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THE ESTIMATION OF PARAMETERS OF MIXED WEIBULL DISTRIBUTION WITH TIME-CENSORED DATA

by

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TABLE OF CONTENTS

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H. Lee

H. Lee

SUMMARY

The problem treated here is concerned with the mixed Weibull distribution which is widely used in life testing. Suppose that the failure population can be divided into subpopulations, each might have a different type of failure. For estimating the parameters of such mixed population model, the moment and the maximum likelihood estimates have been considered. The corresponding asymptotic variance-covariance matrix is given.

Three types of the models have been discussed in Chapters 3, 4 and 5. The methods of obtaining the moment estimators and the maximum likelihood estimators have been derived. Illustrative examples of these three cases (by generating data) are included. Some of the computer flowcharts and programs have been enclosed in the appendices.

CHAPTER 1.

INTRODUCTION

It is our everyday experience that many of the items, systems, or materials which we use suddenly go 'out of order'. For instance, the steel beam under a load may crack or break; the fuse inserted into a circuit may burn out; the wing of the airplane under influence of forces may buckle, or the electronic device may fail to function. Suppose that for any such component (or system) a state we denote as 'failure' can be defined. Each has its own statistical assessment of the life characteristic. Assuming a single homogeneous population of units, the life characteristics have been well developed during the last thirty or more years. When items are subjected to test certain complex situations arise. It is necessary to study such situations, particularly the underlying failure time distribution of items under test.

Reliability theory establishes the regularity of occurrence of defects in devices and methods of prediction. In general, it is a method of evaluating the quality of a unit or a system and is defined by $R(t|\theta) = P(X \geq t)$, where θ is the parameter of the underlying life distribution $f(x|\theta)$. If θ is unknown, the reliability is estimated by different methods.

In order to assess the reliability, we usually start from the observed data — the recorded lives of some or all items subjected to test, or to performance under either actual or simulated conditions.

Assume that the component success or failure data which are used for reliability estimation are governed by some parametric probability distribution. There are certain well-known families of failure time distributions which have been successfully used in life testing problems. So one may find it necessary to identify the family — (a group of related distribution), justify them by experiments, statistical tests or by certain well-defined assumptions. For instance, if the device is a valve which is supposed to open or to close upon demand, we might identify the binomial distribution as our parametric probability distribution. Let us consider the pressure in a boiler; the boiler does not operate if the pressure is too low and it bursts if the pressure is too high, so perhaps the normal distribution should be used. The exponential and Poisson distributions are used when failures occur at a constant level of intensity which does not vary with the accumulated service.

A great deal of modern statistical literature on the exponential distribution is concerned with estimation under conditions of censoring or truncation. It is commonly encountered in the statistical analysis and assessment of data arising from life-tests under laboratory or service condition because it carries strong implication with regard to the underlying model.

The family of exponential distributions is the best known and most thoroughly explored, largely through the work of B. Epstein and his associates (Epstein, 1958). The exponential distribution has a number of desirable mathematical properties. One important property of exponential distribution is that it is 'forgetful', i.e. under the exponential failure time model, if a unit has survived t hours, then the probability of surviving an additional h hours is exactly the same as the probability of surviving h hours of a brand new item. There are structures which have this property. For example, at any hour of time, the future life of an electric fuse (assuming it cannot melt partially) is practically unchanged as long as it does not fail.

The Weibull distribution was suggested by the Swedish physicist Weibull and first used in a paper (1939) dealing with the breaking strength of materials. It is interesting that he proposed this distribution, apparently without recognizing that it is a particular case of the extreme value distribution, considering only the class of failure distributions of the form $F(X) = 1 - \exp\{-\phi(x)\}$, where $\phi(x)$ is a positive non-decreasing function.

The Weibull distribution provides a very general family of distribution in which certain other well known distributions figure as special cases. It is an important model for life-testing problem.

Weibull distributions are closely related to exponential functions. It has one additional parameter called the shape parameter. This maybe the reason why in the past Weibull distributions have been widely used in procedures for the analysis of life-testing data.

During the last few years, Weibull estimation has been greatly aided by the development of linear weighing technique for mixtures of distributions which enables one to estimate Weibull parameters from observed data.

Kao (1959) Mann (1968), Lawless (1972), Mann, Schafer and Singpurwala (1974) have done extensive work on the Weibull distribution. Sinha (1976) studied the Weibull distribution from the Bayesian viewpoint. It is of practical importance to study a situation where the underlying failure time distribution is a mixture of two or more distributions. But it is not easy to estimate the parameters of the mixed distributions, particularly for the mixed Weibull distributions. In 1958, Mendenhall and Hader gave a method for obtaining the estimates of the parameters of mixed exponentially distributed failure time distribution from censored life test data. It is in fact a particular case of the mixed Weibull distributions when the shape parameter is known to be 1. In 1965, Cohen gave a method for estimating Weibull parameters based on complete and censored samples. For the mixed Weibull distribution, there is difficulty in estimating its parameters. Not much work has been done on this problem. It may be useful to try to develop a method for estimating the parameters of a mixed Weibull distribution, although it follows that the computations involved would not be simple.

A study done in this paper is concerned with moment and the maximum likelihood estimation in censored samples from the mixed Weibull distribution. The technique of such procedures has been discussed under the three different types of the model, treated in Chapters 3, 4 and 5. At first, we would like to give a general model for this mixture-distribution.

CHAPTER 2.

THE MODEL AND ITS LIKELIHOOD FUNCTION

2.1 The Model

Assume the parent failure distribution is made up of two subpopulations, each having a cumulative probability distribution defined by:

$$F_i(t) = 1 - \exp\left\{-\left(\frac{t^{p_i}}{\theta_i}\right)\right\} \quad ; \qquad 0 \le t \le \infty \tag{2.1}$$

and the density function:

$$f_i(t|\theta_i,p_i) = \left(\frac{p_i}{\theta_i}\right) t^{p_i-1} \exp\left\{-\left(\frac{t^{p_i}}{\theta_i}\right)\right\} \tag{2.2}$$

$$(\theta_i > 0, \quad p_i > 0 \qquad i = 1,2)$$

Let the two subpopulations be mixed in proportions $\alpha:\beta$ $(\beta = 1 - \alpha)$, $\alpha, \beta \ge 0$. Then the cumulative distribution of the parent population is given by

$$F(t) = \alpha F_1(t) + \beta F_2(t) \tag{2.3}$$

and the density function

$$f(t|\alpha,\theta_1,\theta_2,p_1,p_2) = \alpha\left(\frac{p_1}{\theta_1}\right) t^{p_1-1} \exp\left\{-\left(\frac{t^{p_1}}{\theta_1}\right)\right\} + (1-\alpha)\left(\frac{p_2}{\theta_2}\right) t^{p_2-1} \exp\left\{1-\left(\frac{t^{p_2}}{\theta_2}\right)\right\} \tag{2.4}$$

Let $$G_i(t) = 1 - F_i(t) \tag{2.5}$$

and $$G(t) = 1 - F(t) \tag{2.6}$$

The probability function $G(t)$ is the probability that a unit will survive to time t and is called the survival function.

Assume that, as soon as an item fails, the cause of failure is known and the subpopulation from which the item belongs is identified.

Suppose n items are chosen randomly from the model (2.4) and subjected to the life-test, which terminates at a fixed time, T. During such time r units have failed, r_i from subpopulation (i) and $r_1 + r_2 = r$.

Let T_j be the time passed since the jth item was put to test and t_j be the length of life of that time. If $t_j < T_j$, the jth item is said to have failed, otherwise it is said to have survived. The probability of survival of the jth item is given by

$$Q_j = p(t_j > T_j) = \int_{T_j}^{\infty} f(t|\alpha,\theta_1,\theta_2,p_1,p_2)dt$$

$$= \alpha \exp\left\{-\left(\frac{T_j^{p_1}}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T_j^{p_2}}{\theta_2}\right)\right\} \tag{2.7}$$

For convenience, assume that all measurements of time are in units of

T, the test termination time.

So $$Q = Q_j = \alpha \exp\left\{-\left(\frac{T^{p_1}}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^{p_2}}{\theta_2}\right)\right\} ; \quad j = 1,2,\ldots,n \; . \tag{2.8}$$

Let $x = t/T$ then (2.6) becomes

$$G(xT) = \alpha \exp\left\{- \frac{(xT)^{p_1}}{\theta_1}\right\} + \beta \exp\left\{- \frac{(xT)^{p_2}}{\theta_2}\right\}$$

Particularly, for $x = 1$, we get $G(T) = Q$,

and $$F_i(T) = 1 - \exp\left\{-\left(\frac{T^{p_i}}{\theta_i}\right)\right\} ; \qquad i = 1,2 \; .$$

2.2 Likelihood Function

For a given random sample of n units, the probability of r_1 units failing due to cause (1), r_2 units failing due to cause (2) and $(n-r)$ units surviving is the multinomial:-

$$p(r_1,r_2,n-r|n) = \frac{n!}{r_1!r_2!(n-r)!} [\alpha F_1(T)]^{r_1} [\beta F_2(T)]^{r_2} [G(T)]^{n-r} \tag{2.9}$$

Hence the conditional density of obtaining the ordered observations $x_{i1}, x_{i2},\ldots, x_{ir_i}$, for given r_i and $x_{ij} \le 1$ is

$$p(x_{i1}, x_{i2},\ldots, x_{ir_i}|r_i,x_{ij} \le 1) = \frac{(r_i!) \prod_{j=1}^{r_i} f_i(x_{ij})}{[F_i(T)]^{r_i}} \tag{2.10}$$

The likelihood function L of the sample is given by

$$L = \frac{n!}{(n-r)!}\,[G(T)]^{n-r}\,\alpha^{r_1}\,\beta^{r_2}\prod_{j=1}^{r_1} f_1(x_{1j})\prod_{j=1}^{r_2} f_2(x_{2j})$$

or

$$L = \frac{n!}{(n-r)!}\left[\alpha\exp\left\{-\left(\frac{T^{p_1}}{\theta_1}\right)\right\} + \beta\exp\left\{-\left(\frac{T^{p_2}}{\theta_2}\right)\right\}\right]^{n-r}\alpha^{r_1}\,\beta^{r_2}$$

$$\times\left[\prod_{j=1}^{r_1}\left(\frac{p_1}{\theta_1}\right) x_{1j}^{p_1-1}\,T^{p_1}\,\exp\left\{\left[\frac{(x_{1j}\cdot T)^{p_1}}{\theta_1}\right]\right\}\right]$$

$$\times\left[\prod_{j=1}^{r_2}\left(\frac{p_2}{\theta_2}\right) x_{2j}^{p_2-1}\,T^{p_2}\,\exp\left\{\left[\frac{(x_{2j}\cdot T)^{p_2}}{\theta_2}\right]\right\}\right] \qquad (2.11)$$

The method of moments and the method of maximum likelihood will be used to estimate the parameters of the probability density function (p.d.f.) (2.4) in the following cases

(1) The mixture parameter α known; $p_1=p_2=p$, $\theta_1 \neq \theta_2$ unknown.

(2) α unknown; $p_1=p_2=p$, $\theta_1 \neq \theta_2$ unknown.

(3) α unknown; $p_1 \neq p_2$, $\theta_1 \neq \theta_2$ unknown.

The asymptotic variance-covariance matrices for the maximum likelihood estimates (MLE) will also be obtained.

CHAPTER 3.

THE MIXTURE PARAMETER α IS KNOWN, THE COMMON SHAPE PARAMETER p AND θ_1, θ_2 UNKNOWN

3.1 The Moment Estimation of the Parameters

We consider the density function

$$f(x|\alpha,p,\theta_1,\theta_2) = \alpha(\frac{p}{\theta_1})x^{p-1} \exp\left\{-\left(\frac{x^p}{\theta_1}\right)\right\} + \beta(\frac{p}{\theta_2})x^{p-1} \exp\left\{-\left(\frac{x^p}{\theta_2}\right)\right\} \tag{3.1}$$

Then the sth non-central moment is:

$$\mu_s' = \frac{\alpha p}{\theta_1} \int_0^\infty x^{s+p-1} \exp\left\{-\left(\frac{x^p}{\theta_1}\right)\right\} dx$$

$$+ \frac{(1-\alpha)p}{\theta_2} \int_0^\infty x^{s+p-1} \exp\left\{-\left(\frac{x^p}{\theta_2}\right)\right\} dx$$

$$= \alpha\theta_1^{\frac{s}{p}} \Gamma(\frac{s}{p} + 1) + (1 - \alpha)\theta_2^{\frac{s}{p}} \Gamma(\frac{s}{p} + 1)$$

$$= (\frac{s}{p}) \Gamma(\frac{s}{p}) \left(\alpha\theta_1^{\frac{s}{p}} + \beta\theta_2^{\frac{s}{p}}\right) \tag{3.2}$$

Since the mean and variance are equal to $\mu = \mu_1'$ and $\mathrm{Var}(x) = \mu_2' - (\mu_1')^2$, respectively,

$$\mu = (\frac{1}{p}) \Gamma(\frac{1}{p}) \left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right) \quad \text{and}$$

$$\mathrm{Var}(x) = \left[(\frac{2}{p}) \Gamma(\frac{2}{p}) \left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)\right] - \left[(\frac{1}{p}) \Gamma(\frac{1}{p}) \left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)\right]^2$$

So $$\frac{\mathrm{Var}(x)}{\mu^2} = \frac{(\frac{2}{p})\,\Gamma(\frac{2}{p})\left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)}{\frac{1}{p^2}\,\Gamma^2(\frac{1}{p})\left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)^2} - 1 \qquad (3.3)$$

On taking the square root of (3.3), we have for the coefficient of variation

$$cv = \left(\frac{2p\,\Gamma(\frac{2}{p})\left(\alpha\theta_1^{\frac{2}{p}} + \beta\theta_2^{\frac{2}{p}}\right)}{\Gamma^2(\frac{1}{p})\left(\alpha\theta_1^{\frac{1}{p}} + \beta\theta_2^{\frac{1}{p}}\right)^2} - 1\right)^{\frac{1}{2}} \qquad (3.4)$$

As an example we take α known and equal to $\frac{1}{3}$. Then (3.4) can be reduced as

$$cv = \left(\frac{2p\,\Gamma(\frac{2}{p})\left(\frac{\theta_1^{\frac{2}{p}}}{3} + \frac{2\theta_2^{\frac{2}{p}}}{3}\right)}{\Gamma^2(\frac{1}{p})\left(\frac{\theta_1^{\frac{1}{p}}}{3} + \frac{2\theta_2^{\frac{1}{p}}}{3}\right)^2} - 1\right)^{\frac{1}{2}} \qquad (3.5)$$

Table 3.1 is an abridged table for the coefficient of variation with p, λ_1 and λ_2, where $\lambda_i = \frac{\theta_i}{T^p}$, with known parameter $\alpha = \frac{1}{3}$. (Other ranges for p, λ_1 and λ_2 can be obtained by using the subroutine TABLE which is enclosed).

For a given sample coefficient of variation, we may obtain an approximation to the estimate p^* with corresponding estimates θ_1^* and θ_2^* by comparing the value of the coefficient of variation from table 3.1. Such estimates p^*, θ_1^* and θ_2^* will be called moment estimates.

There is another technique which might be used to obtain the moment estimates. One might let s = 1, 2, 3 successively in equation (3.2) to set up three equations and use them to solve for the three

TABLE 3.1 THE MIXED WEIBULL COEFFICIENT OF VARIATION ($\lambda_i = \theta_i/T^p$)

$\alpha = \frac{1}{3}$

Coefficient of variation	p	λ_1	λ_2
0.50069	2.10	0.600	0.650
0.50100	2.10	0.500	0.450
0.50570	2.10	0.600	0.450
0.51375	2.10	0.400	0.650
0.52315	2.00	0.600	0.650
0.52715	2.00	0.500	0.650
0.53174	2.00	0.500	0.350
0.53712	2.00	0.400	0.650
0.54422	1.95	0.420	0.615
0.55222	1.90	0.500	0.650
0.55725	1.90	0.420	0.615
0.55826	1.90	0.415	0.620
0.57098	1.85	0.420	0.615
0.57537	1.80	0.600	0.650
0.58004	1.80	0.500	0.650
0.58550	1.80	0.500	0.350
0.59157	1.80	0.400	0.650
0.60001	1.80	0.300	0.550
0.61111	1.70	0.500	0.650
0.64745	1.70	0.300	0.650

cv was computed for assigned values of p, θ_1 and θ_2. Table 3.1 was set up in such a way that the coefficients of variation are monotonic.

unknowns p, θ_1 and θ_2. Theoretically, these three unknowns can be obtained, but there are a lot of practical problems when one tries to solve three non-linear equations. Even though it is not hard to eliminate θ_i (i = 1 or 2), the problem is that a unique solution for θ_i (i = 1,2) will not be available for any starting value p. In fact, it is hard to say which of the starting points for p is reasonable to use.

3.2 Maximum Likelihood Estimation of the Parameters

The logarithm of the likelihood function is

$$\ell nL = \ell n\left(\frac{n!}{(n-r)!}\right) + (n-r)\ell n\left[\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right]$$

$$+ r_1 \ell n\alpha + r_2 \ell n\beta$$

$$+ \sum_{j=1}^{r_1}\left[\ell np - \ell n\theta_1 + (p-1)\ell nx_{1j} + p\ell nT - \frac{(x_{1j}T)^p}{\theta_1}\right]$$

$$+ \sum_{j=1}^{r_2}\left[\ell np - \ell n\theta_2 + (p-1)\ell nx_{2j} + p\ell nT - \frac{(x_{2j}T)^p}{\theta_2}\right] \quad (3.6)$$

Taking the first partial derivative of ℓnL with respect to θ_1, θ_2 and p, we have

$$\frac{\partial \ell nL}{\partial \theta_1} = \frac{\alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{T^p}{\theta_1^2}\right) - \frac{r_1}{\theta_1} + \frac{\sum_{j=1}^{r_1}(x_{1j}T)^p}{\theta_1^2}$$

$$= \frac{(n-r)kT^p}{\theta_1^2} - \frac{r_1}{\theta_1} + \frac{\sum_{j=1}^{r_1} (x_{1j}.T)^p}{\theta_1^2} \qquad (3.7)$$

$$\frac{\partial \ell n L}{\partial \theta_2} = \frac{\beta(n-r)\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}\left(\frac{T^p}{\theta_2^2}\right) - \frac{r_2}{\theta_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}.T)^p}{\theta_2^2}$$

$$= \frac{(n-r)(1-k)T^p}{\theta_2^2} - \frac{r_2}{\theta_2} + \frac{\sum_{j=1}^{r_2} (x_{2j}.T)^p}{\theta_2^2} \qquad (3.8)$$

$$\frac{\partial \ell n L}{\partial p} = \left[\alpha(n-r)\exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}\left(-\frac{T^p}{\theta_1}(\ell n T)\right) + (n-r)\beta\exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}\right.$$

$$\left. \times \left(-\frac{T^p}{\theta_2}(\ell n T)\right)\right] \times \frac{1}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}$$

$$+ \frac{r_1}{p} + \sum_{j=1}^{r_1} [\ell n(x_{1j}.T)] - \left(\frac{1}{\theta_1}\right) \sum_{j=1}^{r_1} (x_{1j}.T)^p [\ell n(x_{1j}.T)]$$

$$+ \frac{r_2}{p} + \sum_{j=1}^{r_2} [\ell n(x_{2j}.T)] - \left(\frac{1}{\theta_2}\right) \sum_{j=1}^{r_2} (x_{2j}.T)^p [\ell n(x_{2j}.T)] \quad .$$

$$= -(n-r)T^p(\ell nT)\left(\frac{k}{\theta_1} + \frac{1-k}{\theta_2}\right) + \frac{r_1}{p} + \frac{r_2}{p} + \sum_{j=1}^{r_1} [\ell n(x_{1j}.T)]$$

$$+ \sum_{j=1}^{r_2} [\ell n(x_{2j}.T)] - \frac{1}{\theta_1} \sum_{j=1}^{r_1} (x_{1j}.T)^p \, [\ell n(x_{1j}.T)]$$

$$- \frac{1}{\theta_2} \sum_{j=1}^{r_2} (x_{2j}.T)^p \, [\ell n(x_{2j}.T)] \tag{3.9}$$

where

$$k = \frac{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\}}{\alpha \exp\left\{-\left(\frac{T^p}{\theta_1}\right)\right\} + \beta \exp\left\{-\left(\frac{T^p}{\theta_2}\right)\right\}}$$

$$= \frac{1}{1 + \left(\frac{\beta}{\alpha}\right)\exp\left\{T^p\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\}} \tag{3.10}$$

When the partial derivatives are equated to zero, the estimating equations are

$$\hat{\theta}_1 = \frac{(n-r)T^{\hat{p}}}{r_1}\hat{k} + \frac{\sum_{j=1}^{r_1} (x_{1j}.T)^{\hat{p}}}{r_1} \tag{3.11}$$

$$\hat{\theta}_2 = \frac{(n-r)T^{\hat{p}}}{r_2}(1-\hat{k}) + \frac{\sum_{j=1}^{r_2} (x_{2j}.T)^{\hat{p}}}{r_2} \tag{3.12}$$

and $\hat{p}$ is the solution of

$$g(\hat{p}) \equiv -(n-r)T^{\hat{p}}(\ln T)\left(\frac{\hat{k}}{\hat{\theta}_1} + \frac{1-\hat{k}}{\hat{\theta}_2}\right) + \frac{r_1}{\hat{p}} + \frac{r_2}{\hat{p}} + \sum_{j=1}^{r_1} [\ln(x_{1j}T)]$$

$$+ \sum_{j=1}^{r_2} [\ln(x_{2j}T)] - \frac{1}{\hat{\theta}_1} \sum_{j=1}^{r_1} (x_{1j}T)^{\hat{p}} [\ln(x_{1j}T)]$$

$$- \frac{1}{\hat{\theta}_2} \sum_{j=1}^{r_2} (x_{2j}T)^{\hat{p}} [\ln(x_{2j}T)] = 0 \qquad (3.13)$$

where

$$\hat{k} = \frac{1}{1 + (\frac{\beta}{\alpha}) \exp\left\{T^{\hat{p}}\left(\frac{1}{\hat{\theta}_1} - \frac{1}{\hat{\theta}_2}\right)\right\}} \qquad (3.14)$$

If $\alpha = \frac{1}{3}$,

$$\hat{k} = \frac{1}{1 + 2\exp\left\{T^{\hat{p}}\left(\frac{1}{\hat{\theta}_1} - \frac{1}{\hat{\theta}_2}\right)\right\}}$$

By setting the expression (3.7) - (3.9) equal to zero, we have three equations with three unknown. Theoretically, these three unknowns can be solved, which will maximize the likelihood, at least locally.

We would like to point out that when $\alpha = 1$, the sample becomes a censored sample from a single two parameter Weibull distribution. In this case, equation (3.4) can be simplified as equation (27) in A.C. Cohen's paper (1965). Equation (3.11) and (3.13) are the same as the MLE (11) and (12) in the same paper. So it is reasonable to try to use similar methods to iterate p in equation (3.9). Substituting the moment estimators θ_1^*, θ_2^* into equation (3.9), we have

$$g(p) \equiv -(n-r)T^p \left[\ln\left(\frac{k^*}{\theta_1^*} + \frac{(1-k^*)}{\theta_2^*}\right)\right] + \frac{r_1}{p} + \frac{r_2}{p}$$

$$+ \sum_{j=1}^{r_1} [\ln(x_{1j}.T)] + \sum_{j=1}^{r_2} [\ln(x_{2j}.T)]$$

$$- \frac{1}{\theta_1^*} \sum_{j=1}^{r_1} (x_{1j}.T)^p [\ln(x_{1j}.T)] - \frac{1}{\theta_2^*} \sum_{j=1}^{r_2} (x_{2j}.T)^p [\ln(x_{2j}.T)]$$

$$= 0 \qquad (3.15)$$

where

$$k^* = \frac{1}{1 + \left(\frac{(1-\alpha)}{\alpha}\right) \exp\left\{T^{p^*}\left(\frac{1}{\theta_1^*} - \frac{1}{\theta_2^*}\right)\right\}} \qquad (3.16)$$

and here θ_1^*, θ_2^* (θ_i^* already obtained) and α (= $\frac{1}{3}$, known) are treated as constants. The value of p^* read from table (3.1) should provide a starting point to use in the iterative solution of the equation (3.15).

After we obtained the value $\hat{p}_0$ (say), the iterative process of Mendenhall and Hader (1958) can be used. We obtain $\hat{\theta}_1$ and $\hat{\theta}_2$ from (3.11) and (3.12) in terms of k, and substitute them in (3.14). (3.14) is now of the from $h(\hat{k}, \hat{p}) = \hat{k}$, where $h(\hat{k}, \hat{p})$ is essentially a non-negative function of $(\hat{p}, \hat{k})$. Also, for any fixed $\hat{p} = \hat{p}_0$, $\hat{k}$ is bounded between 0 and 1. The correct value of $\hat{k}$ is the solution of the equation $h(\hat{k}, \hat{p}_0) - \hat{k} = 0$.

Let $\hat{\lambda}_i = \hat{\theta}_i / T^{\hat{p}}$, and $\bar{y}_i = \sum_{j=1}^{r_1} (x_{ij}^{\hat{p}} / r_i)$. So the equation (3.11), (3.12) and (3.14) become:

$$\hat{\lambda}_1 = \frac{(n-r)\hat{k}}{r_1} + \bar{y}_1 \qquad (3.17)$$

$$\hat{\lambda}_2 = \frac{(n-r)(1-\hat{k})}{r_2} + \bar{y}_2 \qquad (3.18)$$

$$\hat{k} = \frac{1}{1 + (\frac{\beta}{\alpha}) \exp\left\{ \frac{1}{\hat{\lambda}_1} - \frac{1}{\hat{\lambda}_2} \right\}} \qquad (3.19)$$

Then by using similar techniques as Deemer and Votaw (1955) the good approximation to $\hat{k}$ (for which $\hat{p} = \hat{p}_0$) can be obtained. Since the distribution is assumed to be censored at time T, and

$$f(t|p, \theta_i) = (\frac{p}{\theta_i}) t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\}$$ we have

$$F(t) = \frac{p}{\theta_i} \int_0^T t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\} dt$$

$$= 1 - \exp\left\{-\left(\frac{T^p}{\theta_i}\right)\right\}$$

Then $$f(t|t \le T) = \frac{\frac{p}{\theta_i} t^{p-1} \exp\left\{-\left(\frac{t^p}{\theta_i}\right)\right\}}{1 - \exp\left\{-\left(\frac{T^p}{\theta_i}\right)\right\}}$$

Put $y = x^p$ and $\lambda_i = \theta_i/T^p$ where $x = \frac{t}{T}$. We get

$$f(t|t \le T) = \frac{\frac{1}{\lambda_i}\exp\{-(\frac{y}{\lambda_i})\}}{1 - \exp\{-(\frac{1}{\lambda})\}} \quad .$$

For this model the likelihood

$$L \propto \frac{\exp\left\{-\left(\frac{r_i \bar{y}_i}{\lambda_i}\right)\right\}}{\lambda_i^{r_i}\left(1 - \exp\{-(\frac{1}{\lambda_i})\}\right)^{r_i}}$$

where $$\bar{y}_i = \frac{\sum_{j=1}^{r_i} y_{ij}}{r_i} = \frac{\sum_{j=1}^{r_i} x_{ij}^p}{r_i}$$

$$\ell nL = \text{constant} - r_i(\ell n\lambda_i) - \frac{r_i\bar{y}_i}{\lambda_i}$$
$$- r_i \ell n\left(1 - \exp\{-(\frac{1}{\lambda_i})\}\right)$$

and $$\frac{\partial \ell nL}{\partial \lambda_i} = -\frac{r_i}{\lambda_i} + \frac{r_i\bar{y}_i}{\lambda_i^2} + \frac{r_i}{1 - \exp\{-(\frac{1}{\lambda_i})\}} \frac{1}{\lambda_i^2} \exp\{-(\frac{1}{\lambda_i})\}$$

setting this expression equal to zero, we get

$$1 = \frac{\bar{y}_i}{\lambda_i} + \frac{1}{\lambda_i \exp\{+(\frac{1}{\lambda_i})\}-1}$$

i.e. the MLE of λ_i is the solution of the equation

$$(\lambda_i - \bar{y}_i)(\exp\{\frac{1}{\lambda_i}\}-1) = 1 \tag{3.20}$$

So we may consider $\hat{\lambda}_i$ as a function of $\bar{y}_i$, for a given value $\hat{p} = \hat{p}_0$.

Table 3.2 gives the value of $\bar{y}_i$'s for arbitrary λ_i's . Let $\bar{y} = \min\{\bar{y}_1, \bar{y}_2\}$ and for convenience let us label it as sub-population (1). From our sample we pick $\bar{y}$ as defined above and obtain the corresponding λ graphically from fig. 3.1. Let us represent this λ by $\hat{\lambda}_{01}$. We now solve for $\hat{k}_0$, the starting value of k from (3.17), viz.,

$$\hat{\lambda}_{01} = \bar{y} + \hat{k}_0 \frac{n-r}{r_1} \tag{3.21}$$

If $D_0 = h(\hat{k}_0, \hat{p}_0) - \hat{k}_0$ equals zero, we are done. If not, consider the sign of D_0. If $D_0 < 0$, then the value of $\hat{k}$ which satisfies $D \equiv h(\hat{k}, \hat{p}) - \hat{k} = 0$ and provides a solution to equations (3.17), (3.18) and (3.19) must be such that $\hat{k} < \hat{k}_0$; similarly $\hat{k} > \hat{k}_0$ if $D_0 > 0$.

Letting $h(k, p_0) = \frac{1}{1+v}$

where $$v = (\frac{\beta}{\alpha}) \exp\{\frac{1}{\hat{\lambda}_1} - \frac{1}{\hat{\lambda}_2}\} \tag{3.22}$$

Since $v = \frac{1}{h(\hat{k},\hat{p}_0)} - 1$, so $\frac{dv}{d\hat{k}} = - \frac{1}{h^2(\hat{k}, \hat{p}_0)} \times \frac{d\left(h(\hat{k}, \hat{p}_0)\right)}{d\hat{k}}$

TABLE 3.2 THE VALUE OF λ_i AND $\bar{y}_i$ FOR THE EQUATION:

$$(\lambda_i - \bar{y}_i)\,(\exp\{\frac{1}{\lambda_i}\}-1) = 1$$

λ_i-value	$\bar{y}_i$-value	λ_i-value	$\bar{y}_i$-value	λ_i-value	$\bar{y}_i$-value
0.01	0.01000	0.29	0.25715	0.49	0.34068
0.07	0.07000	0.31	0.26863	0.51	0.34620
0.13	0.12954	0.33	0.27925	0.56	0.35854
0.15	0.14873	0.35	0.28907	0.61	0.36914
0.17	0.16720	0.37	0.29816	0.66	0.37832
0.19	0.18479	0.39	0.30659	0.71	0.38633
0.21	0.20138	0.41	0.31441	0.81	0.39964
0.23	0.21090	0.43	0.32169	0.91	0.41022
0.25	0.23134	0.45	0.32846	1.01	0.41881
0.27	0.24475	0.47	0.33478	1.21	0.43190

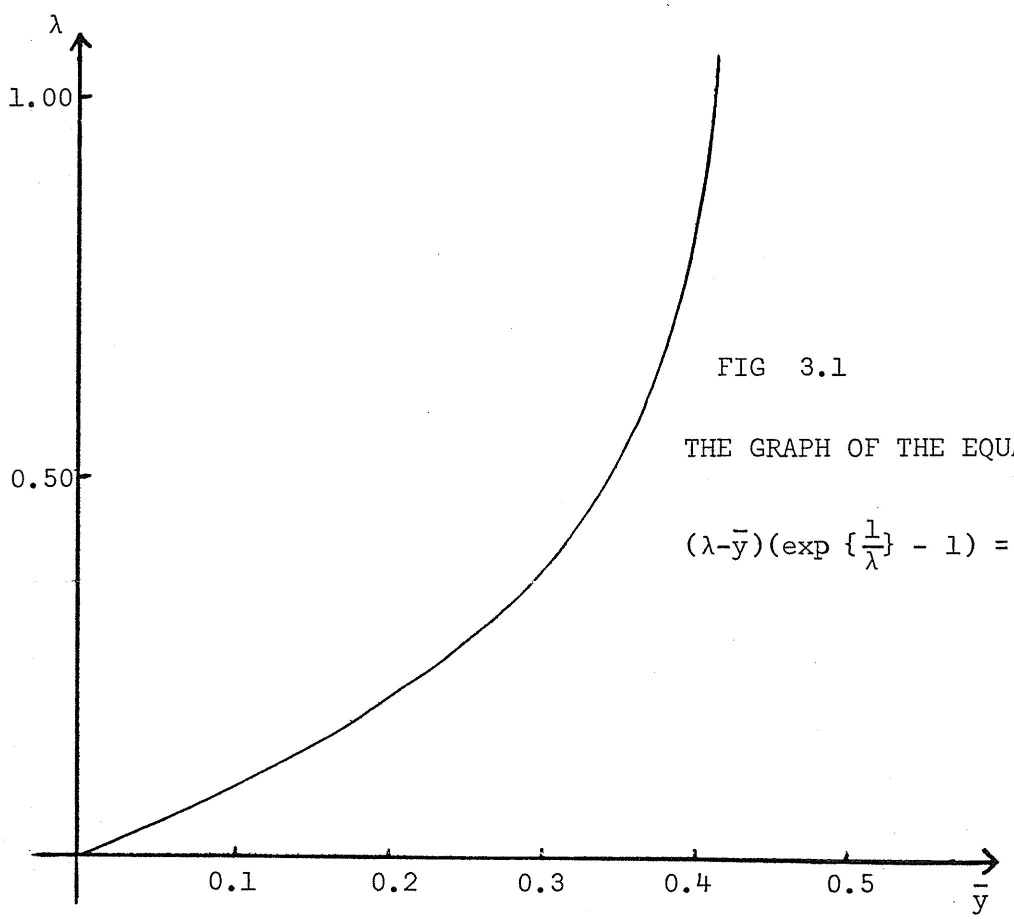

FIG 3.1

THE GRAPH OF THE EQUATION:

$$(\lambda-\bar{y})(\exp\{\frac{1}{\lambda}\} - 1) = 1$$