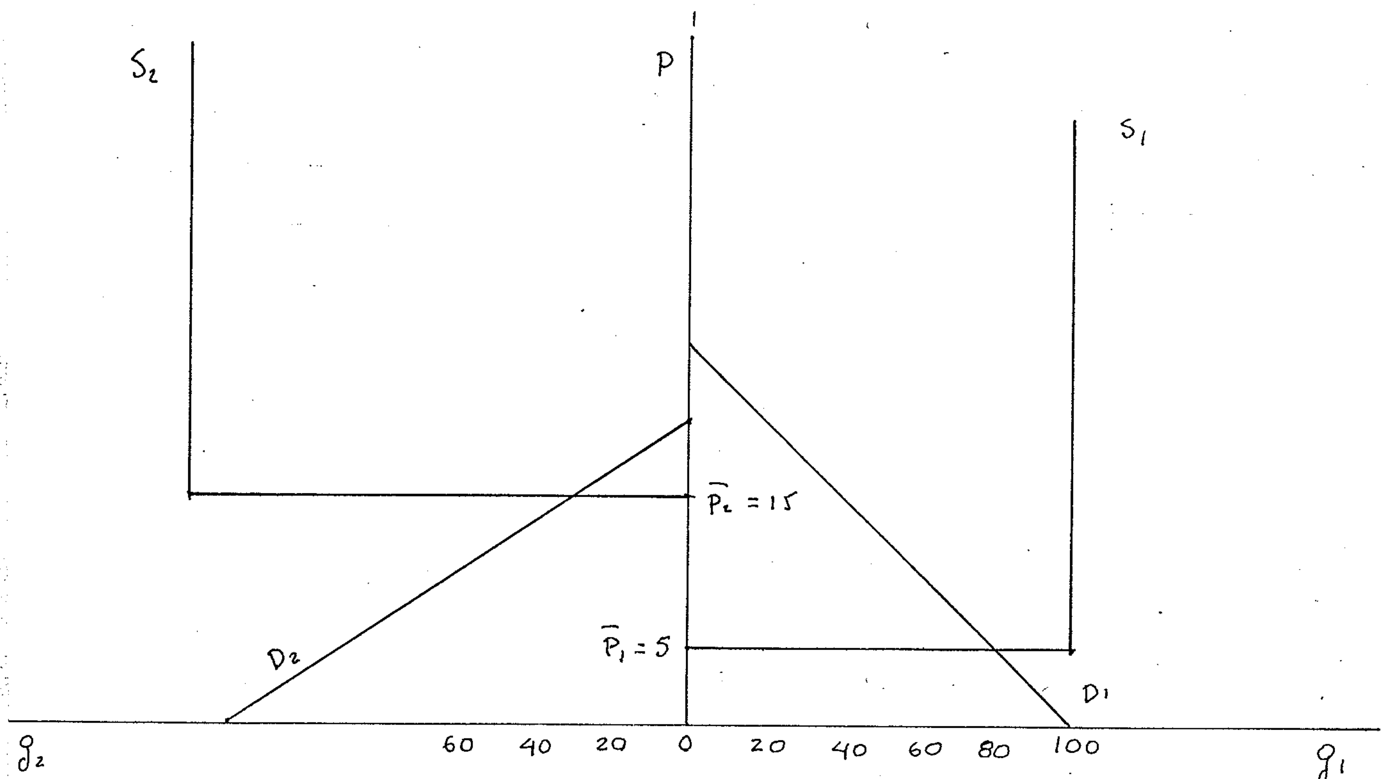

S2
P
S1
P̄2 = 15
D2
P̄1 = 5
D1
q2
60 40 20 0 20 40 60 80 100
q1

Fig 4.1

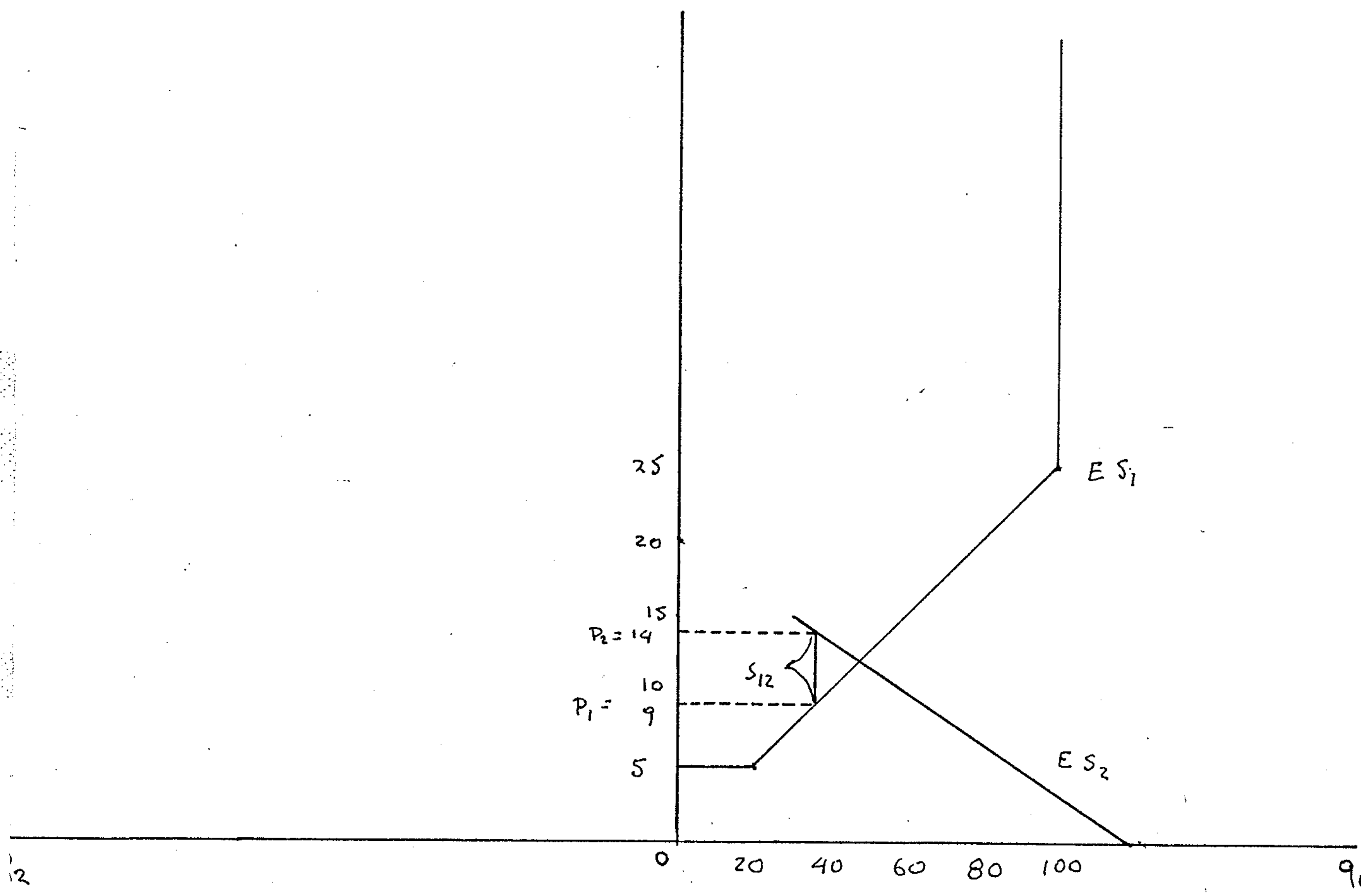

25
E S1
20
15
P2 = 14
S12
10
P1 = 9
5
E S2
q2
0 20 40 60 80 100
q1

Fig 4.2

$$
\begin{aligned}
-x_{11} \quad -x_{12} \quad -x_{13} &= -100 \\
-x_{21} \quad -x_{22} \quad -x_{23} &= -150 \\
x_{11} + x_{21} &= 64 \\
x_{12} + x_{22} &= 36 \\
x_{13} + x_{23} &= 150
\end{aligned}
\qquad (4\text{-}26)
$$

$$x_{ij} \geqslant 0 \qquad \text{for all } i, j \qquad (4\text{-}27)$$

where a fictitious destination 3 has been added to convert inequalities to equalities, as in 3.0.2. The optimal solution is

$$x_{11} = 64 \qquad x_{12} = 36 \qquad x_{22} = 0 \qquad x_{23} = 150 \qquad (4\text{-}28)$$

100

and (4-28) enumerates basic variables only. The minimum value of C is 680. The model is thus seen to choose not the interregional shipments x_{ij} which satisfy the demands at minimum total transportation cost (if it was to do this it would yield $x_{11} = 64$, $x_{22} = 36$, $x_{13} = 36$, $x_{23} = 114$), but the interregional shipments which satisfy the demands at minimum total production plus transportation cost. In the process, it determines importing and exporting regions, too.

The dual to this problem is:

$$\text{Maximize } 64v_1 + 36v_2 + 150v_3 + - 100u_1 - 150u_2 \qquad (4\text{-}29)$$

Subject to

$$
\begin{aligned}
v_1 - u_1 &\leqslant 5 && \text{(a)} \\
v_1 - u_2 &\leqslant 16 && \text{(b)} \\
v_2 - u_1 &\leqslant 10 && \text{(c)} \\
v_2 - u_2 &\leqslant 15 && \text{(d)} \\
v_3 - u_1 &\leqslant 0 && \text{(e)} \\
v_3 - u_2 &\leqslant 0 && \text{(f)}
\end{aligned}
\qquad (4\text{-}30)
$$

$$\text{and } v_j,\ u_i \geqslant 0 \qquad (4\text{-}31)$$

Since the optimal solution to the primal involves x_{11}, x_{12}, x_{22} and x_{23}, constraints (a), (c), (d) and (f) will be satisfied with an equality in the dual. We have:

$$\begin{aligned} v_1 - u_1 &= 5 \\ v_2 - u_1 &= 10 \\ v_2 - u_2 &= 15 \\ v_3 - u_2 &= 0 \end{aligned} \qquad (4\text{-}32)$$

and since the fictitious destination 3 does not have any actual demand at all, we set $v_3 = 0$ and obtain

$$u_1 = 5 \quad u_2 = 0 \quad u_1 = 10 \quad u_2 = 15 \quad u_3 = 0 \qquad (4\text{-}33)$$

with the objective function of the dual attaining the value 680, as expected.

One direct interpretation of the u_i values is the fourth offered in section 3.0.4, namely: $u_1 = 5 > 0 = u_2$ shows that, given the technology, the structure of transportation costs and the demand, region 1 satisfies that demand most economically from the point of view of total production and transportation costs. It should thus be considered first when additions to capacity (due to increased demand) are considered. To see this, consider a shift in the demand of region 2, shifting the curve ES_2 in Fig. 4.2 to the right so that it does not intersect ES_1: this will mean that production in region 2 will have to start to satisfy some of the region's demand at $p_2 = 15$ per unit. But if the capacity of region 1 had increased, production in region 2 would not be required. Total system costs would increase by 10 per unit (5 for production+5 for transportation) and this would represent a saving of 5 per unit, which is equal to u_1. Be it noted again that this interpretation is in the "marginal" sense. Moreover, the values of u_i depend on (among other factors) the technology available in each region as given by the production costs. Their use as indicators for investment is then strictly dependent on the requirement that investment will change only the capacity but not the unit costs. Obviously, if we could invest in region 2 and create capacity to produce at costs lower than 15 per unit, say 9 per unit, the ranking of regions according to the u_i found would not be helpful in telling us whether this investment is to be considered desirable. But then, this difficulty can be overcome by resolving the problem as if this investment had been done and comparing the first to the second situation from the point of view of total system costs. On the other hand, $u_i = 0$ means that, given the data, investment in region 2 will not help in reducing system costs.

When it comes to the v_j, this interpretation can only refer to them as the delivered <u>prices</u>[101] in regions 1 and 2. Given the general problem (4-13) to (4-23), this creates some doubt: price differentials are as they should be, that is

$$v_2 - v_1 = p_2 - p_1 = s_{12}$$

but the v_j are different from the p_j by one unit. This needs some elaboration.[102]

The model (4-13) to (4-17) differs from the linear programming formulation (4-25) to (4-27) in only one essential respect - the treatment of demand. The first, more general, model treats demand as variable with price and determines the equilibrium quantities demanded, while the linear programming model needs these equilibrium quantities as data and considers them as given and constant irrespective of prices. In diagrammatic terms, the linear programming model considers Fig. 4.3 instead of Fig. 4.1 and satisfies demands by using activities (at feasible levels) that involve the lowest total system costs. Demands are then considered independent of price and this makes for the discrepancy between the v_j and p_j.

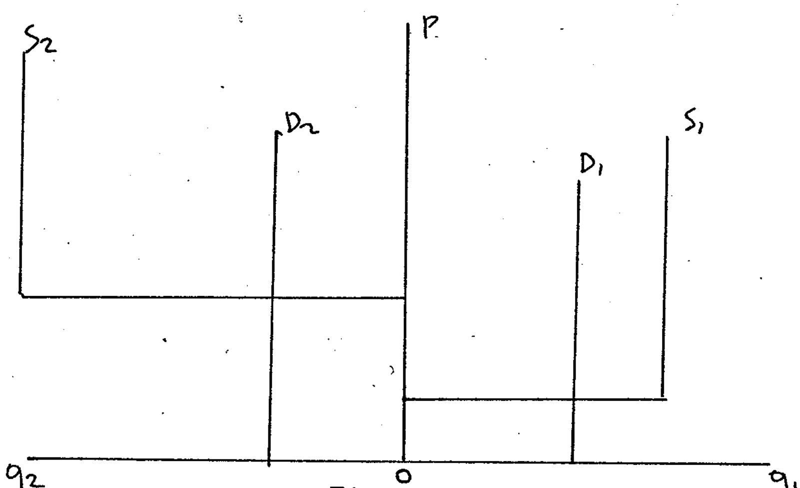


Fig. 4.3

Since, however, the price-differentials yielded by the linear programming model are consistent with those of the more general problem, and since equations (4-32) have one degree of freedom, we may consider setting one v_j equal to its respective p_j and recalculate v_j and u_i:

$$v_1 = p_1 = 9 \text{ yields } v_2 = 14 \quad u_1 = 4 \quad u_2 = 1 \quad v_3 = -1$$

and v_1, v_2 can now be directly interpreted as the delivered prices in regions 1 and 2. The objective function of the dual retains the same maximum value, and the u_i seem to me to have an interpretation as the excesses over cost assigned to the scarce capacities. Producers in region 1, that is, receive an additional 4 per unit of the commodity at equilibrium due to relatively scarce capacities. Consider the case of Figs. 3.3-3.5 of chapter 3, section 3.0.4: there, u_1 would be zero and $u_2 > 0$ with the same interpretation.

The meaning of v_3 in this interpretation is not clear, however. I was not able to pursue this further and I submit it as it stands.

The problem of delivered prices raised above has some interesting implications when one comes to the question of applicability of the linear programming model. For suppose we have good reasons to believe that for a certain commodity,

(a) there are numerous producers in each region,

(b) the production-cost behaviour is close to that depicted by infinitely elastic supply curves, and

(c) transportation costs per unit are constant.

Application of the simple linear programming model of this chapter might then be tried and demands would logically be the observed quantities demanded during the period under consideration. But if the actual situation in the economy is close to that depicted by Fig. 4.1, the linear programming model would not be able to yield the actual commodity prices unless one of them were given from outside. To the extent that demands are dependent on price, that is, the computed prices will differ from those observed. All this apart from other discrepancies that might arise because (a), (b) and (c) may not be good representations of the situation.

What is rather unfortunate in this connection is the fact that possible discrepancies of the computed values from the observed ones cannot be assigned unequivocally to each discrepancy of the model from the actual situation. Thus, a difference between computed and observed price differentials may be due to non-constant transport costs, but also to non-competitive elements. I shall return to this subject in the last chapter.

4.1 A GENERAL MULTI-REGION MULTI-COMMODITY MODEL

4.1.1 Generalizing Model IV

One major handicap of the model of section 4.0 from the point of view of general equilibrium analysis is,of course, that it deals with one commodity only. Thus it may be used in studies of one particular commodity market of a multiregional economy. One "extension" of this model that comes to mind is to use one such model for each final commodity. This "extension" does not make the model more general, however, for the following reasons.

The implicit assumption about given production costs c_i in region i of the model of 4.0 was indeed that all other industries in the economy were at equilibrium. Thus, our one-commodity producing industry was facing given prices for all its inputs and this fact, together with constant returns to scale in its production, yielded the special form of the supply curve of the previous section. The sources of supply of inputs were chosen to minimize input costs. If we "extend" the model to comprise more commodities, we still need the unit costs of production of each commodity in each region as a parameter of the model. But our assumption of equilibrium in the rest of the economy now becomes untenable, since we are in fact to find this equilibrium from the solution to our model. If this assumption is dropped, then the prices and sources of inputs of each industry are to be determined from the solution of the model, and they will in turn determine the costs of production. We are in a vicious circle, that is: we cannot start solving the problem without the unit costs of production, but we must solve it first if we are to find these costs. To show what the problem exactly amounts to I will first construct a highly hypothetical situation that avoids this impasse, and then proceed to drop the simplifying assumptions.

Consider a world with two[103] primary factors and no intermediate goods needed in production. Each region has a pool of these factors available to <u>its own</u> two[104] industries for use in the production of final goods. The primary factors, that is, are non-transportable between regions. The supply of these factors is perfectly elastic at a given price for each, with an upper quantity limit imposed by availability. Each industry uses these factors in fixed quantitative ratios to produce its single output (no joint production, one technique) and all industries enjoy constant returns to scale.

In this highly simplified situation, the unit costs of production in each industry are given when the unit prices of the primary factors are given. The general formula would be:

$$ {}_jc_i = {}_jb'_{1i} \cdot {}_jp_1 + {}_jb'_{2i} \cdot {}_jp_2 $$

where ${}_jc_i$ = unit cost of production of commodity i (by industry i)

in region j,

${}_jb'_{ki}$ = unit physical requirement of resource k ($k = 1, 2, 3$) for the production of commodity i in region j, and

${}_jP_k$ = unit price of resource k in region j.

Differences in production costs between industries of the same region j would then arise because of differences in ${}_jb'_{ki}$. Differences in costs between the same industry in different regions would arise because of differences in both ${}_jb'_{ki}$ and ${}_jP_k$.

Now drop the assumption of non-existence of intermediate commodities and concentrate in one region assuming it as closed. Its technology, resource availability and demand for final use are given by (4-34), (4-35), and (4-36):

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad (4\text{-}34) \qquad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad (4\text{-}35) \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (4\text{-}36)$$

In (4-34), a_{ik} denotes the quantity of commodity i needed as intermediate input in the production of one unit of commodity k, b_{ik} denotes the quantity of resource i needed as input in the production of one unit of commodity k, R_i represents the quantity of resource i available, and B_i is the demand for commodity i for final use (consumption, etc.).

In equilibrium, the following relation must hold (where I_i

is "intermediate" demand for the product of industry i):

$$X_i = B_i + I_i \quad (i = 1, 2) \qquad (4\text{-}37)$$

that is $X_i - I_i = B_i \quad (i = 1, 2) \qquad (4\text{-}38)$

But $I_i = a_{i1} X_1 + a_{i2} X_2$, and by substitution,

$$X_i - \sum_{k=1}^{2} a_{ik} X_k = B_i \quad (i = 1, 2) \qquad (4\text{-}39)$$

This will be recognized as an input-output system. Given the vector of final demand (4-36) one may seek to find the required gross (including intermediate uses) outputs of the industries that satisfy this demand. The solution is unique, since there is only one way of producing each commodity. This absence of choice makes possible the "netting out" of intermediate commodities by the following procedure: system (4-39) can be written:

$$\begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad (4\text{-}40)$$

that is (presupposing that the system is soluble)

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad (4\text{-}41)$$

or,

$$\begin{aligned} X_1 &= E_{11} B_1 + E_{12} B_2 \\ X_2 &= E_{21} B_1 + E_{22} B_2 \end{aligned} \qquad (4\text{-}42)$$

Where E_{ik} are the elements of the inverse matrix of (4-41). One unit of commodity 1 for final use is then seen to require E_{11} units of gross output of sector 1, and E_{21} of output of Sector 2. With technologies unchanged these requirements do not change. But one unit of gross output of industry i requires b_{1i} units of the first resource. Thus, the direct and indirect resource requirements for one unit of final commodity 1 are:

$$b_{11}E_{11} + b_{12}E_{21} \equiv b'_{11} \qquad (4\text{-}43)$$

and similarly for the other commodity,

$$b_{11}E_{12} + b_{12}E_{22} \equiv b'_{12} \qquad (4\text{-}44)$$

and for the other resource,

$$b_{21}E_{11} + b_{22}E_{21} \equiv b'_{21}$$

$$b_{21}E_{12} + b_{22}E_{22} \equiv b'_{22} \qquad (4\text{-}45)$$

Since the b_{ik} and E_{ik} depend only on technology which is assumed unchanging, we can "net out" intermediate commodities from 4-34 and write,

$$\begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{bmatrix} \qquad (4\text{-}46)$$

the b'_{ik} in (4-46) representing the direct and indirect resource needs of industry k from resource i in order that the industry produce one unit of <u>final</u> output. This last technology matrix is of the same form as that used in the example of section 2.2.0.1. It is thus seen that assuming away intermediate commodities in that case did not really cause any problem presupposing that we interpret the b'_{ik} correctly.[105]

It cannot be overemphasised that "netting out" of intermediate commodities was possible because of the basic assumption that there exists only one **activity** for the production of each commodity. If more than one activity exists for the same commodity, the solution to the problem (4-39) to (4-42) involves choice, even with a given final demand vector. This means that we cannot follow the above "netting" procedure from the beginning; intermediate commodities will have to be kept in the problem throughout. Since intermediate commodities require primary factors for their production, an appropriate optimizing criterion will make sure that no waste takes place. More on this last point will be said below.

The above example of "netting out" the intermediate commodities was done, it should be noted, by concentrating in one region and assuming it closed. Hence, given the other assumptions, there was no choice involved in satisfying the final demand of the region. When this intermediate assumption of closedness is dropped, however, and the possibility of transportation of commodities (intermediate and final, but not primary) is recognised to be possible at a transportation cost s_{ij}, there are obviously more than one ways of satisfying final demand in a region. Hence, intermediate goods must be kept in the model. Since they require primary factors for their production, the logical step is to consider primary factor costs as the only costs of production for the system. Intermediate commodities, that is, are not "netted out" from the beginning but during the choice process according to the criterion of minimizing costs. This means that the b_{ik} of (4-34) will be used, not the b'_{ik} of (4-46) and the technology will be in the form of (4-34).

For three regions, two commodities and two primary factors we have, for region 1,

$$\begin{bmatrix} {}_1a_{11} & {}_1a_{12} \\ {}_1a_{21} & {}_1a_{22} \\ {}_1b_{11} & {}_1b_{12} \\ {}_1b_{21} & {}_1b_{22} \end{bmatrix} \quad (4\text{-}47a) \qquad \begin{bmatrix} {}_1R_1 \\ {}_1R_2 \end{bmatrix} \quad (4\text{-}48a) \qquad \begin{bmatrix} {}_1B_1 \\ {}_1B_2 \end{bmatrix} \quad (4\text{-}49a)$$

for region 2,

$$\begin{bmatrix} {}_2a_{11} & {}_2a_{12} \\ {}_2a_{21} & {}_2a_{22} \\ {}_2b_{11} & {}_2b_{12} \\ {}_2b_{21} & {}_2b_{22} \end{bmatrix} \quad (4\text{-}47b) \qquad \begin{bmatrix} {}_2R_1 \\ {}_2R_2 \end{bmatrix} \quad (4\text{-}48b) \qquad \begin{bmatrix} {}_2B_1 \\ {}_2B_2 \end{bmatrix} \quad (4\text{-}49b)$$

and for region 3

$$\begin{bmatrix} {}_3a_{11} & {}_3a_{12} \\ {}_3a_{21} & {}_3a_{22} \\ {}_3b_{11} & {}_3b_{12} \\ {}_3b_{21} & {}_3b_{22} \end{bmatrix} \quad (4\text{-}47c) \qquad \begin{bmatrix} {}_3R_1 \\ {}_3R_2 \end{bmatrix} \quad (4\text{-}48c) \qquad \begin{bmatrix} {}_3B_1 \\ {}_3B_2 \end{bmatrix} \quad (4\text{-}49c)$$

The subscripts on the left refer to the region and imply the possibility of different technology, resource availability and demand per region. Transportation (minimum) unit costs of commodity k from region i to region j are denoted by ${}_{ij}s_k$ and the price of primary factor r in region i is denoted by ${}_ie_r$. Both are given:

$$\begin{bmatrix} {}_{11}s_1 & {}_{12}s_1 & {}_{13}s_1 & {}_{11}s_2 & {}_{12}s_2 & {}_{13}s_2 \\ {}_{21}s_1 & {}_{22}s_1 & {}_{23}s_1 & {}_{21}s_2 & {}_{22}s_2 & {}_{23}s_2 \\ {}_{31}s_1 & {}_{32}s_1 & {}_{33}s_1 & {}_{31}s_2 & {}_{32}s_2 & {}_{33}s_2 \end{bmatrix} \quad (4\text{-}50) \qquad \begin{bmatrix} {}_1e_1 & {}_2e_1 & {}_3e_1 \\ {}_1e_2 & {}_2e_2 & {}_3e_2 \end{bmatrix} \quad (4\text{-}51)$$

The objective is the satisfaction of final demand in all regions at minimum total production plus transportation cost.

Each region i still has one actual activity to produce commodity k, namely,

$$\left[{}_i a_{1k} \quad {}_i a_{2k} \quad {}_i b_{1k} \quad {}_i b_{2k}\right]' \qquad i = 1, 2, 3 \qquad k = 1, 2;$$

with transportation to other regions now possible, however, the unit costs associated with each activity change with the destination of its output because of different transportation costs. For example, for activity 2 in region 1 we find that the total unit costs are $\left[{}_1 b_{12}\right]\left[{}_1 c_1\right] + \left[{}_1 b_{22}\right]\left[{}_1 e_2\right] + s_{11}$ if the product is made available in the same region that it was produced. If the commodity is transported to region 2 costs become $\left[{}_1 b_{12}\right]\left[{}_1 e_1\right] + \left[{}_1 b_{22}\right]\left[{}_1 e_2\right] + s_{12}$, and similarly for transportation to region 3. It is thus found that activity 2 of region 1 has to be made into three separate activities according to destination of its output. We can then write:

$$\begin{bmatrix} -{}_1 a_{12} \\ 1-{}_1 a_{22} \\ -{}_1 b_{12} \\ -{}_1 b_{22} \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -{}_1 a_{12} \\ -{}_1 a_{22} \\ -{}_1 b_{12} \\ -{}_1 b_{22} \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -{}_1 a_{12} \\ -{}_1 a_{22} \\ -{}_1 b_{12} \\ -{}_1 b_{22} \\ 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{matrix} \qquad (4\text{-}52)$$

$$ {}_{11}X_2 \qquad {}_{12}X_2 \qquad {}_{13}X_2 \qquad (7)$$

$$\begin{matrix} \left[{}_1 b_{12}\right]\left[{}_1 e_1\right] + & \left[{}_1 b_{12}\right]\left[{}_1 e_1\right] + & \left[{}_1 b_{12}\right]\left[{}_1 e_1\right] + \\ \left[{}_1 b_{22}\right]\left[{}_1 e_2\right] + {}_{11}S_2 & \left[{}_1 b_{22}\right]\left[{}_1 e_2\right] + {}_{12}S_2 & \left[{}_1 b_{22}\right]\left[{}_1 e_2\right] + {}_{13}S_2 \end{matrix} \qquad (8)$$

The symbol ${}_{ij}X_k$ in the seventh row is to be interpreted as the level of activity producing commodity k in region i and making it available in region j. Row (8) lists the unit costs associated with each activity. Row (1) represents the inputs of commodity 1 required for a unit of production of commodity 2 in region 1. Row (2) lists the unit of commodity 2 made available in region 1 (the unity in the second row and first column) minus what is needed of same for its production ($_1a_{22}$). Rows (3) (4) are obvious. Row (5) represents commodity 2 produced in region 1 and made available in region 2. Row (6) represents commodity 2 produced in region 1 and made available in region 3.

The locution "made available in region ...." does not specify the use of the commodity in the receiving region. The commodity may be used as an intermediate input and/or as final output to satisfy the demand in that region. Thus, trade is made up of both intermediate and final commodities.

That there is more than one way of satisfying final demand in each region is now seen by the fact that, e.g., region 2 has at least three obvious ways of satisfying its demand for commodity 2. One choice is to produce it itself (using the technology of the second column of (4-47b), with the associated costs), another to import it from region 1 (using the second column of (4-52)) and another to import it from region 3. Actually, the problem of choice is more complicated than that, since each of the above choices involves other choices about the sources of intermediate inputs to be used in each activity.

4.1.2 Model V[106]

The discussion in the previous section points towards a number of generalizations of the transportation model of section 4.0 one of which proceeds as follows:

Suppose that the primary factors in the various regions are relatively abundant in relation to regional demand. Thus they do not constrain the levels of activities in each region. Their supply is still perfectly elastic at a given price, however (this is somewhat awkward under competitive conditions: see section 4.1.3). On the other hand, there is a maximum capacity in each industry of each region, associated with its fixed capital stock.

Since primary factors do not constitute constraints any more, the technology of each region (for three regions, two commodities, and two primary factors)[107] changes from that of (4-47) to (4-49) of the previous section, to the one below.

For region i (i = 1, 2, 3)

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -{}_i a_{11} & -{}_i a_{12} \\ -{}_i a_{21} & -{}_i a_{22} \end{bmatrix} \qquad \begin{bmatrix} {}_i A_1 \\ {}_i A_2 \end{bmatrix} \qquad \begin{bmatrix} {}_i B_1 \\ {}_i B_2 \end{bmatrix} \tag{4-53}$$

Where the -1 in the first row and column of the 2x2 matrix indicates that to produce commodity 1 in region i one unit of capacity ${}_i A_1$ is needed; similarly for the second row. The rest have already been explained.

Distinguishing activities by destination of output we write, for region 1

$$\begin{array}{cccccc} \multicolumn{6}{c}{\left[\begin{array}{cccccc} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1-{}_1a_{11} & -{}_1a_{11} & -{}_1a_{11} & -{}_1a_{12} & -{}_1a_{12} & -{}_1a_{12} \\ -{}_1a_{21} & -{}_1a_{21} & -{}_1a_{21} & 1-{}_1a_{22} & -{}_1a_{22} & -{}_1a_{22} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]} \\ {}_{11}x_1 & {}_{12}x_1 & {}_{13}x_1 & {}_{11}x_2 & {}_{12}x_2 & {}_{13}x_2 \end{array} \qquad (4\text{-}54)$$

with the explanation of rows as follows:

(1) Capacity of industry 1

(2) Capacity of industry 2

(3) Commodity 1 (see explanation of respective item in (4-52))

(4) Commodity 2, same explanation as in (3)

(5) Commodity 1 produced in region 1 and transported to region 2

(6) Same as above, for transportation to region 3

(7) Same as in (5), for commodity 2

(8) Same as in (6), for commodity 2 .

The primary factor and transportation costs associated with each activity are as in (4-52) of the previous section. As for the technological matrices of the two other regions they are similar to (4-54). The primal and the dual are presented in tabular form in Table 1.

Each column of Table 1 represents one activity that is to be interpreted as the activities in 4-54. At the bottom of each column is the symbol representing the level of each activity. At the top of each column are the costs associated with unit level operation of each activity. For example column seven represents the activity which produces commodity 1 in region 2 and transports it to region 1 ($_{21}X_1$). At the top of the column are the unit costs of this activity. Within the column, since this activity requires one unit of capacity for commodity 1 in region 2 ($_2A_1$), we find -1 in the respective row. The activity yields one unit of commodity 1 in region 1 (available for final demand $_1B_1$ or for intermediate use in region 1) and a +1 is found in the respective row. The unit requirements of the activity for intermediate inputs 1 and 2 are drawn from the respective pools in region 2 and are shown in the appropriate rows.

The extreme right column lists the capacities of the regions and the final demands to be satisfied. The negative signs in capacities and in the respective requirements of the activities carry the same meaning as in the transportation model, with the direction of inequality reversed. This will also be seen below, in the formal presentation of the model.

The primal can be read off Table 1 in the following way: Let X be the column vector of $_{ij}X_k$ (at the bottom of the Table), A the 12 x 18 matrix (which constitutes the main body of the table), C the row vector with elements the costs of each activity (at the top of the table) and B the column vector of capacities and final demand requirements (at the extreme right-hand side of the Table). The problem is, then,

TABLE I: MODEL V

$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{11}S_1$	$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{12}S_1$	$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{13}S_1$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{11}S_2$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{12}S_2$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{13}S_2$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{21}S_1$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{22}S_1$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{23}S_1$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{21}S_2$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{22}S_2$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{23}S_2$	$[{}_3b_{11}][{}_3e_1]+[{}_3b_{21}][{}_3e_2]+{}_{31}S_1$	$[{}_3b_{11}][{}_3e_1]+[{}_3b_{21}][{}_3e_2]+{}_{32}S_1$	$[{}_3b_{11}][{}_3e_1]+[{}_3b_{21}][{}_3e_2]+{}_{33}S_1$	$[{}_3b_{12}][{}_3e_1]+[{}_3b_{22}][{}_3e_2]+{}_{31}S_2$	$[{}_3b_{12}][{}_3e_1]+[{}_3b_{22}][{}_3e_2]+{}_{32}S_2$	$[{}_3b_{12}][{}_3e_1]+[{}_3b_{22}][{}_3e_2]+{}_{33}S_2$	
-1	-1	-1																$-{}_1A_1$
			-1	-1	-1													$-{}_1A_2$
						-1	-1	-1										$-{}_2A_1$
									-1	-1	-1							$-{}_2A_2$
												-1	-1	-1				$-{}_3A_1$
															-1	-1	-1	$-{}_3A_2$
$1-{}_1a_{11}$	$-{}_1a_{11}$	$-{}_1a_{11}$	$-{}_1a_{12}$	$-{}_1a_{12}$	$-{}_1a_{12}$	1						1						${}_1B'_1$
$-{}_1a_{21}$	$-{}_1a_{21}$	$-{}_1a_{21}$	$1-{}_1a_{22}$	$-{}_1a_{22}$	$-{}_1a_{22}$				1						1			${}_1B_2$
	1					$-{}_2a_{11}$	$1-{}_2a_{11}$	$-{}_2a_{11}$	$-{}_2a_{12}$	$-{}_2a_{12}$	$-{}_2a_{12}$		1					${}_2B_1$
				1		$-{}_2a_{21}$	$-{}_2a_{21}$	$-{}_2a_{21}$	$-{}_2a_{22}$	$1-{}_2a_{22}$	$-{}_2a_{22}$					1		${}_2B_2$
		1						1				$-{}_3a_{11}$	$-{}_3a_{11}$	$1-{}_3a_{11}$	$-{}_3a_{12}$	$-{}_3a_{12}$	$-{}_3a_{12}$	${}_3B_1$
					1						1	$-{}_3a_{21}$	$-{}_3a_{21}$	$-{}_3a_{21}$	$-{}_3a_{22}$	$-{}_3a_{22}$	$1-{}_3a_{22}$	${}_3B_2$
${}_{11}X_1$	${}_{12}X_1$	${}_{13}X_1$	${}_{11}X_2$	${}_{12}X_2$	${}_{13}X_2$	${}_{21}X_1$	${}_{22}X_1$	${}_{23}X_1$	${}_{21}X_2$	${}_{22}X_2$	${}_{23}X_2$	${}_{31}X_1$	${}_{32}X_1$	${}_{33}X_1$	${}_{31}X_2$	${}_{32}X_2$	${}_{33}X_2$	

$$\text{Minimize} \quad C \quad X \tag{4-55a}$$

$$\text{subject to} \quad A \quad X \geq B \tag{4-56a, 4-57a}$$

$$\text{and} \quad X \geq 0$$

or, in formal notation

$$\text{minimize} \sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{2} \left[{}_{ij}s_k\right]\left[{}_{ij}X_k\right] + \sum_{4}^{2}\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{2}\left[{}_{i}e_r\right]\left[{}_{i}b_{rk}\right]\left[{}_{ij}X_k\right] \tag{4-55}$$

subject to

$$-\sum_{j}^{3} {}_{ij}X_k \geq {}_{i}\bar{A}_k \quad (i = 1,2,3 \quad k=1,2) \tag{4-56}$$

$$\sum_{i}^{3} {}_{ip}X_m - \sum_{k}^{2}\sum_{j}^{3}\left[{}_{p}a_{mk}\right]\left[{}_{pj}X_k\right] \geq {}_{p}B_m \quad \begin{pmatrix} m = 1,2 \\ p = 1,2,3 \end{pmatrix} \tag{4-57}$$

$$\text{and} \quad {}_{ij}X_k \geq 0 \quad \text{for all } i, j \qquad *4\text{-}58)$$

where i, j, p, stand for regions and vary from 1 to 3,

r stands for resources and varies from 1 to 2,

k, m stand for commodities and vary from 1 to 2.

The linear forms (4-55a) and (4-55) are identical, stating the total transportation and primary factor costs that are to be minimized. Inequality (4-56) states that, for each region and commodity, the levels of the three activities associated with one specific commodity should not be greater (in absolute value) than the (absolute value of) the productive capacity available in the region. For example, for region 3($i=3$) and commodity 2($k=2$) (4-56) reads (multiplied by -1)

$$_{31}X_2 + {}_{32}X_2 + {}_{32}X_2 \leq {}_{3}A_2$$

This can be read off Table 1 by multiplying the sixth row of the matrix by the vector X, and setting this greater than the capacity listed at the extreme right:

$$-{}_{31}X_2 \quad -{}_{32}X_2 \quad -{}_{33}X_2 \qquad \geq \qquad -{}_3A_2$$

Inequality (4-57) will be explained in parts: $\sum_i {}_{ip}X_m$ represents the pool of commodity m made available in region p for intermediate and final use. For commodity 2 and region 3 this reads

$${}_{13}X_2 + {}_{23}X_2 + {}_{33}X_2$$

$\sum_k^2 \sum_j^3 \left[{}_pa_{mk}\right]\left[{}_{pj}X_k\right]$, on the other hand, represents the total needs of the activities of region p for commodity m as an intermediate. For the same region and commodity as above this reads

$${}_3a_{21}\left[{}_{31}X_1 + {}_{32}X_1 + {}_{33}X_1\right] + \quad {}_3a_{22}\left[{}_{31}X_2 + {}_{32}X_2 + {}_{33}X_2\right]$$

Inequality (4-57) then says that the pool of commodity 2 made available in region 3 minus the amount used up in production of all commodities should be greater or equal to the final demand for that commodity in the region.

Obviously, there are 2x3 restrictions on capacities (the first six rows of the matrix in Table 1) and 2x3 restrictions on final demand (the last six rows of the table) each referring to one commodity and one region. Inequality (4-58) is self-explanatory.

The optimal solution to this model will determine the regional output of each commodity $\sum_j {}_{ij}X_k$ (i = 1,2,3, k = 1,2), the interregional commodity movements ${}_{ij}X_k$ (i = 1,2,3 j=1,2,3, i$\neq$j, k=1,2) and the local production ${}_{ij}X_k$ (i=j) that will satisfy the regional demands ${}_iB_k$ at the minimum total transportation and primary factor costs, given the regional constraints of capacity. For a solution to exist, we must clearly have

$$\sum_i {}_iA_k > \sum_i {}_iB_k \qquad (k = 1,2)$$

the difference being at least equal to the needs of the system for commodity k as an intermediate. I shall not examine the potential applicability of this model here. This will be done in Chapter 6 in connection with a still more complicated model of the next section.

The dual of this model is also presented in tabular form in Table 1. Turn the table ninety degrees clockwise and let ${}_iV_k$ represent the price of commodity k in region i (the price refers to the commodity both as intermediate and final), and ${}_iU_k$ the shadow price of capacity for commodity k in region i. The ${}_iV_k$ and ${}_iU_k$ will be reoognized to have the same meaning as the v_j and u_i in Model IV. Let V represent the column vector of ${}_iV_k$ and ${}_iU_k$ (at the top of the table) and C , A B the same vectors as in the primal. The dual, then, is

$$\text{maximize} \quad B' \quad V \qquad (4\text{-}59a)$$

subject to

$$A' \; V \leq C' \qquad (4\text{-}60a)$$

and $$V \geq 0 \qquad (4\text{-}61a)$$

or, in formal notation,

$$\text{maximize} \sum_k^2 \sum_j^3 \left[{}_jV_k\right]\left[{}_jB_k\right] - \sum_k^2 \sum_i^2 \left[{}_iU_k\right]\left[{}_iA_k\right] \quad (4\text{-}59^*$$

subject to

$$ {}_{j}V_{k} - \sum_{m}^{2} \left[{}_{i}a_{mk}\right]\left[{}_{i}V_{m}\right] - {}_{i}U_{k} \leq {}_{ij}\hat{s}_{k} + \sum_{r}^{2} \left[{}_{i}b_{rk}\right]\left[{}_{i}e_{r}\right] \qquad (4\text{-}60) $$

$$ (j = 1, 2, 3 \quad i = 1, 2, 3 \quad k = 1, 2) $$

and $\quad {}_{i}V_{k},\ {}_{i}U_{k} \geqslant 0 \quad$ for all i, k $\qquad$ (4-61)

The objective of the dual is to maximize the difference between the delivered value of final commodities and the "shadow" value of capacities. The interpretation of ${}_{i}V_{k}$ and ${}_{i}U_{k}$ is the same as in Model IV (section 4.@). The dual constraints (4-60) also have the same interpretation as in that model and in models of Chapter 3 (see 3.0.4) though they are a little more complicated. For commodity 2 delivered in region 3, from region 1, (4-60) reads

$$ {}_{3}V_{2} - \left\{\left[{}_{1}a_{12}\right]\left[{}_{1}V_{1}\right] + \left[{}_{1}a_{22}\right]\left[{}_{1}V_{2}\right]\right\} - \left[{}_{1}U_{2}\right] \leq {}_{13}\hat{s}_{2} + $$
$$ + \left[{}_{1}b_{12}\right]\left[{}_{1}e_{1}\right] + \left[{}_{1}b_{22}\right]\left[{}_{1}e_{2}\right] \qquad (4\text{-}62) $$

The inequality then states[108] that the price of commodity 2 in region 3 and of the same commodity in region 1 must not differ by more than the relevant transportation cost, ${}_{13}s_{2}$. This is the usual interregional equilibrium condition. To see that this is actually what the inequality says consider the price of commodity 2 in region 1 in a situation of perfectly competitive equilibrium: ${}_{1}V_{2}$ will then consist of (a) the primary factor costs per unit of commodity 2, (b) the costs of intermediate inputs per unit, and (c) the "costs" of capacity per unit of the commodity. Now, (a) is the bracketed term on the right hand side of (4-62), and (b) is the braced term on the left hand side; (c) is ${}_{1}U_{2}$. Bringing them all on the left-hand side and substituting ${}_{1}V_{2}$ for them we find,

$$ {}_3V_2 - {}_1V_2 \leqslant {}_{13}s_2 \qquad (4\text{-}62a) $$

which verifies the above remarks.

At this point, it may be useful to attempt a connection between the model of this section and previous transportation models. Let us consider

$$ v_5 - u_1 \leqslant s_{15} \qquad (3\text{-}11(b) \text{ of } 3.0.1) \text{ Model II} $$

$$ v_2 - u_1 \leqslant c_2 + s_{12} \qquad (4\text{-}30c) \text{ Model IV} $$

and $$ {}_2V_1 - \left\{ \left[{}_1a_{21}\right]\left[{}_1V_2\right] + \left[{}_1a_{11}\right]\left[{}_1V_1\right] \right\} - \left[{}_1U_1\right] \leqslant {}_{12}s_1 + \left[{}_1b_{11}\right]\left[{}_1e_1\right] + \left[{}_1b_{21}\right]\left[{}_1e_2\right] \qquad (4\text{-}60) $$

(4-60) can be written: ${}_2V_1 - {}_1V_1 \leqslant {}_{12}s_1$ (4-60a) where ${}_1V_1$ stands for the bracketed terms in (4-60). It will be remembered that in the model of 3.0.1 we had no means of calculating absolute prices unless one such price was given from the solution of the general problem of 3.0.3 (see section 3.0.4 above). In short, it determined only equilibrium price-differentials. In the first interpretation of this model (in 3.0.4) one price was given from outside and the dual determined the absolute price structure; in 3.11(b), u_1 can then be thought as equivalent to ${}_1V_1$ of 4-60(a).

On the other hand 4-30(c) can be written:

$$ v_2 - (u_1 + c_2) \leqslant s_{12} \qquad (4\text{-}30c)' $$

and thus $u_1 + c_2 = 10$ (see 4-24 and 4-31) can be considered equivalent to ${}_1V_1$ of (4-60a). It will be remembered that absolute price determination did not need an outside price in Model IV of 4.0. The problem arose, however, of discrepancy between prices computed from the dual and prices computed from the general problem in which demands were dependent on price.

Finally, (4-60) can be written:

$$ {}_{2}V_{1} - \left\{ \left[{}_{1}a_{21}\right]\left[{}_{1}V_{2}\right] + \left[{}_{1}a_{11}\right]\left[{}_{1}V_{1}\right] + \left[{}_{1}b_{11}\right]\left[{}_{1}e_{1}\right] + \left[{}_{1}b_{21}\right]\left[{}_{1}e_{2}\right] + \right. $$
$$ \left. + \left[{}_{1}U_{1}\right] \right\} \leqq {}_{12}S_{1} $$

and the terms in brackets are equivalent to ${}_{1}V_{1}$ of (4-60a) again.

Thus, one can see how the increasing complexity of the model pays in terms of results. Model II of 3.0.1 was given constant supplies and demands and unit transportation costs and yielded only price differentials (by itself). Model IV of 4.0 was given constant demands and capacities, unit variable costs of production and unit transportation costs and yielded equilibrium prices under the assumption that demand is completely insensitive to price; it also yielded values of capacities in a competitive equilibrium. Finally, the model of this section was given constant demands and capacities, intermediate input requirements, primary factor costs and unit transportation costs and determined equilibrium prices of commodities (under the same assumption about demand as in Model IV of 4.0) and an <u>almost</u> complete breakdown of these F.O.B. prices in terms of costs of intermediate inputs, primary inputs and capacity inputs. "Almost" in this connection means that the model did not determine primary factor costs but it took them as given.

If, however, we seek a generalization of the model that would also yield primary factor prices in its dual solution, problems arise. It will be remembered that the objective function of the primal needed these prices as data; thus, we cannot consider them as variables.

This difficulty is not unique with the model of this section, however. It runs through every linear programming formulation and derives from the fact that linear programming models cannot handle the demand side of the economy successfully, neither can they handle curvilinear resource supply functions.[109] The linearity assumption, which is basic to the linear programming technique and lends its immense computational advantages exacts its price by precluding a sensible[110] consideration of these two aspects of a general economic system.

The above question of prices did not arise with the simpler models in this thesis since they were not so general as the model of this section. This indicates that we have almost reached the present limits of the linear programming technique in analyzing general economic problems.

The problem of prices has been handled in a number of different ways by various authors. The next section deals with one such way in connection with the complete model of A. P. Hurter.[111]

4.1.3 MODEL VI: A.HURTER

One approach to the resource-price problem discussed in the last paragraphs of the previous section is to drop the assumption of relative abundance of primary factors in relation to regional demand. This will be remembered to be the assumption on which Model V was constructed from the material of section 4.1.1. Dropping this assumption is then seen to necessitate the re-introduction of the resource coefficient of the activities and the availability limits of the resources (4-34 and 4-35 of section 4.1.1), since the possibility of resources constraining production cannot be ruled out any longer. We can keep, however, the capacity restrictions and specify that they refer to the availability of capital stock. This makes clear the static short-run character of the model once again. With the same symbols for resources as in section 4.1.1, the model is presented in tabular form in Table 2. The symbols in parentheses represent another complication irrelevant to the problem being discussed, and should be disregarded for the moment (see below, in this section). The model is then seen to differ from Model V only as to additional resource constraints (I have assumed two resources for expositional purposes; the number of resources assumed is immaterial to the argument). Table 2 is to be read in exactly the same way as Table 1. In formal notation the primal is:

$$\text{Minimize} \sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{2} \left[{}_{ij}s_k\right]\left[{}_{ij}X_k\right] + \sum_{r}^{2}\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{2} \left[{}_{i}e_r\right]\left[{}_{i}b_{rk}\right]\left[{}_{ij}X_k\right] \qquad (4\text{-}55)$$

subject to

$$-\sum_{j}^{3} {}_{ij}X_k \geq -{}_{i}A_k \quad (i=1,\ 2,\ 3 \quad k=1,\ 2) \qquad (4\text{-}56)$$

$$-\sum_{k}^{2}\sum_{j}^{3} \left[{}_{i}b_{rk}\right]\left[{}_{ij}X_k\right] \geq -{}_{i}R_r \quad (+\,{}_{i}G_r)$$

$$(r=1,\ 2 \quad i=1,\ 2,\ 3) \qquad (4\text{-}63)$$

$$\sum_{t}^{3} {}_{tp}X_m - \sum_{k}^{2}\sum_{j}^{3} \left[{}_{p}a_{mk}\right]\left[{}_{pj}X_k\right] \geq {}_{p}B_m \ (+\,{}_{p}F_m),$$

$$(m = 1,\ 2 \quad p = 1,\ 2,\ 3) \qquad (4\text{-}57)$$

TABLE 2 MODEL VI

$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{11}S_1$	$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{12}S_1$	$[{}_1b_{11}][{}_1e_1]+[{}_1b_{21}][{}_1e_2]+{}_{13}S_1$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{11}S_2$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{12}S_2$	$[{}_1b_{12}][{}_1e_1]+[{}_1b_{22}][{}_1e_2]+{}_{13}S_2$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{21}S_1$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{22}S_1$	$[{}_2b_{11}][{}_2e_1]+[{}_2b_{21}][{}_2e_2]+{}_{23}S_1$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{21}S_2$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{22}S_2$	$[{}_2b_{12}][{}_2e_1]+[{}_2b_{22}][{}_2e_2]+{}_{23}S_2$	$[{}_3b_{11}][{}_3e_1]+[{}_3b_{21}][{}_3e_2]+{}_{31}S_1$	$[{}_3b_{11}][{}_3e_1]+[{}_3b_{21}][{}_3e_2]+{}_{32}S_1$
-1	-1	-1											
			-1	-1	-1								
						-1	-1	-1					
									-1	-1	-1		
												-1	-1
$-{}_1b_{11}$	$-{}_1b_{11}$	$-{}_1b_{11}$	$-{}_1b_{12}$	$-{}_1b_{12}$	$-{}_1b_{12}$								
$-{}_1b_{21}$	$-{}_1b_{21}$	$-{}_1b_{21}$	$-{}_1b_{22}$	$-{}_1b_{22}$	$-{}_1b_{22}$								
						$-{}_2b_{11}$	$-{}_2b_{11}$	$-{}_2b_{11}$	$-{}_2b_{12}$	$-{}_2b_{12}$	$-{}_2b_{12}$		
						$-{}_2b_{21}$	$-{}_2b_{21}$	$-{}_2b_{21}$	$-{}_2b_{22}$	$-{}_2b_{22}$	$-{}_2b_{22}$		
												$-{}_3b_{11}$	$-{}_3b_{1}$
												$-{}_3b_{21}$	$-{}_3b_{2}$
$1-{}_1a_{11}$	$-{}_1a_{11}$	$-{}_1a_{11}$	$-{}_1a_{12}$	$-{}_1a_{12}$	$-{}_1a_{12}$							1	
$-{}_1a_{21}$	$-{}_1a_{21}$	$-{}_1a_{21}$	$1-{}_1a_{22}$	$-{}_1a_{22}$	$-{}_1a_{22}$								
	1					$-{}_2a_{11}$	$1-{}_2a_{11}$	$-{}_2a_{11}$	$-{}_2a_{12}$	$-{}_2a_{12}$	$-{}_2a_{12}$		1
				1		$-{}_2a_{21}$	$-{}_2a_{21}$	$-{}_2a_{21}$	$-{}_2a_{22}$	$1-{}_2a_{22}$	$-{}_2a_{22}$		
		1						1				$-{}_3a_{11}$	$-{}_3a$
					1						1	$-{}_3a_{21}$	$-{}_3a$
${}_{11}X_1$	${}_{12}X_1$	${}_{13}X_1$	${}_{11}X_2$	${}_{12}X_2$	${}_{13}X_2$	${}_{21}X_1$	${}_{22}X_1$	${}_{23}X_1$	${}_{21}X_2$	${}_{22}X_2$	${}_{23}X_2$	${}_{31}X_1$	${}_{32}X$

and ${}_{i\,j}X_k \geqslant 0$ for all i, j (4-58)

Still disregarding the symbols in parentheses, everything is the same as in Model V except for (4-63): this simply states that the amounts of resources used up by the various activities of the region should not exceed the availability limits of the resources in that region. This implies the stated assumption of resources being immobile between regions (though the assumption could easily be removed by summing 4-63 over i = 1, 2, 3, it is retained for reasons that will be explained in Chapter 6). For region 2 and resource 1 (4-63) reads

$$-{}_2b_{11}\,({}_{21}X_1 + {}_{22}X_1 + {}_{23}X_1) - {}_2b_{12}\,({}_{21}X_2 + {}_{22}X_2 + {}_{23}X_2) \geqslant -{}_2R_1$$

The dual to this model is

$$\text{maximize} \sum_k^2 \sum_j^3 \left[{}_jV_k\right]\left[{}_jB_k\,(+\,{}_jF_k)\right] - \sum_k^2 \sum_i^2 \left[{}_iU_k\right]\left[{}_iA_k\right] - $$

$$-\sum_r^2 \sum_i^3 \left[{}_iW_r\right]\left[{}_iR_r\quad(-\,{}_iG_r)\right] \qquad (4\text{-}64)$$

subject to

$${}_jV_k - \sum_m^2 \left[{}_ia_{mk}\right]\left[{}_iV_m\right] - {}_iU_k - \sum_r^2 \left[{}_ib_{rk}\right]\left[{}_iW_r\right] \leqq {}_{ij}s_k +$$

$$+ \sum_r^2 \left[{}_ib_{rk}\right]\left[{}_ie_r\right] \qquad (4\text{-}65)$$

and ${}_iV_k$, ${}_iU_k$, ${}_iW_r \geqslant 0$ for all i, k, r (4-66)

The new symbol ${}_iW_r$ represents the "price" of the resource r in region i to be determined. When the relevant constraint in the primal is binding in the optimal solution, the price ${}_iW_r$ is found positive in the dual, as already stated in Chapter II.

The objective function of this model includes the value of the resources (third term) as was to be expected, while the constraints contain one additional cost item, referring to the resources. The original resource cost is also included, however, and this needs interpretation.

A. Hurter[112] defines ${}_ie_r$ as the minimum price of resource r in region i and argues that the introduction of such price as externally given minimum resource cost to producers, irrespective of the actual forces of the market, tends to lead the system to choices not necessarily the best from the social economic point of view. Suppose, for example, that the minimum price of a resource is higher than what the market forces would determine if left unrestrained. The imposition of this minimum price then leads to underutilization of the resource and drives the economy away from its production-possibility frontier. Hurter points out the descriptive justification of this imperfection and explains the relationship between ${}_iW_r$ and ${}_ie_r$ as follows:

(a) ${}_ie_r = {}_iW_r = 0$ means excess supply of resource,

(b) ${}_ie_r > 0$, ${}_iW_r = 0$: the minimum price is too high relative to supply and demand; there exists inefficiency in the system,

(c) ${}_ie_r = 0$, ${}_iW_r > 0$: the primary factor is fully utilized in the region . An increase in its availability will lower total system costs, and

(d) ${}_ie_r > 0$, ${}_iW_r > 0$: the resource is fully utilized despite its minimum price (i.e., the minimum price is too low relative to that determined by the market. The latter is ${}_ie_r + {}_iW_r$).

I shall now proceed to explain the symbols introduced in parentheses in Table 2 and equations (4-63), (4-57), (4-64), (4-65).

Though the model of this section is quite general in terms of activities and commodities, transportation appears nowhere as an activity while its costs are given externally. Since activities are distinguished by origin and destination (and, of course, by commodity produced) transportation inputs could easily be introduced among the inputs of each activity, in which case we would face the same problem as with the prices of resources. Hurter's approach, however, is to construct a model suitable for government planning. The transportation sector is assumed to becompletely controlled by the government which regulates transport rates and purchases resources and commodities needed as inputs by the transportation sector. ${}_iG_r$ in (4-63) is the amount of resource r bought by the government in region i. Thus, the availability of the resource r is decreased by this amount (${}_iG_r$) from the start. The government, that is, is assumed to have priority over the private sector in buying resources at the going market price. On the other hand, ${}_pF_m$ is the amount of commodity m required by the government in region p. In this respect, the government stands equal to any other final user of output. The interpretation of the dual follows directly from what was said about the primal.

CHAPTER V

MODEL VII

This model is basically an extension of Model I to include intermediate commodities and many regions. Variations of it have been published by W. Isard[113] and B. Stevens[114]. The exposition in this thesis follows M. Harwitz.[115]

5.0 Extension of Model I

With one region, two resources and three commodities the technological matrix and the resource availability vector of Model I was (cf. 2.2.0.1 (ii), (i))

$$\begin{array}{l} \text{Resource 1} \\ \text{Resource 2} \\ \text{Commodity 1} \\ \text{Commodity 2} \\ \text{Commodity 3} \end{array} \begin{bmatrix} -b'_{11} & -b'_{12} & -b'_{13} \\ -b'_{21} & -b'_{22} & -b'_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5\text{-}1) \qquad \begin{bmatrix} -R_1 \\ -R_2 \end{bmatrix} \quad (5\text{-}2)$$

where the primes on the resource coefficients of the activities are intended to show that intermediate commodities have been "netted out" by the technique described in 4.1.1. Let us drop one commodity from this scheme to align the dimensions of the model of this section to those of the previous chapter, and let us reintroduce intermediate commodities. This latter step is necessary if we are to extend the model to many regions, as has already been explained in section 4.1.1. After both steps have been taken (5-1) is changed to (5-3) below.

$$\begin{bmatrix} -b_{11} & -b_{12} \\ -b_{21} & -b_{22} \\ 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \quad (5\text{-}3) \qquad \begin{bmatrix} -R_1 \\ -R_2 \end{bmatrix} \quad (5\text{-}2)$$

Introduction of capacity requirements (in the same vein as in section 4.1.2, model V, and section 4.1.3, model VII) and labelling of coefficients to show that they refer to activities in region (e.g.) 1 yields (5-4) and (5-5).

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ -{}_1b_{11} & -{}_1b_{12} \\ -{}_1b_{21} & -{}_1b_{22} \\ 1-{}_1a_{11} & -{}_1a_{12} \\ -{}_1a_{21} & 1-{}_1a_{22} \end{bmatrix} \quad (5\text{-}4) \qquad \begin{bmatrix} -{}_1A_1 \\ -{}_1A_2 \\ -{}_1R_1 \\ -{}_1R_2 \end{bmatrix} \quad (5\text{-}5)$$

Finally, we have to distinguish activities 1 and 2 of (5-4) according to destinations of their output, and also include transportation in the model. Let us define

${}_{ij}t_{rk}$ = the quantity of primary factor r required for transportation of one unit of commodity k from region i to region j

${}_{ij}v_{mk}$ = the quantity of commodity m required (as an intermediate input) to transport one unit of commodity k from region i to region j.

Consider now activity 1 of region 1 when it produces and "transports" its output to region 1. Its requirements will be

$$\left[-1,\ 0,\ -({}_1b_{11}+{}_{11}t_{11}),\ -({}_1b_{21}+{}_{11}t_{21}),\ 1-{}_1a_{11}-{}_{11}v_{11},\ -{}_1a_{21}-{}_{11}v_{21}\right]'$$

and its level will be represented by ${}_{11}X_1$. On the other hand, when it produces and transports its output to region 2 its requirements will be

$$\left[-1,\ 0,\ -({}_1b_{11}+{}_{12}t_{11}),\ -({}_1b_{21}+{}_{12}t_{21}),\ -{}_1a_{11}-{}_{12}v_{11},\ -{}_1a_{21}-{}_{12}v_{21},\ 1\right]'$$

and its level will be denoted by ${}_{12}X_1$. As before, the production method (that is, the b's and the a's) does not change with the destination of output. Only transportation requirements of the activity (the t's and v's) vary with it, and the unit of output of the activity has been transferred to another column to show the change in destination of output. Out of each original activity of region 1, then, we created 3, according to destination. The relevant symbols for the activity levels are

$${}_{11}X_1,\ {}_{12}X_1,\ {}_{13}X_1,\ {}_{11}X_2,\ {}_{12}X_2,\ {}_{13}X_2$$

and are to be interpreted in the same way as in Model V. Similarly, for the other two regions:

$${}_{21}X_1,\ {}_{22}X_1,\ {}_{23}X_1,\ {}_{21}X_2,\ {}_{22}X_2,\ {}_{23}X_2, \text{ for region 2}$$

and

$${}_{31}X_1,\ {}_{32}X_1,\ {}_{33}X_1,\ {}_{31}X_2,\ {}_{32}X_2,\ {}_{33}X_2, \text{ for region 3}$$

The objective in this model is the same as in Model I: maximization of total income, but for all regions taken together. We thus need the prices of the final commodities in each region as data:

$$\left[{}_1V_1,\ {}_1V_2,\ {}_2V_1,\ {}_2V_2,\ {}_3V_1,\ {}_3V_2\right] \tag{5-6}$$

and we have to find a way of introducing only final (and not intermediate) outputs in the objective function. Obviously, the variables ${}_{ij}X_k$ will not do for this purpose since they include intermediate production. In other words, $\sum_j {}_{ij}X_k$ denotes the total production of commodity k in region i, including that part which is to be used as intermediate inputs in the production of other commodities.

It is then necessary to introduce a number of "dummy" activities. A "dummy" activity is one which does not actually produce anything, but simply takes one unit of a commodity from a pool and transfers it for a special use. In the particular case of this model the "dummy" activities take one unit of each commodity in each region for final use in that region (when they are operated at unit level). Their only element is thus one unit of output in the appropriate row of the technological matrix.

Up to this point I have used a minus sign for an input and a plus sign for an output of an activity. As stated in Chapter 2, this is simply a convention. It will be found useful to use exactly the opposite convention in this model: minus signs for outputs and plus signs for inputs.[116] With the new convention and the introduction of the dummy activities the technological matrix of the system is as in Table 3, made in the same way as Tables 1 and 2. ${}_iY_k$ is the level of the "dummy" activity in region i taking commodity k for final use.

The model introduces government as a demander of final goods and resources in the various regions. However, the government here does not regulate either the rates or the inputs of transportation sector: this is because of the fact that transportation has been included in the tech-

TABLE 3 MODEL

	$X_{11}1$	$X_{12}1$	$X_{13}1$	$X_{11}2$	$X_{12}2$	$X_{13}2$	$X_{21}1$	$X_{22}1$	$X_{23}1$	$X_{21}2$	$X_{22}2$	$X_{23}2$	$X_{31}1$	$X_{32}1$	$X_{33}1$	$X_{31}2$	$X_{32}2$	$X_{33}2$	${}_1Y_1$	${}_1Y_2$	${}_2Y_1$	${}_2Y_2$	${}_3Y_1$	${}_3Y_2$		
${}_1U_1$	1	1	1																						${}_1A_1$	
${}_1U_2$				1	1	1																			${}_1A_2$	
${}_2U_1$							1	1	1																${}_2A_1$	
${}_2U_2$										1	1	1													${}_2A_2$	
${}_3U_1$													1	1	1										${}_3A_1$	
${}_3U_2$																1	1	1							${}_3A_2$	
${}_1W_1$	${}_1b_{11}+{}_{11}t_{11}$	${}_1b_{11}+{}_{12}t_{11}$	${}_1b_{11}+{}_{13}t_{11}$	${}_1b_{12}+{}_{11}t_{12}$	${}_1b_{12}+{}_{12}t_{12}$	${}_1b_{12}+{}_{13}t_{12}$																			${}_1R_1$	$-{}_1G_1$
${}_1W_2$	${}_1b_{21}+{}_{11}t_{21}$	${}_1b_{21}+{}_{12}t_{21}$	${}_1b_{21}+{}_{13}t_{21}$	${}_1b_{22}+{}_{11}t_{22}$	${}_1b_{22}+{}_{12}t_{22}$	${}_1b_{22}+{}_{13}t_{22}$																			${}_1R_2$	$-{}_1G_2$
${}_2W_1$							${}_2b_{11}+{}_{21}t_{11}$	${}_2b_{11}+{}_{22}t_{11}$	${}_2b_{11}+{}_{23}t_{11}$	${}_2b_{12}+{}_{21}t_{12}$	${}_2b_{12}+{}_{22}t_{12}$	${}_2b_{12}+{}_{23}t_{12}$													${}_2R_1$	$-{}_2G_1$
${}_2W_2$							${}_2b_{21}+{}_{21}t_{21}$	${}_2b_{21}+{}_{22}t_{21}$	${}_2b_{21}+{}_{23}t_{21}$	${}_2b_{22}+{}_{21}t_{22}$	${}_2b_{22}+{}_{22}t_{22}$	${}_2b_{22}+{}_{23}t_{22}$													${}_2R_2$	$-{}_2G_2$
${}_3W_1$													${}_3b_{11}+{}_{31}t_{11}$	${}_3b_{11}+{}_{32}t_{11}$	${}_3b_{11}+{}_{33}t_{11}$	${}_3b_{12}+{}_{31}t_{12}$	${}_3b_{12}+{}_{32}t_{12}$	${}_3b_{12}+{}_{33}t_{12}$							${}_3R_1$	$-{}_3G_1$
${}_3W_2$													${}_3b_{21}+{}_{31}t_{21}$	${}_3b_{21}+{}_{32}t_{21}$	${}_3b_{21}+{}_{33}t_{21}$	${}_3b_{22}+{}_{31}t_{22}$	${}_3b_{22}+{}_{32}t_{22}$	${}_3b_{22}+{}_{33}t_{22}$							${}_3R_2$	$-{}_3G_2$
${}_1\Pi_1$	${}_1a_{11}-1+{}_{11}y_{11}$	${}_1a_{11}+{}_{12}y_{11}$	${}_1a_{11}+{}_{13}y_{11}$	${}_1a_{12}+{}_{11}y_{12}$	${}_1a_{12}+{}_{12}y_{12}$	${}_1a_{12}+{}_{13}y_{12}$	-1						-1						1						F	
${}_1\Pi_2$	${}_1a_{21}+{}_{11}y_{21}$	${}_1a_{21}+{}_{12}y_{21}$	${}_1a_{21}+{}_{13}y_{21}$	${}_1a_{22}-1+{}_{11}y_{22}$	${}_1a_{22}+{}_{12}y_{22}$	${}_1a_{22}+{}_{13}y_{22}$		-1						-1						1					F	
${}_2\Pi_1$		-1					${}_2a_{11}+{}_{21}y_{11}$	${}_2a_{11}-1+{}_{22}y_{11}$	${}_2a_{11}+{}_{23}y_{11}$	${}_2a_{12}+{}_{21}y_{12}$	${}_2a_{12}+{}_{22}y_{12}$	${}_2a_{12}+{}_{23}y_{12}$			-1						1				F	
${}_2\Pi_2$					-1		${}_2a_{21}-1+{}_{21}y_{21}$	${}_2a_{21}+{}_{22}y_{21}$	${}_2a_{21}+{}_{23}y_{21}$	${}_2a_{22}+{}_{21}y_{22}$	${}_2a_{22}+{}_{22}y_{22}$	${}_2a_{22}+{}_{23}y_{22}$				-1						1			F	
${}_3\Pi_1$			-1						-1				${}_3a_{11}+{}_{31}y_{11}$	${}_3a_{11}+{}_{32}y_{11}$	${}_3a_{11}-1+{}_{33}y_{11}$	${}_3a_{12}+{}_{31}y_{12}$	${}_3a_{12}+{}_{32}y_{12}$	${}_3a_{12}+{}_{33}y_{12}$					1		F	
${}_3\Pi_2$						-1						-1	${}_3a_{21}+{}_{31}y_{21}$	${}_3a_{21}+{}_{32}y_{21}$	${}_3a_{21}+{}_{33}y_{21}$	${}_3a_{22}+{}_{31}y_{22}$	${}_3a_{22}+{}_{32}y_{22}$	${}_3a_{22}-1+{}_{33}y_{22}$						1	F	

nology. The symbols for the government are as in Model VI.

As discussed above, the only variables that enter the objective function with a non-zero coefficient (price) are the levels of dummy activities ${}_iY_k$ ($i = 1, 2, 3 \quad k = 1, 2$). The variables are at the bottom and the respective prices at the top of the table, as usual.

The dual prices refer to the constraining factors of the problem and are as follows:

${}_iU_k$ = the rent on capacity to produce commodity k in region i

${}_iW_r$ = the rent on the primary factor r in region i

${}_i\Pi_k$ = the shadow price of commodity k (as an intermediate) in region i

As with Tables 1, 2, the model can be read off Table 3 in exactly the same way. In formal notation the primal is:

$$\text{maximize} \quad \sum_{i}^{3}\sum_{k}^{2} {}_iV_k \;\; {}_iY_k \tag{5-7}$$

subject to

$$\sum_{j}^{3} {}_{ij}X_k \leq {}_iA_k \quad (i = 1, 2, 3 \quad k = 1, 2) \tag{5-8}$$

$$\sum_{j}^{3}\sum_{m}^{2} ({}_ib_{rm} + {}_{ij}t_{rm}) \; {}_{ij}X_m \leq {}_iR_r - {}_iG_r \quad \begin{pmatrix} i=1,2,3 \\ r=1,2 \end{pmatrix} \tag{5-9}$$

$${}_iY_k + \sum_{j}^{3}\sum_{m}^{2} ({}_ia_{km} + {}_{ij}v_{km}) \; {}_{ij}X_m - \sum_{j}^{3} {}_{ji}X_k \leq -{}_iF_k \tag{5-10}$$

$$(i = 1, 2, 3 \quad k = 1, 2)$$

and ${}_{i}Y_{k}$, ${}_{ij}X_{k} \geqslant 0$ for all i, j, k. (5-11)

The objective function (5-7) shows the total value of final commodities in all regions. (5-8) is the usual constraint on capacity (with the new sign convention). (5-9) is the constraint on availability of resources. For region 1 and resource 2 it reads:

$$({}_{1}b_{21} + {}_{11}t_{21})\ {}_{11}X_{1} + ({}_{1}b_{21} + {}_{12}t_{21})\ {}_{12}X_{1} + ({}_{1}b_{21} + {}_{13}t_{21})\ {}_{13}X_{1} +$$
$$({}_{1}b_{22} + {}_{11}t_{22})\ {}_{11}X_{2} + ({}_{1}b_{22} + {}_{12}t_{22})\ {}_{12}X_{2} + ({}_{1}b_{22} + {}_{13}t_{22})\ {}_{13}X_{2} \leqslant$$
$$\leqslant {}_{1}R_{2} - {}_{1}G_{2}$$

Finally, rewrite (5-10) as follows:

$$\sum_{j}^{3} {}_{ji}X_{k} \geqslant {}_{i}Y_{k} + {}_{i}F_{k} + \sum_{j}^{3}\sum_{m}^{2} ({}_{i}a_{xm} + {}_{1j}v_{km})\ {}_{ij}X_{m} \qquad (5\text{-}10a)$$

The meaning of this constraint is that the total pool of commodity k coming to region i from all sources should at least be equal to that drawn from the pool for (a) final demand (b) government demand and (c) intermediate input demand.

The dual to this model is, in turn:

$$\text{minimize} \quad \sum_{i}^{3}\sum_{r}^{2} {}_{i}W_{r}\,({}_{i}R_{r} - {}_{i}G_{r}) + \sum_{i}^{3}\sum_{k}^{2} \left[{}_{i}U_{k}\right]\left[{}_{i}A_{k}\right] -$$
$$- \sum_{i}^{3}\sum_{k}^{2} \left[{}_{i}\Pi_{k}\right]\left[{}_{i}F_{k}\right] \qquad (5\text{-}12)$$

subject to

$$\sum_{k}^{2} \left[{}_{i}\Pi_{k}\right] \left[{}_{i}a_{km} + {}_{ij}V_{km}\right] + \sum_{r}^{2} \left[{}_{i}W_{r}\right]\left[{}_{i}b_{rm} \quad {}_{ij}t_{rm}\right] + {}_{i}U_{m} -$$

$$ {}_{j}\Pi_{m} \geqslant 0 \quad (i,\ j = 1, 2, 3 \quad m=1, 2) \qquad (5\text{-}13)$$

$$ {}_{i}\Pi_{k} \geqslant {}_{i}V_{k} \quad (i = 1, 2, 3 \quad k = 1, 2) \qquad (5\text{-}14)$$

and $\quad {}_{i}W_{r},\ {}_{i}U_{k},\ {}_{i}\Pi_{k} \geqslant 0 \quad$ for all i, k, r $\qquad$ (5-15)

(5-13) states that the cost of each commodity in each region (irrespective of where the commodity comes from) should be at least equal to the price of the commodity ${}_{j}\Pi_{n}$ in that region. (5-14) fixes the necessary relationship between the price of the commodity as an intermediate, Π, and the price of the commodity as final, V.

The primal of this model functions on the same principles as the much simpler model 1: given the final commodity prices it chooses a point on the boundary of the combined production possibility frontier of all regions together so that total national product is maximized for the system. It may perhaps be useful to point out that the combined production possibility frontier is net of real costs of transport (the resources absorbed by transport are taken into account). The optimal solution represents a competitive equilibrium at given prices of final commodities and is, of course, an efficient combination in the sense of section 2.2.1.1.

Given the commodity prices the dual imputes the value of the maximum output (attained in the optimal solution of the primal) back to the resources and capacities. In the process, however, the dual determines (intermediate) commodity prices also. Since the price of one and the same commodity cannot be different when it is used as an intermediate from when it is used as final, we get inequality (5-14). If Π turns out greater than V then the commodity is not put in final use but is only used as intermediate.

The last point about the ${}_i\Pi_k$ and ${}_iV_k$ brings into mind the "price" problem encountered in Model VI. Indeed, the model of this section faces the same problem, perhaps in a greater degree. To elaborate, the dual sets out to find prices of commodities but is restrained by the fact that these computed prices, ${}_i\Pi_k$, must bear a certain relationship with the exogenously given commodity prices ${}_iV_k$. The inconsistency is clear: what is a variable is also a datum.

One way out of this impasse is suggested by B. Stevens[117] and M. Harwitz[118]. Impose on the primal an additional set of restrictions

$$ {}_iY_k \geqslant {}_iB_k \quad (i=1,\ 2,\ 3 \qquad k=1,2) $$

where ${}_iB_k$ is interpreted by Stevens as minimum consumption requirements of commodity k in region i. If, in its objective of maximization of the value of final output, the primal allocates a quantity of a final commodity exactly equal to the regional requirement ${}_iB_k$, this means that the price ${}_iV_k$ set for the commodity in that region is too low (and, roughly speaking, the model avoids the region). If this is the case, the dual variable associated with the constraint will be positive. This variable could then be added to the exogenous price ${}_iV_k$ and the problem re-solved with the new commodity price, as if the latter were the actual price. Or, the dual value could be considered as a subsidy that should be paid to producers to bring the necessary final output to the region.

It may be noted that the above approaches have not been examined adequately in the literature.

CHAPTER VI

FORMAL COMPARISONS, EVALUATION AND CONCLUSION

In this chapter an attempt is made at formal comparison of the two basic models in this thesis (VI and VII) with each other and with other similar models. Section 6.0 is devoted to this task. Further, an evaluation of interregional linear programming models and some conclusions are attempted.

6.0 FORMAL COMPARISONS

6.0.1 Model VI (A. Hurter) and Model VII (M. Harwitz)

The formal similarity of the two models can be seen by examination of Tables 2 (Model VI) and 3 (Model VII). The matrix of constraints is seen to differ between the two models in three respects: (a) signs, (b) transport input coefficients, and (c) the last six columns of Table 3. The difference in signs is purely a conventional one as was explained in Chapter V. The transport input coefficients in Model VII are included because transportation is just another activity in that model, consuming resources and intermediate inputs per unit of its output (this latter being defined as transportation of one unit of commodity k from region i to region j). This treatment of transportation allows for the multidimensionality of the industry's output, that is, it can take into account differences in input requirements as the specific destination of output to be transported changes. On the other hand, in Model VI transportation is handled exogenously. The implicit assumption there is that its intermediate and primary inputs are included in the government's constant demands for primary factors and commodities.

Difference (c) arises because of the different objective of the two models: Model VI takes regional final demands given and proceeds to minimization of total production and transportation costs. The unit production costs are given (by technology and the minimum resource prices) and so are the unit transportation costs (by the regulatory agency of the government). Thus the variables needed are only the ${}_{ij}x_k$. Model VII however, takes prices of final commodities given and proceeds to maximize the value of final output. Final demand by the private sector thus becomes a variable (${}_iB_k$ of Model VI is dropped) and six new activities are introduced to transfer output to final use.

Basically, then, the formal difference between Models VI and VII boils down to a price assumption.[119] The prices assumed as given, that is, differ. Model VI assumes input prices given (not those of intermediate inputs, though) and thus has to define as objective the minimization of costs. In its optimal solution it determines the final commodity prices and the rents to capacities that are consistent with the technology and the constraints (of capacity and demand) in a competitive equilibrium situation, and runs into the problem of having to determine prices of resources also, which were assumed given when the problem was posed. On the other hand Model VII takes final commodity prices as given and has to define the objective in terms of maximization of the value of final output. In its optimal solution it determines the resource and capacity prices consistent with the constraints (of resources and demand) in a competitive equilibrium situation, and runs into the equivalent problem to that of Model VI, in that it has to determine final commodity prices which were assumed given when the problem was posed.

6.0.2 L. Moses' Model[120] and Model VI

L. Moses' model is, in its theoretical form[121], very similar to that of A. Hurter. The main differences are: (a) introduction of transport input coefficients into the matrix of technology, (b) introduction of capacities in the transportation sector and (c) the quantity to be minimized.

The introduction of transport input coefficients was done by L. Moses in exactly the same way as in Model VII.[122] The introduction of transport capacities is similar to the introduction of capacities for the producing industries of each region. Finally, the objective function minimizes the quantity of primary inputs required to satisfy the given final demand. Moses assumes one primary input, labour, and thus his objective function is

$$\text{minimize } Z = \sum_k \sum_i \sum_j \left({}_i b_{1k} + {}_{ij} t_{1k}\right) {}_{ij}X_k$$

where the subscript 1 is used to denote labour. Moses' way of stating the objective function seems to avoid the price problem referred to in the previous section, but this is done at the cost of having to assume[123] one primary factor, so that Z has meaning as the total quantity of labour required by the system to satisfy the given final demands.

6.0.3 The Isard[124] - Stevens[125] Model and Model VII

There seems to be one essential difference between the Isard-Stevens model and model VII: model VII assumes resources immobile and considers all final commodities as potentially intermediate but does not consider intermediate commodities that are not final, e.g., raw nickel ore at the mine head. This reveals an implicit assumption about mobility of strictly intermediate commodities: namely, they are considered immobile. This needs more elaboration.[126]

Consider a resource in the ground, e.g., nickel ore. The relevant constraint on resource capacity will then refer to the rate at which this ore can be extracted during the period of examination and not to the total nickel ore reserve. But in this case, there must exist a mining activity to carry out its task. Its output will be a strictly intermediate good, for which there is no final demand. But the model assumes that each intermediate commodity considered is also demanded by final users, and this can be seen by the fact that dummy activities for transferring to final demand are defined for every produced commodity. If we keep this interpretation, the only way out is to assume that productive processes are sufficiently integrated so that none produces a strictly intermediate commodity. This, in turn, implies that the inputs needed for extraction of the resource are embodied in the input coefficients of the integrated activity. Nickel ore at the minehead, then, does not appear as a product of any activity, which implies that it is not transportable between regions but has to be further processed into another commodity before it becomes mobile.

Another alternative, however, would be to keep the activity that extracts the ore separate in the model. This would entail (a) either dropping the relevant ${}_{i}Y_{k}$ or considering its price ${}_{i}V_{k}$ equal to zero, as we actually do with ${}_{ij}X_{k}$, and (b) putting the relevant government demand ${}_{i}F_{k}$ equal to zero (since the government is not assumed to operate an industrial enterprise outside the model).

The above second alternative interpretation enables the model to consider all strictly intermediate commodities (except transportation services) as mobile between regions, and thus the choice procedure is freed from an artificial constraint. By the last statement it is meant that, if we do not adopt the second interpretation but keep the first, we require from the model to produce smelted ore only in the regions that mines exist. Suppose, for example, that only one region in the system has a nickel mine. Then, even if nickel ore at the minehead could be transported to another region and smelted there at much lower unit total (unit production plus unit transportation) costs, the model cannot consider this possibility, as it is excluded by assumption.

It was noted above that the second alternative interpretation of model VII still considers that the transportation services needed to transport all unit of commodity y from region i to region j <u>must</u> be produced in region i, that is, in the region of origin. This is because the transportation input coefficients are embodied (or rather, added) with the production coefficients of each activity. This treatment has two consequences:

(a) the dual will not give a shadow price for the intermediate good "transportation services" and

(b) the model is restricted from choosing to allocate the production of these services to these regions that can produce them most economically.

The Isard-Stevens model is a way out of this difficulty: transportation services are explicitly considered an intermediate commodity produced by a separate activity. Also, intermediate and final commodities are kept strictly separate in the model. This is achieved by postulating that the outcome of every activity is an intermediate commodity. We thus have, for each region:[127]

(a) Productive Activities: these absorb primary inputs and intermediate goods (except transport services) and produce intermediate goods (except transport services).

(b) One activity producing transport services (intermediate) by absorbing primary and intermediate inputs (except transport services).

(c) Shipment activities: these absorb one unit of an intermediate commodity and transport services and yield that unit of intermediate as an output in another region. Among them, there are shipment activities transferring the intermediate good "transportation services".

(d) Final good activities: these absorb one unit of an intermediate commodity and yield one unit of same as output. They are dummy activities, that is. In number, they are as many as the number of final commodities.

This treatment requires additional constraints on the output of intermediate commodities. The constraints require that the total of each intermediate commodity produced in a region plus the quantity shipped into the region from other regions, minus the quantity used in the region minus what was shipped from this region to others, should be equal to zero.

In all other basic respects the Isard-Stevens model is the same as Model VII.

6.1 An Attempt at Evaluation

I shall devote the first subsection below to some words of caution in connection with the models considered in this thesis. The next subsection will then be devoted to an examination of the potential of these models in an application.

6.1.1 Methodology and Assumptions

First, all models are static. Their solutions, then, give us no clue about what the future path of the economy will be. Any interpretations towards this direction should be made with extreme caution.

Second, the solutions to the models describe a perfectly competitive equilibrium. Since not many are prepared to assume that the actual imperfect economy operates in this way, the usual justification for using competitive equilibrium models in actual applications is that an examination of differences between the model's solution and that of the actual economy will reveal the parts of the latter where imperfections exist. Acceptance of this justification does not solve all the problems, though, the reason being that the differences referred to above cannot safely be ascribed to non-competitive elements only: other assumptions in the model (e.g., the linearity of production functions and the fact that the solution values refer to an equilibrium) may be responsible for discrepancies, apart from discrepancies due to inaccuracy of the data.

Third,[128] since optimal solutions of the models usually differ from actual market solutions, the question arises as to the usefulness of the comparative static theorems derived from the models.

It should be noted, however, that all three points considered above are not peculiar to the models of this thesis: the first two run through the most part of economic theory and the third will apply to all models which do not estimate relationships, but are based on direct statistical data. The answer to such criticisms will then probably be that, given the present state of theoretical knowledge in economics, we shall have to do with models having these basic deficiencies.

There are two points, however, that are peculiar to general models cast in linear programming terms: the linearity assumption in production and the treatment of demand.

The linearity assumption is of course indispensable and provides us with actual solutions to complicated models. Until specific methods of solution of non-linear programming models have been found we shall have to do with models assuming constant returns to scale in production.

The same remarks as above hold with respect to the treatment of demand (see also section 4.1.2). We simply do not have a way of introducing it into the linear programming model, because of the linearity assumption. Connected with the deficient treatment of demand is the "price" problem that arises in all these models. As noted above, Moses' model avoids it but puts another assumption in its place. And moreover, the discrepancy between observed and computed prices noted at the end of section 4.0 is not avoided by his assumption.

6.1.2 <u>The Potential of Models VI and VII</u>

I shall confine my remarks in this subsection to Models VI and VII so that I can be specific. In any case, the formal similarity of the models justifies this approach.

Model VI is more of a planning model than VII since it isolates the transportation sector outside the model and can thus provide some answers to policy questions related to transportation and regional development. The effects of changing transportation rates on the outputs of commodities in each region can be calculated by resolving the problem with the proposed changes in rates as data. The new solution will show the effects that these changes will likely have on regional outputs and employment, as well as on the total output of the transportation sector.

The effects of a change in the regional distribution of government purchases of final goods and resources can also be shown by resolving the model with the proposed ${}_iF_k$ and ${}_iG_k$. Obviously such changes will affect regional outputs, employment and resource use, and the output of the transportation sector.

In terms of the dual, such changes as those examined above will change the dual prices of commodities, capacities and resources. Comparison of the new prices to those computed in the initial solution will show the effects that government policy may have in changing the relative attractiveness of regions to investors (by changing the ${}_iV_K$), and mobile labour (by changing the ${}_iW_r$). Whether or not investment and labour will follow the pattern dictated by the dual prices is uncertain, however, because of imperfections that exist in the economy.

The initial solution to the primal will show the optimal shipping pattern: notwithstanding the remarks made in 6.1.1, some remarks about non-competitiveness could be made, by comparing the optimal to the actual pattern.

The shadow prices of resources in the intial optimal solution will also show the regions where some resources are in short supply (${}_1W_r > 0$) and will thus give indications for government policy on this matter.

Finally, the introduction of a new productive activity in a region could be tested by inserting engineering estimates of its technology and capacity and resolving the model. The new solution would show the effects of such change on regional outputs, etc. The above are instances of application of sensitivity analysis and parametric programming.

Model VII is more of a descriptive than a planning model. The aim of the government should be defined as maximization of private income. Parametric programming can also be made in this model, except referring to transportation rates. The uses are thus very similar to those of Model VI.

6.2 Conclusion

It is the opinion of the author of this thesis that the two basic models examined herein, as well as those formally similar to them (section 6.0) provide quite useful tools for analysis of interregional policies. Up to now, we have not been able to solve some basic problems noted above. But then, science is bound to proceed forward. In the meantime, these models are among the best we have available for analysis and policy.

FOOTNOTES

1. These initials will be used for "Linear Programming".

2. Sections 2.2.1, 2.2.2.

3. See Kuhn and Tucker (14) Paper No. 4, or Hadley (5) Ch. 8.

4. Given positive contributions of activities to it.

5. In mathematical terms, there is no finite maximum.

6. This assumes divisibility. See 2.2.0.0.

7. Since this smaller number indicates the maximum feasible level of the acitivity.

8. In mathematical terms the matrix A in (2d) of 2.0.1 is decomposable.

9. This is relevant to case (c) above, and is also implicit in the objective of the problem. The aim is to maximize the sum $p_1x_1+\dots+p_n x_n$, not any particular $p_i x_i$. It is thus seen that the activities constitute alternative ways of satisfying an aim.

10. The fact that one can count and add five units of resource 1 directly implies homogeneity of the units of this resource. By the same token, final commodities are also assumed homogeneous.

11. It is perfectly legitimate, of course, to define an infinite number of activities in such a way that their totality gives the usual neo-classical isoquant. See Koopmans (11) and Dorfman, Samuelson, Solow (3).

12. Since there no more exists any need for differentiation between primary factors and final commodities, we can also dispose of the minus signs.

13. Some other details should be borne in mind: the coefficient of x_4 must be positive and $1x_4$ must be added to the left hand side of a $\leq$ inequality. Otherwise we have the same problems as when x_4 is allowed to take negative values.

    On the other hand, if the inequality is in the $\geqslant$ direction everything else is done the same way except that the expression $1x_4$ is subtracted from the left hand side. x_4 is then called a surplus variable.

14. In mathematical terms, the solution set of (2c), (3c) bears a one-to-one correspondence with the solution set of (2d), (3d). Also, the slack variables are unit vectors which can be thought of as units of measurement (or as a basis) in the two dimensional space of the example.

15. On instances of using the free disposal assumption in models more general than that of the example see Koopmans (11) and Debreu (2).

16. A feasible solution (i.e. a non-negative solution) exists for a system $Ax = b$ if and only if the vector b is an element of the convex polyhedral cone generated by the columns of A.

17. Suppose for example that we have max. $f(x) = c_1 x_1 + c_2 x_2$

subject to

$$a_{11} x_1 + a_{12} x_2 \leq k_1$$
$$a_{21} x_1 + a_{22} x_2 \leq k_2$$
$$x_i > 0 \qquad i = 1, 2$$

Addition of the slack variables will produce more unknowns (four) than equations (two).

18. With m equations and n unknowns, the matrix of a system of equations can have maximum rank equal to m. Since $m < n$, there will be an infinite number of solutions (it is assumed, of course, that in the system $Ax = b$ $\quad r(A) = r(A_b)$

where $\quad A_b \equiv (A, b)$. See Hadley, (5), p. 52)

19. See Hadley, (5), Ch. 2,3.

20. For a precise definition see Hadley (5), p. 54.

21. See Hadley (5), p. 99.

22. This is the most obvious basic solution. It is chosen for computational convenience, since it is not going to be optimal unless the commodity prices are negative, or zero.

23. In mathematical terms, a subsystem of equations

(2) and (3) $$\underset{A}{\begin{bmatrix} 5 & 6 & 7 & 1 & 0 \\ 2 & 15 & 1 & 0 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}} \equiv \underset{b}{\begin{bmatrix} 60 \\ 18 \end{bmatrix}}$$

is obtained by setting $x_1 = x_2 = x_3 = 0$. We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 60 \\ 18 \end{bmatrix}$$

$$A_1 \quad x \quad = \quad b$$

Since rank of A_1 = rank of (A_1, b) = number of unknowns we have a unique solution. The solution is non-negative because b lies in the non negative orthant of the two-dimensional space, and the columns of matrix A_1 generate a convex polyhedral cone which is the non-negative orthant.

24. In mathematical terms, the vectors of the matrix A_1 of footnote 23 form a basis (in this case, an orthogonal basis) of the two-dimensional Euclidean space. Hence, any vector $x \in E^2$ can be represented as a linear combination of x_4 and x_5. In the case of x_1 we have,

$$5x_4 + 2x_5 = x_1$$

The two sides of this equation represent alternative ways of using 5 units of resource 1 and 2 units of resource 2. The right-hand-side choice yields \$1. The difference 0-1 shows what is lost if we choose the left-hand-side, while the difference 1-0 shows what we gain if we implement the right-hand choice instead.

25. This can be given an alternative explanation: activity 1 at level 9 uses 9x2 = 18 units of resource 2, i.e. fully employs this resource. This means that the slack variable corresponding to this resource (x_5) must be zero.

26. If $x_1 = 9$, resource 2 is not enough (see previous footnote). In other words, constraint 2 is violated.

27. In mathematical terms what we have done amounts to the following: From the matrix

$$A = \begin{bmatrix} 5 & 6 & 7 & 1 & 0 \\ 2 & 1.5 & 1 & 0 & 1 \end{bmatrix}$$

of footnote 23 we chose the submatrix

$$A_2 = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{i.e., we set } x_2 = x_3 = x_5 = 0$$

and solved the system

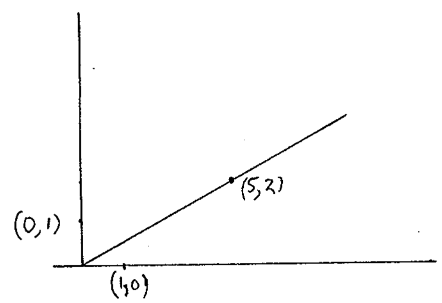
$$\begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 18 \end{bmatrix}$$

to find $x_1 = 9$ $x_4 = 15$. The iterative procedure described in the text helped in determining which submatrix to choose by specifying that $x_5 = 0$. Now, A_1 was $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. To find A_2 we determine which vector is to enter the basis (activity 1) and which is to go (activity 5). The iterative procedure does not necessarily take us through all possible submatricices of A, and is thus seen to economize in computations.

28. In mathematical terms, the column-vectors (points) of the matrix A_2 of footnote 27 are linearly independent and thus from another basis of E^2. Since $x_2, x_3 \in E^2$, x_2 and x_3 can be (uniquely) expressed as linear combinations of the basis vectors. The rest is the same as in footnote 24.

29. Actually the economic interpretation of this case is somewhat awkward since there is no economic sense in saying that we operate activity 4 at a negative level. Mathematically the solution to the system $\begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(i.e. the linear combination of activities 1and 4 that gives activity 5) cannot be non-negative, since the vector (0, 1)' lies outside the convex polyhedral cone generated by the vectors of the matrix as can be seen from the diagram below:



30. The words "at face value" imply that the criterion used for identification of the most "profitable" activity refers to <u>unit</u> "profitability" only, while we are interested in maximizing <u>total</u> revenue.

31. All this, of course, is found by solving

$$\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 18 \end{bmatrix}$$

32. This is because

(a) activities 4 and 5 contribute zero revenue
(b) activities 1 and 3 do not cause the revenue to diminish

33. If there are more than one, there are infinite optimum solutions. See Hadley (5).

34. By solving $\begin{bmatrix} 5 & 6 \\ 2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 18 \end{bmatrix}$

35. Every point on the segment ab is represented by the linear combination $ma + (1-m)\,b$, $1 \geqslant m \geqslant 0$ or $ka + (1-k)\,c$, $1 \geqslant k \geqslant 0$

36. See below, section 2.3.1.

37. The unit vectors (slack variables) are part of the problem. They span the whole non-negative octhant. Hence, any point in Oabc is feasible.

38. See fn. 32.

39. This need not always be the case: see section 2.3.

40. As usual, we solve $\begin{bmatrix} 5 & 7 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

On the economic interpretation of the solution see fn. 29.

41. By solving

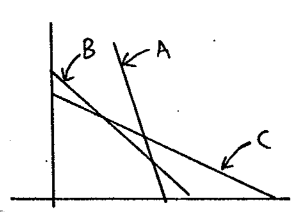
$$\begin{bmatrix} 5 & 7 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

42. In the standard neoclassical theory, this is indeed what is implied. Full employment of all factors is always the case on the frontier, and is achieved through factor substitution on the "well-behaved" production functions of y_1, y_2.

43. Section 2.2.0.0, (i), (ii) (iii) (iv) (vi).

44. To find bundle c, solve $\begin{bmatrix} 5 & 12.5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 250 \\ 120 \end{bmatrix}$

45. Indeed there are cases where full employment of <u>all</u> resources is plainly impossible. Consider



47. The notion of efficient point sets, however, has come out of deep analysis, while the notion of production function seems to have been taken for granted by economists. The former seems superior to the latter, in many respects. See Koopmans, (11) (12).

48. The price ratio, is of course, equal to the social marginal rate of substitution between the two commodities, as well as to the technical rate of their transformation, in this continuous "model".

49. This was to be expected. Commodity 1 uses less of both resources and bears a higher price than commodity 2.

50. Actually, the choice set in L.P. usually consists of an area containing efficient and inefficient bundles. For normal conditions, however, the technique concentrates on the efficient point set, and, in fact, only on extreme points, i.e. on basic feasible solutions as we saw earlier. An exception to this statement is the first basic solution. Note, however, that an inefficient point is a candidate for optimality if prices are negative or all zero.

51. We talk, then, about, "activity analysis". See Koopmans (12).

52. In quite a number of cases: see Samuelson (19), Chapters II, III.

53. Especially in its more general form of activity analysis. See Debreu (2).

54. See (3).

55. This is true only "in the limit", i.e. as long as the composition of the optimal solution does not change. See 2.3.

56. See 2.0.2.

57. Equation: $5u_1 + 2u_2 = 1$.

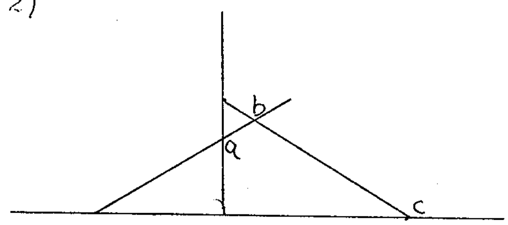
58. From $60u_1 + 18u_2 = C$, for $C = 18$.

59. Not always: consider
Max $5x_1 + 6x_2$

$$\text{Subject to } \begin{bmatrix} -5 & 4 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$x_i \geq 0 \quad i=1,2$

The frontier (assuming activity 1 produces y_1 and activity 2 produces y_2) is abc



but only the part bc is efficient, not ab.

60. This is not always true: it holds only when in the optimal solution all resources are fully employed.

61. The reason for dropping these constraints will be explained in the next section.

62. In mathematical terms, the activities that break even will be operating at non-zero levels in the optimal solution to the primal, and the resources with positive dual prices will be fully employed, that is, the respective constraints will be satisfied with an equality sign in the optimal solution. The original problem is then transformed to that in the text. Since the solution to the system (2)' is unique, we need not use the Lagrange-multiplier technique. We need only solve (2)' and substitute the values of x_1, x_3 in the objective function. If, however, one resource had zero price we would have been left with a meaningful constrained maximum problem, to be solved by Lagrangean technique.

63. See, however, fn. 60.

64. See Dorfman, Samuelson-Solow (3), Henderson-Quandt (8).

65. More or less a "novel" treatment.

66. This is the convex polyhedral cone generated by the half-lines Op_1 Op_3 (Op_2 is redundant from this point of view).

67. See 2.3.1.

68. See fn. 37 of section 2.2.

69. This points towards an aggregate model for an economy, where unemployment could exist because of this reason.

70. See 2.2.0.1.

71. See section 2.1 for another example.

72. It should be noted in passing that the "transportation" model can be used in problems irrelevant to transportation. This is because its definition pertains to the structure of its matrix of constraints, and not to its interpretation, which can be varied depending on the problem. The specific name "transportation model" is due to its first applications, which did refer to space. See F. L. Hitchcock "The Distribution of a Product from Several Sources to Numerous Localities", Journal of Mathematics and Physics, Vol. 20, 1941, pp. 224-30.

73. Problems associated with the notion of a region will not concern me in this thesis. The basic assumption throughout will be equivalent to considering a region as a point in economic space (which in turn means that transportation does not arise within the region).

74. "Potentially" covers the possibility of zero supply or demand in a region.

75. Identical conditions of supply and demand in all regions are not allowed since there will be no basis for interregional trade in this case.

76. "Minimum" implies solution of the so-called "transhipment" problem. See Hadley (5), p. 368.

77. A region cannot do both: see the section 30.3. Also regions not participating in trade can be ignored.

78. All feasible solutions will have to satisfy

$$\sum_i x^o_{ij} = X_j \qquad (i = 1, \ldots m)$$

$$\sum_i x^o_{ij} = B_j \qquad (j = m+1, \ldots n)$$

where x^o_{ij} are the elements of a feasible solution vector.

79. The matrix of equality constraints (3-2) and (3-3) has rank n - 1. See Hadley , (5), p. 275.

80. When an x_{ij} appears in the optimal solution, the constraint which has the respective s_{ij} in the dual is satisfied with an equality.

81. The exposition here approaches the solution of the dual indirectly through the solution of the primal, by using the symmetry properties of duality. See Chapter 2.

82. See Samuelson (20).

83. One method is that provided by S. Enke (4).

84. Except in the case where trade "branches off", that is, one exporting region monopolizes a number of importing regions and is cut off from the rest. In this case, two prices, one from each branch, will if given determine the rest. See (20).

85. See Samuelson (20).

86. For a proof based on a transformation of the general problem into a maximum problem see Samuelson, (20).

87. Cf. Isard and Ostroff, (10).

88. This is correct only under the assumption of non-existence of external economies and diseconomies. See also Chapter 2.

89. The proof relates to game theory. See Kuhn and Tucker (14) pp. 76-8.

90. See Chapter II.

91. Cf. Stevens, (22).

92. Cf. (3), pp. 125-6.

93. If, by setting a $u_i = 0$ there arise some negative values, the smallest-value u is set equal to zero and the computation repeated from the beginning. This procedure does not affect the analysis since the dual only determines relative and not absolute values, in this interpretation. cf. Stevens (22).

94. B. Stevens (22).

95. The number 4 is reserved for introduction of another warehouse location, below.

96. Cf. B. Stevens (21).

97. In (7).

98. Indeed, more than that is required for a meaningful transportation model: at least four regions must be considered. Cf. P. Samuelson (20).

99. "Novel" with this thesis.

100. The specific example involves degeneracy.

101. Cf. (7).

102. "Novel" with this thesis.

103. More than two factors do not change the argument.

104. The number of industries (commodities) is also immaterial to the argument. There should be at least two, however, for the notion of an intermediate commodity to be discussed.

105. Though there are some problems concerning the assumption that "changing prices will not change the technology". This is because we have two primary factors. See Koopmans (11) Chs. VII - IX.

106. Basically A. Hurter's model, simplified in a number of respects. See section 4.1.3.

107. The model can be extended to any number of regions, commodities and primary factors. The particular numbers have been here chosen for expositional simplicity.

108. The following three paragraph explanation is novel with this thesis.

109. Cf. (21) p. 62.

110. Simple, linear functions can of course be considered. Cf. (21) p. 62 and (11) Ch. 3.

111. (7), Model III.

112. (7), pp. 53-55.

113. (9).

114. (21).

115. (7), Model IV.

116. It is usual for the inequality signs to be in the " $\leq$ " direction in a maximization problem. This is readily achieved by changing the sign convention, as will be seen later.

117. (21). See also Isard, (9).

118. (7).

119. Cf. M. Harwitz and A. Hurter, (7), p. 60.

120. L. Moses, (16).

121. L. Moses has used a model much more similar to Model VI in an application. See (16).

122. M. Harwitz, A. Hurter and L. Moses have worked with the research division of the Transportation Center of Northwestern University, Evanston, Illinois.

123. This necessity is not explicitly recognized by Moses.

124. W. Isard, (9).

125. B. Stevens, (21).

126. "Novel" with this thesis.

127. Cf. B. Stevens (21), and R. Kuenne (13), pp. 429-30.

128. This is based on an argument suggested to me by my adviser, Dr. P. S. Dhruvarajan.

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