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SOME SYNTHESIS METHODS FOR MULTIVARIABLE NETWORK FUNCTIONS

BY

CHANH SIEU PHAN

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A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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To my parents

ABSTRACT

This thesis is concerned with the synthesis of multivariable network functions.

The realizations of multivariable network functions by simple decomposition techniques are investigated. The conditions for a multivariable rational function to be realizable in certain simple structures with constituent building blocks involving functions of reduced complexity are derived. Three different configurations are considered:

(1) A sum connection of immittances which are functions of mutually disjoint sets of variables.

(2) A cascade connection of single-variable passive lumped networks, the cascaded subnetworks are also assumed lossless except the last termination.

(3) An extended Bott-Duffin type structure.

Apart from the general formulations in terms of the multivariable positive reality condition, more direct and explicit alternative approaches are also presented.

The synthesis of independent zeros of the even part of a multivariable positive real function is studied. In addition to the usual cascade extraction by the basic sections, viz., the Richards', Brune, type C, type E and type D sections, removal methods without resorting to gyrators and transformers are presented. The developments of the latter are primarily based on Miyata's separation concept of the even part function in single variable synthesis theory.

The problem of synthesizing a class of networks composed of cascaded noncommensurate transmission lines separated by passive lumped lossless two-ports and terminated by a passive lumped network is considered. A new set of realizability conditions is presented. The proposed set of conditions, which is simple in application, circumvents the difficulty associated with the test of multivariable positive reality. Several interesting special cases are also considered and the realizability conditions are accordingly modified to produce much simpler synthesis procedures.

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CONTENTS

CHAPTER I

INTRODUCTION

The concept of multivariable networks was first introduced by Ozaki and Kasami [15] arising from their work on variable-parameter networks. The theory has since been investigated extensively in the study of analysis and synthesis of many other classes of networks [20,27,33], in particular, the class of mixed lumped-distributed networks.

One of the prominant features of a multivariable formulation is that the resulting network functions are rational functions of a set of complex variables. Each of these variables characterizes a special type of component. For example, consider a class of networks made up of mixed lumped elements and noncommensurate transmission lines [25,27]. The associated network functions are not rational in the complex frequency variable p. However, such functions may be conveniently expressed as multivariable rational functions by characterizing the lumped RLC elements by the frequency variable p and the i-th type transmission lines by its Richards' [25] variable $\mu_i = \tanh \tau_i p$, where τ_i is the basic electrical length of the i-th type line.

It should be noted that the independent variables in a multivariable formulation, in general, are not necessarily physically independent. For instance, in the example given above, the Richards' variables μ_i's are in fact functions of the frequency variable p.

Furthermore, it is also not necessary to require that each variable be a function of the frequency p; as in the case of variable-parameter networks where some of the variables could be functions of some outside factors such as temperature, a control setting, etc..

Similar to single variable theory, the concept of multivariable positive reality is of paramount importance in multivariable synthesis theory. The following are the fundamental definitions:

A multivariable rational function $Z(\underline{p})$ of a set of complex variables $\underline{p} = (p_1, p_2, \ldots, p_n)$ is said to be a multivariable positive (m.p.) function if and only if (iff)

$$\mathrm{Re}\, Z(\underline{p}) \geq 0 \quad \text{for} \quad \mathrm{Re}\, p_i \geq 0, \quad i = 1,2,\ldots,n,$$

where Re denotes "The real part of". An m.p. function $Z(\underline{p})$ is said to be multivariable positive and real (m.p.r.) iff $Z(\underline{p})$ is real when all the variables are real.

An m.p. function $Z(\underline{p})$ is said to be multivariable para-odd iff

$$Z(\underline{p}) + Z_*(\underline{p}) = 0,$$

where $Z_*(\underline{p})$ is the para-conjugate of $Z(\underline{p})$ and is defined as $Z_*(\underline{p}) = Z^*(-\underline{p}^*)$, where the upper asterisk denotes the conjugate operation. For real rational functions, one has $Z_*(\underline{p}) = Z(-\underline{p})$, and an m.p.r. function $Z(\underline{p})$ is said to be a multivariable reactance function iff

$$Z(\underline{p}) + Z(-\underline{p}) = 0.$$

A multivariable rational function $s(\underline{p})$ of a set of complex variables $\underline{p} = (p_1, p_2, \ldots, p_n)$ is said to be a multivariable bounded function iff

$$|s(\underline{p})| \leq 1 \quad \text{for} \quad \mathrm{Re}\, p_i \geq 0, \quad i = 1,2,\ldots,n.$$

A multivariable bounded function $s(\underline{p})$ is said to be multivariable bounded real iff $s(\underline{p})$ is also real when all the variables are real.

Since its introduction [15] in 1960, a substantial amount of work on multivariable synthesis has been reported in the literature. Detailed reviews of the early developments have been given by Scanlan [27] and Youla [33], and recently a comprehensive bibliography has been presented by Ramachandran and Rao [20]. In spite of the extensive developments in the past, the multivariable synthesis proved to be unwieldy, and consequently more straightforward synthesis techniques are expected to emerge. This study is concerned with the development of simple and straightforward special synthesis methods. Furthermore, it is noted that although the multivariable positive reality is a compact gauge for the measurement of the realizability of a multivariable function, the verification of such a property is difficult and laborious. In this study, a special emphasis is also placed upon deriving possible alternative explicit realizability conditions, which would replace this prerequisite condition by some simpler conditions.

In Chapter II, the realizations of multivariable rational functions, in the forms of certain simple structures with component building blocks involving functions of reduced complexity, are investigated. Three different configurations are considered:

(1) A sum connection of immittances which are functions of mutually disjoint sets of variables.

(2) A cascade of single-variable blocks.

(3) A Bott-Duffin type structure.

The realizability conditions are formulated in terms of the decomposability of the given function into certain special forms. In addition to the general formulations based on the multivariable positive reality condition, more direct and explicit approaches are also discussed.

Chapter III is the study of the removal of independent zeros of the even part of a multivariable positive real function. Apart from the discussion of the cascade extraction by the basic sections, viz., the Richards', Brune, type C, type E and type D sections, realization methods without resorting to gyrators and transformers are also presented.

In Chapter IV, the problem of synthesizing a class of networks comprising cascaded noncommensurate transmission lines separated by passive lumped lossless two-ports and terminated by a passive lumped network is considered. A new set of realizability conditions is presented. The advantage of the proposed set of conditions is that it replaces the multivariable reality test and facilitates the synthesis procedure in a straightforward manner. Several interesting special cases are also considered and the realizability conditions are duly modified into much simpler forms.

CHAPTER II

SYNTHESIS OF MULTIVARIABLE NETWORK FUNCTIONS BY SIMPLE DECOMPOSITION METHODS

The synthesis of general m.p.r. functions was first proposed by Koga [13]. He proved that the multivariable positive reality is a sufficient condition for realizability. However, his approach, involving certain factorization processes of multivariable matrices, is known to be difficult and laborious. Furthermore, the validity of his result has been questioned recently by Bose [37], who provides a counter example indicating that his method does not always work. To circumvent the inherent difficulties of the general synthesis problem, some workers [3,4,30] have recently developed special techniques for certain classes of functions. The essential idea of these developments is to derive simple criteria for the decomposition of a given m.p.r. function into a sum of single variable p.r. functions so that the synthesis may be performed by the well-established single variable methods. In this chapter, we consider a more general aspect of synthesizing multivariable network functions in the forms of certain simple structures with constituent building blocks involving functions of reduced complexity. Three different configurations are considered:

(1) A sum connection of immittances being functions of mutually disjoint sets of variables.

(2) A cascade connection of single variable subnetworks, which are lossless except the last termination.

(3) An extended Bott-Duffin type structure.

The realizability conditions are formulated in terms of the decomposability of the given function into certain special forms. Furthermore, since the verification of the multivariable positive reality, in general, is rather unwieldy and intricate, we shall, in the following, also develope possible alternative explicit formulations which remove this prerequisite condition in favour of some one-variable type conditions.

2.1 SUM DECOMPOSITION

The concept of realizing a class of multivariable reactance functions in terms of single variable reactance functions in a sum form was first advanced by Soliman and Bose [30]. Recently, Bose [3] extended the method by presenting a revised version for the previous result. However, the above work mainly dealt with the complete decomposability of an m.p.r. function into a sum of single variable p.r. functions; moreover, the decomposition algorithm for the case of reactance functions involves laborious steps of extracting various constants. In this section, we consider the more general problem of decomposing a class of m.p.r. functions into a sum of such functions each having a smaller number of variables than the original one. In particular, the decomposition of the class of multivariable reactance functions into a sum of single variable reactance functions is reinvestigated. Results are presented in Section 2.1.1 and illustrated by examples in Section 2.1.3. Explicit formulations are discussed in Section 2.1.2.

2.1.1 General MPR Approach

In the following, we first establish a simple criterion, stated in the form of Theorem 2.1, for the decomposability of a given m.p.r. function into a sum of m.p.r. functions with fewer variables. Several interesting consequences of the theorem are then discussed. For the special class of multivariable reactance functions, an extremely simple decomposition method is given in Theorem 2.2[1] [16] which eliminates the laborious steps of constant extractions as required by the algorithm given in [3].

Theorem 2.1

Let $Z(\underline{p})$ be a multivariable positive real function of a set of complex variables $\underline{p} = (p_1,p_2,\ldots,p_n)$. Then $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = Z_1(p_1,p_2,\ldots,p_\ell) + Z_2(p_{\ell+1},p_{\ell+2},\ldots,p_n)\ , \quad \ell<n \qquad (2.1)$$

where $Z_1(p_1,p_2,\ldots,p_\ell)$ is m.p.r. in $p_1,p_2,\ldots,p_\ell$ and $Z_2(p_{\ell+1},p_{\ell+2},\ldots,p_n)$ is m.p.r. in $p_{\ell+1},p_{\ell+2},\ldots,p_n$, if and only if

$$Z(\underline{p}) - Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1) \qquad (2.2)$$

is not a function of $p_1,p_2,\ldots,p_\ell$.

Proof: The necessity is evident. We shall show the sufficiency.

Since $Z(\underline{p})$ is m.p.r., $Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1)$ is also m.p.r. Let $p_i = j\omega_{io}$, $i = 1,2,\ldots,\ell$, be the minimum point of $\mathrm{Re}\ Z(j\omega_1,j\omega_2,\ldots,j\omega_\ell,1,1,\ldots,1)$ with the minimum value K, where $\mathrm{Re}\ Z$ denotes

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1 Independently, a similar result was also reported recently in [23].

the real part of Z. By repeated applications of the maximum modulus theorem of a function of a complex variable, it can be shown that

$$\text{Re } Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1) - K \geq 0 \qquad (2.3)$$

$$\text{for Re } p_i \geq 0, \quad i = 1,2,\ldots,\ell.$$

Let Z_1 be defined as

$$Z_1 = Z(p_1,p_2,\ldots,p_\ell,1,\ldots,1) - K , \qquad (2.4)$$

then with (2.3), it follows immediately from the definition of an m.p.r. function that Z_1 is m.p.r. in $p_1,p_2,\ldots,p_\ell$.

Now, let Z_2 be defined as

$$Z_2 = Z(\underline{p}) - Z_1(p_1,p_2,\ldots,p_\ell) . \qquad (2.5)$$

By hypothesis (2.2), it is apparent that Z_2 so defined is not a function of $p_1,p_2,\ldots,p_\ell$. Hence, by selecting $p_i = j\omega_{io}$, $i=1,2,\ldots,\ell$, and taking the real part of both sides of (2.5) it is seen that

$$\text{Re } Z_2 = \text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{\ell 0},p_{\ell+1},\ldots,p_n) \geq 0 ,$$

$$\text{for Re } p_i \geq 0, \quad i = \ell+1,\ell+2,\ldots,n.$$

Therefore, Z_2 is also m.p.r. in $p_{\ell+1},p_{\ell+2},\ldots,p_n$. The sufficiency thus follows from (2.5).

Note that apart from an additive constant the component functions Z_1 and Z_2 are completely defined by (2.4) and (2.5). For $\ell = 1$, Theorem 2.1 yields the following useful corollary which enables us to detect the possibility of extracting a single variable p.r. function from a given m.p.r. function while leaving the remaining function still m.p.r. and having one variable less.

Corollary 2.1.1

A necessary and sufficient condition for an m.p.r. function $Z(\underline{p})$ to be decomposed as

$$Z(\underline{p}) = Z_1(p_1) + Z_2(p_2,p_3,\ldots,p_n) ,$$

where $Z_1(p_1)$ is a single variable p.r. function in p_1 and $Z_2(p_2,p_3,\ldots,p_n)$ is m.p.r. in $p_2,p_3,\ldots,p_n$, is that

$$Z(\underline{p}) - Z(p_1,1,1,\ldots,1)$$

is not a function of p_1. (The component function $Z_1(p_1)$ may be determined according to (2.4) as follows

$$Z_1(p_1) = Z(p_1,1,1,\ldots,1) - \underset{\omega_1}{\text{Min}}\ \text{Re}\ Z(j\omega_1,1,1,\ldots,1)) \quad . \qquad (2.6)$$

By repeated applications of Corollary 2.1.1, we obtain the following corollary, which corresponds to the special case considered in [3].

Corollary 2.1.2

A necessary and sufficient condition for an m.p.r. function $Z(\underline{p})$ to be decomposed as

$$Z(\underline{p}) = \sum_{i=1}^{n} Z_i(p_i) ,$$

where $Z_i(p_i)$ is a single variable p.r. function in p_i, is that

$$Z(\underline{p}) - Z(1,\ldots,1,p_i,1,\ldots,1)$$

is not a function of p_i, for $i = 1,2,\ldots,n-1$.

As in (2.6), the sub-functions $Z_i(p_i)$ may be derived from (2.4) as follows

$$Z_i(p_i) = Z(1,\ldots,1,p_i,1,\ldots,1) - K_i, \quad i = 1,2,\ldots,n-1, \tag{2.7}$$

and

$$Z_n(p_n) = Z(\underline{p}) - \sum_{i=1}^{n-1} Z_i(p_i), \tag{2.8}$$

where

$$K_i = \min_{\omega_i} \mathrm{Re}\, Z(1,\ldots,1,j\omega_i,1,\ldots,1) . \tag{2.9}$$

Note that Corollary 2.1.2 is essentially equivalent to the main theorem given in [3]. However, it may be noted that the condition that $F^{(n-1)}(p_n)$ in [3] ($Z_n(p_n)$ of (2.8) above) be p.r. is superfluous as far as the realizability is concerned. The fact is that the p.r. nature of $F^{(n-1)}(p_n)$ is automatically satisfied from the hypothesis that the given function is m.p.r.. As evident from Corollary 2.1.2 above, no further p.r. test at any stage is necessary provided that the given function is m.p.r..

From (2.7), it is observed that the determination of the sub-functions $Z_i(p_i)$'s involves the extraction of a maximum possible positive constant from a p.r. function. For reactance functions, however, such laborious steps may be avoided by fully exploiting the property of reactance functions. The improved result is summarized in the following theorem.

Theorem 2.2

A necessary and sufficient condition for a multivariable reactance function

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})}$$

to be decomposed as

$$Z(\underline{p}) = \sum_{i=1}^{n} Z_i(p_i) ,$$

where $Z_i(p_i)$ is a single variable reactance function in p_i, for $i = 1,2,\ldots,n,$ is that the denominator $Q(\underline{p})$ can be factored as

$$Q(\underline{p}) = \prod_{i=1}^{n} q_i(p_i) , \qquad (2.10)$$

where $q_i(p_i)$ is a single variable polynomial in p_i, $i = 1,2,\ldots,n.$ Furthermore,

(i) If $q_i(p_i)$ does not vanish at the origin, then

$$Z_i(p_i) = Z(0,\ldots,0,p_i,0,\ldots,0). \qquad (2.11)$$

(ii) If $q_i(p_i)$ vanishes at the origin, defining a new function

$$\hat{Z}(\underline{p}) = Z(\underline{p}) - \sum_{i=1}^{n} \frac{A_i}{p_i} , \qquad (2.12)$$

where

$$A_i = p_i Z(\underline{p})\Big|_{p_i=0}$$

is the residue of $Z(\underline{p})$ at $p_i = 0$, then

$$Z_i(p_i) = \hat{Z}(0,\ldots,p_i,0,\ldots,0) + \frac{A_i}{p_i} . \qquad (2.13)$$

Proof: The necessity is evident. We shall prove the sufficiency.

It is observed that a multivariable reactance function $Z(\underline{p})$ is an m.p.r. function satisfying the following additional condition

$$Z(\underline{p}) + Z(-\underline{p}) \equiv 0 . \qquad (2.14)$$

With (2.14), it can be shown that a pole of a multivariable reactance function in a p_i-plane, independent of all other variables, lies on the imaginary axis of the p_i-plane.

Also, according to a result due to Ozaki and Kasami [15], the residue of an imaginary axis independent pole including the origin and infinity of a m.p.r. function is a positive constant, and the removal of such a pole yields a remaining function which is also m.p.r..

By hypothesis, the denominator of $Z(\underline{p})$ can be factored in the form of (2.10), therefore all the poles of $Z(\underline{p})$ are independent poles. Consequently, it follows from the above two results that $Z(\underline{p})$ can be expressed in the form

$$Z(\underline{p}) = A_{1\infty}p_1 + \frac{A_{10}}{p_1} + \sum_{\ell} \frac{A_{1\ell}p_1}{p_1^2 + \omega_{1\ell}^2} + Z_2(\underline{p}) , \qquad (2.15)$$

where $A_{1\infty}$, A_{10} and $A_{1\ell}$'s are non-negative and $Z_2(\underline{p})$ is m.p.r..

It is apparent from (2.15) that the denominator of $Z_2(\underline{p})$ is free of the variable p_1. Therefore, it can be shown from the degree property of m.p.r. functions that $Z_2(\underline{p})$ is no more a function of p_1.

Applying the analogous procedure successively, we can decompose $Z(\underline{p})$ as

$$Z(\underline{p}) = \sum_{i=1}^{n} \left(A_{i\infty}p_i + \frac{A_{io}}{p_i} + \sum_{\ell} \frac{A_{i\ell}p_i}{p_i^2 + \omega_{i\ell}^2} \right)$$

$$= \sum_{i=1}^{n} Z_i(p_i) , \qquad (2.16)$$

where the $Z_i(p_i)$'s are obviously single variable reactance functions.

Now, if $q_i(p_i)$ does not vanish at the origin, then $A_{io} = 0$ for every i. By setting $p_k = 0$ for every k except $k = i$, it follows from (2.16) that

$$Z_i(p_i) = Z(0,\ldots,0,p_i,0,\ldots,0) \ .$$

Moreover, if $q_i(p_i)$ has p_i as a factor, we can always remove the terms $\frac{A_{io}}{p_i}$'s by inspection in advance to yield a new function which satisfies the same condition as the previous one. Hence, we can write (2.13) as a consequence of (2.11) and (2.12). Q.E.D.

2.1.2 Explicit Approach

Due to the particular nature of the problems considered in Corollary 2.1.2 and Theorem 2.2, the requirement of the multivariable positive reality on the given function can therefore be relaxed by reformulating the propositions into the following alternative forms.

Theorem 2.3

Let $Z(\underline{p})$ be a multivariable rational function. The following two conditions

(i) $Z(1,\ldots,1,p_i,1,\ldots,1)$ is a single variable p.r. function of p_i, $i = 1,2,\ldots,n-1$.

(ii) The function

$$Z_n = Z(\underline{p}) - \sum_{i=1}^{n-1} Z_i(p_i) \tag{2.17}$$

is a single variable p.r. function in p_n, where

$$Z_i(p_i) = Z(1,\ldots,1,p_i,1,\ldots,1) - K_i \tag{2.18}$$

with

$$K_i = \min_{\omega_i} \operatorname{Re} Z(1,\ldots,1,j\omega_i,1,\ldots,1) \ .$$

are necessary and sufficient for $Z(\underline{p})$ to be a member of a subclass of m.p.r. functions, which can be decomposed as a sum of single variable p.r. functions.

Proof: The proof is straightforward.

Necessity: Since $Z(\underline{p})$ is m.p.r., condition (i) is obviously satisfied. Next, we shall show that Z_n defined by (2.17) is a single variable p.r. function of p_n.

Since $Z(1,\ldots,1,p_i,1,\ldots,1)$ is p.r. in p_i, the functions $Z_i(p_i)$'s defined by (2.18) are also p.r. and minimum, viz.,

$$\text{Re } Z_i(j\omega_{io}) = 0 , \quad i = 1,2,\ldots,n-1 \tag{2.20}$$

where ω_{io} is the minimum point of (2.19).

Since $Z(\underline{p})$ is decomposable into a sum of single variable p.r. functions, it can be shown that Z_n defined by (2.17) is solely a function of p_n. Consequently, by selecting $p_i = j\omega_{io}$ for $i = 1,2,\ldots,n-1$ and taking the real parts of both sides of (2.17) then comparing with (2.20), we have

$$\text{Re } Z_n(p_n) = \text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{n-10},p_n) . \tag{2.21}$$

Since $Z(\underline{p})$ is m.p.r.,

$$\text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{n-10},p_n) \geq 0 \quad \text{for} \quad \text{Re } p_n \geq 0 . \tag{2.22}$$

Therefore,

$$\text{Re } Z_n(p_n) \geq 0 \quad \text{for} \quad \text{Re } p_n \geq 0 . \tag{2.23}$$

With (2.23), it is apparent from the definition of p.r. function that $Z_n(p_n)$ is p.r. in p_n.

Sufficiency: As shown above condition (i) coupled with equation (2.18) indicates that the $Z_i(p_i)$'s, $i = 1,2,\ldots,n-1$, are also p.r..

The sufficiency is thus evident from (2.17). Q.E.D.

Since independent poles at the origin of p_i-plane can easily be removed by inspection in advance, in the following theorem, we shall assume without lack of generality that the given function does not possess poles at the origin.

Theorem 2.4

Let

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})}$$

be an irreducible multivariable rational function having no poles at the origin. The necessary and sufficient conditions for $Z(\underline{p})$ to be decomposed into a sum of single variable reactance functions in the form of

$$Z(\underline{p}) = \sum_{i=1}^{n} \frac{\alpha_i(p_i)}{q_i(p_i)}$$

are

(i) The denominator can be factored as

$$Q(\underline{p}) = \prod_{i=1}^{n} q_i(p_i),$$

where $q_i(p_i)$ are even polynomials of p_i.

(ii) The numerator can be expressed as

$$P(\underline{p}) = \sum_{i=1}^{n} \alpha_i(p_i) \prod_{\ell \neq i} q_\ell(p_\ell) ,$$

where

$$\alpha_i(p_i) = \frac{P(0,\ldots,0,p_i,0,\ldots,0)}{\prod_{\ell \neq i} q_\ell(0)} .$$

(iii) $\alpha_i(p_i)/q_i(p_i)$ is a reactance function of p_i.

Proof: It is noted that the first two conditions assure the separability of the given function into a sum form; while the third condition attests to the reactance nature of the component functions. The proof is straightforward and thus omitted for brevity.

2.1.3 Examples

Example 2.1.1: Consider the following m.p.r. function in $\underline{p} = (p_1,p_2,p_3,p_4)$

$$Z(\underline{p}) = \frac{p_1p_2p_3p_4 + p_1p_3p_4 + 3p_1p_2p_3 + 2p_1p_2 + p_3p_4 + 2p_1 + 3p_3 + 2}{(p_1p_2 + 1)\ (p_3p_4 + 2)} . \tag{2.24}$$

It is required to determine the decomposability of (2.24) into a sum of m.p.r. functions having fewer variables.

It is observed that

$$Z(\underline{p}) - Z(p_1,p_2,1,1) = \frac{3p_3 - p_3p_4 - 2}{p_3p_4 + 2}$$

is not a function of p_1 and p_2, therefore, it follows from Theorem 2.1 that $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = Z_1(p_1,p_2) + Z_2(p_3,p_4) . \tag{2.25}$$

Using (2.4) we obtain

$$Z_1(p_1,p_2) = Z(p_1,p_2,1,1) - \min_{\omega_1,\omega_2} \text{Re } Z(j\omega_1,j\omega_2,1,1)$$

$$= \frac{p_1}{p_1p_2 + 1} .$$