THE UNIVERSITY OF MANITOBA

WAVE PROPAGATION IN RODS WITH MULTIPLE WAVE SPEEDS

by
ROBERT CLAYTON BACHMAN

A Thesis
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN CIVIL ENGINEERING

WINNIPEG, MANITOBA
May, 1978

WAVE PROPAGATION IN RODS WITH MULTIPLE WAVE SPEEDS

BY

ROBERT CLAYTON BACHMAN

A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

MASTER OF SCIENCE

ABSTRACT

A method of solving for the wave mode vector for nth order waves in linear hyperelastic rods is determined for the case of multiple wave speeds. The solution is carried out for both unconstrained and constrained rods. Special cases of the theory, for instance uniform rods, are considered, and lead to previous researcher's results. An example problem having a diagonal constitutive and inertia matrix is considered. Solutions to problems with the following geometries are considered: 1) straight untwisted rods, 2) twisted but uncurved rods, 3) plane curved but untwisted rods. The solutions obtained are consistent with physical intuition.

ACKNOWLEDGEMENTS

The guidance and assistance of Dr. H. Cohen, in pursuing this research is very gratefully acknowledged. The financial assistance of the National Research Council of Canada, without whom this thesis would not have been presented, is also recognized.

Finally, I would like to thank Mrs. V. Ring for typing such a difficult manuscript.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This thesis concerns itself with the subject of linear wave propagation in hyperelastic rods. The governing constitutive and balance relationships are developed by means of a direct approach. A direct approach considers the rod as a one-dimensional oriented continuum defined by a spatial curve and a number of deformable vectors called directors. In our formulation a two director model is used. These directors characterize the size and shape of the rod cross-section. If these directors are allowed to deform the theory will be called unconstrained. However, if the directors are allowed to undergo only rigid rotations, the theory will be called constrained. Clearly constrained theories have fewer kinematical degrees of freedom.

The notion of a wave as a propagating discontinuity in one of the field variables is considered. In general, a shock wave is defined as a wave involving discontinuities in the first derivative of a generalized displacement vector. An nth order wave is defined in terms of a discontinuity in the nth derivative of a displacement vector. The theory to be developed establishes the so-called speed of propagation of the wave and the wave mode vector, quantities to be defined in the body of the thesis.

We start in chapter two by developing the nonlinear balance and constitutive equations for an unconstrained rod.

Our approach is very similar to that of Whitman and De Silva [1]. The method consists of setting up a general Hamilton's principle which the rod must satisfy. We then postulate two invariance principles pertaining to all rods. These are the principle of material frame indifference and the principle of Euclidean invariance as defined by Truesdell and Noll [2]. The resulting modified equations are reformulated in terms of a higher order vector space using an idea formulated by Ericksen [3]. The resulting equations are in a concise algebraic form easy to manipulate. The equations are then linearized in a very specific manner leading to what Cohen [4] calls an exact linear theory.

Chapter three concerns itself with developing the constrained theory of rods. The results of chapter two are modified to give a nonlinear theory of constrained rods. Linearization about a point of finite deformation is done in a similar manner to Bansal [5]. The resulting equations are further simplified by considering the finitely deformed state to be the undeformed state.

In chapter four the nth order compatibility equations are derived for quantities at the wave front. A generalization of Bansal's work [5], who only considered a finite number of discontinuities and Cohen [6], who only considered constant wave speeds as applied to shells, is undertaken.

Chapter five leads to the development of the nth order propagation and decay-induction equations for both unconstrained and constrained rods. This is really only an extension of Cohen's results in [7] and [8] for shock (n=1) and acceleration (n=2) waves. A general orthogonality principle is discovered which all solutions to the propagation and decay-induction equation must satisfy.

In chapter six the form of the solution for the constrained and unconstrained rods is found. This solution can be decomposed into a combination of two mappings. The first maps vectors from the initial eigenspace at $s=s_o$ to the eigenspace at arbitrary s. The second mapping can be viewed as a transformation in the eigenspace at arbitrary s. By the polar decomposition theorem we find these mappings can be viewed as a general rotation and deformation of the initial wave mode vectors. The analysis performed allows the solution of distinct or multiple wave speed problems. For multiple wave speeds the governing set of equations are a coupled set of ordinary linear differential equations. A basis is established which allows these equations to be uncoupled.

Chapter seven specializes the general theory developed in the previous chapter to various simplified rods. The nature of the solution for uniform rods as developed by Cohen [4] is just such a specialization. Also the cases of the wave speeds and inertia tensor $\underset{\sim}{K}$ being independent of s are considered. The case of $\underset{\sim}{K}$ being in a special diagonal form is also examined.

Chapter eight concerns itself with solving an actual example. For simplicity a constrained rod satisfying some of the properties of chapter seven is considered. The problem in a slightly modified form was examined by Cohen [7] who considered only straight untwisted rods with distinct eigenvalues. In our examination we consider straight untwisted rods, twisted but uncurved rods, and plane curved but untwisted rods, all with various multiple wave speeds. We find the results consistent with physical intuition.

CHAPTER 2

FORMULATION OF UNCONSTRAINED THEORY

The objective of this chapter is to develop the equations for a general nonlinear isothermal hyperelastic rod. The model will then be linearized in a specific manner to give an exact linear theory of rods.

We start by modelling a rod as a one-dimensional directed continuum in a three-dimensional euclidean point space. The rod is characterized by an associated translation space V. This space contains a curve C representing the rod axis and a two-dimensional subspace $\bar{V}$ representing the cross-section. Two directors $\underset{\sim}{d}_\alpha$ $(\alpha = 1,2)$[1], contained in $\bar{V}$ are established which characterize the geometry of the cross-section. The rod axis and the directors are in general functions of the material coordinate s and time t. They are represented respectively by

$$\underset{\sim}{r} = \underset{\sim}{r}(s,t) \quad , \quad \underset{\sim}{d}_\alpha = \underset{\sim}{d}_\alpha(s,t) \tag{2.1}$$

The initialized values of these quantities are defined as

$$\underset{\sim}{R}(s) = \underset{\sim}{r}(s,0) \quad , \quad \underset{\sim}{D}_\alpha(s) = \underset{\sim}{d}_\alpha(s,0) \quad . \tag{2.2}$$

We require that $\underset{\sim}{d}_\alpha$ constitute a basis in $\bar{V}$ and that $\underset{\sim}{d}_1, \underset{\sim}{d}_2, \hat{\underset{\sim}{r}}$ define a basis in V, where ^ denotes $(\frac{\partial}{\partial s})_t$. We will also use $\cdot$ to represent $(\frac{\partial}{\partial t})_s$. S is assumed to represent arc

---

[1] Unless otherwise specified Greek and Latin indices consist of the ranges, 1,2 and 1,2,3 respectively.

length at t = 0. A real valued action density or Lagrangian is now assumed to exist, being a function of the following quantities

$$L = L(\hat{\underset{\sim}{r}}, \underset{\sim}{d}_\alpha, \hat{\underset{\sim}{d}}_\alpha, \dot{\underset{\sim}{r}}, \dot{\underset{\sim}{d}}_\alpha, S_o; s) \;,\; S_o = (\hat{\underset{\sim}{R}}, \underset{\sim}{D}_\alpha, \hat{\underset{\sim}{D}}_\alpha, \dot{\underset{\sim}{R}}, \dot{\underset{\sim}{D}}_\alpha) \;. \tag{2.3}$$

The specification of this quantity will govern the dynamical behavior of the rod. The action a, is defined by

$$a = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \rho L \; ds \; dt \quad , \tag{2.4}$$

where the intervals on space and time are arbitrary. A virtual impulse I is defined by

$$I = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \rho(\underset{\sim}{f} \cdot \delta \underset{\sim}{r} + \underset{\sim}{f}^\alpha \cdot \delta \underset{\sim}{d}_\alpha) ds dt + \int_{t_1}^{t_2} (\underset{\sim}{\tilde{\tilde{n}}} \cdot \delta \underset{\sim}{r} + \underset{\sim}{\tilde{\tilde{n}}}^\alpha \cdot \delta \underset{\sim}{d}_\alpha) dt \Big|_{s_1}^{s_2}$$
$$- \int_{s_1}^{s_2} (\underset{\sim}{\tilde{\tilde{p}}} \cdot \delta \underset{\sim}{r} + \underset{\sim}{\tilde{\tilde{p}}}^\alpha \cdot \delta \underset{\sim}{d}_\alpha) ds \Big|_{t_1}^{t_2} \;[2] \quad , \tag{2.5}$$

$\underset{\sim}{f}$ represents a body force, $\underset{\sim}{\tilde{\tilde{n}}}$ the stress vector, $\underset{\sim}{\tilde{\tilde{p}}}$ the generalized momentum. The quantities $\underset{\sim}{f}^\alpha$, $\underset{\sim}{\tilde{\tilde{n}}}^\alpha$, $\underset{\sim}{\tilde{\tilde{p}}}^\alpha$ represent the corresponding director quantities. ρ is the mass per unit length at t = 0, $\delta \underset{\sim}{r}$ and $\delta \underset{\sim}{d}_\alpha$ are general virtual displacements. The ordinary dot product is symbolized by expressions of the form $\underset{\sim}{f} \cdot \delta \underset{\sim}{r}$.

---

[2] The summation convention occurs over the full range of indices for terms like $\underset{\sim}{\tilde{\tilde{p}}}^\alpha \cdot \delta \underset{\sim}{d}_\alpha$.

Since general virtual displacements of the directors are allowed, it is apparent that the model being developed allows for director and hence cross-sectional deformation. Explicit expressions for the equations of motion and the constitutive relationships are now found by stating Hamilton's principle as

$$I = \delta a \quad . \tag{2.6}$$

The variation described by (2.6) is restricted by the principle of mass conservation which is expressible as

$$\delta(\rho ds) = 0 \quad . \tag{2.7}$$

The variation of the action can now be determined by using (2.3), (2.4), (2.7), integration by parts, and the relations

$$\delta(\dot{\underset{\sim}{d}}_\alpha) = \dot{\overline{\delta \underset{\sim}{d}_\alpha}} \quad , \quad \delta(\hat{\underset{\sim}{d}}_\alpha) = \widehat{\overline{\delta \underset{\sim}{d}_\alpha}} \quad .$$

$$\delta a = \int_{t_1}^{t_2} (\underset{\sim}{n} \cdot \delta \underset{\sim}{r} + \underset{\sim}{n}^\alpha \cdot \delta \underset{\sim}{d}_\alpha) dt \Big|_{s_1}^{s_2} - \int_{t_1}^{t_1} \int_{s_1}^{s_2} \{ (\hat{\underset{\sim}{n}} - \dot{\underset{\sim}{p}}) \cdot \delta \underset{\sim}{r} + (\hat{\underset{\sim}{n}} - \underset{\sim}{l}^\alpha - \dot{\underset{\sim}{p}}^\alpha)$$

$$\cdot \delta \underset{\sim}{d}_\alpha \} ds dt - \int_{s_1}^{s_2} (\underset{\sim}{p} \cdot \delta \underset{\sim}{r} + \underset{\sim}{p}^\alpha \cdot \delta \underset{\sim}{d}_\alpha) ds \Big|_{t_1}^{t_2} \quad , \tag{2.8}$$

where we have defined

$$\underset{\sim}{n} = \rho \frac{\partial L}{\partial \hat{\underset{\sim}{r}}}, \; \underset{\sim}{n}^\alpha = \rho \frac{\partial L}{\partial \hat{\underset{\sim}{d}}_\alpha}, \; l^\alpha = \rho \frac{\partial L}{\partial \underset{\sim}{d}_\alpha}, \; \underset{\sim}{p} = -\rho \frac{\partial L}{\partial \dot{\underset{\sim}{r}}} \;\; \underset{\sim}{p}^\alpha = -\rho \frac{\partial L}{\partial \dot{\underset{\sim}{d}}_\alpha} \quad . \tag{2.9}$$

In developing (2.8) we have assumed that all quantities on the right hand side are continuous over s and t. With this continuity requirement we may now express the equations developed from our Hamilton's principle (2.6) in integral or differential form. Using (2.5) and (2.8) we develop the integral form of these equations

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{s_1}^{s_2} \{(\hat{\mathbf{n}}+\rho\mathbf{f}-\dot{\mathbf{p}})\cdot\delta\mathbf{r}+(\hat{\mathbf{n}}^{\alpha}-\mathbf{l}^{\alpha}+\mathbf{f}^{\alpha}-\dot{\mathbf{p}}^{\alpha})\cdot\delta\mathbf{d}_{\alpha}\}\,ds\,dt \\
&+\int_{t_1}^{t_2} \{(\tilde{\tilde{\mathbf{n}}}-\mathbf{n})\cdot\delta\mathbf{r}-(\mathbf{n}^{\alpha}-\tilde{\tilde{\mathbf{n}}}^{\alpha})\cdot\delta\mathbf{d}_{\alpha}\}\,dt\,\Big|_{s_1}^{s_2} \qquad (2.10)\\
&-\int_{s_1}^{s_2} \{(\tilde{\tilde{\mathbf{p}}}-\mathbf{p})\cdot\delta\mathbf{r}+(\tilde{\tilde{\mathbf{p}}}^{\alpha}-\mathbf{p})\cdot\delta\mathbf{d}_{\alpha}\}\,ds\,\Big|_{t_1}^{t_2} \quad .
\end{aligned}
$$

Since (2.10) must hold for arbitrary limits of integration we conclude that the quantities being integrated must be identically zero. This lets us exhibit the differential form of Hamilton's principle by

$$
\begin{aligned}
&\hat{\mathbf{n}} + \rho\mathbf{f} = \dot{\mathbf{p}} \;,\; \hat{\mathbf{n}}^{\alpha}-\mathbf{l}^{\alpha} + \mathbf{f}^{\alpha} = \dot{\mathbf{p}}^{\alpha} \quad , \\
&\mathbf{n} = \tilde{\tilde{\mathbf{n}}} \;,\; \mathbf{n}^{\alpha} = \tilde{\tilde{\mathbf{n}}}^{\alpha} \;,\; S = S_1 \;,\; S_2 \quad , \qquad (2.11)\\
&\mathbf{p} = \tilde{\tilde{\mathbf{p}}}^{\alpha} ,\; \mathbf{p}^{\alpha} = \tilde{\tilde{\mathbf{p}}}^{\alpha} \;,\; t = t_1 \;,\; t_2 \quad .
\end{aligned}
$$

The Lagrangian is now assumed to be made up of two distinct parts, a strain energy component W and a kinetic energy component T in a specific form

$$
\begin{aligned}
&L = W(\hat{\mathbf{r}},\mathbf{d}_{\alpha},\hat{\mathbf{d}}_{\alpha},M_o;S)-T(\dot{\mathbf{r}},\dot{\mathbf{d}}_{\alpha}) \quad , \qquad (2.12)\\
&T(\dot{\mathbf{r}},\dot{\mathbf{d}}_{\alpha}) = \tfrac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \tfrac{1}{2}y^{\alpha\beta}\dot{\mathbf{d}}_{\alpha}\cdot\dot{\mathbf{d}}_{\beta} \;,\; \dot{y}^{\alpha\beta} = 0 \;,\; M_o = (\hat{\mathbf{R}},\mathbf{D}_{\alpha},\hat{\mathbf{D}}_{\alpha}) \;.
\end{aligned}
$$

The definition of T ensures that $y^{\alpha\beta} = y^{\beta\alpha}$. Making use of (2.12) the constitutive equations (2.9) may be written as

$$\underset{\sim}{n} = \rho\frac{\partial W}{\partial \underset{\sim}{\hat{r}}},\ \underset{\sim}{n}^{\alpha} = \rho\frac{\partial W}{\partial \underset{\sim}{\hat{d}}_{\alpha}},\ \underset{\sim}{l}^{\alpha} = \rho\frac{\partial W}{\partial \underset{\sim}{d}_{\alpha}},\ \underset{\sim}{p} = \rho\dot{\underset{\sim}{r}},\ \underset{\sim}{p}^{\alpha} = \rho y^{\alpha\beta}\dot{\underset{\sim}{d}}_{\beta}\ . \quad (2.13)$$

The equations of motion $(2.11)_{1,2}$ may be rewritten as

$$\hat{\underset{\sim}{n}} + \rho\underset{\sim}{f} = \rho\ddot{\underset{\sim}{r}}\ ,\ \hat{\underset{\sim}{n}}^{\alpha} - \underset{\sim}{l}^{\alpha} + \underset{\sim}{f}^{\alpha} = \rho y^{\alpha\beta}\ddot{\underset{\sim}{d}}_{\beta}\ . \quad (2.14)$$

These equations represent the balance of linear and director momentum respectively. Continuity of L in its arguments has been assumed in developing (2.13) and (2.14). This restriction will now be somewhat relaxed.

A single moving point of discontinuity of stress and momentum like terms will be allowed to traverse the rod. The quantities $\underset{\sim}{f}$, $\underset{\sim}{f}^{\alpha}$ and $\underset{\sim}{l}^{\alpha}$ in (2.11) are seen not to be transmitted across a cut in the rod. Their behavior is therefore not that of stress or momentum like terms. They may be considered as generalized body forces. Discontinuities in $\underset{\sim}{f}$, $\underset{\sim}{f}^{\alpha}$ and $\underset{\sim}{l}^{\alpha}$ will therefore not be allowed. The terms $\underset{\sim}{n}$, $\underset{\sim}{p}$, $\underset{\sim}{n}^{\alpha}$, and $\underset{\sim}{p}^{\alpha}$ are alone allowed to have discontinuities.

At the moving point of discontinuity the material coordinate is a function of time

$$S = P(t)\ . \quad (2.15)$$

The location of the point of discontinuity and the speed of propagation V may be defined as

$$\underset{\sim}{r} = \underset{\sim}{r}(P(t),t) \quad , \quad V = \frac{dP}{dt} \quad . \tag{2.16}$$

With these preliminaries Cohen [9] gives the equations of motion across a discontinuity as

$$[\underset{\sim}{n}] = \rho V[\dot{\underset{\sim}{r}}] \quad , \quad [\underset{\sim}{n}^{\alpha}] = -\rho y^{\alpha\beta} V[\dot{\underset{\sim}{d}}_{\beta}] \quad , \tag{2.17}$$

where [n] is defined as follows

$$[\underset{\sim}{n}] = \lim_{h \to o}\{\underset{\sim}{n}(P(t)+h,t) - \underset{\sim}{n}(P(t)-h,t)\}, h > 0 \quad . \tag{2.18}$$

The Lagrangian and strain energy functions that have been considered are not arbitrary. The Lagrangian is made to satisfy the principle of Euclidean invariance, and the strain energy the principle of material frame indifference.

The principle of Euclidean invariance will be discussed first. This is a requirement that the variation of the Lagrangian must be zero for arbitrary rigid rotations of the entire rod.

The position of the rod and directors is initially given in terms of a reference state, represented by $\check{\underset{\sim}{r}}$ and $\check{\underset{\sim}{d}}_{\alpha}$ respectively. This state is fixed and is not allowed any possible variation. The vectors $\underset{\sim}{r}$ and $\underset{\sim}{d}_{\alpha}$ give the rotated positions of the rod as

$$\underset{\sim}{r} = \underset{\sim}{O}\check{\underset{\sim}{r}} \quad , \quad \underset{\sim}{d}_{\alpha} = \underset{\sim}{O}\check{\underset{\sim}{d}}_{\alpha} \quad . \tag{2.19}$$

$\underset{\sim}{O}$ is a proper constant orthogonal tensor. A proper orthogonal tensor $\underset{\sim}{O}$ is defined as

$$\underset{\sim}{O}\,\underset{\sim}{O}^T = \underset{\sim}{O}^T\,\underset{\sim}{O} = \underset{\sim}{I}\ ,\quad \underset{\sim}{u}\cdot\underset{\sim}{O}\underset{\sim}{v} = \underset{\sim}{O}^T\underset{\sim}{u}\cdot\underset{\sim}{v}\ ,\quad \underset{\sim}{u},\underset{\sim}{v}\in\mathcal{V}\ . \tag{2.20}$$

$\underset{\sim}{O}^T$ is the transpose of $\underset{\sim}{O}$. The variation of $\underset{\sim}{r}$ and $\underset{\sim}{d}_\alpha$ can be expressed as

$$\delta\underset{\sim}{r} = \delta\underset{\sim}{L}^A\underset{\sim}{r}\ ,\quad \delta\underset{\sim}{d}_\alpha = \delta\underset{\sim}{L}^A_\alpha\underset{\sim}{d}_\alpha\ ,\quad \delta\underset{\sim}{L}^A = \delta\underset{\sim}{O}\,\underset{\sim}{O}^T\ .^3 \tag{2.21}$$

$\delta\underset{\sim}{L}^T$ is a constant antisymmetric tensor.

Mathematically the principle of Euclidean invariance can be stated

$$\rho\delta L = \rho\frac{\partial L}{\partial\hat{\underset{\sim}{r}}}\cdot\delta\hat{\underset{\sim}{r}} + \rho\frac{\partial L}{\partial\hat{\underset{\sim}{d}}_\alpha}\cdot\delta\hat{\underset{\sim}{d}}_\alpha + \rho\frac{\partial L}{\partial\underset{\sim}{d}_\alpha}\cdot\delta\underset{\sim}{d}_\alpha + \rho\frac{\partial L}{\partial\dot{\underset{\sim}{r}}}\cdot\delta\dot{\underset{\sim}{r}} + \rho\frac{\partial L}{\partial\dot{\underset{\sim}{d}}_\alpha}\cdot\delta\dot{\underset{\sim}{d}}_\alpha = 0, \tag{2.22}$$

$$\text{for } \delta\underset{\sim}{r} = \delta\underset{\sim}{L}^A\underset{\sim}{r}\ ,\quad \delta\underset{\sim}{d}_\alpha = \delta\underset{\sim}{L}^A\underset{\sim}{d}_\alpha\ . \tag{2.23}$$

We pause at this point to present some useful results from linear algebra. If $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{I}$ are arbitrary vectors and tensors in $\mathcal{V}$ and $\mathcal{V}\otimes\mathcal{V}$ respectively

$$\begin{aligned} &\underset{\sim}{u}\cdot\underset{\sim}{T}\underset{\sim}{v} = \mathrm{tr}(\underset{\sim}{S}\underset{\sim}{T})\ ,\quad \underset{\sim}{S} = \underset{\sim}{v}\otimes\underset{\sim}{u}\ ,\\ &\mathrm{tr}(\underset{\sim}{S}\underset{\sim}{T}^A) = \mathrm{tr}(\underset{\sim}{S}^A\underset{\sim}{T}^A)\ ,\quad \mathrm{tr}(\underset{\sim}{S}^S\underset{\sim}{T}^A) = 0\ , \end{aligned} \tag{2.24}$$

where tr reads trace and $\otimes$ reads tensor product. Using the constitutive equations (2.9), $(2.13)_{4,5}$ and the relations (2.23) and (2.24) the statement of Euclidean invariance (2.22) becomes

---

[3] A general tensor is composed of symmetric and antisymmetric parts $\underset{\sim}{A} = \underset{\sim}{A}^S + \underset{\sim}{A}^A$ defined by $\underset{\sim}{A}^S = \frac{\underset{\sim}{A}+\underset{\sim}{A}^T}{2}$, $(\underset{\sim}{A}^S)^T = \underset{\sim}{A}^S$, $\underset{\sim}{A}^A = \frac{\underset{\sim}{A}-\underset{\sim}{A}^T}{2}$, $(\underset{\sim}{A}^A)^T = -\underset{\sim}{A}^A$.

$$\mathrm{tr}((\hat{\underset{\sim}{r}}\otimes\underset{\sim}{n} + \hat{\underset{\sim}{d}}_\alpha\otimes\underset{\sim}{n}^\alpha + \underset{\sim}{d}_\alpha\otimes\underset{\sim}{l}^\alpha)^A\delta\underset{\sim}{L}^A) = \underset{\sim}{0} \ . \tag{2.25}$$

It is noted that the last two terms in (2.22) corresponding to the inertia terms are symmetric. Since (2.25) holds for all $\delta\underset{\sim}{L}^A$

$$(\hat{\underset{\sim}{r}}\otimes\underset{\sim}{n})^A + (\hat{\underset{\sim}{d}}_\alpha\otimes\underset{\sim}{n}^\alpha)^A + (\underset{\sim}{d}_\alpha\otimes\underset{\sim}{l}^\alpha)^A = \underset{\sim}{0} \ . \tag{2.26}$$

This is a system of tensor equations which is equivalent to three scalar equations. Through $(2.13)_{1,2,3}$ equation (2.26) represents a restriction upon the form of the strain energy function.

The principle of material frame indifference requires that the strain energy be invariant under rigid body motions. A rigid body motion is defined by

$$\overset{*}{\underset{\sim}{r}} = \underset{\sim}{r}_o(t) + \underset{\sim}{Q}(t)\underset{\sim}{r} \ , \ \overset{*}{\underset{\sim}{d}}_\alpha = \underset{\sim}{Q}(t)\underset{\sim}{d}_\alpha \ , \tag{2.27}$$

where $\underset{\sim}{Q}(t)$ is an arbitrary proper orthogonal tensor. This principle is expressed by

$$W(\hat{\overset{*}{\underset{\sim}{r}}},\overset{*}{\underset{\sim}{d}}_\alpha,\hat{\overset{*}{\underset{\sim}{d}}}_\alpha,M_o;s) = W(\hat{\underset{\sim}{r}},\underset{\sim}{d}_\alpha,\hat{\underset{\sim}{d}}_\alpha,M_o;s) \ . \tag{2.28}$$

Substitution of (2.27) into (2.28) results in

$$W(\underset{\sim}{Q}\hat{\underset{\sim}{r}},\underset{\sim}{Q}\underset{\sim}{d}_\alpha,\underset{\sim}{Q}\hat{\underset{\sim}{d}}_\alpha,M_o;s) = W(\hat{\underset{\sim}{r}},\underset{\sim}{d}_\alpha,\hat{\underset{\sim}{d}}_\alpha,M_o;s) \tag{2.29}$$

A scalar valued function satisfying (2.29) is said to be invariant under the full orthogonal group.

A representation theorem attributed to Cauchy by Truesdell and Noll states that a scalar valued vector function is invariant if, and only if, the scalar is a function of the dot products of the arguments. Since $\mathcal{W}$ must satisfy the additional constraint (2.26) imposed by the principle of Euclidean invariance, all of the above mentioned dot products are not independent of one another. Equation (2.26) represents a set of three linear homogeneous partial differential equations in fifteen independent variables, the components of $\hat{\underset{\sim}{r}}$, $\underset{\sim}{d}_\alpha$ and $\hat{\underset{\sim}{d}}_\alpha$. The theory of such equations states that there exist twelve independent solutions in terms of which $\mathcal{W}$ may be expressed. For convenience we let

$$\underset{\sim}{d}_3 = \hat{\underset{\sim}{r}} \; . \tag{2.30}$$

The independent quantities we choose are $\underset{\sim}{d}_i \cdot \underset{\sim}{d}_j$, $\hat{\underset{\sim}{d}}_\alpha \cdot \underset{\sim}{d}_i$. These terms satisfy the necessary criterion for the principle of material frame indifference to apply.

The strain energy function satisfying all invariance requirements is now expressed as

$$\mathcal{W} = \mathcal{W}(U;\ H_{ij}, \Omega_{\alpha i}, s) \; , \tag{2.31}$$

where

$$U: \gamma_{ij}, \kappa_{\alpha i} \; , \tag{2.32}$$

are the relative strain measures given by

$$\gamma_{ij} = h_{ij} - H_{ij}, \quad \kappa_{\alpha i} = \mathcal{W}_{\alpha i} - \Omega_{\alpha i} \; ,$$

$$h_{ij} = \underset{\sim}{d}_i \cdot \underset{\sim}{d}_j, \quad H_{ij} = \underset{\sim}{D}_i \cdot \underset{\sim}{D}_j, \quad \mathcal{W}_{\alpha i} = \hat{\underset{\sim}{d}}_\alpha \cdot \underset{\sim}{d}_i \; , \quad \Omega_{\alpha i} = \hat{\underset{\sim}{D}}_\alpha \cdot \underset{\sim}{D}_i \; . \tag{2.33}$$

For simplicity the rod equations will now be expressed in a highly compressed format. To do so it is necessary to define a higher order vector space. The vector space $\mathcal{U}$ is a direct sum, $\mathcal{U} = \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{V}$ where elements $\underset{\sim}{u} \in \mathcal{U}$ are constructed by the rule

$$\underset{\sim}{u} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{u}_3) \ , \ \underset{\sim}{u}_i \in \mathcal{V} \ , \ i = 1,2,3 \ . \tag{2.34}$$

The standard Euclidean inner product on $\mathcal{V}$, henceforward denoted by $\langle \ , \ \rangle$, implies a Euclidean inner product on $\mathcal{U}$, defined by $\{\underset{\sim}{u}, \underset{\sim}{v}\} = \langle \underset{\sim}{u}_i, \underset{\sim}{v}_i \rangle$. The component subspaces of $\mathcal{U}$ are thus seen to be orthogonal complements of one another.

The position and geometry of the rod are defined by (2.1). This description can be modified such that the rod may equivalently be considered as a moving curve in $\mathcal{U}$. Doing so requires the definition of a generalized displacement or coordinate vector $\underset{\sim}{p} = (\underset{\sim}{r}, \underset{\sim}{d}_\alpha)$. The system (2.1) may now be replaced by

$$\underset{\sim}{p} = \underset{\sim}{p}(s,t) \ . \tag{2.35}$$

In a similar manner the following quantities, all vectors in $\mathcal{U}$ are defined

$$\underset{\sim}{N} = (\underset{\sim}{n}, \underset{\sim}{n}^\alpha) \ , \ \underset{\sim}{M} = (\underset{\sim}{l}, \underset{\sim}{l}^\alpha) \ , \ \underset{\sim}{F} = (\underset{\sim}{f}, \underset{\sim}{f}^\alpha) \ . \tag{2.36}$$

The quantity $\underset{\sim}{l}$ is given by

$$\underset{\sim}{l} = \rho \frac{\partial W}{\partial \underset{\sim}{r}} = \underset{\sim}{0} \ , \tag{2.37}$$

since W by $(2.12)_1$ is independent of $\underset{\sim}{r}$.

In terms of this new notation the equations of motion (2.14), (2.17) and the constitutive relationships (2.13) may be written as

$$\hat{\underset{\sim}{N}} - \underset{\sim}{M} + \rho\underset{\sim}{F} = \underset{\sim}{K}\ddot{\underset{\sim}{p}} \ , \ [\underset{\sim}{N}] = -V\underset{\sim}{K}[\dot{\underset{\sim}{p}}] \ , \ \underset{\sim}{N} = \rho\frac{\partial W}{\partial\hat{\underset{\sim}{p}}} \ , \ \underset{\sim}{M} = \rho\frac{\partial W}{\partial\underset{\sim}{p}} \ , \tag{2.38}$$

where

$$\underset{\sim}{K} = \rho\begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{K}_{\alpha\beta} \end{bmatrix} \ , \quad \underset{\sim}{K}_{\alpha\beta} = y^{\alpha\beta}\underset{\sim}{I} \ , \ \underset{\sim}{K} = \underset{\sim}{K}^{T} \ , \tag{2.39}$$

The mappings $\underset{\sim}{I}$, $\underset{\sim}{O}$ are the identity and zero operators on V respectively.

The principle of material frame indifference expressed by (2.28) can be redefined in terms of the nine-dimensional space U. We let $\underset{\sim}{Q}$ be an arbitrary proper orthogonal transformation on V and $\underset{\sim}{O}$ be an orthogonal transformation on $\underset{\sim}{U}$ defined by

$$\underset{\sim}{O} = \underset{\sim}{Q} \oplus \underset{\sim}{Q} \oplus \underset{\sim}{Q} \ . \tag{2.40}$$

The tensors $\underset{\sim}{O}$ denote a subgroup O_g of the full orthogonal group $O_{g}^{\underset{\sim}{I}}$ on U. The system (2.27) may be reformulated in terms of our higher order space by

$$\overset{*}{\underset{\sim}{p}} = (\underset{\sim}{r}_o,\underset{\sim}{O},\underset{\sim}{O}) + \underset{\sim}{O}\underset{\sim}{p} \ , \ \hat{\overset{*}{\underset{\sim}{p}}} = \underset{\sim}{O}\hat{\underset{\sim}{p}} \ . \tag{2.41}$$

With these definitions (2.28) becomes

$$W(\overset{*}{\underset{\sim}{p}},\hat{\overset{*}{\underset{\sim}{p}}},M_o;s) = W(\underset{\sim}{p},\hat{\underset{\sim}{p}},M_o;s) \ . \tag{2.42}$$

It may also be verified that (2.39) satisfies

$$\underset{\sim}{K} = \underset{\sim}{O}\,\underset{\sim}{K}\,\underset{\sim}{O}^{T} \quad . \tag{2.43}$$

$\underset{\sim}{K}$ is therefore isotropic with respect to $\underset{\sim}{O}$.

In order to satisfy the principle of material frame indifference in U, W must be a function of the inner products between $\underset{\sim}{p}$ and $\hat{\underset{\sim}{p}}$ both at the initial state and the deformed state.

The equations of motion and the constitutive equations developed are highly nonlinear. There are a number of ways in which an approximate set of linear equations might be developed about a suitably defined equilibrium point. This approach will not be taken here. Instead we will assume that the strain energy W is a quadratic in $\underset{\sim}{p}$, $\hat{\underset{\sim}{p}}$ only. This is a tremendous specialization of the nonlinear theory. Not only does this state that the rod motion is independent of the initial or reference state but that third order derivatives of W with respect to $\underset{\sim}{p}$, $\hat{\underset{\sim}{p}}$ vanish. The Taylor series expansion for $\underset{\sim}{N}$ and $\underset{\sim}{M}$ about p, $\hat{p} = 0$ are therefore finite series

$$\begin{aligned}
\underset{\sim}{M} &= \rho\frac{\partial W}{\partial \underset{\sim}{p}} = \rho\frac{\partial^2 W}{\partial \underset{\sim}{p}\,\partial \underset{\sim}{p}}\bigg|_{\substack{\underset{\sim}{p}=\underset{\sim}{0}\\ \hat{\underset{\sim}{p}}=\underset{\sim}{0}}} \underset{\sim}{p} + \rho\frac{\partial^2 W}{\partial \hat{\underset{\sim}{p}}\,\partial \underset{\sim}{p}}\bigg|_{\substack{\underset{\sim}{p}=\underset{\sim}{0}\\ \hat{\underset{\sim}{p}}=\underset{\sim}{0}}} \hat{\underset{\sim}{p}} \quad , \\
\underset{\sim}{N} &= \rho\frac{\partial W}{\partial \hat{\underset{\sim}{p}}} = \rho\frac{\partial^2 W}{\partial \underset{\sim}{p}\,\partial \hat{\underset{\sim}{p}}}\bigg|_{\substack{\underset{\sim}{p}=\underset{\sim}{0}\\ \hat{\underset{\sim}{p}}=\underset{\sim}{0}}} \underset{\sim}{p} + \rho\frac{\partial^2 W}{\partial \hat{\underset{\sim}{p}}\,\partial \hat{\underset{\sim}{p}}}\bigg|_{\substack{\underset{\sim}{p}=\underset{\sim}{0}\\ \hat{\underset{\sim}{p}}=\underset{\sim}{0}}} \hat{\underset{\sim}{p}} \quad .
\end{aligned} \tag{2.44}$$

Since third order derivatives of W with respect to $\underset{\sim}{p}$, $\hat{\underset{\sim}{p}}$ vanish it follows that second order derivatives are functions only of s. Therefore, it is immaterial that the quantities $\frac{\partial^2 W}{\partial \underset{\sim}{p} \partial \underset{\sim}{p}}$ etc. are evaluated at $\underset{\sim}{p}$, $\hat{\underset{\sim}{p}} = \underset{\sim}{0}$. We have

$$\begin{aligned} \underset{\sim}{M} &= \underset{\sim}{T}\underset{\sim}{p} + \underset{\sim}{H}^{T}\hat{\underset{\sim}{p}} \ , \quad \underset{\sim}{N} = \underset{\sim}{H}\underset{\sim}{p} + \underset{\sim}{A}\hat{\underset{\sim}{p}} \ , \\ \underset{\sim}{T} &= \rho\frac{\partial^2 W}{\partial \underset{\sim}{p} \partial \underset{\sim}{p}} \ , \quad \underset{\sim}{H} = \rho\frac{\partial^2 W}{\partial \underset{\sim}{p} \partial \hat{\underset{\sim}{p}}} \ , \quad \underset{\sim}{A} = \rho\frac{\partial^2 W}{\partial \hat{\underset{\sim}{p}} \partial \hat{\underset{\sim}{p}}} \ . \end{aligned} \tag{2.45}$$

This linear theory of rods is called an exact linear theory of rods as once the specified form of the strain energy function is chosen no additional assumptions are necessary. No higher order terms are neglected as recourse to small deformation theory has not been appealed to.

CHAPTER 3

FORMULATION OF CONSTRAINED THEORY

The theory developed in the previous chapter concerned itself with general rod and director (cross-sectional) deformation. This chapter contains a specialization of those results, no director (cross-sectional) deformation. Known as the constrained theory only rigid rotations of the directors are allowed. A general nonlinear theory is developed and by a process known as small deformations superimposed on finite deformations an approximate linear theory is established.

The constrained theory is developed by requiring the directors to be rigid. Without loss of generality it will be assumed that the directors constitute an orthonormal triad. To form a triad a third director is defined by

$$\underset{\sim}{d}_3 = \underset{\sim}{d}_1 \times \underset{\sim}{d}_2 \neq \hat{\underset{\sim}{r}} \quad . \tag{3.1}$$

To behave as an orthonormal triad

$$\underset{\sim}{d}_i \cdot \underset{\sim}{d}_j = \delta_{ij} \quad . \tag{3.2}$$

It is to be noted that (2.30) is not valid in this chapter.

A fixed orthonormal basis $\underset{\sim}{e}_i$ is assumed to exist in $\mathcal{V}$. The motion of the directors can be expressed as a function of $\underset{\sim}{e}_i$ as

$$\underset{\sim}{d}_i = \underset{\sim}{Q}\underset{\sim}{e}_i \quad . \tag{3.3}$$

where $\underset{\sim}{Q} = \underset{\sim}{Q}(s,t)$ is an arbitrary proper orthogonal tensor. The material derivative of $\underset{\sim}{d}_i$ is found to be

$$\hat{\underset{\sim}{d}}_i = \underset{\sim}{Y}^A \underset{\sim}{d}_i \ , \ \underset{\sim}{Y}^A = \hat{\underset{\sim}{Q}} \underset{\sim}{Q}^T \ . \tag{3.4}$$

For every antisymmetric tensor $\underset{\sim}{Y}^A$ of rank 3 there corresponds an axial vector defined by $\underset{\sim}{Y}_A^A$ such that

$$\hat{\underset{\sim}{d}}_i = \underset{\sim}{Y}^A \underset{\sim}{d}_i = \underset{\sim}{\varepsilon} \times \underset{\sim}{d}_i \ , \ \underset{\sim}{\varepsilon} = \underset{\sim}{Y}_A^A \ . \tag{3.5}$$

Similarily by taking time derivatives of (3.3)

$$\dot{\underset{\sim}{d}}_i = \underset{\sim}{W}^A \underset{\sim}{d}_i = \underset{\sim}{\Omega} \times \underset{\sim}{d}_i \ , \ \underset{\sim}{W}^A = \dot{\underset{\sim}{Q}} \underset{\sim}{Q}^T \ , \ \underset{\sim}{\Omega} = \underset{\sim}{W}_A^A \ . \tag{3.6}$$

Physical meanings are easily attributed to the symmetric and antisymmetric parts of a tensor. It is apparent from (3.6) that $\underset{\sim}{W}^A$ corresponds to a rate of rotation tensor. The term $\underset{\sim}{W}^S$, which for the constrained theory is identically zero, can be shown to be the rate of deformation tensor.

To develop the constrained equations of motion we must see what effect (3.6) has upon our formulation of the variational principle. For L continuous in all of its arguments this means a modification of (2.10). Since that equation holds for arbitrary variations $\delta \underset{\sim}{d}_\alpha$ it must hold for

$$\delta \underset{\sim}{d}_\alpha = \dot{\underset{\sim}{d}}_\alpha \ . \tag{3.7}$$

For a constrained rod $\dot{\underset{\sim}{d}}_\alpha$ is given by $(3.6)_1$. Therefore the variational principle must be satisfied for

$$\delta \underset{\sim}{d}_\alpha = \underset{\sim}{W}^A \underset{\sim}{d}_\alpha \ . \tag{3.8}$$

Following the argument used to develop (2.11) from (2.10) all terms under the integral sign in (2.10) upon substitution of (3.8) must be made identically zero. The equations affected by (3.8) alone will be developed. Using (3.8), $(2.24)_{1,3}$ and $(2.13)_5$ in (2.10) we have

$$
\begin{aligned}
&\mathrm{tr}((\underset{\sim}{d}_\alpha \otimes \hat{\underset{\sim}{n}}^\alpha - \underset{\sim}{d}_\alpha \otimes \underset{\sim}{l}^\alpha + \underset{\sim}{d}_\alpha \otimes \underset{\sim}{f}^\alpha - \underset{\sim}{d}_\alpha \otimes \rho y^{\alpha\beta} \ddot{\underset{\sim}{d}}_\beta)^A \underset{\sim}{W}^A) = 0 \,, \\
&\mathrm{tr}((\underset{\sim}{d}_\alpha \otimes \overset{\approx}{\underset{\sim}{n}}{}^\alpha - \underset{\sim}{d}_\alpha \otimes \underset{\sim}{n}^\alpha)^A \underset{\sim}{W}^A) = 0 \,, \quad s = s_1, s_2 \,, \qquad (3.9) \\
&\mathrm{tr}((\underset{\sim}{d}_\alpha \otimes \overset{\approx}{\underset{\sim}{p}}{}^\alpha - \underset{\sim}{d}_\alpha \otimes \underset{\sim}{p}^\alpha)^A \underset{\sim}{W}^A) = 0 \,, \quad t = t_1, t_2 \,.
\end{aligned}
$$

These relationships must hold for arbitrary $\underset{\sim}{W}^A$. Using this information and integration by parts

$$
\begin{aligned}
&(\overset{\wedge}{\overline{\underset{\sim}{d}_\alpha \otimes \underset{\sim}{n}^\alpha}})^A - (\hat{\underset{\sim}{d}}_\alpha \otimes \underset{\sim}{n}^\alpha)^A - (\underset{\sim}{d}_\alpha \otimes \underset{\sim}{l}^\alpha)^A + \rho(\underset{\sim}{d}_\alpha \otimes \underset{\sim}{f}^\alpha)^A = \rho y^{\alpha\beta} (\dot{\overline{\underset{\sim}{d}_\alpha \otimes \dot{\underset{\sim}{d}}_\beta}})^A \,, \\
&(\underset{\sim}{d}_\alpha \otimes \overset{\approx}{\underset{\sim}{n}}{}^\alpha)^A = (\underset{\sim}{d}_\alpha \otimes \underset{\sim}{n}^\alpha)^A \,, \quad s = s_1, s_2 \,, \qquad (3.10) \\
&(\underset{\sim}{d}_\alpha \otimes \overset{\approx}{\underset{\sim}{p}}{}^\alpha)^A = (\underset{\sim}{d}_\alpha \otimes \underset{\sim}{p}^\alpha)^A \,, \quad t = t_1, t_2 \,,
\end{aligned}
$$

where it is found $\rho y^{\alpha\beta}(\dot{\underset{\sim}{d}}_\alpha \otimes \dot{\underset{\sim}{d}}_\beta)$ is symmetric. The relationships unaffected by (3.8) and hence still applicable to the constrained rod are $(2.14)_1$ and $(2.11)_{3,4}$.

Upon substitution of the invariance relation (2.26) into $(3.10)_1$ a modified equation of motion corresponding to the classical balance of angular momentum results

$$
(\overset{\wedge}{\overline{\underset{\sim}{d}_\alpha \otimes \underset{\sim}{n}^\alpha}})^A + (\hat{\underset{\sim}{r}} \otimes \underset{\sim}{n})^A + \rho(\underset{\sim}{d}_\alpha \otimes \underset{\sim}{f}^\alpha)^A = \rho y^{\alpha\beta} (\dot{\overline{\underset{\sim}{d}_\alpha \otimes \dot{\underset{\sim}{d}}_\beta}})^A \,. \qquad (3.11)
$$

This equation is put in more recognizable form if the following relation is used

$$-2(\underset{\sim}{a}\otimes\underset{\sim}{b})_A^A = \underset{\sim}{a}\times\underset{\sim}{b} \ , \ \underset{\sim}{a},\underset{\sim}{b}\in V \ ; \tag{3.12}$$

Taking -2 times the axial vector of (3.11) and utilizing (3.12) causes

$$\begin{gathered} \hat{\underset{\sim}{m}} + \hat{\underset{\sim}{r}} \times \underset{\sim}{n} + \rho \underset{\sim}{q} = \rho \dot{\underset{\sim}{\sigma}} \ , \\ \underset{\sim}{m} = \underset{\sim}{d}_\alpha \times \underset{\sim}{n}^\alpha \ , \ \underset{\sim}{q} = \underset{\sim}{d}_\alpha \times \underset{\sim}{f}^\alpha \ , \ \underset{\sim}{\sigma} = y^{\alpha\beta} \underset{\sim}{d}_\alpha \times \dot{\underset{\sim}{d}}_\beta \ . \end{gathered} \tag{3.13}$$

For our constrained theory (3.13) replaces $(2.14)_2$ the more general expression for rods with deformable cross-sections. The number of vector differential equations to be solved is two, one less than required for the more general rod. The balance laws across a discontinuity for the constrained theory are easily computed by taking the cross product of $\underset{\sim}{d}_\alpha$ with $(2.17)_2$ and recognizing that $(2.17)_1$ is unchanged

$$[\underset{\sim}{n}] = -\rho V[\dot{\underset{\sim}{r}}] \ , \ [\underset{\sim}{m}] = -\rho V[\underset{\sim}{\sigma}] \ . \tag{3.14}$$

$(3.14)_2$ depends on the fact $[\underset{\sim}{d}_\alpha] = \underset{\sim}{0}$.

The strain energy function (2.31) developed in chapter two is not directly applicable. Relationship (2.31) is a function of 12 independent quantities. The constrained theory is subject to six additional constraints. These involve the directors and the material derivatives of the directors. The constraints are

$$\begin{gathered} \underset{\sim}{d}_1 \cdot \underset{\sim}{d}_1 = 1 \ , \ \underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2 = 0 \ , \ \underset{\sim}{d}_2 \cdot \underset{\sim}{d}_2 = 1 \ , \\ \hat{\underset{\sim}{d}}_1 \cdot \underset{\sim}{d}_2 = -\underset{\sim}{d}_1 \cdot \hat{\underset{\sim}{d}}_2 \ , \ \hat{\underset{\sim}{d}}_1 \cdot \underset{\sim}{d}_1 = 0 \ , \ \hat{\underset{\sim}{d}}_2 \cdot \underset{\sim}{d}_2 = 0 \ . \end{gathered} \tag{3.15}$$

Actually more geometric constraints such as $\underset{\sim}{d}_2 \cdot \underset{\sim}{d}_3 = 0$ exist but they are irrelevant for the following reason. To satisfy the principle of material frame indifference W is a function of all possible inner products between $\hat{\underset{\sim}{r}}$, $\underset{\sim}{d}_\alpha$, $\hat{\underset{\sim}{d}}_\alpha$, which are all assumed to give new information. It is clear that the six equations of (3.15) do not. Hence W is restricted by (3.15), and the number of independent quantities of which the constrained strain energy is a function of is six. This modified strain energy called ψ is

$$
\begin{aligned}
\psi &= W_c(U; H_i, Y_i, s) \ , \quad u: \gamma_i, t_i \ , \\
\gamma_i &= h_i - H_i \ , \quad t_i = y_i - Y_i \ , \\
h_i &= \tfrac{1}{2} e_{ijk} \underset{\sim}{d}_k \cdot \hat{\underset{\sim}{d}}_j \ , \quad H_i = \tfrac{1}{2} e_{ijk} \underset{\sim}{D}_k \cdot \hat{\underset{\sim}{D}}_j \ , \\
y_i &= \underset{\sim}{d}_i \cdot \hat{\underset{\sim}{r}} \ , \quad Y_i = \underset{\sim}{D}_i \cdot \hat{\underset{\sim}{R}} \ ,
\end{aligned}
\tag{3.16}
$$

where e_{ijk} is the standard permutation symbol.

The constitutive equation for $\underset{\sim}{m}$ is developed from $(2.13)_2$ and $(3.13)_2$ and is

$$\underset{\sim}{m} = \underset{\sim}{d}_\alpha \times \rho \frac{\partial \psi}{\partial \hat{\underset{\sim}{d}}_\alpha} \ . \tag{3.17}$$

This expression can be replaced by the equivalent

$$\underset{\sim}{m} = \rho \frac{\partial \psi}{\partial \underset{\sim}{\varepsilon}} \ . \tag{3.18}$$

In a slightly different form we write the constitutive equations for both $\underset{\sim}{m}$ and $\underset{\sim}{n}$

$$\mathbf{m} = \rho \frac{\partial \psi}{\partial h_i} \frac{\partial h_i}{\partial \boldsymbol{\varepsilon}} = \rho \frac{\partial \psi}{\partial h_i} \mathbf{d}_i \ , \ \mathbf{n} = \rho \frac{\partial \psi}{\partial y_i} \frac{\partial y_i}{\partial \hat{\mathbf{r}}} = \rho \frac{\partial \psi}{\partial y_i} \mathbf{d}_i \ . \qquad (3.19)$$

The fully nonlinear equations of motion and constitutive equations have been developed. We will now linearize these equations by considering small perturbations superposed on a finitely deformed configuration. The finitely deformed configuration will be assumed to be in a state of statical equilibrium. The inertia terms are zero for this condition. Quantities in the perturbed state will be signified by a bar over the desired scalar, vector, or tensor. The linearization occurs by considering only the effect of first order quantities. Second and higher order terms are assumed to be insignificant. The displacement of the rod axis and the directors correct to first order terms are

$$\begin{aligned} \bar{\mathbf{r}}(s,t) &= \mathbf{r}(s,t) + \mathbf{u}(s,t) \ , \\ \bar{\mathbf{d}}_\alpha(s,t) &= \mathbf{d}_\alpha(s,t) + \boldsymbol{\phi}(s,t) \times \mathbf{d}_\alpha(s,t) \ . \end{aligned} \qquad (3.20)$$

$\mathbf{r}(s,t)$ and $\mathbf{d}_\alpha(s,t)$ represent quantities in the finitely deformed state. $\mathbf{u}(s,t)$ and $\boldsymbol{\phi}(s,t)$ are the displacement and rotation vectors from the finitely deformed to the perturbed states. Because of (3.1) we may generalize $(3.20)_2$ by

$$\mathbf{d}_i = \mathbf{d}_i + \boldsymbol{\phi} \times \mathbf{d}_i \ . \qquad (3.21)$$

Since $\mathbf{r}$ and $\mathbf{d}_i$ represent statical quantities

$$\dot{\mathbf{d}}_i = \mathbf{0} \ , \ \dot{\mathbf{r}}_i = \mathbf{0} \ . \qquad (3.22)$$

We now break up all of the quantities in the momentum and constitutive equations for the perturbed state into component parts giving

$$\bar{\underset{\sim}{m}} = \underset{\sim}{m} + \tilde{\underset{\sim}{m}} \ , \ \bar{\underset{\sim}{n}} = \tilde{\underset{\sim}{n}} + \underset{\sim}{n} \ , \ \bar{\underset{\sim}{q}} = \underset{\sim}{q} + \tilde{\underset{\sim}{q}} \ , \ \bar{\underset{\sim}{\sigma}} = \tilde{\underset{\sim}{\sigma}} \ ,$$
$$\bar{\underset{\sim}{f}}^{\alpha} = \underset{\sim}{f} + \tilde{\underset{\sim}{f}}^{\alpha} \ , \ \bar{\underset{\sim}{n}}^{\alpha} = \underset{\sim}{n}^{\alpha} + \tilde{\underset{\sim}{n}}^{\alpha} \ . \tag{3.23}$$

The terms of the form $\underset{\sim}{m}$, $\underset{\sim}{n}$ etc. represent quantities in the finitely deformed state. $\tilde{\underset{\sim}{m}}$, $\tilde{\underset{\sim}{n}}$ etc, are the increments of $\underset{\sim}{m}$, $\underset{\sim}{n}$ required to arrive at the perturbed state. From the definitions of $\underset{\sim}{m}$, $\underset{\sim}{q}$ and $\underset{\sim}{\sigma}$, $(3.13)_{2,3,4}$ we find using $(3.23)_{1,3,4}$ and (3.20) that

$$\tilde{\underset{\sim}{m}} = \underset{\sim}{d}_{\alpha} x \tilde{\underset{\sim}{n}}^{\alpha} + (\underset{\sim}{\phi} x \underset{\sim}{d}_{\alpha}) x \underset{\sim}{n}^{\alpha} \ , \ \tilde{\underset{\sim}{q}} = \underset{\sim}{d}_{\alpha} x \tilde{\underset{\sim}{f}}^{\alpha} + (\underset{\sim}{\phi} x \underset{\sim}{d}_{\alpha}) x \underset{\sim}{f}^{\alpha} \ ,$$
$$\tilde{\underset{\sim}{\sigma}} = \rho y^{\alpha\beta} \underset{\sim}{d}_{\alpha} x (\dot{\underset{\sim}{\phi}} x \underset{\sim}{d}_{\beta}) = \rho \underset{\sim}{B} \dot{\phi} \ , \ \dot{\tilde{\underset{\sim}{\sigma}}} = \rho y^{\alpha\beta} \underset{\sim}{d}_{\alpha} x (\ddot{\underset{\sim}{\phi}} x \underset{\sim}{d}_{\beta}) = \rho \underset{\sim}{B} \ddot{\underset{\sim}{\phi}} \ , \ \underset{\sim}{B} = \underset{\sim}{B}^{T} \tag{3.24}$$

where second order quantities are neglected. By using the definition of transpose it is possible to show $\underset{\sim}{B} = \underset{\sim}{B}^{T}$.

The equations of motion for the perturbed state can now be written as a linear combination of the finitely deformed state and the increments. Since the quantities in the finitely deformed state satisfy a condition of statical equilibrium they may be cancelled out of the perturbed equations of motion. Therefore equations $(2.14)_1$, $(3.13)_1$, $(3.14)_{1,2}$ become

$$\hat{\tilde{\underset{\sim}{n}}} = \rho \tilde{\underset{\sim}{f}} = \rho \ddot{\underset{\sim}{u}} \ , \ \hat{\tilde{\underset{\sim}{m}}} + \hat{\underset{\sim}{r}} x \tilde{\underset{\sim}{n}} + \hat{\underset{\sim}{u}} x \underset{\sim}{n} + \rho \tilde{\underset{\sim}{q}} = \rho \underset{\sim}{B} \ddot{\underset{\sim}{\phi}}$$
$$[\tilde{\underset{\sim}{n}}] = -\rho V [\dot{\underset{\sim}{u}}] \ , \ [\tilde{\underset{\sim}{m}}] = -\rho V \underset{\sim}{B} [\dot{\underset{\sim}{\phi}}] \ . \tag{3.25}$$

The constitutive equations are now linearized by expanding the relations (3.19) as a Taylor series about the deformed equilibrium state A and are given by

$$
\begin{aligned}
\bar{\underset{\sim}{n}} &= \rho\frac{\partial\bar{\psi}}{\partial y_i}\,\bar{\underset{\sim}{d}}_i = \left(\rho\frac{\partial\psi}{\partial y_i}\bigg|_A + \rho\frac{\partial^2\psi}{\partial y_j\partial y_i}\bigg|_A \tilde{y}_j + \rho\frac{\partial^2\psi}{\partial h_j\partial y_i}\bigg|_A \tilde{h}_j\right)(\underset{\sim}{D}_i + (\underset{\sim}{\phi}\times\underset{\sim}{D}_i))\,, \\
\bar{\underset{\sim}{m}} &= \rho\frac{\partial\bar{\psi}}{\partial h_i}\,\bar{\underset{\sim}{d}}_i = \left(\rho\frac{\partial\psi}{\partial h_i}\bigg|_A + \rho\frac{\partial^2\psi}{\partial y_j\partial h_i}\bigg|_A \tilde{y}_j + \rho\frac{\partial^2\psi}{\partial h_j\partial h_i}\bigg|_A \tilde{h}_j\right)(\underset{\sim}{D}_i + (\underset{\sim}{\phi}\times\underset{\sim}{D}_i))\,.
\end{aligned}
\tag{3.26}
$$

$\tilde{y}_j$ and $\tilde{h}_j$ are computed from $(3.16)_{3,5}$, (3.20) and the fact

$$\tilde{y}_j = \bar{y}_j - y_j \;,\; \tilde{h}_j = \bar{h}_j - h_j \;. \tag{3.27}$$

This gives, to first order approximation

$$\tilde{y}_j = \underset{\sim}{z}\cdot\underset{\sim}{D}_j \;,\; \tilde{h}_j = \underset{\sim}{\kappa}\cdot\underset{\sim}{D}_j \;,\; \underset{\sim}{z} = \hat{\underset{\sim}{u}} + \hat{\underset{\sim}{R}}\times\underset{\sim}{\phi} \;,\; \underset{\sim}{\kappa} = \hat{\underset{\sim}{\phi}} \;. \tag{3.28}$$

$\underset{\sim}{z}$ represents the extensional and shearing strain, $\underset{\sim}{\kappa}$ the rotational strain. Substitution of $(3.28)_{1,2}$ into (3.26) and using the definitions for $\underset{\sim}{n}$ and $\underset{\sim}{m}$ $(3.23)_{1,2}$ we have

$$
\begin{aligned}
&\tilde{\underset{\sim}{n}} = \underset{\sim}{\phi}\times\underset{\sim}{n} + \underset{\sim}{E}\underset{\sim}{z} + \underset{\sim}{H}\underset{\sim}{\kappa} \;,\; \tilde{\underset{\sim}{m}} = \underset{\sim}{\phi}\times\underset{\sim}{m} + \underset{\sim}{H}^T\underset{\sim}{z} + \underset{\sim}{G}\underset{\sim}{\kappa}\,, \\
&\underset{\sim}{E} = \rho\frac{\partial^2\psi}{\partial y_j\partial y_i}\bigg|_A \underset{\sim}{d}_i\otimes\underset{\sim}{d}_j \;,\; \underset{\sim}{H} = \rho\frac{\partial^2\psi}{\partial h_j\partial y_i}\bigg|_A \underset{\sim}{d}_i\otimes\underset{\sim}{d}_j \;,\; \underset{\sim}{G} = \rho\frac{\partial^2\psi}{\partial h_j\partial h_i}\bigg|_A \underset{\sim}{d}_i\otimes\underset{\sim}{d}_j \;.
\end{aligned}
\tag{3.29}
$$

The linearized equations of a constrained rod become much simpler if we consider the finitely deformed configuration A to be the undeformed state. For this particular case

$$\underset{\sim}{n} = \underset{\sim}{0} \;,\; \underset{\sim}{m} = \underset{\sim}{0} \;,\; \underset{\sim}{r} = \underset{\sim}{R} \;. \tag{3.30}$$

The system of governing equations (3.25 and $(3.29)_{1,2}$ reduce to

$$\hat{\underset{\sim}{n}} + \rho\underset{\sim}{\tilde{f}} = \rho\underset{\sim}{\ddot{u}} \ , \ \hat{\underset{\sim}{m}} + \underset{\sim}{R}\text{x}\underset{\sim}{\tilde{n}} + \rho\underset{\sim}{q} = \rho\underset{\sim}{B}\underset{\sim}{\ddot{\phi}} \ ,$$

$$[\underset{\sim}{\tilde{n}}] = -\rho V[\underset{\sim}{\dot{u}}] \ , \ [\underset{\sim}{\tilde{m}}] = -\rho V\underset{\sim}{B}[\underset{\sim}{\dot{\phi}}] \ , \tag{3.31}$$

$$\underset{\sim}{\tilde{n}} = \underset{\sim}{E}\underset{\sim}{z} + \underset{\sim}{H}\underset{\sim}{\kappa} \ , \ \underset{\sim}{\tilde{m}} = \underset{\sim}{H}^T\underset{\sim}{z} + \underset{\sim}{G}\underset{\sim}{\kappa} \ .$$

The tensors $\underset{\sim}{E}$, $\underset{\sim}{H}$, and $\underset{\sim}{G}$ are now functions of s alone. This set is slightly modified by replacing the cross product in $(3.28)_3$ and $(3.31)_2$ by the appropriate antisymmetric tensor $\underset{\sim}{T}$

$$\underset{\sim}{z} = \hat{\underset{\sim}{u}} + \underset{\sim}{T}\phi \ , \ \hat{\underset{\sim}{m}} + \underset{\sim}{T}\underset{\sim}{\tilde{n}} + \rho\underset{\sim}{q} = \rho\underset{\sim}{B}\underset{\sim}{\ddot{\phi}} \ . \tag{3.32}$$

The procedure of introducing a higher order vector space to compress the format of the governing equations, as in chapter two, is now accomplished. We define a higher order vector space $\mathcal{Y}$ as a direct sum $\mathcal{Y} = \mathcal{V}\oplus\mathcal{V}$. Elements $\underset{\sim}{u} \in \mathcal{Y}$ are constructed by the rule

$$\underset{\sim 1}{u} = (\underset{\sim 1}{u}, \underset{\sim 2}{u}) \ , \ \underset{\sim\alpha}{u} \in \underset{\sim}{\mathcal{V}}, \ \alpha = 1,2 \ . \tag{3.33}$$

A standard Euclidean product on $\mathcal{Y}$ is defined by $\{\underset{\sim}{u},\underset{\sim}{v}\} = \langle \underset{\sim i}{u}, \underset{\sim i}{v}\rangle$[4].

We now define the generalized strains, displacements, stresses, and body forces by

$$\underset{\sim}{s} = (\underset{\sim}{z},\underset{\sim}{\kappa}) \ , \ \underset{\sim}{d} = (\underset{\sim}{u},\underset{\sim}{\phi}) \ , \ \underset{\sim}{t} = (\underset{\sim}{\tilde{n}},\underset{\sim}{\tilde{m}}) \ , \ \underset{\sim}{h} = (\underset{\sim}{f},\underset{\sim}{q}) \ . \tag{3.34}$$

We also define higher order tensor quantities by

---

[4] It will always be apparent for the problem being discussed whether we mean the Euclidean product on $\mathcal{V}$ or on $\mathcal{Y}$.

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{O} & \underset{\sim}{T} \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix} , \quad \underset{\sim}{\mathcal{D}} = \begin{bmatrix} \underset{\sim}{E} & \underset{\sim}{H} \\ \underset{\sim}{H}^T & \underset{\sim}{G} \end{bmatrix} , \quad \underset{\sim}{K} = \rho \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{B} \end{bmatrix} , \quad \underset{\sim}{K} = \underset{\sim}{K}^T \; {}^{5} \tag{3.35}$$

We may rewrite (3.31) and $(3.28)_{3,4}$ taking into account (3.32) as

$$[\underset{\sim}{t}] = -V\underset{\sim}{K}[\dot{\underset{\sim}{d}}] , \quad \hat{\underset{\sim}{t}} - \underset{\sim}{X}^T\underset{\sim}{t} + \rho\underset{\sim}{h} = \underset{\sim}{K}\,\ddot{\underset{\sim}{d}} , \quad \underset{\sim}{t} = \underset{\sim}{\mathcal{D}}\underset{\sim}{s} , \quad \underset{\sim}{s} = \hat{\underset{\sim}{d}} + \underset{\sim}{X}\underset{\sim}{d} . \tag{3.36}$$

This system of equations is of dimension six. The constrained theory reduces the degrees of freedom for the rod by three.

---

[5] It will always be apparent for the problem being discussed whether we are referring to the $\underset{\sim}{K}$ in $\underset{\sim}{\mathcal{U}} \otimes \underset{\sim}{\mathcal{U}}$ or in $\underset{\sim}{\mathcal{V}} \otimes \underset{\sim}{\mathcal{V}}$.

CHAPTER 4

JUMP COMPATIBILITY RELATIONSHIPS

This chapter concerns itself with deriving the so called jump compatibility equations. The jump compatibility conditions relate time and spatial derivatives of quantities across a moving discontinuity.

We start our discussion by recalling that a moving point of discontinuity in a rod can be represented by

$$\underset{\sim}{r} = \underset{\sim}{r}(s,t) \;,\; s = P(t) \quad . \tag{4.1}$$

At this stage we assume a point of discontinuity is a place where some property of the rod, say Π, suffers a definite jump in value given by

$$[\Pi] = \lim_{h \to o} (\Pi(s+h,t) - \Pi(s-h,t) \neq 0 \;,\; s = P(t) \quad . \tag{4.2}$$

Only one point of discontinuity will be assumed to exist along the rod. The point of singularity is known as the wave front. The wave is a travelling wave front. The speed of propagation is

$$V = \dot{P} = \mathring{P} \quad , \tag{4.3}$$

where $^\circ$ represents a total time derivative. The speed of propagation represents the velocity of the wave front.

At the wave front

$$\mathring{\Pi} = \dot{\Pi} + V\hat{\Pi} \quad . \tag{4.4}$$

Taking jumps in (4.4) and rearranging gives

$$[\dot{\Pi}] = [\mathring{\Pi}] - V[\hat{\Pi}] \quad . \tag{4.5}$$

Hadamard's lemma states that

$$[\overset{n\circ}{\Pi}] = \overset{n\circ}{\overline{[\Pi]}} \quad . \tag{4.6}$$

$\overset{n\circ}{\Pi}$ denotes n total time derivatives. Utilizing (4.6) and (4.5) we have

$$[\dot{\Pi}] = \overset{\circ}{\overline{[\Pi]}} - V[\hat{\Pi}] \quad . \tag{4.7}$$

Replacing Π by $\overset{n\wedge}{\Pi}$ in (4.7) yields

$$[\overset{n\wedge\cdot}{\Pi}] = \overset{\circ}{\overline{[\overset{n\wedge}{\Pi}]}} - V[\overset{(n+1)\wedge}{\Pi}] \quad , \tag{4.8}$$

where $\overset{n\wedge\cdot}{\Pi}$ represents n spatial derivatives and one time derivative. We replace Π by $\dot{\Pi}$ in (4.7)

$$[\overset{2\cdot}{\Pi}] = \overset{\circ}{\overline{[\dot{\Pi}]}} - V[\hat{\dot{\Pi}}] \quad . \tag{4.9}$$

Substituting (4.8) into (4.9) results in

$$[\overset{2\cdot}{\Pi}] = \overset{\circ\circ}{\overline{[\Pi]}} - \overset{\circ}{V}[\hat{\Pi}] - 2V\overset{\circ}{\overline{[\hat{\Pi}]}} + V^2[\overset{2\wedge}{\Pi}] \quad . \tag{4.10}$$

In subsequant analysis we will only be interested in quantities having discontinuities in at most the two highest order differential terms. Assuming the jumps in all lower order derivatives are zero we can rewrite (4.10) as

$$[\overset{2\cdot}{\Pi}] = V^2[\overset{2\wedge}{\Pi}] - 2V\overset{\circ}{\overline{[\hat{\Pi}]}} - \overset{\circ}{V}[\hat{\Pi}] \quad . \tag{4.11}$$

Discontinuities are not allowed in Π for this case.

We will now consider the jump in a quantity with (m+p) derivatives. Discontinuities will only be allowed in the terms with (m+p) and (m+p-1) derivatives. This may be summarized by

$$[\overset{a\wedge b\cdot}{\Pi}] = 0 \ , \ a + b < m + p-1 \ . \tag{4.12}$$

If relationship (4.12) holds it is possible to develop a relationship for $[\overset{m\wedge p\cdot}{\Pi}]$ by

$$[\overset{m\wedge p\cdot}{\Pi}] = (-V)^p[\overset{(m+p)\wedge}{\Pi}] + p(-V)^{p-1}[\overset{\circ}{\overline{\overset{(m+p-1)\wedge}{\Pi}}}] + (p-1)p(-1)^{p+1}V^{p-2}\overset{\circ}{V}[\overset{(m+p-1)\wedge}{\Pi}]/2 \ . \tag{4.13}$$

The proof of (4.13) will be carried out by induction. In the proof we will suppress the dependence of Π on spacial derivatives. Therefore we wish to show that for

$$[\overset{p\cdot}{\Pi}] \neq 0 \ , \ [\overset{(p-1)\cdot}{\Pi}] \neq 0 \ , \ [\overset{(p-r)\cdot}{\Pi}] = 0 \ , \ r = 2 \ \ldots \ p \ , \tag{4.14}$$

the following holds

$$[\overset{p\cdot}{\Pi}]=(-V)^p[\overset{p\wedge}{\Pi}]+p(-V)^{p-1}[\overset{\circ}{\overline{\overset{(p-1)\wedge}{\Pi}}}]+(p-1)p(-1)^{p+1}V^{p-2}\overset{\circ}{V}[\overset{(p-1)\wedge}{\Pi}]/2 \ . \tag{4.15}$$

For the case p=1 (4.15) gives (4.5). Therefore (4.15) is true for p=1. We now assume that it is also true for the case p=c. We must show that equation (4.15) is true for p=c+1. In this case new conditions corresponding to (4.14) must be established. These are

$$[\overset{(c+1)\cdot}{\Pi}] \neq 0, \ [\overset{c\cdot}{\Pi}] \neq 0 \ , \ [\overset{(c+1-r)\cdot}{\Pi}] = 0 \ , \ r = 2 \ \ldots \ c \ . \tag{4.16}$$

It is to be noted that only the two highest order differential terms have non-zero jumps. By replacing Π in (4.15) by $\overset{\circ}{\Pi}$ and using

$$[\overset{c\wedge\cdot}{\Pi}] = \overline{\overset{\circ}{[\overset{c\wedge}{\Pi}]}} - v[\overset{(c+1)\wedge}{\Pi}]\,,\quad [\overset{(c-1)\wedge\cdot}{\Pi}] = -v[\overset{c\wedge}{\Pi}]\,,$$
$$\overline{\overset{\circ}{[\overset{(c-1)\wedge\cdot}{\Pi}]}} = -\overset{\circ}{v}[\overset{c\wedge}{\Pi}] - v\overline{\overset{\circ}{[\overset{c\wedge}{\Pi}]}}\,, \qquad (4.17)$$

we find $[\overset{(c+1)\cdot}{\Pi}]$ attains the form of (4.15) where p has been replaced by (c+1). Equations $(4.17)_{1,2}$ came as a consequence of (4.8) and (4.16). Equation $(4.17)_3$ results by applying the product rule to $(4.17)_2$.

We may conclude that by mathematical induction (4.15) holds for all positive integers p. Replacing Π in (4.15) by $\overset{m\wedge}{\Pi}$ gives (4.13).

CHAPTER 5

DEVELOPMENT OF WAVE EQUATIONS

This chapter concerns itself with developing the so called nth order wave propagation and decay-induction equations for the general rod and the constrained rod. We will initially assume that discontinuities exist in the nth and higher derivatives of the appropriate displacement variables. The systems of equations governing the motion of rods can always be expressed in terms of these quantities. This analysis therefore establishes a method of studying the propagation of nth order waves at the wave front.

There are two equations of motion at the wave front in terms of the displacement variables. These are called the propagation and decay-induction equations. From the latter one is able to generate an equation for the so called wave mode vector. A scalar multiple of this vector is seen to be independent of the order of the waves.

1) Unconstrained Rods

We start by giving the complete set of linear equations for the unconstrained rod as developed in chapter two.

$$\hat{\underset{\sim}{N}} - \underset{\sim}{M} + \rho\underset{\sim}{F} = \underset{\sim}{K}\ddot{\underset{\sim}{p}} \ , \ [\underset{\sim}{N}] = -V\underset{\sim}{K}[\dot{\underset{\sim}{p}}] \ ,$$
$$\underset{\sim}{M} = \underset{\sim}{T}\underset{\sim}{p} + \underset{\sim}{H}^T\hat{\underset{\sim}{p}} \ , \ \underset{\sim}{N} = \underset{\sim}{H}\underset{\sim}{p} + \underset{\sim}{A}\hat{\underset{\sim}{p}} \ , \qquad (5.1)$$

where $\underset{\sim}{T}$, $\underset{\sim}{H}$, $\underset{\sim}{A}$ are functions of s alone.

A shock wave is defined as a wave in which discontinuities occur in first and higher order derivatives of $\underset{\sim}{p}$ or

$$[\underset{\sim}{p}] = 0 \ , \ [\dot{\underset{\sim}{p}}] \neq 0 \ , \ [\hat{\underset{\sim}{p}}] = \underset{\sim}{0} \ . \tag{5.2}$$

Substitution of $(5.1)_4$ into $(5.1)_2$ and utilizing (5.2) results in

$$\underset{\sim}{A}[\hat{\underset{\sim}{p}}] = -V\underset{\sim}{K}[\dot{\underset{\sim}{p}}] \ . \tag{5.3}$$

Using compatibility relationship (4.7) gives

$$(\underset{\sim}{A}-V^2\underset{\sim}{K})[\hat{\underset{\sim}{p}}] = \underset{\sim}{0} \ . \tag{5.4}$$

This equation is known as the propagation equation for weak shock waves.[6]

Substitution of $(5.1)_{3,4}$ into $(5.1)_1$ and taking the jump in the results gives upon using (5.2)

$$\underset{\sim}{A}[\overset{2}{\hat{\underset{\sim}{p}}}] + (\underset{\sim}{A}+\underset{\sim}{H}-\underset{\sim}{H}^T)[\hat{\underset{\sim}{p}}] + \rho[\underset{\sim}{F}] = \underset{\sim}{K}[\overset{2}{\dot{\underset{\sim}{p}}}] \ . \tag{5.5}$$

Physically the quantity $\underset{\sim}{F}$ represents a general body force. We examine problems in which $\underset{\sim}{F}$ is continuous, therefore $[\underset{\sim}{F}] = \underset{\sim}{0}$. Using the compatibility equation (4.11) we find (5.5) becomes

$$(\underset{\sim}{A}-V^2\underset{\sim}{K})[\overset{2}{\hat{\underset{\sim}{p}}}] + (\hat{\underset{\sim}{A}}-\underset{\sim}{B}+\mathring{V}\underset{\sim}{K})[\hat{\underset{\sim}{p}}] + 2V\underset{\sim}{K}\overset{\circ}{\overline{[\hat{\underset{\sim}{p}}]}} = \underset{\sim}{0} \ , \ \underset{\sim}{B} = -(\underset{\sim}{H}-\underset{\sim}{H}^T) \ . \tag{5.6}$$

This equation is called the decay-induction equation for shock waves.

Before going into a discussion of the propagation and decay-induction equations we will develop relationships for a

---

[6] Weak shock waves mean we are considering a linear theory.

general nth order wave. Equations (5.4) and (5.6) will then be shown to be special cases of these relationships.

We define an nth order wave as

$$[\overset{t\wedge q\cdot}{\mathbf{p}}] \neq \mathbf{0}\ , \quad t + q \geq n\ , \quad [\overset{t\wedge q\cdot}{\mathbf{p}}] = \mathbf{0}\ , \quad t + q < n\ . \tag{5.7}$$

It is apparent that for $n > 1$ $(5.1)_2$ is identically satisfied. A method of developing two equations of motion to be satisfied is not obvious. However two equations can be established by the following method. Differentiate $(5.1)_1$ with respect to time (n-2) and (n-1) times. Taking the jump in these two equations and using (5.7) and the continuity of $\mathbf{F}$ gives

$$\begin{aligned}
&[\overset{\wedge(n-2)\cdot}{\mathbf{N}}]-[\overset{(n-2)\cdot}{\mathbf{M}}] = \mathbf{K}[\overset{n\cdot}{\mathbf{p}}]\ , \quad [\overset{\wedge(n-1)\cdot}{\mathbf{N}}]-[\overset{(n-1)\cdot}{\mathbf{M}}] = \mathbf{K}[\overset{(n+1)\cdot}{\mathbf{p}}]\ , \\
&\overset{(n-1)\cdot}{\mathbf{M}} = \mathbf{H}^T \overset{\wedge(n-1)\cdot}{\mathbf{p}}\ , \quad \overset{(n-2)\cdot}{\mathbf{M}} = \mathbf{0}\ , \\
&\overset{\wedge(n-1)\cdot}{\mathbf{N}} = \mathbf{H}\overset{\wedge(n-1)\cdot}{\mathbf{p}} + \hat{\mathbf{A}}\overset{\wedge(n-1)\cdot}{\mathbf{p}} + \mathbf{A}\overset{2\wedge(n-1)\cdot}{\mathbf{p}}\ , \\
&\overset{\wedge(n-2)\cdot}{\mathbf{N}} = \mathbf{A}\overset{2\wedge(n-2)\cdot}{\mathbf{p}}\ .
\end{aligned} \tag{5.8}$$

Substitution of $(5.8)_{3,4,5,6}$ into $(5.8)_{1,2}$ and using the compatibility relationship (4.13) results in

$$\begin{aligned}
&(\mathbf{A}-V^2\mathbf{K})[\overset{n\wedge}{\mathbf{p}}] = \mathbf{0}\ , \\
&(\mathbf{A}-V^2\mathbf{K})[\overset{(n+1)\wedge}{\mathbf{p}}]+\{\hat{\mathbf{A}}-\mathbf{B}+(2n-1)\mathring{V}\mathbf{K}-(n-2)(n-1)V^{-2}\mathring{V}(\mathbf{A}-V^2\mathbf{K})/2\}[\overset{n\wedge}{\mathbf{p}}] \\
&-V^{-1}\{(n-1)(\mathbf{A}-V^2\mathbf{K})-2V^2\mathbf{K}\}[\overset{\circ}{\overline{\overset{n\wedge}{\mathbf{p}}}}]\ .
\end{aligned} \tag{5.9}$$

Equations $(5.9)_1$ and $(5.9)_2$ are the general propagation and decay-induction equations for an nth order discontinuity. For the case n=1 they are identical to (5.4) and $(5.6)_1$. Therefore n=1 is a special case of (5.9).

Equation $(5.9)_1$ is an eigenvalue problem, the solution of which gives the eigenvalues V^2 and the eigenvectors $[{}^{n}\hat{\underset{\sim}{p}}]$. Since $\underset{\sim}{A}$ and $\underset{\sim}{K}$ are functions only of s we conclude that V^2 and $[{}^{n}\hat{\underset{\sim}{p}}]$ are functions of s alone. Therefore (4.4) reduces to

$$\overset{\circ}{\Pi} = V\hat{\Pi} \quad . \tag{5.10}$$

We may also solve for the term $\hat{\underset{\sim}{A}}[{}^{n}\hat{\underset{\sim}{p}}]$ in $(5.9)_2$ by differentiating $(5.9)_1$ and rearranging. This gives

$$\hat{\underset{\sim}{A}}[{}^{n}\hat{\underset{\sim}{p}}] = (2V\hat{V}\underset{\sim}{K}+V^2\hat{\underset{\sim}{K}})[{}^{n}\hat{\underset{\sim}{p}}]-(\underset{\sim}{A}-V^2\underset{\sim}{K})\widehat{[{}^{n}\hat{\underset{\sim}{p}}]} \quad . \tag{5.11}$$

Using the general relationships (5.10) and (5.11) in $(5.9)_2$ gives

$$\begin{aligned} &\underset{\sim}{L}_\lambda \underset{\sim}{b} + \{\lambda \underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}}-\underset{\sim}{K}^{-1}\underset{\sim}{B} + \frac{(2n+1)}{2}\hat{\lambda}\underset{\sim}{I}-(n-2)(n-1)\lambda^{-1}\hat{\lambda}\underset{\sim}{L}_\lambda/4\}\underset{\sim}{a} \\ &-\{n\underset{\sim}{L}_\lambda-2\lambda\underset{\sim}{I}\}\,\hat{\underset{\sim}{a}} = \underset{\sim}{0} \,, \end{aligned} \tag{5.12}$$

$$\underset{\sim}{b} = [{}^{(n+1)}\hat{\underset{\sim}{p}}] \,, \quad \underset{\sim}{a} = [{}^{n}\hat{\underset{\sim}{p}}] \,, \quad \underset{\sim}{L}_\lambda = \underset{\sim}{K}^{-1}\underset{\sim}{A}-\lambda\underset{\sim}{I} \,, \quad \lambda = V^2 \quad .$$

$\underset{\sim}{a}$ is the wave mode vector while $\underset{\sim}{b}$ is the induced wave. $\underset{\sim}{I}$ is the identity map on U. The propagation equation $(5.9)_1$ can be rewritten in terms of our new terminology as

$$\underset{\sim}{L}_\lambda \underset{\sim}{a} = \underset{\sim}{0} \quad . \tag{5.13}$$

We pause now to define a new inner product $\{\underset{\sim}{u},\underset{\sim}{v}\}_{\underset{\sim}{K}}$ on U defined by

$$\{\underset{\sim}{u},\underset{\sim}{v}\}_{\underset{\sim}{K}} = \{\underset{\sim}{u},\underset{\sim}{K}\underset{\sim}{v}\} \,, \quad \underset{\sim}{u},\underset{\sim}{v} \in U \quad . \tag{5.14}$$

This relationship satisfies all the requirements for an inner product. The transpose of a second order tensor $\underset{\sim}{R}$ with respect to the $\underset{\sim}{K}$ inner product will be defined as

$$\{\underset{\sim}{R}\underset{\sim}{u},\underset{\sim}{v}\}_{\underset{\sim}{K}} = \{\underset{\sim}{u},\underset{\sim}{R}^{\bar{T}}\underset{\sim}{v}\}_{\underset{\sim}{K}} \ , \quad \underset{\sim}{u},\underset{\sim}{v} \in V \ . \tag{5.15}$$

The ordinary transpose can be related to this new quantity resulting in

$$\underset{\sim}{R}^{\bar{T}} = \underset{\sim}{K}^{-1}\underset{\sim}{R}^{T}\underset{\sim}{K} \ . \tag{5.16}$$

With respect to the new inner product we have

$$\{\underset{\sim}{u},\underset{\sim}{K}^{-1}\underset{\sim}{A}\underset{\sim}{v}\}_{\underset{\sim}{K}} = \{\underset{\sim}{K}^{-1}\underset{\sim}{A}\underset{\sim}{u},\underset{\sim}{v}\}_{\underset{\sim}{K}} \ . \tag{5.17}$$

Therefore $\underset{\sim}{K}^{-1}\underset{\sim}{A}$ is symmetric with respect to the new inner product. It will now be assumed that $\underset{\sim}{K}^{-1}\underset{\sim}{A}$ is positive definite with respect to $\underset{\sim}{K}$.

Looking at (5.13) we see that it represents the same eigenvalue problem as $(5.9)_1$. The quantity λ representing the eigenvalues is the square of the speed of propagation. The eigenvalue problem (5.13), because of the positive definiteness of $\underset{\sim}{K}^{-1}\underset{\sim}{A}$ with respect to $\underset{\sim}{K}$ has some very special properties.

The characteristic equation

$$\det \underset{\sim}{L}_{\lambda} = 0 \quad , \tag{5.18}$$

associated with the eigenvalue problem has nine real roots

λ_i (i = 1 ... 9)[7], not all of which must necessarily be distinct.

Corresponding to each distinct eigenvalue λ_i there exists an eigenspace $\mathcal{U}_i$. The eigenspace $\mathcal{U}_i$ is defined as the set of all vectors called eigenvectors satisfying the equation

$$\underset{\sim}{L}_{\lambda_i} \; {}_i\underset{\sim}{a} = \underset{\sim}{0} \;, \quad {}_i\underset{\sim}{a} \in \mathcal{U}_i \;. \tag{5.19}$$ [8]

The dimension of the eigenspace is the same as the degree of multiplicity of the eigenvalues. It is also clear that the eigenspaces $\mathcal{U}_i$ are all subspaces of $\mathcal{U}$, and that $\mathcal{U}$ is composed of a direct sum of all $\mathcal{U}_i$.

A theorem in linear algebra states that the eigenvalue problem (5.13) subject to the condition that $\underset{\sim}{K}^{-1}\underset{\sim}{A}$ is symmetric and positive definite with respect to $\underset{\sim}{K}$, has eigenspaces which are orthogonal with respect to $\underset{\sim}{K}$. The mathematical statement of this problem is

$$\{{}_i\underset{\sim}{a}, {}_j\underset{\sim}{a}\}_{\underset{\sim}{K}} = 0 \;, \quad i \neq j \;, \quad {}_i\underset{\sim}{a} \in \mathcal{U}_i \;, \quad {}_j\underset{\sim}{a} \in \mathcal{U}_j \;. \tag{5.20}$$

If the eigenvalues λ_i are all distinct then the dimension of all eigenspaces is one. The direction of the eigenvectors is therefore uniquely specified by a solution of (5.13). It is also apparent that all the eigenvectors are orthogonal with respect to the $\underset{\sim}{K}$ inner product.

---

[7] From now on Latin indices will indicate summation from 1 ... 9 or or 1 ... 6 depending on if we are talking about the vector space $\mathcal{U}$ or $\mathcal{Y}$ respectively.

[8] The summation convention does not apply if one index appears to the left of an argument.

The magnitudes of the eigenvectors, which represent the actual waves at the wave front, are indeterminate by (5.13). These magnitudes will be determined by a solution of the decay-induction equation $(5.12)_1$ subject to initial conditions at $s = s_o$. The decay-induction equation also determines a higher order wave $\underset{\sim}{b}$, which is induced due to the wave $\underset{\sim}{a}$.

The case of multiple eigenvalues is now considered. Let λ_i be an eigenvalue of multiplicity r. The solution of (5.19) gives an eigenspace U_i. We may find a set of r vectors which forms a basis for this subspace. These vectors can never be solved directly from (5.19). Therefore the set of spanning eigenvectors is arbitrary. Once such an arbitrary set is chosen, by any means, a solution of the decay-induction equation will give the magnitudes of the vectors subject to the initial conditions.

We now develop an equation by which the magnitude of $\underset{\sim}{a}$ may be determined. Firstly two operators are defined. $\underset{\sim}{P}_i$ and $\underset{\sim}{P}_i^{\perp}$ are two orthogonal projection operators which take vectors $\underset{\sim}{d} \in U$ into ${}_i\underset{\sim}{d} \in U_i$ and ${}_i\underset{\sim}{d}^{\perp} \in U_i^{\perp}$ the orthogonal complement of U_i.

The eigenspace U_i is now examined. At this stage the dimension of U_i is of no significance. The eigenvalue problem gives λ_i and the space in which ${}_i\underset{\sim}{a} \in U_i$ is contained. Substituting these two values for λ and $\underset{\sim}{a}$ in the decay-induction equation $(5.12)_1$ and taking $\underset{\sim}{P}_i$ of the entire expression gives

$$\underset{\sim}{P}_i(2\lambda_i\, {}_i\hat{\underset{\sim}{a}} + (\lambda_i \underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}} - \underset{\sim}{K}^{-1}\underset{\sim}{B} + (2n+1)\hat{\lambda}_i \underset{\sim}{I}/2)\, {}_i\underset{\sim}{a}) = \underset{\sim}{0} \;, \qquad (5.21)$$

where it is recognized that the various projection operators are such that

$$\underset{\sim}{P}_i \underset{\sim}{L}_{\lambda_i} \underset{\sim}{d} = \underset{\sim}{0} \;, \quad \underset{\sim}{P}_i^{\perp} \underset{\sim}{L}_{\lambda_i} \underset{\sim}{d} = \underset{\sim}{L}_{\lambda_i} \underset{\sim}{d} \;, \quad \underset{\sim}{d} \in U \;. \qquad (5.22)$$

Rearranging (5.21) gives the more convenient form.

$$\underset{\sim}{P}_i({}_i\hat{\underset{\sim}{c}} + \tfrac{1}{2}(\underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}} - \underset{\sim}{K}^{-1}\underset{\sim}{B}/\lambda_i)\, {}_i\underset{\sim}{c}) = \underset{\sim}{0} \;, \quad {}_i\underset{\sim}{c} = \lambda_i^{\frac{2n+1}{4}}\, {}_i\underset{\sim}{a} \;. \qquad (5.23)$$

This is a first order system of ordinary differential equations subject to the initial values of ${}_i\underset{\sim}{c}$ at $s = s_o$.

It is seen that the eigenvalue problem (5.19) is independent of the wave order. Therefore the eigenvalues and the eigenspace are also independent of order. In a similar manner we see that ${}_i\underset{\sim}{c}$ in (5.23) is invariant in regards to the wave order. We therefore see that once a solution to the system (5.19) and (5.23) is established for any order wave the solution to all orders of waves is known.

Assuming that an ${}_i\underset{\sim}{a}$ has been found which satisfies (5.19) and (5.23) we can solve $(5.12)_1$ for the induced wave $\underset{\sim}{b}$. Taking the projection of $(5.12)_1$ into $U_i^{\perp}$ and using $(5.22)_2$ results in

$$\begin{aligned}\underset{\sim}{L}_{\lambda_i}\underset{\sim}{b} = \lambda_i^{\frac{-(2n+1)}{4}}\, \underset{\sim}{P}_i((n\underset{\sim}{L}_{\lambda_i} - 2\lambda_i \underset{\sim}{I})\, {}_i\hat{\underset{\sim}{c}} - (\lambda_i \underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}} - \underset{\sim}{K}^{-1}\underset{\sim}{B} \\ - (n^2+4n-2)\lambda_i^{-1}\hat{\lambda}_i \underset{\sim}{L}_{\lambda_i}/4)\, {}_i\underset{\sim}{c}) \;. \end{aligned} \qquad (5.24)$$

This represents a set of linear equations in the unknown $\underset{\sim}{b}$. Two factors are immediately evident from (5.24). Firstly the induced wave is dependent on the order of the wave being considered. Secondly the component of the induced wave $\underset{\sim}{b}$ in U_i is indeterminate.

We will discontinue studying the induced wave $\underset{\sim}{b}$ and return to the wave vector $_i\underset{\sim}{c}$. The vector differential equation (5.23) can be represented in operator notation as

$$\underset{\sim}{E}\ {}_i\underset{\sim}{c} = \underset{\sim}{0} \quad . \tag{5.25}$$

Let us assume for the moment that two solutions $_i\underset{\sim}{c}^p$ and $_i\underset{\sim}{c}^q$ both satisfy (5.25) and are hence both contained in U_i. Consider performing

$$\{{}_i\underset{\sim}{c}^p, \underset{\sim}{E}\ {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} + \{{}_i\underset{\sim}{c}^q, \underset{\sim}{E}\ {}_i\underset{\sim}{c}^p\}_{\underset{\sim}{K}} = 0 \quad . \tag{5.26}$$

Recalling that $\underset{\sim}{B}$ is antisymmetric, $\underset{\sim}{K}$ and $\hat{\underset{\sim}{K}}$ are symmetric and that

$$\begin{aligned} \widehat{\{{}_i\underset{\sim}{c}^p, {}_i\underset{\sim}{c}^q\}}_{\underset{\sim}{K}} &= \{{}_i\hat{\underset{\sim}{c}}^p, {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} + \{{}_i\hat{\underset{\sim}{c}}^q, {}_i\underset{\sim}{c}^p\}_{\underset{\sim}{K}} + \{{}_i\underset{\sim}{c}^p, \hat{\underset{\sim}{K}}\ {}_i\underset{\sim}{c}^q\} , \\ \{\underset{\sim}{P}_i\ {}_i\underset{\sim}{c}^p, {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} &= \{{}_i\underset{\sim}{c}^p, {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} , \end{aligned} \tag{5.27}$$

equation (5.26) reduces to

$$\widehat{\{{}_i\underset{\sim}{c}^p, {}_i\underset{\sim}{c}^q\}}_{\underset{\sim}{K}} = 0 \quad . \tag{5.28}$$

The solution to this differential equation can be written as

$$\{{}_i\underset{\sim}{c}^p, {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} = e \quad , \tag{5.29}$$

or in terms of $_i\underset{\sim}{a}$ as

$$\{{}_i\underset{\sim}{a}^p, {}_i\underset{\sim}{a}^q\}_{\underset{\sim}{K}} = e\lambda_i^{-\frac{(2n+1)}{2}} \quad . \tag{5.30}$$

Relationship (5.30) shows that solutions of (5.23), which is essentially the decay part of the decay-induction equation, are orthogonal with respect to $\underset{\sim}{K}$ at all s if they are orthogonal with respect to $\underset{\sim}{K}$ at any one point. We will assume that the initial conditions of our problem are such that this type of orthogonality occurs. We will therefore set up U_i such that a maximal set of wave mode vectors corresponding to the dimension of U_i satisfies $(5.23)_1$ and initial conditions, which are orthogonal with respect to $\underset{\sim}{K}$. The wave mode vectors are hence orthogonal with respect to $\underset{\sim}{K}$ at all points of the rod. Now both single and multiple eigenvalue problems have wave mode vectors orthogonal with respect to $\underset{\sim}{K}$.

2) Constrained Rods

The complete set of linear equations for the constrained rod deformed from a point of zero initial stress is

$$\hat{\underset{\sim}{t}}-\underset{\sim}{X}^{T}\underset{\sim}{t}+\rho\underset{\sim}{h} = \underset{\sim}{K}\ddot{\underset{\sim}{d}} \ , \ [\underset{\sim}{t}] = -V[\dot{\underset{\sim}{d}}] \ , \ \underset{\sim}{t} = \underset{\sim}{D}\underset{\sim}{s} \ , \ \underset{\sim}{s} = \hat{\underset{\sim}{d}}+\underset{\sim}{X}\underset{\sim}{d} \ . \quad (5.31)$$

A shock wave for a constrained rod is defined as a wave in which discontinuities occur in first and higher order derivatives of $\underset{\sim}{d}$ or

$$[\underset{\sim}{d}] = \underset{\sim}{0} \ , \ [\dot{\underset{\sim}{d}}] \neq \underset{\sim}{0} \ , \ [\hat{\underset{\sim}{d}}] \neq \underset{\sim}{0} \quad . \qquad (5.32)$$

Substitution of $(5.31)_{3,4}$ into $(5.31)_{1,2}$ and utilizing the compatibility relationship (4.7) results in

$$(\underset{\sim}{D}-V^2\underset{\sim}{K})[\hat{\underset{\sim}{d}}] = \underset{\sim}{0} \quad . \qquad (5.33)$$

This is the propagation equation for constrained rods.

We develop the decay induction equation for shock waves by taking the jump in $(5.31)_{1,3,4}$ and using (5.32) as well as the compatibility relationships. The jump in $(5.31)_1$ then becomes

$$(\mathcal{D}-V^2\mathcal{K})[\overset{2\wedge}{d}]+(\hat{\mathcal{D}}+\mathcal{S}+\mathring{V}\mathcal{K})[\hat{d}]+2V[\overset{\circ}{\overline{\hat{d}}}] = 0 \ , \ \mathcal{S} = -\mathcal{S}^T = \mathcal{D}\mathcal{X}-\mathcal{X}^T\mathcal{D} \ . \quad (5.34)$$

It is seen that the quantities in (5.33) and (5.34) have the exact same mathematical form as the variables in (5.4) and (5.6). The only significant difference is the reduced dimension of the problem when considering the constrained theory.

In an analogous procedure we may define an nth order wave in a constrained theory as

$$[\overset{t\wedge q\cdot}{d}] \neq 0 \ , \ t+q \geq n \ , \ [\overset{t\wedge q\cdot}{d}] = 0 \ , \ t+q < n \ . \quad (5.35)$$

The general shock and decay-induction equations for nth order waves can be shown to be

$$\begin{aligned} &\mathcal{L}_\lambda a = 0 \ , \\ &\mathcal{L}_\lambda b+(\lambda\mathcal{K}^{-1}\hat{\mathcal{K}}+\mathcal{K}^{-1}\mathcal{S}+(2n+1)\hat{\lambda}\mathcal{I}/2-(n-2)(n-1)\lambda^{-1}\hat{\lambda}\mathcal{L}_\lambda/4)a \\ &-(n\mathcal{L}_\lambda-2\lambda\mathcal{I})\hat{a} = 0 \ , \qquad (5.36) \\ &\mathcal{L}_\lambda = \mathcal{K}^{-1}\mathcal{D}-\lambda\mathcal{I}\ ^{9} \ , \ b = [\overset{(n+1)\wedge}{d}] \ , \ a = [\overset{n\wedge}{d}] \ . \end{aligned}$$

This expression is good for all positive integer values of n.

The propagation and decay induction equations for nth order waves are completely analogous to those developed for

---

[9] $\mathcal{L}_\lambda$, $\mathcal{K}$, $\mathcal{I}$ are different quantities than those developed in part A.

the more general unconstrained rod. This is easy to understand when we realize that our linear constrained theory is really a specialization of the exact linear theory for unconstrained rods. All of the properties shown to hold for the more general theory obviously can be carried over to the constrained theory. The only major difference is that the inertia tensor $\underset{\sim}{K}$ in the constrained theory has a slightly different form, and that all mappings are in a six-dimensional space instead of a nine-dimensional one.

We will now record some of the more relevant formulas. Letting λ and $\underset{\sim}{a}$ be replaced by λ_i and ${}_i\underset{\sim}{a}$ respectively, where ${}_i\underset{\sim}{a} \in \mathcal{Y}_i$ ($\mathcal{Y}_i$ is the ith eigenspace of $\mathcal{Y}$), we find

$$\underset{\sim}{P}_i({}_i\underset{\sim}{c} + \tfrac{1}{2}(\underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}} + \underset{\sim}{K}^{-1}\underset{\sim}{S}/\lambda_i)\,{}_i\underset{\sim}{c} = \underset{\sim}{0} \ , \quad {}_i\underset{\sim}{c} = \lambda_i^{\frac{2n+1}{4}}\,{}_i\underset{\sim}{a} \ . \qquad (5.37)$$

$\underset{\sim}{P}_i$ is the orthogonal projection operator defined in the $\mathcal{Y}$ space.

Assuming that two solutions ${}_i\underset{\sim}{c}^p$ and ${}_i\underset{\sim}{c}^q$ exist for (5.37) we can show

$$\{{}_i\underset{\sim}{c}^p \ , \ {}_i\underset{\sim}{c}^q\}_{\underset{\sim}{K}} = e \quad , \qquad (5.38)$$

or alternatively

$$\{{}_i\underset{\sim}{a}^p \ , \ {}_i\underset{\sim}{a}^q\}_{\underset{\sim}{K}} = e\lambda_i^{-\left(\frac{2n+1}{2}\right)} \ . \qquad (5.39)$$

If ${}_i\underset{\sim}{a}^p$ and ${}_i\underset{\sim}{a}^q$ are made orthogonal with respect to $\underset{\sim}{K}$ at one point they are so for all points s.

CHAPTER 6

FORM OF THE SOLUTION TO THE WAVE EQUATIONS

This chapter develops solutions for the propagation and the decay part of the decay-induction equation. Solutions are found assuming multiple eigenvalues (wave speeds) occur. The case of distinct eigenvalues is a special case of this problem. A method of uncoupling the system of vector differential equations which constitute the decay part of the decay-induction equation is then developed for the multiple eigenvalue problem.

We have already seen that if the eigenvalues of (5.13) are distinct the eigenspaces are of dimension one. The direction of the wave mode vector is therefore completely determined. The solution of (5.23) gives the change in magnitude of the wave mode vector subject to the initial conditions. For this problem we find it convenient to use the eigenvectors at s normalized in a specific manner as a basis in $\mathcal{U}$. The solution of (5.23) for all nine subspaces can be viewed as a pure dilatation of the basis vectors in each $\mathcal{U}_i$.

This leads us to a convenient definition for the multiple eigenvalue case: an uncoupled system of vector differential equations is one in which the solution allows only a pure dilatation of the basis vectors in $\mathcal{U}_i$.

For multiple eigenvalues λ_i there is no natural set of basis vectors in $\mathcal{U}_i$. We may therefore choose any linearly independent set which spans $\mathcal{U}_i$. For convenience a basis which is everywhere orthogonal with respect to $\underset{\sim}{K}$ is chosen. Solving (5.23) for this case we will view the results as a deformation and a rotation of our basis. Clearly the vector differential equations are not uncoupled in U_i with respect to an arbitrary basis in U_i.

A method of finding a basis in U_i which undergoes pure dilatation is desired. This method will lead to an uncoupled system of vector differential equations of the same form as the distinct eigenvalue problem.

We begin our study by considering the solution to the system of governing equations (5.19) and (5.23). For convenience we record the equations here as

$$\begin{aligned} &\underset{\sim}{L}_{\lambda_i}\ {}_i\underset{\sim}{a} = \underset{\sim}{0} \ , \\ &\underset{\sim}{P}_i\left({}_i\hat{\underset{\sim}{c}} + \tfrac{1}{2}(\underset{\sim}{K}^{-1}\hat{\underset{\sim}{K}} - \underset{\sim}{K}^{-1}\underset{\sim}{B}/\lambda_i)\,{}_i\underset{\sim}{c}\right) = \underset{\sim}{0} \ , \qquad (6.1) \\ &{}_i\underset{\sim}{c} = \lambda_i^{\frac{2n+1}{4}}\,{}_i\underset{\sim}{a} \ . \end{aligned}$$

It is to be recalled that these equations are subject to initial conditions at $s=s_o$, which are orthogonal with respect to $\underset{\sim}{K}(s_o)$. The quantity $\underset{\sim}{K}_o$ is defined as $\underset{\sim}{K}(s_o)$.

A linear transformation is now defined over the eigenspace. This transformation will take specific vectors from the eigenspace at $s=s_o$ into vectors in the eigenspace at arbitrary s. The specified vectors upon which this transformation will operate are the known initial values of ${}_i\underset{\sim}{a}$. We define these by

$$ {}_{i}\underset{\sim}{a}(S_o) = {}_{io}\underset{\sim}{a} \; , \; {}_{io}\underset{\sim}{a} \in U_{io} \; , \; U_{io} = U_i(S) \quad . \tag{6.2}$$

The mappings are now defined for all U_i by

$$ {}_{i}\underset{\sim}{G} = \left({}_{i}\underset{\sim}{\bar{a}} \in U_i \,\middle|\, {}_{i}\underset{\sim}{\bar{a}} = {}_{i}\underset{\sim}{G}\; {}_{io}\underset{\sim}{a} \text{ for } {}_{io}\underset{\sim}{a} \in U_{io} \right) \quad . \tag{6.3}$$

The bases used to define the mapping are eigenvectors in U_{io} and U_i, since all vectors in these spaces are by necessity eigenvectors.

The direct sum of U_i for all eigenspaces U_i defines a direct sum mapping in U. This process will be applied to ${}_{i}\underset{\sim}{\bar{a}}$ to give

$$ \underset{\sim}{e} = \oplus \; {}_{i}\underset{\sim}{\bar{a}} \; , \; \underset{\sim}{e} \in U \; , \; {}_{i}\underset{\sim}{\bar{a}} \in U_i \quad . \tag{6.4}$$

Using (6.3) which necessitates the use of specific bases we find

$$ \begin{aligned} &\underset{\sim}{e} = \underset{\sim}{G}\,\underset{\sim}{a}_o = \oplus\; {}_{i}\underset{\sim}{G}\; {}_{io}\underset{\sim}{a} \; , \; \underset{\sim}{P}_i\, \underset{\sim}{G}\, \underset{\sim}{a}_o = {}_{i}\underset{\sim}{G}\; {}_{io}\underset{\sim}{a} \; , \\ &\underset{\sim}{P}_{io}\, \underset{\sim}{a}_o = {}_{io}\underset{\sim}{a} \; , \; \underset{\sim}{G} : U \to U \quad . \end{aligned} \tag{6.5}$$

where $\underset{\sim}{P}_{io}$ represents the projection operator on the eigenspace U_{io}. Thus $\underset{\sim}{G}$ is an operator which can be decomposed into direct sums of the mappings ${}_{i}\underset{\sim}{G}$.

If we use an arbitrary basis in U the matrix of the operator $\underset{\sim}{G}$ can no longer be constructed as a block diagonal matrix, where the blocks were of the size of the dimension of the subspace U_i.

We now put some restriction on the mapping ${}_{i}\underset{\sim}{G}$. These are

$$\{{}_{i}\bar{\underset{\sim}{a}},{}_{j}\bar{\underset{\sim}{a}}\}_{\underset{\sim}{K}} = \{{}_{io}\underset{\sim}{a},{}_{jo}\underset{\sim}{a}\}_{\underset{\sim}{K}_o} = 0\ ,\ i \neq j\ ,$$
$$\{{}_{i}\bar{\underset{\sim}{a}}^p,{}_{i}\bar{\underset{\sim}{a}}^q\}_{\underset{\sim}{K}} = \{{}_{io}\underset{\sim}{a}^p,{}_{io}\underset{\sim}{a}^q\}_{\underset{\sim}{K}_o}\ . \tag{6.6}$$

where ${}_{io}\underset{\sim}{a}^p$ and ${}_{io}\underset{\sim}{a}^q$ represent two possible wave mode vectors in U_{io}. They give rise to two vectors in U_i.

Stipulation $(6.6)_1$ satisfies the requirement that the various eigenspaces remain orthogonal to one another with respect to $\underset{\sim}{K}$. Condition $(6.6)_2$ simply states:

1) if ${}_{io}\underset{\sim}{a}^p$ and ${}_{io}\underset{\sim}{a}^q$ are orthogonal with respect to $\underset{\sim}{K}_o$ then ${}_{i}\bar{\underset{\sim}{a}}^p$ and ${}_{i}\bar{\underset{\sim}{a}}^q$ are orthogonal with respect to $\underset{\sim}{K}$.
2) the magnitude of ${}_{io}\underset{\sim}{a}^p$ does not change due to the transformation $\underset{\sim}{G}$ and the change in metric.

In looking at $\underset{\sim}{G}$ we must now distinguish between single and multiple eigenvalues. For single eigenvalues $\underset{\sim}{G}$ is uniquely defined by (6.6). For multiple eigenvalues λ_i, ${}_{i}\underset{\sim}{G}$ hence $\underset{\sim}{G}$ is not uniquely specified. It will be assumed that we have picked a fixed basis in U_{io} and U_i, where one of the basis vectors in each space are multiples of ${}_{io}\underset{\sim}{a}^p$ and ${}_{i}\bar{\underset{\sim}{a}}^p$ respectively. The tensor ${}_{i}\underset{\sim}{G}$ is now fully defined and satisfies $(6.6)_2$. The relationships (6.6) may be expressed in terms of $\underset{\sim}{G}$. From (6.5) it can be seen that

$$\underset{\sim}{G}\ {}_{io}\underset{\sim}{a} = {}_{i}\underset{\sim}{G}\ {}_{io}\underset{\sim}{a}\ . \tag{6.7}$$

Equations (6.6) may be rewritten as

$$\{\underset{\sim}{G}\ {}_{io}\underset{\sim}{a}^p\ ,\ \underset{\sim}{G}\ {}_{io}\underset{\sim}{a}^q\}_{\underset{\sim}{K}} = \{{}_{io}\underset{\sim}{a}^p\ ,\ {}_{io}\underset{\sim}{a}^q\}_{\underset{\sim}{K}_o}\ . \tag{6.8}$$

From this relationship we find

$$\tilde{G}^{T} \tilde{K} \tilde{G} = \tilde{K}_{o} \quad . \tag{6.9}$$

Upon using (5.16) we find

$$\tilde{G}^{\bar{T}} \tilde{G} = \tilde{K}^{-1} \tilde{K}_{o} \quad . \tag{6.10}$$

The polar decomposition of a transformation $\tilde{G}$ with respect to $\tilde{K}$ is now given without proof as

$$\tilde{G} = \tilde{0}^{K} \tilde{U} \ , \ \tilde{U}^{2} = \tilde{G}^{\bar{T}} \tilde{G} \ , \ \tilde{0}^{K} \tilde{0}^{K^{\bar{T}}} = \tilde{I} \quad . \tag{6.11}$$

This theorem simply states that an invertible tensor $\tilde{G}$ can be broken down into an orthogonal tensor with respect to $\tilde{K}$ and a tensor which is symmetric and positive definite with respect to K. These tensors are represented by $\tilde{0}^{K}$ and $\tilde{U}$ respectively. Physically $\tilde{0}^{K}$ and $\tilde{U}$ represent a rotation and a deformation of the vectors being operated on. Substituting (6.10) into $(6.11)_2$ gives the following relationship for the deformation tensor squared

$$\tilde{U}^{2} = \tilde{K}^{-1} \tilde{K}_{o} \quad . \tag{6.12}$$

This concludes our development of the tensor $\tilde{G}$. The formulation of $\tilde{G}$ corresponds to a solution of the eigenvalue problem $(6.1)_1$ in U_i in terms of the initialized wave mode vectors in U_{io}. This solution as has been pointed out is not unique for the multiple eigenvalue case.

We now attempt to set up tensor operators which will satisfy $(6.1)_2$ and hence be solutions to the wave decay problem. Before proceeding with that we will evaluate the constant e in (5.30) by considering conditions at $s=s_o$. Then (5.30) can be expressed as

$$\{{}_i\underset{\sim}{a}^p, {}_i\underset{\sim}{a}^q\}_{\underset{\sim}{K}} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{2}} \{{}_{io}\underset{\sim}{a}^p \; {}_{io}\underset{\sim}{a}^q\}_{\underset{\sim}{K}_o}{}^{10} \; ,$$

$$\lambda_{io} = \lambda_i(s_o) \; , \; {}_{io}\underset{\sim}{a}^p = {}_i\underset{\sim}{a}^p(s_o) \; . \qquad (6.13)$$

We will view the solution of the system (6.1) as an operator which takes the initialized wave mode vector at $s=s_o$ into the wave mode vector at arbitrary s. Clearly this may be viewed as a linear transformation from $\mathcal{U}_{io}$ to $\mathcal{U}_i$. This transformation will be broken up in the following way. We will consider the operator $\underset{\sim}{G}$ to bring the initialized wave mode vector into ${}_i\underset{\sim}{\bar{a}}$. The solution of the differential equation will take ${}_i\underset{\sim}{\bar{a}}$ into the wave mode vector at s. Therefore the solution to $(6.1)_2$ can be expressed as

$$ {}_i\underset{\sim}{a} = {}_i\underset{\sim}{J}\; {}_i\underset{\sim}{\bar{a}} \; , \; {}_i\underset{\sim}{J} : \mathcal{U}_i \to \mathcal{U}_i \; . \qquad (6.14)$$

The bases used to define this mapping are eigenvectors. We may now define a direct sum mapping in $\mathcal{U}$ by

$$\underset{\sim}{g} = \underset{\sim}{J}\,\underset{\sim}{e} = \oplus \; {}_i\underset{\sim}{J}\; {}_i\underset{\sim}{\bar{a}} \; , \; \underset{\sim}{P}_i \; \underset{\approx}{g} = {}_i\underset{\sim}{a} \; ,$$

$$\underset{\sim}{J} : \mathcal{U} \to \mathcal{U} \; , \; \underset{\approx}{g} \in \mathcal{U} \; . \qquad (6.15)$$

---

[10] The summation convention will not apply between terms in the numerator and the denominator.

If an arbitrary basis in U is used the matrix of the operator $\underset{\sim}{J}$ can no longer be constructed as a block diagonal matrix, where the blocks were of the size of the dimension of the subspaces U_i.

The form of ${}_i\underset{\sim}{J}$ as defined in (6.14) is desired. Using $(6.6)_2$ and (6.14) equation $(6.13)_1$ becomes

$$\{{}_i\underset{\sim}{J}\;{}_i\underset{\sim}{\bar{a}}^p, {}_i\underset{\sim}{J}\;{}_i\underset{\sim}{\bar{a}}^q\}_{\underset{\sim}{K}} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{2}} \{{}_i\underset{\sim}{\bar{a}}^p, {}_i\underset{\sim}{\bar{a}}^q\}_{\underset{\sim}{K}} \quad . \tag{6.16}$$

From this we may conclude

$${}_i\underset{\sim}{J}\;{}_i\underset{\sim}{J}^{\bar{T}} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{2}} {}_i\underset{\sim}{I} \;,\; {}_i\underset{\sim}{I} = \underset{\sim}{P}_i\,\underset{\sim}{I} \quad . \tag{6.17}$$

By means of the polar decomposition with respect to $\underset{\sim}{K}$ we find

$${}_i\underset{\sim}{J} = {}_i\underset{\sim}{O}^K\;{}_i\underset{\sim}{U} \;,\; {}_i\underset{\sim}{U} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{4}} {}_i\underset{\sim}{I} \;,\; {}_i\underset{\sim}{O}^K\;{}_i\underset{\sim}{O}^{K^{\bar{T}}} = {}_i\underset{\sim}{I} \;. \tag{6.18}$$

${}_i\underset{\sim}{O}^K$ is seen to be an orthogonal transformation with respect to $\underset{\sim}{K}$. The form of the deformation tensor ${}_i\underset{\sim}{U}$ is significant. Since it is in diagonal form the tensor allows pure dilatation with respect to the $\underset{\sim}{K}$ inner product only. Shearing deformation with respect to $\underset{\sim}{K}$, which would be represented by symmetric off diagonal terms, is nonexistent. The solution to the differential system $(6.1)_{2,3}$ can be written using (6.3), (6.14) and (6.18) as

$${}_i\underset{\sim}{a} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{4}} {}_i\underset{\sim}{O}^K\;{}_i\underset{\sim}{\bar{a}} \;,\; {}_i\underset{\sim}{\bar{a}} = {}_i\underset{\sim}{G}\;{}_{io}\underset{\sim}{a} \quad . \tag{6.19}$$

Since the eigenvalues and the term ${}_{i}\underset{\sim}{\bar{a}}$ are solutions to the eigenvalue problem $(6.1)_1$ it is clear that the solution of $(6.1)_{2,3}$ gives the mapping ${}_{i}\underset{\sim}{O}^{K}$.

For a specific ${}_{i}\underset{\sim}{G}$ the component form of the operator ${}_{i}\underset{\sim}{O}^{K}$ will now be determined. First we find it necessary to define the tensors in equation $(6.1)_{2,3}$ and the operator ${}_{i}\underset{\sim}{O}^{K}$ in terms of a basis. This basis is a set of eigenvectors made everywhere orthogonal with respect to $\underset{\sim}{K}$. The basis vectors represented by $\overset{i}{\underset{\sim}{W}}_m$ and $\overset{\perp}{\underset{\sim}{W}}_m$ are normalized such that

$$\{\overset{i}{\underset{\sim}{W}}_m,\overset{i}{\underset{\sim}{W}}_n\}_{\underset{\sim}{K}} = \delta_{mn} \;,\; \{\overset{i}{\underset{\sim}{W}}_m,\overset{\perp}{\underset{\sim}{W}}_n\}_{\underset{\sim}{K}} = 0 \;,\; \{\overset{\perp}{\underset{\sim}{W}}_m,\overset{\perp}{\underset{\sim}{W}}_n\}_{\underset{\sim}{K}} = \delta_{mn} \quad . \qquad (6.20)$$

Basis vectors of the form $\overset{i}{\underset{\sim}{W}}_m$ are contained in U_i while basis vectors of the form $\overset{\perp}{\underset{\sim}{W}}_n$ are contained in $U_i^{\perp}$. It is apparent that a vector $\underset{\sim}{d}$ can be expressed in terms of this basis by

$$\underset{\sim}{d} = d^m\, \overset{i}{\underset{\sim}{W}}_m + d^n\, \overset{\perp}{\underset{\sim}{W}}_n = d^J\, \underset{\sim}{W}_J \;,\; \underset{\sim}{d} \in U \quad . \qquad (6.21)$$

The small latin indices represent a summation over the subspaces U_i or $U_i^{\perp}$. The capital latin indices represent a summation over U.

We will now modify some of the above statements to develop a convention. Small latin indices will henceforward represent summations over U_i alone. There will therefore be no need for superposed quantities i in defining the basis in U_i. The capital latin indices will always represent a summation over U.

A dual basis $\underset{\sim}{W}^m$ is established such that

$$\{\underset{\sim}{W}^M, \underset{\sim}{W}_J\} = \delta^M_J \quad . \tag{6.22}$$

The projection operator can be written in terms of the dual basis as

$$\underset{\sim}{P}_i \, \underset{\sim}{d} = \underset{\sim}{W}_m \{\underset{\sim}{W}^m, \underset{\sim}{d}\} \ , \ \underset{\sim}{d} \in U \quad . \tag{6.23}$$

The derivatives of the base vectors $\underset{\sim}{W}_M$ can be expressed in terms of the basis $\underset{\sim}{W}_M$ by

$$\hat{\underset{\sim}{W}}_M = \Gamma^{\cdot K}_{M} \, \underset{\sim}{W}_K \quad , \tag{6.24}$$

where $\Gamma^{\cdot K}_{M}$ are called the connection coefficients.

Furthermore we make the following definitions

$$\lambda_i \, \underset{\sim}{B} = \underset{\sim}{C} \ , \ \Gamma^{\cdot M}_{N} \, \delta_{MJ} \, \delta^{NP} = \Gamma^{P}_{\cdot J} \quad . \tag{6.25}$$

By differentiating (6.20) and using $(6.25)_2$ we can show that

$$K^{-1P}_{\ \ N} \, \hat{K}^N_J = -(\Gamma^P_{\cdot J} + \Gamma^{\cdot P}_{J}) \quad . \tag{6.26}$$

The initial value of the wave mode vector in U_i will be expressed in terms of the basis $\underset{\sim}{W}_m$ evaluated at $s=s_o$. This is

$${}_{io}\underset{\sim}{a} = {}_{io}a^{(m)} \, {}_{o}\underset{\sim}{W}_{(m)} \ , \ {}_{o}\underset{\sim}{W}_m = \underset{\sim}{W}(s_o) \quad , \tag{6.27}$$

where brackets about an index indicate no summation occurs. The initial values of the vectors $\underset{\sim}{W}_m$ by (6.20) are clearly orthogonal with respect to $\underset{\sim}{K}_o$. Applying the operator ${}_i\underset{\sim}{G}$ and the results (6.3) and (6.6) to (6.27) we have

$${}_i\bar{\underset{\sim}{a}} = {}_{io}a^{(m)} \underset{\sim}{W}_{(m)} \ ; \ \underset{\sim}{W}_m = {}_i\underset{\sim}{G} \, {}_{o}\underset{\sim}{W}_m \quad . \tag{6.28}$$

We have made ${}_{i}\underset{\sim}{\bar{a}}$ a multiple of one of the base vectors in $\mathcal{U}_i$. The tensor ${}_{i}\underset{\sim}{\mathcal{O}}^{K}$ can be written in component form with respect to a specific set of bases as

$$ {}_{i}\underset{\sim}{\mathcal{O}}^{K} = \mathcal{Q}^{m}_{n}\, \underset{\sim}{\mathcal{W}}_{m} \otimes \underset{\sim}{\mathcal{W}}^{n} \quad . \tag{6.29} $$

Substituting $(6.19)_1$ into $(6.1)_{2,3}$ and using the relationships (6.20) through (6.29) we find the component equation for the mapping ${}_{i}\underset{\sim}{\mathcal{O}}^{K}$ given by

$$ \{\hat{\mathcal{Q}}^{m}_{(d)} + \frac{1}{2}(\Gamma_{n}^{\cdot m} - \Gamma^{m}_{\cdot n} - K^{-1m}_{\;\;J}\, C^{J}_{n})\mathcal{Q}^{n}_{(d)}\} = 0 \quad . \tag{6.30} $$

It is apparent from (6.18) that at $s=s_o$ the mappings $\underset{\sim}{G}$ and ${}_{i}\underset{\sim}{\mathcal{O}}^{K}$ must be identity maps over the appropriate spaces. Therefore in component form the initial conditions on $\mathcal{Q}^{m}_{d}$ are

$$ \mathcal{Q}^{m}_{d}(s_o) = \delta^{m}_{d} \quad . \tag{6.31} $$

An equation of the form (6.30) is known as a matrix differential equation. For ease of analysis all of the quantities in (6.30) are assumed to be continuous functions of s.

Instead of solving equation (6.30) for the unknown matrix $\mathcal{Q}^{m}_{d}$ directly we shall attempt to do so by another method. Writing ${}_{i}\underset{\sim}{c}$ of $(6.1)_2$ in terms of the basis developed in (6.20) gives

$$ {}_{i}\underset{\sim}{c} = c^{m}\, \underset{\sim}{\mathcal{W}}_{m} \quad . \tag{6.32} $$

Substitution of (6.32) into $(6.1)_2$ and utilizing (6.20) through (6.26) gives

$$\{\hat{c}^m + \tfrac{1}{2}(\Gamma^{\cdot m}_{n} - \Gamma^{m}_{\cdot n} - K^{-1m}_{J}\, C^J_n)\, c^n\} = 0 \quad , \qquad (6.33)$$

subject to the known initial conditions

$$c^m(s_o) = {}_{io}c^m \quad . \qquad (6.34)$$

Equation $(6.32)_1$ is known as a vector-matrix differential equation.

Comparing the equations (6.30) and (6.33) the terms inside the curly brackets have much in common. The known quantities in the ( ) brackets are in each case identical. The only difference between the two problems is that in (6.30) the matrix Q^m_d is unknown while in (6.33) the vector c^m is unknown. Barrett and Bradley [10] in chapter six, section nine discuss the relationship between the solutions to these two problems. They conclude that if c^m is any solution to (6.33) then there is a constant vector g^n such that

$$c^m = Q^m_n\, g^n \quad , \qquad (6.35)$$

where Q^m_n satisfies (6.30) and the boundary condition (6.31). If we let $s=s_o$ in (6.35) and use (6.31) and (6.34) we see

$$g^n = {}_{io}c^n \quad . \qquad (6.36)$$

On substitution of (6.36) into (6.35) we get

$$c^m = Q^m_n\, {}_{io}c^n \quad . \qquad (6.37)$$

In solving (6.33) subject to (6.34) we find a solution set of p linearly independent vectors c^m, where p is the dimension of

$\mathcal{U}_i$. Arranging these vectors as columns constructs the matrix $\underset{.}{Q}^m_n$ of (6.37).

A means has been established to determine the tensor ${}_i\underset{\sim}{O}^K$ with respect to a specific basis. Obviously the mapping may now be expressed in terms of any basis.

For distinct eigenvalues the eigenspaces $\mathcal{U}_i$ are of dimension one. The only permissable values of ${}_i\underset{\sim}{O}^K$ for these eigenspaces are $\pm{}_i\underset{\sim}{I}$ by $(6.18)_3$ and (5.16). Clearly if distinct eigenvalues occur the governing differential equation $(6.1)_2$ is uncoupled and it is unnecessary to solve $(6.1)_2$ for ${}_i\underset{\sim}{O}^K$.

For the case of multiple eigenvalues the equations $(6.1)_2$ are in general coupled. We write the solution of the system (6.1) in terms of a basis in $\mathcal{U}_i$. This is done by substituting $(6.28)_1$ into $(6.19)_1$ giving

$$ {}_i\underset{\sim}{a} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{4}} {}_i\underset{\sim}{O}^K \; {}_{io}a^{(m)} \underset{\sim m}{w} \quad . \tag{6.38} $$

To uncouple the system of differential equations $(6.1)_2$ we must find a basis in $\mathcal{U}_i$ such that the wave mode vectors are only dilatations of them. This is equivalent to finding a mapping ${}_i\underset{\sim}{G}$ in $(6.28)_2$, which defines our basis, such that ${}_i\underset{\sim}{O}^K = {}_i\underset{\sim}{I}$ in (6.38). Defining a new basis by

$$ \underset{\sim m}{\bar{w}} = {}_i\underset{\sim}{O}^K \underset{\sim m}{w} \quad , \tag{6.39} $$

we can rewrite (6.38), $(6.28)_2$ and (6.39) as

$$
{}_i\underset{\sim}{a} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{4}} {}_{io}a^{(m)}\underset{\sim}{\bar{W}}_{(m)}, \quad \underset{\sim}{\bar{W}}_m = {}_i\underset{\sim}{\bar{G}}\; {}_o\underset{\sim}{W}_m, \quad {}_i\underset{\sim}{\bar{G}} = {}_i\underset{\sim}{O}^K\underset{\sim}{G} \quad . \qquad (6.40)
$$

The mapping $\underset{\sim}{\bar{G}}$, which is the appropriate one to uncouple the differential equations, is given by

$$
\underset{\sim}{\bar{G}} = \oplus \; {}_i\underset{\sim}{\bar{G}} \quad . \qquad (6.41)
$$

We now summarize the method of finding the uncoupling mapping $\underset{\sim}{\bar{G}}$. Select arbitrary ${}_i\underset{\sim}{G}$ in eigenspaces of dimension greater than one. For this ${}_i\underset{\sim}{G}$ determine the corresponding ${}_i\underset{\sim}{O}^K$ in (6.38) by solving $(6.1)_2$. The uncoupling basis is then found by an application of $(6.40)_2$ and (6.41).

This chapter has dealt exclusively with the unconstrained theory. All of the results are essentially the same for our constrained theory. Only the range of indices, dimension of the total space, and the form of $\underset{\sim}{B}$ in $(6.1)_2$ change.

CHAPTER 7

SPECIAL CASES

This chapter deals with simplifications that occur to the solution of the wave propagation problem developed in chapter six. The simplifications are caused by assumptions regarding the geometry or the material properties of the rod. For convenience we summarize the solution of the wave propagation problem. It is

$$
\begin{aligned}
{}_i\underset{\sim}{a} &= {}_i\underset{\sim}{J}\;{}_i\underset{\sim}{\bar{a}} \;, \quad {}_i\underset{\sim}{\bar{a}} = \underset{\sim}{G}\;{}_{io}\underset{\sim}{a} \;, \quad \underset{\sim}{G} = \underset{\sim}{O}^K \underset{\sim}{U} \;, \\
{}_i\underset{\sim}{J} &= {}_i\underset{\sim}{O}^K\;{}_i\underset{\sim}{U} \;, \quad {}_i\underset{\sim}{U} = \left(\frac{\lambda_{io}}{\lambda_i}\right)^{\frac{2n+1}{4}} {}_i\underset{\sim}{I} \;.
\end{aligned}
\tag{7.1}
$$

Various classes of rods will now be studied.

1) Uniform Rods

A uniform rod must satisfy both geometric as well as material properties. The motion of a uniform rod was introduced by Ericksen [11]. The method of presenting the definition of uniform rods and the results closely resemble that of Cohen [4].

The geometric requirements are

$$
\begin{aligned}
&\underset{\sim}{p} = \underset{\sim}{Q}\,\underset{\sim}{p}_o + \underset{\sim}{t}\,s \;, \quad \underset{\sim}{\hat{p}} = \underset{\sim}{Q}\,\underset{\sim}{\hat{p}}_o \;, \\
&\underset{\sim}{Q} = \underset{\sim}{O} \oplus \underset{\sim}{O} \oplus \underset{\sim}{O} \;, \quad \underset{\sim}{O}(s_o) = \underset{\sim}{I} \;, \quad \underset{\sim}{p}_o = \underset{\sim}{p}(s_o) \;.
\end{aligned}
\tag{7.2}
$$

$\underset{\sim}{O}$ is a proper orthogonal tensor operating on the three dimensional component translation spaces. It satisfies

$$
\underset{\sim}{\hat{O}}\,\underset{\sim}{O}^T = \underset{\sim}{W}_o = \underset{\sim}{O}\,\underset{\sim}{W}_o\,\underset{\sim}{O}^T \;, \tag{7.3}
$$

where $\underset{\sim}{W}_o$ is a fixed antisymmetric tensor. The vector $\underset{\sim}{t}$ in $(7.2)_1$ is defined by

$$\underset{\sim}{t} = (h\,\underset{\sim}{y},\ \underset{\sim}{0},\underset{\sim}{0})\ ,\ \underset{\sim}{W}_o\,\underset{\sim}{y} = \underset{\sim}{0}\ , \tag{7.4}$$

where h is an arbitrary constant, $\underset{\sim}{y}$ is the constant axial vector of $\underset{\sim}{W}_o$. The mathematical requirements (7.2) cause the geometry of the rod to be restricted to (i) straight rods with uniformly twisted cross-sections, (ii) circular rods with untwisted cross-sections, (iii) circular helical rods with cross-sectional twist equal to the torsion of the helix.

The material properties a uniform rod obeys are: 1) the strain energy function is independent of s, 2) $\underset{\sim}{K}$ is a uniform tensor. Mathematically these two conditions are given by

$$\mathcal{W}(\underset{\sim}{p},\hat{\underset{\sim}{p}},M_o;s) = \mathcal{W}(\underset{\sim}{p}_o,\hat{\underset{\sim}{p}}_o,M_o;S_o)\ ,\ \underset{\sim}{K} = \underset{\sim}{Q}\,\underset{\sim}{K}_o\,\underset{\sim}{Q}^T\ . \tag{7.5}$$

$\underset{\sim}{K}$ must satisfy the invariance requirement (2.43) as well as $(7.5)_2$ resulting in

$$\underset{\sim}{K} = \underset{\sim}{K}_o\ . \tag{7.6}$$

An application of the chain rule to $(7.4)_1$ and recognizing the definition of $\underset{\sim}{A}$ $(2.45)_5$ results in

$$\underset{\sim}{A} = \underset{\sim}{Q}\,\underset{\sim}{A}_o\,\underset{\sim}{Q}^T\ ,\ \underset{\sim}{A}_o = \underset{\sim}{A}(s_o)\ . \tag{7.7}$$

Utilizing $(7.5)_2$ and $(7.7)_1$ the eigenvalue problem$(6.1)_1$ gives

$$\lambda_i = \lambda_{io}\ ,\ \mathcal{U}_i = \underset{\sim}{Q}\,\mathcal{U}_{io}\ . \tag{7.8}$$

The tensor $\underset{\sim}{Q}$ is clearly orthogonal with respect to the ordinary inner product. Calculating the quantity $\underset{\sim}{Q}\,\underset{\sim}{Q}^{\bar{T}}$ and using (5.16) and (2.43), since $\underset{\sim}{Q}$ and $\underset{\sim}{O}$ are of the same form, gives

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^{\bar{T}} = \underset{\sim}{I} \quad . \tag{7.9}$$

$\underset{\sim}{Q}$ is therefore orthogonal with respect to the $\underset{\sim}{K}$ inner product. Using (7.6) and (7.8) we may rewrite (7.1) for uniform rods as

$${}_{i}\underset{\sim}{a} = {}_{i}\underset{\sim}{O}^{K}\,{}_{i}\underset{\sim}{\bar{a}} \ , \ {}_{i}\underset{\sim}{\bar{a}} = \underset{\sim}{Q}\,{}_{io}\underset{\sim}{a} \ , \ {}_{i}\underset{\sim}{U} = {}_{i}\underset{\sim}{I} \ , \ \underset{\sim}{U} = \underset{\sim}{I} \quad . \tag{7.10}$$

The solution (7.10) shows that no deformation occurs to the initial wave mode vectors as the metric does not change and the eigenvalues are independent of s. This solution allows only a rotation of the wave mode vectors with respect to the $\underset{\sim}{K}$ inner product.

The direct sum mapping of ${}_{i}\underset{\sim}{a}$ for all U_i was defined in (6.4) Applying this to $(7.10)_1$ we have

$$\underset{\sim}{e} = \oplus\,{}_{i}\underset{\sim}{O}^{K}\,{}_{i}\underset{\sim}{\bar{a}} \quad . \tag{7.11}$$

The results (7.11) correspond to those determined by Cohen [4] in his equations (3.21) and (3.22).

2) $\underset{\sim}{K}$ Diagonal with Respect to the Eigenvectors

This case is defined by a specific material property. The tensor $\underset{\sim}{K}$ is diagonal with respect to the eigenvectors for all s. The various wave mode vectors, which were set up to be orthogonal with respect to the $\underset{\sim}{K}$ inner product, are now orthogonal to the ordinary inner product for all s. This is determined by an expansion of (6.20) with $\underset{\sim}{K}$ in the specified form. The tensors $\underset{\sim}{O}^K$ and ${}_i\underset{\sim}{O}^K$ which are orthogonal with respect to the $\underset{\sim}{K}$ inner product are not in general orthogonal with respect to the ordinary inner product.

3) $\underset{\sim}{K}$ in a Special Diagonal Form

If $\underset{\sim}{K}$ is diagonal with respect to the eigenvectors in the following special way

$$\underset{\sim}{K} = \rho(s)\, \underset{\sim}{I} \quad , \tag{7.12}$$

then the results of section two are still applicable. In addition because of (5.16) and (7.12) we see that

$$\underset{\sim}{R}^{\bar{T}} = \underset{\sim}{R}^{T} \quad , \tag{7.13}$$

for arbitrary tensor $\underset{\sim}{R}$. The transpose with respect to both inner products is identical. The roation tensors with respect to the $\underset{\sim}{K}$ inner product in the U_i and U spaces are now orthogonal with respect to both the inner products.

4) $\underset{\sim}{K} = \underset{\sim}{K}_o$

For this case we see from (6.12) that

$$\underset{\sim}{U} = \underset{\sim}{I} \quad . \tag{7.14}$$

The operator $\underset{\sim}{G}$ given by $(7.1)_3$ reduces to an orthogonal transformation with respect to the $\underset{\sim}{K}$ inner product.

5) $\underline{\lambda = \lambda_{io}}$

The operator ${}_i\underset{\sim}{U}$ in $(7.1)_5$ reduces to

$$ {}_i\underset{\sim}{U} = {}_i\underset{\sim}{I} \tag{7.15}$$

Therefore ${}_i\underset{\sim}{J}$ given by $(7.14)_4$ reduces to an orthogonal transformation with respect to the $\underset{\sim}{K}$ inner product. The solution of the differential equation $(6.1)_2$ allows no deformation due to ${}_i\underset{\sim}{J}$. If $\lambda_i = \lambda_{io}$ for all eigenspaces U_i the same reasoning holds for $\underset{\sim}{J}$.

Obviously various combinations of these cases can occur. The rod will then satisfy criterion in all cases in which it belongs.

CHAPTER 8

AN EXAMPLE

An illustration of the foregoing analysis will now be undertaken. For simplicity only the constrained theory of rods will be considered. Constitutive equations of a particularly simple form are selected. This allows explicit determination of some of the desired quantities.

We will assume that at least one basis of eigenvectors normalized in the sense of (6.20) exists such that the following properties hold:[11]

A1: $\underset{\sim}{\mathcal{D}}\,\underset{\sim}{K} = \underset{\sim}{K}\,\underset{\sim}{\mathcal{D}}$,

A2: $\underset{\sim\alpha}{P}\,\underset{\sim i}{\mathcal{W}} = \underset{\sim}{O}$, $\alpha = 1,\ i = 4,5,6$ and $\alpha = 2,\ i = 1,2,3$,

A3: $\underset{\sim 1}{\mathcal{W}}$, $\underset{\sim 2}{\mathcal{W}}$, $\underset{\sim 3}{\mathcal{W}}$ are parallel to $\underset{\sim 4}{\mathcal{W}}$, $\underset{\sim 5}{\mathcal{W}}$, $\underset{\sim 6}{\mathcal{W}}$ respectively ,

A4: $\underset{\sim 3}{\mathcal{W}}$ is parallel to $\underset{\sim}{\hat{R}}$,

A5: $\underset{\sim}{K} = \rho \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{B} \end{bmatrix}$, $\underset{\sim}{I}$, $\underset{\sim}{O} \in \mathcal{V} \otimes \mathcal{V}$,

$$\underset{\sim}{B} = B^m_n\, \underset{\sim m}{\mathcal{W}} \otimes \underset{\sim}{\mathcal{W}}^n \ , \quad (m,n = 4,5,6) \ , \tag{8.1}$$

$$B^m_n = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} ,$$

---

[11] It is to be noted that this basis does not necessarily cause uncoupling of the vector differential equations governing the rod. If the eigenvalues are distinct this is the only allowable basis of eigenvectors and it uncouples the governing vector differential equations.

where d_i $(i = 1,2,3)$ are functions of s.

This example in a slightly modified form was considered by Cohen [7]. Assumptions A1 through A4 are identical to his example. He treated the case of straight untwisted rods with distinct eigenvalues. In our problem the cases of different geometries and possible multiple eigenvalues are considered.

Assumption A1 states that the constitutive tensor $\underset{\sim}{\mathcal{D}}$ and the inertia tensor $\underset{\sim}{K}$ commute. Mathematically this is equivalent to stating that $\underset{\sim}{\mathcal{D}}$ and $\underset{\sim}{K}$ are simultaneously diagonal with respect to the eigenvectors. For this case the eigenvalues given by a solution of $(5.36)_1$ are given by

$$\lambda_i = \mathcal{D}^{(i)}{}_{(i)}/K^{(i)}{}_{(i)} \quad . \tag{8.2}$$

The results of chapter seven section two are applicable because of the diagonal form of $\underset{\sim}{K}$.

For assumption A2 the quantity $\underset{\sim}{P}_\alpha$ $(\alpha=1,2)$ represents projection of a vector into a component translation space associated with

a) linear (extension and shear) effects, $\alpha= 1$;

b) angular (bending and twisting) effects, $\alpha=2$.

Assumptions A1 and A2 taken together indicate that the set of bases eigenvectors $\underset{\sim}{W}_M$ can be considered as being composed of purely linear or angular effects and not a combination of both.[12]

---

[12] For the constrained theory capital Latin indices indicate summation over the space $\mathcal{Y}$ while small Latin indices indicate summation over the space $\mathcal{Y}_i$.

The base vectors $\underset{\sim}{W}_M$ can therefore be considered to be contained in $\mathcal{Y}$ or the appropriate component translation space.

Assumptions A3 and A4 show the rod axis and the cross-section perpendicular to it are the directions associated with the basis of eigenvectors. In addition the directions for linear and angular effects are coincident. We can associate $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$ with pure shearing waves in the cross-section while $\underset{\sim}{W}_3$ is a pure axial extension wave. Moreover $\underset{\sim}{W}_4$ and $\underset{\sim}{W}_5$ are pure bending waves in the cross-section and $\underset{\sim}{W}_6$ is a pure twisting wave. For convenience $\underset{\sim}{W}_1$, $\underset{\sim}{W}_2$, and $\underset{\sim}{W}_3$ will be assumed to form a right handed system.

Assumption A5 gives the specific form of the inertia tensor $\underset{\sim}{K}$. The tensor $\underset{\sim}{B}$ in A5 gives the moments of inertia about the three basis vectors. The special diagonal form of $\underset{\sim}{K}$ shows that these moments of inertia are principle moments of inertia.

We now proceed to an actual determination of the map $\underset{\sim}{G}$. We can always write $\underset{\sim}{G}$ in the following component form

$$\underset{\sim}{G} = \delta^M_N \, \underset{\sim}{W}_M \otimes {}_o\underset{\sim}{W}^N \;, \quad {}_o\underset{\sim}{W}^N = \underset{\sim}{W}^N(s_o) \quad . \tag{8.3}$$

The question remains of how we can get the matrix of $\underset{\sim}{G}$ with respect to a different basis, for example

$$\underset{\sim}{G} = G^M_N \; {}_o\underset{\sim}{W}_M \otimes {}_o\underset{\sim}{W}^N \quad . \tag{8.4}$$

A method for determining the G^M_N's in terms of the Γ's is now found. Firstly by (6.7) and (6.28) we find

$$\underset{\sim}{W}_N = \underset{\sim}{G}\ {}_o\underset{\sim}{W}_N \quad . \tag{8.5}$$

In terms of the basis utilized in (8.4) we have

$$\underset{\sim}{W}_N = G^M_N\ {}_o\underset{\sim}{W}_M \quad . \tag{8.6}$$

Differentiation of (8.6) gives

$$\hat{\underset{\sim}{W}}_N = \hat{G}^M_N\ {}_o\underset{\sim}{W}_M \quad . \tag{8.7}$$

An alternative expression for $\hat{\underset{\sim}{W}}_N$ can be had from the substitution of (8.6) into (6.24) resulting in

$$\hat{\underset{\sim}{W}}_N = \Gamma^{\cdot L}_{N}\ G^M_L\ {}_o\underset{\sim}{W}_M \quad . \tag{8.8}$$

Comparing (8.7) and (8.8) gives a relationship between G^M_N's and the Γ's given by

$$\hat{G}^M_N = \Gamma^{\cdot L}_{N}\ G^M_L \quad . \tag{8.9}$$

The initial conditions for the $\underset{\sim}{G}$ mapping are obviously

$$G^M_N\ (S_o) = \delta^M_N \quad . \tag{8.10}$$

We now establish a method for determining the mapping ${}_i\underset{\sim}{\mathcal{O}}^K$ when multiple eigenvalues occur. For the unconstrained theory equation $(6.1)_2$ lead to the component equation (6.33). The solution of (6.33) subject to (6.34) leads to the solution of the component form of ${}_i\underset{\sim}{\mathcal{O}}^K$ by means of (6.37). For the constrained theory relating $(5.37)_1$ to $(6.1)_2$ we can write the component equation corresponding to (6.33). It is

$$\hat{C}^m + \frac{1}{2}(\Gamma^{\cdot m}_{n} - \Gamma^{m}_{\cdot n} + \lambda^{-1}_i\ K^{-1m}_{J}\ S^J_n)C^n = 0 \quad . \tag{8.11}$$

By the definition of $\underset{\sim}{S}$ $(5.34)_2$ and the projection operator (6.23) we can show with the help of $(5.36)_1$ that

$$\lambda_i^{-1} K^{-1m}_{\ \ J} S^J_n C^n = (X^m_n - X^{\bar{T}m}_{\ \ n})C^n \quad . \tag{8.12}$$

Substitution of (8.12) into (8.11) results in

$$\hat{C}^m + \frac{1}{2}(\Gamma^{\cdot m}_{\ \ n} - \Gamma^m_{\ \cdot n} + X^m_n - X^{\bar{T}m}_{\ \ n})C^n = 0 \quad . \tag{8.13}$$

The initial conditions are

$$C^m(s_o) = {}_{io}C^m \quad . \tag{8.14}$$

By a relationship of exactly the same form as (6.37) we can determine the component form of the mapping ${}_i\underset{\sim}{O}^K$ for multiple eigenvalues. The development of the $\underset{\sim}{X}$ tensor will now be undertaken. It will be calculated in the form

$$\underset{\sim}{X} = X^M_N \underset{\sim}{W}_M \otimes \underset{\sim}{W}^N \quad . \tag{8.15}$$

From the definition of $\underset{\sim}{X}$, equation $(3.35)_1$, and $\underset{\sim}{T}$ we find that only two nonzero components exist. These two terms are

$$X^1_5 = -d_2^{-\frac{1}{2}} \ , \ X^2_4 = d_1^{-\frac{1}{2}} \quad . \tag{8.16}$$

We now write the tensor $\underset{\sim}{X}^{\bar{T}}$ as

$$\underset{\sim}{X}^{\bar{T}} = X^{\bar{T}M}_{\ \ N} \underset{\sim}{W}_M \otimes \underset{\sim}{W}^N \quad . \tag{8.17}$$

Upon using relationship (5.15) in component form we find

$$X^{\bar{T}M}_{\ \ N} = \delta_{NL} X^L_J \delta^{JM} \quad . \tag{8.18}$$

Applying (8.18) to (8.16) we find

$$X^{\bar{T}1}_{\ 5} = X^5_1 = -d_2^{-\frac{1}{2}} \ , \ X^{\bar{T}2}_{\ 4} = X^4_2 = d_1^{-\frac{1}{2}} \quad . \tag{8.19}$$

The uncoupling mapping $\bar{\underset{\sim}{G}}$ may be calculated by means of $(6.40)_3$ and (6.41). It can be represented in terms of the various bases as

$$\bar{\underset{\sim}{G}} = \delta^M_N \ \bar{\underset{\sim}{W}}_M \otimes \underset{o\sim}{W}_N = \tilde{\tilde{G}}^M_N \ \underset{\sim}{W}_M \otimes \underset{o\sim}{W}^N = \bar{G}^M_N \ \underset{o\sim}{W}_M \otimes \underset{o\sim}{W}^N \quad . \tag{8.20}$$

By applying (8.6) to (8.18) we can show

$$\bar{G}^M_N = G^M_J \ \tilde{\tilde{G}}^J_N \quad . \tag{8.21}$$

Various geometries of rods will now be examined.

1) Straight Untwisted Rods

Straight untwisted rods are defined by

$$\Gamma^{\cdot M}_{\ N} = 0 \ , \ M \neq N \quad . \tag{8.22}$$

The matrix of $\underset{\sim}{G}$ in the form of (8.4) can be determined by a solution of (8.9) subject to (8.10). This results in

$$G^M_N = b \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_3 \end{bmatrix} , \tag{8.23}$$

$$b = (\rho(S_o)/\rho(S))^{\frac{1}{2}} \ , \ f \ = (d_i(S_o)/d_i(S))^{\frac{1}{2}}, \ i = 1,2,3 \quad .$$

A) Distinct Eigenvalues

For distinct eigenvalues there is only one allowable mapping $\underset{\sim}{G}$. Therefore we have

$$\bar{\underset{\sim}{G}} = \underset{\sim}{G} \quad . \tag{8.24}$$

B) All Eigenvalues the Same

We find from a solution of (8.13) subject to (8.16), (8.19), (8.22), and (8.14) a component form of the tensor ${}_{i}\underset{\sim}{O}^{K}$ through (6.37). Using these results plus (6.29), (8.3), $(6.40)_3$, and (6.41) we find the component form of the mapping $\bar{\underset{\sim}{G}}$ given by (8.20) as

$$\tilde{G}^M_N = \begin{bmatrix} g_2 & 0 & 0 & 0 & h_2 & 0 \\ 0 & g_1 & 0 & -h_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & h_1 & 0 & g_1 & 0 & 0 \\ -h_2 & 0 & 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{8.25}$$

where

$$g_\alpha = \cos m_\alpha,\ h_\alpha = \sin m_\alpha,\ m_\alpha = \frac{1}{2}\int_{s_o}^{s} d_\alpha^{\ -\frac{1}{2}} ds,\ (\alpha=1,2) \ . \tag{8.26}$$

There are some interesting features to be discussed about the matrices of $\underset{\sim}{G}$ and $\bar{\underset{\sim}{G}}$. In our problem $\underset{\sim}{U}$, the deformation due to the change in metric, is not the identity map. The terms b and f_i in $(8.23)_1$ give a measure of the deformation. In (8.25) we see no terms which can be associated with a deformation. Clearly the deformation is masked by the different bases we have used to define the mappings.

The physical significance of the results will now be discussed. The uncoupling basis given in component form by $(8.25)_1$ may be viewed as a rotation in six dimensional space. Actually a simpler view of the matter is possible. It can be seen by observing $(8.25)_1$ that coupling exists between the one-five mode and separately in the two-four mode. Thus the uncoupling basis may be viewed as two separate rotations in two-dimensional subspaces of $\mathcal{V}$. These two-dimensional spaces are not subspaces of a component translation space V. Therefore the uncoupling basis cannot be considered a pure linear or angular wave as our convenient basis $\underset{\sim}{W}_M$ was. Assumption A2 does not apply to the uncoupling basis.

The coupling that occurs in the one-five and two-four modes is not unexpected. The coupling is between a shear wave in one direction and a bending wave in a perpendicular direction. Since in a statical situation a shearing force applied to a straight rod will in general induce a bending vector perpendicular to it these two phenomena are linked. It seems logical therefore that when coupling occurs it will be between these types of modes.

C) Other Possibilities

The case of all eigenvalues being the same represents the problem with the absolutely highest degree of coupling. If this problem can be solved it is relatively easy to solve problems having the same geometry but fewer muliple eigenvalues. This is a general criterion and will not be restricted to just straight rods.

The procedure is as follows: Firstly it will be assumed that the problem of all eigenvalues being equal has been solved.

The matrix $\tilde{\tilde{G}}^M_N$ is therefore known. If fewer eigenvalues are identical eliminate rows and columns of the distinct eigenvalues location in $\tilde{\tilde{G}}^M_N$ and place one's on the diagonal. Also eliminate Γ's and X's which link multiple eigenvalue terms to distinct eigenvalue terms.

For our problem of straight rods we see that the three and six modes can never be coupled to the other modes, as these terms are already in the indicated diagonal form. Therefore section B would have identical results if λ_3 and λ_6 were distinct.

2) Twisted but Uncurved Rods

Twisted but uncurved rods are specified by the geometric properties $\Gamma^{\cdot 2}_{1}$, $\Gamma^{\cdot 1}_{2}$, $\Gamma^{\cdot 5}_{4}$, and $\Gamma^{\cdot 4}_{5}$. To be twisted and un-uncurved all other $\Gamma^{\cdot N}_{M} = 0$ for $M \neq N$. For simplicity we will make the assumption

$$d_1 = d_2 \quad . \tag{8.27}$$

By substituting m = 1, n = 2 into $(6.20)_1$ and differentiating we find

$$\Gamma^{\cdot 2}_{1} = -\Gamma^{\cdot 1}_{2} \quad . \tag{8.28}$$

By substituting m = 4, n = 5 into $(6.20)_1$ differentiating and using (8.27) we find

$$\Gamma^{\cdot 5}_{4} = -\Gamma^{\cdot 4}_{5} \quad . \tag{8.29}$$

We may now determine the matrix G^M_N of (8.4) by solving (8.9) subject to (8.10). This results in

$$G^M_N = b \begin{bmatrix} m & -n & 0 & 0 & 0 & 0 \\ n & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_1 p & -f_1 q & 0 \\ 0 & 0 & 0 & f_1 q & f_1 p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{8.30}$$

where

$$m = \cos r,\ n = \sin r,\ p = \sin t,\ q = \sin t,$$
$$r = \int_{s_o}^{s} \Gamma^{\cdot\,2}_{\ 1}\, ds,\ t = \int_{s_o}^{s} \Gamma^{\cdot\,5}_{\ 4}\, ds \quad . \tag{8.31}$$

The quantities f_i and b are defined by $(8.23)_{2,3}$.

A) Distinct Eigenvalues

For distinct eigenvalues there is only one allowable mapping $\underset{\sim}{G}$. Therefore we have

$$\bar{\underset{\sim}{G}} = \underset{\sim}{G} \quad . \tag{8.32}$$

Physically (8.32) states that the set of shearing and bending wave mode vectors rotate along the rod at the same rate as the twist of the rod. This rotation can be viewed as a rotation in the full six-dimensional space or in a subspace of a component translation space V. Rods with equal principle moments of inertia in the cross-section but having distinct wave speeds are anisotropic. From (8.2) we may conclude

$$D^1_1 \neq D^2_2 \ , \quad D^4_4 \neq D^5_5 \quad . \tag{8.33}$$

B) $\underline{\lambda_4 = \lambda_5}$

We find from a solution of (8.13) subject to (8.28), (8.29) and (8.14) a component form of the tensor ${}_{i}\underset{\sim}{O}^{K}$ through (6.37). Using these results plus (6.29), (8.3), $(6.40)_3$, and (6.41) we find the component form of the mapping $\underset{\sim}{\bar{G}}$ given by (8.20) as

$$\tilde{\bar{G}}^{M}_{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q & 0 \\ 0 & 0 & 0 & -q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \qquad (8.34)$$

By using (8.30), (8.34) and (8.21) we have

$$\bar{G}^{M}_{N} = b \begin{bmatrix} m & -n & 0 & 0 & 0 & 0 \\ n & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_3 \end{bmatrix} . \qquad (8.35)$$

The physical interpretation of the results is significant. If the matrix of $\underset{\sim}{\bar{G}}$ is computed in the form of (8.34) the uncoupling basis is a rotation in a two-dimensional subspace of one of the component translation subspaces V of $\mathcal{V}$. The component translation space is the one corresponding to angular effects. However since the basis of bending eigenvectors $\underset{\sim}{W}_M$ is rotating about the axis of the rod as one progresses down the rod a simpler description is available. Matrix (8.35) shows that if the initial values of the eigenvectors are used as a basis, only a dilation of these vectors occurs in the four-five mode in giving the wave mode vectors at arbitrary s. The four-five wave mode vectors therefore remain parallel to their initialized values as the waves progress down the rod. This is equivalent to stating that the four-five system of wave mode vectors behave as if the rod were straight.

The rod can be viewed as an isotropic rod in the four-five mode with equal moments of inertia. The rod is a circular rod of arbitrary twist. The wave mode vectors would be in the direction of the principle moments of inertia. However as all directions are principle directions the rod could just as easily be modelled without twist. The results of considering the rod with or without twist in the four-five mode will give the same answers.

C) Other possibilities

The most general case of all eigenvalues being identical turns out to have the same form as $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5$. Therefore as with straight rods the three-six modes never couple with the other terms.

The solution of the uncoupling tensor $\underset{\sim}{\bar{G}}$ is not too difficult. However coupling between the modes is heightened by the presence of X^1_5 and X^2_4. An easy physical interpretation of the results was not found.

In section B we considered the case $\lambda_4 = \lambda_5$. For a completely isotropic rod this implies that $\lambda_1 = \lambda_2$ also. The discussion in section B is then relevant to the one-two mode as well.

3) <u>Plane Curved but Untwisted Rods</u>

This section considers plane curved but untwisted rods of a specific form. The nature of such rods is specified by $\Gamma^{\cdot 3}_{2}$, $\Gamma^{\cdot 2}_{3}$, $\Gamma^{\cdot 6}_{5}$, and $\Gamma^{\ 6}_{\cdot 5}$. To be plane curved but untwisted all other $\Gamma^{\cdot N}_{M} = 0$ for $M \neq N$. Specialization of the more general problem occurs by setting

$$d_2 = d_3 \quad . \qquad (8.36)$$

By a process similar to the development of (8.28) and (8.29) we can show that

$$\Gamma^{\cdot 2}_{3} = -\Gamma^{\cdot 3}_{2} \ , \ \Gamma^{\cdot 5}_{6} = -\Gamma^{\cdot 6}_{5} \quad . \qquad (8.37)$$

The form of the uncoupling basis for some particularily simple multiple eigenvalue cases is similar to that of the twisted but uncurved rod.

Only one case will be considered here. We examine a rod with

$$\lambda_2 = \lambda_3 \ , \ \lambda_5 = \lambda_6 \quad . \qquad (8.38)$$

The uncoupling basis is the initial values of the eigenvectors affected at most by a dilatation. The waves in the two-three mode and the five-six mode are unable to detect any curvature. Mathematically, if not physically, the rod may be considered straight.

Cases distinct in form from the twisted but uncurved rod are possible, but the results are difficult to interpret physically.

REFERENCES

[1] Whitman, A.B., DeSilva, C.N., A Dynamical Theory of Elastic Directed Curves, Z.A.M.P., Vol. 20 (1969).

[2] Truesdell, C., Noll, W., The Non-Linear Field Theories of Mechanics, Handbuch der Physik, Vol. III/3, Springer-Verlag (1965).

[3] Ericksen, J.L., Wave Propagation in Thin Elastic Shells, Arch. Rat. Mech. Anal., Vol. 43 (1971).

[4] Cohen, H., Acceleration Waves in Elastic Rods, Arch. Rat. Mech. Anal., Vol. 67 (1978).

[5] Bansal, S.C., Wave Propagation in Finitely Deformed Thermoelastic Rods, Ph.D. thesis, Wayne State University, Detroit, Michigan (1977).

[6] Cohen, H., Axi-Symmetric Waves in Shells of Revolution, Proc. Int. Ass. Shell Structures, Calgary (1972).

[7] Cohen, H., Wave Decay in Nonhomogeneous Elastic Rods, Arch. of Mechanics, Vol. 30 (1978).

[8] Cohen, H., Shock Wave Decay in Elastic Rods, Iranian J. of Science Techn., to appear (1978).

[9] Cohen, H., Whitman, A.B., Waves in Elastic Rods, J. Sound and Vibr., Vol. 51 (1977).

[10] Barrett, J.H., Bradley, J.S., Ordinary Differential Equations, Intext Educational Publishers, Toronto (1972).

[11] Ericksen, J.L., Simpler Static Problems in Non-Linear Theories of Rods, Int. J. Solids Structures, Vol. 6 (1970).

[12] Cohen, H., A Non-Linear Theory of Elastic Directed Curves, Int. J. Engng. Sci., Vol. 4 (1966).

[13] Protter, M.H., Morrey, C.B., Modern Mathematical Analysis, Addison-Wesley Ltd., Don Mills, Ont. (1966).

[14] Leigh, D.C., Nonlinear Continuum Mechanics, McGraw-Hill Book Co., Toronto (1968).

[15] Greub, W.H., Linear Algebra, Springer-Verlag, New York (1975).