THE UNIVERSITY OF MANITOBA

VERTICALLY LOADED HYPERBOLIC PARABOLOID CONCRETE SHELLS

by

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A Thesis

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ABSTRACT

The theory of shallow shells and membranes is discussed with respect to the analysis of hyperbolic paraboloid shells. The exact solution of the membrane equations for a vertically loaded hypar shell is studied. Finite Element and Finite Difference analyses of hypar shells are discussed. A Finite Element analysis of a hypar shell is compared to a variational method of analysis of the same shell.

A large scale model of a concrete hypar shell was built and tested in the laboratory. The shell was found to be too stiff to get accurate strain and deflection measurements. The cracking pattern of the shell is discussed and qualitatively compared with results concluded from the Finite Element analysis of the same shell. Finally, a number of recommendations concerning the testing of shallow shells are made.

ACKNOWLEDGEMENTS

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To Mr. Lemke and the staff in the structures laboratory, especially Mr. Brian Turnbull, for their help in the building and testing of the hypar shell.

To my friends and fellow students whose encouragement and friendship helped me through the difficult times.

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CHAPTER 1

INTRODUCTION

Thin shell structures carry loads applied to them by making use of form, rather than mass, to resist the internal forces. A thin curved surface can be designed to span large areas free from intermediate columns or large beams. Besides making efficient use of materials, shells are usually architecturally interesting.

This thesis is primarily interested in shallow hyperbolic-paraboloid shells. A hyperbolic-paraboloid, sometimes referred to as a hypar or h.p., is classed as a translational surface. A hypar surface can be generated by translating a concave upward parabola, called a generator, parallel to itself along a concave downward parabola, called a director. (See Fig. 1.1)

Hypar surfaces also can be thought of as warped parallelograms. Consider the rectangle IJKL in Figure 1.2. When the corner K is dropped to K' a hyperbolic paraboloid surface is formed by the family of straight lines parallel to the y - z plane joining points on the x-axis and line LK'. Similarly the surface can be generated by the lines parallel to the x - z plane joining the y-axis and the line JK'. The equation for this surface is:

$$z = \frac{c}{ba} xy = kxy \ . \qquad \qquad (1)$$

The quantity $k = c/ba$ is called the twist. Hypars are ruled surfaces; however, they cannot be flattened out onto a plane without tear or stretching.

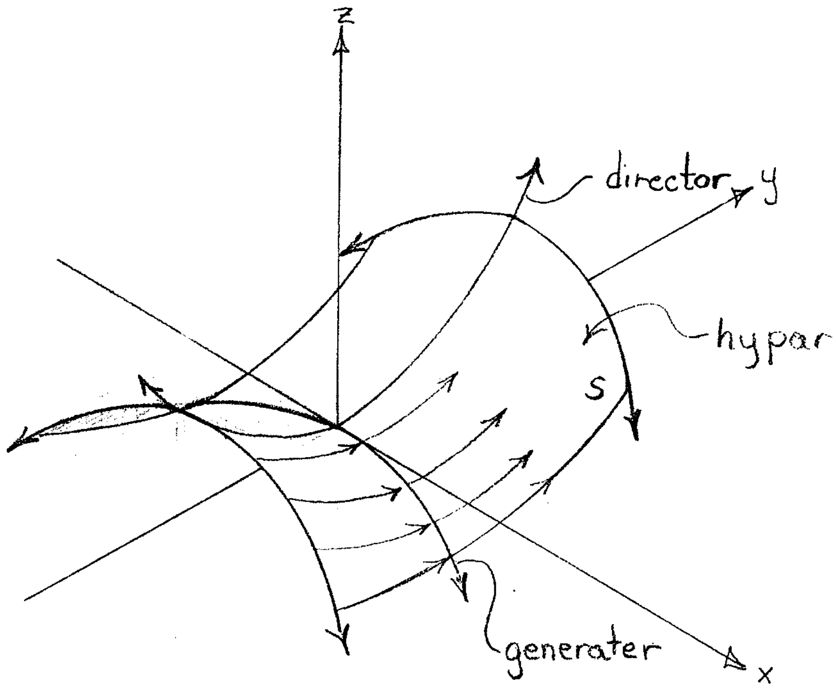

FIGURE 1.1 Hyperbolic Paraboloid

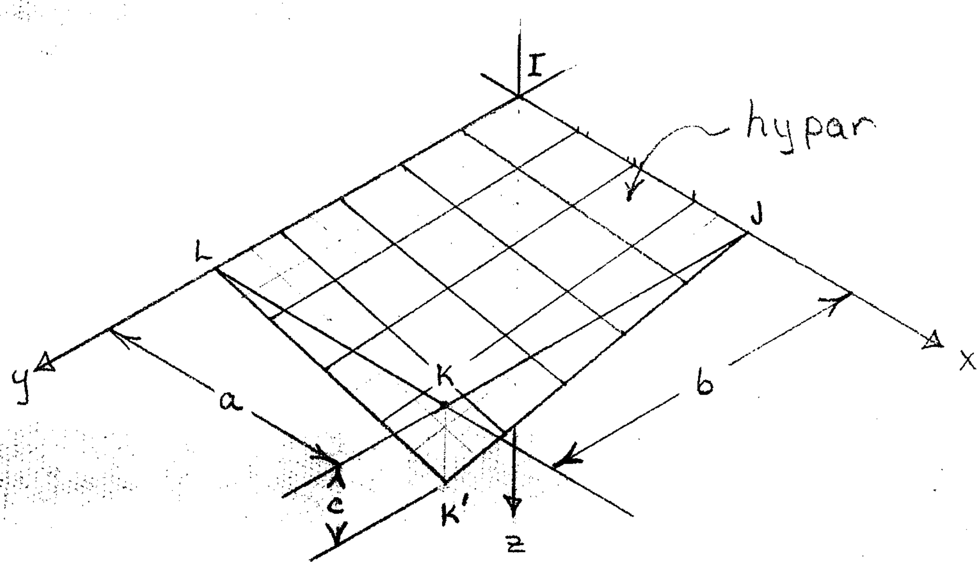

FIGURE 1.2 Hypar Generated From Straight Lines

A ruled surface makes the construction of a concrete hypar shell more attractive to the builder. The formwork can be constructed of straight members and the reinforcing need not be bent or curved, unlike other shell geometry which may demand more complicated formwork and reinforcing.

1.1 ANALYSIS OF SHELLS

Before the advent of the Finite Element method hypar shells were usually analized by membrane theory [4]. Some numerical analyses using the bending equations for hyperbolic paraboloid shells had been made. Nearly all these methods made use of finite difference techniques [7,16,8,1].

Some of the methods and analysis that have been developed are aimed at designers. Russell and Gerstle [16] give the result of their finite difference solutions in a non-dimensional form, intended as a design aid. M. Liebour [13] suggests a method of determining the ultimate strength of a cantilevered hypar on two supports, which could be used by a designer.

With the increased accessibility of computers, designers are using many package programs which can analyze shells with their support structures. These programs if used intelligently can give realistic answers quickly. Designers are able to alter and re-analyze the structure quickly and economically to obtain more effective designs.

1.2 PREVIOUS TESTING

During the early 1960's there was more interest in model testing of shells than at present. Shells of all types and materials were tested [4]. Tests of hypar shells on two supports have been done by a number of workers and reported in the literature [12,18,14,17]. Load tests on umbrella hypar shells have been done by some experimenters [9].

CHAPTER 2

THEORY

2.1 GENERAL

In general, there exist two approaches to the analysis of shells. The most basic and simplest approach deals only with in-plane force stress resultants. This is usually referred to as membrane theory. A more sophisticated approach to the shell analysis also considers transverse shear, bending and twisting moments.

Both approaches to shell analysis will be outlined in the following sections of this chapter. The membrane equations for translational shells and the bending equations for shallow shells will be discussed.

2.2 BASIC MEMBRANE EQUATIONS FOR TRANSLATIONAL SHELLS

Consider the small differential element in Figure 2.1 on which external loads p_i and membrane stress resultants $N_{\alpha\beta}$ act. The membrane stress resultants are projected onto the horizontal x - y plane. The projected stress resultants $\bar{N}_{\alpha\beta}$ in terms of the $N_{\alpha\beta}$ can be found from:

$$\bar{N}_x \, dy = N_x \cos\phi \, dq \; . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.1)$$

By substituting $\cos\phi \, dq = dy$ into (2.1) we find:

$$\bar{N}_x = \frac{\cos\phi}{\cos\phi} N_x \; . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.2)$$

Similarly:

$$\bar{N}_y = \frac{\cos\phi}{\cos\phi} N_y \; . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.3)$$

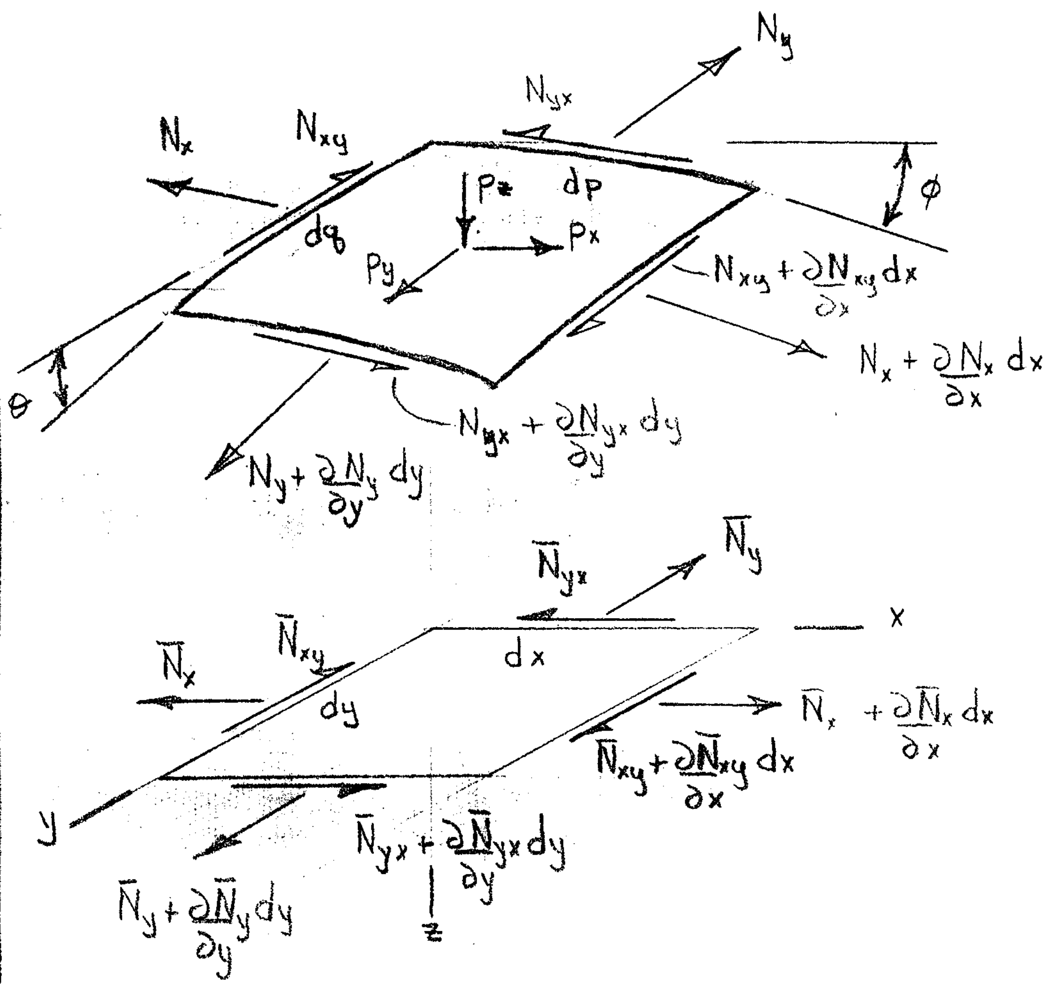

FIGURE 2.1 Differential Element of a Translational Shell

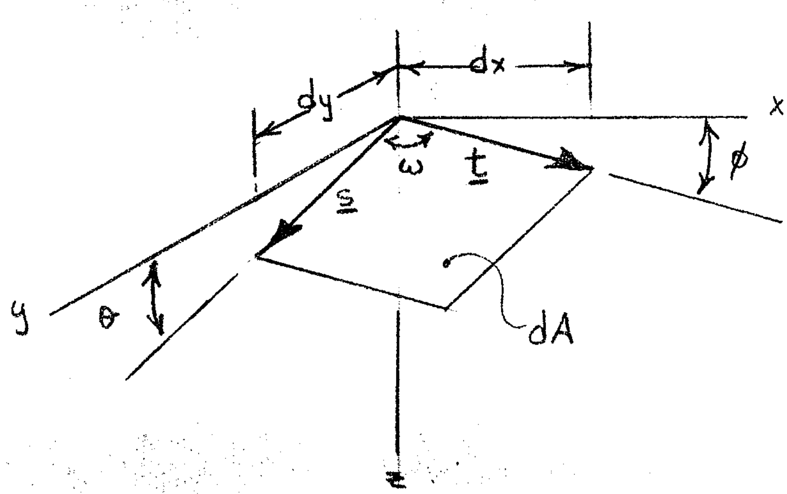

FIGURE 2.2 Area of Differential Element

$$\bar{N}_{xy} = N_{xy} \ , \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.4)$$

The components of the load must also be expressed in terms of the projected area. To this end we must find the elemental area dA in terms of dx and dy. This can be seen from Figure 2.2 as:

$$dA = |\underline{s} \times \underline{t}| = \{|(0,1,\tan\theta) \times (1,0,\tan\phi)|\} dx\ dy$$

$$= (\tan^2\theta + \tan\phi + 1)^{\frac{1}{2}}\ dx\ dy$$

$$= \sqrt{\frac{1 - \sin^2\phi\ \sin^2\theta}{\cos\theta\ \cos\phi}}\ dx\ dy \ . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.5)$$

Equation (2.5) is used to relate projected loads $\bar{p}_i$ to the surface loads p_i by:

$$\bar{p}_i = p_i \sqrt{\frac{1 - \sin^2\phi\ \sin^2\theta}{\cos\theta\ \cos\phi}} \ . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.6)$$

Using the quantities defined by expressions (2.2) to (2.4) and (2.6) the equilibrium equations can be written for the shell element. For equilibrium in the x - y plane we find:

$$\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{yx}}{\partial y} + \bar{p}_x = 0 \ , \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.7)$$

$$\frac{\partial \bar{N}_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + \bar{p}_y = 0 \ . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.8)$$

The components of the forces acting in the z-direction can be expressed in terms of the barred stress resultants as:

$$N_x \sin\phi\ dq = \bar{N}_x \tan\phi\ dy = \bar{N}_x \frac{\partial z}{\partial x}\ dy$$

$$N_x \sin\theta\ dp = \bar{N}_y \frac{\partial z}{\partial y}\ dy$$

$$N_{xy} \sin\theta\ dq = \bar{N}_{xy} \tan\theta\ dy = \bar{N}_{xy} \frac{\partial z}{\partial y}\ dy$$

$$N_{yx} \sin\phi\ dp = \bar{N}_{yx} \frac{\partial z}{\partial x}\ dx$$

Equilibrium in the z-direction yields:

$$\frac{\partial}{\partial x}(\bar{N}_x \frac{\partial z}{\partial x}) + \frac{\partial}{\partial y}(\bar{N}_y \frac{dz}{\partial y}) + \frac{\partial}{\partial x}(\bar{N}_{xy} \frac{\partial z}{\partial y}) + \frac{\partial}{\partial y}(\bar{N}_{yx} \frac{\partial z}{\partial x}) + \bar{p}_z = 0 \ . \qquad (2.9)$$

Differentiating (2.9) and recognizing that $\bar{N}_{xy} = \bar{N}_{yx}$ we obtain:

$$\frac{\partial \bar{N}_x}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial \bar{N}_y}{\partial y}\frac{\partial z}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial y}\frac{\partial z}{\partial x} + \bar{N}_x \frac{\partial^2 z}{\partial x^2}$$

$$+ \bar{N}_y \frac{\partial^2 z}{\partial y^2} + \bar{N}_{xy}\frac{\partial^2 z}{\partial x \partial y} + \bar{p}_z = 0 \ . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.10)$$

Reduction of the three equilibrium equations (2.7), (2.8) and (2.10) to one equation is achieved by substituting (2.7) and (2.8) into (2.10). This yields:

$$\bar{N}_x \frac{\partial^2 z}{\partial x^2} + 2\bar{N}_{xy}\frac{\partial^2 z}{\partial x \partial y} + \bar{N}_y \frac{\partial^2 z}{\partial y^2} = \bar{p}_x \frac{\partial z}{\partial x} + \bar{p}_y \frac{\partial z}{\partial y} - p_z \ . \quad \ldots \quad (2.11)$$

The solution of the above equation is more conveniently handled by introducing a stress function F defined such that:

$$\bar{N}_x = \frac{\partial^2 F}{\partial y^2} - \int \bar{p}_x \, dx$$

$$\bar{N}_y = \frac{\partial^2 F}{\partial x^2} - \int \bar{p}_y \, dy$$

$$\bar{N}_{xy} = \frac{-\partial^2 F}{\partial x \partial y} \ . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.12)$$

Introducing (2.12) into (2.11) results in:

$$\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = q \ , \ \ldots\ldots\ldots \qquad (2.13)$$

where q is defined as:

$$q = -\bar{p}_z + \bar{p}_x \frac{\partial z}{\partial x} + \bar{p}_y \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x^2} \int p_x \, dx - \frac{\partial^2 z}{\partial y^2} \int \bar{p}_y \, dy \quad . . . \quad (2.14)$$

Under a vertical load the quantity q reduces to $-\bar{p}_z$, since $\bar{p}_x = \bar{p}_y = 0$.

2.3 COMPLETE EQUATIONS FOR SHALLOW SHELLS

2.3.1 General

In general, the load on a shell is not carried by membrane forces alone. Stress gradients through the thickness of the shell result in bending and twisting stress resultants. In fact, a complete analysis of a shell must take in account both flexure and in-plane forces and deformations. Since shells have curvature, membrane and flexure effects interact even when a small deflection theory is used.

The equations of classical shell theory are derived with the aid of the following assumptions:

1) Small deflections. Deflections are assumed not to change the shell geometry enough to alter static equilibrium.

2) The material has linear elastic properties. Young's law holds.

3) Conservation of normals. Material points in the shell which lie on normals to the middle surface before bending remain on the same normal after bending. Shear deformations are neglected.

In addition to the above assumptions shallow shell theory makes use of a number of simplifications which can be made due to the geometry of a shallow shell. These are:

1) The slope is small compared to some reference plane (i.e. the

horizontal plane, in the case of a roof).

2) The curvature of the surface is small.

3) Transverse deflections are much greater than in-plane deflections.

4) Changes in curvature of the surfaces are small.

2.3.2 Stress Resultants

Consider the shell element, S, in Figure 2.3. The membrane stress resultants, $N_{\alpha\beta}$, acting on ,S, can be defined by:

$$N_{xx} = \int_{h/2}^{-h/2} \sigma_{xx}\, dz$$

$$N_{xy} = N_{yx} = \int_{h/2}^{-h/2} \sigma_{xy}\, dz \quad , \qquad \ldots\ldots\ldots\ldots \quad (2.15)$$

$$N_{yy} = \int_{h/2}^{-h/2} \sigma_{yy}\, dz$$

where the shell thickness, h, is much smaller than the radius of curvature.

In the same way used for membrane forces, bending and shear stress resultants, illustrated in Figure 2.3, are defined as

$$M_{xx} = -\int_{-h/2}^{h/2} \sigma_{xx}\, zdz$$

$$M_{yy} = -\int_{-h/2}^{h/2} \sigma_{yy}\, zdz$$

$$M_{xy} = -M_{yx} = \int_{-h/2}^{h/2} \tau_{xy}\, zdz \qquad \ldots\ldots\ldots\ldots \quad (2.15)$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz$$

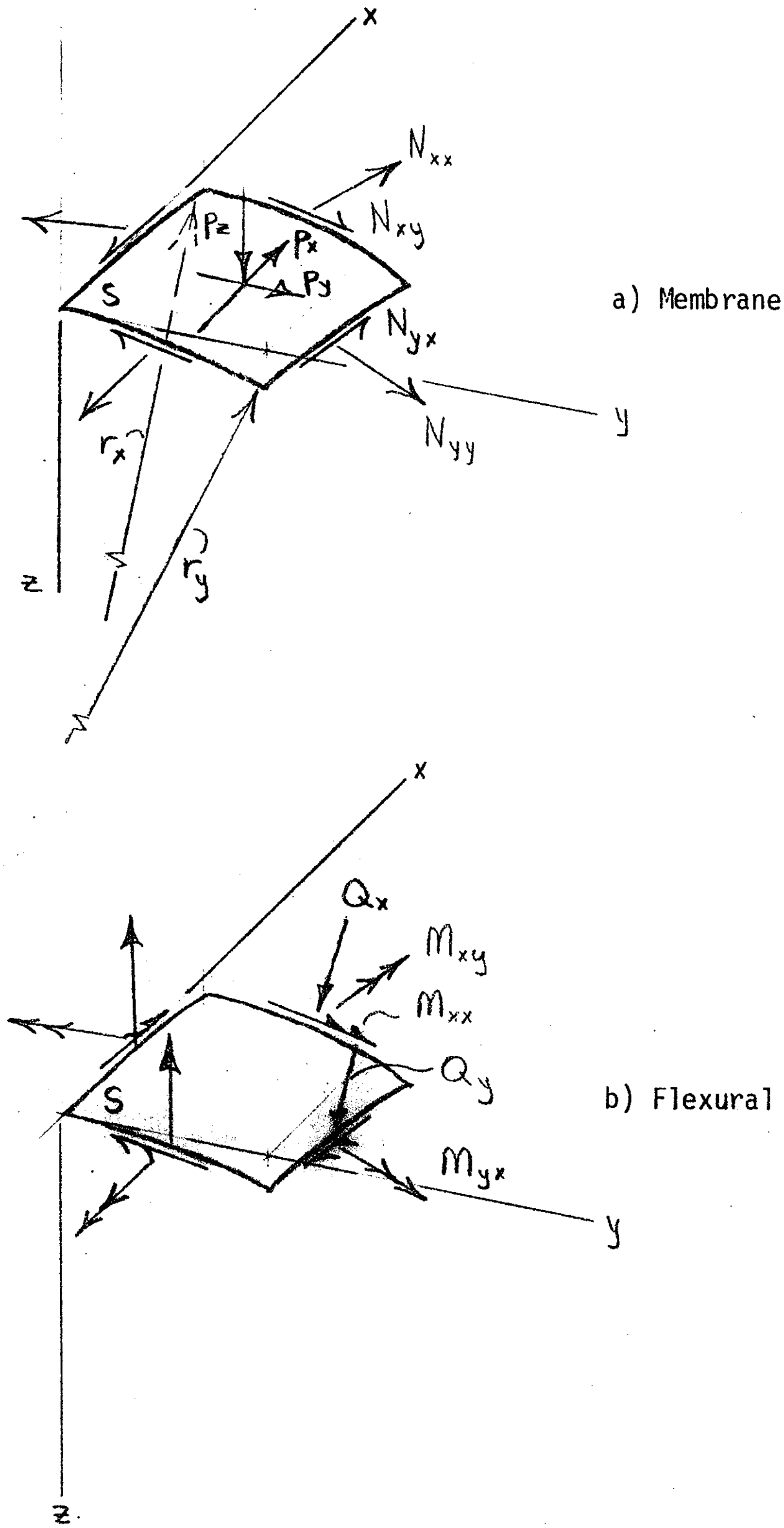

FIGURE 2.3 Stress Resultants

2.3.3 Equilibrium and Constitutive Equations

By considering the equilibrium of the shell element the following equations result:

$$N_{xx,x} + N_{yx,y} + p_x = 0$$

$$N_{yy,y} + N_{xy,x} + p_y = 0$$

$$Q_{x,x} + Q_{y,y} + \frac{N_x}{r_x} + \frac{2N_{xy}}{r_{xy}} + \frac{N_y}{r_y} + p_z = 0 \quad \ldots\ldots\ldots \quad (2.16)$$

$$- M_{y,y} + M_{xy,x} + Q_y = 0$$

$$- M_{x,x} - M_{yx,y} + Q_x = 0$$

Where $1/r_x = \partial^2 z/\partial x^2$, $1/r_y = \partial^2 z \; \partial y^2$ and $1/r_{xy} = \partial^2 z/\partial_x \partial_y$.

The first two equations are simply the membrane force equilibrium equations. The third equation expresses force equilibrium in the z-direction. The last two equations are found by taking moments about the in-plane x and y axis.

The constitutive equations for a shallow shell, subject to the assumptions in section 2.3.1, are:

$$N_{xx} = K \left\{ u,_{xx} - \frac{w}{r_x} + \nu(v,_y - \frac{w}{r_y}) \right\}$$

$$N_{yy} = K \left\{ v,_y - \frac{w}{r_y} + \nu(u,_x - \frac{w}{r_x}) \right\}$$

$$N_{xy} = Gh \left\{ v,_x + u,_y - \frac{2w}{r_{xy}} \right\}$$

$$M_x = - D \left\{ w,_{xx} - \nu \, w,_{yy} \right\} \quad \ldots\ldots\ldots \quad (2.17)$$

$$M_y = - D \left\{ w,_{yy} - \nu \, w,_{xx} \right\}$$

$$M_{xy} = D(1-\nu) \, w,_{xy}$$

where ν is Poisson's ratio, E and G are the elastic and shear moduli and K and D are the extensional and flexural rigidities for the shell defined as:

$$K = \frac{Eh}{1-\nu^2} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

2.3.4 Reduction Into a Pair of Coupled Fourth Order Differential Equations

Beginning with the first three equations of (2.17) we combine these to form:

$$\begin{aligned} N_x - \nu N_y &= K(1-\nu^2)(u,_x - \frac{w}{r_x}) \\ N_y - \nu N_x &= K(1-\nu^2)(v,_y - \frac{w}{r_y}) \qquad \ldots\ldots \quad (2.18) \\ 2(1+\nu)N_{xy} &= K(1-\nu^2)(v,_x + v,_y - \frac{2w}{r_{xy}}) \ . \end{aligned}$$

By taking the appropriate second derivatives and combining again:

$$N_{x,yy} - \nu N_{y,yy} - 2(1+\nu)\, N_{xy,xy} + N_{y,xx} - \nu N_{x,xx} = K(1-\nu^2)$$
$$[u,_{xyy} - \frac{w,_{yy}}{r_x} - v,_{xxy} - u,_{yxy} + \frac{2w,_{xy}}{r_{xy}} + v,_{yxx} - \frac{w,_{xx}}{r_y}] \ . \ \ldots \quad (2.19)$$

By introducing a stress function ϕ such that the first two equations of (2.16) are identically satisfied, we have:

$$N_x = \phi,_{yy} - \int p_x\, dx \ ; \ N_y = \phi,_{xx} - \int p_y\, dy \ ; \ N_{xy} = -\phi,_{xy} \ . \qquad (2.20)$$

Taking the appropriate derivatives of (2.20) equation (2.19) can be rewritten as:

$$\phi,_{yyyy} + 2\phi_{xx,yy} + \phi,_{xxxx} = -K(1-\nu^2)(\frac{w,_{xx}}{r_x} - \frac{2w,_{xy}}{r_{xy}} + \frac{w,_{yy}}{r_y})$$
$$+ \ p_{x,yy}\, dx - \nu\int p_{y,yy}\, dy + \int p_{y,xx}\, dy - \nu\int p_{x,xx}\, dx$$

or:

CHAPTER 3

NUMERICAL METHODS USED FOR SHELL ANALYSIS

3.1 INTRODUCTION

A number of numerical methods have been used to solve shell analysis problems. All the methods except the finite element method (F.E.M.) deal strictly with shells having a geometry that can be expressed in an analytical form. Finite difference methods have been used for shells with elastic boundaries (i.e. edge beams), but for the most part methods of analysis other than the F.E.M. are usually restricted to unrealistic boundary conditions. For example: Infinitely rigid clamped edges or knife edge supports. Only the finite difference and finite element methods will be discussed in any detail, since these methods are by far the most common and most important.

3.2 THE FINITE DIFFERENCE METHOD APPLIED TO SHALLOW SHELLS

The finite differences can be used to solve the partial differential equations developed in (2.3.4). Consider the case when $p_x = p_y = 0$ (i.e. lateral load only) then:

$$\nabla^4 w - \frac{1}{D}\nabla_R^2\phi = \frac{p_z}{D} \quad . \qquad \ldots\ldots\ldots\ldots\ldots \quad (2.25)$$

$$\nabla^4\phi + K(1-\nu^2)\nabla_R^2 w = 0 \quad . \qquad \ldots\ldots\ldots\ldots\ldots \quad (2.21)$$

When the shell surface is divided into a regular rectangular grid, the values, at grid intersection points of ϕ and w can be solved for by writing the above equations in different form.

Consider the equation (2.25) for a point with grid coordinates (i,j) (see Figure 3.1). Points with i constant lie on curves with x constant similarly with j constant and y constant. Writing the required derivatives in different form for point i,j:

$$\frac{\partial^4 w}{\partial x^4} \cong \frac{1}{h_x^4}\{6\, w_{i\ j} - 4(w_{i-1\ j} + w_{i+1\ j}) - w_{i+2\ j} + w_{i-2\ j}\}$$

$$\frac{\partial^4 w}{\partial y^4} \cong \frac{1}{h_y^4}\{6\, w_{i\ j} - 4(w_{i\ j-1} + w_{i\ j+1}) + w_{i\ j+2} + w_{i\ j-2}\}$$

$$\frac{\partial^4 w}{\partial y^2 \partial x^2} \cong \frac{1}{h_y^2 h_x^2}[w_{i+1\ j+1} + w_{i-1\ j-1} + w_{i+1\ j-1} + w_{i-1\ j-1}$$

$$+ 4\, w_{i\ j} - 2(w_{i\ j+1} + w_{i\ j-1} + w_{i+1\ j} + w_{i-1\ j})]$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{1}{h_y h_x}(\phi_{i+1\ i+1} + \phi_{i-1\ j+1} + \phi_{i+1\ j-1} + \phi_{i-1\ j-1})$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{h_x^2}(\phi_{i-1\ j} - 2\phi_{i\ j} + \phi_{i+1\ j})$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{h_y^2}(\phi_{i\ j-1} - 2\phi_{i\ j} + \phi_{i\ j+1})$$

When the simplifying assumption that $h_x = h_y = h$ and the above are substituted into (2.25^1):

$$\frac{1}{h^4}\{16\, w_{i\ j} - 6[w_{i\ j+1} + w_{i\ j-1} + w_{i+1\ j} + w_{i-1\ j}]$$

$$+ w_{i+1\ j+1} + w_{i+1\ j-1} + w_{i-1\ j+1} + w_{i-1\ j-1} + w_{i+2\ j}$$

$$+ w_{i\ j-2} + w_{i\ j+2}\} - \frac{1}{h^2 D}\ \frac{1}{r_y}[\phi_{i-1\ j-2} - 2\phi_{i\ j} + \phi_{i-1\ j}]$$

$$- \frac{2}{r_{xy}}[\phi_{i+1\ j+1} + \phi_{i-1\ j+1} + \phi_{i-1\ j-1} + \phi_{i-1\ -1}] + \frac{1}{r_x}[\phi_{i\ j+1}$$

$$+ w_{i\ j+1}]\} = 0\ . \qquad (3.2)$$

At each grid point both equations (3.1) and (3.2) can be written. For convenience, the following notation is introduced.

$$W = \begin{Bmatrix} w \\ \\ \\ 0 \end{Bmatrix} \text{ and } \underline{P} = \begin{Bmatrix} \frac{p_z}{D} \\ \vdots \\ 0 \end{Bmatrix} \qquad 0 ,$$

where $\underline{M}$ is a column vector containing w_{ij} and ϕ_{ij} for each point, ij, that equations (3.1) and (3.2) are written. $\underline{P}$ is a column vector containing $p_z ij/D$ for each point equation (3.1) is written, and a 0 for each point equation (3.2) is written. Thus the complete set of equations can be written.

$$A \underline{W} = \underline{P} , \qquad \ldots\ldots\ldots\ldots\ldots \qquad (3.3)$$

where A is a coefficient matrix obtained from equations (3.1) and (3.2). Boundary conditions can be applied and equation (3.3) can be rewritten:

$$\begin{bmatrix} A_{xx} & A_{xk} \\ A_{kx} & A_{kk} \end{bmatrix} \begin{Bmatrix} W_x \\ W_k \end{Bmatrix} = \begin{Bmatrix} p_k \\ p_x \end{Bmatrix} , \qquad \ldots\ldots\ldots\ldots\ldots \qquad (3.4)$$

where W_x are the unknown values of the stress function and displacements. Solving equation (3.4) for W_x:

$$\{W_x\} = A_{xx}^{-1} \left\{ p_k - A_{xk}\{W_k\} \right\} . \qquad \ldots\ldots\ldots\ldots\ldots \qquad (3.5)$$

Once {W} is known the stress resultants at the grid points can be found from the different forms of the flexural constitutive equations and the stress function - membrane force relations.

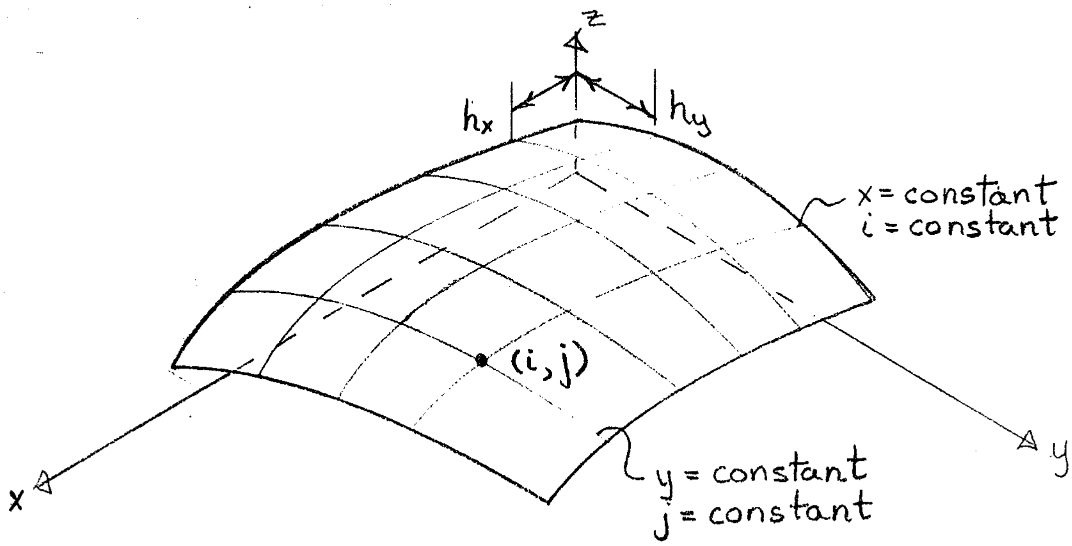

FIGURE 3.1 Grid System for Finite Differences

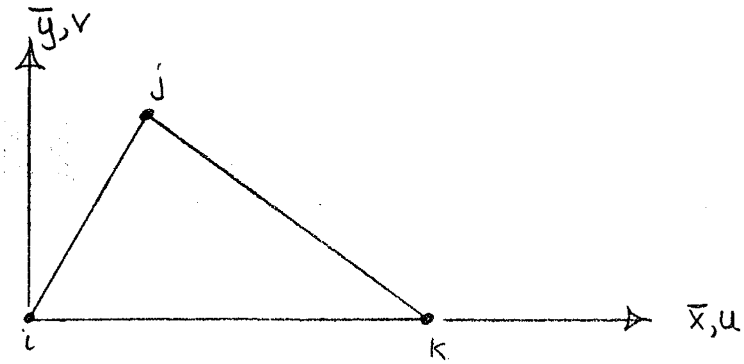

FIGURE 3.2 Coordinate System for Constant Strain Triangle

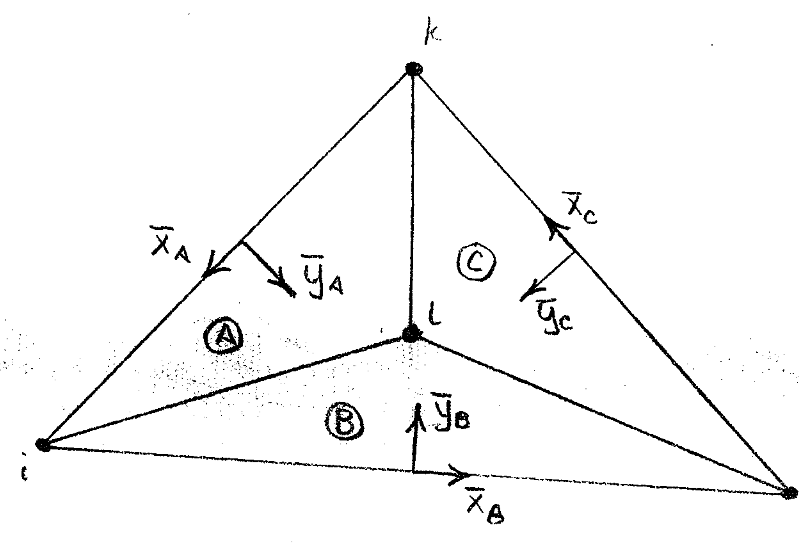

FIGURE 3.3 Plate Bending Triangular Element

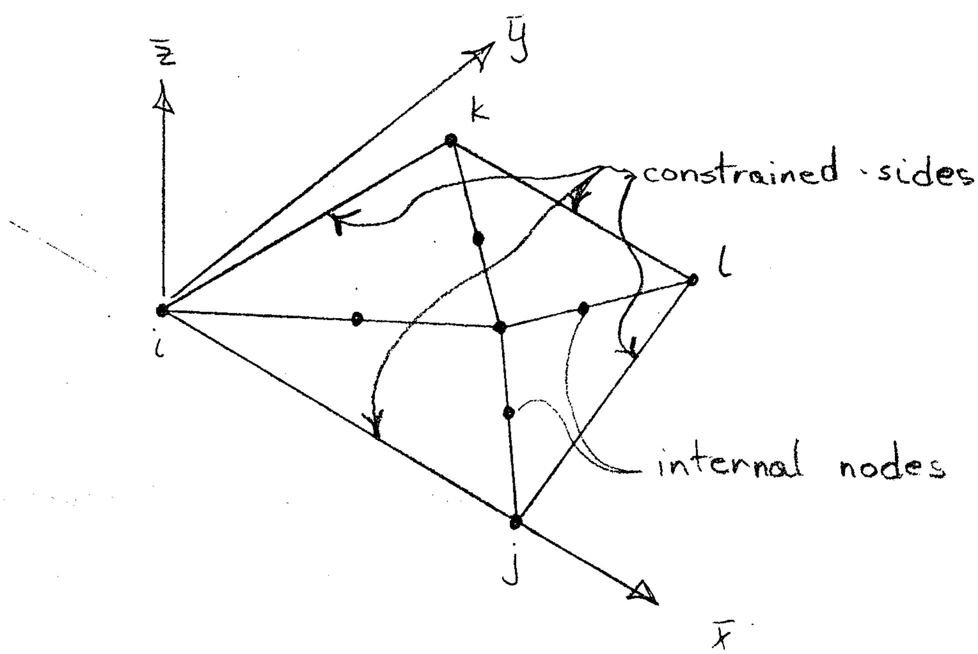

FIGURE 3.4 Quadrilateral Membrane Element

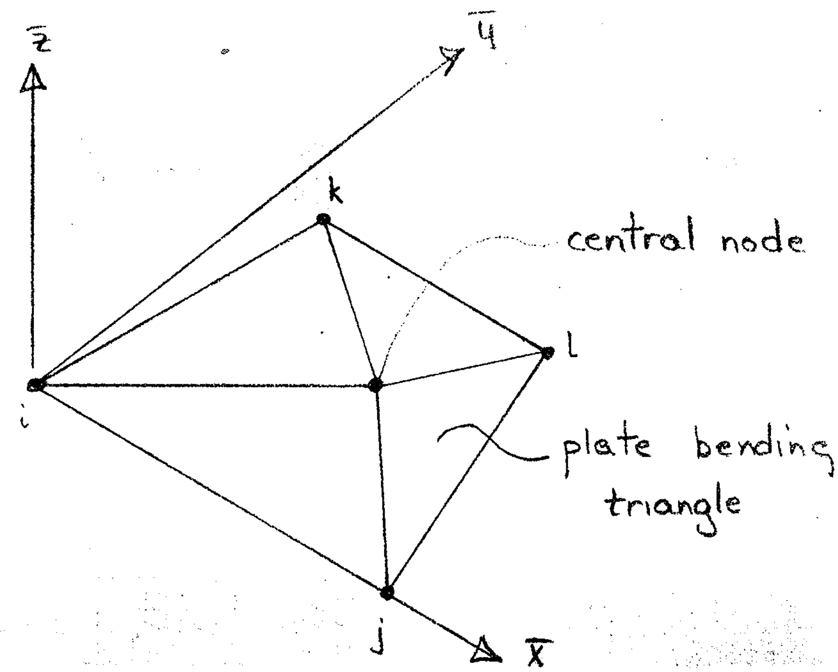

FIGURE 3.5 Quadrilateral Bending Element

3.3 THE FINITE ELEMENT METHOD

3.3.1 General Remarks

Many different finite elements are available for shell analysis. For the most part, they fall into three major categories:

1) Flat thin shell elements, which neglect curvature within each element. The shell is modelled as an assembly of small flat plates.

2) Curved thin shell elements, which account for curvature efforts within the element.

3) Thick shell elements, which do not neglect shearing deformations as do thin shell elements.

The finite element method offers the greatest amount of flexibility. Unlike the finite difference method, or other methods, finite elements can be used to model shells of a purely arbitrary shape. Ribs, stiffeners and varying thickness can be modeled more easily. Realistic boundary conditions can be used in analysis by supporting the shell on boundary elements or on beam or diaphragm elements which simulate the real support structure. A finite element program is possibly the most important tool of analysis available to a shell designer.

A finite element analysis was done on both the shell analyzed by Tottenham and Chetty [5], and on the shell tested in the laboratory. The analyses were performed using a shell finite element program called 'DSHELL' at the University of Manitoba. The program was developed at the University of California (Berkeley) in 1968 by Johnson and Smith [11]. The basic element used in the program was developed by Clough and Johnson [6]. What follows is a description of that element.

3.3.2 Element Description

'DSHELL' uses both triangular and quadrilateral elements. The triangular element is a hybrid of a planar membrane element and a planar plate bending element. The quadrilateral element is made up of four triangular plate bending elements.

i) Triangular Element

The basic membrane element is a constant strain triangle (CST). There are two in-plane displacements at each mode making a total of six degrees of freedom. The element is developed by assuming the displacements are linear throughout the element, which gives rise to a constant strain.

From the coordinate system defined in Figure 3.2 the displacement function is:

$$\begin{bmatrix} 1,\bar{x},\bar{y},0,0,0 \\ 0,0,0,1,\bar{x},\bar{y} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \end{Bmatrix} = \begin{Bmatrix} u \\ y \end{Bmatrix} \quad , \qquad \ldots\ldots\ldots\ldots \quad (3.6)$$

where the α_i's are constants.

Omitting the details the nodal forces and nodal displacements for the i^{th} element are related by:

$$[K_m^i] \begin{Bmatrix} \beta_m^i \\ \beta_m^j \\ \beta_m^k \end{Bmatrix} = \begin{Bmatrix} P_m^i \\ P_m^j \\ P_m^k \end{Bmatrix} \quad , \qquad \ldots\ldots\ldots\ldots \quad (3.7)$$

where $\beta_m^\ell = \begin{Bmatrix} U^\ell \\ V^\ell \end{Bmatrix}$ and $P_m^\ell = \begin{Bmatrix} P_{\bar{x}}^\ell \\ P_{\bar{y}}^\ell \end{Bmatrix}$ and

$[K_m^i]$ is the membrane stiffness matrix.

The flexural element is a fully compatible triangular plate element, which relates the in-plane nodal notations and lateral deflections with nodal bending moments and shears. The element is developed by dividing the triangle into three sub-triangles which all share a central node. The displacement function (see Figure 3.3) is of the form:

$$\begin{bmatrix} <\xi_A> & 0 & 0 \\ 0 & <\xi_B> & 0 \\ 0 & 0 & <\xi_C> \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{27} \end{Bmatrix} = \begin{Bmatrix} W_A \\ W_B \\ W_C \end{Bmatrix} , \qquad \ldots\ldots\ldots\ldots\ldots \quad (3.8)$$

where $<\xi_A> = <1, x_A, y_A, x_A^2, x_A, y_A, y_A^2, x_A^3, x_A, y_A^2, y_A^3>$ and W_A is the lateral displacement in sub-triangle A. Similarly for ξ_B, ξ_C, W_B and W_C.

There are nine independent degrees of freedom, three at each of the exterior nodes. The lateral displacement, W, varies cubically within the triangle and varies quadratically at the exterior boundaries. The stiffness matrix is found by enforcing compatibility at the central node and condensing out the three internal degrees of freedom. Symbolically:

$$\begin{bmatrix} K_e & K_{ie}^T \\ K_{ie} & K_i \end{bmatrix} \begin{Bmatrix} \beta_e \\ \beta_i \end{Bmatrix} = \begin{Bmatrix} P_e \\ 0 \end{Bmatrix} . \qquad \ldots\ldots\ldots\ldots \quad (3.9)$$

Solving for β_i in the second equation of 3.9 and the substitution into first equation of (3.9) results in:

$$[K_e - K_{ie}^T K_i^{-1} K_{ie}]\{\beta_e\} = \{P_e\} . \qquad \ldots\ldots\ldots\ldots \quad (3.10)$$

Let: $$[K_f] = [K_e - K_{ie}^T K_i K_{ie}] , \qquad \ldots\ldots\ldots\ldots \quad (3.11)$$

then: $$[K_f]\{\beta_e\} = \{P_e\} . \qquad \ldots\ldots\ldots\ldots \quad (3.12)$$

where K_f is the flexural stiffness matrix, K_e is the sub-matrix relating external displacements to external nodal forces and K_{ie}^T and K_{ie} relate external displacements to internal nodal forces and vice versa. The flexural stiffness matrix K_f relates two rotations $\theta_{\bar{x}}^k$, $\theta_{\bar{y}}^k$, and one translation W^k to two moments and a shear at each node k. Hence, for the n^{th} element:

$$[K_f^n] \begin{Bmatrix} \beta_f^i \\ \beta_f^j \\ \beta_f^k \end{Bmatrix} = \begin{Bmatrix} p_f^i \\ p_f^j \\ p_f^k \end{Bmatrix} , \qquad \text{............} \quad (3.13)$$

where $\beta_f^i = \begin{Bmatrix} w^i \\ \theta_{\bar{x}}^i \\ \theta_{\bar{y}}^i \end{Bmatrix}$ and $p_f^i = \begin{Bmatrix} p_z^i \\ m_{\bar{x}}^i \\ m_{\bar{y}}^i \end{Bmatrix}$

The complete triangular element is obtained by combining the flexural and membrane elements. For the i^{th} element:

$$\begin{bmatrix} K_m & 0 \\ 0 & K_f \end{bmatrix} \begin{Bmatrix} \beta_m \\ \beta_f \end{Bmatrix} = \begin{Bmatrix} p_m \\ p_f \end{Bmatrix} . \qquad \text{............} \quad (3.14)$$

When expressed in the local frame, the membrane effects are uncoupled from the flexural effects.

ii) The Quadrilateral Element

The quadrilateral element is a hybrid of four linear strain triangles and four triangular plate bending elements as described in i). The membrane element (Figure 3.4) has quadratic displacements within each of the sub-elements but linear displacements on the exterior boundaries, as required by inter-element compatibility. The element has five interior nodes which are condensed out in the same manner as described for the central node in the triangular plate bending element.

The flexural element is simply developed from four of the triangular plate elements sharing a single central node. The interior node is condensed out to yield the quadrilateral flexural element.

The complete quadrilateral element is formed in the same way as described in i), yielding, as before, a stiffness matrix for which the membrane and flexural parts are uncoupled.

3.3.3 Element Assembly

The individual element stiffness matrices are in terms of the local element coordinate system. Assembly of the individual element stiffnesses requires that each of the element stiffnesses be in terms of global coordinates. To achieve this result let:

$$\bar{T}: \{\bar{\xi}\} \rightarrow \{\bar{x}\} \quad T\{\bar{\xi}\} = \{\bar{x}\} \,, \qquad \ldots\ldots\ldots\ldots \quad (3.15)$$

and

$$T_{\xi}: \{\xi\} \rightarrow \{\bar{\xi}\} \quad T_{\xi}\{\xi\} = \{\bar{\xi}\} \,, \qquad \ldots\ldots\ldots\ldots \quad (3.16)$$

where $\{\bar{\xi}\}$ are surface coordinates, $\{\bar{\xi}\}$ are global coordinates and $\{\bar{x}\}$ are element coordinates. All the coordinate systems are in terms of an orthonormal basis. Using equations (3.15) and (3.16) the following is obtained:

$$\bar{T}\, T_{\xi}\{\xi\} = T\{\xi\} = \{\bar{x}\} \,, \qquad \ldots\ldots\ldots\ldots \quad (3.17)$$

where:

$$T = \bar{T}\, T_{\xi} \,, \qquad \ldots\ldots\ldots\ldots \quad (3.18)$$

and

$$T^{-1}\{\bar{x}\} = T^{T}\{\bar{x}\} = \{\xi\} \,. \qquad \ldots\ldots\ldots\ldots \quad (3.19)$$