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TRANSIENTS IN PLATES AND SHELLS OF REVOLUTION

by

DAVID PATHMASEELAN THAMBIRATNAM

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A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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TO MY PARENTS

MR. AND MRS. P.J. THAMBIRATNAM

ABSTRACT

The propagation of transient waves in linear, elastic, isotropic and homogeneous plates and shells of revolution is treated in this thesis. The analysis is based on the concept of a wave as a carrier of discontinuities in the field variable and/or its derivatives. The one to one relationship that exists between a particular transient problem and the corresponding time harmonic problem is first established and then exploited. This relationship makes it possible to deal with transient problems in terms of asymptotic series expansions, thereby making the analysis very much simpler than the usual method of discontinuity analysis.

The transient problems considered are due to impulsive loads acting at the boundaries of structures and specified in the form of strain, velocity or acceleration boundary conditions. Several numerical examples are solved to illustrate the method of solution as well as to establish its validity. The results are compared with existing solutions, wherever possible, and we obtain excellent agreement. A numerical superposition technique is developed which makes it possible to treat transient problems due to boundary loads of longer duration. This technique is applied to solve the problems of transient wave propagation in cylindrical shell structures subjected to ground excitation.

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CHAPTER I

INTRODUCTION

The problems of the propagation of transient waves in linear, elastic, isotropic and homogeneous plates and shells are treated in this dissertation. In the case of shells, consideration is restricted to shells of revolution with straight line generators. The transient waves that we consider are due to time dependent loads acting at the boundaries of the structure and specified in the form of strain, velocity or acceleration boundary conditions. Though the treatment stems from the concept of the propagation of discontinuities, the method of solution is somewhat different from the usual method of discontinuity analysis.

The equations of Naghdi [1] are employed in this thesis and we obtain a set of coupled displacement equations of motion for each case considered. Naghdi's equations are based on the Cosserat theory and include the effects of transverse shear, transverse normal stress and strain and rotatory inertia. Due to the presence of lower order derivatives, the governing displacement equations of motion are dispersive [2] causing the distortion of transient waves and the phase velocities of time harmonic waves to be frequency dependent. However, finite wave front speeds are assured due to the hyperbolic nature of these equations [3]. This is a primary requirement for solution by the method of discontinuity analysis.

The method of discontinuity analysis is well known and described in detail in [4], [5] and [6]. According to this method a wave is considered as a carrier of discontinuities in the field variable and/or its derivatives. The order of a wave is defined as the order of the lowest derivative of the field variable that is discontinuous across the wavefront. The discontinuities satisfy certain conditions across the wavefront from which it is possible to obtain a set of recursive relations known as transport-induction equations [5], [7]. These equations which govern the propagation of discontinuities, can be solved together with the specified time-dependent boundary condition to determine the discontinuities of all order at the wavefront. The transient solution is then represented in terms of a Taylor series expansion behind the wavefront, where the coefficients involved are the very discontinuities discussed above. Such expansions are suggested in the monographs of Achenbach [4] and Friedlander [8]. This method of solution will be known as the direct method.

Certain transient problems that we consider involve boundary loads that act for a finite time. For such cases the direct method if possible, will be tedious and it is preferable to adapt the Green's function concept. We define the unit pulse solution as the transient solution to the problem with a boundary condition involving the Heaviside unit function. This solution is easy to determine by the direct method and upon differentiation with respect to time gives the Green's function for the problem concerned [9]. The Green's function, together with the Duhamel integral, will yield the required transient solution [8].

The corresponding time harmonic problem can be solved by the Karal-Keller technique [10], where we formally assume asymptotic time harmonic series solutions to the equations of motion. The equations governing the variation of the coefficients in these series turn out to be exactly the transport-induction equations for the unit pulse problem. There is thus a one to one correspondence between the unit pulse problem and the corresponding time harmonic problem. For a given set of equations of motion it is much simpler to generate the transport-induction equations by the Karal-Keller technique than by the method of discontinuity analysis. This fact is first established and then exploited in this thesis.

In Chapter II the aim is twofold; viz to obtain the solutions to all the possible wave types in a plate and to establish the relationship between the unit pulse solution and the corresponding time harmonic solution. Earlier in this chapter the method of discontinuity analysis is described and applied to obtain the transport-induction equations necessary for a transient solution. To this end the field equations are cast into integral form in space-time allowing us to extract the form of the field equations when derivatives of the displacement are discontinuous [5]. The analysis yields a classification of the possible wave types in a plate together with their speeds and propagation conditions. For each wave type the transport-induction equations governing the propagation of an arbitrary order displacement discontinuity are obtained. These results are an extension to those presented by Cohen [11] who dealt with the geometric acoustics case, which is the value of the disturbance at the wavefront.

Later on in the same chapter, the Karal-Keller technique is used to obtain general steady state time-harmonic solutions to the plate equations. The coefficients in these series expansions are found to satisfy a set of recurrence relations from which we obtain the very same classification of the wave types. It is here that we establish the definite relationship that exists between the unit pulse solution and the corresponding time harmonic solution. The results obtained turn out to be in complete agreement with those of Kline and Kay [5] who considered the analogous problem for the electromagnetic field equations by a somewhat different approach.

In general the waves of the various types become coupled together in a fashion governed by the induction equations. We consider certain special types of wave motion in which there is no coupling between wave types and refer to these as pure wave motions. Some of these motions require constraining body forces or couples in order to be maintained. Finally in this chapter we consider the wave propagation problems corresponding to (i) a shear stress applied to a circular cavity in an unbounded plate, and (ii) a bending moment applied to a straight edge in an unbounded plate whose faces are constrained between two rigid plates. The results obtained are compared with existing closed form solutions [12], [13].

In Chapter III the propagation of axi-symmetric transients in shells of revolution with straight line generators is considered. The Karal-Keller technique is used, firstly to obtain the classification of the possible wave types together with their speeds and propagation conditions. The results so obtained are in agreement with those of Cohen [14] who proceeded along somewhat different lines. The transport-induction equations for the various wave types are then obtained. The prescribed boundary conditions together with the appropriate transport-induction equations can be used to obtain the solution to the given problem.

The series solutions obtained by our method are found to converge slowly, especially at large values of T, the time elapsed after the wavefront. The problem is similar as in the evaluation of the exponential of negative T using its Taylor series expansion, when T is large. Mainardi and Turchetti [15] used Padē approximants to accelerate the convergence of these series solutions. We present a simple numerical superposition technique as an alternative means of overcoming the same difficulty. The results obtained by using this technique agree with those of Mainardi and Turchetti who used Padē approximants.

Later on in the chapter we solve several numerical examples and discuss the results. The first example deals with the longitudinal impact of a conical shell. Herein we not only illustrate our technique of solution, but also verify them by comparing the results obtained with those obtained by using Laplace transforms [16]. The next two examples treat the propagation of axi-symmetric transients in a cylindrical shell due to velocity and acceleration boundary conditions. We also demonstrate how the response due to certain ground motions resulting from earthquake and blast loading may be obtained by incorporating the superposition technique. The effect of shell location and the effect of the thickness of a cylindrical shell on the response are next studied. Finally, in this chapter we discuss the approximate rod theories available for treating longitudinal transients in a cylinder.

The problem of general transient waves in cylindrical shells is treated in Chapter IV. The various displacement components are expressed in the form of Fourier series in θ (the circumferential coordinate) and the displacement equations of motion are written for each harmonic. The Karal-Keller technique is used as before to obtain the classification, speeds and propagation conditions of the possible wave types. Once again the results are in agreement with those obtained by Cohen [14]. Two of the possible wave types are coupled and as a result we obtain a coupled transport equation for these two waves and coupled induction equations for the other wave types. The prescribed boundary conditions, together with the transport-induction equations, can be used to obtain the solution to the given transient problem.

The approximate rod and beam theories available for treating transients in a cylinder are next discussed. This is followed by three numerical examples. In the first example the lateral impact of a cylinder is treated and the results obtained are compared with those obtained by using Laplace transforms [17]. The other two examples deal with the flexural and torsional problems pertaining to a cylindrical tank whose base is subjected to horizontal ground excitation. In the flexural problem we compare the solutions obtained by using the shell and beam theories. The effects of higher order waves induced due to homogeneous boundary conditions are illustrated and discussed in the first two examples.

CHAPTER II

TRANSIENT AND TIME HARMONIC WAVES IN PLATES

2.1 Equations of Motion

We consider the propagation of waves in linear, isotropic and homogeneous elastic plates. The plate equations that we utilize are those of linearised Cosserat plate theory as developed by Naghdi [1]. These equations developed from a *direct* two-dimensional approach are based on a director model and are equivalent to those developed from three-dimensional considerations, and include the effects of transverse shear deformation, transverse normal stress and strain and rotatory inertia. The displacement equations of motion separate into two sets governing the *extensional* and *bending* motions, respectively [11]. These are

$$\mu\nabla^2\underset{\sim}{u} + (\lambda+\mu)\nabla(\nabla.\underset{\sim}{u}) + \lambda\nabla\delta^3 + \frac{\rho}{h}\underset{\sim}{F} = \frac{\rho}{h}\ddot{\underset{\sim}{u}} \quad , \tag{2.1}$$

$$\alpha_8\nabla^2\delta^3 - (\lambda+2\mu)h\delta^3 - \lambda(\nabla.\underset{\sim}{u})h + \rho L^3 = \rho\alpha\ddot{\delta}^3 \quad , \tag{2.2}$$

for the extensional theory, and

$$\nabla^2\underset{\sim}{\delta} + \frac{(3\lambda+2\mu)}{(\lambda+2\mu)}\nabla(\nabla.\underset{\sim}{\delta}) - \frac{\alpha_3}{\mu h\alpha}(\underset{\sim}{\delta}+\nabla u^3) + \frac{\rho}{\mu h\alpha}\underset{\sim}{L} = \frac{\rho}{\mu h}\ddot{\underset{\sim}{\delta}} \quad , \tag{2.3}$$

$$\nabla.\underset{\sim}{\delta} + \nabla^2 u^3 + \frac{\rho}{\alpha_3}F^3 = \frac{\rho}{\alpha_3}\ddot{u}^3 \quad , \tag{2.4}$$

for the bending theory.

In the above equations the displacement of the Cosserat plane is given by $\underset{\sim}{u}^* = (\underset{\sim}{u}, u^3)$ and the displacement of the director by $\underset{\sim}{\delta}^* = (\underset{\sim}{\delta}, \delta^3)$. The vectors $\underset{\sim}{u}, \underset{\sim}{\delta}$ represent the displacements parallel to the plane of the plate, while u^3, δ^3 represent the displacement normal to the plate. From the three-dimensional point of view, the assumed displacement $\underset{\sim}{U}^*$ across the plate space is given by [11],[1]

$$\underset{\sim}{U}^* = \underset{\sim}{u}^* + z\underset{\sim}{\delta}^* , \tag{2.5}$$

where z is the co-ordinate along the normal to the plate mid-surface. ∇ is the two-dimensional gradient operator in the plane of the plate. Also λ, μ are Lame's constants, $\underset{\sim}{F}, F^3$ are body forces, $\underset{\sim}{L}, L^3$ are body couples, $\alpha = \frac{h^2}{12}$, while the constitutive coefficients α_3 and α_8 are taken as constants and could take on values depending on the problem at hand [11],[1]. The mass per unit area is ρ while h is the plate thickness.

The plate equations (2.1) - (2.4) being hyperbolic, ensure finite wave front velocities for the propagation of disturbances [2]. In this respect they are similar to the equations of motion in three-dimensional elasticity and are suitable for studying the dynamic response in plates. However due to the presence of terms of lower order differentiation, these plate equations are *dispersive* [3]. Thus a pulse will suffer distortion and the phase velocity of a harmonic wave-train will depend on the frequency ω.

Introducing the notations

$$\underset{\sim}{w}_1 = (\underset{\sim}{u},\delta^3),\ \underset{\sim}{w}_2 = (\underset{\sim}{\delta},u^3),\ \underset{\sim}{f}_1 = -(\underset{\sim}{F},L^3),\ \underset{\sim}{f}_2 = -(\underset{\sim}{L},F^3) \quad , \qquad (2.6)$$

equations (2.1) - (2.4) can be conveniently written as

$$\underset{\sim}{L}_\alpha \underset{\sim}{w}_\alpha = \rho \underset{\sim}{f}_\alpha \ , \ \alpha = 1,2 \quad , \qquad (2.7)$$

where $\underset{\sim}{L}_\alpha$ is a suitably defined linear second order differential operator.

2.2 Transient Waves and Discontinuities

Consider a source of disturbance acting over some curve in a homogeneous isotropic elastic plate as shown in Figure 1a.[1] If the source begins to act at time t = 0, then for t>0 this disturbance will spread into the plate with a constant wave front velocity G. The wave front will constitute a family of parallel curves $\psi(x,y) = Gt$ in the x,y plane of the plate while sweeping out a hypercone $\phi(x,y,t) = 0$ in space-time. The value of the field at a point $P_o(x_o,y_o,t_o)$ on the wave front is called the *geometrical acoustic* field by analogy to the geometrical optics situation arising in [5].

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[1] The results to follow are readily generalized to nonhomogeneous plates. The general features of the analysis are analogous to those presented here. The complication appears as an algebraic one, due to the fact that the speed of propagation is no longer constant.

The value of this field at any point $P(x_o, y_o, t)$, $t>t_o$, behind the wave front will constitute the so-called *transient* or *pulse* solution to the disturbance problem.

We assume a transient solution to equation (2.7) in the form of a Taylor's series expansion [4],[8] at the wave front into the region behind it. Thus we write

$$\underset{\sim}{w}_\alpha = \sum_{n=o}^{\infty} \underset{\sim}{[}\frac{\partial^n \underset{\sim}{w}_\alpha}{\partial t^n}\underset{\sim}{]}_{t=t_o} \frac{<t-t_o>^n}{n!} , \tag{2.8}$$

where $<,> = 0$ if the argument is negative while $\underset{\sim}{[}\ \underset{\sim}{]}$ indicates the *discontinuity* or *jump* of the argument across the wave front. These discontinuities occur at the wave front since the region ahead of the wave is undisturbed. The wave is thus naturally a carrier of discontinuities. The lowest order derivative of $\underset{\sim}{w}_\alpha$ having a discontinuity defines the order of the wave. A first order wave is called a shock or strain wave and waves of this type will constitute the subject matter dealt with herein. Higher order waves yield results which are completely analogous to those for first order waves. For first order waves, a knowledge of the first order discontinuities on the wave front will constitute the geometric acoustics solution, while a knowledge of the higher order discontinuities will allow calculation of the transient solution from equation (2.8).

Associated with the geometry of the wave front at any point are its unit tangent $\underset{\sim}{\lambda}$ and unit normal $\underset{\sim}{\nu}$. We use ℓ and s to denote arc lengths along the wave curve, and perpendicular to it, respectively. Thus s measures distance along the rays defined as the orthogonal trajectories to the wave curves, which for homogeneous isotropic plates are straight lines. We find it convenient to define the vector differential operator ∇_D by

$$\nabla_D = \nabla + \frac{\underset{\sim}{\nu}}{G}\frac{\partial}{\partial t} \quad . \tag{2.9}$$

This operator allows calculation of all one-sided directional derivatives along the wave surface $\phi(x,y,t) = 0$. In particular the operator $\frac{D}{Dt}$ is defined by

$$\frac{D}{Dt} = G\underset{\sim}{\nu}.\nabla_D = \frac{\partial}{\partial t} + G\underset{\sim}{\nu}.\nabla \quad . \tag{2.10}$$

This is the so-called *displacement* derivative [7] and calculates rate of change as seen by an observer moving along the rays with the wave speed G, i.e. moving with the wave front. Applying equation (2.10) to the wave surface equation $\phi(x,y,t) = \psi(x,y) - Gt = 0$, we compute that

$$\underset{\sim}{\nu} = \nabla\phi = \nabla\psi \ , \ \kappa = -\nabla^2\psi \quad , \tag{2.11}$$

where κ is the curvature of the wave front.

In terms of the operator ∇_D the so-called Hadamard's Lemma [7] takes the form

$$[\nabla_D \underset{\sim}{w}_\alpha] = \nabla_D [\underset{\sim}{w}_\alpha] \quad . \tag{2.12}$$

In what is to follow it is convenient to represent vector quantities in terms of components tangent and normal to the wave front. Thus we write

$$\underset{\sim}{w}_{\alpha} = w_{\alpha}^{\lambda}\underset{\sim}{\lambda} + w_{\alpha}^{\nu}\underset{\sim}{\nu} \quad , \tag{2.13}$$

and moreover define the directional derivatives

$$\frac{d}{d\ell} = \underset{\sim}{\lambda}.\nabla \quad , \quad \frac{d}{ds} = \underset{\sim}{\nu}.\nabla \quad . \tag{2.14}$$

In particular we obtain from equations (2.12), (2.9) and $(2.14)_2$ the compatibility relations

$$[\underset{\sim}{\underset{\sim}{w}}_{\alpha,n+1}] = -\underset{\sim}{G}[\frac{d\underset{\sim}{w}_{\alpha,n}}{ds}] + \frac{D}{Dt}[\underset{\sim}{\underset{\sim}{w}}_{\alpha,n}] \quad , \quad n \geq 0 \quad , \tag{2.15}$$

where the comma followed by the subscript n indicates an n^{th} order time derivative.

In order to determine the possible types of discontinuities and their behaviour at the wave front we must utilize the field equations (2.7) to obtain the appropriate governing discontinuity equations. Since we are dealing with first order waves we shall require discontinuity equations of a lower order than can be obtained by taking the jumps of an nth order $(n\geq 0)$ time derivative of equation (2.7). These lower order equations are obtained by following the procedure utilised in [5] to deal with Maxwell's equations. We introduce a *testing* function Ω, which possesses derivatives of all order. The testing function and its derivatives vanish on and outside the boundary ∂R of a domain R in space-time. Multiplying the extensional field equation (2.1) and (2.2) by Ω, integrating over R and utilising integration by parts we obtain

$$\int_R \{\mu(\nabla\Omega.\nabla)\underset{\sim}{u}+(\lambda+\mu)\nabla\Omega(\nabla.\underset{\sim}{u})+\lambda\nabla\Omega\delta^3-\frac{\rho}{h}\dot{\Omega}\dot{\underset{\sim}{u}}-\Omega\frac{\rho}{h}\underset{\sim}{F}\} = 0 \quad , \qquad (2.16)$$

$$\int_R \{\alpha_8(\nabla\Omega.\nabla)\delta^3+(\lambda+2\mu)h\Omega\delta^3-\lambda(\nabla\Omega.\underset{\sim}{u})h-\rho\alpha\dot{\Omega}\dot{\delta}^3-\Omega\rho L^3\} = 0 \ . \qquad (2.17)$$

Equations (2.16) and (2.17) are integral forms of the field equations (2.1) and (2.2) respectively, and these are mathematically equivalent to one another in regions where the derivatives involved are continuous. We also define $\dot{u} = u_{,1}$ etc.

We now assume the surface of discontinuity $\phi(x,y,t) = 0$ to pass through the region R, dividing it into regions R_1 and R_2 as in Figure 1b. Reversing the procedure used to obtain equations (2.16) and (2.17) with appropriate integration by parts over the domains R_1 and R_2, inside of which the necessary derivatives are continuous, we find[2]

$$\underset{\sim}{\mu}[(\nabla\phi.\nabla)\underset{\sim\sim}{u}] + (\lambda+\mu)\nabla\underset{\sim}{\phi}[\nabla.\underset{\sim\sim}{u}] + \lambda\nabla\underset{\sim}{\phi}[\delta^3] + \frac{\rho}{h}G[\dot{\underset{\sim\sim\sim}{u}}] = 0 \qquad (2.18)$$

---

[2] In this analysis we assume that the body forces $\underset{\sim}{f}_\alpha$ are C^∞ and note that the continuity of $\underset{\sim}{f}_\alpha$ has been used in obtaining (2.18) and (2.19)

$$\alpha_8[(\nabla\phi.\nabla)\delta^3] - \lambda h[\nabla\phi.u] + \rho\alpha G[\dot{\delta}^3] = 0 \; . \qquad (2.19)$$

Equations (2.18) and (2.19) are the required discontinuity equations and are a consequence of the field equations (2.1) and (2.2) at a surface of discontinuity. In an analogous fashion, the plate bending equations (2.3) and (2.4) lead to the discontinuity equations.

$$[(\nabla\phi.\nabla)\delta] + \frac{(3\lambda+2\mu)}{(\lambda+2\mu)}\nabla\phi[\nabla.\delta] - \frac{\alpha_3}{\mu h\alpha}\nabla\phi[u^3] + \frac{\rho}{\mu h}G[\dot{\delta}] = 0 \; , \qquad (2.20)$$

$$[\nabla\phi.\delta] + [(\nabla\phi.\nabla)u^3] + \frac{\rho}{\alpha_3}G[\dot{u}^3] = 0 \; . \qquad (2.21)$$

In order to obtain discontinuity equations of higher order, i.e. governing jumps in higher order derivatives, we need only take the jumps of any n^{th} order $(n \geq 0)$ time derivative of equation (2.7). This leads to

$$\mu[(\nu.\nabla)u,_n] + (\lambda+\mu)\nu[\nabla.u,_n] + \lambda\nu[\delta^3,_n] + G\frac{\rho}{h}[u,_{n+1}]$$
$$= G\{\mu\nabla_D.[\nabla u,_{n-1}] + (\lambda+\mu)\nabla_D[\nabla.u,_{n-1}] + \lambda\nabla_D[\delta^3,_{n-1}]\} \; , \qquad (2.22)$$

$$\alpha_8[(\nu.\nabla)\delta^3,_n] - \lambda h[\nu.u,_n] + G\rho\alpha[\delta^3,_{n+1}]$$
$$= G\{\alpha_8\nabla_D.[\nabla\delta^3,_{n-1}] - (\lambda+2\mu)h[\delta^3,_{n-1}] - \lambda h\nabla_D.[u,_{n-1}]\} \; , \qquad (2.23)$$

for the extensional theory, and

$$[(\nu.\nabla)\delta,_n] + \frac{(3\lambda+2\mu)}{(\lambda+2\mu)}\nu[\nabla.\delta,_n] + \frac{\alpha_3}{\mu h\alpha}\nu[u^3,_n] + \frac{G\rho}{\mu h\alpha}[\delta,_{n+1}]$$
$$= G\{\nabla_D.[\nabla\delta,_{n-1}] + \frac{(3\lambda+2\mu)}{(\lambda+2\mu)}\nabla_D[\nabla.\delta,_{n-1}] - \frac{\alpha_3}{\mu h\alpha}[\delta,_{n-1}] - \frac{\alpha_3}{\mu h\alpha}\nabla_D[u^3,_{n-1}]\} \; , \qquad (2.24)$$

$$[\underset{\sim}{\nu}\cdot\underset{\sim}{\delta},_{n}]+[(\underset{\sim}{\nu}\cdot\nabla)u^{3},_{n}]+\frac{\rho G}{\alpha_{3}}[u^{3},_{n+1}] = G\{\nabla_{D}\cdot[\underset{\sim}{\delta},_{n-1}]+\nabla_{D}\cdot[\nabla u^{3},_{n-1}]\} , \tag{2.25}$$

for the bending theory. We note that in writing equations (2.22) - (2.25) we have modified their forms by use of equation (2.9).

I *Extensional Waves*

We now turn our attention to the consequences of the above discontinuity equations for the case of extensional waves. Equations (2.18) and (2.19) determine the *propagation conditions* for extensional waves. These conditions determine the possible types of waves which can propagate, as well as their associated speeds of propagation. If we take the scalar product of equation (2.18) with $\underset{\sim}{\lambda}$ and $\underset{\sim}{\nu}$, utilise equation (2.13) as well as the appropriate first order compatibility relation obtained from equation (2.15) by putting n = 0 and $[\underset{\sim}{w}_{1}] = 0$, we find from equations (2.18) and (2.19) that

$$(G_{T}^{2}-G^{2})[u^{\lambda},_{1}] = 0,\ (G_{L}^{2}-G^{2})[u^{\nu},_{1}] = 0,\ (G_{S}^{2}-G^{2})[\delta^{3},_{1}] = 0, \tag{2.26}$$

where

$$G_{T}^{2} = \frac{\mu h}{\rho},\ G_{L}^{2} = \frac{(\lambda+2\mu)h}{\rho},\ G_{S}^{2} = \frac{\alpha_{8}}{\rho\alpha}, \tag{2.27}$$

Equations (2.26) will define the three types of waves. For each of these, if the values of the possible jumps are given on an initial curve, their variations as they move with the wave front will be governed by equations (2.22) and (2.23).

If we take the scalar product of equation (2.22) with $\underset{\sim}{\lambda}$ and $\underset{\sim}{\nu}$, use equations (2.12)-(2.15), we obtain as a consequence of equations (2.22) and (2.23), a system of three first order differential equations which involve jumps of order n,(n-1) and (n-2) in the field quantities. These equations determine the *transport-induction* equations for each type of wave by substitution of the appropriate solution to equations (2.26) into them. We now proceed to give a classification of the wave types along with their transport-induction equations.

(i) *Longitudinal Wave*

$$[u^{\nu}_{,1}] \neq 0 \ , \ [u^{\lambda}_{,1}] = [\delta^{3}_{,1}] = 0 \ , \ G^2 = G_L^2 \ , \tag{2.28}$$

$$2\frac{d}{ds}[u^{\nu}_{,n}] - \kappa[u^{\nu}_{,n}] = -\frac{1}{2(1-\nu)}\left\{\frac{d}{d\ell}[u^{\lambda}_{,n}] - G_L \frac{d}{ds}(\nabla\cdot[\underset{\sim}{u}_{,n-1}])\right\} - \frac{\nu}{(1-\nu)}\left([\delta^{3}_{,n}] - G_L\frac{d}{ds}[\delta^{3}_{,n-1}]\right) + \frac{G_T^2}{G_L}\underset{\sim}{\nu}\cdot\nabla^2[\underset{\sim}{u}_{,n-1}] \ , \tag{2.29}$$

$$[u^{\lambda}_{,n}] = -(1-2\nu)G_L\left(2\frac{d}{ds}[u^{\lambda}_{,n-1}] - \kappa[u^{\lambda}_{,n-1}] - G_L\underset{\sim}{\lambda}\cdot\nabla^2[\underset{\sim}{u}_{,n-2}]\right) + G_L\frac{d}{d\ell}\left(G_L \ \nabla\cdot[\underset{\sim}{u}_{,n-2}] + 2\nu G_L[\delta^{3}_{,n-2}] - [u^{\nu}_{,n-1}]\right) \ , \tag{2.30}$$

$$(G_L^2 - G_S^2)[\delta^{3}_{,n}] = \frac{2\nu}{(1-2\nu)}\cdot\frac{G_T^2 G_L}{\alpha}\left([u^{\nu}_{,n-1}] - G_L\nabla\cdot[\underset{\sim}{u}_{,n-2}]\right) - G_L G_S^2\left(2\frac{d}{ds}[\delta^{3}_{,n-1}] - \kappa[\delta^{3}_{,n-1}]\right) + G_L^2 G_S^2\nabla^2[\delta^{3}_{,n-1}] - \frac{G_L^4}{\alpha}[\delta^{3}_{,n-2}] \ . \tag{2.31}$$

(ii) *Shear Wave*

$$[u^{\lambda}_{,1}] \neq 0 \ , \ [u^{\nu}_{,1}] = [\delta^{3}_{,1}] = 0 \ , \ G^2 = G_T^2 \ , \tag{2.32}$$

$$2\frac{d}{ds}[\underset{\sim}{u},^{\lambda}_{n}] - \kappa[\underset{\sim}{u},^{\lambda}_{n}] = -\frac{1}{(1-2\nu)}\frac{d}{d\ell}\left([\underset{\sim}{u},^{\nu}_{n}] - G_T(\nabla\cdot[\underset{\sim\sim}{u},_{n-1}])\right.$$
$$\left. - 2\nu G_T[\delta^3,_{n-1}]\right) - G_T\underset{\sim}{\lambda}\cdot\nabla^2[\underset{\sim\sim}{u},_{n-1}] \quad , \tag{2.33}$$

$$[\underset{\sim}{u},^{\nu}_{n}] = G_T\frac{d}{d\ell}[\underset{\sim}{u},^{\lambda}_{n-1}] + 2(1-\nu)G_T\left(2\frac{d}{ds}[\underset{\sim}{u},^{\nu}_{n-1}] - \kappa[\underset{\sim}{u},^{\nu}_{n-1}]\right)$$
$$-(1-2\nu)G_T^2\underset{\sim}{\nu}\cdot\nabla^2[\underset{\sim\sim}{u},_{n-2}] + 2\nu G_T\left([\delta^3,_{n-1}] - G_T\frac{d}{ds}[\delta^3,_{n-2}]\right) - G_T^2\frac{d}{ds}(\nabla\cdot[\underset{\sim\sim}{u},_{n-2}]) \quad , \tag{2.34}$$

$$(G_T^2-G_S^2)[\delta^3,_{n}] = \frac{2\nu}{(1-2\nu)}\frac{G_T^3}{\alpha}\left([\underset{\sim}{u},^{\nu}_{n-1}] - G_T\nabla\cdot[\underset{\sim\sim}{u},_{n-2}]\right) - G_TG_S^2\left(2\frac{d}{ds}[\delta^3,_{n-1}]\right.$$
$$\left. - \kappa[\delta^3,_{n-1}]\right) + G_T^2G_S^2\nabla^2[\delta^3,_{n-2}] - \frac{G_T^2G_L^2}{\alpha}[\delta^3,_{n-2}] \quad . \tag{2.35}$$

(iii) *Squeeze-gradient wave*

$$[\delta^3,_1] \neq 0 \ , \quad [\underset{\sim\sim}{u},_1] = 0 \ , \quad G^2 = G_S^2 \quad , \tag{2.36}$$

$$2\frac{d}{ds}[\delta^3,_{n}] - \kappa[\delta^3,_{n}] = \frac{2\nu}{(1-2\nu)}\frac{G_T^2}{\alpha G_S^2}\left([\underset{\sim}{u},^{\nu}_{n}] - G_S\nabla\cdot[\underset{\sim\sim}{u},_{n-1}]\right)$$
$$+ G_S\nabla^2[\delta^3,_{n-1}] - \frac{G_L^2}{\alpha G_S^2}[\delta^3,_{n-1}] \quad , \tag{2.37}$$

$$(G_L^2-G_S^2)[\underset{\sim}{u},^{\nu}_{n}] = G_L^2G_S\left(2\frac{d}{ds}[\underset{\sim}{u},^{\nu}_{n-1}] - \kappa[\underset{\sim}{u},^{\nu}_{n-1}]\right) - G_T^2G_S^2\underset{\sim}{\nu}\cdot\nabla^2[\underset{\sim\sim}{u},_{n-2}]$$
$$+ \frac{G_S}{2(1-\nu)}\left\{G_L^2\frac{d}{d\ell}[\underset{\sim}{u},^{\lambda}_{n-1}] - G_L^2G_S\frac{d}{ds}(\nabla\cdot[\underset{\sim\sim}{u},_{n-2}])\right\} \tag{2.38}$$
$$+ \frac{2\nu}{(1-2\nu)}G_T^2G_S\left([\delta^3,_{n-1}] - G_S\frac{d}{ds}[\delta^3,_{n-2}]\right) \quad ,$$