TWO-DIMENSIONAL MATRICES (MODULO 3) FOR THE CUBIC GROUPS

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by

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ABSTRACT

The success of Einstein's theory of gravitation as a geometric effect, has made it plausible that the surprising phenomena encountered in modern physics may be explained by assuming a new geometry. A geometry suggested for this purpose is what in mathematics is known as finite geometry. This thesis reviews some of the analytical aspects of this geometry and where possible, touches upon the implications of assuming such a geometry. In this direction, the rotation group of finite geometry is treated at some length and its homomorphism with 2 x 2 matrix group is worked out. Although our approach keeps a parallelism with the treatment of the matter in the usual Euclidean geometry, nevertheless new features arise. In particular not only do we get the homomorphism with the 2 x 2 "complex" matrices, but also a homomorphism with the 2 x 2 "real" matrices.

To illustrate some features of rotation in finite geometry a small model (a finite geometry of 27 points) is provided. The model geometry as well serves another purpose:

It provides a simple 2-dimensional, 2-valued "real" representation of the cubic symmetries (rotational). The simplicity is in that, with this representation at hand there is no need of a written list of matrices or a group multiplication table.

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C H A P T E R 1

THE (T) AND (O) GROUPS AS FINITE GEOMETRY ROTATION GROUP

INTRODUCTION

The Tetrahedral (T) and Octahedral (O) groups arise in the quantum mechanical study of certain molecules (e.g. CH_4) and electron wave functions of eigenstates with cubic symmetry.

In this thesis the main aim is to present a simple "real" 2-dimensional 2-valued) representation of T and O groups over F_3, the field of order 3, which seems not to have been given before. The matrices can be read easily from a simple formula (easy to remember) and conversely by a simple procedure each matrix can be decoded back into the group spectrum. The matrix multiplications are very direct and simple and not tedious at all. There is no need of a written list of the matrices or of a large multiplication table such as appears in some texts.

Secondly there is a theory which conjectures that physical space is what in mathematics is termed a finite geometry [1,2]. In this connection, the geometry of 27 points constructed over F_3 not only leads us to the desired representation of T and O groups, but also serves as a small model illustrating some features of 'rotation' in the conjectured finite geometry.

1.1. Finite geometry over a finite field

A finite geometry, obviously consisting of a finite number of points, is analytically constructed by taking coordinates of a point to be elements of a finite field. Familiar terms of ordinary geometry such as line, plane, vector, etc. are conveniently employed in finite geometry where for them the analytical expressions in coordinates and parameters are defined over a finite field rather than over the reals.

The definition of a field and the theory of finite fields can be found in most books on modern algebra [3]. A finite field, that is a field with a finite number of elements, is characterized by the property that rational operations of algebra may be performed on the elements of the field and they lead in every case to the elements of the field. It is from this point that terms of algebra and corresponding notations could be taken over and employed for finite fields. In particular, letting $C_o, C_1, \ldots, C_{K-1}$ be the elements of F_K i.e. a finite field of order K:

a) Addition and multiplication of any two elements of the field, say, C_i and C_j, are denoted respectively by $C_i + C_j$ and $C_i\, C_j$.

b) The unique element, say C_o, having the property of 'zero' under addition and multiplication $(C_i + C_o = C_i \quad C_oC_i = C_iC_o = C_o \; \forall \; C_i)$ is also denoted by 0, and the unique element giving 0 when added to C_i is denoted by $-C_i$.

c) The unique element, say C_1, having the property of 'unity' under multiplication is also denoted by 1 and the unique element giving 1 when multiplied by C_i is denoted by $1/C_i$ or C_i^{-1}.

The element of F_K defined by the sequence,

$$1, \quad C_{(2)} = 1 + 1 \quad C_{(3)} = 1 + 1 + 1, \; \ldots \;, \tag{1.1}$$

are called 'integral marks' of the field. Since the field is finite, for the above sequence there must be a least integer (P); such that $C_{(P)} = 0$. Then $0, 1, C_{(2)}, C_{(3)}, \ldots, C_{(P-1)}$ are all the distinct integral marks of field. Moreover it may be shown that P is a prime number [4]. Now as a convenient representation for these integral marks we write,

$$0, 1, 2, \ldots P-1 \tag{1.2}$$

in the order given above. The rule of composition for these integral marks is ordinary addition and multiplication with a reduction module P. Moreover it can be easily verified that the integral marks form a field (called the prime field) of order P, which we denote by F_P. As for the order K of F_K it may be shown that $K = P^n$ where n is some positive integer [5]. Also it should be noted that a finite field of every order P^n, where P is a prime and n is a positive integer, exists and that all finite fields of the same order are isomorphic [6]. Hence, F_{P^n} is uniquely determined by the number P^n of its elements.

In view of the above paragraph, for the sum:

$$X + X + X + \ldots + X \tag{1.3}$$

where X is any element of the field and it is repeated m times in the sum, we will have:

$$\begin{aligned} X + X + X + \ldots + X &= (1 + 1 + 1 + \ldots + 1)X \\ &= C_{(m)}X \\ &= m\,X \end{aligned} \tag{1.4}$$

In particular, when X is repeated P times:

$$X + X + X + \dots + X = P\,X$$
$$= 0\,X$$
$$= 0, \quad \text{or}$$
$$P\,X = 0 \qquad (1.5)$$

Eqn. (1.5) holds in every F_{P^n} for a given prime P and any positive integer $n \geq 1$. In this sense it is said F_{P^n} is of characteristic P.

Any element X of a finite field of order P^n satisfies the relation (Fermat theorem) [7].

$$X^{P^n} = X \,, X \in F_{P^n} \qquad (1.6)$$

We remark that a relation $X^{\ell} = X$ with $\ell < P^n$ may be fulfilled for suitable X; however in any F_{P^n} there exists at least one element ε (called a primitive root) [8] such that $0, \varepsilon, \varepsilon^2, \dots \varepsilon^{P^n-1}$ are all distinct and span the whole field. The elements of the field which are even or odd powers of the primitive root ε are respectively called 'squares' or 'not-squares'. The zero element is not considered either as a square or not-square. Then the number of squares is $1/2(P^n-1)$ and so is the number of not-squares.

Now in choosing a field to replace the real numbers for our co-ordinates in physics, we note that the prime field F_P is analogous to the reals and that F_{P^n} with $n \geq 2$ is related to F_P in much the same way that the complex numbers are related to the reals. We therefore use the field F_P to co-ordinate our geometry.

Also in constructing a new physical geometry on finite fields, we have to bear in mind a correspondence principle with ordinary physical schemes. Specifically we must introduce some analogue of positive and negative and we must in some approximate sense ensure the pythagorean property. For this purpose, we recall that in F_p, half the non-zero elements are squares and half are not-squares with the multiplicative properties of positive and negative respectively. The product of a square and not-square is a not-square, etc. In this respect, we may define an element as being 'positive' if it is a square and 'negative' if it is a not-square. To ensure consistency with the sign rules when transposing monomials from one side of an inequality to the other side, we further require the minus of a square element of the field, $-X \in F_p$ to be a not-square (and vice versa). Hence the element $-1 \equiv P-1$ (mod P) should be a not-square. This requirement is satisfied by taking P of the form [9];

$$P = 4n-1 \quad (n \ \text{integer}) \tag{1.7}$$

We take this to be the case hereafter.

Henceforth, we can speak of "greater than" (>) and "smaller than" (<) with the meaning that $X > Y$ or $X < Y$ according to whether $X - Y$ is square ("positive") or not-square ("negative"). However, we realize that there is still a difficulty: finite fields are non-pythagorean —— the sum of two squares can not be always square, so that there is a lack of transitivity for inequalities. As a consequence, the geometry whose points have coordinates in F_p would be deprived of usual metric relations.

For such a geometry to be valid as physical geometry, we must ensure the pythagorean property i.e. order relations including transitivity over a large subset E of the field. The requirement is fulfilled by taking an enormous prime (estimated by Jarnefelt [10] to be roughly $10^{10^{81}}$) of the form [11]

$$P = 8X \prod_{i=1}^{K} q_i - 1 \tag{1.8}$$

where X is an odd integer and $\Pi\, q_i$ is the product of the first K odd primes. Then $N(\sim q_K \sim \log P \sim 10^{81})$ consecutive elements of F_P obey ordinary arithmetic (the sum of "positives" is "positive"). With these as co-ordinates the corresponding region, called pythagorean region behaves like ordinary observed physical space.

In the next section we develop the notion of "rotation" for finite geometry in analogy with ordinary rotation. We denote the finite rotation group by $R(3,F_p)$. In Chapter II and III the homomorphism of $R(3,F_p)$ with 2×2 matrices will be worked out. We will see that not only do we have the homomorphism with $SU(2,F_{p^2})$ (analogue of $SU(2)$) but also a homomorphism with the group of 2×2 "real" matrices, $SL(2,F_p)$ which has no analogue in ordinary geometry. We only mention here that the desired representation of T and O groups is provided by $SL(2,F_3)$.

To end this section we quote some results about quadratic equations [12] which are used later. The number ν of sets of solutions $(X_1, X_2, \ldots X_{2m})$ in F_P of the equation;

$$C_1X_1^2 + C_2X_2^2 + \ldots + C_{2m}X_{2m}^2 = d \tag{1.9}$$

with m integer, C_i and d non-zero elements of the field F_P is given by

$$\nu = P^{2m-1} - g\, P^{m-1} \tag{1.10}$$

where

$$g = \begin{cases} 1 & \text{if } (-1)^m C_1 C_2 \ldots C_{2m} \text{ is square} \\ -1 & \text{if } (-1)^m C_1 C_2 \ldots C_{2m} \text{ is not square} \end{cases}$$

Similarly for:

$$C_1 X_1{}^2 + \ldots + C_{2m+1} X_{2m+1}^2 = d \tag{1.11}$$

the number of sets of solutions is given by:

$$\nu = P^{2m} + r\, P^m \tag{1.12}$$

where $r = \begin{cases} +1 & \text{if } (-1)^m d\, C_1 C_2 \ldots C_{2m+1} \text{ is square} \\ -1 & \text{if } (-1)^m d\, C_1 C_2 \ldots C_{2m+1} \text{ is not-square} \end{cases}$

1.2. Rotation in finite geometry

The notions of "rotation", "rotation" operator and "rotation" group can be defined for finite geometry along the same lines as in ordinary geometry. The algebraic procedure is not dissimilar to that of ordinary geometry except that here components of any relation take their values from the field F_p. With these considerations, 'rotation' is an operation which takes a vector $\vec{V}$ into $\vec{V}'$ endowed with the properties that:

1. inner products of vectors are invariant under rotation,
2. there is a direction $\vec{n}$ ("axis of rotation") such that under rotation $\vec{n} = \vec{n}'$.

The above-said statements can be summarized as:

$$\vec{V}' \cdot \vec{V}' = \vec{V} \cdot \vec{V} \tag{1.13}$$

$$\vec{n} \cdot \vec{V}' = \vec{n} \cdot \vec{V} \tag{1.14}$$

From (1.13) and (1.14) for the inner product $(\vec{n} \cdot \vec{V}_{CR})$ it follows that

$$\vec{n} \cdot \vec{V}_{CR} = 0 \tag{1.15}$$

where V_{CR} denotes the vector which under rotation carries $\vec{V}$ into $\vec{V}'$ i.e.:

$$\vec{V}_{CR} = \vec{V}' - \vec{V} \tag{1.16}$$

Now out of the two vectors $\vec{n}$ and $\vec{V}$ we can construct two new distinct directions, i.e. $\vec{n} \times \vec{V}$ and $\vec{n} \times (\vec{n} \times \vec{V})$; where X stands for the usual cross product. Equation (1.15) guarantees that we can write $\vec{V}_{CR}$ as the linear combination of $\vec{n} \times \vec{V}$ and $\vec{n} \times (\vec{n} \times \vec{V})$:

$$\vec{V}_{CR} = \alpha\, \vec{n} \times \vec{V} + \beta\, \vec{n} \times (\vec{n} \times \vec{V}) \tag{1.17}$$

Then from (1.16) and (1.17) it follows that:

$$\vec{V}' = \vec{V} + \alpha\, \vec{n} \times \vec{V} + \beta\, \vec{n} \times (\vec{n} \times \vec{V}) \tag{1.18}$$

Equation (1.18) can be written in matrix form.

To do this we denote:

$$[\vec{V}] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad [\vec{n}X] = M$$

with $M_{ik} = \varepsilon_{ijk}\, n_j$ and ε_{ijk} the unit anti-symmetric tensor. With this notation (1.18) will be written as:

$$[\vec{V}'] = R_{\vec{n}}^{\alpha,\beta}\ [V] \tag{1.19}$$

where $R_{\vec{n}}^{\alpha,\beta}$ is given explicitly by

$$R_{\vec{n}}^{\alpha,\beta} = 1 + \alpha \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}^2 \tag{1.20}$$

or in compact form

$$R_{\vec{n}}^{\alpha,\beta} = 1 + \alpha M + \beta M^2 \tag{1.21}$$

Now equation (1.13) in matrix language is equivalent to $R_{\vec{n}}^{\alpha,\beta}$ being orthogonal, i.e.

$$R_{\vec{n}}^{\alpha,\beta}\ \widetilde{R_{\vec{n}}^{\alpha,\beta}} = 1 \qquad (\sim\ = \text{transpose}) \tag{1.22}$$

or

$$(1 + \alpha M + \beta M^2)(1 - \alpha M + \beta M^2) = 1$$

It may be shown that:

$$M^3 = -\vec{n} \cdot \vec{n}\ M,$$

and hence;

$$M^4 = -\vec{n} \cdot \vec{n}\ M^2 \tag{1.23}$$

Using Eqn. (1.23) in (1.22), the orthogonality condition becomes:

$$\alpha^2 + \beta^2(\vec{n} \cdot \vec{n}) - 2\beta = 0 \tag{1.24}$$

The set of the "rotation" operators defined by (1.21) and (1.24) form the "rotation" group. The order of the group Ω_R is finite and is determined by (1.24). To see this, we take $\beta = \pm C^2$, where C^2 is any square element of the field. Then (1.24) can be written as:

$$(\frac{\alpha}{C})^2 + (C\vec{n}, C\vec{n}) = \pm 2$$

We let $X_o = \frac{\alpha}{C}$ and $\vec{X} = C\vec{n}$. Whence in terms of X_o and $\vec{X}(X_1, X_2, X_3)$ (1.24) becomes:

$$X_o^2 + X_1^2 + X_2^2 + X_3^2 = \pm 2 \qquad (1.25)$$

Now the number of sets of solutions (X_o, X_1, X_2, X_3) of (1.25) in F_P is found from (1.10) to be $2(P^3 - P)$. We notice that equation (1.21) is invariant under $\alpha \to -\alpha$ and $\vec{n} \to -\vec{n}$, hence the numbers of distinct matrices $R^{\alpha,\beta}_{\vec{n}}$ i.e. $\Omega_R = P^3 - P$. It is to be noted that $\vec{n} \cdot \vec{n}$, is not restricted to have the value 1, but can take any value in the field, even zero or "negative". Consequently we will want to modify somewhat the usual notion of "unit vector".

We recall that for the type of finite field we are dealing with, P the order of the field is such that -1 is a not-square element of the field. Now consider a scalar quantity $\vec{V} \cdot \vec{V}$ in this field, where except the zero element, half of the elements are square and the other half not-square, then if k^2 is any square element of the field:

$$\vec{V} \cdot \vec{V} = \pm k^2, 0$$

or

$$\frac{\vec{V}}{k} \cdot \frac{\vec{V}}{k} = \pm 1, 0$$

where, k being the analogue of the length of the vector $\vec{V}$, $\frac{\vec{V}}{k}$ denotes its "unit vector". In this way three types of directions will arise:

(1) $\vec{n}_o \cdot \vec{n}_o = 0$

(2) $\vec{n}_1 \cdot \vec{n}_1 = 1$

(3) $\vec{n}_{-1} \cdot \vec{n}_{-1} = -1$

Combining the above considerations with the equation (1.24), the "rotation" group will split in three types, each type having its own orthogonality condition derived from (1.24) i.e.:

$R^{\alpha,\beta}_{\vec{n}_o}$, $R^{\alpha,\beta}_{\vec{n}_1}$, $R^{\alpha,\beta}_{\vec{n}_{-1}}$, where the index of $\vec{n}$ refers to its type and the orthogonality conditions for these types respectively will be

$$\alpha^2 - 2\beta = 0$$

$$\alpha^2 + \beta^2 - 2\beta = 0$$

$$\alpha^2 - \beta^2 - 2\beta = 0$$

the above types can be summarized as follows:

$$\vec{n}_o \cdot \vec{n}_o = 0$$

$$\beta = \alpha^2/2 \tag{1.26}$$

$$R^{\alpha}_{\vec{n}_o} = 1 + \alpha M_{\vec{n}_o} + \frac{\alpha^2}{2} M^2_{\vec{n}_o}$$

$$\vec{n}_1 \cdot \vec{n}_1 = 1$$

$$\alpha^2 + \lambda^2 = 1 \tag{1.27}$$

$$R^{\alpha,\lambda}_{\vec{n}_1} = 1 + \alpha M_{\vec{n}_1} + (1 - \lambda)\, M^2_{\vec{n}_1}$$

$$\vec{n}_{-1} \cdot \vec{n}_{-1} = -1$$

$$\lambda^2 - \alpha^2 = +1 \tag{1.28}$$

$$R^{\alpha,\lambda}_{\vec{n}_{-1}} = 1 + \alpha M_{\vec{n}_{-1}} + (\lambda - 1)M^2_{\vec{n}_{-1}}$$

It is worth noting that in form $R^{\alpha}_{\vec{n}_o}$ has the appearance of an infinitesimal rotation operator, exponential expansion, but truncated; $R^{\alpha,\beta}_{\vec{n}_1}$ that of the classical case where α and λ look like sine and cosine of the angle of rotation respectively and $R^{\alpha,\lambda}_{\vec{n}_{-1}}$ has no analog in Euclidean space.

1.3. The groups (O) and (T)

As we well know, the Octahedral group (O) refers to the set of rotational symmetries exhibited by a cube and the Tetrahedral group (T) a subgroup of O refers to the set of the rotational symmetries enjoyed by a tetrahedron. The fact that T is a subgroup of O is well understood once we realize that any tetrahedron is inscribeable in a cube such that its centroid and vertices are respectively coincident with the centre and alternative corners of the cube. For a cube there are 4,3,6 symmetry axes of order, respectively; 3,4,2 giving rise to the 24 rotational symmetry elements. In the following we show that the "rotation" group of the finite geometry over the field F_3 i.e. $R(3,F_3)$ is isomorphic to the group O and provides us with a representation of it.

A convenient way to designate the 24 rotational symmetry elements of the cube is to introduce the indices h_1, h_2, h_3 [13]:

a) $(h_1\ h_2\ h_3)$ stands for an axis $\vec{h}$ having directional ratios h_1, h_2, h_3 with respect to the three mutually perpendicular edges of the cube.

b) A bar (-) on top of the index represents the sign of the relevant index.

c) $(h_1\ h_2\ h_3)^{\theta/2\pi}$ stands for the matrix operator describing rotation through angle θ about $(h_1\ h_2\ h_3)$ in the sense of a right hand rotation.

Now considering a cube (see Figure 1) it is easy to designate the relevant set of 24 symmetry elements as follows:

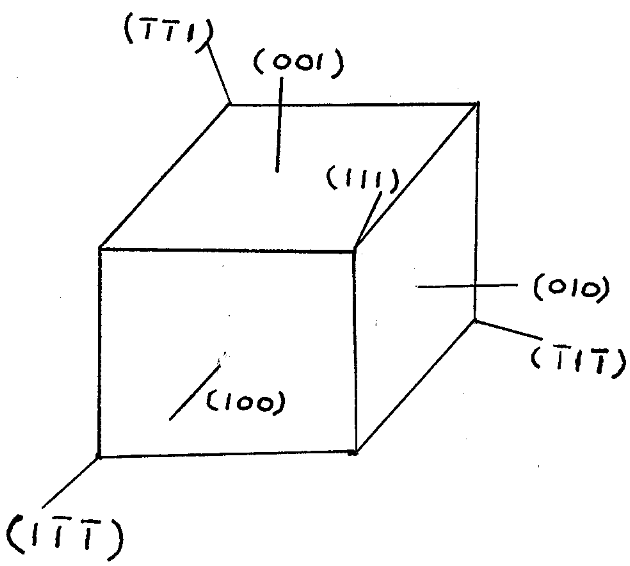

Fig. 1.

TABLE I

1 & $(111)^{1/3}$ $(\bar{1}\bar{1}1)^{1/3}$ $(\bar{1}1\bar{1})^{1/3}$ $(1\bar{1}\bar{1})^{1/3}$

$(111)^{-1/3}$ $(\bar{1}\bar{1}1)^{-1/3}$ $(\bar{1}1\bar{1})^{-1/3}$ $(1\bar{1}\bar{1})^{-1/3}$

TABLE II

$(100)^{1/4}$ $(010)^{1/4}$ $(001)^{1/4}$

$(100)^{1/2}$ $(010)^{1/2}$ $(001)^{1/2}$

$(100)^{-1/4}$ $(010)^{-1/4}$ $(001)^{-1/4}$

TABLE III

$(011)^{1/2}$ $(101)^{1/2}$ $(110)^{1/2}$

$(01\bar{1})^{1/2}$ $(10\bar{1})^{1/2}$ $(1\bar{1}0)^{1/2}$

Note that for axes of I, II, and III we have respectively,

$$h_1^2 + h_2^2 + h_3^2 = 3$$

$$h_1^2 + h_2^2 + h_3^2 = 1$$

$$h_1^2 + h_2^2 + h_3^3 = 2$$

Considering rotation of a vector around an axis with unit vector $\vec{n}$ through an angle θ, we have for the rotation matrix operator

$$R_{\vec{n}}(\theta) = 1 + \sin\theta[\vec{n}\ X] + (1 - \cos\theta)[\vec{n}\ X]^2 \qquad (1.29)$$

the above matrix for axis $\vec{h}$ with directional ratios h_1, h_2 and h_3 is

$$R_{\vec{h}}(\theta) = (h_1\ h_2\ h_3)^{\theta/2\pi}$$

$$= 1 + \frac{\sin\theta}{h} M_{\vec{h}} + \left(\frac{1 - \cos\theta}{h^2}\right) M_{\vec{h}}^2 \qquad (1.30)$$

where $h^2 = h_1^2 + h_2^2 + h_3^2$, $M_{\vec{h}} = [\vec{h}\ X]$ and $[\vec{h}\ X]_{ik} = \varepsilon_{ijk} h_j$. Now in Table I we have that $\theta = 0,\ 2\pi,\ \pm\frac{2\pi}{3}$. Then for the relevant matrices we will have:

$$(h_1\ h_2\ h_3)^0 = (h_1\ h_2\ h_3)^{2\pi} = 1 \qquad (1.31a)$$

$$(h_1\ h_2\ h_3)^{\pm 1/3} = 1 \pm \frac{1}{2} M_{\vec{h}} + \frac{1}{2} M_{\vec{h}}^2 \qquad (1.31b)$$

(Note that in (1.31b) correspondence of sign is in the order written) Furthermore, the matrices in (1.31b) have only $0, \pm 1$ for their entries. This can be seen by writing (1.31b) explicitly in terms of h_1, h_2 and h_3:

$$(h_1\ h_2\ h_3)^{\pm 1/3} = 1 \pm 1/2 \begin{bmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{bmatrix}$$

$$+ 1/2 \begin{bmatrix} -(h_2^{\ 2} + h_3^{\ 2}) & h_1h_2 & h_1h_3 \\ h_2h_1 & -(h_1^2 + h_3^2) & h_2h_3 \\ h_3h_1 & h_2h_3 & -(h_1^{\ 2} + h_2^{\ 2}) \end{bmatrix} \tag{1.32}$$

From (1.32) the diagonal and off-diagonal entries are:

$$(h_1\ h_2\ h_3)^{\pm 1/3}_{ii} = 1 + \frac{1}{2}\,(h_i^{\ 2} - h^2)$$

$$(h_1\ h_2\ h_3)^{\pm 1/3}_{ij} = \frac{1}{2}(\mp\ \varepsilon_{ijk}\ h_k + h_i h_j) \tag{1.33}$$

Now in Table I h_1, h_2, h_3 are all ± 1, $h_1^{\ 2} = h_2^{\ 2} = h_3^{\ 2} = 1$ and $h^2 = h_1^{\ 2} + h_2^{\ 2} + h_3^{\ 2} = 3$. Then it is clear that:

$$(h_1h_2h_3)^{\pm 1/3}_{ii} = 1 + \frac{1}{2}\,(1 - 3) = 0 \qquad \text{and}$$

$$(h_1h_2h_3)^{\pm 1/3}_{ij} = 0, \pm 1$$

We recall that according to our representation of F_p (see eqn. 1.2), F_3 is represented by numbers modulo 3 i.e. $F_3 = \{0,1,2\}$ or equivalently $F_3 = \{0,1,-1\}$, since $2 \equiv -1$ (mod. 3). Taking this to be our view of the numbers 0,1 and -1, then with regard to our discussion about matrices in (1.31), axes and matrices of Table I can just as well be taken as over F_3, moreover the following correspondence is immediate:

$\vec{h}$ of Table I $\longrightarrow$ $\vec{n}_o$ of the finite geometry over F_3,

(since $h_1{}^2 + h_2{}^2 + h_3{}^2 = 3 \equiv 0$)

$$M_{\vec{h}} \longrightarrow M_{\vec{n}_o}$$

$(h_1h_2h_3)^0 = 1 \longrightarrow R^{0,0}_{\vec{n}_o}$ (i.e. $\alpha = o$ in $R^{\alpha,\beta}_{\vec{n}_o}$)

$$(h_1h_2h_3)^{\pm 1/3} \longrightarrow 1 \pm 1/2\, M_{\vec{n}_o} + 1/2\, M^2_{\vec{n}_o}$$

$$\equiv 1 \mp M_{\vec{h}_o} - M^2_{\vec{n}_o} = R^{\mp 1,-1}_{\vec{n}_o}$$ (i.e. $\alpha = \mp 1$ and $\beta = -1$ in $R^{\alpha,\beta}_{\vec{n}_o}$; $\alpha = -1, +1$ corresponds to $+\frac{2\pi}{3}$, $-\frac{2\pi}{3}$ respectively).

In a similar way it can be shown that matrices in Table II and III also have for their entries only $0, \pm 1$. Consequently, axes and matrices of Table II (Table III) are respectively $\vec{n}_1$ and $R^{\alpha,\beta}_{\vec{n}_1}$ ($\vec{n}_{-1}$ and $R^{\alpha,\beta}_{\vec{n}_{-1}}$) of the finite geometry over F_3. The above results can be summarized as:

$$\vec{n}_o \cdot \vec{n}_o = 0$$

$$(h_1h_2h_2)^{1/3} \equiv R^{-1,-1}_{\vec{n}_o} \qquad (1.33a)$$

$$(h_1h_2h_3)^{-1/3} \equiv R^{1,-1}_{\vec{n}_o}$$

$$\vec{n}_1 \cdot \vec{n}_1 = 1$$

$$(h_1h_2h_3)^{1/4} \equiv R^{1,1}_{\vec{n}_1} \qquad (1.33b)$$

$$(h_1h_2h_3)^{1/2} \equiv R^{0,-1}_{\vec{n}_1}$$

$$(h_1h_2h_3)^{-1/4} \equiv R^{-1,1}_{\vec{n}_1}$$

$$\vec{n}_{-1} \cdot \vec{n}_{-1} = -1 ; \quad (h_1 h_2 h_3)^{1/2} \equiv R^{0,1}_{\vec{n}_{-1}}$$

$$\vec{n} \cdot \vec{n} = 0, \pm 1 \ ; \quad 1 \equiv R^{0,0}_{\vec{n}} \tag{1.33c}$$

CHAPTER II

REPRESENTATION OF THE FINITE GEOMETRY ROTATION GROUP OVER THE "COMPLEX" FINITE FIELD

INTRODUCTION

In ordinary geometry complex numbers and complex matrices have been very useful. In particular, the set of unitary matrices with determinant +1 (denoted by $D^j_{m'm}(U)$, where $U \in SU(2)$, $-j \leq m \leq j$ [14]) most conveniently present all the $(2j + 1)$ dimensional irreducible representations of the rotation group $(j = 0, 1/2, 1, 3/2, \ldots)$. In quantum mechanics the irreducible representations of the rotation group find a direct application: the angular momentum operators are identified with the generators of rotation and the dimension $(2j + 1)$ of the representation is identified with $2j + 1$ degenerate states of the angular momentum.

Analogous techniques can be developed in finite geometry, where one is dealing with a finite field F_p instead of the reals. An analogous extension of F_p is considered as the "complex" finite field. Unitary matrices and unitary transformation are then defined as usual. Moreover the irreducible representations of the finite rotation group $R(3,F_p)$ are developed in much the same way as for the ordinary rotation group [15].

In this Chapter, "complexification" of F_p will be introduced and in section (2) the finite geometry's version of $D^j_{m'm}(U)$ for $j = 1/2$ will be presented. This will be denoted by $SU(2,F_{p^2})$ in analogy with the well known special unitary group SU(2)[16].

2.1. The complex finite field

By the "complexification" of the finite field F_P we mean the extension of F_P to F_{P^2}, which can be considered as Cartesian product of the field F_P with itself, i.e. $F_{P^2} = F_P \otimes F_P$. We recall that for the type of prime P we are dealing with ($P = 4n - 1$, n integer); -1 is not a square element in F_P. Then we can consider the complex field F_{P^2} to be the "real" field F_P with $i = \sqrt{-1}$ adjoined. Then the usual complex notation can be used conveniently: an element $Z \varepsilon F_{P^2}$ can be written as $Z = X + iy$ with X and $y \varepsilon F_P$.

The algebra of the finite complex field is in many ways similar to that of the ordinary complex numbers[17]. However, due to the finiteness of the field, some new features arise:

(a) For $P = 4n - 1$, we have:

$$\begin{aligned} i^P &= i^{4n-1} = (i)^{4n}\frac{1}{i} \\ &= (+1)\frac{1}{i} = \frac{i}{i^{-2}} = -i \end{aligned} \tag{2.1}$$

(b) For any $Z = X + iy \varepsilon F_{P^2}$ we have that:

$$\begin{aligned} Z^P &= (X + iy)^P \\ &= \sum_{K=o}^{P} \{^P_K\}\, x^{P-K}\, (iy)^K , \end{aligned}$$

where

$$\{^P_K\} = \frac{P!}{(P-K)!K!}$$

We recall that every F_{P^n} is of characteristic P (i.e. $PX = 0$ for all $X \in F_{P^n}$ eqn. 1.5), so the only non-zero terms in the above binomial expansion are for $K = 0$ and $K = P$. Hence,

$$Z^P = x^P + (iy)^P$$

$$= x^P - iy^P \qquad (2.3)$$

Now using Fermat's theorem (ref. [7]), we have:

$$x^P = x$$

$$y^P = y$$

Then eqn. (2.3) will be written as:

$$Z^P = x - iy \equiv Z^* \qquad (2.4)$$

From eqn. (2.4) it follows that:

$$Z^{P + 1} = ZZ^* = x^2 + y^2 \qquad (2.5)$$

We draw attention to the fact that $Z^{P + 1}$ while being the analogue of the modulus-squared of an ordinary complex number is not necessarily "positive" but it can be "negative" as well. This is understood once we realize that for the finite field F_P the sum of two squares ("positives") can be either "positive" or "negative". In fact in F_P a quadratic equation of the form $x_1^2 + x_2^2 = \pm d$ (d any square element of the field) has solutions for both "signs" and in each case there are $P + 1$ solutions to it (eqn. 1.10). Now in ordinary complex number theory, a complex number of unit modulus represents a unit vector or direction in the complex plane. Then in this view for F_{p^2} there are two sets of directions:

1. directions which may be considered as usual directions;
   $\omega = \lambda + i\mu$ with $\lambda^2 + u^2 = 1$.
2. directions which have no analogue in ordinary complex planes/
   $\xi = \nu + i\phi$ with $\nu^2 + \phi^2 = -1$.

(c) The ω's and ξ's defined above can be used as the basis of a polar form for finite complex numbers. In this sense any element Z of F_{p^2} can be written as one of the forms: $Z = \rho\omega$ or $Z = \rho\xi$; where $\rho \in F_p$ is zero or some "positive" element of F_p and $\omega^{P+1} = \omega\omega^* = 1$; $\xi^{P+1} = \xi\xi^* = -1$. The set of all ω's and ξ's give all possible directions of the finite "complex plane", analogous to the set of $e^{i\phi}$ for all angles in the ordinary complex plane. We remark that the product of two ξ's is an ω and the product of a ξ by an ω is another ξ.

2.2 The group $SU(2,F_{p2})$

In ordinary geometry the set of 3 x 3 - real orthogonal matrices with determinant +1 form the rotation group (O_3^+). Under the operation of any rotation group element on a vector $\vec{V}$: $[\vec{V}] \rightarrow [\vec{V}']$ such that; the $\widetilde{[\vec{V}']}[\vec{V}'] = \widetilde{[\vec{V}]}[\vec{V}]$, i.e. the length of the vector remains invariant. A homomorphic representation of the rotation group can be furnished by the 2 x 2 unitary matrices (SU(2)). For this purpose, then a vector V is represented by a matrix of the form

$$T = \vec{\sigma} \cdot \vec{V} = \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3$$

$$= \begin{bmatrix} V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_3 \end{bmatrix} \qquad (2.6)$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Under a unitary tranformation $U \varepsilon SU(2)$; (det. $U = 1$, $UU^\dagger = 1$) i.e. $T' = UTU^\dagger$:

(a) the trace of the matix is invariant. Then, with T and hence T' being Hermitian, T' must have the same form as T, i.e.

$$T' = \vec{\sigma} \cdot \vec{V}' = \begin{bmatrix} V'_3 & V'_1 - iV_2 \\ V'_1 + iV'_2 & -V'_3 \end{bmatrix} \qquad (2.7)$$

(b) Also the determinant is invariant, i.e.

$$\vec{V} \cdot \vec{V} = \vec{V}' \cdot \vec{V}' \qquad (2.8)$$

Note that also $\pm U$ both imply the same transformation.

The above suggests that $SU(2)$ and 0_3^+ are homomorphic and where $R_{\vec{n}}(\psi)$ is given by,

$$R_{\vec{n}}(\psi) = 1 + \sin\psi[\vec{n}\ X] + (1 - \cos\psi)[\vec{n}\ X][\vec{n}\ X] \qquad (2.9)$$

The corresponding 2 x 2 unitary matrix is given [18] by

$$U_{\vec{n}}(\psi) = \pm (\mathrm{Cos}\psi/2\ 1 + i \sin\psi/2\ \vec{\sigma} \cdot \vec{n}) \qquad (2.10)$$

In the following we establish in a not dissimilar way the correspondence of $R^{\alpha,\beta}_{\vec{n}} \varepsilon R(3,F_p)$ with $U^{\alpha,\beta}_{\vec{n}} \varepsilon SU(2,F_{p^2})$. We recall that in the classical case, a general group element of $SU(2)$ is of the form:

$$U = \begin{bmatrix} \eta & \nu \\ -\nu^* & \eta^* \end{bmatrix}$$ with $\det U = 1$, η, ν ordinary complex numbers.

Analogously, in finite geometry a general group element of $SU(2,F_{p^2})$ can be written as:

$$U = \begin{bmatrix} \eta & \nu \\ -\nu^* & \eta^* \end{bmatrix}, \eta \text{ , and } \nu \in F_{p^2}.$$

Now in connection with the condition $\det U = 1$, the equation

$\eta\, \eta^* + \nu\, \nu^* = 1$ (this is an equation of the form

$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1$ with

X_1, X_2, X_3 and $X_4 \in F_p$) in a finite field F_p has $P^3 - P = \Omega_R$ solutions (see eqn. 1.10) instead of $2\Omega_R$ (Ω_R denotes the order of the finite geometry's rotation group). This then shows that it is impossible to have, according to the simplest analogy with the classical case discussed above, a 1 to 2 homomorphism of $R(3,F_p)$ into $SU^+(2,F_{p^2})$, (+ refers to det $U = +1$). Note that, however in a finite field where the sum of squares can be "negative" we can have det $U = -1$. We will see that, infact we have the homorphism of $R(3,F_p)$ into $SU(2,F_{p^2})$; the group of matrices U with determinants of either $+1$ or -1. In view of the above consideration, the group of matrices U is denoted by $SU^{\pm}(2,F_{p^2}) \equiv SU(2,F_{p^2})$ where + and - refers to the fact that the group is formed of matrices with determinant of either +1 or -1.

Let us recall that in finite geometry, the rotation of a vector $\vec{V}$ in vector notation is expressed by (eqn. 1.18):

$$\vec{V}' = \vec{V} + \alpha\, \vec{n} \times \vec{V} + \beta\, \vec{n} \times (\vec{n} \times \vec{V}) \qquad (2.11)$$

where $\vec{n}$ is the axis of rotation with α,β and $\vec{n}$ satisfying the equation:

$$\alpha^2 + \beta^2(\vec{n} \cdot \vec{n}) = 2\beta \tag{2.12}$$

Now, the 2 x 2 matrix representation of $\vec{V}'$ is furnished by the Pauli matrices:

$$\vec{\sigma} \cdot \vec{V}' = \vec{\sigma} \cdot \vec{V} + \alpha\, \vec{\sigma} \cdot \vec{n} X \vec{V} + \beta\, \vec{\sigma} \cdot \vec{n} X (\vec{n} X \vec{V}) \tag{2.13}$$

Taking advantage of the identity:

$$(\vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{v}) = \vec{n} \cdot \vec{v} + i\, \vec{\sigma} \cdot \vec{n}\, X \vec{V} \quad , \tag{2.14}$$

equation (2.13) is then transformed to:

$$\begin{aligned} \vec{\sigma} \cdot \vec{V}' = {} & (1 - \beta/2\, \vec{n} \cdot \vec{n})\, \vec{\sigma} \cdot \vec{V} \\ & + i\alpha/2\, [(\vec{\sigma} \cdot \vec{V})(\vec{\sigma} \cdot \vec{n}) - (\vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{V})] \\ & + \beta/2\, (\vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{V})(\vec{\sigma} \cdot \vec{n}). \end{aligned} \tag{2.15}$$

On the other hand the transformation: $\vec{\sigma} \cdot \vec{V} \to \vec{\sigma} \cdot \vec{V}'$ equivalently can be written as:

$$\vec{\sigma} \cdot \vec{V}' \equiv A(\vec{\sigma} \cdot \vec{V})A^{\dagger} \quad \text{where } AA^{\dagger} = 1 \tag{2.16}$$

In analogy with the classical unitary transformation matrix $e^{i\theta}U_{\vec{n}}(\psi)$ (where $e^{i\theta}$ is a phase factor and $U_{\vec{n}}(\psi)$ is the matrix given by eqn. 2.10) the matrix A will be written as:

$$A = a + ib\, \vec{\sigma} \cdot \vec{n} \quad \text{where } a \text{ and } b \in F_p 2 \tag{2.17}$$

Using (2.17) in (2.16) we will have that

$$\vec{\sigma} \cdot \vec{V}' \equiv (a + ib\, \vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{V})(a^* - ib^*\, \vec{\sigma} \cdot \vec{n})$$

$$= aa^*\, \vec{\sigma} \cdot \vec{V}$$

$$+ i[a^*b(\vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{V}) - ab^*(\vec{\sigma} \cdot \vec{V})(\vec{\sigma} \cdot \vec{n})]$$

$$+ bb^*\, (\vec{\sigma} \cdot \vec{n})(\vec{\sigma} \cdot \vec{V})(\vec{\sigma} \cdot \vec{n}) \tag{2.18}$$

It is evident that if $R_{\vec{n}}^{\alpha,\beta}$ is to correspond to A the right-hand side of equation (2.15) must be equal to that of equation (2.18). For this to be so the following equalities must be satisfied:

$$\begin{aligned} a\, a^* &= 1 - \beta/2\, \vec{n} \cdot \vec{n} \\ a\, b^* &= ba^* = -\alpha/2 \\ b\, b^* &= \beta/2 \end{aligned} \tag{2.19}$$

where $\alpha^2 + \beta^2(\vec{n} \cdot \vec{n}) = 2\beta$ and α,β and $n_i \in F_p$.

In solving the above equations for a and b; two cases are considered:

CASE (I): $\beta/2$ is square - as it was mentioned earlier any element Z of F_{p^2} could be written in one of the forms:

$$\begin{aligned} Z &= \rho\omega \\ & \qquad \rho \in F_p \\ Z &= \rho\xi \end{aligned} \tag{2.20}$$

where ω's and ξ's were defined as polar form analogues of $e^{i\phi}$'s. Now considering eqn. (2.19) we can write:

$$\begin{aligned} b &= \rho\omega \\ a &= \rho'\omega' \end{aligned} \tag{2.21}$$

then $\quad b\, b^* = \rho^2\, \omega\, \omega^* = \rho^2 = \beta/2 \quad$ or

$\rho = \sqrt{\beta/2}$ i.e. "positive" root of $\beta/2$ also

$$ab^* = ba^* = \rho\rho'\omega^*\omega'$$
$$= \rho\rho'\omega\omega'^*$$
$$= -\alpha/2$$

(note that: $\omega\omega'^* = \omega^*\omega' \equiv$ ("real" $\varepsilon\ F_p) = \pm 1$) Whence:

$$\rho' = -\frac{\alpha}{2}\sqrt{\frac{2}{\beta}}\ \frac{1}{\omega'\omega^*}\ ,$$ and we can then write A as:

$$A_\omega^{\vec{n},\alpha,\beta} = \left(-\frac{\alpha}{2}\ \sqrt{\frac{2}{\beta}}\ \frac{1}{\omega'\omega^*}\,\omega' + i\sqrt{\frac{\beta}{2}}\ \omega\ \vec{\sigma}\cdot\vec{n}\right)$$
$$= \omega\left(-\frac{\alpha}{2}\sqrt{\frac{2}{\beta}} + i\sqrt{\frac{\beta}{2}}\ \vec{\sigma}\cdot\vec{n}\right)\ . \qquad (2.22)$$

Now let us denote:

$$\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta} = \left(-\frac{\alpha}{2}\sqrt{\frac{2}{\beta}}\ + i\ \sqrt{\frac{\beta}{2}}\ \vec{\sigma}\cdot\vec{n}\right), \qquad (2.23)$$

Where the $+$ sign on the U refers to the determinant of the matrix $\overset{+}{U}$ which is $\dfrac{\alpha^2 + \beta^2(n.n)}{2\beta} = 1$, then

$$A_\omega^{\vec{n},\alpha,\beta} = \omega\ \overset{+}{U}{}_{\vec{n}}^{\ \alpha,\beta} \qquad (2.24)$$

Note that, in (2.19), when $\beta/2 = 0$; then $\alpha = 0$ and hence

$$A_\omega^{\vec{n},o,o} = \omega 1 \quad \text{i.e.} \quad \overset{+}{U}{}_{\vec{n}}^{\ 0,0} \equiv 1.$$

CASE (II): $\beta/2$ is not-square —— To solve eqn. (2.19) in this case, we should write a and b as:

$$a = \rho'\xi'$$
$$b = \rho\xi$$

then

$$b\,b^* = \rho^2\,\xi\xi^*$$

$$= -\,\rho^2 = \beta/2$$

or

$$\rho = \sqrt{-\beta/2}$$

also

$$\rho' = -\frac{\alpha}{2}\sqrt{-2/\beta}\;\frac{1}{\xi'\xi^*}$$

(note that again: $\xi\,\xi'^* = \xi^*\xi' = \pm 1$)

As a result, A for this case will be given by:

$$A_{\xi}^{\vec{n},\alpha,\beta} = \left(-\frac{\alpha}{2}\sqrt{-2/\beta}\;\frac{1}{\xi'\xi^*}\;\xi' + i\sqrt{-\beta/2}\;\xi\;\vec{\sigma}\cdot\vec{n}\right)$$

$$= \xi\left(+\frac{\alpha}{2}\sqrt{-2/\beta} + i\sqrt{-\beta/2}\;\vec{\sigma}\cdot\vec{n}\right) \qquad (2.25)$$

Again, we denote:

$$\bar{U}_{\vec{n}}^{\alpha,\beta} = \frac{\alpha}{2}\sqrt{-2/\beta} + i\sqrt{-\beta/2}\;\vec{\sigma}\cdot\vec{n} \qquad (2.26)$$

where the - sign refers to the determinant of the matrix being -1

$$\left(\det \bar{U}_{\vec{n}}^{\alpha,\beta} = -\frac{\alpha^2 + \beta^2(n.n)}{2\beta} = -1\right), \quad \text{then}$$

$$A_{\xi}^{\vec{n},\alpha,\beta} = \xi\;\bar{U}_{\vec{n}}^{\alpha,\beta} \qquad (2.27)$$

Now the unitary transformation:

$$A\,\vec{\sigma}\cdot\vec{V}\;A^{\dagger} \quad \text{with} \quad A\,A^{\dagger} = 1,$$

in view of the above discussion, will become

$$(A_{\omega}^{\vec{n},\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(A_{\omega}^{\vec{n},\alpha,\beta})^{\dagger} = \omega\omega^{*}(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger}$$

$$= (+1)(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger}$$

$$= (\det \overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger} \tag{2.29}$$

and

$$(A_{\xi}^{\vec{n},\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(A_{\xi}^{\vec{n},\alpha,\beta})^{\dagger} = \xi\xi^{*}(\bar{U}{}_{\vec{n}}^{\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(\bar{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger}$$

$$= (-1)(\bar{U}{}_{\vec{n}}^{\alpha,\beta})(\vec{\sigma}\cdot\vec{V})(\bar{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger}$$

$$= (\det \bar{U}{}_{\vec{n}}^{\alpha,\beta})(\bar{U}{}_{\vec{n}}^{\alpha,\beta})(\sigma\cdot V)(\bar{U}{}_{\vec{n}}^{\alpha,\beta})^{\dagger} \tag{2.30}$$

The set of all matrices $\overset{+}{U}{}_{\vec{n}}^{\alpha,\beta}$ and $\bar{U}{}_{\vec{n}}^{\alpha,\beta}$ defined by (2.23) and (2.26) for all possible α,β and $\vec{n}$ is indeed what we denoted by $SU^{\pm}(2,F_{p^2})$. Also according to the above considerations to every "rotation" operator $R \in R(3,F_p)$ there corresponds some $\pm U \in SU^{\pm}(2,F_{p^2})$, where the relevant "rotation" in terms of U is expressed by:

$$(\vec{\sigma}\cdot\vec{V}') = (\det U)\,U(\vec{\sigma}\cdot\vec{V})U^{\dagger} \tag{2.31}$$

Moreover, if R_1, R_2 and $R_3 = R_1R_2 \in R(3,F_p)$ correspond respectively to $\pm U_1$, $\pm U_2$ and $U_3 \in SU^{\pm}(2,F_{p^2})$, from (2.31) it then follows that:

$$U_3 = U_1U_2 \quad \text{i.e.} \quad R_1R_2 \longrightarrow \pm U_1U_2$$

Therefore the set of all matrices $\overset{+}{U}_{\vec{n}}^{\alpha,\beta}$ and $\overset{-}{U}_{\vec{n}}^{\alpha,\beta}$ i.e. $(SU^{\pm}(2,F_{p^2})$ provides a homomorphic representation of the "rotation" group $R(3,F_p)$. Also according to our notation in cases (I) and (II) of this section, the correspondence between $R_{\vec{n}}^{\alpha,\beta}$ and $U_{\vec{n}}^{\alpha,\beta}$ can be summarized as:

$$\begin{aligned} R_{\vec{n}}^{\alpha,\beta} \quad (\beta/2 \quad \text{"positive"}) &\to \pm \overset{+}{U}_{\vec{n}}^{\alpha,\beta} \\ R_{\vec{n}}^{\alpha,\beta} \quad (\beta/2 \quad \text{"negative"}) &\to \pm \overset{-}{U}_{\vec{n}}^{\alpha,\beta} \end{aligned} \tag{2.32}$$

2.3. Modular Irreducible representations of $SU^{\pm}(2,F_{p^2})$

To end this Chapter, we note that all the irreducible representations of $SU^{\pm}(2,F_{p^2})$ are constructed in a way not dissimilar to that of SU(2) [15] —— The homogenous monomials

$$f_m^j(u,v) = N_m^j \; u^{j+m} \; v^{j-m} \tag{2.33}$$

(where m and j are both integers or half integers, $-j \le m \le j$, and N_m^j, u, $v \in F_{p^2}$) are taken as a basis of the carrier space. Then for the group $SU(2,F_{p^2})$ with a general group element

$$U = \begin{bmatrix} \eta & \nu \\ -\nu^* & \eta^* \end{bmatrix} \qquad \det U = \pm 1,$$

the transformation

$$\begin{aligned} T^{(0)}(U)\; f_m^j(u,v) &= f_m^j(U^{-1}\binom{u}{v}) = N_m^j(\eta^* u - \nu v)^{j+m} \; (\nu^* u + \eta v)^{j-m} \\ &= \sum_{m'} D_{m'm}^{(1,0)}(U)\; f_m^j(u,v) \end{aligned} \tag{2.34}$$

defines one set of $(2j + 1)$ dimensional representations of $SU^{\pm}(2,F_{p^2})$ with $D^{(j,o)}_{m'm}(U)$ given by [19]

$$D^{(j,o)}_{m'm}(U) = \frac{N^j_m}{N^j_{m'}} \sum_{k=\max(0,\, m-m')}^{\min(j+m,\, j-m')} C^{j+m}_k \; C^{j-m}_{k+m'-m}$$

$$X\ \eta^{j-m'-k}\eta^{*j+m-k}\ \nu^k\ (-\nu^*)^{k-m+m'} \qquad (2.35)$$

It may be shown that the representations $\{D^{(j,0)}(U)\,|\,j=0,1/2,1,\ldots\ldots\}$ are irreducible if and only if $j \leq \frac{P-1}{2}$ i.e. $(j=0,1/2,\ldots\ldots\frac{P-1}{2})$ [15,20]. Moreover the appearance of matrices U of determinants -1 in $SU^{\pm}(2,F_{p^2})$ leads to the possibility of another set of representations for $SU^{\pm}(2,F_{p^2})$:

$$\{D^{(j,1)}(U)\,|\,D^{(j,1)}_{m'm}(U) = \det U\, D^{(j,0)}_{m'm} \quad j = 0,1/2,1 \ldots \frac{P-1}{2}\},$$

$$U \in SU^{\pm}(2,F_{p^2}) \qquad (2.36)$$

The representation $D^{(j,1)}(U)$ is inequivalent to $D^{j,0}(U)$ and likewise is irreducible if and only if $j \leq \frac{P-1}{2}$ [15, 20]. In view of the above discussion the label j will no longer suffice to specify all the irreducible representations of $SU^{\pm}(2,F_{p^2})$, introducing a new two-valued label e, all the irreducible representations found so far will be given by the set:

$$\{D^{(j,e)}(U)\,|\,D^{j,e}_{m'm}(U) = (\det U)^{e}\, D^{(j,0)}_{m'm}(U)\}, \qquad (2.37)$$

where

$$j = 0,\ 1/2,1 \ldots \frac{P-1}{2}$$

$$e = 0,1$$

It is worth noting that from a quantum mechanical point of view, e is a new quantum number: an interpretation of e supported by experimental evidence has been given by Beltrametti, where he has associated it with Leptons [15, 20].

CHAPTER III

THE "REAL" 2-DIMENSIONAL REPRESENTATION OF THE FINITE GEOMETRY ROTATION GROUP

SUMMARY

In previous chapters we studied the feature of geometrical concepts in the framework of finite geometry. We saw that in such a geometry, although its analytical machinery is not dissimilar to that of ordinary geometry, due to the very fact that the field is finite and the sum of its square-elements may be a not-square element; some new features arise. For example, in Chapter (I), we saw that, besides the usual type of direction $\vec{n}_{+1}(\vec{n}_{+1} \cdot \vec{n}_{+1} = 1)$ and its relevant rotation operator $R_{\vec{n}_{+1}}$ we have directions $\vec{n}_o(\vec{n}_o \cdot \vec{n}_o = 0)$ and $\vec{n}_{-1}(\vec{n}_{-1} \cdot \vec{n}_{-1} = -1)$ with their corresponding rotation operators $R_{\vec{n}_o}$ and $R_{\vec{n}_{-1}}$. Also in Chapter (II), when dealing with all the irreducible representations of rotation group $R(3,F_p)$, we saw that the label j does no longer suffice to specify the representation, but a new double valued label $(e = 0,1)$, besides j is indeed necessary. Furthermore, in the following we will see that for the finite geometry's version of $SU(2)$: i.e. $SU^{\pm}(2,F_{p^2})$, as against the classical case, there exists a similarity transformation under which any element U of $SU^{\pm}(2,F_{p^2})$ will transform into a "real" 2×2 matrix. The set of such transformed matrices is indeed what we mean by $SL^{\pm}(2,F_p)$. It is to be understood that the "real" representation $SL^{\pm}(2,F_p)$ is isomorphic to $SU^{\pm}(2,F_{p^2})$ and consequently homomorphic to the "rotation" group $R(3,F_p)$. In the following, after saying

as much as needed to specify the group $SL(2,F_p)$, the special case $P = 3$ will be considered (Sec. 2 of this chapter) and thereby the simple representation of the Octahedral group (O) will be presented. The simplicity of this representation will be illustrated by few examples.

3.1. The group $SL^{\pm}(2,F_p)$

To specify the group $SL^{\pm}(2,F_p)$ it is sufficient to find the similarity transformation between $SU^{\pm}(2,F_{p^2})$ and $SL^{\pm}(2,F_p)$. In doing so, we consider the similarity transformation of U the general element of the group $SU^{\pm}(2,F_{p^2})$ by a matrix S with $\det S \neq 0$.

Considering the equations (2.23) and (2.26) the general element of the group $SU^{\pm}(2,F_{p^2})$ is written as:

$$U = t_o + i\,\vec{\sigma} \cdot \vec{t} \tag{3.1}$$

Where σ_i $(i = 1, 2, 3)$ are the Pauli matrices,

$t_o,\ t_i (i = 1, 2, 3) \in F_p$ and

$$t_o^2 + \vec{t} \cdot \vec{t} = \pm 1 \tag{3.2}$$

Under a similarity transformation S; U transforms to

$$SUS^{-1} = t_o + i\,\vec{T} \cdot \vec{t} \tag{3.3}$$

where

$$T_i = S\,\sigma_i\,S^{-1} \tag{3.4}$$

It is convenient to introduce τ_i matrices as:

$$T_i = i\,\tau_i \tag{3.5}$$

The τ_i matrices satisfy the following relation;

$$\begin{aligned}\tau_i\tau_j &= S(\frac{1}{i^2}\,\sigma_i\,\sigma_j)S^{-1}\\ &= -\,S(\delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k)S^{-1}\\ &= -\,\delta_{ij} + \varepsilon_{ijk}\,\tau_k\end{aligned} \tag{3.6}$$

where

$$\delta_{ij} = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix}$$

Moreover, since under a similarity transformation the trace of a matrix remains invariant, τ_i matrices are of trace zero (trace $\sigma_i = 0$), and hence can be written as:

$$\tau_i = \vec{\sigma} \cdot \vec{I}_i \tag{3.7}$$

Using (3.7) in (3.6) we have:

$$(\vec{\sigma} \cdot \vec{I}_i)(\vec{\sigma} \cdot \vec{I}_j) = -\delta_{ij} + \varepsilon_{ijk}(\vec{\sigma} \cdot \vec{I}_k) \tag{3.8}$$

The left-hand side of (3.8) can be expressed as:

$$(\vec{\sigma} \cdot \vec{I}_i)(\vec{\sigma} \cdot \vec{I}_j) = \vec{I}_i \cdot \vec{I}_j + i\,\vec{\sigma} \cdot \vec{I}_i \times \vec{I}_j \tag{3.9}$$

Comparing the right-hand sides of (3.8) and (3.9) we may set up the following equalities:

$$\begin{aligned}\vec{I}_i \cdot \vec{I}_j &= -\,\delta_{ij}\\ i(\vec{I}_i \times \vec{I}_j) &= \varepsilon_{ijk}\,\vec{I}_k\end{aligned} \tag{3.10}$$

Now to find S such that:

$$SUS^{-1} = t_o - \vec{\tau}\cdot\vec{t} \equiv L$$

where

$$L \in SL^{\pm}(2,F_p),$$

we need to choose the matrices τ_i such that while they satisfy the relation:

$$\tau_i\tau_j = -\delta_{ij} + \varepsilon_{ijk}\,\tau_k\,,$$

they are also "real", i.e., the matrix entries $(\tau_i)_{mm'} \in F_p$. This requirement can be fulfilled by an appropriate choice of the vectors $\vec{I}_i$. It is evident from (3.7) and the first equation in (3.10) that the choice

$$[\vec{I}_3] = \begin{bmatrix} 0 \\ -i \\ 0 \end{bmatrix} \tag{3.11}$$

is a suitable one, since we will then have:

$$\tau_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.12}$$

With $\vec{I}_3$ as in (3.11), the other two vectors i.e. $\vec{I}_1$ and $\vec{I}_2$ in (3.10) are of the form:

$$[\vec{I}_1] = \begin{bmatrix} \delta \\ 0 \\ \gamma \end{bmatrix}, \quad [\vec{I}_2] = \begin{bmatrix} \gamma \\ 0 \\ -\delta \end{bmatrix}$$

and as a result:

$$\tau_1 = \begin{bmatrix} \gamma & \delta \\ \delta & -\gamma \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} -\delta & \gamma \\ \gamma & \delta \end{bmatrix} \tag{3.13}$$

where

$$\gamma^2 + \delta^2 = -1 \tag{3.14}$$

We remark that; in a finite field F_p, there are $P + 1$ solutions to an equation of the form $X_1{}^2 + X_2{}^2 = -1$, (see Eqn. 1.10). Thus we may take γ and δ in (3.14) to be elements of F_p. In this sense, then the matrices τ_i and hence $SUS^{-1} = t_o - \vec{\tau} \cdot \vec{t}$ are "real". Now, the transformation matrix S can be found by solving, simultaneously, two of the matrix equations in:

$$S\, \sigma_i\, S^{-1} = i\, \tau_i$$

where

σ_i are the Pauli matrices and

τ_i are given in (3.12) and (3.13).

The solution for S is:

$$S = \begin{bmatrix} -i & \gamma + i\delta \\ 1 & \delta - i\gamma \end{bmatrix} \quad \text{where} \quad \begin{array}{l} \gamma \;\&\; \delta \in F_p \\ \gamma^2 + \delta^2 = -1 \end{array} \tag{3.15}$$

Then with S as specified in (3.15), the group $SL^{\pm}(2,F_p)$ is given by the set; $\{SUS^{-1} : \forall\ U \in SU^{\pm}(2,F_{p^2})\}$. To be more specific about the labels $(\pm)$ in $SL^{\pm}(2,F_p)$, we remind that under a similarity transformation the determinant of a matrix is invariant and hence:

$$\begin{aligned} SL^{+}(2,F_p) &= \{SU^{+}S^{-1} : \forall\ U^{+} \in SU^{+}(2,F_{p^2})\ \det U^{+} = 1\} \\ SL^{-}(2,F_p) &= \{S\bar{U}S^{-1} : \forall\ \bar{U} \in S\bar{U}(2,F_{p^2})\ \det U^{-} = -1\} \end{aligned} \tag{3.16}$$

We remark that for U; an element of $SU^{\pm}(2,F_{p^2})$, the transformation corresponding to the rotation of a vector $\vec{V} \to \vec{V}'$ is expressed by:

$$\sigma \cdot V' = \det U \; U(\sigma \cdot V)U^{\dagger} \tag{3.17}$$

Now under a similarity transformation by S (S as in (3.15)) equation (3.17) becomes:

$$S(\vec{\sigma} \cdot \vec{V}')S^{-1} = \det U \; S \; U(\vec{\sigma} \cdot \vec{V}) \; U^{\dagger} \; S^{-1}$$

$$= \det U \; (SUS^{-1})[S(\vec{\sigma} \cdot \vec{V})S^{-1}](SU^{\dagger}S^{-1}) \tag{3.18}$$

where we have that

$$SUS^{-1} = L \in S^{+}[(2,F_p)],$$

$$S \; \vec{\sigma} \; S^{-1} = i \; \vec{\tau}$$

and

$$\det U(SU^{\dagger}S^{-1}) = L^{-1}$$

Then (3.18) is written as:

$$\vec{\tau} \cdot \vec{V}' = L(\vec{\tau} \cdot \vec{V}) \; L^{-1} \tag{3.19}$$

From (3.19) it follows that the "real" matrix $L = SUS^{-1}$, with the transformation rule as specified by the right-hand side of (3.19), describes equivalently the same rotation as U in (3.17) does, and likewise if $\pm$ U correspond to R some element of $R(3,F_p)$, equivalently we have the correspondence: $R \to \pm L$. In view of the above discussion, the correspondences in (2.32) now read:

$$R_{\vec{n}}^{\alpha,\beta}\ (\beta/2\ \text{"positive"}) \rightarrow \pm\, S\overset{+}{U}_{\vec{n}}^{\alpha,\beta}\, S^{-1} = \pm\, \overset{+}{L}_{\vec{n}}^{\alpha,\beta} \in SL(2,F_p)$$

$$R_{\vec{n}}^{\alpha,\beta}(\beta/2\ \text{"negative"}) \rightarrow \pm\, S\overset{-}{U}_{\vec{n}}^{\alpha,\beta} S^{-1} = \pm\, \overset{-}{L}_{\vec{n}}^{\alpha,\beta} \in SL(2,F_p)$$

(3.20)

Using equations (2.23) and (2.26), the "real" matrices L will be given in terms of α, β and $\vec{n}$ as:

$$\overset{+}{L}_{\vec{n}}^{\alpha,\beta} = -(\frac{\alpha}{2}\sqrt{\frac{2}{\beta}} + \sqrt{\frac{\beta}{2}}\ \vec{\tau}\cdot\vec{n})\ , \quad \beta/2\ \text{"positive"}$$

$$\overset{-}{L}_{\vec{n}}^{\alpha,\beta} = (\frac{\alpha}{2}\sqrt{-\frac{2}{\beta}} - \sqrt{-\frac{\beta}{2}}\ \vec{\tau}\cdot\vec{n}),\quad \beta/2\ \text{"negative"} \qquad (3.21)$$

where. as before $\alpha^2 + \beta^2(\vec{n}\cdot\vec{n}) = 2\beta$.

In the following section, $SL(2,F_p)$ for $P = 3$ will be treated. The correspondence of the octahedral group (O) and $SL(2,F_3)$ will then be incorporated in a single formula.

3.2. Simple representation of the groups (T) and (O)

In section (4) of Chapter (I), when discussing the rotational symmetries enjoyed by a cube i.e. the group (O) and its subgroup (T), we showed that the "rotation" group of the finite geometry over F_3, i.e. $R(3,F_3)$, is isomorphic to the group (O) and provides us with a 3-dimensional representation of it. Also in the previous section of this Chapter, we saw that the group $SL(2,F_p)$, i.e., the set of all matrices $L_{\vec{n}}^{\alpha,\beta}$ in (3.21) for all possible "rotation" paramters α, β and $\vec{n}$, is homomorphic to the group $R(3,F_p)$. Therefore for $P = 3$, the "real" matrices $L_{\vec{n}}^{\alpha,\beta}$ as well will provide us with a representation of the group (O). In this representation each one of the 24 symmetry elements tabulated in Tables (I) to (III) of Chapter (I), is represented by an element of $SL(2,F_3)$ up to a sign ambiguity, i.e.,

$$(h_1\ h_2\ h_3)^{\theta/2\pi} \rightarrow \pm L \in SL(2,F_p)$$

We recall that according to our representation of F_p (see Eqn. (1.2) F_3 is represented by numbers modulo 3 i.e. $F_3 = \{0 \equiv 3, 1, -1 \equiv 2\}$ where (1) and (-1) are respectively the only "positive" and "negative" elements of F_p. This being the case, the matrices $L_{\vec{n}}^{\alpha,\beta}$ in (3.21) now become:

$$\frac{\beta}{2} = 1\ (\beta = -1) \rightarrow \overset{+}{L}{}_{\vec{n}}^{\alpha,-1} = (\alpha - \vec{\tau} \cdot \vec{n}) \in SL^{\pm}(2,F_3)$$

$$\frac{\beta}{2} = -1 (\beta = 1) \qquad \overset{-}{L}{}_{\vec{n}}^{\alpha,1} = -(\alpha + \vec{\tau} \cdot \vec{n}) \in SL^{\pm}(2,F_3)$$

$$\alpha = 0, \quad \beta = 0 \qquad L_{\vec{n}}^{0,0} = 1 \qquad (3.22),$$

where the matrices τ_i, are obtained by solving the equation $\gamma^2 + \delta^2 = -1$ in F_3. The $P + 1 = 4$ solutions to this equation are:

$$\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad (3.23)$$

Now it is convenient to choose the solution $\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then from (3.12) and (3.13) the τ_i matrices are:

$$\tau_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (3.24)$$

The correspondence of the group O and the group $R(3,F_3)$ was established in section (3) of Chapter (I), the results are given through (1.33a to 1.33c). Below we reproduce these results and in junction to them we give the corresponding $L^{\alpha,\beta}_{\vec{n}}$'s obtained from (3.22).

$$\vec{n}_o \cdot \vec{n}_o = o$$

$$(h_1\ h_2\ h_3)^{1/3} \equiv R^{-1,-1}_{\vec{n}_o} \rightarrow \pm (1 + \vec{\tau} \cdot \vec{n}_o)$$

$$(h_1\ h_2\ h_3)^{2/3} \equiv R^{1,-1}_{\vec{n}_o} \rightarrow \pm (-1 + \vec{\tau} \cdot \vec{n}_o) \tag{3.25a}$$

$$\vec{n}_1 \cdot \vec{n}_1 = 1$$

$$(h_1\ h_2\ h_3)^{1/4} \equiv R^{1,1}_{\vec{n}_1} \rightarrow \pm (1 + \vec{\tau} \cdot \vec{n}_1)$$

$$(h_1\ h_2\ h_3)^{1/2} \equiv R^{0,-1}_{\vec{n}_1} \rightarrow \pm \vec{\tau} \cdot \vec{n}_1$$

$$(h_1\ h_2\ h_3)^{3/4} \equiv R^{-1,1}_{\vec{n}_1} \rightarrow (-1 + \vec{\tau} \cdot \vec{n}_1) \tag{3.25b}$$

$$\vec{n}_{-1} \cdot \vec{n}_{-1} = -1; (h_1 h_2 h_3)^{1/2} \equiv R^{0,1}_{\vec{n}_{-1}} \rightarrow \pm \vec{\tau} \cdot \vec{n}_{-1}$$

$$\vec{n} \cdot \vec{n} = 0, \pm 1\ ;\ 1 \equiv R^{0,0}_{\vec{n}} \rightarrow \pm 1 \tag{3.25c}$$

Now we remark that, according to our consideration in section (3) of Chapter (I), $\vec{h}(h_1\ h_2\ h_3)$ in the above relations in each case on the left is representatively the same as $\vec{n}$ on the right (i.e. when h_1, h_2, and h_3 are considered to be mod. 3). Also in the notation used to designate the cubic symmetry elements (rotational), we replace ( )

by [ ], i.e. $(h_1\ h_2\ h_3)^{\theta/2\pi} \to [h_1\ h_2\ h_3]^{\theta/2\pi}$, where the change is only meant to emphasize our reference to the 2-dimensional "real" representation of the relevant symmetry operator. Then in view of the above discussion and relations (3.25a) to (3.25c), the "real" 2-valued 2-dimensional representation of the group (O) is expressed as:

$$[h_1\ h_2\ h_3]^{1/3} \equiv \pm (1 + \vec{\tau} \cdot \vec{n}_o) \equiv \pm (1 + 3 + \vec{\tau} \cdot \vec{n}_o)$$

$\vec{n}_o \cdot \vec{n}_o = 0$

$$[h_1\ h_2\ h_3]^{2/3} \equiv \pm (-1 + \vec{\tau} \cdot \vec{n}_o) \equiv \pm(2 + 3 + \tau \cdot \vec{n}_o)$$

$$[h_1\ h_2\ h_3]^{1/4} = \pm (1 + \vec{\tau} \cdot \vec{n}_1) \equiv \pm (-1-4 + \vec{\tau} \cdot \vec{n}_1) \equiv \mp (1+4 - \vec{\tau} \cdot \vec{n}_1)$$

$\vec{n}_1 \cdot \vec{n}_1 = 0$ $\quad [h_1\ h_2\ h_3]^{1/2} \equiv \pm \vec{\tau} \cdot \vec{n}_1 \equiv \mp (1+2 - \vec{\tau} \cdot \vec{n}_1)$

$$[h_1\ h_2\ h_3]^{3/4} \equiv \pm(-1 + \vec{\tau} \cdot \vec{n}_1) \equiv \pm (-3-4 + \vec{\tau} \cdot \vec{n}_1) \equiv \mp (3+4 - \vec{\tau} \cdot \vec{n}_1)$$

$\vec{n}_{-1} \cdot \vec{n}_{-1} = -1$ $\quad [h_1\ h_2\ h_3]^{1/2} \equiv \pm \vec{\tau} \cdot \vec{n}_{-1} \equiv \mp (1+2 - \vec{\tau} \cdot \vec{n}_{-1})$,

and the unit element $\pm 1 = \pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Now in general denoting any of the above operators by $[h_1\ h_2\ h_3]^{i/j}$, where i/j is the order of rotation and depending on the relevant operator it has one of the values; 1/3, 2/3, 1/4, 1/2, 3/4, the corresponding 2 x 2 "real" matrix up to a sign ambiguity is written as:

$$[h_1\ h_2\ h_3]^{i/j} = \pm [i + j + (-1)^k \vec{\tau} \cdot \vec{n}_k] \tag{3.26}$$

where (i) and (j) in the right-hand side of (3.26) are to be considered numbers modulo 3, moreover as we mentioned earlier h_1, h_2

and h_3, are considered to be elements of F_3 (mod. 3) and hence $\vec{h} \equiv \vec{n}_k$. Equation (3.26) is infact the simple formula we promised to give. Clearly with this formula at hand, the explicit 2 x 2 matrix corresponding to any of the cubic symmetry elements (rotational) is easily determined. It is from this point that there is no need of a written list of the matrices or a table.

We can simply ask for the desired symmetry element, then, h_1, h_2, h_3 and i/j are decided and hence the relevant matrix. For example, let us ask for $[111]^{1/3}$, then we will have

$$[111]^{1/3} = \pm [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{note} \quad 1 + 3 \equiv 1 \quad (\text{mod. } 3)$$

To give another example let us ask for $[110]^{1/2}$:

$$[110]^{1/2} = \pm (1 + 2 + (-1)^{-1} \vec{\tau} \cdot \vec{n}_{-1})$$

$$\equiv \pm \vec{\tau} \cdot \vec{n}_{-1} \equiv \pm [\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}]$$

$$\equiv \pm \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \equiv \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From the appearance of the matrices in the above example, it is seen that they are in form simple; i.e. they are small and for their entries they have nothing except 0, 1, or -1. Clearly this kind of simplicity is common to all of the matrices obtained in the way prescribed. For them the matrix multiplication is direct and simple and not tedious at all. Moreover, to end our discussion we show that there is no need of the group multiplication table, since once we have the explicit matrix i.e. the compact form of the right-hand side of (3.26), we can decode it back into the group spectrum. Decoding back simply means to find h_1, h_2 h_3 and i/j, this in other words means to determine for any two or more consecutive rotations an equivalent rotation. Let $\pm$ L be the matrix that is to be decoded back into the group spectrum. We then write L in the form:

$$L = t_o + \vec{\tau} \cdot \vec{t}, \tag{3.27}$$

where t_o and t_i are found from:

$$\begin{aligned} t_o &= 1/2 \ \text{Trace } L = - \text{Trace } L \\ t_i &= -1/2 \ \text{Trace } \tau_i L = \text{Trace } \tau_i L \end{aligned} \tag{3.28}$$

(note that: Trace $\tau_i \ \vec{\tau} \cdot \vec{t}$ = Trace $(-1t_i) = t_i$).

Now from comparison of (3.27) with (3.26) we have

$$\begin{aligned} t_o &= i + j \\ \vec{t} &= (-1)^k \ \vec{n}_k \equiv (-1)^k \ \vec{h} \end{aligned} \tag{3.29}$$

therefore, once t_1, t_2 and t_3 are obtained (eqn. 3.28) the type of axis i.e. $k(k = \vec{t} \cdot \vec{t})$ and hence $\vec{h}$, j (for k = o, 1 or -1 respectively j = 3, 4 or 2) and then i will be all determined. To give an example we consider rotations $[111]^{1/3}$ and $[110]^{1/2}$ of the previous examples and find the equivalent rotation of $[111]^{1/3}$ $[110]^{1/2}$:

$$[111]^{1/3} \ [110]^{1/2} = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \pm \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we have; $t_o = \frac{1}{2} = -1$. Also using τ_i given in (3.24) we find $[\vec{t}] = (0, 1, 0)$. Then:

$$[111]^{1/3}\ [110]^{1/2} = \pm\ [-1 + \widetilde{[\vec{\tau}]} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}]$$

$$= \pm\ [+1 - \widetilde{[\vec{\tau}]} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}]$$

$$= \pm\ [3 + 4 - \widetilde{[\vec{\tau}]} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}] \equiv [010]^{3/4}$$

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