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A SYSTEMS ANALYSIS OF HOLOGRAPHIC IMAGING WITH DIFFUSED ILLUMINATION FROM A STATISTICAL POINT OF VIEW

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ABSTRACT

Holography with diffused light illumination is studied from a statistical communication point of view. The final amplitude transmittance of the hologram is considered to result from some combination of linear and nonlinear transformation of a nonstationary incoherent Gaussian process plus a sine wave. Formation of a hologram is modelled as a five-stage systems configuration. Each stage of the model is described. The problem is considered analogous to passing white noise and a sine wave through linear and nonlinear electrical devices, in terms of well-known concepts in communication theory. Statistical properties of the outputs of the different stages are considered. For large, thin Fresnel holograms of fine grain recording materials, expressions are obtained for the mean values, variances and autocorrelation functions. In the general case of arbitrary reference and object irradiances, the autocorrelation function of the amplitude transmittance of the hologram is obtained with the aid of the characteristic function method. Several examples of practical use are worked out. As an example of the direct method of obtaining the correlation function of the output of a nonlinear device, we apply this method to a linear phase hologram of exponential transmittance. Although the direct method seems unmanageable in general cases, in the special case of a very weak object irradiance compared with that of the reference's, this method of analysis is employed without complications. In both cases, it is shown that the autocorrelation function of the amplitude transmittance can be expressed as a power series of the Fourier transform of the object irradiance with the major difference that in the general case, some distorting factors are involved.

Interpretation of the results lead to conclusions concerning the image irradiance distribution in the presence of intermodulation noise and non-linear distortions; the intermodulation noise being the result of higher order image correlations and convolutions. Finally, the likely effects on image quality due to polarization change of the illuminating beam after scattering by the diffuse object are considered. The autocorrelation function of the amplitude transmittance in this more general case is obtained. A method of improving the effects of depolarization on the image is discussed.

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TABLE OF CONTENTS

CHAPTER ONE
INTRODUCTION

1.1 History

The links between optics and electrical engineering have become so strong that a unified treatment of problems in the two fields is now a promising possibility. Now, in very many cases, one could treat a problem in one field, in a rather straight-forward manner, by applying the methods already used to study an analogous problem in the other.

It has become a common experience to encounter in literature such similarities as between a 'lens' and a 'linear FM generator', or 'Fresnel diffraction' and input-output relation of a 'quatrature phase filter', or the 'light concentration' of a 'lens' and a 'pulse compression process', etc.

The interaction of these two disciplines is not just in the methods of treatment, but the optical devices themselves are now indispensable in almost all communication systems. They are being used extensively as vital parts of the information processing instruments.

Fourier analysis which has been used widely in many branches of science and engineering might be thought of as the first and strongest stimulus of this interaction, since the Fourier integral works with any physical system in which *cause* and *effect* are linearly related. Therefore, we might say that the first step in unification of optics and electrical engineering was taken by P.M. Duffieux by his introduction of Fourier transform methods and its application in optics, which appeared in his book *L'Integral de Fourier et ses Application à l'Optique* in 1946 [1].

Of course Duffieux's step was taken on a road already paved by other pioneers late in the 19th and early in the 20th century. To this period belongs the Abbe theory of image formation, a very important theory which, for instance, made "spatial filtering" a realizable practice.

There has been great progress in the studies of object-image problems since Duffieux's work; for example, the work of Otto H. Schade, which appeared in a series of papers [2,3] between 1948 and 1958. By applying the transform methods already used in electrical systems and radio propagation, Schade analysed the process of image formation in television cameras and subsequently suggested some techniques to improve their functioning.

Now it was time for other participants to follow and merge the two disciplines and unify the approach. Considerable work has been done and much remains to be done. Nevertheless, the unification is not distant.

1.2 This Thesis

In this treatment, the process of holography is studied from the viewpoint of the statistical communication systems theory. To deal with the process of holography in the framework of communication theory, was first suggested by Leith and Upatnieks[4]. In their treatment, a hologram is made in two stages: 1) the defocusing or spatial-frequency dispersion of the image; 2) the hologram recording, which is similar to a square-law or non-linear detection. In the following, the process is viewed as the result of the input signal being sent through a five-stage model. This approach was originally used by D.H. Kelly, reported in a paper titled "Systems Analysis of the Photographic Process" [5]. Kelly considers a three-stage model for the process of photography; a nonlinearity located in series between two isotropic linear systems.

The components of the five-stage model that we choose to work with in the process of making a hologram are as follows:

1) The projection process of the field's function onto the hologram is accounted for by: a) a quadrature phase filter, or, b) if the finite dimension of the hologram or some other optical element in the path of the signal is a frequency limiter, by a low pass filter with quadratic phase.

2) A square-law envelope detector in which the recording medium detects the signal output of the first stage by being sensitive to a time average of the energy received.

3) A low pass filter, which accounts for the frequency filtering effect of the recording medium due to the optical diffusion.

4) A nonlinearlity, which is due to the nonlinear characteristic of the recording medium in the conversion of the time-averaged intensity to an optical transmittance.

5) A third linear filter which is introduced because of a non-uniformity in the developing process.

For the nonlinear part of his model, Kelly uses the "H & D" characteristic of the recording medium (Hurter-Driffield curve, which is a plot of the photographic density vs. the logarithm of exposure) as a point-by-point nonlinear scale conversion. The H & D curves have been used most commonly in classical photography. However, for holography and most other coherent optical processing techniques a better and more convenient description of the nonlinearity is the T_a-E (amplitude Transmittance vs. Exposure) curve. This was first used by Leith and Upatnieks [6] in their early holographic experiments, and later by Vander Lught [7] for spatial filtering purposes. Kozma [8] used the T_a-E curves in exactly the same way as the characteristic curves of nonlinear electronic devices are being treated (e.g. characteristic curve of a vacuum tube). A working point is usually chosen on the center portion of the smoothest part of the curve and a small variational analysis is applied when the input variation is not exceeding the limits of this smooth part. However, when the variation exceeds the linear limits, first a function (e.g. an error function limiter, a polynomial, a νth law function) is chosen that fits the curve best on both sides of the working point. Then considering the chosen function as the transfer function of a nonlinear device, one of the techniques already used in electrical engineering to study nonlinearities could be employed to characterize the output. The above-mentioned method has been the one used by most workers to study the effects of film nonlinearity on signals. Among those are Kozma [8], Friesem and Zelenka [9], Bryngdhal and Lohmann [10], and Lee and Greer [11], in whose treatments input is considered to be deterministic. Goodman and Knight [12], and Kozma, Jull and Hill [13], studied the problem for random inputs.

The method of analysis in the following study is statistical as in references [12] and [13], although the approach is a bit different. The working point and the variation around it are not considered; input to the nonlinearity is the whole exposure. The mathematical treatment is more general. In reference [12] it is shown that when the irradiance of the reference wave is much stronger than that of the object's, the autocorrelation function of the variation of the transmittance of the recording medium can be expressed in a power series of the autocorrelation function of the exposure variation, the latter being proportional to the Fourier transform of the object irradiance. However, in the general case of arbitrary irradiances of object and reference, the autocorrelation function of the output remains without a general expression. In the following, using the characteristic function method, it will be shown that in the general case, too, the autocorrelation function of the transmittance can be expressed in a power series of the Fourier transform of the object irradiance. The treatment, with some slight modifications,can also be applied to cases in which the phase of the reference wave over the hologram plane happens to be random. Fresnel holography is considered rather than the more special case of Fourier transform holography. Besides, in some cases, we also use the *direct method* of finding the autocorrelation function of the output of a nonlinearity [14] in addition to the well-known *characteristic function method* [14,15] used in several treatments, e.g. Ref. [12]. The advantages of the direct method are that, first there is no need to find the Fourier or Laplace transform of the transfer function of the nonlinearity. Secondly, in cases where the transfer function does not have a Fourier transform but a Laplace transform, a contour integration technique must be used to evaluate the integral involved in the characteristic function method. This contour integration may be troublesome. Using the direct method may eliminate this problem too.

Some preliminary considerations are given in *Chapter Two* and *Appendix B*. These include statistics of diffuse objects, the coherence requirement, the definition of spatial coherence as a space average correlation and not a time average one, the autocorrelation of diffuse objects, some properties of autocorrelation functions, and the use of autocorrelation functions in determination of the irradiance distribution of the diffracted waves.

The different stages of our systems model are then discussed in the following chapters. Some statistics of the signal at the output of each stage are considered. The autocorrelation function of the signal is followed through the model to study some relevant characteristics of images in the reconstruction process, e.g., their irradiance distribution.

The last chapter is devoted to a brief consideration of the depolarization of the illuminating beam by a diffuse object. Some probable effects of the cross-polarized component on the image degradation are given. Some ways of reducing these effects in order to record holograms with improved quality are discussed.

CHAPTER TWO
FUNDAMENTALS

2.1 Introduction

The concept of diffused or scattered light illumination in holography was first introduced in 1964 by G.W. Stroke [16], and by E.N. Leith and J. Upatnieks [17] almost at the same time. Before the introduction of diffused illumination, holograms were made only of transparent subjects. In diffused illumination holography the property of high spatial and temporal coherence of gas lasers are exploited to make holograms of diffusely reflecting three-dimensional objects.

Diffused illumination holography was first achieved by using a diffusing element such as a ground glass and a subject transparency. The ground glass is placed in the path of a coherent illuminating beam to illuminate the transparency with diffused coherent light. Without the ground glass, light from each portion of the transparency reaches the recording medium in a specular direction (the direction of the illuminating source and that portion) resulting in a *small region* to *small region* correspondence mapping of the subject in a recorded intensity. In the reconstruction process, when viewing the virtual image, the observer sees only the portion which lies between the illuminating source and his eyes. To see another portion, he has to move his head to intercept the light coming from that portion. That is to say, the observer has to move his head to scan the different small portions of the subject's virtual image. The real image could be viewed on a diffusing screen placed at the real image plane.

On the other hand, if a diffusing screen is introduced in the way of the illuminating beam when the hologram is recorded, the light is scattered in a myriad of directions. Now each portion of the subject transparency is receiving and transmitting light in all of these directions. Therefore, each portion of the hologram receives light from all portions of the subject, since each portion of the subject has sent light over the entire hologram area. In the reconstruction process, then, no matter what portion of the diffracted light is intercepted, an aspect of the whole subject transparency is viewed. The real image could be viewed on an ordinary screen, e.g. a card board, placed at the plane of the real image.

A further advantage of diffused-illumination holography over non-diffused holography is that the dynamic range of the recording medium is increased more effectively and the distortion of the image due to non-linearity of the recording medium is to some extent reduced. Without a diffuser, very bright and very weak areas in the subject might give exposures which exceed the almost linear limits of the T-E characteristics of the recording medium, resulting in a nonuniformity in the recording of different portions and therefore a distortion of the images. In diffused illumination holography, the energy of each small portion is no longer concentrated on its corresponding small area of the hologram; rather, it shares the whole area of the hologram with the other portions by spreading its energy all over the hologram. Whereas in non-diffused holography, some portions of the recording medium may be exposed to very strong and some to very weak intensity, in diffused holography, the whole area is exposed to the almost uniform mean intensity; the dynamic range of the recording medium is effectively increased and image distortion is greatly reduced.

Another deficiency of non-diffused holography is the problem of ring-like noise associated with the real image. Any imperfection in the path of the high coherent subject beam causes annoying diffraction patterns of circular fringes. These circular fringes become apparent on the transparency and would be recorded as if they were part of the subject details. Diffused illumination removes this problem too, due to the destruction of the diffraction patterns by the diffuser, although it produces a speckled or granular appearance of the image instead.

In making holograms of three-dimensional reflecting objects, the process could be thought of as diffused illumination holography, since most objects scatter light in much the same way as a diffusing screen. The reason is that most objects have rough surfaces at the visible portion of the electromagnetic spectrum. Many of these might be considered as configurations of very many individual scatterers with random positions and orientations. Light on the surface of the object can be regarded as a collection of secondary point sources. Each source appears to emit a spherical wave of random phase.

In this study the term 'diffuse object' refers to either a subject transparency backed by a diffusing screen, or a reflecting object with a very rough surface.

In the following sections a brief study of the statistics of coherently illuminated diffuse objects will be given. Some parts of the background to the following subjects can be found in Appendices A and B. For a more detailed and complete description of the theorems and definitions, the interested reader is referred to Ref.'s [18, 19, 20].

2.2 Autocorrelation of the diffuse object field function

The time-averaged mutual coherence function of a coherent quasi-monochromatic field, $\Gamma_{12}(\tau)$, as in the case of an ideal monochromatic field, can be expressed in the form [21, 22]:

$$\Gamma_{12}(\tau) = \tilde{U}(P_1)\, \tilde{U}^*(P_2)\, e^{-2\pi j\bar{\nu}\tau} \tag{2.1}$$

where $\tilde{U}(P_1)$ is the field function evaluated at P_1 and $\tilde{U}^*(P_2)$ is the complex conjugate of the field function evaluated at P_2. As can be seen, the space and time dependent parts of $\Gamma_{12}(\tau)$ are separated. Thus, $\Gamma_{12}(\tau)$ could be written in the form:

$$\Gamma_{12}(\tau) = \Gamma_{12}(0)\Gamma_{11}(\tau). \tag{2.2}$$

In the Young double-beam interference experiment, for instance, it can be shown that $\Gamma_{11}(\tau)$ is a measure of "temporal" and $\Gamma_{12}(0)$ of "spatial" coherence of the source [23].

If a diffuse object is illuminated by a coherent quasi-monochromatic field, the space-dependent part of the mutual coherence function, $\Gamma_{12}(0)$, of the resultant field will be a random function of space, since the random optical path of the diffuse object impresses a random phase on the incident field. As in the case of any random function, a meaningful quantity for $\Gamma_{12}(0)$ will be its ensemble average. This ensemble average is called the autocorrelation function of the resultant field. The autocorrelation function in this case, then, could be thought of as a "spatial coherence function" [24] when compared to $\Gamma_E(\vec{r}_1,t_1,\vec{r}_2,t_2)$, the time coherence function [c.f. Appendix A].

The ensemble average of $\Gamma_{12}(0)$ is defined as [25]:

$$R(\vec{r}_1,\vec{r}_2) = E\,\{\Gamma_{12}(0)\} = E\,\{\tilde{U}(\vec{r}_1)\tilde{U}^*(\vec{r}_2)\} =$$

$$\int_{-\infty}^{+\infty} \tilde{U}(\vec{r}_1)\tilde{U}^*(\vec{r}_2)\; p(\tilde{U}_1,\tilde{U}_2)\; d\tilde{U}_1 d\tilde{U}_2 \qquad (2.3)$$

where $\tilde{U}(\vec{r})$ is the complex field function of the diffuse object with a deterministic amplitude $A(\vec{r})$, but a random phase $\phi(\vec{r})$. $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of two points on the object surface and $p(\tilde{U}_1,\tilde{U}_2)$ is the joint probability density function of $\tilde{U}(\vec{r}_1)$ and $\tilde{U}(\vec{r}_2)$.

With $\tilde{U}(\vec{r}) = A(\vec{r})\, e^{j\phi(\vec{r})}$ the autocorrelation function of the diffuse object field could be written as

$$E\,\{A(\vec{r}_1)\, e^{j\phi(\vec{r}_1)}\, A^*(\vec{r}_2)\, e^{-j\phi(\vec{r}_2)}\} = A(\vec{r}_1)\, A^*(\vec{r}_2) E\,\{e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\}$$

$$= A(\vec{r}_1)\, A^*(\vec{r}_2) \int_{-\infty}^{+\infty} e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\; p(\phi_1,\phi_2)\; d\phi_1\, d\phi_2 \qquad (2.4)$$

since $A(\vec{r})$ is deterministic.

It was noted above that light on the surface of the diffuse object may be regarded as a configuration of very many secondary sources. The random phase of each secondary source is considered in most cases to be uniformly distributed between 0 and 2π or in general, between some constant C and $C+2\pi$, i.e. [19]

$$p(\phi) = \frac{1}{2\pi} \quad (C < \phi < C + 2\pi). \qquad (2.5)$$

Of course, this assumption is only an approximation, but if the configuration of the scatterers is random and sufficiently dispersed to give a wide phase distribution, then it could be said that (2.5) holds effectively* [19]. Furthermore, if there is a very large number of scatterers, we might also assume that the space phase fluctuations for any two points of the object are statistically independent. Several investigators have used this assumption of independent secondary sources [26,27,28]. Of course, this assumption is not true either because there is always some degree of correlation between very close points. Suzuki and Hioki [29] showed that the correlation area of phase is of the order of λ^2 at least. However, as we will see later, under some conditions it is a reasonable assumption to consider that the individual scatterers are statistically independent. (Perhaps a more reasonable assumption would be that the correlation areas are statistically independent and should be considered as individual independent scatterers).

On the basis of the above assumption, it follows that:

$$p(\phi_1,\phi_2) = p(\phi_1)\, p(\phi_2) = \frac{1}{2\pi} \times \frac{1}{2\pi} = \frac{1}{4\pi^2}$$

and

$$E\{e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\} = \frac{1}{4\pi^2} \int_0^{2\pi} e^{j\phi(\vec{r}_1)}\, d\phi_1 \int_0^{2\pi} e^{-j\phi(\vec{r}_2)}\, d\phi_2$$

$$= \begin{cases} 1 \ ; \ \vec{r}_1 = \vec{r}_2 \\ 0 \ ; \ \text{otherwise}. \end{cases} \qquad (2.6)$$

A process with such autocorrelation as in (2.6) is termed "incoherent", or by its analogy in communication theory, "white noise" [c.f. Section 2.3]. The autocorrelation function of a nontrivial white noise must be of the

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* For a detailed discussion of the validity of such an assumption the interested reader is referred to Ref. 19, Section 7.3, p. 146-151.

form [30]:

$$R(\vec{r}_1,\vec{r}_2) = I(\vec{r}_1) \quad \delta(\vec{r}_2-\vec{r}_1) \tag{2.7}$$

where we choose to show $A(\vec{r})\ A^*(\vec{r}) = |A(\vec{r})|^2 = \tilde{U}(\vec{r})\tilde{U}^*(\vec{r}) = |\tilde{U}(\vec{r})|^2$, the object irradiance, by $I(\vec{r})$.

Very often one assumes an ergodic type hypothesis and equates the spatial and ensemble averages. Then for a stationary ergodic process the autocorrelation function would be defined as

$$R(\vec{\tau}_r) = \int_{-\infty}^{\infty} \tilde{f}(\vec{u})\ \tilde{f}^*(\vec{u}-\vec{\tau}_r)d\vec{u}\ . \tag{2.8}$$

This function is related to the Fourier transform of $\tilde{f}(\vec{u})$, $\tilde{F}(\vec{s})$, by the autocorrelation theorem [31], which states that the Fourier transform of $R(\vec{\tau}_r)$ is equal to $|\tilde{F}(\vec{s})|^2$.

$$R(\vec{\tau}_r) = \mathcal{F}^{-1}[|\tilde{F}(\vec{s})|^2] = M\delta(\vec{\tau}_r) + \sum_{\substack{m,n=1 \\ m\neq n}}^{M} \sum^{M} \delta[\vec{\tau}_r-(\vec{\tau}_{r_m}-\vec{\tau}_{r_n})] \exp [j(\phi_m-\phi_n)] \tag{2.9}$$

Eq. (2.9) is in accordance with our previously mentioned assumption that when there is a large number of scatterers it is reasonable to assume they are independent. As M becomes larger, the correlation function approaches a delta function or shows the tendency toward a state of non-correlation of phase [32].

Another way of analyzing this problem is by modeling the random optical path of the diffuse object by a random function $Z(x,y)$, defined in a domain of the xy plane*. The coordinate system is defined in such

*The background to this material can be found in Ref. 19, Chapter 5, p. 70-98.

a way as to make the xy plane the mean plane of the surface, so that

$$E\,\{Z(x,y)\} = 0 \,. \tag{2.10}$$

It is assumed that the surface is isotropically rough, i.e. that Z is distributed with the same statistical distribution in all directions over the surface [33]. The diffuse object field function $\tilde{U}(\vec{r})$ could be written in the form:

$$\tilde{U}(\vec{r}) = A(\vec{r})\; e^{jkZ(\vec{r})} \tag{2.11}$$

where $k = \frac{2\pi}{\lambda}$ is the wave number.

It follows that the autocorrelation function $R(\vec{r}_1,\vec{r}_2)$ is

$$R(\vec{r}_1,\vec{r}_2) = A(\vec{r}_1)A^*(\vec{r}_2)\; E\{e^{jkZ(\vec{r}_1)}\; e^{-jkZ(\vec{r}_2)}\} = A(\vec{r}_1)A^*(\vec{r}_2)\; \cdot$$

$$\cdot \int_{-\infty}^{\infty} e^{jk[Z(\vec{r}_1)-Z(\vec{r}_2)]}\; p(Z_1,Z_2)\; dZ_1\; dZ_2 \tag{2.12}$$

where $p(Z_1,Z_2)$ is the joint probability density of $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$. But by definition [34]

$$E\,\{e^{jv_1\xi}\; e^{jv_2\eta}\} = M(v_1,v_2) \tag{2.13}$$

is the joint characteristic function of the distribution $p(\xi,\eta)$ of two random variables ξ and η. Therefore, $E\,\{e^{jk[Z(\vec{r}_1) - Z(\vec{r}_2)]}\}$ is the characteristic function of $p(Z(\vec{r}_1), Z(\vec{r}_2))$ evaluated at $v_1 = k$ and $v_2 = -k$ [35].

A model that is used very often for many rough surfaces is the Gaussian or the normal distribution model [35]. The joint probability density for two normally distributed random functions with mean value zero, variance σ^2 and correlation coefficient ρ, is [36]

$$p(\xi,\eta) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{\xi^2-2\rho\,\xi\eta+\eta^2}{2\sigma^2(1-\rho^2)}\right] \tag{2.14}$$

and its characteristic function is

$$M(v_1,v_2) = \exp\left[-\frac{1}{2}\sigma^2(v_1^2 + 2\rho\, v_1 v_2 + v_2^2)\right]. \tag{2.15}$$

For a normally distributed surface

$$M(k,-k) = \exp\left[-k^2\sigma_Z^2(1-\rho_Z)\right] \tag{2.16}$$

ρ_Z, the correlation coefficient, is the normalized correlation function of $Z(\vec{r})$ and is defined as [37]

$$\rho_Z(\vec{\tau}_r) = \frac{R_Z(\vec{\tau}_r) - m_Z^2}{\sigma_Z^2} = \frac{R_Z(\vec{\tau}_r) - [E\{Z(\vec{r})\}]^2}{E\{|Z(\vec{r})|^2\} - [E\{Z(\vec{r})\}]^2}$$

$$= \frac{R_Z(\vec{\tau}_r)}{\sigma_Z^2}. \tag{2.17}$$

m_Z is the mean value of $Z(\vec{r})$ which is zero. $\vec{\tau}_r$ is the space difference $\vec{r}_2 - \vec{r}_1$. $R_Z(\vec{\tau}_r)$ is the correlation function of $Z(\vec{r})$ [38]:

$$R_Z(\vec{\tau}_r) = E\{Z(\vec{r})\, Z(\vec{r} + \vec{\tau}_r)\} \tag{2.18}$$

For large values of the separation parameter, $\vec{\tau}_r$, $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$ are independent. But when $\vec{\tau}_r$ is small, $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$ are correlated; when $\vec{\tau}_r = 0$ they will be identical. $\rho_Z(\vec{\tau}_r)$ will decrease monotonously from its maximum value $\rho_Z(0) = 1$ to $\rho_Z(\infty) = 0$. Let the "correlation distance", defined as the distance over which $\rho_Z(\vec{\tau}_r)$ drops to the value e^{-1}, be T_Z. Then a sufficiently general autocorrelation coefficient $\rho_Z(\vec{\tau}_r)$ could be taken to be the function [39]

$$\rho_Z(\vec{\tau}_r) = e^{-\tau_r^2/T_Z^2} . \tag{2.19}$$

This assumption gives

$$M(k,-k) = \exp\left[-k^2\sigma_Z^2\left(1-\exp-\frac{\tau_r^2}{T_Z^2}\right)\right] . \tag{2.20}$$

The correlation distance T_ϕ for the phase function $e^{jkZ(\vec{r})}$ may be obtained by equating $(1-\exp-\tau_r^2/T_Z^2)$ with the inverse of $k^2\sigma^2$. Thus

$$1 - \exp[-T_\phi^2/T_Z^2] = \left(\frac{\lambda}{2\pi\sigma_Z}\right)^2 . \tag{2.21}$$

$\rho_Z(\vec{\tau}_r)$ is between -1 and $+1$, therefore, $\exp(-T_\phi^2/T_Z^2)$ could be well approximated by its expansion to the second term. Hence, one could write

$$1 - (1 - T_\phi^2/T_Z^2) \simeq \left(\frac{\lambda}{2\pi\sigma_Z}\right)^2 \tag{2.22}$$

from which it follows that:

$$T_\phi \simeq \frac{\lambda T_Z}{2\pi\sigma_Z} . \tag{2.23}$$

It can be seen that T_ϕ can be made negligibly small by increasing the standard deviation of the surface, σ_Z. Actually, the condition $\sigma_Z >> \lambda$ implies a very rough surface [40] for which most of the diffracted light is diffused and the non-diffused part (in the specular direction) is negligible [41]. Therefore, for a very rough surface, the following approximation can be assumed to be fairly reasonable:

$$M(k,-k) \begin{cases} = 1 \; ; \; \vec{\tau}_r = 0 \\ \simeq 0 \; ; \; \text{otherwise} . \end{cases}$$

We conclude that the autocorrelation function of our diffuse object could be well approximated by the irradiance of the object at $\vec{r}_1$ times the delta function $\delta(\vec{r}_2 - \vec{r}_1)$:

$$R_{UU}(\vec{r}_1,\vec{r}_2) = I(\vec{r}_1)\ \delta(\vec{r}_2-\vec{r}_1).$$

In Appendix B, the input-output relationships of the autocorrelation functions for a linear shift invariant system are briefly studied. Then the relationships are used in the following chapters to obtain the autocorrelation function of the actual transmittance of the hologram. For once the autocorrelation function of the hologram transmittance is obtained, with the help of the input-output relationships of the autocorrelation functions, some important properties of the hologram images can be predicted.

2.3 Non-correlation of Phase and Its Implications

Eqn. (2.6&7) of the last section look like the mutual coherence function for an incoherent object [42]. But it should be kept in mind that we have obtained it after arranging a space averaging process over the mutual coherence function. Nevertheless, a process with such space autocorrelation, as in (2.6), is also termed "incoherent". The analogy of the incoherent process in communication theory is termed "white noise". The terms "incoherent" or "white noise" refer to the assumption of non-correlation of phases of the process, i.e.,

$$E\{e^{j(\phi(\vec{r}_1) - \phi(\vec{r}_2))}\} = \begin{cases} 1 ; & \vec{r}_1 = \vec{r}_2 \\ 0 ; & \text{otherwise} \end{cases}.$$

The fact that the process of coherent holography remains unchanged despite such unrelated space phases as in the diffuse object field function is not surprising. For the spatial random nature of the phase distribution on the object surface (which is introduced by a stationary diffuser) is time independent; in other words, the time variation of the phase difference of each point of the object and the reference wave remains the same as if there were no diffuser. The properties of temporal and spatial coherence of the source (in the common use definitions) are not altered by the stationary diffuser. Each point of the object remains capable of interfering with the reference wave if it was capable of such an interference before introducing the diffuser. Therefore, the single coherence requirement necessary for the recording of a hologram in coherent light is unchanged [43].

2.4 Probability Distribution of Diffuse Object Field Function

The assumption of the many scatterers and their independence enable the Central Limit Theorem [44] to be applied.

Roughly speaking, the Central Limit Theroem states that whenever a random process can be represented by a linear superposition of a large number M of essentially independent random effects, its statistics will asymptotically approach normal (or Guassian).

Therefore, on the basis of our assumptions, it follows that the diffuse object field function has a normal distribution. The assumption of a uniformly distributed phase between 0 and 2π makes the real and imaginary parts of the field function independent [45]. The mean value of the process is assumed to be zero, since

$$E\,\{A(\vec{r})\,e^{j\phi(\vec{r})}\} = A(r)\,E\,\{e^{j\phi(\vec{r})}\} \doteq A(\vec{r}) \times \frac{1}{2\pi}\int_0^{2\pi} e^{j\phi}.d\phi = 0\,.$$

CHAPTER THREE

DISPERSION AND DETECTION

3.1 Dispersion

3.1.1 Diffraction as a Linear Shift-Invariant System

The problem of diffraction has been treated extensively. Very often the analyses have been confined to the solution of the homogeneous Helmholtz equation using Green's function approach together with the aid of some boundary conditions, e.g., Kirchhoff's or Sommerfeld's extended boundary conditions [46].

On the other hand, in a treatment by Sherman [47] the problem of diffraction of monochromatic scalar waves from plane diffracting screens is approached from a linear shift-invariant systems transformation point of view. The derivation of the transform formulas is independent of the boundary conditions and therefore the necessity for pre-assuming them to obtain any diffraction formula is removed.

The integral transform equations {Eqns. (13) and (14) in Ref. [47]} are given as:

$$\tilde{f}(x,y;0) = \iint_{-\infty}^{\infty} \tilde{g}(x',y';d)\tilde{h}_C(x-x',y-y';d)\,dx'dy' \qquad \text{for } d \geq 0 \tag{3.1}$$

$$\tilde{g}(x,y;d) = \iint_{-\infty}^{\infty} \tilde{f}(x',y';0)\tilde{h}_D(x'-x,y'-y;d)dx'dy' \qquad \text{for } d \geq 0 . \tag{3.2}$$

Eqn. (3.2) defines the diffraction-transform operation and yields the solution to the diffraction problem with a known boundary value on the Z=0 plane. Eqn. (3.1) defines the inverse diffraction transform operation and yields the solution to the inverse diffraction problem with a known diffracted field and an unknown boundary condition. $\tilde{h}_C$ and $\tilde{h}_D$, the transform kernels of the transformation integrals, are evaluated by expressing the wave field as an angular spectrum of plane waves and are orthogonal to each other. $\tilde{h}_D$ is the point divergence, the point spread, or the impulse response of the diffraction transformation, i.e. the field distribution on the plane $Z=d$ due to a point source $\delta(x,y)$ on the plane $Z=0$, and it is evaluated as

$$\tilde{h}_D(x,y;d) = -\frac{1}{2\pi}\frac{\partial}{\partial d}\frac{\exp(jkr)}{r} \tag{3.3}$$

where $r=(x^2+y^2+d^2)^{1/2}$. $\tilde{h}_C$ is the point convergence of the inverse diffraction transformation, i.e. the field $\tilde{h}_C$ on $Z=0$ produces a delta function singularity on the plane $Z=d$. $\tilde{h}_C(x,y;d)$ cannot be expressed by an ordinary function and must be defined as a functional [47].

$\tilde{h}_D(x,y;d)$ is an even function [c.f. Eqn. 3.3) and therefore Eqn. (3.2) can be written as:

$$\tilde{g}(x,y;d) = \iint_{-\infty}^{\infty} \tilde{f}(x',y';0)\tilde{h}_D(x-x',y-y';d)\,dx'dy' . \tag{3.4a}$$

The above is a convolution integral

$$\tilde{g}(x,y;d) = \tilde{f}(x,y;0)**\tilde{h}_D(x,y;d) \tag{3.4b}$$

where $**$ stands for a two dimensional convolution. Therefore the field on a plane $Z=d$ due to an aperture function on the plane $Z=0$ can be considered as the output of a linear shift invariant system with input $\tilde{f}(x,y;0)$ and impulse response $\tilde{h}_D(x,y;d)$.

To obtain expressions of more practical use we perform the following operations:

$$\tilde{h}_D(x-x',y-y';d) = \frac{\partial}{\partial d}\, \frac{e^{jkR}}{R} = \frac{e^{jkR}(jk-\frac{1}{R})\,\frac{\partial R}{\partial d}}{R}$$

$$= \frac{e^{jkR}(jk - \frac{1}{R})}{R} \cos(\vec{R},\vec{n})$$

where $R = [(x-x')^2 + (y-y')^2 + d^2]^{1/2}$; $\vec{R}$ is the position vector connecting a point on plane $Z=0$ with coordinates (x,y) to a point on plane $Z=d$ with coordinates (x',y'); $\vec{n}$ is the direction of propagation; and $\frac{\partial R}{\partial d} = \frac{d}{R} = \cos(\vec{R},\vec{n})$. At the visible portion of the electromagnetic spectrum

$$k = \frac{2\pi}{\lambda} >> \frac{1}{R}$$

and therefore

$$\tilde{h}_D(x-x',y-y';d) \simeq -\frac{jk}{2\pi}\, \frac{e^{jkR}}{R} \cos(\vec{R},\vec{n}). \tag{3.5}$$

Substituting for $\tilde{h}_D$ in (3.4a) gives

$$\tilde{g}(x,y;d) \simeq -\frac{jk}{2\pi} \iint_{-\infty}^{\infty} \tilde{f}(x',y';0)\, \frac{e^{jkR}}{R} \cos(\vec{R},\vec{n})\, dx'dy' \tag{3.6}$$

which is the Fresnel-Kirchhoff integral [e.g. Ref.[46]]with $\cos(\vec{R},\vec{n})$ as the obliquity factor.

If d is large with respect to the dimensions of the diffracting aperture and diffraction pattern, a good approximation for R will be

$$R \simeq d + \frac{(x-x')^2 + (y-y')^2}{2d} \tag{3.7}$$

which is an approximation of a binomial expansion of

$$R = d\,[1 + \frac{(x-x')^2 + (y-y')^2}{d^2}]^{1/2}$$

by using the first two terms while neglecting the higher order terms. A pattern for which the above-mentioned approximation is justified is referred to as Fresnel diffraction. If d is not large enough, i.e. in the case of very near field patterns, some higher order terms,(depending on the desired accuracy of calculation), must be considered in order to have a justified approximation. These patterns are sometimes referred to as the "Sommerfeld" diffraction. On the other hand, if d is very large with respect to the dimension of the diffracting aperture so that in (3.7) the term $\frac{x'^2 + y'^2}{2d}$ is also negligible, the pattern is called the "Fraunhofer" diffraction pattern. The well-known Fourier transform formulation of the Fraunhofer diffraction is obtained by using the latter approximation for R in the Fresnel-Kirchhoff integral (3.6).

In the following we use the approximation (3.7). Now, when substituting for R in (3.5) we note that a change in R in the denominator is not very significant compared with a change in the value of R in the exponent. So, while putting $R = d + \frac{(x-x')^2 + (y-y')^2}{2d}$ in the argument of the exponent, we put $R=d$ in the denominator. Then,

$$\tilde{h}_D(x-x',y-y';d) \simeq \frac{-jk\,\exp(jkR)}{2\pi R} \simeq \frac{-jk}{2\pi} \cdot \frac{\exp\{jk[d + \frac{(x-x')^2 + (y-y')^2}{2d}]\}}{d}$$

$$= \frac{-jk}{2\pi} \cdot \frac{\exp(jkd)}{d} \cdot \exp\{\frac{jk}{2d}[(x-x')^2 + (y-y')^2]\} \tag{3.8}$$

where we also put

$$\frac{\partial R}{\partial d} = \cos(\vec{R},\vec{n}) \simeq 1 \quad .$$

Substituting for $\tilde{h}_D(x-x',y-y';d)$ in (3.6) and omitting the proportionality factor $\frac{-jk}{2\pi}\,\frac{\exp(jkd)}{d}$ gives:

$$\tilde{g}(x,y;d) = \iint_{-\infty}^{\infty} \tilde{f}(x',y';0) \exp\{j\gamma[(x-x')^2 + (y-y')^2]\} \qquad (3.9a)$$

or

$$\tilde{g}(x,y;d) = \tilde{f}(x,y;0) ** \exp[j\gamma(x^2 + y^2)] \qquad (3.9b)$$

where $\gamma = \frac{k}{2d}$. Under the above mentioned approximations, the impulse response of the system is

$$\tilde{h}(x,y) = \exp[j\gamma(x^2 + y^2)] \qquad (3.10)$$

where we choose to show $\tilde{h}_D(x,y;d)$ with $\tilde{h}(x,y)$. $\tilde{h}(x,y)$ is analogous to the impulse response of a quadratic phase filter in electrical communication [48]: $h(t) = \exp(j\Gamma t^2)$. The corresponding transfer function, i.e., the Fourier transform of $\tilde{h}(x,y)$ is given by:

$$\tilde{H}(\xi,\eta) = e^{j\pi/4} \sqrt{\frac{\pi}{\gamma}} \exp[-j(\xi^2 + \eta^2)/4\gamma] \qquad (3.11)$$

where ξ and η are the spatial frequency variables corresponding to x and y, respectively. Thus the filter is all-pass, but it gives a quadratic dispersion (or phase-shift) to the spatial frequency components of the object signal, equal to:

$$\theta(\xi,\eta) = \frac{\xi^2 + \eta^2}{4\gamma} - \frac{\pi}{4}\,. \qquad (3.12)$$

The dispersion factors [49] (or the group delays [48]) are:

$$d\theta/d\xi = \xi/2\gamma \ , \quad d\theta/d\eta = \eta/2\gamma \ . \tag{3.13}$$

Above we considered the diffraction of light by a plane diffracting screen as an all-pass quadratic phase filter. However, it might happen that some optical element in the path of the diffracted field acts as an aperture stop which limits the high spatial frequency components. That is, because of its finite dimensions, the aperture stop blocks the field components of too high spatial frequencies. The finite optical element, then, acts almost like an ideal low-pass filter; the only recordable field components are those which lie within the limits of the dimensions of the projected aperture stop on the hologram plane. So, in this case, instead of an all-pass quadratic phase filter, a low-pass filter with quadratic phase should be considered. However, in the following, we confine our study to an all-pass filter by assuming that the optical elements in the path of the object signal are not limiting the field diffracted by the object, at least up to very high frequencies.

From now on, without limiting the generality of application, we will consider one-dimensional processes to simplify the mathematical notation; the extension to two dimensions is straightforward. Using a one dimensional notation, Eqn. (3.10) will be

$$\tilde{h}(r) = \exp(j\gamma r^2) \ . \tag{3.14}$$

3.1.2 Autocorrelation of the Field Diffracted by a Diffuse Object

In Section 2.2 we obtained the autocorrelation of the diffuse object as

$$R_{UU}(\rho_1;\rho_2) = I_U(\rho_1)\delta(\rho_2 - \rho_1) .$$

To find the autocorrelation function of the diffracted field on the hologram , $\tilde{V}(r) = V(r)\, e^{j\psi(r)}$, one may use Eqns. (B-5) and (B-6)

$$\tilde{R}_{UV}(r_1;r_2) = I_U(r_1)\, h^*(r_2-r_1)$$

$$\tilde{R}_{VV}(r_1;r_2) = [I_U(r_1)\tilde{h}^*(r_2-r_1)]*\tilde{h}(r_1)$$

with $\tilde{h}(r)$ given as in (3.14). Hence

$$\tilde{R}_{VV}(r_1;r_2) = \int_{-\infty}^{+\infty} I_U(\rho_1)\, \exp[-j\gamma(r_2-\rho_1)^2]$$

$$\times \exp[j\gamma(r_1-\rho_1)^2]\, d\rho_1$$

$$= \int_{-\infty}^{+\infty} I_U(\rho_1)\, \exp[j\gamma(r_1^2-r_2^2)]\, \exp[-j2\gamma(r_1-r_2)\rho_1]\, d\rho_1$$

$$= \exp\, [j\gamma(r_1^2-r_2^2)] \int I_U(\rho_1)\exp[-j2\gamma(r_1-r_2)\rho_1]\, d\rho_1 \, . \quad (3.15)$$

If we denote the Fourier transform of the diffuse object irradiance, $I_U(\rho)$, with $\tilde{\mathcal{F}}_U(\cdot)$ and $2\gamma(r_1-r_2)$ with τ_r, then Eqn. (3.12) may be written as

$$\tilde{R}_{VV}(r_1;r_2) = \exp[j\gamma(r_1^2 - r_2^2)]\; \tilde{\mathcal{F}}_U(\tau_r). \quad (3.16)$$

That is, the autocorrelation function of the diffracted field of the diffuse object on the hologram plane is proportional to a Fourier transform of the irradiance of the object.

3.1.3 Other Statistics of the Diffracted Field

If the statistics of a random input to a linear system are normal, the statistics of the output are also normal [e.g, Ref;s 50,51 and 52]. Therefore we conclude that the field diffracted by our diffuse object has normal real and imaginary components with zero means which are independent at any single point [c.f. Section 2.4 and Eqn. B-9].

The mean irradiance of the diffracted field, M_I, can be obtained by the substitution $r_1 = r_2 = r$ in $R_{VV}(r_1;r_2)$[c.f. Eqn. (B-10) and (3.16)]:

$$M_I = R_{VV}(r;r) = \tilde{\mathcal{I}}_U(0) \tag{3.17}$$

or by using Eqn. (B-11):

$$M_I = I_U(r) * |h(r)|^2 = \int_{-\infty}^{\infty} I_U(\rho)d\rho = \tilde{\mathcal{I}}_U(0) \tag{3.18}$$

which is constant.

The variance of the field is [c.f. Eqn. (B-13)]:

$$\sigma_V^2 = I_U(r) * |h(r)|^2 - m_V^2(r) = M_I = \tilde{\mathcal{I}}(0) . \tag{3.19}$$

The autocorrelation of the diffracted field irradiance will be [c.f. Eqn. (B-15)]:

$$R_{II}(\tau_\rho) = \sigma_V^4 + |\tilde{\mathcal{I}}_U(\tau_\rho)|^2 . \tag{3.20}$$

The mean value of the square of the field irradiance is [c.f. Eqn. (B-16)]:

$$M_{I^2} = 2\sigma_V^4 . \tag{3.21}$$

The variance of the field irradiance is:

$$\sigma_I^2 = \sigma_V^4 . \tag{3.22}$$

3.2 Detection

3.2.1 Square-Law Envelope Detector

We have used complex variables to represent the scalar quasi-monochromatic fields, e.g., the diffracted field by the diffuse object as $\tilde{V}(r,t) = V(r)\ e^{j\psi(r)}\ e^{j\omega t}$. The time harmonic part was suppressed since it is a common factor unchanged in the process of diffraction by a stationary diffuse object. However, the electromagnetic field is a real physical quantity. The reason for employing a complex representation is mostly its mathematical simplicity. One can go from this convenient mathematical description to the real field quantity by taking the real (or imaginary) part of the complex quantity. For example, the field diffracted by the diffuse object is

$$\mathrm{Re}[\tilde{V}(r,t)] = \mathrm{Re}[V(r)\ \exp j[\psi(r) + \omega t]]$$

$$= V(r)\cos(\omega t + \psi(r)) \tag{3.23a}$$

or

$$\mathrm{Re}[\tilde{V}(r,t)] = \mathrm{Re}\ \{V(r)\ [\cos\psi(r) + j\sin\psi(r)][\cos\omega t + j\sin\omega t]\}$$

$$= V(r)\cos\psi(r)\cos\omega t - V(r)\sin\psi(r)\sin\omega t$$

$$= C(r)\cos\omega t - S(r)\sin\omega t \tag{3.23b}$$

where "Re" reads as "the real part of" (and "Im" as "the imaginary part of") and we choose to show $V(r)\cos\psi(r)$ by $C(r)$ and $V(r)\sin\psi(r)$ by $S(r)$. $V(r)$ is called the *envelope* and $\psi(r)$ the *phase* of the diffracted field. $V(r)$, the envelope of the real field function is the modulus of the complex representation. When detecting such a signal the quantity which is measurable is a time average of the incident energy, since the optical field is oscillating too fast with respect to the response time of the detector [53]. In other words, the detectors are responsive to a time average of the square of the field amplitude, e.g., $\langle[V(r)\cos(\omega t+\psi(r))]^2\rangle$ which turns out to be $\approx V^2(r)$, i.e., the square modulus of the complex field function or the square of the envelope of the real field function. Now, let us assume we are to detect two coherent quasi-monochromatic fields of the angular frequency $\bar{\omega}$ of the forms $a_1(r)\cos(\bar{\omega}t+\theta_1)$ and $a_2(r)\cos(\bar{\omega}t+\theta_2)$. The resultant field can be written as follows:

$$\begin{aligned} a_1(r)\cos(\bar{\omega}t+\theta_1) + a_2(r)\cos(\bar{\omega}t+\theta_2) &= (a_1(r)\cos\theta_1+a_2(r)\cos\theta_2)\cos\bar{\omega}t \\ &- (a_1(r)\sin\theta_1+a_2(r)\sin\theta_2)\sin\bar{\omega}t \\ &= A(r)\cos\Theta\cos\bar{\omega}t - A(r)\sin\Theta\sin\bar{\omega}t \\ &= A(r)\cos(\bar{\omega}t+\theta) \end{aligned} \tag{3.24}$$

where $A(r) = \{[a_1(r)\cos\theta_1 + a_2(r)\cos\theta_2]^2 + [a_1(r)\sin\theta_1 + a_2(r)\sin\theta_2]^2\}^{1/2}$

(3.25a)

and

$$\theta = \tan^{-1}\frac{a_1(r)\sin\theta_1 + a_2(r)\sin\theta_2}{a_1(r)\cos\theta_1 + a_2(r)\cos\theta_2} \tag{3.26}$$

The time average irradiance is

$$A^2(r) = a_1^2(r) + a_2^2(r) + 2a_1(r)a_2(r)\cos(\theta_1-\theta_2) \tag{3.25b}$$

in which the phase information is seen to be preserved.

In terms of the complex representation, the two disturbances may be shown as:

$$a_1(r)\ \exp(j\theta_1) \cdot \exp\ j\bar{\omega}t$$

$$a_2(r)\ \exp(j\theta_2) \cdot \exp\ j\bar{\omega}t\ . \tag{3.27}$$

The complex amplitude of the resultant is given by

$$[a_1(r)\ \exp(j\theta_1) + a_2(r)\ \exp(j\theta_2)]\ \exp\ j\bar{\omega}t \tag{3.28}$$

which could be reduced to

$$A(r)\ \exp(j\Theta)\exp(j\bar{\omega}t) \tag{3.29}$$

where $A(r)$ and Θ are given by (3.25) and (3.26). If (3.28) is multiplied by its conjugate we will have

$$[a_1(r)\ \exp(j\theta_1) + a_2(r)\ \exp(j\theta_2)][a_1(r)\exp(-j\theta_1) + a_2(r)\exp(-j\theta_2)]$$

$$= a_1^2(r)+a_2^2(r)+a_1(r)a_2(r)\ [\exp j(\theta_1-\theta_2) + \exp\ -j(\theta_1-\theta_2)]$$

$$= a_1^2(r)+a_2^2(r)+2a_1(r)a_2(r)\cos(\theta_1-\theta_2) = A^2(r)\ . \tag{3.30}$$

That is, the time average irradiance of the resultant field can be obtained by multiplying the sum of the complex fields by its conjugate.

In the process of holography the same procedure of recording the time average irradiance of two coherent quasi-monochromatic fields is followed in order to preserve the phase information of the fields. (The reconstruction process consists of illuminating the hologram by a replica of one of the two fields to reconstruct a replica of the other. In other words, a hologram may be considered as a physical realization of a boundary condition that forces one field to assume also the values of a second field in order to create the second field from the first [54]). The detector, the recording medium, is responsive to the time average energy of the sum of the two fields, i.e., the square of the envelope of the resultant field. This clarifies the reason for calling the first stage of the recording process a *square-law envelope detector* (also c.f. Ref. 4 and 55].

3.2.2 Autocorrelation Function of the Exposure

Let the reference wave at the hologram plane be:

$$\begin{aligned} \mathrm{Re}[\tilde{K}(r,t)] &= \mathrm{Re}[\tilde{K}(r)\,\exp j\bar{\omega}t] \\ &= \mathrm{Re}[K\exp(\eta r)\exp(j\bar{\omega}t)] \\ &= K\cos(\bar{\omega}t + \eta r) \end{aligned} \tag{3.31}$$

Where K is a constant. (3.31) represents a uniform plane wave incident at the hologram plane at an angle α given by :

$$\alpha = \sin^{-1}\frac{\lambda\eta}{2\pi}\,, \tag{3.32}$$

The time average irradiance of the sum of the reference wave and the field diffracted by the object at the hologram plane will be [c.f. Eqn. (3.30)]:

$$I_{sum}(r) = |\tilde{K}(r)+\tilde{V}(r)|^2 = K^2+V^2(r) + 2KV(r)\cos(\eta r + \psi(r)) \tag{3.33}$$

This times the exposure time gives the exposure E:

$$E = T|\tilde{K}(r) + \tilde{V}(r)|^2 \tag{3.34}$$

where T is the exposure time.

For the autocorrelation of the exposure E we have

$$R_{\mathcal{EE}}(r_1;r_2) = E\{\mathcal{E}(r_1)\mathcal{E}(r_2)\} =$$

$$= \mathcal{T}^2\, E\{|\tilde{K}(r_1) + \tilde{V}(r_1)|^2 \cdot |\tilde{K}(r_2) + \tilde{V}(r_2)|^2\}. \qquad (3.35)$$

Considering that $\tilde{K}(r)$ and $\tilde{V}(r)$ are independent of each other and each has a zero mean, the result will be

$$\begin{aligned} R_{\mathcal{EE}}(r_1;r_2) = \mathcal{T}^2[&E\{|\tilde{K}(r_1)|^2|\tilde{K}(r_2)|^2\} + E\{|\tilde{K}(r_1)|^2\} \cdot E\{|\tilde{V}(r_2)|^2\} \\ &+ E\{|\tilde{V}(r_1)|^2\} \cdot E\{|\tilde{K}(r_2)|^2\} \\ &+ E\{|\tilde{V}(r_1)|^2 \cdot |\tilde{V}(r_2)|^2\} \\ &+ E\,\{\tilde{K}(r_1)\tilde{K}^*(r_2)\} \cdot E\{\tilde{V}^*(r_1)\tilde{V}(r_2)\} \\ &+ E\{\tilde{K}^*(r_1)\tilde{K}(r_2)\} \cdot E\{\tilde{V}(r_1)\tilde{V}^*(r_2)\} \\ &+ E\{\tilde{K}(r_1)\tilde{K}(r_2)\tilde{V}^*(r_1)\tilde{V}^*(r_2) + \tilde{K}^*(r_1)\tilde{K}^*(r_2) \\ &\quad \cdot \tilde{V}(r_1)\tilde{V}(r_2)\}]. \end{aligned} \qquad (3.36)$$

The last term in the above expression is identical to zero. To show this, we note that

$$\mathrm{Re}[\tilde{A}] = \frac{1}{2}(\tilde{A} + \tilde{A}^*).$$

So, the last term in (3.36) can be written as:

$$\begin{aligned} &2E\{\mathrm{Re}[\tilde{K}(r_1)\tilde{K}(r_2)\tilde{V}^*(r_1)\tilde{V}^*(r_2)]\} \\ &\quad = 2[(E\{C(r_1)C(r_2)\} - E\{S(r_1)S(r_2)\}) \\ &\quad \cdot (E\{c(r_1)c(r_2)\} - E\{s(r_1)s(r_2)\}) \end{aligned}$$

$$+ (E\{S(r_1)C(r_2)\} + E\{S(r_1)C(r_1)\})$$

$$.(E\{s(r_1)c(r_2)\} + E\{s(r_1)c(r_2)\})] \qquad (3.37)$$

where $s(r) = K\sin \eta r$, $c(r) = K\cos\eta r$, $S(r) = V(r) \sin\psi(r)$, and $C(r) = V(r)\cos\psi(r)$. But, $E\{c(r_1)c(r_2)\} = E\{s(r_1)s(r_2)\}$, and $E\{c(r_1)s(r_2)\} = -E\{s(r_1)\ c(r_2)\}$[56,57], which makes (3.37) equal to zero. It can also be shown that [c.f. Ref. 58]

$$E\{C(r_1)C(r_2)\} = E\{S(r_1)S(r_2)\} \qquad (3.38)$$

and

$$E\{C(r_1)S(r_2)\} = -E\{S(r_1)C(r_2)\} \qquad (3.39)$$

which are assumed to be true also by Kozma *et al.*,[13].

On the basis of the above mentioned assumptions, the expression for $R_{EE}(r_1,r_2)$ may be written as follows [c.f. Eqn. (3.19),(3.20)]:

$$R_{EE}(r_1;r_2) = T^2\{K^4+2K^2\tilde{\mathcal{I}}_U(0) + \tilde{\mathcal{I}}_U^2(0)$$

$$+ |\tilde{\mathcal{I}}_U(\tau_r)|^2 + K^2\exp[j\eta(r_1-r_2)]$$

$$\cdot\ \tilde{R}^*_{VV}(r_1;r_2)+K^2\exp[-j\eta(r_1-r_2)]$$

$$\cdot\ \tilde{R}_{VV}(r_1;r_2)\} \qquad (3.40)$$

where, by Eqn. (3.16), $\tilde{R}_{VV}(r_1;r_2) = \exp[j\gamma(r_1^2-r_2^2)]\tilde{\mathcal{I}}_U(\tau_r)$.

For the time being, let us assume that the recording process is linear, i.e., there is a linear mapping of the exposure into the amplitude transmittance of the processed recording medium:

$$T_a(E) \propto E \qquad (3.41)$$

where $T_a(E)$ is the amplitude transmittance of the recording medium.

In the reconstruction process, we let the plane of the hologram be $Z=0$. A plane wave illuminating this hologram, at the normal angle for example, will be diffracted by the hologram. The autocorrelation of the diffracted field can be obtained by using the input-output relationships of autocorrelation functions outlined in Appendix B. The input is the field immediately behind the hologram which is proportional to E. The autocorrelation function of the input then can be obtained from (3.40):

$$\tilde{R}_{ibH}(r_1;r_2) \propto \{C + |\tilde{\mathcal{I}}_U(\tau_r)|^2 + K^2\exp[j\eta(r_1-r_2)]\tilde{R}^*_{VV}(r_1;r_2) + K^2\exp[-j\eta(r_1-r_2)]\tilde{R}_{VV}(r_1,r_2)\} \quad (3.42)$$

where $R_{ibH}(r_1;r_2)$ is the autocorrelation of the field immediately behind the hologram and $C = K^2 + 2K^2\mathcal{I}_U(0) + \mathcal{I}_U^2(0)$ [c.f. Eqn. (3.40)].

The important terms in (3.42) are the third and the fourth ones. The second term $|\mathcal{I}_U(\tau_r)|^2$ is called the *ambiguity term.* When the reference wave is much stronger than the object wave, the ambiguity term may be neglected when compared with other terms. The contribution of the ambiguity term to the autocorrelation function of the field diffracted by the hologram in the Fraunhofer region will be:

$$R_{aF}(\rho_1,\rho_2) \propto \mathcal{F}_{\rho_2}\{|\tilde{I}_U(\tau_r)|^2\}\delta(\rho_1-\rho_2)\} \quad (3.43)$$

Thus the average irradiance is:

$$E\{|\tilde{I}_U(\tau_r)|^2\} \propto \mathcal{F}_\rho\{|\tilde{I}_U(\tau_r)|^2\} . \quad (3.44)$$

But, by the autocorrelation theorem [31]

$$F_{\rho}\{|\tilde{\mathcal{I}}_U(\tau_r)|^2\} = \int I_U(\rho+r)I_U(r)dr \tag{3.45}$$

which is an autoconvolution of the object irradiance. Therefore, the ambiguity term gives rise to an incoherent field [c.f. Eqn. (2.7)] in the Fraunhofer region with an average irradiance which is an autoconvolution of the object irradiance.

The last terms in (3.42) are responsible for the so-called "real" and the "virtual" images, respectively. These images are located at planes $Z=d$ and $Z=-d$ respectively. However, it should be noted that if the hologram is turned 180° around, the two images will change position. That is, the one which was termed the virtual image now is a real image at $Z=d$, and vice versa. In the following, we will check for the contribution of the third term to the autocorrelation function of the diffracted field at plane $Z=d$. The impulse response is again as in (3.14). The autocorrelation of the input is proportional to $K^2\exp[j\eta(r_1-r_2)]\tilde{R}^*_{VV}(r_1;r_2)$ and the following transformations should be performed:

a) $$\tilde{R}^*_{UV}(r_1;\rho_2) = \int_{-\infty}^{\infty} K^2\exp[-j\eta(r_2-r_1)]\ \tilde{R}^*_{VV}(r_1;r_2)\tilde{h}^*(\rho_2-r_2)dr_2$$

b) $$\tilde{R}^*_{UU}(\rho_1;\rho_2) = \int_{-\infty}^{+\infty}\tilde{R}^*_{UV}(r_1,r_2)\tilde{h}(\rho_1-r_1)dr_1 \ .$$

Substituting for $\tilde{R}^*_{VV}(r_1;r_2)$ in (a) one arrives at:

$$\tilde{R}^*_{UV}(r_1;\rho_2) = K^2\int_{-\infty}^{+\infty}\exp[j\eta(r_1-r_2)]\ \tilde{\mathcal{I}}^*_U[2\gamma(r_1-r_2)]$$

$$\cdot\ \exp[-j\gamma(r_1^2-r_2^2)]\ .\ \exp[-j\gamma(\rho_2-r_2)^2]\ dr_2\ .$$

A change of variable $2\gamma(r_1-r_2) = \tau_r$ gives:

$$\tilde{R}^{*}_{UV}(r_1;\rho_2) = \frac{K^2}{2\gamma}\exp[-j\gamma(r_1^2+\rho_2^2 - 2r_1\rho_2)]$$

$$\cdot \int \exp(j\frac{n}{2\gamma}\tau_r)\,\tilde{r}^{*}_{U}(\tau_r)\exp(j\tau_r\rho_2)d\tau_r$$

$$= \frac{K^2}{2\gamma}\exp[-j\gamma(r_1^2+\rho_2^2 - 2r_1\rho_2)] \cdot I_U(-\rho_2 - \frac{n}{2\gamma}) \,. \qquad (3.46)$$

The tranformation (b) will be:

$$R^{*}_{UU}(\rho_1;\rho_2) = \int_{-\infty}^{\infty} \frac{K^2}{2\gamma}\exp[-j\gamma(r_1^2+\rho_2^2 - 2r_1\rho_2)]\, I_U(-\rho_2 - \frac{n}{2\gamma})\exp[j\gamma(\rho_1-r_1)^2]\, dr_1$$

$$= \frac{K^2}{2\gamma}\exp[-j\gamma(\rho_2^2-\rho_1^2)]I_U(-\rho_2 - \frac{n}{2\gamma}) \int_{-\infty}^{\infty}\exp[j2\gamma r_1(\rho_2-\rho_1)]\, dr_1$$

$$= K^2\exp[-j\gamma(\rho_2^2-\rho_1^2)]\, I_U(-\rho_2 - \frac{n}{2\gamma})\, \delta(\rho_2 - \rho_1)$$

$$= K^2\, I_U(-\rho_2 - \frac{n}{2\gamma})\delta(\rho_1-\rho_2) \,. \qquad (3.47)$$

The result shows that at the plane $Z=d$ the contribution of the term $K^2\exp[\,jn(r_1-r_2)]R^{*}(r_1;r_2)$ is a replica of the object inverted and shifted sidewise by $\frac{n}{2\gamma}$. The same operations as in (a) and (b) performed over the fourth term in (3.42), this time with a negative d in the impulse response will give

$$R_{UU}(\rho_1;\rho_2) = K^2 I_U(\rho_2 - \frac{n}{2\gamma})\; \delta(\rho_1-\rho_2) \qquad (3.48)$$

which is proportional to the object field autocorrelation function shifted sidewise by $\frac{n}{2\gamma}$. The diffracted field is, of course, travelling in the positive Z direction. Therefore, to have an image at the plane $Z=-d$ implies that the image is virtual. That is, it is obtained as a result of a fictitious backward continuation of that part of the diffracted field which corresponds to the fourth term in (3.42).

3.2.3 Other Statistics of the Exposure

Since $\tilde{K}(r)$ and $\tilde{V}(r)$ are independent, the mean of the output is

$$m_{E} = E\{E(r)\} = T[E\{|\tilde{K}(r)|^2\} + E\{|\tilde{V}(r)|^2\}]$$

$$= T[K^2 + \sigma_V^2] = T[K^2 + \tilde{I}_U(0)] \ . \tag{3.49}$$

The mean square value will be

$$E\{E^2(r)\} = E\{|\tilde{K}(r)|^4\} + 4E\{|\tilde{K}(r)|^2\} \ . \ E\{|\tilde{V}(r)|^2\} + E\{|\tilde{V}(r)|^4\}$$

$$= T^2[K^4 + 4K^2\sigma_V^2 + 2\sigma_V^4] = T^2[K^4 + 4K^2 \tilde{I}_U(0) + 2\tilde{I}^2(0)] \ . \tag{3.50}$$

The variance of E now can be obtained as:

$$\sigma_E^2 = E\{E^2(r)\} - (E\{E(r)\})^2 = T^2[2K^2 \tilde{I}_U(0) + \tilde{I}_U^2(0)] \ . \tag{3.51}$$

The probability density function of the exposure can be obtained when we know that the statistics of $\tilde{V}(r)$ are normal. The procedure will be the same as that which is usually followed to find the probability density of the envelope of the sum of a normal noise and a sine wave [59,60, 61] or, as in Ref [62], the sum of a constant and a Rayleigh phasor. In order to follow the former, first we write the sum of the reference wave and the object wave as

$$\mathrm{Re}[\tilde{K}(r,t)] + \mathrm{Re}[\tilde{V}(r,t)] = K\cos(\overline{\omega}t + \eta r) + V(r)\cos(\overline{\omega}t + \psi(r))$$

$$= X_c(r)\cos\omega t - X_s(r)\sin\overline{\omega}t \tag{3.52}$$

where

$$X_c(r) = V(r)\cos\psi(r) + K\cos\eta r = C(r) + c(r) \tag{3.53}$$

and

$$X_s(r) = V(r)\sin\psi(r) + K\sin\eta r = S(r) + s(r). \tag{3.54}$$

If we express the resultant field in terms of an envelope and phase

$$\mathrm{Re}[\tilde{K}(r,t) + \tilde{V}(r,t)] = B(r)\cos(\bar{\omega}t + \theta(r)) \tag{3.55}$$

it follows that

$$X_c(r) = B(r)\cos\theta(r) \tag{3.56}$$

and

$$X_s(r) = B(r)\sin\theta(r) \tag{3.57}$$

and hence that

$$B(r) = [X_c^2(r) + X_s^2(r)]^{1/2} \tag{3.58}$$

and

$$\theta(r) = \tan^{-1}\frac{S(r) + s(r)}{C(r) + c(r)}\ . \tag{3.59}$$

The random variables $C(r)$ and $S(r)$ are independent normal random variables with zero means and variance $\frac{1}{2}\sigma_V^2$, where $\sigma_V^2 = E\{|\tilde{V}(r)|^2\}$ $= \tilde{\mathcal{I}}_U(0)$. The joint probability density function of $X_c(r)$ and $X_s(r)$ is therefore

$$p(X_c(r),X_s(r)) = (\pi\sigma_V^2)^{-1/2}\exp[-(X_c-c)^2/\sigma_V^2]\ .\ (\pi\sigma_V^2)^{-1/2}\exp[-X_s-s)^2\sigma_V^2]$$

$$= (\pi\sigma_V^2)^{-1}\exp -[X_c^2+X_s^2+K^2-2(X_c c+X_s s)/\sigma_V^2]. \tag{3.60}$$

The Jacobian of the transformation from $X_c(r)$ and $X_s(r)$ to $\mathcal{E}(r) = B^2(r)$ and $\theta(r)$ is

$$\frac{\partial(X_c,X_s)}{\partial(E,\theta)} = \begin{vmatrix} \partial X_c/\partial E & \partial X_c/\partial E \\ \partial X_s/\partial E & \partial X_s/\partial E \end{vmatrix} = \begin{vmatrix} \cos\theta/2\sqrt{TE} & -\sqrt{E}\,\sin\theta/\sqrt{T} \\ \sin\theta/2\sqrt{TE} & \sqrt{E}\,\cos\theta/\sqrt{T} \end{vmatrix} \tag{3.61}$$

$$= 1/2T\ .$$

Therefore

$$p(E,\theta) = \begin{cases} (1/2\pi T\sigma_V^2)\exp[\dfrac{E+TK^2-2K\sqrt{TE}\cos(\theta-\eta r)}{T\sigma_V^2} \\ \qquad\qquad E \geq 0 \quad \text{and } 0 \leq \theta \leq 2\pi \\ 0 \qquad\qquad \text{otherwise .} \end{cases} \tag{3.62}$$

We can now determine the probability density function of E by integrating this over θ . Thus, we have

$$p(E) = (1/T\sigma_V^2)\ \exp[-(E+TK^2)/T\sigma_V^2]$$

$$\cdot\ (1/2\pi)\int_0^{2\pi} \exp[2K\sqrt{TE}\ \cos(\theta-\eta r)/T\sigma_V^2]\ d\theta\ . \tag{3.63}$$

The integral can be expressed by means of a Bessel function:

$$(1/2\pi)\int_0^{2\pi} \exp(x\cos\theta)\ d\theta = I_o(x) = J_o(jx) \tag{3.64}$$

where $I_o(x)$ is the modified Bessel function of order zero. Using (3.64) in (3.63), we finally arrive at:

$$p(E) = (1/T\sigma_V^2)\exp[-(E+TK^2)/T\sigma_V^2].\ I_o(2K\sqrt{TE}/T\sigma_V^2) \tag{3.65}$$

$$\text{for } (E \geq 0)\ .$$

Knowing the probability density of the envelope of the sum of a normal process and a sine wave, i.e.,

$$P(B) = (2B/\sigma_V^2)\ \exp[-(B^2+K^2)/\sigma_V^2]\ I_o(2KB/\sigma_V^2) \qquad (B \geq 0) \qquad (3.66)$$

(3.65) could have been obtained by the tranformation

$$E = TB^2 \qquad (3.67)$$

(3.66) is called the "Rice-Nakagami" distribution [63]. An expression for the joint probability density of $P(B)$ is given by Middleton [58] as:

$$P_B(B_1,B_2) = \begin{cases} \dfrac{4B_1B_2\ \exp\{[\sigma_V^2/(\sigma_V^2 - |R_{VV}|^2)](B_1^2+B_2^2)\}}{\sigma_V^4 - |R_{VV}|^2} \\ \cdot\ \exp[-2K^2(\sigma_V^2 - |R_{VV}|\ \cos\phi_o)/(\sigma_V^4 - |R_{VV}|^2)] \\ \cdot \sum_\ell \varepsilon_\ell I_\ell\left(\dfrac{2|R_{VV}|}{\sigma_V^4 - |R_{VV}|^2}\ B_1B_2\right) I_\ell\left(\dfrac{2\ \Omega_1\sigma_V^2}{\sigma_V^4 - |R_{VV}|^2}\ B_1\right) \\ \cdot\ I_\ell\left(\dfrac{2\Omega_2\sigma_V^2}{\sigma_V^4 - |R_{VV}|^2}\ B_2\right)\ \cos\ell\phi_o \qquad \text{for} \quad 0 \leq B_1,B_2 < \infty \\ 0 \qquad\qquad \text{otherwise} \end{cases} \qquad (3.68)$$

where I_ℓ is the modified Bessel function, $\phi_o=\tan^{-1}\left(\dfrac{\mathrm{Im}[R_{VV}]}{\mathrm{Re}[R_{VV}]}\right) + \eta(\mathbf{r}_1-\mathbf{r}_2)$ (3.69);

$$\Omega_i = \frac{K}{\sigma_V^2}\sqrt{\sigma_V^4 + |R_{VV}|^2 - 2\sigma_V^2|R_{VV}|\cos\phi_i},\ i=1,2,;\ \phi_1=\phi_o,\ \phi_2=\tan^{-1}\left(\frac{\mathrm{Im}[R_{VV}]}{\mathrm{Re}[R_{VV}]}\right) - \eta(r_1-r_2) \qquad (3.70)$$

ε_ℓ is the Neumann factor $\begin{array}{l}\varepsilon_o = 1 \\ \varepsilon_\ell = 2 \quad \ell > 0.\end{array}$

The relationship between the exposure E and the envelope B is

$$E = TB^2 .$$

Therefore, the joint probability density of E_1 and E_2 will be [64]:

$$P_E(E_1,E_2) = (1/T^2) \cdot (1/4\sqrt{E_1E_2}/T)$$
$$\cdot [P_B(\sqrt{E_1/T}, \sqrt{E_2/T}) + P_B(-\sqrt{E_1/T}, \sqrt{E_2/T})$$
$$+ P_B(\sqrt{E_1/T}, -\sqrt{E_2/T}) + P_B(-\sqrt{E_1/T}, -\sqrt{E_2/T})] . \quad (3.71)$$

The only non-zero term is the first term, thus the joint probability density of $P(E)$ will be:

$$p(E_1,E_2) = \begin{cases} \dfrac{\exp\{[-\sigma_V^2/T(\sigma_V^4 - |R_{VV}|^2)](E_1+E_2)\}}{T^2(\sigma_V^4 - |R_{VV}|^2)} \\ \cdot \exp[-2K^2(\sigma_V^2 - |R_{VV}| \cos\phi_o)/(\sigma_V^4 - |R_{VV}|^2)] \\ \cdot \sum_\ell \varepsilon_\ell I_\ell\left(\dfrac{2|R_{VV}|}{T(\sigma_V^2 - |R_{VV}|^2)} \sqrt{E_1E_2}\right) \\ \cdot I_\ell\left(\dfrac{2\Omega_1\sigma_V^2}{\sqrt{T}(\sigma_V^2 - |R_{VV}|^2)} \sqrt{E_1}\right) \\ \cdot I_\ell\left(\dfrac{2\Omega_2\sigma_V^2}{\sqrt{T}(\sigma_V^2 - |R_{VV}|^2)} \sqrt{E_2}\right) \cos \ell\phi_o \\ \qquad\qquad \text{for } 0 \leq E_1, E_2 < \infty \\ 0 \qquad\qquad \text{otherwise} . \end{cases} \quad (3.72)$$

3.2.4 Statistics of the Exposure when $K^2 >> E\{V^2(r)\}$

An asymptotic series of the modified Bessel function, valid for large values of argument is [59,65]

$I_o(x) = (2\pi x)^{-1/2} \exp(x) \cdot (1 + 1/8x + 9/128x^2 + \ldots)$. It therefore follows that when the reference wave is much stronger than the object wave, i.e., $K^2 >> \sigma_V^2$ (or equivalently $K\sqrt{E} >> \sigma_V^2$), Eqn. (3.65) may be approximated by

$$p(E) \cong (1/T\sigma_V)(\sqrt{T}/4\pi K\sqrt{E})^{1/2}\exp[-\sqrt{E} - \sqrt{T}K)^2/T\sigma_V^2]. \tag{3.73}$$

On the basis of the above assumption, $\sqrt{E} - \sqrt{T}K$ can be well approximated by

$$\sqrt{E} - \sqrt{T}\,K \cong \sqrt{T}\,V(r)\cos(\psi(r) + \eta r). \tag{3.74}$$

But, on the other hand

$$E - TK^2 \cong 2KTV(r)\cos(\psi(r) + \eta r) \quad ; \tag{3.75}$$

hence,

$$(\sqrt{E} - \sqrt{T}K)^2 \cong (E - TK^2)^2/2(2K^2T^2\sigma_V^2). \tag{3.76}$$

Substituting for $(\sqrt{E} - \sqrt{T}K)^2$ from (3.76) in (3.73) and letting $\sigma_E = \sqrt{2}\,KT\sigma_V$ and $m_E = TK^2$ [c.f. Eqn.'s (3.49) and (3.51)] gives:

$$p(E) \cong (1/\sqrt{2\pi}\,\sigma_E)\exp[-(E - m_E)^2/2\,\sigma_E^2]. \tag{3.77}$$

That is, when the reference wave is much stronger than the object wave, the probability density function of the expsoure is approximately normal. This is, of course, what is to be expected. When in Eqn. (3.33) $V^2(r)$ in the presence of K^2 can be neglected, E will be approximately

$$E \cong T[K^2 + 2KV(r)\cos(\psi(r) + \eta r)]. \tag{3.78}$$

$V(r)$ is Rayleigh distributed and $\psi(r)$ is uniformly distributed between 0 and 2π, and hence they could be thought of as the envelope and phase respectively, of a normal process [66] which could be, for instance:

$$V(r)\cos(\eta r + \psi(r))$$

with mean zero and variance $1/2\ \sigma_V^2$. Therefore, E, given by (3.78), will be a normal process with mean TK^2 and variance $2T^2K^2\sigma_V^2$ [67].

3.2.5 Optical Diffusion in Recording Media and Its Frequency Filtering Effects

In holography we are to record the interference pattern of a reference and an object wave. The very fine microscopic structure of the fringes (usually of the order of magnitude of the wavelength of the field used for exposure), places a high demand on the resolution capability of the recording medium. This becomes more critical particularly when diffuse illumination and/or off axis geometry is used. The maximum fringe frequency which must be recorded is determined by the maximum angle which light rays from the object make with those from the reference. For example, when the angle is 80° and a red laser light is used (helium neon laser) the frequency is close to 1600 lines per millimeter. The diffuser scatters light over a wide angular range which results in a substantial increase in the maximum angle of the object and reference waves. Therefore, a very high resolving power is an essential requirement of the recording medium if it is to be useful for the purpose of holography. Too low a resolving power can result in loss of image resolution and limitation of the image field [68].

During the process of recording a hologram the optical transmission properties of the recording medium must change in proportion to the local variations of the radiation pattern. If the new optical properties are changed in correspondence to the local variation of the diffraction pattern of the object and reference, modulation of the illuminating beam in the reconstruction process will correspond exactly to the original pattern. Therefore, after meeting the hologram, the illuminating beam will carry some additional information which resembles that carried by the object and reference waves. The changes in the optical transmission properties of the medium are either the variations of the transmittance (amplitude modulating or absorption type) or the variations of the thickness or the index of refraction (phase modulating) [69]. However, these local variations, naturally, will be possible if the incident energy has enough power to effectively enforce them upon the medium. That is, the local changes will occur in the regions where there is sufficient energy to surmount the potential energy barrier of the medium. Therefore, the level of the potential energy barrier can be a measure of the sensitivity of the recording medium. A lower level barrier means a lower minimum detectable signal power and, therefore, a higher sensitivity. Different materials have different sensitivities, and for a particular material, the sensitivity is different for different wavelengths. So, a high sensitivity to the wavelength of the exposing radiation is another requirement of the recording medium to allow shortest possible exposure time to avoid possible destruction of the fringes due to imperfect stability of the set up and the surrounding medium.

Holograms have been recorded in a variety of materials [69], e.g. silver halide emulsions and photochromic materials (for absorption and bleached phase holograms), dichromated gelatin, electro-optical crystals and magneto-optic materials (for thick phase holograms), photopolymers and photoresist materials (for thin phase holograms). Among these, despite some deficiencies, silver halide photographic emulsion has been used most commonly, because of its high sensitivity. The sensitivity of the other materials is typically much lower than that of the silver halide emulsion.

The photographic recording medium usually consists of a large number of small grains of photosensitive silver halide, which by the aid of a layer of gelatin have been spread almost uniformly over a base of transparent acetate film or glass plate. Usually, some sensitizing agent is also added to the gelatin.

The granular nature of the emulsion layer causes a loss of the modulation associated with the exposure. The reason for this is an optical diffusion [5,70] which occurs in the layer because of multiple scattering of light by the photosensitive grains. The distribution of the exposure is modified by this optical diffusion, bringing about a distinction between the original exposure and an *effective exposure* [5] which produces the latent image. The loss of modulation is spatial frequency dependent and is described by a function termed the "Modulation Transfer Function", or in short, MTF.

The original exposure and the effective exposure may be regarded as, respectively, the input and output of a linear filter [5] with the MTF as its "transfer function". Since there is no preferred direction of scattering in the recording medium, the filter can be considered "isotropic" [5]. The well-known Fourier transform relationship of the impulse response and the transfer function may be used in order to obtain one from the other if one of the two is already determined. A convolution of the original exposure with the impulse response of the system will give the effective exposure.

The theoretical explanations of the problems involved in the phenomenon of optical diffusion can be found in the advanced theories of scattering and multiple scattering of suspended particles [71,72,73,74,75,76,77]. It is shown, practically and theoretically, that the finer the grain size, the less the optical diffusion effects will be. Since the loss of modulation is frequency dependent and increases with the spatial frequency, a higher resolving power will be obtained with a finer grain size. Therefore, a proper recording medium for holography should have a very fine grain structure, which usually can be obtained at the expense of the speed or sensitivity of the recording medium. However, since in holography a high speed is also desirable to allow short exposures, a compromise should be made for the highest possible efficiency.

Kodak Spectroscopic Plate, Type 649-F is probably the type of reording medium which was used most commonly in original investigations of laser holography, though it was not originally made for this purpose. The published resolving power for this type is at least 2000 lines/mm. [78] with an almost flat spectral response. Kodak Eastman Co. has manufactured some other recording media particularly for the purpose of holography, [79] including Kodak Holographic plate, Type 120-02 and Kodak Holographic film, (Estar Base) SO-173, which have high sensitivity in the region 600 to 750 nm, High Resolution plates and films are also good media for holography with a high sensitivity in the region up to 560 nm.

Agfa-Gevaert has also introduced a number of emulsions to meet various practical demands [80], namely 8E75 and 10E75 for red laser light, and 10E56 for blue and green laser light. The published resolving power for these types are about 3000 lines/mm. A micro-fine grain material is Pan 300 [81] which has an MTF almost completely independent of spatial frequency up to 4000 lines/mm.

A satisfactory analytical expression for the impulse response of weakly scattering photographic materials was first introduced by Frieser [81a]. A slightly changed version of Freiser's expression in a more tractable form is [81]:

$$h(r) = T\delta(r) + (1 - T)\,\frac{2 \cdot 3}{\kappa}\,\exp\left(-4.6|r|/\kappa\right). \qquad (3.79)$$

The first term describes the unscattered absorbed light and the second term the light diffusion caused by scattering. κ is Frieser's κ-value and is used as an index of the light-diffusion properties of the emulsion. T can be defined as the ratio of the unscattered absorption to total absorption and is higher for emulsions with finer grain size.

Frieser's formula is satisfactory for weakly scattering photographic materials. For the fine-grain materials made specially for recording holograms, the second term in (3.79) is actually negligible due to the very fine grain size and high absorptive emulsions. That is, the impulse response of good holographic materials may be well approximated by a delta function, i.e., almost a perfect filter. Therefore, the statistics of the effective exposure will be approximately the same as those of the original exposure.

CHAPTER FOUR

NONLINEARITY AND ADJACENCY EFFECTS

4.1 Methods of Analysis

The problem which will concern us in this chapter is that of determining the autocorrelation function of the transmittance of the processed hologram.

As was noted before, in the process of recording a hologram, the exposure should result in a change of optical transmission properties of the recording medium so that in the reconstruction process the hologram can generate a replica of the original wave field by modulating the reconstructing wave. To have an exact replica a linear mapping of the exposure into a transmittance is needed. But, in practice, such linear transformation does not exist. This is obvious, for instance from the T_a-E curves.

The nonlinear effects on image reconstruction has been studied by several authors [e.g. c.f. Ref's [8] - [13]]. The basic analytic procedure has been the *characteristic function method* (or the *transform method*)[14,15] of obtaining the correlation function of the output of a nonlinear device. Here we use this method and the *direct method* [Ref. 14, Chapt. 12, p. 250-278] wherever possible. The difference between the two methods is that in the direct one the transfer function of the nonlinearity and the joint probability density of the input are directly used in the following formula [Ref. 14, p. 250-252]:

$$R_{N-L}(r_1,r_2) = \iint_{-\infty}^{\infty} T(E_1)T(E_2)\ p(E_1,E_2)\ dE_1dE_2 \tag{4.1}$$

where $E_1 = E(r_1)$ and $E_2 = E(r_2)$ and $T(E)$ is the transfer function of the nonlinearity. Now, let $L(S)$ be the Laplace transform (or if applicable as in most practical cases, the Fourier transform) of $T(E)$. Then by definition:

$$T(E) = \frac{1}{2\pi j} \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(S)\, e^{SE}\, dS. \tag{4.2}$$

Substitution of this in Eq. (4.1) gives

$$R_{N-L}(r_1,r_2) = \frac{1}{(2\pi j)^2} \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(S_1)\, dS_1 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(S_2)\, dS_2$$

$$\cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(E_1,E_2) \cdot \exp(S_1E_1 + S_2E_2)\, dE_1E_2 \,. \tag{4.3}$$

The double integral with respect to E_1 and E_2 is the joint characteristic function of E_1 and E_2 expressed as a function of the complex variables S_1 and S_2 [14]. Hence

$$R_{N-L}(\vec{r}_1,\vec{r}_2) = \frac{1}{(2\pi j)^2} \int_C L(S_1)\, dS_1 \int_C L(S_2) dS_2\, \tilde{M}(S_1,S_2)\,. \tag{4.4}$$

Eqn. (4.4) represents the characteristic function method of analysis of nonlinear devices in response to random inputs.

In the following sections, we shall find the autocorrelation function of the amplitude transmittance of the hologram for: a) the special case in which the reference wave is much stronger than the object's,b) the general case of arbitrary irradiances of reference and object waves. Some examples of solutions to the problem, assuming some approximately fitting functions for the nonlinear T-E characteristics are given.

4.1.1 Autocorrelation Function of Transmittance when $K^2 >> \sigma_V^2$

In section 3.2 it was concluded that when the reference wave is much stronger than the object wave the exposure may be considered to have approximately normal statistics with mean $m_E = TK^2$ and variance $\sigma_E^2 = 2T^2 K^2\sigma_V^2$. The general normal joint probability density function of the two real random variables y_1 and y_2 with mean m_1 and m_2, variances σ_1 and σ_2 and correlation coefficients ρ_y is [Ref. 14, p. 149]:

$$p(y_1,y_2) = [2\pi\sigma_1\sigma_2(1-\rho_y^2)]^{-1/2}$$

$$\exp[-\frac{\sigma_2^2(y_1-m_1)^2-2\sigma_1\sigma_2\rho_y(y_1-m_1)(y_2-m_2)+\sigma_1^2(y_2-m_2)^2}{2\sigma_1^2\sigma_2^2(1-\rho_y^2)}] . \tag{4.5}$$

Thus, in our case, for the joint probability density of E_1 and E_2 we can write

$$p(E_1,E_2) = [4\pi\sigma^2(1-\rho_E^2)]^{-1/2}$$

$$\exp[-\frac{(E_1-m_E)^2-2\rho_E(E_1-m_E)(E_2-m_E)+(E_2-m_E)^2}{2\sigma_E^2(1-\rho_E^2)} \tag{4.6}$$

where $\rho_E = \dfrac{R_{EE}(r_1,r_2)-m_E^2}{\sigma_E^2}$, and $R_{EE}(r_1,r_2)$ is given by Eqn. (3.40).

Thus, for $R_{N-L}(r_1,r_2)$ we have

$$R_{N-L}(r_1,r_2) = (2\pi\sigma_E^2)^{-1} \iint_{-\infty}^{+\infty} T(E_1)T(E_2)(1-\rho_E^2)^{-1/2}$$

$$\exp[-\frac{(E_1-m_E)^2-2\rho_E(E_1-m_E)(E_2-m_E)+(E_2-m_E)^2}{2\sigma_E^2(1-\rho_E^2)}] dE_1dE_2 . \tag{4.7}$$

Letting $(E_1-m_E)/\sigma_E=y_1$ and $(E_2-m_E)/\sigma_E=y_2$

as new variables,

$$R_{N-L}(\vec{r}_1,\vec{r}_2) = (2\pi)^{-1} \iint_{-\infty}^{+\infty} T(\sigma_E y_1+m_E)T(\sigma_E y_2+m_E)(1-\rho_E^2)^{-1/2}$$

$$\exp[- \frac{y_1^2-2\rho_E y_1 y_2+y_2^2}{2(1-\rho^2)}]\, dy_1 dy_2 \tag{4.8}$$

An application of Mehler's formula [82], namely

$$(1-\rho^2)^{-1/2}\exp[- \frac{(x_1^2+x_2^2)-2\rho x_1 x_2}{2(1-\rho^2)}] = \sum_{n=0}^{\infty} \frac{H_n(x_1)H_n(x_2)}{n!} \rho^n \tag{4.9}$$

gives

$$R_{N-L}(r_1,r_2) = (2\pi)^{-1} \sum_{n=0}^{\infty} \frac{\rho_E^n}{n!} \int_{-\infty}^{+\infty} T(\sigma_E y_1+m_E)\exp-\frac{y_1^2}{2} H_n(y_1)dy_1$$

$$\times \int_{-\infty}^{+\infty} T(\sigma_E y_2+m_E)\exp - \frac{y_2^2}{2} H_n(y_2)dy_2 \tag{4.10}$$

or in short

$$R_{N-L}(r_1,r_2) = \sum_{n=0}^{\infty} C_n\rho_E^n , \tag{4.11}$$

where

$$C_n = (2\pi .\ n!)^{-1} [\int_{-\infty}^{+\infty} T(\sigma_E x+m_E)\exp- \frac{x^2}{2} H_n(x)\, dx]^2. \tag{4.12}$$

It can be seen that the autocorrelation function of the transmittance can be expressed in a power series of the autocorrelation of the exposure. The term corresponding to $n=1$ of the above expression gives a faithful replica of the object, as was mentioned in Section 3.2.2.

4.1.2 Some Examples

Example 1 - A Gaussian Transfer Function.

If we approximate the T_a-E curve of the photographic emulsion by a normal function

$$T_a(E) = \exp[-\frac{E^2}{2\Pi^2}] \tag{4.13}$$

C_n's will be

$$C_n = (2\pi n!)^{-1}\{\int_{-\infty}^{+\infty} \exp[-\frac{(\sigma_E x+m_E)^2}{2\Pi^2} - \frac{x^2}{2}]H_n(x)\, dx\}^2$$

$$= (2\pi n!)^{-1}\{\int_{-\infty}^{+\infty} \exp[-\frac{(\sigma_E^2+\Pi^2)x^2+2m_E\sigma_E x+m_E^2}{2\Pi^2}]\, H_n(x)\, dx\}^2$$

$$= (2\pi n!)^{-1}\,\{\exp[\frac{m_E^2}{2\Pi^2}(\frac{\sigma_E^2}{\sigma_E^2+\Pi^2}-1)]\int_{-\infty}^{+\infty}\exp-[\sqrt{\frac{\sigma_E^2+\Pi^2}{2\Pi^2}}\, x$$

$$-\frac{m_E\sigma_E}{-\sqrt{2\Pi^2(\sigma_E^2+\Pi^2)}}]^2 \cdot H_n(x)\, dx\}^2$$

$$= 2\Pi^2[\pi\ n!\ 2(\sigma_E^2+\Pi^2)] \quad \exp(-\frac{m_E^2}{\sigma_E^2+\Pi^2}) \cdot$$

$$\cdot \{\int_{-\infty}^{+\infty}\exp-(Z-\beta)^2 H_n(\alpha Z)dZ\}^2 \tag{4.14}$$

where we have let

$$\sqrt{\frac{\sigma_E^2+\Pi^2}{2\Pi^2}}\, x = Z, \quad \sqrt{\frac{2\Pi^2}{\sigma_E^2+\Pi^2}} = \alpha \quad \text{and} \quad -\frac{m_E\sigma_E}{\sqrt{2\Pi^2(\sigma_E^2+\Pi^2)}} = \beta \ .$$

Now, by Eq. 8, p. 837 of Ref. [83] we may write

$$C_n = 2\Pi^2[\pi n!\ 2(\sigma_E^2+\Pi^2)]^{-1}\exp[\frac{-m_E^2}{\sigma_E^2+\Pi^2}]\ .\{\pi^{1/2}(1-\alpha^2)^{n/2}.H_n[\frac{\alpha\beta}{(1-\alpha^2)^{1/2}}]\}^2$$

$$= \frac{4}{n!}(\sigma_E^2+\Pi^2)^{-(n+1)}(\sigma_E^2-\eta^2)^n\exp[\frac{-m_E^2}{\sigma_E^2+\Pi^2}][H_n(\frac{m_E}{\Pi^2-\sigma_E^2})]^2 . \tag{4.15}$$

The larger the n, the smaller the coefficient C . Thus, the irradiance of the images will become negligibly small from some high order terms on.

Example 2 - A Linear Phase Hologram

For a linear phase hologram the transmittance transfer function may be shown by an exponential [84,85]

$$\tilde{T}(E) = \exp(-\tilde{\Lambda}E) \qquad (4.16)$$

where $\tilde{\Lambda}$ is a complex number with a positive real part, i.e.,

$$\tilde{\Lambda} = p + jq \quad \text{with } p > 0 \, . \qquad (4.17)$$

For this example the coefficient C_n's are

$$\tilde{C}_n = (2\pi \quad n!)^{-1} \left| \int_{-\infty}^{+\infty} \exp[-\tilde{\Lambda}(\sigma_E x + m_E) \exp - \frac{X^2}{2} H_n(x)dx \right|^2 . \qquad (4.18)$$

The integral can be evaluated using Eqn. 1,7.376, p. 838 of Ref. [83]. Hence,

$$\tilde{C}_n = -(n!)^{-1} \exp[-2pm_E + \sigma_E^2(p^2 - q^2)] \quad H_n(\sigma_E \tilde{\Lambda}) \; H_n(\sigma_E \tilde{\Lambda}^*) \, . \qquad (4.19)$$

Example 3 - $T_a(E) = (1+2.1E)\exp(-2E)$

Another example we employ is the function

$$T_a(E) = (1+2.1E)\exp(-2E) \qquad (4.20)$$

given by Velzel [85]. (4.20) is actually a crude approximation given by Velzel as an example of a more general model for the T_a-E curves. This model, which is a sum of Lagurre polynomials in the form [Eqn. (2.6) of Ref. [85]]:

$$T_a(E) = \exp(-\mu E) \sum_n c_n L_n(\mu E) \qquad (4.21)$$

is proposed to replace the commonly used polynomial approximations [10, 12, 13, 86], since these lead to physically unrealizable results of a transmittance value greater than unity [Ref. 85, p. 585]. The function (4.20) is obtained by letting the values of the coefficients c_n be:

$c_o = 1.05$, $c_1 = -1.05$; $c_n = 0$ for $n > 1$; with $\mu = 2$.

Fig. 1 shows the function $T_a(E)$ given by (4.20) as well as the T_a-E characteristic curve measured experimentally for Kodak 649F spectroscopic plates given in Ref. [12].

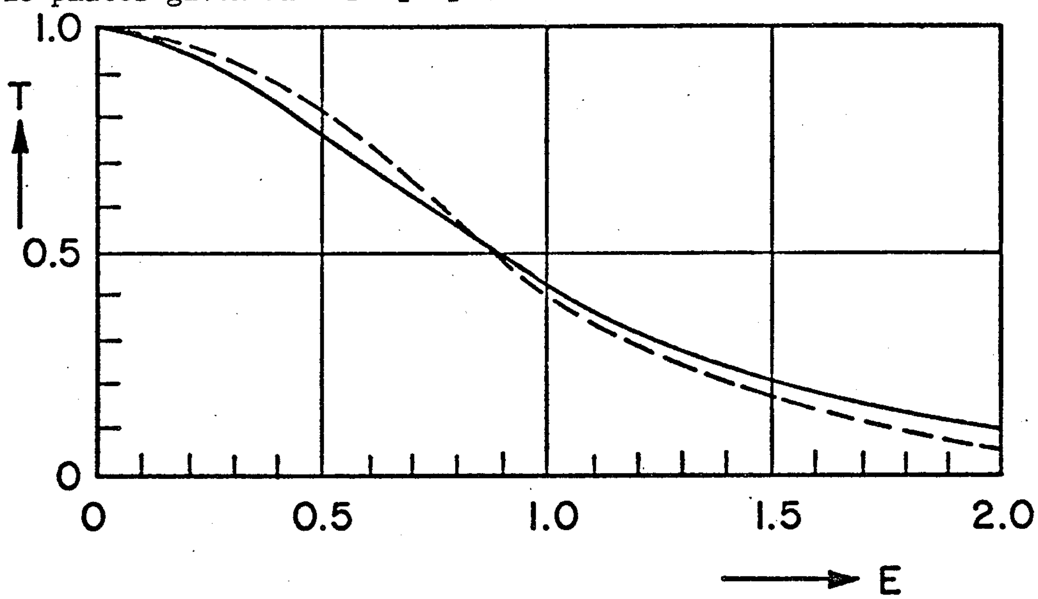

Fig. 1 [85] The broken curve respresents the measured characteristic, while the continuous curve gives a model of the characteristic according to the Eqn. (4.20).

For this example the C_n s will be

$$C_n = (2\pi n!)^{-1} \{ \int_{-\infty}^{+\infty} [1 + 2.1(\sigma_E x + m_E)] \exp[-2(\sigma_E x + m_E)] \exp -\frac{x^2}{2} H_n(x) dx \}^2$$

$$= (2\pi n!)^{-1} \exp(-4m_E) \{ \int_{-\infty}^{+\infty} [(1+2.1m_E) + 2.1\sigma_E x] \exp(-2\sigma_E x) \exp- \frac{x^2}{2} H_n(x) dx \}^2. \quad (4.22)$$

Changing the variable x to $\sqrt{2}\, y$ and considering the fact that

$H_o(\sqrt{2}y)=1$ and $2\sqrt{2}y=H_1(\sqrt{2}y)$ give

$$C_n = (\pi n!)\ \exp(-4m_E)\{\int_{-\infty}^{+\infty}[(1+2.1m_E)H_o(\sqrt{2}y)+\frac{2.1}{2}\sigma_E H_1(\sqrt{2}y)]$$

$$\cdot \exp[-(y+\sqrt{2}\sigma_E)^2]H_n(\sqrt{2}y)\ dy\}^2. \tag{4.23}$$

The above integral may be evaluated by Eqn. 9,7.374,p. 837 of Ref. [83] and we obtain

$$C_o = \exp[4(\sigma_E^2-m_E)]\ [1-2.1(\sigma_E^2-m_E)]^2 \tag{4.24}$$

$$C_n = (n!)\ \exp[4(\sigma_E^2-m_E)]\ \{(1+2.1m_E)H_n(2\sigma_E)-\frac{2.1\sigma_E}{2}[H_{n+1}(2\sigma_E)$$

$$+ 2nH_{n-1}(2\sigma_E)]\}^2 \qquad \text{for } n > 0 \tag{4.25}$$

Using the recurrence relation [Ref. 87, p. 252]

$$H_{n+1}(x) + 2nH_{n-1}(x) = 2xH_n(x) \qquad \text{for } n > 0 \tag{4.26}$$

one finally arrives at

$$C_n = (n!)^{-1}\exp[4(\sigma_E^2-m_E)][1 - 2.1(\sigma_E^2-m_E)]^2\ H_n^2(2\sigma_E) \tag{4.27}$$

$$n = 0,1,2,3,\ldots$$

Note:

In the above evaluation of the autocorrelation function we used an approximation by letting the lower limit of the integral in Eqn. (4.7) be $-\infty$ instead of 0. The error involved is $\int_{-\infty}^{-m_E/\sigma_E} T(\sigma_E y-m_E)\ e^{-y^2/2} \cdot H_n(y)\ dy$. Since $\frac{m_E}{\sigma_E} = \frac{K}{\sqrt{2}\sigma_V} >> 1$, for $y < -\frac{m_E}{\sigma_E}$ we are well down on the tail of $e^{-y^2/2}$, i.e., close to zero and the contribution of the integral in the region $-\infty$ to $-\frac{m_E}{\sigma_E}$ is actually negligible.

4.2 Autocorrelation Function of Transmittance for General Cases of Arbitrary Irradiances of Reference and Object Waves

In cases where the reference wave is no longer much stronger than the object wave and therefore the statistics of the exposure can no longer be considered normal, the mathematics involved is generally very complicated. Nevertheless, it can be shown that the autocorrelation of the output of the nonlinearity can always be expressed by a power series of the Fourier transform of the object irradiance. This will be shown in the following sections first by applying the direct method to an example and then in general by applying the characteristic function method. Several examples are also worked out in the latter case.

4.2.1 Direct Method - An Example

As an example we will evaluate the integral (4.1) for a linear phase hologram with the transmittance transfer function as in (4.16). Substituting for $\tilde{T}(E)$ from (4.16) and $p(E_1,E_2)$ from (3.72) in (4.1) gives

$$R_{N-L}(r_1,r_2) = \frac{N}{T^2\sigma_V^4}\iint_0^\infty \exp(-\tilde{\Lambda}E_1-\tilde{\Lambda}^*E_2)\sum_\ell \varepsilon_\ell \cos\ell\phi_o \exp\left[\frac{-(E_1+E_2)}{T\sigma_V^2(1-|\rho_{VV}|^2)}\right]$$
$$\cdot\frac{1}{(1-|\rho_{VV}|^2)} I_\ell\left(2\frac{|\rho_{VV}|}{(1-|\rho_{VV}|^2)}\sqrt{\frac{E_1}{T\sigma_V^2}\cdot\frac{E_2}{T\sigma_V^2}}\right)$$
$$\cdot I_\ell\left(\frac{2\Omega_1}{\sigma_V(1-|\rho_{VV}|^2)}\sqrt{\frac{E_1}{T\sigma_V^2}}\right)\cdot I_\ell\left(\frac{2\Omega_2}{\sigma_V(1-|\rho_{VV}|^2)}\sqrt{\frac{E_2}{T\sigma_V^2}}\right)dE_1dE_2, \tag{4.28}$$

where $N = \exp\left[\frac{-2K^2(1-|\rho_{VV}|\cos\phi_o)}{\sigma_V^2(1-|\rho_{VV}|^2)}\right]$, and $\tilde{\rho}_{VV} = \frac{\tilde{R}_{VV}}{\sigma_V^2}$ is the correlation coefficient or the normalized autocorrelation function with the property that $|\rho_{VV}| < 1$[e.g., c.f. Ref. 14, p. 59]. The above integral may be evaluated as follows. First, one may use the identity [c.f. Ref. 82a, p. 190 or Ref. 83, Eqn. 1, 8.976, p. 1038 or Ref. 87, p. 242]

$$\exp\{-\frac{1}{2}(x+y)\frac{1+t}{1-t}\} \cdot I_\ell\left(\frac{2\sqrt{(xyt)}}{1-t}\right) = \sum_{n=0}^{\infty} t^n \frac{n!(xy)^{\ell/2}\exp[-(x+y)/2]}{\Gamma(n+\ell+1)} \cdot$$

$$\cdot\ L_n^\ell(x)\ L_n^\ell(y) \qquad \text{for } |t|<1 \tag{4.29}$$

to separate the variables E_1 and E_2 in integral (4.28) and obtain:

$$\tilde{R}_{N-L}(r_1,r_2) = N\sum_\ell\sum_n \frac{n!\ \varepsilon_\ell\cos\ell\phi_o}{\Gamma(n+\ell+1)}(|\rho_{VV}|)^{2n+\ell}\prod_{i=1}^{2}\int_0^\infty (X)^{\ell/2}$$

$$\cdot\ \exp[-(1+T\sigma_V^2\tilde{\Lambda}_i)X] \cdot I_\ell\left[\frac{2\Omega_i}{\sigma_V(1-|\rho_{VV}|^2)}\sqrt{X}\right] L_n^\ell(X)\,dX \quad .$$

$$\tilde{\Lambda}_1 = \tilde{\Lambda}, \quad \tilde{\Lambda}_2 = \tilde{\Lambda}^* \tag{4.30}$$

This integral, after the change of variable $X = Y^2$, can be evaluated using Eqn. 4, 7.421, p. 847 of Ref. [83]. The final result is:

$$\tilde{R}_{N-L}(r_1,r_2) = N\sum_\ell\sum_n \frac{n!\ \varepsilon_\ell\cos\ell\phi_o}{\Gamma(n+\ell+1)} \cdot (\sigma_V^2)^{2n-\ell}(|\rho_{VV}|)^{2n+\ell}|(1+T\sigma_V^2\tilde{\Lambda})|^{-2(\ell+n+1)}$$

$$\cdot\,|(T\tilde{\Lambda})|^{2n}\left[\frac{\sqrt{\Omega_1\Omega_2}}{(1-|\rho_{VV}|^2)}\right]^{2\ell}\exp\left[\frac{1}{\sigma_V^2(1-|\rho_{VV}|^2)^2}\right.\cdot$$

$$\cdot\left.\left(\frac{\Omega_1^{\,2}}{(1+T\sigma_V^2\tilde{\Lambda})}+\frac{\Omega_2^{\,2}}{(1+T\sigma_V^2\tilde{\Lambda}^*)}\right)\right] \cdot L_n^\ell\left[\frac{-\Omega_1^{\,2}}{T\sigma_V^4(1-|\rho_{VV}|^2)^2(1+T\sigma_V^2\tilde{\Lambda})\tilde{\Lambda}}\right]$$

$$\cdot\ L_n^\ell\left[\frac{\Omega_2^{\,2}}{T\sigma_V^4(1-|\rho_{VV}|^2)^2(1+T\sigma_V^2\tilde{\Lambda}^*)\tilde{\Lambda}^*}\right] \tag{4.31}$$

As can be seen, the autocorrelation function of the recorded exposure can be expressed as a power series of the Fourier transform of the object irradiance multiplied by some phase and amplitude distorting factors. Ignoring the distorting factors, it is evident that the first order term, i.e. the term corresponding to $n=0$ and $\ell=1$ will be proportional to

$$2|\rho_{VV}|\cos\phi_o = \tilde{\rho}_{VV} e^{-j\eta(r_1,r_2)} + \tilde{\rho}^*_{VV}\, e^{j\eta(r_1-r_2)} \qquad (4.32)$$

or in terms of $\tilde{R}_{VV}(r_1,r_2)$

$$\tilde{R}_{VV} e^{-j\eta(r_1-r_2)} + \tilde{R}^*_{VV}\, e^{+j\eta(r_1-r_2)}\,. \qquad (4.33)$$

The first order images in this case are amplitude and phase distorted by the factor $N[\frac{\sqrt{\Omega_1\ \Omega_2}}{(1-|\rho_{VV}|^2)}]^2 \exp[-\frac{1}{\sigma_V^2(1-|\rho_{VV}|^2)^2}(\frac{\Omega_1^2}{1+T\sigma_V^2\tilde{\Lambda}} + \frac{\Omega_2^2}{1+T\sigma_V^2\tilde{\Lambda}^*})]$. Since $|\rho_{VV}| < 1$, the terms with third or higher powers are relatively unimportant. Nevertheless, these terms which are the result of higher order correlations and convolutions lead to higher order and highly distorted images of which the irradiance distributions become a background noise to the already distorted first order image.

4.2.2 Characteristic Function Method: A General Treatment

In the characteristic function method, first we need to know the joint characteristic function of the joint propability density $P(E_1,E_2)$. This can be obtained by using the formula [Ref. 14, p. 53]

$$M(S_1,S_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(S_1)^p(S_2)^q}{p!\ q!} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^p y^q p(x,y)\ dxdy. \quad (4.34)$$

The double integral in this equation is the joint moment $E\{x^p y^q\}$; hence for the joint characteristic function of the exposure

$$M(S_1,S_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} E\{E_1^p E_2^q\} \frac{(S_1)^p(S_2)^q}{p!\ q!}\,. \qquad (4.35)$$

So, the first problem reduces to evaluation of joint moments $E\{E_1^p E_2^q\}$.

The joint probability density of E_1 and E_2 was obtained in Section 3.2.3 and is given by Eqn. (3.72). The joint moments $E\{E_1^p E_2^q\}$ then may be evaluated as follows:

$$E\{E_1^p E_2^q\} = \iint_{-\infty}^{\infty} E_1^p E_2^q p(E_1,E_2)\, dE_1\, dE_2$$

$$= Q \iint_0^{\infty} E_1^p E_2^q \exp[-G(E_1+E_2)] \cdot \sum_\ell \varepsilon_\ell I_\ell\left(\frac{2|R_{VV}|}{\sigma_V^2} G\sqrt{E_1 E_2}\right)$$

$$\cdot\, I_\ell(D_1 G\sqrt{E_1}) \cdot I_\ell(D_2 G\sqrt{E_2}) \cdot \cos\ell\phi_o \quad dE_1\, dE_2 \qquad (4.36)$$

where $G = \dfrac{\sigma_V^2}{T(\sigma_V^4 - |R_{VV}|^2)}$, $D_i = 2\Omega_i\sqrt{T}$, $i=1,2$; and $Q = \dfrac{G}{T\sigma_V^2} \exp\left[\dfrac{-2K^2(\sigma_V^2 - |R_{VV}|\cos\phi_o)}{\sigma_V^4 - |R_{VV}|^2}\right]$.

The above integral can be evaluated with the help of the transformations

$$E_1 = \frac{1}{G} Z^{1/2} e^{\zeta/2} \text{ and } E_2 = \frac{1}{G} Z^{1/2} e^{-\zeta/2} \quad \text{[c.f. Ref. 15, p. 404]}$$

where the domain of E_1 and E_2, i.e., $0 \le E_1, E_2 < \infty$ goes over into $(0 \le Z < \infty)$, $(-\infty \le \zeta < \infty)$ for the new variables (Z and ζ). The magnitude of the Jacobian is found to be $1/(2G^2)$. Thus, for Eqn. (4.36), we arrive at

$$E\{E_1^p E_2^q\} = \frac{Q}{2G^{p+q+2}} \int_0^{\infty}\int_{-\infty}^{\infty} Z^{\frac{p+q}{2}} \exp[-Z^{1/2}(e^{\zeta/2}+e^{-\zeta/2}) + (p-q)\zeta/2]$$

$$\cdot \sum_\ell \varepsilon_\ell I_\ell\left(\frac{2|R_{VV}|}{\sigma_V^2} Z^{1/2}\right) \cdot I_\ell(aZ^{1/4}) I_\ell(bZ^{1/4})$$

$$\cdot \cos\ell\phi_o\, dZ\, d\zeta \qquad (4.37)$$

where $a = D_1 G^{1/2}\exp(\zeta/4)$ and $b = D_2 G^{1/2}\exp(-\zeta/4)$.

Substituting $e^{\ell\pi j/2} J_\ell(jX)$ for $I_\ell(X)$, and the applications of the result for the product of two Bessel functions [c.f. Ref. [88], p. 148, Eqn. (2)] gives

$$I_\ell(aZ^{1/4})\, I_\ell(bZ^{1/4}) = (-1)^\ell J_\ell(jaZ^{1/4})\, J_\ell(jbZ^{1/4})$$

$$= \frac{(D_1D_2G/4)^\ell\, Z^{\ell/2}}{\Gamma(\ell+1)} \cdot$$

$$\cdot \sum_{m=0}^{\infty} \frac{(D_1^2G/4)^m\, e^{m\zeta/2}\, Z^{m/2}\, {}_2F_1(-m,-\ell-m;\ell+1;\, \frac{D_2^2}{D_1^2}\, e^{-\zeta})}{m!\,\Gamma(\ell+m+1)} \tag{4.38}$$

where ${}_kF_i(\alpha_1,\alpha_2,\ldots\alpha_k;\ \beta_1,\beta_2,\ldots,\beta_i;\ X)$ is the hypergeometric series given by [e.g. Ref. 88, p. 100]:

$${}_kF_i(\alpha_1,\alpha_2,\ldots,\alpha_k;\beta_1,\beta_2,\ldots,\beta_i;X) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_k)_n}{n!(\beta_1)_n(\beta_2)_n\ldots(\beta_i)_n}\, X^n$$

and

$$(\alpha)_n = (\alpha+1)(\alpha+2)\ldots(\alpha+n-1),\ (\alpha)_o = 1.$$

After the substitution of the expansion of ${}_2F_1(-m,-\ell-m;\ell+1;\ \frac{D_2^2}{D_1^2}\, e^{-\zeta})$ in (4.38), the integral (4.37) will be

$$E\{E_1^pE_2^q\} = \frac{Q}{G^{p+q+2}} \sum_\ell \sum_m \sum_n \varepsilon_\ell\, \frac{(-m)_n[-(m+\ell)]_n}{n!\ (\ell+1)_n}\ \frac{(D^2G)^{m+2\ell}}{m!\Gamma(\ell+m+1)}\ \frac{\cos\ell\phi_o}{2^{2(m+\ell)}}$$

$$\cdot \int_0^\infty dZ\ I_\ell(\frac{2|R_{VV}|}{\sigma_V^2}\, Z^{1/2})\ Z^{(p+q+\ell+m)/2}$$

$$\cdot \int_{-\infty}^{+\infty} \exp\,[-2Z^{1/2}\cosh\zeta/2 - (2n-m-p+q)\zeta/2]\ d\zeta \tag{4.39}$$

but

$$K_k(X) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-X\cosh t\ -\ kt}\ dt \tag{4.40}$$

[c.f. Ref. 88, p. 182, Eqn. (7)],

where K_k is the modified Bessel function of the second kind and order k. Thus, substituting for the ζ integral in (4.39) reduces this double integral to a single integral which, after we make the change of variable $Z^{1/2} = Y$, will be in the form:

$$4 \int_0^\infty I_\ell\left(\frac{2|R_{VV}|}{\sigma_V^2} Y\right) K_{2n-m-p+q}(2Y)\, Y^{\ell+m+p+q+1}\, dY \tag{4.41}$$

which can be evaluated by Watson's [88] equation (1), p. 410. The final result is

$$E\{E_1^p E_2^q\} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{Q\, D_2^{\ell+2n}\, D_1^{2(m-n)}}{G^{p+q+2-(m+\ell)}\, 2^{2(m+\ell)}} \cdot \frac{(\ell+q+n)!(\ell+p+m-n)!}{p!q!n!(m-n)!(\ell+m-n)!(\ell+n)!}$$

$$\cdot\ {}_2F_1\left(\ell+q+n+1,\ \ell+p+m-n+1; \ell+1; \frac{|R_{VV}|^2}{\sigma_V^4}\right) \left(\frac{|R_{VV}|}{\sigma_V^2}\right)^\ell \cos\ell\phi_Q\ . \tag{4.42}$$

Thus

$$R_{N-L}(r_1,r_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} E\{E_1^p E_2^q\}(1/j2\pi)^2 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(s_1)(s_1)^p\, ds_1$$

$$\cdot \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(s_2)\ (s_2)^q\, ds_2 = \sum_p \sum_q E\{E_1^p\ E_1^q\}\ h_{pq}. \tag{4.43}$$

As can be seen [c.f. Eqn. (4.42)], except for some amplitude distorting factors, the autocorrelation function may be expressed in a power series of the Fourier transform of the object irradiance. The term

$$|R_{VV}| \cos\phi_o = \tilde{R}_{VV}\, e^{-jn(r_1-r_2)} + \tilde{R}^*_{VV}\, e^{jn(r_1-r_2)},$$

corresponds to the first order image and is obtained by letting $\ell=1$ while considering the zero order term in ${}_2F_1$.

The coefficients $h_{pq} = (1/2\pi j)^2 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_1\ L(S_1)S_1^p \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_2$. $L(S_2)\ S_2^q$ may be evaluated once the Laplace transform of the $T(E)$ curve is known,

$$L(S) = \int_0^{\infty} T(E)\ e^{-jSE}\ dE \quad . \tag{4.44}$$

However, instead of a Laplace transform we may employ a Fourier transform of $T(E)$. $T(E)$ is defined for $E \geq 0$, but for the sake of mathematical simplicity we may assume that the function is symmetric; that is, it has the same form in $-\infty$ to 0 as from 0 to ∞. This fictitious assumption does not affect in any way the results, since $p(E_1,E_2)$ is zero in the region $E_1,E_2 < 0$. However, in the following examples we will employ both Fourier and one sided Laplace transforms of $T(E)$ to get the coefficient h'_{pq}s.

Example 1. Gaussian $T_a(E)$

a) Using Fourier Transform of a Symmetric $T_a(E)$

Let us again approximate the T_a-E curve with a Gaussian

$$T_a(E) = \exp(-\frac{E^2}{2\Pi^2}) \cdot$$

Since this function has a Fourier transform which is also Gaussian, h'_{pq}s are as follows [e.g. c.f. Ref. 83, Eqn. 4, 3.462, p. 338]:

$$h_{pq} = \frac{\Pi^2}{(2\pi)} \int_{-\infty}^{+\infty} (jv_1)^p \exp(-\frac{\Pi^2 v_1^2}{2})\ dv_1 \int_{-\infty}^{+\infty} (jv_2)^q \exp(-\frac{\Pi^2 v_2^2}{2})\ dv_2$$

$$= (\sqrt{2}\Pi)^{-p-q} H_p(0)H_q(0) = \begin{cases} (-1)^{\frac{p+q}{2}} \dfrac{p!q!}{(\frac{p}{2})!(\frac{q}{2})!} (\sqrt{2}\Pi)^{-p-q} & \text{for p and q both even} \\ 0 & \text{otherwise} \end{cases}$$

(4.45)

b) Using One Sided Laplace Transform of $T_a(E)$

The Laplace transform of $\exp\left(-\frac{E^2}{2\Pi^2}\right)$ is [e.g. Ref. 89, Eqn. 2, 3.1, p. 20]:

$$1/2\ (2\pi\Pi^2)^{1/2}\ \exp(\Pi^2 S^2/2)\ \mathrm{Erfc}(\Pi S/\sqrt{2}) \tag{4.46}$$

where Erfc(x) is the complementary error function [e.g., c.f. Ref. 87, Sec. 9.2.3, p. 349].

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{4.47}$$

Coefficients h_{pq}'s are

$$h_{pq} = (1/2\pi j)^2 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_1\ \frac{1}{2}(2\pi\Pi^2)^{1/2} \exp(\Pi^2 S_1^2/2)\ \mathrm{Erfc}(\Pi S_1/\sqrt{2})\ S_1^p$$

$$\cdot \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_2\ \frac{1}{2}(2\pi\Pi^2)^{1/2} \exp(\Pi^2 S_2^2/2)\ \mathrm{Erfc}(\Pi S_2/\sqrt{2})\ S_2^q\ . \tag{4.48}$$

The above integral may be evaluated using Eqn. (1,16.3.2, p. 316)and Eqn. (12, p. 170) of Ref. [89]. We have

$$h_{pq} = (\sqrt{2}\Pi)^{-p-q} \frac{d^p}{dt^p}(e^{-t^2}) \cdot \frac{d^q}{dt^q}(e^{-t^2})\Big|_{t=0} = (\sqrt{2}\Pi)^{-p-q} H_p(0) H_q(0)$$

$$= \begin{cases} (-1)^{\frac{p+q}{2}} \cdot \dfrac{p!q!}{(\frac{p}{2})!(\frac{q}{2})!} (\sqrt{2}\Pi)^{-p-q} & \text{for p and q both even} \\ 0 & \text{otherwise} \end{cases} \tag{4.49}$$

which is the same as when using a Fourier transform of the $T(E)$ function defined for the whole region $-\infty$ to ∞ rather than 0 to ∞ [c.f. Eqn. 4.45].

Example 2: Linear Phase Hologram

For a linear phase hologram:

a) $T(E) = \exp(-\Lambda E)$

The Fourier transform of an exponential is a delta function. The coefficients h_{pq}'s are given by

$$\tilde{h}_{pq} = \int_{-\infty}^{+\infty} \delta(v_1 - j\tilde{\Lambda})(jv_1)^p \, dv_1 \int_{-\infty}^{+\infty} \delta(v_2 - j\tilde{\Lambda})(jv_2)^p \, dv_2$$

$$= (-1)^{p+q} \tilde{\Lambda}^p \cdot \tilde{\Lambda}^{*q} . \tag{4.50}$$

b) The one sided Laplace transform of $T(E)$ is [e.g. c.f. Ref. 89, Eqn. 3.1, p. 20]

$$L(S) = \frac{1}{S+\tilde{\Lambda}} .$$

The coefficients h_{pq}'s are

$$\tilde{h}_{pq} = \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \frac{S_1^p}{S_1+\tilde{\Lambda}} dS_1 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \frac{S_2^q}{S_2+\tilde{\Lambda}^*} dS_2 .$$

The above integral may be evaluated using Eqn. (76, 1, p. 189) and Eqn. (12, p. 170) of Ref. [89]:

$$\tilde{h}_{pq} = \frac{d^p}{dt^p} (e^{-\Lambda t}) \cdot \frac{d^q}{dt^q} (e^{-\Lambda^* t}) \Bigg|_{t=0} = (-1)^{p+q} \tilde{\Lambda}^p . \tilde{\Lambda}^{*q} . \tag{4.51}$$

Example 3:

Another example of practical use could be the function [11]

$$T_a(E) = \beta^2/(E^2+\beta^2) . \tag{4.52}$$

where β is a parameter used to fit $T(E)$ to the experimentally derived data [Fig. 2].

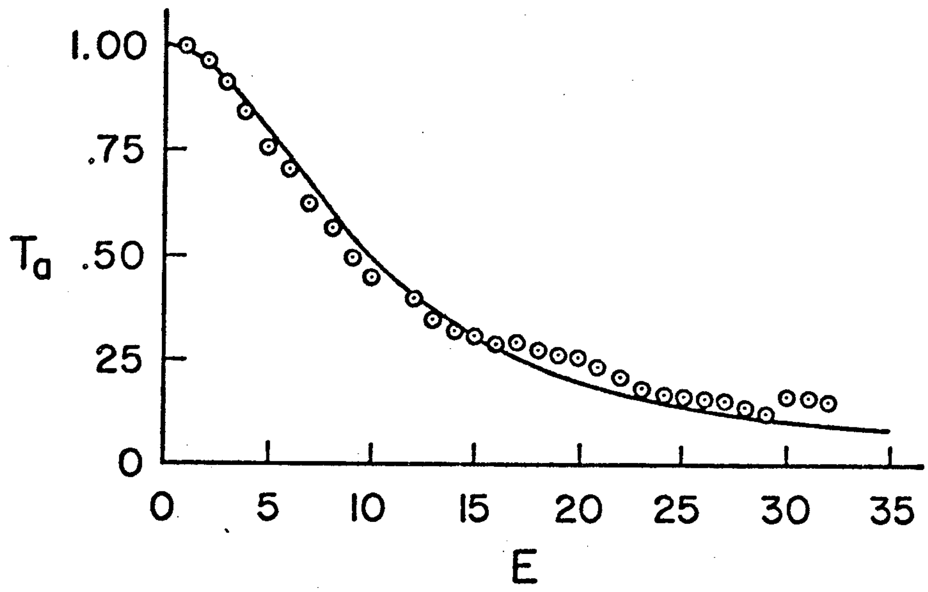

Fig. 2. [11] Amplitude transmittance vs. exposure for Kodak 649 F plate. ⊙ experimental results, ——— calculated T_a vs. E curve.

a) The Fourier transform of $T_a(E)$ in this case is $\pi\beta e^{-|v|\beta}$ [e.g.Ref. 90, Eqn. 5, 3.2, p. 118]. Thus, h_{pq}'s will be:

$$h_{pq} = (1/2)^2\ \beta^2 \int_{-\infty}^{+\infty} e^{-|v_1|\beta} (jv_1)^p\, dv_1 \int_{-\infty}^{\infty} e^{-|v_2|\beta} (jv_2)^q\, dv_2\ , \quad (4.53)$$

which, with the aid of Eqn. (3,3.341, p. 310) of Ref. [83] may be evaluated as:

$$h_{pq} = \begin{cases} (-1)^{\frac{p+q}{2}}\, p!q!\beta^{-p-q} & \text{for p and q both even} \\ 0 & \text{otherwise .} \end{cases} \quad (4.54)$$

b) The Laplace transform of $T_a(E)$, in this case, is [e.g. c.f. Ref. 88, Eqn. 8, 1, p. 12]:

$$-\ \beta[\mathrm{ci}(\beta S)\sin(\beta S) + \mathrm{si}(\beta S)\cos(\beta S)]\ ,$$

where ci(x) is the cosine integral defined as $\int_x^\infty \frac{\cos u}{u}\, du$, and si(x) is the sine integral defined as $\int_x^\infty \frac{\sin u}{u}\, du$.

Thus, the coefficients h_{pq}'s are given by

$$h_{pq}=(1/2\pi j)^2 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \beta[ci(\beta S_1)\sin(\beta S_1)+si(\beta S_1)\cos(\beta S_1)]\ (S_1)^p\ dS_1$$

$$\cdot \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \beta[ci(\beta S_2)\sin(\beta S_2)+si(\beta S_2)\cos(\beta S_2)]\ (S_2)^q\ dS_2 \ . \qquad (4.55)$$

The above integral may be evaluated using Eqn. (1, 18.5, p. 321) and Eqn. (12,p. 170) of Ref. 89. We arrive at

$$h_{pq} = \beta^2 \cdot \frac{d^p}{dt^p}\left(-\frac{\beta}{t^2+\beta^2}\right) \cdot \frac{d^q}{dt^q}\left(-\frac{\beta}{t^2+\beta^2}\right)\Bigg|_{t=0} \qquad (4.56)$$

and hence

$$h_{pq} = \begin{cases} (-1)^{\frac{p+q}{2}}\ p!q!\ \beta^{-p-q} & \text{for } p \text{ and } q \text{ both even} \\ 0 & \text{otherwise .} \end{cases}$$

Example 4:

As the last example let us consider the function

$T_a(E) = (1+2.1E)\exp(-2E)$ [c.f. Section 4.1.2, Ex. 3].

a) The Fourier transform of $T_a(E)$ will be:

$$2\pi\ [\delta(v-2j) + j\ 2.1\ \delta'(v-2j)]$$

where $\delta'(x)$ is the first derivitive of $\delta(x)$, $\frac{d}{dx}\delta(x)$. Thus h_{pq}'s are:

$$h_{pq} = \int_{-\infty}^{+\infty} [\delta(v_1-2j) + j\delta'(v_1-2j)](jv_1)^p\ dv_1$$

$$\cdot \int_{-\infty}^{\infty} [\delta(v_2-2j) + j\delta'(v_2-2j)](jv_2)^q\ dv_2 \quad . \qquad (4.57)$$

The following sifting property of $\delta'(x)$ [c.f., Ref. 31,p. 82]

$$\delta' * f = \int_{-\infty}^{\infty} \delta'(x-x')f(x')dx' = f'(x) \qquad (4.58)$$

may be used to evaluate the h_{pq}'s given by Eqn. (4.57)

$$h_{pq} = (-1)^{p+q} \, 2^{p+q} \, (1-1.05p)(1-1.05q) \, . \qquad (4.59)$$

b) The Laplace transform of $(1+2.1E) \exp(-2E)$ [Ref. 89, Eqn. 1,3.1 p. 20]

$$\frac{1}{S+2} + \frac{2.1}{(S+2)^2}$$

The h_{pq}'s, then, are

$$h_{pq} = \left(\frac{1}{2\pi j}\right)^2 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \left[\frac{1}{S_1+2} + \frac{2.1}{(S_1+2)^2}\right] S_1^p \, dS_1$$

$$\cdot \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} \left[\frac{1}{S_2+2} + \frac{2.1}{(S_2+2)^2}\right] S_2^q \, dS_2 \, ,$$

which with the aid of Eqn. (76F, 1, p. 189), Eqn. (106G, 1, p. 193) and Eqn. (12, p. 170) of Ref.[89] can be evaluated as:

$$h_{pq} = \left[\frac{d^p}{dt^p}\left(e^{-2t}+2.1\, te^{-2t}\right)\right].\left[\frac{d^q}{dt^q}\left(e^{-2t}+2.1\, te^{-2t}\right)\right]\Bigg|_{t=0}$$

$$= (-1)^{p+q} \, 2^{p+q} \, (1-1.05p)(1-1.05q) \, . \qquad (4.60)$$

4.3 Adjacency Effects

During exposure those silver halide grains which have been exposed to some radiant energy and have absorbed a suffucient amount of it become *on the average developable;* i.e., the probability of development for them increases to 1/2 or better*. (What is meant by development is actually the reduction of the silver halide to opaque metallic silver during the developing process). This definition of developability, which is rather an arbitrary probability criterion, is actually a conclusion of a realistic consideration that "not all grains which are on average *developable* will develop, and not all grains which will develop are *developable*" Ref. 70, p. 90].

The developer potential, the height of the potential barrier at the site of development and the development time are important factors in the determination of developability [Ref. 70, p. 88]. For example, even the silver halide of an unexposed emulsion will be eventually reduced to silver metallic if in contact with a developer solution for a long enough time. The effects of the above mentioned factors are also evidenced by the different development effects of different developing chemicals or developers diluted to varying degrees [91]. Besides, in a particular developer solution, the development effects are *space-dependent*, since the chemical change in the solution which is due to the chemical reaction of the developer with the exposed emulsion, is not uniform, though isotropic.

This chemical change, which increases with the amount of silver developed, is not uniform for two reasons. First, the strongly exposed areas react with their surrounding solution much more than the weakly exposed areas, resulting in an inhomogeneous solution, i.e., more exhausted around strongly exposed areas and fresher around weakly exposed ones. This results in a tone-reversed [5] exposure mapping. The second reason for the

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* For a complete and detailed explanation of the mechanism of development and the definition of developability the interested reader is referred to Ref.[70] Chapter 5 "The Mechanism of the Formation of the Latent Image", by Hamilton, J.F. and Urbach, F., p. 87-119.

non-uniformity is *chemical diffusion,* which is due to an exchange of fresh developer and reaction products between the borders of weakly exposed and strongly exposed areas. In other words, reaction products which usually tend to inhibit development diffuse from the strongly exposed areas into the borders of weakly exposed areas and vice versa. This exchange is responsible for the effects of "overshoot" and "undershoot" at the border of two adjacent areas with different exposures. Actually, the inhibition of development by reaction products concentrated along the border of little exposed areas results in a retardation of the development process in these regions, which in turn causes these parts to be underdeveloped (undershoot). On the other hand, on the borders of strongly exposed areas, too much fresh developer causes these portions to be overdeveloped (overshoot).

The above mentioned development effects are also known as *adjacency effects*, since the formation of the changes in the optical properties of the medium at a certain point is determined not only by the exposure at that point, but also by the exposure at adjacent regions*. Following Kelly's [5] suggestion, one may regard the output of the nonlinearity of the recording medium, which is a scale conversion of the effective exposure, as representing the number of grains which would be developed in the absence of nonuniform chemical changes. If we regard this as the input of a linear filter, the output represents the proportion of these grains actually developed.

* We have chosen the photographic films as the recording media; however, similar effects are observed with other recording materials [92], e.g., with photopolymers due to monomer diffusion [93].

The development effects would have been almost eliminated if somehow the concentration of the chemical species could be kept fixed and uniform during the development. In that case, each exposed grain would be equally likely to be developed and therefore, the spectral response of the filter would be flat up to spatial frequencies of the order of the resolving power of the medium [5]. For very thin emulsions and/or concentrated developers, this is often an adequate approximation. The nonuniformity of the developer may be compensated for to a large extent by agitating the hologram during the development process. We will be content with this last approximation and assume that the emulsion is very thin, the developer concentrated and the hologram is being agitated through the development.

CHAPTER FIVE

DEPOLARIZATION EFFECTS

So far we have considered the interacting wave amplitudes as scalar quantities. In other words, the incident wave has been considered to be linearly polarized and the question of whether the polarization of the incident wave is changed in the process of scattering has been ignored, i.e., it has been supposed that the scattered field is also linearly polarized and parallel to the reference wave. Under these assumptions we were able to write the vector form of the time average irradiance of the interference of two coherent waves, i.e.,

$$I_{tot}(r) = \vec{K}.\vec{K} + \vec{V}(r) \, . \, \vec{V}(r) + 2\vec{K}.\vec{V}(r) \cos(\phi_r - \phi_o) \qquad (5.1)$$

in the scalar form of Eq. (3.33). But, in general, we are not allowed to make this assumption and the exact form of (5.1) must be used, since the scattered field in general is not linearly polarized and/or parallel to the incident field. Then, if we let the source beam be linearly polarized, the scattered complex field vector $\underset{\rightarrow}{\tilde{V}}(r)$ may be decomposed into two orthogonal components, $\tilde{V}_{\parallel}(r)\vec{e}_{\parallel}$ and $\tilde{V}_{\perp}(r)\vec{e}_{\perp}$, polarized parallel and polarized orthogonal to the field $\underset{\rightarrow}{\tilde{K}}(r) = \tilde{K}_{\parallel}(r)\,\vec{e}_{\parallel}$, where $\vec{e}_{\parallel}$ and $\vec{e}_{\perp}$ are unit vectors in $\parallel$ and $\perp$ directions, respectively. Then, Eqn. (5.1) may be written in the following form:

$$I_{tot}(r) = [K^2 + V_{\parallel}^2(r) + 2KV_{\parallel}(r) \cos(\phi_r - \phi_o)] + V_{\perp}^2(r) \, . \qquad (5.2)$$

The total exposure then is

$$E_{tot}(r) = T[K^2 + V_{\parallel}^2(r) + 2KV_{\parallel}(r)\cos(\phi_r-\phi_o) + V_{\perp}^2(r)]$$

$$= E_{\parallel}(r) + E_{\perp}(r) \quad . \qquad (5.3)$$

where we choose to show $TV_{\perp}^2(r) = E_{\perp}(r)$ and the rest by $E_{\parallel}(r)$. As can be seen, the background noise power $T[V_{\parallel}^2(r) + V_{\perp}^2(r)] = T|\underset{\rightarrow}{V}(r)|^2$ remains the same while the signal $2TV_{\parallel}(r)K_{\parallel}\cos(\phi_r-\phi_o)$ is reduced in power with respect to the case that the scattered field was not depolarized, since $V(r) \leq |\underset{\rightarrow}{V}(r)| = \sqrt{V_{\parallel}^2(r) + V_{\perp}^2(r)}$. Furthermore, as we know, [Ref. 94a, p. 192], "in the backscattering of linearly polarized waves at an extended object the scattered field in parallel polarization is determined mainly by the segments of the surface of the object having a small curvature, whereas the cross-polarization component of the scattered field is caused by the scattering of the waves at segments of the surface with large curvature (edges, discontinuities)"[94]. Therefore, the phase information of the latter ones is lost in the process of holography with a linearly polarized reference wave, since two light waves which are polarized in mutually perpendicular directions contribute only the sum of their individual irradiances [95] and can not produce an interference pattern which actually bears the phase information. So, the only useful informative part is in $E_{\parallel}(r)$, i.e., $V(r)\cos(\phi_r-\phi_o)$, which carries the phase information of the segments of the surface having small curvatures. To get a better image we have to look for a way of recording the phase information of the other segments, too. A method which may be useful is using a reference wave which also could have two components in $\parallel$ and $\perp$ directions, respectively. Therefore, the part of the source beam which is going to be used as the reference wave can be passed through a polarizing plate, e.g., a quarter-wave plate to give a circularly polarized wave. But, there also remains the question of signal power increase. For example, if the cross-polarized component is not of a significant magnitude we lose a part of the reference wave's power and instead of $\approx 2KV(r)\cos(\phi_r-\phi_o)$ we will have $\approx 2K_{\parallel}V_{\parallel}(r)\cos(\phi_r-\phi_o)$. Therefore, we have to make sure whether we have a cross-polarized field with significant power and then use part of the reference power in recording the phase information delivered by that part. So the best method might be to measure the power of each component of the scattered field and then divide the power of the reference wave accordingly to have the best fringe visibility (the best condition for fringe visibility is the equal power of two interacting waves [c.f. e.g., Ref. 96]). In the following section, however, we shall give the mathematical procedure to find the autocorrelation function of the output of the recording medium nonlinearity in the actual and more general situation when the scalar form of (3.33) is no longer considered and Eqn. (5.1) is considered instead.

Varshavchuk and Kobak [94] show that when the elements of the statistical scattering matrix are normally distributed (which applies to our case) and the incident wave is linearly (or circularly) polarized, the scattered field components $\tilde{V}_{\parallel}(r)$ and $\tilde{V}_{\perp}(r)$ are uncorrelated, and since they are normally distributed they are statistically independent. The autocorrelation of the output of the nonlinearity is, then, given by [Ref. 14, p. 289]:

$$R_{N-L}(\vec{r}_1,\vec{r}_2) = \frac{1}{(2\pi j)^2}\int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(S_1)dS_1 \int_{\varepsilon-j\infty}^{\varepsilon+j\infty} L(S_2)dS_2 M_E^{\parallel}(S_1,S_2)M_E^{\perp}(S_1,S_2), \tag{5.4}$$

where $M_E^{\parallel}(S_1,S_2)$ was obtained and is given by Eqn.'s(4.35 & 42)of Section 4.2.2. So, it remains to obtain an expression for $M_E^{\perp}$, which we shall do in the following.

On the basis of the assumption about the statistics of a diffuse object [c.f. Chapter 3] $\tilde{V}_{\parallel}(r)$ and $\tilde{V}_{\perp}(r)$ have normal statistics [c.f. Section 3.1.3] with means zero and variances $\sigma_V^{\parallel 2}$ and $\sigma_V^{\perp 2}$. Thus, $V_{\parallel}(r) = |\tilde{V}_{\parallel}(r)|$ and $V_{\perp}(r) = |\tilde{V}_{\perp}(r)|$ are Rayleigh distributed [Ref. 14, p. 160, or Ref. 15, p. 366, or Ref. 19, p. 128]. Therefore, the probability density function of $E_{\perp}(r) = TV_{\perp}^2(r)$ is exponential [e.g., Ref. 14, p. 254]

$$p(E_{\perp}) = \begin{cases} (2\sigma_{\perp}^2)^{-1} \exp(-E_{\perp}/2\sigma_{\perp}^2) & E_{\perp} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

where $2\sigma_{\perp}^2 = T\sigma_V^{\perp 2}$. The joint probability density function of $V_{\perp}(r_1)$ and $V_{\perp}(r_2)$ is shown [97] to be

$$P(V_{\perp 1},V_{\perp 2}) = \frac{4V_{\perp 1}V_{\perp 2}}{\sigma_V^{\perp 4}(1-|R_{VV}^{\perp}|^2)} \; I_0\left(\frac{|R_{VV}^{\perp}|}{\sigma_V^{\perp 4}-|R_{VV}^{\perp}|^2} V_{\perp 1} V_{\perp 2}\right)$$
$$\cdot \exp\left[-\frac{\sigma_V^{\perp 2}}{\sigma_V^{\perp 2}-|R_{VV}^{\perp}|^2} (V_{\perp 1}^2+V_{\perp 2}^2)\right] \quad \text{for } V_{\perp 1}, V_{\perp 2} \geq 0 \,. \tag{5.6}$$

With transformations $E_{\perp 1} = TV_{\perp 1}^2$ and $E_{\perp 2} = TV_{\perp 2}^2$, and with the aid of Eqn. (3.71) one arrives at

$$P(E_{\perp 1},E_{\perp 2}) = \frac{1}{T^2(\sigma_V^{\perp 4}-|R_{VV}^{\perp}|^2)} \; I_o\left[\frac{|R_{VV}^{\perp}|}{T(\sigma_V^{\perp 4}-|R_{VV}^{\perp}|^2)} \sqrt{E_{\perp 1}E_{\perp 2}}\right]$$
$$\cdot \exp\left[-\frac{\sigma_V^{\perp 2}}{T(\sigma_V^{\perp 4}|R_{VV}^{\perp}|^2)} (E_{\perp 1}+E_{\perp 2})\right] . \tag{5.7}$$

Now the identity [c.f. Eqn. (4.29)]

$$(1-t)^{-1}\exp\{-(x+y)\frac{t}{1-t}\} \quad I_o\{\frac{2\sqrt{xyt}}{1-t}\} = \sum_{n=0}^{\infty} t^n L_n(x)\, L_n(y) \qquad (5.8)$$

can be used to give

$$P(E_{\perp 1},E_{\perp 2}) = \frac{1}{4\sigma_{\perp}^4}\exp[-\frac{E_{\perp 1}+E_{\perp 2}}{2\sigma_{\perp}^2}]\sum_{k=0}^{\infty}(\frac{|R_{VV}^{\perp}|^2}{\sigma_V^{\perp 4}})^k L_k(\frac{E_{\perp 1}}{2\sigma_{\perp}^2}) \cdot L_k(\frac{E_{\perp 2}}{2\sigma_{\perp}^2})$$

$$= P(E_{\perp 1})P(E_{\perp 2})\sum_{k=0}^{\infty}(\frac{|R_{VV}^{\perp}|^2}{\sigma_V^{\perp 4}})^k L_k(\frac{E_{\perp 1}}{2\sigma_{\perp}^2})\, L_k(\frac{E_{\perp 2}}{2\sigma_{\perp}^2}) \qquad (5.9)$$

$M_E^{\perp}(S_1,S_2)$ is the Laplace transform of $P(E_{\perp 1},E_{\perp 2})$, and it may be obtained with the aid of Eqn. (5, 9.3.2, p. 55) of Ref. [89]:

$$M_E^{\perp}(S_1,S_2) = \sum_{k=0}^{\infty}(T|R_{VV}^{\perp}|)^{2k}\frac{S_1^k S_2^k}{(2\sigma_{\perp}^2 S_1+1)^{k+1}(2\sigma_{\perp}^2 S_2+1)^{k+1}} \qquad (5.10)$$

Substituting for $M_E^{\perp}(S_1,S_2)$ in (5.4) gives:

$$R_{N-L}(r_1,r_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} E\{E_{\parallel 1}^p\, E_{\parallel 2}^q\} \cdot h_{pqk}\,, \qquad (5.11) \qquad (5.11)$$

where h_{pqk} is given by

$$h_{pqk} = \frac{1}{(2\pi j)^2}\sum(T|R_{VV}^{\perp}|)^{2k}\int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_1\, L(S_1)\frac{S_1^{p+k}}{(2\sigma_{\perp}^2 S_1+1)^{k+1}}$$

$$\cdot\int_{\varepsilon-j\infty}^{\varepsilon+j\infty} dS_2\, L(S_2)\frac{S_2^{q+k}}{(2\sigma_{\perp}^2 S_2+1)^{k+1}} \qquad (5.12)$$

As can be seen, the coefficient h_{pqk} is a function of $R_{VV}^{\perp}(\tau_r)$ and therefore, unlike h_{pq}, it is not a constant. Thus, h_{pqk} is another distoring factor which fortifies the distortion due to the nonlinearity.

CHAPTER SIX

SUMMARY

The applications of holography and optical data processing have increased vastly in many different fields, and particularly in electrical engineering. As was implicitly mentioned in CHAPTER ONE, most of the recent advances in modern optics have become realizable by the rapid progress and advancements in different branches of electrical engineering, especially communication sciences. A reason for this is the ingenious recognition of the possibility of similar treatment of the problems in these two once separate disciplines of optics and electrical engineering. This was briefly discussed in CHAPTER ONE, which also contained some examples of the similarities in the two fields.

CHAPTER TWO explains in detail the necessary assumptions involved in considering a diffuse object field analogous to a non-stationary "white noise" (of normal statistics).

The first part of CHAPTER THREE was devoted to an explanation of scalar diffraction theory from a systems point of view. The analogy between the concept of spatial information carried by wavefronts and the more familiar term of temporal information carried by time signals was explained in terms of the similarity between a quadratic phase filter and Fresnel diffraction. The concepts of "black box", "operational notation", or "systems" approach, which are familiar to the electrical engineer, can be used in optics for the study of elements operating on spatial information. This was explained in the consideration of the scalar diffraction theory, and later in the discussion of optical diffusion by the grains of the recording medium and the development effects. The factual concept of the invariance of a normal process with respect to a linear transformation was used to conclude that the statistics of the diffracted field are also normal. Using the input-output relationships of autocorrelation functions in shift-invariant linear systems, the autocorrelation function of the diffracted field over hologram plane was obtained. This was shown to be proportional to a spatial Fourier transform of the diffuse object irradiance.

In the second part of CHAPTER THREE, some of the concepts and techniques used extensively in detection and processing of temporal information were applied to the process of recording and detection of spatial information. Recording the hologram was considered as analogous to the square-law envelope detection of a normal noise and a sine wave. By performing a simple transformation on the already obtained joint probability density function of the envelope of the sum of a normal noise and a sinusoidal signal, we were able to obtain the joint probability density of the exposure. Our calculations confirmed that when the reference wave is much stronger than the object wave, the statistics of the exposure are approximately normal. We also calculated the autocorrelation function of the exposure, two terms of which were shown to predict the so-called virtual and real images. A third term was shown to be responsible for the so-called ambiguity term which gives rise to an image in the Fraunhofer region with an average irradiance distribution determined by an autoconvolution of the object irrandiance. At the end of CHAPTER THREE the optical diffusion and its spatial frequency filtering effects were considered. It was mentioned that the transfer function of this stage can be approximated to be constant for very fine grain absorptive materials.

CHAPTER FOUR was devoted to the effects of the nonlinearity of the recording medium on the reconstructed images. This study reconfirmed that the first order image is in general a distorted image and with it are associated irradiance distributions of extra images in higher diffraction orders. The irradiance distribution of these false higher order images comprise a background noise for the first order image. This background noise, which is also called the intermodulation noise, is a result of multiple autoconvolutions of the object irradiance. In the last section of CHAPTER FOUR, the space dependent effects of development or the so-called adjacency effects were explained. It was mentioned that for a very thin hologram developed in a concentrated developer with agitation, the effects of development may be ignored.

In CHAPTER FIVE it was shown that the change of polarization of the illuminating beam after scattering may be another factor in image degradation, since it causes a loss of some information. It was suggested that in these cases using a polarizing plate in the way of the linearly polarized reference wave might to some extent eliminate this degradation factor. If possible, the change of polarization of the reference wave so as to produce components with approximately the same power as their corresponding components in the diffracted field will give the best improved image.

We have examined the process of diffused illumination holography by applying the already familiar concepts in electrical communication. This was shown to be a convenient method which fully explicates and elucidates the process.

CHAPTER SEVEN

CONCLUSIONS

In the foregoing treatment a systems approach was employed to study holography with diffused illumination. It was shown that formation of a hologram might be modelled as a five-stage systems configuration, which was then simplified to a three-stage model by confining the analysis to thin holograms of fine grain recording material. This study focussed on the application of this three-stage model to large Fresnel holograms, yielding a quadratic phase filter, a square-law envelop detector, and a nonlinearity.

It was explained that in most cases the diffuse object's field function, which was considered as a random process, is analogous to a Gaussian (normal) white noise encountered in electrical communication. The problem of finding the quantity in question, i.e., the autocorrelation function of the amplitude transmittance of the hologram, was solved by analogy to the problem of passing white noise and a sine wave through linear and nonlinear electrical devices, in terms of well-known concepts in communication theory. Unlike the conventional method, which considers the "variation of exposure" as the input to a nonlinearity, in this approach the "whole exposure" would be the input to the nonlinearity.

Previous analyses have been confined to cases of large reference-to-object irradiance ratios, or particular forms of nonlinearities. Our approach of sinusoidal plus white noise inputs proved more convenient and easier to handle; it permitted a general expression for the autocorrelation function of the amplitude transmittance of large, thin

Fresnel holograms of fine grain material to be obtained with arbitrary reference and object irradiances, regardless of the particular nature of the nonlinearity. This expression, except for some distoring factors, was shown to be a power series of the Fourier transform of the object irradiance, the first power of which predicts the virtual and the real images, their positions and their average irradiances. The higher powers contribute to the intermodulation noise.

Finally, when considering the vector nature of the diffracted fields to better explicate the process of making a hologram, we showed that a change of the polarization of the illuminating beam after being scattered by the object would result in a reduction of the signal-to-noise ratio and the degradation of the images. In this more general case, too, a general expression was obtained for the autocorrelation function of the amplitude transmittance of the hologram. This expression was similar, in body, to the one obtained for the scalar case except for different coefficients which were denoted by h_{pq} in the scalar case and h_{pqk} in the vector one. h_{pqk} was expressed in terms of a summation of even powers of the Fourier transform of the irradiance of the cross-polarized component of the diffuse object and, therefore, unlike h_{pq} is a function of space. This fortifies the distortion of already distorted images due to the nonlinearity.

The degradation of the images due to the change of polarization of the illuminating beam after scattering, was shown to be correctable by changing the polarization of the reference wave, accordingly.

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Appendix A

The Coherence of Light

In this appendix the coherence of light is briefly considered. Most of the definitions and theorems are as introduced by Beran and Parrent [A1].

The definition of coherence was arrived at through observations of phenomena of interference and diffraction of light. For example, in Young's double-beam interference experiment, fringes of high visibility were the sign of coherence between the two beams and no fringes the sign of incoherence.

For many years the realistic situation of the partially coherent states did not seem to be of any interest, although as early as in 1869 Verdet referred to the inadequecy of such descriptions as "strictly coherent" or "strictly incoherent" light. However, the need for some more profound definitions led to a lot of research which was actually initiated in the 1900's by Von Laue and later on continued by many other investigators resulting in more rigorous definitions*.

The theory of partial coherence has been developed on the basis of two general concepts. The first is based on the classical electromagnetic theory governed by Maxwell's equations and the theory of stochastic processes; the second is based on the theory of quantum electro-

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* For a detailed survey of research, the interested reader is referred to Ref.[A1], Chapter 1, p. 4-7, or *The Principles of Optics,* Born and Wolf, Chapter 10 [A2].

dynamics. However, in both cases a statistical aspect is inevitable, since the properties of most physical phenomena can be detected only by an averaging procedure. Let us, for instance, consider the light from a thermal source. The resultant radiation field is regarded as superposition of the light waves from myriads of elementary radiators (atoms). Each of these radiators, which somehow has become excited at some random instant to some random state of excitation, will emit radiation at some random instant for a random finite period of time and might be re-excited and re-emit in the same random manner, and so on. Therefore, we are led to specify the radiation in a statistical sense, and to determine the quantities of interest by statistical averages. Furthermore, we are to define some random stochastic processes. For example, the scalar light amplitude at a point $\vec{r}_1$, $V(\vec{r}_1, t)$ can be regarded as a random process. For a given t, t_1, we obtain a "static" random variable with some probability density $p(V_1, t_1)$. As t changes, the distribution $p(V_1; t)$ may change. The expected value of the static random variable $V(\vec{r}_1, t_i)$ is given by

$$\eta_i = \int_{-\infty}^{+\infty} V(\vec{r}_1, t_i)\, p(V_1; t_i)\, dV_{1i} \,. \tag{A-1}$$

So, the expected value of $V(\vec{r}_1, t)$ in general is a function of time

$$\eta(t) = \int_{-\infty}^{+\infty} V(\vec{r}_1, t)\, p(V_1; t)\, dV_1 \,. \tag{A-2}$$

If the statistics of the processes $V(\vec{r}, t)$ are stationary, which means that averages do not depend on the origin of time, then

$$E\ \{V(\vec{r}_1,t)\} = \eta \tag{A-3}$$

and

$$\Gamma_E(\vec{r}_1,\vec{r}_2,\tau\ ) = E\ \{V(\vec{r}_1,t_2 + \tau)\ V(\vec{r}_2,t_2)\}$$

$$= \int_{-\infty}^{\infty} V_1 V_2\ p\ (V_1,V_2;\tau)\ dV_1\ dV_2 \tag{A-4}$$

where $\tau = t_1 - t_2$ and t_2 may have any value.

Now, if the stationary processes $V(\vec{r},t)$ satisfy certain conditions (ergodicity) then the time averages $\langle V(\vec{r},t)\rangle$ of almost all the members of each process ought to exist and "almost certainly" be equal to a constant ensemble average $E\{V(\vec{r},t)\}$ of the corresponding process. Under such a hypothesis, one can assume that

$$\Gamma_E(\vec{r}_1,\vec{r}_2,\tau) = \Gamma_t(\vec{r}_1,\vec{r}_2,\tau) = \langle V(\vec{r}_1,t_2 + \tau)\ V(\vec{r}_2,t_2)\rangle\ . \tag{A-5}$$

The time autocorrelation or self-coherence function of the process $V(r_1,t)$ is defined as the joint moment of the random variables $V(r_1,t_1)$ and $V(\vec{r}_1,t_2)$:

$$\Gamma_E(\vec{r}_1,t_1;\vec{r}_1,t_2) = E\{V\ (\vec{r}_1,t_1)\ V\ (\vec{r}_1,t_2)\}$$

$$= \int_{-\infty}^{\infty} V_{11}\ V_{12}\ p(V_{11},\ V_{12};\ t_1,t_2)\ dV_{11}\ dV_{12} \tag{A-6}$$

which is a function of t_1 and t_2.

The mutual coherence function $\Gamma_E(\vec{r}_1,t_1;\vec{r}_2,t_2)$ is defined as the joint moment of the random variables $V(\vec{r}_1,t_1)$ and $V(\vec{r}_2,t_2)$

$$\Gamma_E(\vec{r}_1,t_1;\vec{r}_2,t_2) = E\{V(\vec{r}_1,t_1)V(\vec{r}_2,t_2)\}$$

$$= \int_{-\infty}^{\infty} V_{11}\, V_{22}\, p\,(V_{11},V_{22};\, t_1,t_2)\, dV_{11}\, dV_{22}\ . \tag{A-7}$$

Since we are dealing essentially with time averaging procedures, it seems much more convenient to use a complex representation in terms of the analytic signal [A3]. Thus, the time-averaged mutual coherence function for complex functions could be defined as:

$$\Gamma_{12}(\tau) = \langle\tilde{V}(\vec{r}_1,t+\tau)\ \tilde{V}^*\ (\vec{r}_2,t)\rangle \tag{A-8}$$

where $\tilde{V}(\vec{r},t)$ is the analytic signal associated with $V(\vec{r},t)$, a cartesian component of the electric field vector. Subscript 12 shows that a time cross-correlation between the two time functions $\tilde{V}(\vec{r}_1,t)$ and $\tilde{V}(\vec{r}_2,t)$ is involved. The self coherence function in this case will be

$$\Gamma_{11}(\tau) = \langle\tilde{V}(\vec{r}_1,t+\tau)\ \tilde{V}^*(\vec{r}_1,t)\rangle \tag{A-9}$$

which reduces to the intensity at point $\vec{r}_1$ when the time delay is zero. So,

$$I(r_1) = \Gamma_{11}(0)\ . \tag{A-10}$$

The normalized form of $\Gamma_{12}(\tau)$, $\gamma_{12}(\tau)$, is termed the complex degree of coherence and is defined as

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}}\ . \tag{A-11}$$

Both the modulus and phase of $\gamma_{12}(\tau)$ are measurable [A4]. In the quasi-monochromatic approximation, the modulus of $\gamma_{12}(\tau)$ is related to the fringe visibility, which is defined by Michelson [A5]

$$V = |\gamma_{12}(0)| = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} . \tag{A-12}$$

From Schwartz' inequality and the fact that $\Gamma(\vec{r},\vec{r},\tau)$ is maximum for $\tau=0$ it follows that $0 \leq |\gamma_{12}(\tau)| \leq 1$. Only in the case of a strictly monochomatic field could $|\gamma_{12}(\tau)| = 1$ and only in the cases of the non-existance of any field in free space could $|\gamma_{12}(\tau)| = 0$. Proofs of these theorems are given by Beran and Parrent [A1].

Light from a real physical source, however, is never strictly monochromatic, since all fields have some spectral width. There is no strictly incoherent source either, since there is always some degree of correlation between some points of the field.

To give a physical realization to the idea of coherent fields, sources of very sharp spectral width could be used. The spectral distribution function for such a source is peaked about some frequency $\bar{\nu}$(mean frequency) and tends to zero rather rapidly as $\bar{\nu}$ departs from the vicinity of $\bar{\nu}$. Such sources are called quasi-monochromatic sources.

A monochromatic wave field might be represented by a complex disturbance $\tilde{V}(t)$, (or rather in actual fact, with either the real part, or the imaginary part of $\tilde{V}(t)$), of the form

$$\tilde{V}(t) = A\, e^{-j(2\pi\nu_o t + \beta)} \tag{A-13}$$

where A and β can be functions of position only. The amplitude of the vibrations at any point $\vec{r}$ is constant while the phase varies linearly with time. On the other hand, in the wave field of a real source, the amplitude and phase undergo irregular fluctuations, the rapidity of which depends essentially on the effective width of $\Delta\nu$ of the spectrum [A2]. Now the field may be represented in the form of

$$\tilde{V}(t) = A(t)\, e^{j(2\pi\bar{\nu}t + \beta(t))} \tag{A14}$$

where $A(t)$ and $\beta(t)$ are functions of time. For a quasi-monochromatic field $\Delta\nu/\bar{\nu} << 1$, and $A(t)$ and $\beta(t)$ are slowly varying functions of time (compared to $1/\bar{\nu}$). For a quasi-monochromatic field to be coherent we require that the time variation of $A(t)$ and $\beta(t)$ vary in a like manner.

The condition $\Delta\nu/\bar{\nu} << 1$ implies that the mutual power spectrum of radiation, $\hat{\Gamma}_{12}(\nu)$, is appreciably different from zero only for the spectral components ν which satisfy the inequality $|\nu-\bar{\nu}| < \Delta\nu$.

$\hat{\Gamma}_{12}(\nu)$, the Fourier time transform of $\Gamma_{12}(\tau)$, is given as

$$\begin{aligned} \hat{\Gamma}_{12}(\nu) &= \int_{-\infty}^{\infty} \Gamma_{12}(\tau)\, e^{2\pi j\nu\tau}\, d\tau \qquad && \nu > 0 \\ &= 0 && \nu < 0 \end{aligned} \tag{A-15}$$

The mutual coherence function $\Gamma_{12}(\tau)$, then, is obtained from the relation

$$\Gamma_{12}(\tau) = \int_{0}^{\infty} \hat{\Gamma}_{12}(\nu) e^{-2\pi j\nu\tau}\, d\nu \tag{A-16}$$

which may be written as

$$\Gamma_{12}(\tau) = e^{-2\pi j\bar{\nu}\tau} \int_{0}^{\infty} \hat{\Gamma}_{12}(\nu)\, e^{-2\pi j(\nu-\bar{\nu})\tau}\, d\nu\ . \tag{A-17}$$

If we now focus our attention on small τ such that the inequality $\Delta\nu\ |\tau| << 1$ holds, then the exponential factor is approximately 1 and we find that

$$\Gamma_{12}(\tau) = e^{-2\pi j\bar{\nu}\tau} \int_0^{\infty} \hat{\Gamma}_{12}(\nu)\, d\nu = e^{-2\pi j\bar{\nu}\tau}\, \Gamma_{12}(0)\,. \qquad \text{(A-18)}$$

The choice of $\tau=0$ is, of course, arbitrary and we may find an expression for $\Gamma_{12}(\tau)$ about some other point τ_o if we choose to do so. We then find that

$$\Gamma_{12}(\tau_o+\tau') = e^{-2\pi j\bar{\nu}\tau'}\Gamma_{12}(\tau_o) \qquad \text{(A-19)}$$

where $\tau = \tau_o+\tau'$ and $\Delta\nu|\tau'| << 1$.

By choosing to limit the value of $|\tau|$ in any particular problem, it becomes physically meaningful to speak of coherent fields in a restricted sense. We now call a field "coherent" if for all pairs of points of interest $(\vec{r}_1,\vec{r}_2)$ there exists a τ such that $|\gamma_{12}(\tau)| = 1$ if $|\tau'|\,\Delta\nu << 1$. If $\Delta\nu/\bar{\nu}$ is very small, the field may act like a monochromatic field for all times of interest in any physical problem, and similarly it can be shown that

$$\Gamma_{12}(\tau_o) = A_1\, A_2\, e^{j\phi_{12}} \qquad \text{(A-20)}$$

and

$$\Gamma_{12}(\tau) = \tilde{U}(P_1)\, \tilde{U}^*(P_2)\, e^{-2\pi j\bar{\nu}\tau'}$$

c.f. Eqns. A-18
A-19
A-20

where $\tilde{U}(\vec{P}_1) = A_1\, e^{j\phi_1}$ is the wave function evaluated at $\vec{r}_1$ and $\tilde{U}^*(\vec{r}_2) = A_2\, e^{-j\phi_2}$ is the complex conjugate of the wave function evaluated at $\vec{r}_2$.

Appendix B

Input-Output Relationships of Autocorrelation Functions in Linear Systems*

Let us assume that $\tilde{f}(x,y)$ is the input of a linear shift-invariant system with the impulse response $\tilde{h}(x,y)$. The well-known convolution theorem, then, determines the output $\tilde{g}(x,y)$ as:

$$\tilde{g}(x,y) = \iint_{-\infty}^{\infty} \tilde{f}(u,v)\, \tilde{h}(x-u,\, y-v)\, du\, dv\, . \tag{B-1}$$

Knowing the impulse response of the system and the autocorrelation of the input, the autocorrelation of the output is obtained by:

$$\begin{aligned}
\tilde{R}_{gg}(x_1,y_1;x_2,y_2) &= E\{\tilde{g}(x_1,y_1)\, \tilde{g}^*(x_2,y_2)\} \\
&= E\,\{[\iint_{-\infty}^{\infty} \tilde{f}(u,v)\, \tilde{h}(x_1-u,\, y_1-v)\, du\, dv] \times \\
&\quad [\iint_{-\infty}^{\infty} \tilde{f}^*(\xi,\eta)\, \tilde{h}^*(x_2-\xi,\, y_2-\eta)\, d\xi\, d\eta]\} \\
&= E\{\iiiint_{-\infty}^{\infty} \tilde{f}(u,v)\tilde{f}^*(\xi,\eta)\tilde{h}(x_1-u,y_1-v)\tilde{h}^*(x_2-\xi,y_2-\eta)dudvd\xi d\eta\} \\
&= \iiiint_{-\infty}^{\infty} E\{\tilde{f}(u,v)\tilde{f}^*(\xi,\eta)\}\, \tilde{h}(x_1-u,y_1-v)\tilde{h}^*(x_2-\xi,y_2-\eta)dudvd\xi d\eta \\
&= \iiiint_{-\infty}^{\infty} \tilde{R}_{ff}(u,v;\xi,\eta)\, \tilde{h}^*(x_2-\xi,y_2-\eta)\tilde{h}(x_1-u,y_1-v)d\xi d\eta dudv \\
&= \tilde{R}_{ff}(x_1,y_1;x_2,y_2)*\tilde{h}^*(x_2,y_2)*\tilde{h}(x_1,y_1)
\end{aligned} \tag{B-2}$$

which symbolically means a convolution between $\tilde{R}_{ff}(x_1,y_1;x_2,y_2)$ and $\tilde{h}^*(x_2,y_2)$ considering x_1,y_1 as parameters and x_2,y_2 as variables, and another convolution between the resultant function and $\tilde{h}(x_1,y_1)$, this time

* The material in this Appendix can be found in Refs. [B1,B2,B3].

with parameters x_2 and y_2 and variables x_1,y_1.

$\tilde{R}_{ff}(x_1,y_1;x_2,y_2)*\tilde{h}^*(x_2,y_2)$ in (B-2) is the cross-correlation between the input and the output, $\tilde{R}_{fg}(x_1,y_1;x_2,y_2)$, since one can write:

$$\tilde{R}_{gg}(x_1,y_1;x_2,y_2) = \tilde{E}\{\tilde{g}(x_1,y_1)\ \tilde{g}^*(x_2,y_2)\}$$

$$= E\{[\iint_{-\infty}^{\infty} \tilde{f}(u,v)\tilde{h}(x_1-u,y_1-v)dudv]\ \tilde{g}^*(x_2,y_2)\}$$

$$= \iint_{-\infty}^{\infty} E\{\tilde{f}(u,v)\tilde{g}^*(x_2,y_2)\}\ \tilde{h}(x_1-u,y_1-v)\ dudv$$

$$= \iint_{-\infty}^{\infty} \tilde{R}_{fg}(u,v;x_2,y_2)\tilde{h}(x_1-u,y_1-v)\ dudv$$

$$= \tilde{R}_{fg}(x_1,y_1;x_2,y_2)*\tilde{h}(x_1,y_1) \tag{B-3}$$

When input is an incoherent process, in which

$$R_{ff} = I_f(x_1,y_1)\delta(x_2-x_1)\delta(y_2-y_1) \tag{B-4}$$

one obtains:

$$\tilde{R}_{fg}(x_1,y_1;x_2,y_2) = \tilde{I}(x_1,y_1)\tilde{h}^*(x_2-x_1,y_2-y_1) \tag{B-5}$$

and

$$\tilde{R}_{gg}(x_1,y_1;x_2,y_2) = [I(x_1,y_1)\tilde{h}^*(x_2-x_1;y_2-y_1)]*\tilde{h}(x_1,y_1) \tag{B-6}$$

If the input to the system is stationary, where the autocorrelation depends on $\tau_x = x_2-x_1$ and $\tau_y = y_2-y_1$ only, one can obtain:

$$\tilde{R}_{fg}(\tau_x;\tau_y) = E\{\tilde{f}(x+\tau_x,y+\tau_y)\tilde{g}^*(x,y)\}$$

$$= \tilde{R}_{ff}(\tau_x;\tau_y)*\tilde{h}^*(-\tau_x;-\tau_y) \tag{B-7}$$

and

$$\tilde{R}_{gg}(\tau_x;\tau_y) = \tilde{R}_{fg}(\tau_x;\tau_y)*\tilde{h}(\tau_x;\tau_y) . \qquad \text{(B-8)}$$

By applying the autocorrelation theorem [B4], one could write:

$$\tilde{S}_{fg}(\alpha,\beta) = \tilde{S}_{ff}(\alpha,\beta)\tilde{H}*(\alpha,\beta)$$

$$\tilde{S}_{gg}(\alpha,\beta) = \tilde{S}_{fg}(\alpha,\beta)\tilde{H}(\alpha,\beta) = \tilde{S}_{gg}(\alpha,\beta)|\tilde{H}(\alpha,\beta)|^2$$

where $\tilde{S}_{ff}$, $\tilde{S}_{fg}$, $\tilde{S}_{gg}$ are the Fourier transforms of $\tilde{R}_{ff}$, $\tilde{R}_{fg}$, $\tilde{R}_{gg}$. $\tilde{H}(\alpha,\beta)$, the Fourier transform of $\tilde{h}(x,y)$ is the system function.

The mean value of the output of the system is given by

$$\tilde{M}_g(x,y) = E\{\tilde{g}(x,y)\} = \iint_{-\infty}^{\infty} E\{\tilde{f}(u,v)\}\ \tilde{h}(x-y,y-v)\ dudv$$

$$= \tilde{M}_f(x,y)\ *\tilde{h}(x,y) \qquad \text{(B-9)}$$

where $\tilde{M}_f(x,y)$ is the mean value of the input. It can be seen that if the input has a zero mean, the mean value of the output, $\tilde{M}_g(x,y)$, will also be zero.

To obtain the output mean irradiance one may put $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in $\tilde{R}_{gg}(x_1,y_1;x_2,y_2)$.

$$M_I(x,y) = R_{gg}(x,y;x,y) = E\{\tilde{g}(x,y)\tilde{g}*(x,y)\} = E\{|\tilde{g}(x,y)|^2\}$$

$$= E\{I_g(x,y)\}. \qquad \text{(B-10)}$$

When the input process is incoherent as in (B-4) we have

$$M_I = I_f(x,y)*|\tilde{h}(x,y)|^2 . \qquad \text{(B-11)}$$

The variance of the output is given by

$$\sigma_g^2(x,y) = E\{|\tilde{g}(x,y) - \tilde{M}_g(x,y)|^2\} = E\{|\tilde{g}(x,y)|^2\} - |\tilde{M}_g(x,y)|^2 \ . \tag{B-12}$$

Using (B-11) and (B-12) for the case of an incoherent input, the variance of output is:

$$\sigma_g^2(x,y) = I_f(x,y)*|\tilde{h}(x,y)|^2 - |\tilde{M}_g(x,y)|^2. \tag{B-13}$$

If the incoherent input has a zero mean, (B-13) will be the same as (B-11), i.e., $M_I = \sigma_g^2 = I_f(x,y)*|h(x,y)|^2$.

The autocorrelation of the output irradiance is given by

$$R_{II}(x_1,y_1,;x_2,y_2) = E\{|g(x_1,y_1)|^2 \ |g(x_2,y_2)|^2\}. \tag{B-14}$$

For zero mean normal real $g(x_1,y_1)$ and $g(x_2,y_2)$, or normal complex $\tilde{g}(x_1,y_1)$ and $\tilde{g}(x_2,y_2)$ with independent real and imaginary parts, R_{II} will be in the form

$$R_{II}(x_1,y_1;x_2,y_2) = \sigma_g^2(x_1,y_1)\sigma_g^2(x_2,y_2) + |\tilde{R}_{gg}(x_1,y_1;x_2,y_2)|^2 . \tag{B-15}$$

The mean value of the square of the output irradiance is obtained by putting $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in $R_{II}(x_1,y_1;x_2,y_2)$. Hence, if $R_{II}(x_1,y_1;x_2,y_2)$ is given by (B-15), one can obtain

$$E\{I_g^2(x,y)\} = 2\sigma_g^4(x,y) . \tag{B-16}$$

The variance of the output irradiance is defined as

$$\sigma_I^2 = E\{I_g^2(x,y)\} - M_I^2 \tag{B-17}$$

which in the case of (B-15) will be:

$$\sigma_I^2 = 2\sigma_g^4(x,y) - [\sigma_g^2(x,y)]^2 = \sigma_g^4(x,y) \; . \qquad \text{(B-18)}$$