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A SYSTEMS ANALYSIS OF HOLOGRAPHIC IMAGING WITH DIFFUSED ILLUMINATION FROM A STATISTICAL POINT OF VIEW

HOSSEIN GHANDEHARIAN

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ABSTRACT

Holography with diffused light illumination is studied from a statistical communication point of view. The final amplitude transmittance of the hologram is considered to result from some combination of linear and nonlinear transformation of a nonstationary incoherent Gaussian process plus a sine wave. Formation of a hologram is modelled as a five-stage systems configuration. Each stage of the model is described. The problem is considered analogous to passing white noise and a sine wave through linear and nonlinear electrical devices, in terms of well-known concepts in communication theory. Statistical properties of the outputs of the different stages are considered. For large, thin Fresnel holograms of fine grain recording materials, expressions are obtained for the mean values, variances and autocorrelation functions. In the general case of arbitrary reference and object irradiances, the autocorrelation function of the amplitude transmittance of the hologram is obtained with the aid of the characteristic function method. Several examples of practical use are worked out. As an example of the direct method of obtaining the correlation function of the output of a nonlinear device, we apply this method to a linear phase hologram of exponential transmittance. Although the direct method seems unmanageable in general cases, in the special case of a very weak object irradiance compared with that of the reference's, this method of analysis is employed without complications. In both cases, it is shown that the autocorrelation function of the amplitude transmittance can be expressed as a power series of the Fourier transform of the object irradiance with the major difference that in the general case, some distorting factors are involved.

Interpretation of the results lead to conclusions concerning the image irradiance distribution in the presence of intermodulation noise and non-linear distortions; the intermodulation noise being the result of higher order image correlations and convolutions. Finally, the likely effects on image quality due to polarization change of the illuminating beam after scattering by the diffuse object are considered. The autocorrelation function of the amplitude transmittance in this more general case is obtained. A method of improving the effects of depolarization on the image is discussed.

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CHAPTER ONE
INTRODUCTION

1.1 History

The links between optics and electrical engineering have become so strong that a unified treatment of problems in the two fields is now a promising possibility. Now, in very many cases, one could treat a problem in one field, in a rather straight-forward manner, by applying the methods already used to study an analogous problem in the other.

It has become a common experience to encounter in literature such similarities as between a 'lens' and a 'linear FM generator', or 'Fresnel diffraction' and input-output relation of a 'quatrature phase filter', or the 'light concentration' of a 'lens' and a 'pulse compression process', etc.

The interaction of these two disciplines is not just in the methods of treatment, but the optical devices themselves are now indispensable in almost all communication systems. They are being used extensively as vital parts of the information processing instruments.

Fourier analysis which has been used widely in many branches of science and engineering might be thought of as the first and strongest stimulus of this interaction, since the Fourier integral works with any physical system in which *cause* and *effect* are linearly related. Therefore, we might say that the first step in unification of optics and electrical engineering was taken by P.M. Duffieux by his introduction of Fourier transform methods and its application in optics, which appeared in his book *L'Integral de Fourier et ses Application à l'Optique* in 1946 [1].

Of course Duffieux's step was taken on a road already paved by other pioneers late in the 19th and early in the 20th century. To this period belongs the Abbe theory of image formation, a very important theory which, for instance, made "spatial filtering" a realizable practice.

There has been great progress in the studies of object-image problems since Duffieux's work; for example, the work of Otto H. Schade, which appeared in a series of papers [2,3] between 1948 and 1958. By applying the transform methods already used in electrical systems and radio propagation, Schade analysed the process of image formation in television cameras and subsequently suggested some techniques to improve their functioning.

Now it was time for other participants to follow and merge the two disciplines and unify the approach. Considerable work has been done and much remains to be done. Nevertheless, the unification is not distant.

1.2 This Thesis

In this treatment, the process of holography is studied from the viewpoint of the statistical communication systems theory. To deal with the process of holography in the framework of communication theory, was first suggested by Leith and Upatnieks[4]. In their treatment, a hologram is made in two stages: 1) the defocusing or spatial-frequency dispersion of the image; 2) the hologram recording, which is similar to a square-law or non-linear detection. In the following, the process is viewed as the result of the input signal being sent through a five-stage model. This approach was originally used by D.H. Kelly, reported in a paper titled "Systems Analysis of the Photographic Process" [5]. Kelly considers a three-stage model for the process of photography; a nonlinearity located in series between two isotropic linear systems.

The components of the five-stage model that we choose to work with in the process of making a hologram are as follows:

1) The projection process of the field's function onto the hologram is accounted for by: a) a quadrature phase filter, or, b) if the finite dimension of the hologram or some other optical element in the path of the signal is a frequency limiter, by a low pass filter with quadratic phase.

2) A square-law envelope detector in which the recording medium detects the signal output of the first stage by being sensitive to a time average of the energy received.

3) A low pass filter, which accounts for the frequency filtering effect of the recording medium due to the optical diffusion.

4) A nonlinearlity, which is due to the nonlinear characteristic of the recording medium in the conversion of the time-averaged intensity to an optical transmittance.

5) A third linear filter which is introduced because of a non-uniformity in the developing process.

For the nonlinear part of his model, Kelly uses the "H & D" characteristic of the recording medium (Hurter-Driffield curve, which is a plot of the photographic density vs. the logarithm of exposure) as a point-by-point nonlinear scale conversion. The H & D curves have been used most commonly in classical photography. However, for holography and most other coherent optical processing techniques a better and more convenient description of the nonlinearity is the T_a-E (amplitude Transmittance vs. Exposure) curve. This was first used by Leith and Upatnieks [6] in their early holographic experiments, and later by Vander Lught [7] for spatial filtering purposes. Kozma [8] used the T_a-E curves in exactly the same way as the characteristic curves of nonlinear electronic devices are being treated (e.g. characteristic curve of a vacuum tube). A working point is usually chosen on the center portion of the smoothest part of the curve and a small variational analysis is applied when the input variation is not exceeding the limits of this smooth part. However, when the variation exceeds the linear limits, first a function (e.g. an error function limiter, a polynomial, a νth law function) is chosen that fits the curve best on both sides of the working point. Then considering the chosen function as the transfer function of a nonlinear device, one of the techniques already used in electrical engineering to study nonlinearities could be employed to characterize the output. The above-mentioned method has been the one used by most workers to study the effects of film nonlinearity on signals. Among those are Kozma [8], Friesem and Zelenka [9], Bryngdhal and Lohmann [10], and Lee and Greer [11], in whose treatments input is considered to be deterministic. Goodman and Knight [12], and Kozma, Jull and Hill [13], studied the problem for random inputs.

The method of analysis in the following study is statistical as in references [12] and [13], although the approach is a bit different. The working point and the variation around it are not considered; input to the nonlinearity is the whole exposure. The mathematical treatment is more general. In reference [12] it is shown that when the irradiance of the reference wave is much stronger than that of the object's, the autocorrelation function of the variation of the transmittance of the recording medium can be expressed in a power series of the autocorrelation function of the exposure variation, the latter being proportional to the Fourier transform of the object irradiance. However, in the general case of arbitrary irradiances of object and reference, the autocorrelation function of the output remains without a general expression. In the following, using the characteristic function method, it will be shown that in the general case, too, the autocorrelation function of the transmittance can be expressed in a power series of the Fourier transform of the object irradiance. The treatment, with some slight modifications,can also be applied to cases in which the phase of the reference wave over the hologram plane happens to be random. Fresnel holography is considered rather than the more special case of Fourier transform holography. Besides, in some cases, we also use the *direct method* of finding the autocorrelation function of the output of a nonlinearity [14] in addition to the well-known *characteristic function method* [14,15] used in several treatments, e.g. Ref. [12]. The advantages of the direct method are that, first there is no need to find the Fourier or Laplace transform of the transfer function of the nonlinearity. Secondly, in cases where the transfer function does not have a Fourier transform but a Laplace transform, a contour integration technique must be used to evaluate the integral involved in the characteristic function method. This contour integration may be troublesome. Using the direct method may eliminate this problem too.

Some preliminary considerations are given in *Chapter Two* and *Appendix B*. These include statistics of diffuse objects, the coherence requirement, the definition of spatial coherence as a space average correlation and not a time average one, the autocorrelation of diffuse objects, some properties of autocorrelation functions, and the use of autocorrelation functions in determination of the irradiance distribution of the diffracted waves.

The different stages of our systems model are then discussed in the following chapters. Some statistics of the signal at the output of each stage are considered. The autocorrelation function of the signal is followed through the model to study some relevant characteristics of images in the reconstruction process, e.g., their irradiance distribution.

The last chapter is devoted to a brief consideration of the depolarization of the illuminating beam by a diffuse object. Some probable effects of the cross-polarized component on the image degradation are given. Some ways of reducing these effects in order to record holograms with improved quality are discussed.

CHAPTER TWO
FUNDAMENTALS

2.1 Introduction

The concept of diffused or scattered light illumination in holography was first introduced in 1964 by G.W. Stroke [16], and by E.N. Leith and J. Upatnieks [17] almost at the same time. Before the introduction of diffused illumination, holograms were made only of transparent subjects. In diffused illumination holography the property of high spatial and temporal coherence of gas lasers are exploited to make holograms of diffusely reflecting three-dimensional objects.

Diffused illumination holography was first achieved by using a diffusing element such as a ground glass and a subject transparency. The ground glass is placed in the path of a coherent illuminating beam to illuminate the transparency with diffused coherent light. Without the ground glass, light from each portion of the transparency reaches the recording medium in a specular direction (the direction of the illuminating source and that portion) resulting in a *small region* to *small region* correspondence mapping of the subject in a recorded intensity. In the reconstruction process, when viewing the virtual image, the observer sees only the portion which lies between the illuminating source and his eyes. To see another portion, he has to move his head to intercept the light coming from that portion. That is to say, the observer has to move his head to scan the different small portions of the subject's virtual image. The real image could be viewed on a diffusing screen placed at the real image plane.

On the other hand, if a diffusing screen is introduced in the way of the illuminating beam when the hologram is recorded, the light is scattered in a myriad of directions. Now each portion of the subject transparency is receiving and transmitting light in all of these directions. Therefore, each portion of the hologram receives light from all portions of the subject, since each portion of the subject has sent light over the entire hologram area. In the reconstruction process, then, no matter what portion of the diffracted light is intercepted, an aspect of the whole subject transparency is viewed. The real image could be viewed on an ordinary screen, e.g. a card board, placed at the plane of the real image.

A further advantage of diffused-illumination holography over non-diffused holography is that the dynamic range of the recording medium is increased more effectively and the distortion of the image due to non-linearity of the recording medium is to some extent reduced. Without a diffuser, very bright and very weak areas in the subject might give exposures which exceed the almost linear limits of the T-E characteristics of the recording medium, resulting in a nonuniformity in the recording of different portions and therefore a distortion of the images. In diffused illumination holography, the energy of each small portion is no longer concentrated on its corresponding small area of the hologram; rather, it shares the whole area of the hologram with the other portions by spreading its energy all over the hologram. Whereas in non-diffused holography, some portions of the recording medium may be exposed to very strong and some to very weak intensity, in diffused holography, the whole area is exposed to the almost uniform mean intensity; the dynamic range of the recording medium is effectively increased and image distortion is greatly reduced.

Another deficiency of non-diffused holography is the problem of ring-like noise associated with the real image. Any imperfection in the path of the high coherent subject beam causes annoying diffraction patterns of circular fringes. These circular fringes become apparent on the transparency and would be recorded as if they were part of the subject details. Diffused illumination removes this problem too, due to the destruction of the diffraction patterns by the diffuser, although it produces a speckled or granular appearance of the image instead.

In making holograms of three-dimensional reflecting objects, the process could be thought of as diffused illumination holography, since most objects scatter light in much the same way as a diffusing screen. The reason is that most objects have rough surfaces at the visible portion of the electromagnetic spectrum. Many of these might be considered as configurations of very many individual scatterers with random positions and orientations. Light on the surface of the object can be regarded as a collection of secondary point sources. Each source appears to emit a spherical wave of random phase.

In this study the term 'diffuse object' refers to either a subject transparency backed by a diffusing screen, or a reflecting object with a very rough surface.

In the following sections a brief study of the statistics of coherently illuminated diffuse objects will be given. Some parts of the background to the following subjects can be found in Appendices A and B. For a more detailed and complete description of the theorems and definitions, the interested reader is referred to Ref.'s [18, 19, 20].

2.2 Autocorrelation of the diffuse object field function

The time-averaged mutual coherence function of a coherent quasi-monochromatic field, $\Gamma_{12}(\tau)$, as in the case of an ideal monochromatic field, can be expressed in the form [21, 22]:

$$\Gamma_{12}(\tau) = \tilde{U}(P_1)\ \tilde{U}^*(P_2)\ e^{-2\pi j\bar{\nu}\tau} \tag{2.1}$$

where $\tilde{U}(P_1)$ is the field function evaluated at P_1 and $\tilde{U}^*(P_2)$ is the complex conjugate of the field function evaluated at P_2. As can be seen, the space and time dependent parts of $\Gamma_{12}(\tau)$ are separated. Thus, $\Gamma_{12}(\tau)$ could be written in the form:

$$\Gamma_{12}(\tau) = \Gamma_{12}(0)\Gamma_{11}(\tau). \tag{2.2}$$

In the Young double-beam interference experiment, for instance, it can be shown that $\Gamma_{11}(\tau)$ is a measure of "temporal" and $\Gamma_{12}(0)$ of "spatial" coherence of the source [23].

If a diffuse object is illuminated by a coherent quasi-monochromatic field, the space-dependent part of the mutual coherence function, $\Gamma_{12}(0)$, of the resultant field will be a random function of space, since the random optical path of the diffuse object impresses a random phase on the incident field. As in the case of any random function, a meaningful quantity for $\Gamma_{12}(0)$ will be its ensemble average. This ensemble average is called the autocorrelation function of the resultant field. The autocorrelation function in this case, then, could be thought of as a "spatial coherence function" [24] when compared to $\Gamma_E(\vec{r}_1,t_1,\vec{r}_2,t_2)$, the time coherence function [c.f. Appendix A].

The ensemble average of $\Gamma_{12}(0)$ is defined as [25]:

$$R(\vec{r}_1,\vec{r}_2) = E\,\{\Gamma_{12}(0)\} = E\,\{\tilde{U}(\vec{r}_1)\tilde{U}^*(\vec{r}_2)\} =$$

$$\int_{-\infty}^{+\infty} \tilde{U}(\vec{r}_1)\tilde{U}^*(\vec{r}_2)\; p(\tilde{U}_1,\tilde{U}_2)\; d\tilde{U}_1 d\tilde{U}_2 \qquad (2.3)$$

where $\tilde{U}(\vec{r})$ is the complex field function of the diffuse object with a deterministic amplitude $A(\vec{r})$, but a random phase $\phi(\vec{r})$. $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of two points on the object surface and $p(\tilde{U}_1,\tilde{U}_2)$ is the joint probability density function of $\tilde{U}(\vec{r}_1)$ and $\tilde{U}(\vec{r}_2)$.

With $\tilde{U}(\vec{r}) = A(\vec{r})\, e^{j\phi(\vec{r})}$ the autocorrelation function of the diffuse object field could be written as

$$E\,\{A(\vec{r}_1)\, e^{j\phi(\vec{r}_1)}\, A^*(\vec{r}_2)\, e^{-j\phi(\vec{r}_2)}\} = A(\vec{r}_1)\, A^*(\vec{r}_2) E\,\{e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\}$$

$$= A(\vec{r}_1)\, A^*(\vec{r}_2) \int_{-\infty}^{+\infty} e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\, p(\phi_1,\phi_2)\, d\phi_1\, d\phi_2 \qquad (2.4)$$

since $A(\vec{r})$ is deterministic.

It was noted above that light on the surface of the diffuse object may be regarded as a configuration of very many secondary sources. The random phase of each secondary source is considered in most cases to be uniformly distributed between 0 and 2π or in general, between some constant C and $C+2\pi$, i.e. [19]

$$p(\phi) = \frac{1}{2\pi} \quad (C < \phi < C + 2\pi). \qquad (2.5)$$

Of course, this assumption is only an approximation, but if the configuration of the scatterers is random and sufficiently dispersed to give a wide phase distribution, then it could be said that (2.5) holds effectively* [19]. Furthermore, if there is a very large number of scatterers, we might also assume that the space phase fluctuations for any two points of the object are statistically independent. Several investigators have used this assumption of independent secondary sources [26,27,28]. Of course, this assumption is not true either because there is always some degree of correlation between very close points. Suzuki and Hioki [29] showed that the correlation area of phase is of the order of λ^2 at least. However, as we will see later, under some conditions it is a reasonable assumption to consider that the individual scatterers are statistically independent. (Perhaps a more reasonable assumption would be that the correlation areas are statistically independent and should be considered as individual independent scatterers).

On the basis of the above assumption, it follows that:

$$p(\phi_1,\phi_2) = p(\phi_1)\, p(\phi_2) = \frac{1}{2\pi} \times \frac{1}{2\pi} = \frac{1}{4\pi^2}$$

and

$$E\{e^{j\phi(\vec{r}_1)}\, e^{-j\phi(\vec{r}_2)}\} = \frac{1}{4\pi^2} \int_0^{2\pi} e^{j\phi(\vec{r}_1)}\, d\phi_1 \int_0^{2\pi} e^{-j\phi(\vec{r}_2)}\, d\phi_2$$

$$= \begin{cases} 1 \;;\; \vec{r}_1 = \vec{r}_2 \\ 0 \;;\; \text{otherwise.} \end{cases} \qquad (2.6)$$

A process with such autocorrelation as in (2.6) is termed "incoherent", or by its analogy in communication theory, "white noise" [c.f. Section 2.3]. The autocorrelation function of a nontrivial white noise must be of the

* For a detailed discussion of the validity of such an assumption the interested reader is referred to Ref. 19, Section 7.3, p. 146-151.

form [30]:

$$R(\vec{r}_1,\vec{r}_2) = I(\vec{r}_1)\ \delta(\vec{r}_2-\vec{r}_1) \tag{2.7}$$

where we choose to show $A(\vec{r})\ A^*(\vec{r}) = |A(\vec{r})|^2 = \tilde{U}(\vec{r})\tilde{U}^*(\vec{r}) = |\tilde{U}(\vec{r})|^2$, the object irradiance, by $I(\vec{r})$.

Very often one assumes an ergodic type hypothesis and equates the spatial and ensemble averages. Then for a stationary ergodic process the autocorrelation function would be defined as

$$R(\vec{\tau}_r) = \int_{-\infty}^{\infty} \tilde{f}(\vec{u})\ \tilde{f}^*(\vec{u}-\vec{\tau}_r)d\vec{u}\ . \tag{2.8}$$

This function is related to the Fourier transform of $\tilde{f}(\vec{u})$, $\tilde{F}(\vec{s})$, by the autocorrelation theorem [31], which states that the Fourier transform of $R(\vec{\tau}_r)$ is equal to $|\tilde{F}(\vec{s})|^2$.

$$R(\vec{\tau}_r) = \mathcal{F}^{-1}[|\tilde{F}(\vec{s})|^2] = M\delta(\vec{\tau}_r) + \sum_{\substack{m,n=1\\ m\neq n}}^{M}\sum^{M} \delta[\vec{\tau}_r-(\vec{\tau}_{r_m}-\vec{\tau}_{r_n})]\ \exp\ [j(\phi_m-\phi_n)] \tag{2.9}$$

Eq. (2.9) is in accordance with our previously mentioned assumption that when there is a large number of scatterers it is reasonable to assume they are independent. As M becomes larger, the correlation function approaches a delta function or shows the tendency toward a state of non-correlation of phase [32].

Another way of analyzing this problem is by modeling the random optical path of the diffuse object by a random function $Z(x,y)$, defined in a domain of the xy plane*. The coordinate system is defined in such

*The background to this material can be found in Ref. 19, Chapter 5, p. 70-98.

a way as to make the xy plane the mean plane of the surface, so that

$$E\{Z(x,y)\} = 0 . \tag{2.10}$$

It is assumed that the surface is isotropically rough, i.e. that Z is distributed with the same statistical distribution in all directions over the surface [33]. The diffuse object field function $\tilde{U}(\vec{r})$ could be written in the form:

$$\tilde{U}(\vec{r}) = A(\vec{r})\, e^{jkZ(\vec{r})} \tag{2.11}$$

where $k = \frac{2\pi}{\lambda}$ is the wave number.

It follows that the autocorrelation function $R(\vec{r}_1,\vec{r}_2)$ is

$$R(\vec{r}_1,\vec{r}_2) = A(\vec{r}_1)A^*(\vec{r}_2)\, E\{e^{jkZ(\vec{r}_1)}\, e^{-jkZ(\vec{r}_2)}\} = A(\vec{r}_1)A^*(\vec{r}_2) \cdot$$

$$\cdot \int_{-\infty}^{\infty} e^{jk[Z(\vec{r}_1)-Z(\vec{r}_2)]}\, p(Z_1,Z_2)\, dZ_1\, dZ_2 \tag{2.12}$$

where $p(Z_1,Z_2)$ is the joint probability density of $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$. But by definition [34]

$$E\{e^{jv_1\xi}\, e^{jv_2\eta}\} = M(v_1,v_2) \tag{2.13}$$

is the joint characteristic function of the distribution $p(\xi,\eta)$ of two random variables ξ and η. Therefore, $E\{e^{jk[Z(\vec{r}_1) - Z(\vec{r}_2)]}\}$ is the characteristic function of $p(Z(\vec{r}_1), Z(\vec{r}_2))$ evaluated at $v_1 = k$ and $v_2 = -k$ [35].

A model that is used very often for many rough surfaces is the Gaussian or the normal distribution model [35]. The joint probability density for two normally distributed random functions with mean value zero, variance σ^2 and correlation coefficient ρ, is [36]

$$p(\xi,\eta) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{\xi^2-2\rho\ \xi\eta+\eta^2}{2\sigma^2\ (1-\rho^2)}\right] \tag{2.14}$$

and its characteristic function is

$$M(v_1,v_2) = \exp\left[-\frac{1}{2}\sigma^2(v_1{}^2 + 2\rho\ v_1v_2 + v_2{}^2)\right]. \tag{2.15}$$

For a normally distributed surface

$$M(k,-k) = \exp\left[-k^2\sigma_Z{}^2(1-\rho_Z)\right] \tag{2.16}$$

ρ_Z, the correlation coefficient, is the normalized correlation function of $Z(\vec{r})$ and is defined as [37]

$$\rho_Z(\vec{\tau}_r) = \frac{R_Z(\vec{\tau}_r) - m^2{}_Z}{\sigma_Z^2} = \frac{R_Z(\vec{\tau}_r) - [E\{Z(\vec{r})\}]^2}{E\{|Z(\vec{r})|^2\} - [E\{Z(\vec{r})\}]^2}$$

$$= \frac{R_Z(\vec{\tau}_r)}{\sigma_Z^2}. \tag{2.17}$$

m_Z is the mean value of $Z(\vec{r})$ which is zero. $\vec{\tau}_r$ is the space difference $\vec{r}_2 - \vec{r}_1$. $R_Z(\vec{\tau}_r)$ is the correlation function of $Z(\vec{r})$ [38]:

$$R_Z(\vec{\tau}_r) = E\{Z(\vec{r})\ Z(\vec{r} + \vec{\tau}_r)\} \tag{2.18}$$

For large values of the separation parameter, $\vec{\tau}_r$, $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$ are independent. But when $\vec{\tau}_r$ is small, $Z(\vec{r}_1)$ and $Z(\vec{r}_2)$ are correlated; when $\vec{\tau}_r = 0$ they will be identical. $\rho_Z(\vec{\tau}_r)$ will decrease monotonously from its maximum value $\rho_Z(0) = 1$ to $\rho_Z(\infty) = 0$. Let the "correlation distance", defined as the distance over which $\rho_Z(\vec{\tau}_r)$ drops to the value e^{-1}, be T_Z. Then a sufficiently general autocorrelation coefficient $\rho_Z(\vec{\tau}_r)$ could be taken to be the function [39]

$$\rho_Z(\vec{\tau}_r) = e^{-\tau_r^2/T_Z^2} \quad . \tag{2.19}$$

This assumption gives

$$M(k,-k) = \exp\,[-k^2\sigma_Z^2(1-\exp- \frac{\tau_r^2}{T_Z^2})] \, . \tag{2.20}$$

The correlation distance T_ϕ for the phase function $e^{jkZ(\vec{r})}$ may be obtained by equating $(1-\exp-\tau_r^2/T_Z^2)$ with the inverse of $k^2\sigma^2$. Thus

$$1 - \exp[- T_\phi^2/T_Z^2] = (\frac{\lambda}{2\pi\sigma_Z})^2 \quad . \tag{2.21}$$

$\rho_Z(\vec{\tau}_r)$ is between -1 and $+1$, therefore, $\exp\,(- T_\phi^2/T_Z^2)$ could be well approximated by its expansion to the second term. Hence, one could write

$$1 - (1 - T_\phi^2/T_Z^2) \simeq (\frac{\lambda}{2\pi\sigma_Z})^2 \tag{2.22}$$

from which it follows that:

$$T_\phi \simeq \frac{\lambda T_Z}{2\pi\sigma_Z} \quad . \tag{2.23}$$

It can be seen that T_ϕ can be made negligibly small by increasing the standard deviation of the surface, σ_Z. Actually, the condition $\sigma_Z >> \lambda$ implies a very rough surface [40] for which most of the diffracted light is diffused and the non-diffused part (in the specular direction) is negligible [41]. Therefore, for a very rough surface, the following approximation can be assumed to be fairly reasonable:

$$M(k,-k) \begin{cases} = 1 \; ; \; \vec{\tau}_r = 0 \\ \simeq 0 \; ; \; \text{otherwise} . \end{cases}$$

We conclude that the autocorrelation function of our diffuse object could be well approximated by the irradiance of the object at $\vec{r}_1$ times the delta function $\delta(\vec{r}_2 - \vec{r}_1)$:

$$R_{UU}(\vec{r}_1,\vec{r}_2) = I(\vec{r}_1)\ \delta(\vec{r}_2-\vec{r}_1).$$

In Appendix B, the input-output relationships of the autocorrelation functions for a linear shift invariant system are briefly studied. Then the relationships are used in the following chapters to obtain the autocorrelation function of the actual transmittance of the hologram. For once the autocorrelation function of the hologram transmittance is obtained, with the help of the input-output relationships of the autocorrelation functions, some important properties of the hologram images can be predicted.

2.3 Non-correlation of Phase and Its Implications

Eqn. (2.6&7) of the last section look like the mutual coherence function for an incoherent object [42]. But it should be kept in mind that we have obtained it after arranging a space averaging process over the mutual coherence function. Nevertheless, a process with such space autocorrelation, as in (2.6), is also termed "incoherent". The analogy of the incoherent process in communication theory is termed "white noise". The terms"incoherent" or "white noise" refer to the assumption of non-correlation of phases of the process, i.e.,

$$E\{e^{j(\phi(\vec{r}_1) - \phi(\vec{r}_2))}\} = \begin{cases} 1\ ;\ \vec{r}_1 = \vec{r}_2 \\ 0\ ;\ \text{otherwise} \end{cases}.$$

The fact that the process of coherent holography remains unchanged despite such unrelated space phases as in the diffuse object field function is not surprising. For the spatial random nature of the phase distribution on the object surface (which is introduced by a stationary diffuser) is time independent; in other words, the time variation of the phase difference of each point of the object and the reference wave remains the same as if there were no diffuser. The properties of temporal and spatial coherence of the source (in the common use definitions) are not altered by the stationary diffuser. Each point of the object remains capable of interfering with the reference wave if it was capable of such an interference before introducing the diffuser. Therefore, the single coherence requirement necessary for the recording of a hologram in coherent light is unchanged [43].

2.4 Probability Distribution of Diffuse Object Field Function

The assumption of the many scatterers and their independence enable the Central Limit Theorem [44] to be applied.

Roughly speaking, the Central Limit Theroem states that whenever a random process can be represented by a linear superposition of a large number M of essentially independent random effects, its statistics will asymptotically approach normal (or Guassian).

Therefore, on the basis of our assumptions, it follows that the diffuse object field function has a normal distribution. The assumption of a uniformly distributed phase between 0 and 2π makes the real and imaginary parts of the field function independent [45]. The mean value of the process is assumed to be zero, since

$$E\,\{A(\vec{r})\,e^{j\phi(\vec{r})}\} = A(r)\,E\,\{e^{j\phi(\vec{r})}\} \doteq A(\vec{r}) \times \frac{1}{2\pi}\int_0^{2\pi} e^{j\phi}.d\phi = 0\,.$$