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LATERAL STABILITY OF UNRESTRAINED REINFORCED CONCRETE BEAMS

BY

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ABSTRACT:

The lateral stability of reinforced concrete beams was investigated experimentally and analytically. A total of 14 small scale rectangular beams were load tested with support conditions including unrestrained cantilever, simple-support and overhung. Of the 14 test specimens, 11 failed by lateral buckling.

It is concluded that current building code provisions which use the slenderness ratio L/b are incorrect and that the ratio Ld/b^2 should be used. For unrestrained cantilevers, a maximum value of $Ld/b^2 = 600$ is recommended pending further investigation.

ACKNOWLEDGEMENTS

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NOTATION:

X, Y, Z	- Global axes of a beam, corresponding to the unloaded position
ξ, η, ζ	- Local axes of a beam cross-section in the deformed position
A	- Area
A_s	- Area of tensile reinforcement
A_s'	- Area of compressive reinforcement
A_t	- Area of one leg of a vertical stirrup
B	- Lateral flexural rigidity ($B = EI_\eta$)
B_1	- Vertical flexural rigidity ($B_1 = EI_\xi$)
C	- Torsional rigidity
C_w	- Warping rigidity
E	- Young's Modulus (Elastic modulus)
E_s	- Elastic modulus of steel
E_{sec}	- Secant modulus of concrete
F	- Force
G	- Modulus of rigidity
G_c	- Modulus of rigidity of concrete
G_c'	- Reduced modulus of rigidity of concrete
G_s	- Modulus of rigidity of steel
I	- Moment of inertia (second moment of area)
I_{sy}	- Moment of inertia of the reinforcement about the Y Axis

I_1, I_2 - Moment of inertia of the reinforcement (about the Y Axis) at one face (top or bottom) of the beam ($I_{sy} = I_1 + I_2$)

I_ξ - Moment of inertia in the strong direction of the beam

I_ζ - Moment of inertia in the weak direction

L - Length of a beam

M - Bending moment

M_X, M_Y, M_Z - Bending moments in the directions of the global axes of the beam

M_ξ M_η M_ζ - Bending moments with respect to the local axes

M_e - Moment required to produce an extreme fiber stress of $0.85f'_c$ on a transformed elastic section having a modular ratio n = 15

M_u - Ultimate flexural moment

P - Load

P_{cr} - Critical lateral buckling load of a beam

P^*_{cr} - Ideal critical load, at $0.85f'_c$

S - Scale factor, ratio of prototype size to model size

a - Depth of the rectangular concrete stress block at ultimate moment

b - Width of a rectangular beam

b_s - Width out to out of reinforcement

b_1 - Width enclosed by a stirrup

c	- Depth from compressive face to neutral axis at ultimate moment
d	- Depth of a rectangular beam from compressive face to centroid of tensile reinforcement
d'	- Total depth of a rectangular beam
d_1	- Depth enclosed by a stirrup
f_c'	- Compressive strength of a 3" x 6" concrete cylinder specimen
f_T	- Stress in the tensile reinforcement in an over-reinforced flexural member at ultimate moment
f_{yC}	- Yield stress in the compressive reinforcement
f_{yT}	- Yield stress in the tensile reinforcement
h_1	- Height of load application above or below the centroid of a beam
h	- Distance between centroids of tensile and compressive reinforcement
k_1, k_2 --	- Constants
n	- Modular ratio of concrete ($n = E_s/E_{sec}$)
p	- Tensile reinforcement ratio ($p = (A_s - A_s')/bd$)
p_s	- Pitch of vertical stirrups
q	- A numerical factor dependent upon f_c' in the expression for "a"
s	- A measure of length
t_s	- Thickness of reinforcement parallel to Y direction
u, v	- Deflections of a beam in the X and Y directions respectively

$\bar{y}$	- Distance from the compressive face of a beam to the neutral axis of its transformed section
σ	- Stress
σ_{cr}	- Critical stress at lateral buckling
σ_s	- Average stress in the tensile reinforcement in a transformed section
ε	- Strain
ε_s	- Tensile reinforcement strain
ε_s'	- Compressive reinforcement strain
$\emptyset$	- Twist angle about the Z axis
Δ	- Deflection
ν	- Poisson's ratio
β	- A factor dependent upon d'/b
γ	- A factor dependent upon d_1/b_1

CHAPTER 1

INTRODUCTION

The introduction of high strength concrete and reinforcement have made the use of deep, slender concrete beam elements practical and economical in many structures. At some point, the slenderness of such beams is limited by lateral instability, but the phenomenon is referred to only briefly in available building codes(1)(2). The commentary on the ACI Code suggests that there is no problem of lateral instability for beams of reasonable proportions. The code writers appear, in this regard, to be thinking in terms of beams supported at two or more points and provided with ample lateral restraint. In the case of cantilevered beams, the restraint is considerably less and lateral instability may become a problem. This study was undertaken to explore the effects of lateral instability on the load-carrying capacity of slender cantilevered concrete beams.

A review of available literature indicates that very little has been written on the subject of lateral instability of concrete beams in general and almost nothing on cantilevered concrete beams. All of the experimental data reported have been for simply-supported beams tested under various loading conditions. As a result of the lack of experimental data for cantilevered beams, it was decided to proceed with an experimental program with the objective of measuring the lateral buckling loads for cantilevered concrete beams and correlating these buckling loads with parameters related to the beam geometry.

The study was carried out using fourteen small-scale reinforced concrete beam specimens. Ten specimens were loaded as laterally-unsupported cantilevers. Two were tested as overhung beams (laterally unsupported beams with loaded overhanging ends) to illustrate the most unstable case likely to be found in practice. Finally, two were tested as center-point-loaded simply-supported beams (as models of a larger beam referred to in the literature).

Initially, the study was to be confined to the consideration of under-reinforced beams, to be consistent with the prevailing ultimate strength design philosophy that all flexural members should fail in a ductile manner. A shortage of mild steel reinforcement forced a change, however, and some of the test specimens were over-reinforced. As a second objective, an attempt was made to establish geometric limits (in the form of a slenderness ratio) which would preclude a lateral buckling failure before a flexural failure occurred.

1.1 Relationship to Previous Studies

A survey of literature on the subject of lateral instability of beams was prepared by Lee[(3)]. This literature survey listed 142 articles beginning with a thesis written by Prandtl in 1899, considered to be the original work on the subject, and continued to the end of 1959. The survey dealt with metal beams under various loadings but did not consider concrete beams.

The first paper dealing with the stability of concrete beams, Marshall[4] gave a theoretical treatment of the problem without experimental results. Marshall concluded that code provisions which treat lateral instability of beams on the basis of the ratio L/b are inadequate; instead, the ratio Ld/b^2 should be used. The most recent codes[1][2] use the L/b criterion. (L = Length, d = depth, b = width)

The first record of test data for slender concrete beams was published in 1954 by Vasarhelyi and Turkalp[5]. Three beams with L/b ratios of 36 were tested. Although lateral deflections were observed, Vasarhelyi and Turkalp concluded that the strengths of their beams were not reduced by lateral instability.

Hansell and Winter[6] published the results of tests of ten beams with L/b ratios in the range 28.8 to 86.4. The specimens were tested as simple beams, loaded at their quarter points. The load points were designed to allow lateral deflection of the beam. No reduction was found in the ultimate strengths of the beams, and Hansell and Winter concluded that no lateral instability had occurred. They demonstrated that lateral instability occurred at an L/b ratio of 106 and concluded that the then current code provisions, requiring L/b not to exceed 32, were too restrictive. They further demonstrated that the secant modulus E_{sec}, corresponding to the extreme fibre strain, should be used for lateral instability computations for concrete beams.

Smith[7], in a discussion of Hansell and Winter's paper, reported on the testing of twelve simply-supported center-point-loaded microconcrete beams. His discussion was the first published report of concrete beam specimens which failed by lateral buckling. In addition, he discussed lateral stability of the test specimens in terms of the slenderness ratio Ld/b^2 proposed by Marshall[4].

Siev[8] reported on the testing of ten beams, six of which were rectangular and four of which were L-shaped in cross-section. Lateral instability was observed in all of the tests. In addition, Siev carried out a theoretical investigation of the flexural and torsional rigidity of beams in various states of loading, ranging from elastic uncracked to plastic cracked sections.

Sant and Bletzacker[9] tested eleven reinforced concrete beams with L/b ratios of 96 and d/b ratios in the range 3.78 to 12.45. The specimens were simply-supported and center-point-loaded, with rotation and lateral deflection allowed at the load point. Sant and Bletzacker concluded that there was a critical slenderness ratio of the form Ld/b^2 (and dependent upon the material properties and loading configuration) above which lateral buckling occurred. They suggested that the slenderness of the beam should be limited in terms of the maximum reinforcement ratio and plotted this criterion in the form of an interaction curve relating the slenderness ratio Ld/b^2 to the reinforcement ratio and yield strength, $(p)(f_{sy})$. Sant and Bletzacker plotted their results as well as those of Smith[7] and Hansell and Winter[6] on the interaction diagram and added those of Siev[8] in the discussion in a later publication[10].

Two specimens tested in the present study were scale models of Sant and Bletzacker's specimen B_{24}-1. They were used to determine the similarity between the results of model tests and prototype behaviour of laterally buckling beams.

In 1967, Massey(11) published the results of the tests of thirteen specimens which were simply-supported and loaded in pure bending. Of the thirteen specimens, eleven buckled laterally. Massey developed analytical expressions for the lateral flexural rigidity, the torsional rigidity and the warping rigidity of slender rectangular concrete beams and determined the reduced modulus of rigidity of concrete as a function of the bending moment. Massey's procedures are used later in this study to compute the theoretical buckling loads of the test specimens.

In a literature survey(12) published in 1969, Marshall analyzed the published results of Hansell and Winter(6), Sant and Bletzacker(9), Massey(11) and others not previously published. Marshall concluded, as he had done earlier(4), that lateral stability was a function of the slenderness ratio Ld/b^2 and, in addition, that the bending moment at lateral buckling was a function of the ratio $f_c' b^3 d/L$ (in which f_c' is the compressive strength of the concrete).

Marshall(4)(12), Smith(7) and Sant and Bletzacker(9) have pointed out that the slenderness ratio for the lateral buckling of rectangular beams should be written in the form Ld/b^2 rather than in the form L/b used in current building codes(1)(2). Their argument is as follows:

In simplified form, ignoring the warping rigidity, the elastic critical load, at which a rectangular beam buckles laterally, is given by[13]:

$$P_{cr} = K_1 \sqrt{BC}/L^2 \quad (1.1)$$

The constant k_1 is dependent upon the loading condition and the amount of lateral restraint, having the value 4.013 for an unrestrained tip-loaded cantilever and 16.94 for an unrestrained center-point-loaded simply-supported beam. The lateral flexural rigidity is given by:

$$B = k_2 d\ b^3 E_{sec} \quad (1.2)$$

The constant k_2 has the value 1/12 for an elastic uncracked section in which the reinforcement is ignored. The cross-sectional dimensions are given by b and d, and the secant modulus E_{sec} is used after Hansell and Winter[6]. The torsional rigidity is given by:

$$C = k_3 d\ b^3 G \quad (1.3)$$

The constant k_3 has the value 1/3 for a very narrow rectangular cross-section and G is the modulus of rigidity. For simplicity, the modulus of rigidity is assumed to equal:

$$G = k_4 E_{sec} \quad (1.4)$$

In which k_4 is an arbitrary constant. By substituting Equations (1.2) through (1.4) into Equation (1.1), and collecting all constant terms into one term, (k_5), the following is obtained.

$$P_{cr} = k_5 b^3 d \, E_{sec}/L^2 \quad (1.5)$$

If the usual expression for flexural stress at any point in the beam (My/I) is written in terms of the buckling load, the following expression results:

$$\sigma_{cr} = k_6 P_{cr} L/k_7 bd^2 \quad (1.6)$$

The constant k_6 depends on the loading configuration, and is 1.0 for a cantilever. The denominator is in the form of a section modulus (k_7 reflects the amount of cracking). By combining Equations (1.5) and (1.6) and writing all constant terms into one constant, (k_8), the stress at lateral buckling is given by:

$$\sigma_{cr} = \frac{k_8 E_{sec}}{Ld/b^2} \quad (1.7)$$

Equation (1.7) indicates that the critical stress at lateral buckling is a function of the secant modulus of the concrete and the loading and restraint geometry (as the summation of the effects of the constant terms) and the slenderness ratio Ld/b^2. It is based upon assumed linear behaviour, which assumption is inaccurate for reinforced concrete. It does, however, point out the nature of the governing geometric parameter for the discussion of lateral stability of rectangular beams.

Sant and Bletzacker assumed an under-reinforced rectangular cross-section and argued that the member should reach its ultimate flexural moment at or before the instant of lateral buckling. For these conditions, Equation (1.6) is rewritten with the reinforcement yield stress as the critical stress and the section modulus in terms of the tensile reinforcement, as follows:

$$f_y = k_5 PL/(k_9 d)(bdp) \qquad (1.8)$$

In Equation (1.8), the term $(k_9 d)$ is the distance between the tensile and compressive stress resultants and the parameter p is the tensile reinforcement ratio (as a fraction of the cross-section area). Combining Equations (1.5) and (1.8):

$$pf_y = \frac{k_{10} E_{sec}}{Ld/b^2} \qquad (1.9)$$

Equation (1.9) relates the reinforcement ratio and yield strength to the slenderness ratio of the beam. The term k_{10} reflects the loading and restraint geometry and the term E_{sec} reflects the concrete properties. This equation provides the basis for a code provision which limits the tensile reinforcement as a function of the beam slenderness.

In contrast to Sant and Bletzacker, Marshall(4)(12) approached the problem on the assumption of a balanced section and argued that the moment in the member should be limited to the point at which the concrete reached its compressive strength at the instant of lateral buckling. This approach is not well suited to use with the current ultimate strength design philosophy which requires that flexural members be under-reinforced.

Some work has been carried out to determine the laterally deflected shape of unrestrained elastic cantilevers. Piotrowski and Zihrul[14] used Moiré photography techniques and found that their results compared favourably with a numerical example published by Swann and Godden[15]. Woolcock and Trahair[16] published deflection measurements for a narrow rectangular cantilever, an I-shaped cantilever and an I-shaped simply-supported beam, all of steel. They also published theoretical results which agreed closely with the measured deflections.

Numerical and analytical analyses of the problem of lateral buckling of elastic beams have been published by several authors[13][15][16]. The linear-elastic analysis given by Timoshenko and Gere[13] was used as a basis for the theoretical discussion in this study.

1.2 Assumptions and Limitations in the Present Study

The experimental portion of this study consisted of the load testing of ten small-scale reinforced concrete members as laterally unsupported cantilevers and two as overhung beams. The cross-sections were kept constant and only the length was changed. The experimental work was, therefore, limited to a very small portion of the large range of possibilities of unstable beams.

The experimental data used in this study are based upon the testing of small-scale beams. It is assumed that the conclusions drawn from these data apply to full-sized members with similar loading and restraint conditions. Such an assumption is valid only if the test specimens are flexurally similar models of the full-sized (prototype) members. The general theory of model similitude has been presented by Charlton[17] and the modelling of reinforced concrete structures has been discussed by Harris, Sabnis and White[18] and Petri[19].

In a direct model, similar (or identical) materials are used and all linear dimensions are scaled by the same factor. That is:

$$L_i = SL_i' \qquad (1.10)$$

In Equation (1.10), S is the scale factor, the primed (') symbol refers to the model and the subscript i refers to any particular dimension. For flexural similarity, the deflections of the model and prototype are related by the same scale factor:

$$\Delta_i = S\Delta_i' \qquad (1.11)$$

From model similitude theory[17], it can be demonstrated that the condition of flexural similarity occurs only when the strain in the model equals the prototype strain at every point.

$$\varepsilon_i = \varepsilon_i' \qquad (1.12)$$

For Equation (1.12) to be valid, the elastic modulii (Young's modulus, the modulus of rigidity, the secant modulus and Poisson's ratio) and the tensile and compressive strengths for the model and prototype materials must be equal. In other words, the materials must have identical stress-strain behaviour and strength. With the exception of the concrete stress-strain curve, in which case the model concrete curve was flatter than generally observed prototype curves, the properties of the materials used in this study approximated these requirements. It can be shown that, to a first approximation, the conditions of flexural similarity are met with stress-strain curves of similar shape. Thus the differences in the concrete stress-strain curves have not been considered, and flexural similarity has been assumed.

It can also be shown that the model and prototype loads (forces) are related by:

$$P = S^2P' \quad (1.13)$$

By using Equations (1.10) and (1.13), and by applying dimensional analysis to the relationships previously developed it is possible to relate the data derived from small-scale tests to full-sized structures. In particular, Equation (1.9) will be used in Chapter 6 in the discussion of a design criterion. Since the left side of this equation has dimensions of force/length^2, the scale factor S cancels out when Equations (1.10) and (1.13) are applied. Thus, Equation (1.9) is unaltered by scale and the experimental data can be applied to any size of member, assuming the conditions of flexural similarity are met.

CHAPTER 2

THEORETICAL CONSIDERATIONS

This chapter discusses the elastic lateral stability of an unrestrained cantilevered beam and the ultimate flexural strength of a reinforced concrete beam.

In Section 2.1, the equation giving the lateral buckling load of an unrestrained, homogeneous, linearly elastic cantilever is developed. In Section 2.2, the rigidity co-efficients required to compute the buckling load are developed for a reinforced concrete member. The combined result of Sections 2.1 and 2.2 is an elastic analysis modified to approximate non-homogeneous non-linear inelastic behaviour. Such an approach has the disadvantage that the computed buckling loads will be inaccurate to the extent that the simplifying assumptions are inaccurate. The analysis of the actual buckling behaviour of a reinforced concrete beam is complex in the extreme and has not been attempted in this study. The modified elastic analysis is used, in Chapter 5, to compute buckling loads for the test specimens.

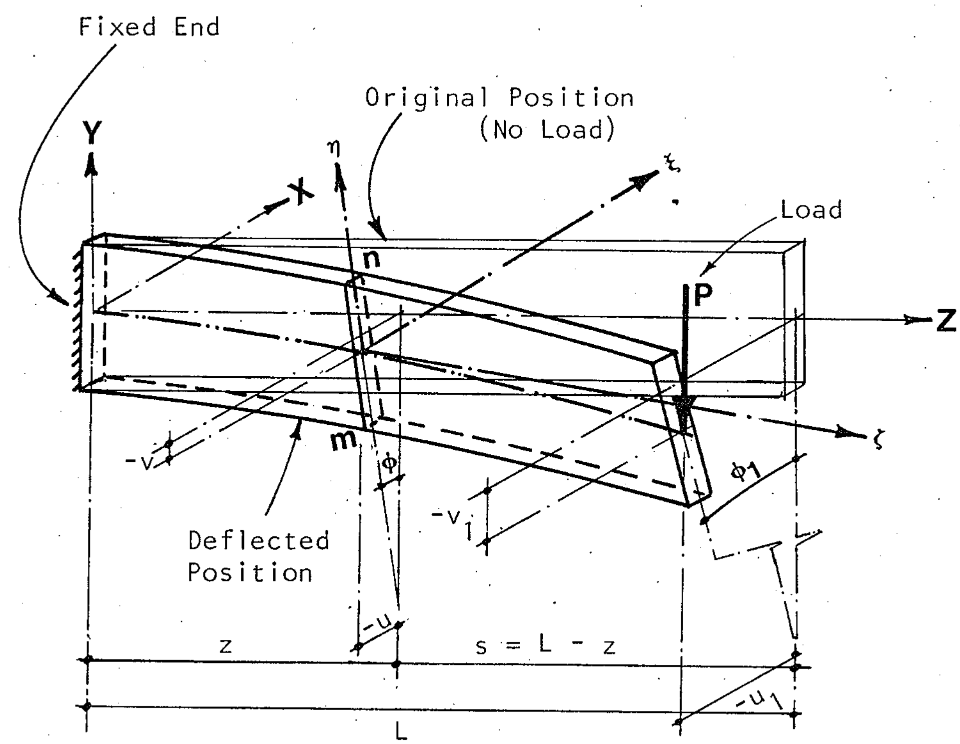
The final section in the chapter presents the analysis of the ultimate flexural strength of a reinforced concrete beam to give a basis for the computation, in Chapter 5, of the ultimate flexural capacity of the test specimens.

2.1 Lateral Buckling of an Unrestrained Cantilevered Beam

An unrestrained cantilevered beam is one which is fixed at one end and free to translate and rotate along the remainder of its length. This discussion is limited to the case of an elastic beam of uniform cross-section which is loaded through its centroid at the free end and which is initially straight, as shown in Figure 2.1. The co-ordinate system shown has its origin at the centroid of the cross-section at the end. The Z-axis is coincident with the longitudinal axis of the beam and the X and Y axes are coincident with the principal axis of the cross-section.

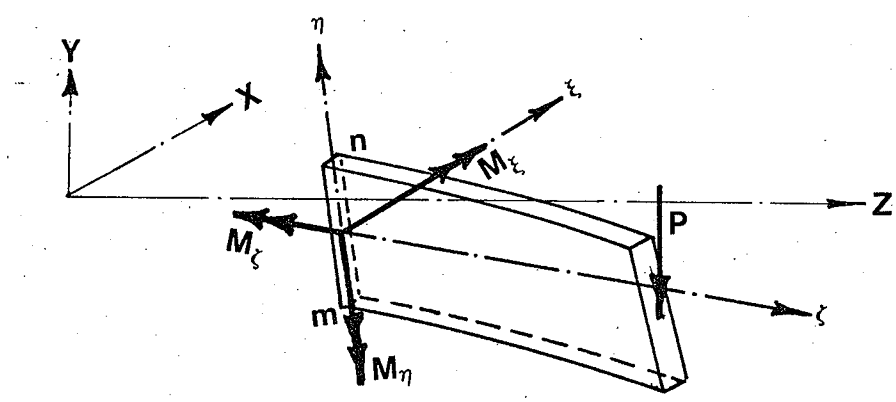
The beam is subjected to a load P at its free end and a small lateral deflection is assumed, as shown in the figure. At some distance z from the origin, a section mn is taken and a local co-ordinate system drawn with the ζ-axis tangent to the deflected beam axis and the ξ and η axes coincident with the principal axes of the cross-section. The twist angle ϕ about the Z-axis is shown positive and, as a result, the deflection components u and v in the X and Y directions respectively are negative.

A free body of the beam segment to the right of section mn is shown in Figure 2.1(b). The moments M_ξ , M_η , and M_ζ acting on the left end of the segment are shown in the positive directions assumed by Timoshenko and Gere[13]. Since the deflections are assumed to be small, the curvature in the ξ-ζ plane is assumed equal to the curvature in the X-Z plane and the curvature in the η-ζ plane is assumed equal to that in



DEFLECTED CONFIGURATION

(a)



FORCES ACTING ON A BEAM SEGMENT

(b)

LATERAL BUCKLING OF AN UNRESTRAINED CANTILEVERED BEAM

FIGURE 2.1

the Y-Z plane. The equations of bending of the beam segment in Figure 2.1(b) then become[13]:

$$B_1 \frac{d^2v}{dz^2} = M_\xi \qquad (2.1)$$

$$B \frac{d^2u}{dz^2} = M_\eta \qquad (2.2)$$

$$C \frac{d\phi}{dz} - C_w \frac{d^3\phi}{dz^3} = M_\zeta \qquad (2.3)$$

Equations (2.1) and (2.2) are derived from beam bending theory. The term B_1 is the strong-direction bending rigidity (EI_ξ) and the term B is the weak-direction bending rigidity (EI_η). Equation (2.3) is derived from torsional theory. The term C is the St Venant torsional rigidity and the term C_w is the warping rigidity.

Considering the free body in Figure 2.1(b) and taking moments in the directions of the X, Y and Z axes at the centroid of the cut cross-section, the following are obtained:

$$M_X = -P(L - z) \qquad (2.4)$$

$$M_Y = 0 \qquad (2.5)$$

$$M_Z = P(-u_1 + u) \qquad (2.6)$$

Again, the moments are taken positive in the directions assumed by Timoshenko and Gere, in which M_X and M_Y are positive when they produce curvatures such that the normal to the concave side points in the positive direction of the respective axis and M_Z is positive when it rotates the cross-section clockwise. For small values of $\emptyset$, u and v, the X-, Y- and Z- axes are related to the ξ-, η- and ζ- axes by the direction cosines given in Table 2.1.

Table 2.1

Direction Cosines Relating Global and Local Axes

	X	Y	Z
ξ	1	$\emptyset$	$-\frac{du}{dz}$
η	$-\emptyset$	1	$-\frac{dv}{dz}$
ζ	$\frac{du}{dz}$	$\frac{dv}{dz}$	1

Using these direction cosines, considering the assumed positive directions and ignoring second-order small quantities, the following expressions are obtained for the moment components in the local co-ordinate system:

$$M_\xi = -P(L - z) \qquad (2.7)$$

$$M_\eta = -P(L - z)\emptyset \qquad (2.8)$$

$$M_\zeta = P(L - z)\frac{du}{dz} - P(u_1 - u) \qquad (2.9)$$

Substituting Equations (2.7) to (2.9) into Equations (2.1) to (2.3), the following expressions are obtained:

$$B_1 \frac{d^2v}{dz^2} + P(L - z) = 0 \qquad (2.10)$$

$$B \frac{d^2u}{dz^2} + P(L - z)\emptyset = 0 \qquad (2.11)$$

$$C_w \frac{d^3\phi}{dz^3} - C \frac{d\phi}{dz} + P(L - z) \frac{du}{dz} - P(u_1 - u) = 0 \quad \text{----------------} \quad (2.12)$$

To obtain an expression for the angle of twist, Equation (2.12) is differentiated with respect to "z" and the result combined with Equation (2.11). The result is as follows:

$$C_w \frac{d^4\phi}{dz^4} - C \frac{d^2\phi}{dz^2} + P(L - z) \frac{d^2u}{dz^2} = 0 \quad \text{--------------------------} \quad (2.13)$$

From Equation (2.11):

$$\frac{d^2u}{dz^2} = - \frac{P(L - z)\phi}{B} \quad \text{--} \quad (2.14)$$

Substituting (2.14) into (2.13), the following expression results:

$$C_w \frac{d^4\phi}{dz^4} - C \frac{d^2\phi}{dz^2} - \frac{P^2(L - z)^2\phi}{B} = 0 \quad \text{---------------------------} \quad (2.15)$$

To simplify Equation (2.15), the variable $s = (L - z)$ is introduced, resulting in:

$$\frac{d^4\phi}{ds^4} - \frac{C}{C_w} \frac{d^2\phi}{ds^2} - \frac{P^2 s^2 \phi}{C_w B} = 0 \quad \text{-----------------------------------} \quad (2.16)$$

For the case of the unrestrained cantilevered beam shown in Figure 2.1(a), the following boundary conditions apply to Equation (2.16):

$$\text{At } s = 0; \quad M_\zeta = 0 = C \frac{d\phi}{ds} - C_w \frac{d^3\phi}{ds^3} \quad \text{---------------------------} \quad (2.17)$$

$$\frac{d^2\phi}{ds^2} = 0 \quad \text{--} \quad (2.18)$$

At $s = L$; $\phi = 0$ -- (2.19)

$$\frac{d\phi}{ds} = 0 \qquad (2.20)$$

To obtain the critical load at which the beam buckles laterally, Equation (2.16) must be solved for the boundary conditions in Equations (2.17) to (2.20). Timoshenko and Gere[13] give an approximate solution, for large values of $\frac{L^2C}{C_w}$ as:

$$P_{cr} = \frac{4.013\sqrt{BC}}{\left\{1-\sqrt{\frac{C_w}{L^2C}}\right\}^2 L^2} \qquad (2.21)$$

Equation (2.21) gives the load P_{cr} at which an unrestrained beam buckles laterally, in terms of the length L, the lateral flexural rigidity B, the torsional rigidity C and the warping rigidity C_w. Expressions for the three rigidity terms are developed in the following section.

For the case of the overhung beams, Equation (2.21) may be used to compute the buckling load by introducing the concept of effective length. Kerensky, Flint and Brown[20] give the effective lengths of cantilevered beams with various types of restraint using the effective length of an unrestrained cantilever as 1.0L. For the case of the overhung beam, they give an effective length of 2.7L based upon a theoretical analysis and of

2.95L based upon test measurements.

For the simply-supported beam, the development of the critical load equation is similar to that for the cantilevered beam. Timoshenko and Gere[13] give the following solution, for a narrow rectangular cross-section in which the warping rigidity is assumed to be zero, and for loading through the centroid at mid-span as:

$$P_{cr} = 16.94 \frac{\sqrt{BC}}{L^2} \qquad (2.22)$$

For non-zero values of the warping rigidity, Timoshenko and Gere provide a table of factors to be used in place of the 16.94 multiplier.

2.2 Parameters in the Critical Load Equation

To determine the critical buckling load for any given beam, it is necessary to compute the values of the three rigidity terms, B, C and C_w for substitution into Equation (2.21) or (2.22). In the sections which follow, expressions developed by Massey[11] for the three terms are used.

2.2.1 Lateral Flexural Rigidity - B

On the assumption that the concrete below the neutral axis does not contribute to the lateral flexural rigidity, Massey[11] gives the following expression for B:

$$B = \frac{\bar{y}b^3}{12} E_{sec} + \sum I_{sy} E_s \qquad (2.23)$$

Where: