THE UNIVERSITY OF MANITOBA

A CONJUGACY PROBLEM IN THE THEORY OF FINITE SOLVABLE GROUPS

by

Jack B. Feldman

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfillment

of the Requirements for the Degree

of Doctor of Philosophy

Department of Mathematics

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ABSTRACT

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A finite group G is solvable if and only if G has a finite normal series terminating in the identity in which each factor is nilpotent. The most rapidly descending such series is called the lower nilpotent series of G. For finite solvable G, the last non-trivial term of the lower nilpotent series is called the lower residual of G and is denoted by $L(G)$. Graham Higman has proved that if $L(G)$ is abelian then $L(G)$ is complemented and all complements are conjugate. What happens if $L(G)$ is not abelian? In this dissertation we give some partial answers to this question.

Let G be a finite group and N a normal subgroup of G. A subgroup H of G is a cover of N if $G = HN$; a minimal cover of N is called a ϕ-cover. Subgroups H and K of G are π-conjugate, π a set of primes, if the Sylow π-subgroups of H and K are conjugate in G. The group G is p-restricted, p a prime, if the Sylow p-subgroups of G have nilpotency class ≤ 2; G is π-restricted if G is p-restricted for each p in π.

Let G be a finite solvable π-restricted group. We obtain necessary and sufficient conditions that all ϕ-covers of $L(G)$ be π-conjugate. These

necessary and sufficient conditions enable us to prove the following result: Let G be a finite solvable group which is p-restricted for all primes p. Then all ϕ-covers of L(G) are conjugate if and only if they are p-conjugate for all primes p.

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

All groups occurring in this dissertation are assumed to be finite.

Conjugacy Problems

Let G be a group and $\mathfrak{M}$ a given family of subgroups of G . The conjugacy problem for the family $\mathfrak{M}$ can be stated as follows :

> Are there conditions under which all the members of $\mathfrak{M}$ are conjugate ?

The best known solution of a conjugacy problem is the result due to L. Sylow [12] :

> In a finite group G any two maximal p - subgroups of G are conjugate.

More recent results include those of P. Hall [7] :

> In a solvable group any two maximal π - subgroups are conjugate.

and of R. Carter [1] :

> In a solvable group any two nilpotent self - normalizing subgroups are conjugate.

In the class of solvable groups , these results are special cases of a general theorem in the theory of formations.

Solvable Groups

Recall that a group is solvable if it has a normal series in which each factor is abelian. If G is a solvable group then it has the following properties :

Subgroups and homomorphic images of G are solvable.

A minimal normal subgroup of G is an elementary abelian p - group, for some prime p.

Proofs of these results may be found in [6] , [9] or [11] .

Let π be a set of primes. Denote by π' the set of all primes not in π . In particular, if p is a prime then $\{p\}'$ will be denoted simply by p' . An integer n is called a π - number if every prime divisor of n belongs to π . A group G is called a π - group if $|G|$, the order of G , is a π - number.

A subgroup H of G is called a Sylow π - subgroup (or a Hall π - subgroup) of G if H is a π - group and $|G : H|$, the index of H in G , is a π' - number. Denote by $Syl_{\pi}(G)$ the set of all Sylow π - subgroups of G . The theorem of P. Hall mentioned in the preceding section can then be stated as :

LEMMA 1.1

If G is solvable and π a set of primes, then $Syl_{\pi}(G)$ is not empty and any two members of $Syl_{\pi}(G)$ are conjugate.

Formations

A formation $\mathfrak{F}$ is a class of solvable groups which satisfies the two conditions :

(1) If $G \in \mathfrak{F}$ and $N \triangleleft G$ then $G/N \in \mathfrak{F}$

(2) If $M , N \triangleleft G$ and $G/M , G/N \in \mathfrak{F}$ then $G/M \cap N \in \mathfrak{F}$

A formation $\mathfrak{F}$ is said to be saturated if it also satisfies the condition :

(3) If $G/\Phi(G) \in \mathfrak{F}$ then $G \in \mathfrak{F}$ where $\Phi(G)$ denotes the Frattini subgroup.

If $\mathfrak{F}$ is a formation and G a group then the $\mathfrak{F}$ - residual of G is defined to be the subgroup

$$\mathfrak{F}(G) = \cap \{ M \triangleleft G \mid G/M \in \mathfrak{F} \}$$

From (2) it follows that if N is a normal subgroup of G, then $G/N \in \mathfrak{F}$ if and only if $\mathfrak{F}(G)$ is contained in N.

Let E be a subgroup of the group G . Then E is called an $\mathfrak{F}$ - covering subgroup of G for the formation $\mathfrak{F}$ if

(1) $E \in \mathfrak{F}$

(2) $E \leq H \leq G$ implies $H = \mathfrak{F}(H) E$

The conjugacy results given in the opening section of this chapter can all be obtained , for a solvable group G , from the following theorem of Gaschutz [5], by a suitable choice of the formation $\mathfrak{F}$.

LEMMA 1.2

Let $\mathfrak{F}$ be a saturated formation and G a solvable group. Then G has $\mathfrak{F}$ - covering subgroups, and they are all conjugate.

For example, let $\mathfrak{N}$ denote the class of all nilpotent groups. It follows from the elementary properties of nilpotent groups that $\mathfrak{N}$ is a formation (see [9]). In [13] Wielandt has proved that a group G is nilpotent if $G/\Phi(G)$ is nilpotent. Thus we have the following :

LEMMA 1.3

The class $\mathfrak{N}$ of all nilpotent groups is a saturated formation.

The $\mathfrak{N}$ - covering subgroups of a solvable group G are the nilpotent self - normalizing subgroups of G (see Carter [1], Gaschutz [5]).

Nilpotent Series and Nilpotent Length

A descending sequence of subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_i \trianglerighteq \cdots$$

is called a descending nilpotent series for G if each factor G_i/G_{i+1} is nilpotent. It is immediate that G has a descending nilpotent series terminating with $G_n = 1$, for some n, if and only if G is solvable. For any group G we may define a canonical descending nilpotent series as follows :

$$R_0 = G \,, \quad R_{i+1} = \mathfrak{N}(R_i)$$

where $\mathfrak{N}(R_i)$ is the $\mathfrak{N}$ - residual of R_i. The series

$$G = R_0 \geqslant R_1 \geqslant R_2 \geqslant \cdots \geqslant R_i \geqslant \cdots$$

of characteristic subgroups of G is the most rapidly descending nilpotent series for G in the following sense :

<u>LEMMA 1.4</u>

For any descending nilpotent series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_i \trianglerighteq \cdots$$

we have $R_i \leqslant G_i$ for all i.

<u>Proof.</u> The lemma is true for $i = 0$.

Assume $R_j \leqslant G_j$ and apply induction.

$$G_j/G_{j+1} \geqslant R_j G_{j+1}/G_{j+1} \cong R_j/R_j \cap G_{j+1}$$

Since $G_j/G_{j+1} \in \mathfrak{N}$ we also have $R_j/R_j \cap G_{j+1} \in \mathfrak{N}$

so $R_{j+1} \leqslant R_j \cap G_{j+1}$. Hence $R_{j+1} \leqslant G_{j+1}$.

The series $G = R_0 \geqslant R_1 \geqslant R_2 \geqslant \cdots$

will be called the <u>lower nilpotent series</u> for G. If G is solvable then this series terminates in the identity. The least integer n such that $R_n = 1$, for the solvable group G, is called the nilpotent length of G and is denoted by m(G).

We define $(i)\mathcal{N}$ to be the class of all solvable groups G such that $m(G) \leq i$. In [5] it is proved that

LEMMA 1.5

$(i)\mathcal{N}$ is a saturated formation.

Existence and Conjugacy of Complements of L(G)

If G is solvable and $m(G) = n > o$ then R_{n-1} will be called the lower residual of G and will be denoted by L(G).

Let G be a solvable group with $m(G) = n$.
Let $\mathcal{F} = (n-1)\mathcal{N}$ and let $X = \mathcal{F}(G)$. Since $G/L(G) \in \mathcal{F}$ it follows that $X \leq L(G)$. But G has a descending nilpotent series which passes through X. Hence $L(G) \leq X$ by lemma 4.
Therefore $L(G) = X = \mathcal{F}(G)$.

In 1955 Schenkman [10] proved

LEMMA 1.6

Let G be solvable. If $\mathcal{N}(G)$ is abelian then it is complemented in G and any two complements are conjugate.

In 1956 G. Higman [8] extended this to

LEMMA 1.7

Let G be solvable. If L(G) is abelian then it is complemented in G and any two complements are conjugate.

This was generalized by Carter and Hawkes [3] :

LEMMA 1.8

Let $\mathcal{F}$ be a saturated formation, and G a solvable group. If $\mathcal{F}(G)$ is abelian then it is complemented in G, and the complements are the $\mathcal{F}$- covering subgroups of G. Thus any two complements are conjugate (by lemma 2).

Statement of the Problem and Results

Let G be a group and N a normal subgroup of G. A subgroup H of G is called a cover of N in G if $G = NH$. We call H a minimal cover of N if no proper subgroup of H is a cover of N.

Let M and N be subgroups of a solvable group G. Let π be a set of primes and let $P \in Syl_{\pi}(M)$, and $Q \in Syl_{\pi}(N)$. We say that M and N are π- conjugate in G if and only if P and Q are conjugate in G. This definition is independent of the choice of P and Q, by lemma 1.

The problem to be investigated in this dissertation is the following :

> Are there conditions under which all minimal covers of L(G) will be π- conjugate in G for a given set of primes π ?

The results we obtain are based on the concept of a group which is π- restricted. Let p be a prime. We say that G is p - restricted if the nilpotency class of a Sylow p - subgroup of G is not greater than two. If π is a set of primes then G is called π - restricted if G is p - restricted for each prime p in π.

The problem will now be stated in its specific form. Let π be a set of primes and let G be a π - restricted solvable group. If Y is a minimal cover of $L(G)$, under what conditions will Y be π - conjugate to all other minimal covers of $L(G)$? In chapter 2 we describe conditions that are both necessary and sufficient for Y to have this property. The proofs are given in chapter 4.

The conditions for the π - conjugacy of all minimal covers of $L(G)$ as given in chapter 2 require some results from the theory of p' - groups of automorphisms of abelian p - groups. However, it is possible to give a brief summary of these conditions without reference to this material.

Let G be a solvable group, $X = L(G)$ and Y a minimal cover of X in G. Let p be a prime. A <u>p - mapping</u> from X to Y is a triple $(\theta, N, K/L)$ such that

(1) N is a normal p - subgroup of G which is contained in X' but not contained in Y, and $N/\Phi(N)$ is a chief factor of G

(2) K/L is a complemented p - chief factor of Y

(3) $\theta : N \longrightarrow K/L$ is a Y - homomorphism of N onto K/L, that is, for all elements $g \in N$, $y \in Y$ we have

$$\theta(g^y) = \theta(g)^y$$

If π is a set of primes and G is π - restricted, we define a <u>π - mapping</u> from X to Y as a p - mapping from X to Y for some prime p in π. The main result of this dissertation can be stated as follows :

<u>Theorem</u>

Let π be a set of primes and let G be a π - restricted solvable group. Set $X = L(G)$ and let Y denote some fixed minimal cover of X in G. Then all minimal covers of X in G are π - conjugate if and only if there do not exist any π - mappings from X to Y.

There is an interesting consequence of this theorem :

<u>Corollary</u>

Let G be solvable and p - restricted for all primes p. Then all minimal covers of L(G) are conjugate in G if and only if they are all p - conjugate for all primes p.

<u>Note</u> : The curved brackets $\{ \quad \}$ will be used to denote sets , and the straight line brackets $\langle \quad \rangle$ will be used to denote subgroups.

Notation and Terminology

The capitals A , B , C , ... will always denote groups and subgroups.

$N \leqslant G$	N is a subgroup of G
$N < G$	N is a proper subgroup of G
$N \lhd G$	N is normal in G
N char G	N is characteristic in G
$\lvert G \rvert$	the order of G
$\lvert G : H \rvert$	the index of H in G
Z(G)	the center of G
$[g, h]$	$g^{-1}h^{-1}gh = g^{-1}g^{h}$

Let A and B be contained in G

$[A, B]$	$\langle [a, b] \mid a \in A, b \in B \rangle$
$C_B(A)$	the centralizer of A in B
$N_B(A)$	the normalizer of A in B
G'	$[G, G]$ the commutator subgroup
$O_{\pi}(G)$	maximal normal π- subgroup
$\Phi(G)$	Frattini subgroup
F(G)	Fitting subgroup of G, the maximal normal nilpotent subgroup
$N \cdot\lhd G$	N is a minimal normal subgroup of G

Chapter 2

CONDITIONS FOR π - CONJUGACY

The material in this chapter is applied to give an alternative definition of a p - mapping. This definition makes direct use of the fact that the group G is p - restricted.

Abelian p - groups Acted on by p' - groups

Let G and A be groups. We define an action of A on G as a homomorphism σ of A into the automorphism group of G.

$$\sigma : A \longrightarrow \mathrm{Aut}(G)$$

Let $\alpha \in A$ and $g \in G$. Then we write

$$g^{\alpha} \equiv g^{\sigma(\alpha)}$$

A subgroup H of G is called A - invariant if $H^{\alpha} = H$ for all elements $\alpha \in A$. If N is a normal A - invariant subgroup of G then A also acts on G/N by :

$$(gN)^{\alpha} = g^{\alpha}N$$

If A acts on groups G and H, then a homomorphism

$$\theta : G \longrightarrow H$$

is an A - homomorphism if for all $g \in G$, $\alpha \in A$

$$(g^{\alpha})^{\theta} = (g^{\theta})^{\alpha}$$

In this case we write $\theta : G \xrightarrow[A]{} H$

An isomorphism $\theta : G \longrightarrow H$ which is an A - homomorphism will be called an A - isomorphism. If G and H are A - isomorphic we will write $G \underset{A}{\cong} H$

The group G is A - indecomposable if G cannot be written as a direct product of two nontrivial A - invariant subgroups. The group G is A - basic if it cannot be written as the direct product of two nontrivial subgroups, at least one of which is A - invariant.

We are interested here in the action of a p' - group A on an abelian p - group P. Throughout the remainder of this section A and P will denote such groups.

LEMMA 2.1 [6 , p. 69]

If E is an A - invariant direct factor of P then

$$P = E \times F$$

where F is also A - invariant.

This lemma states that , with the fixed hypothesis that A is a p' - group and P an abelian p - group, the concepts of A - indecomposable and A - basic are equivalent.

COROLLARY 2.2

It is possible to write P as the direct product of subgroups, each of which is A - basic.

LEMMA 2.3 [6 , p. 176]

If P is A - basic then P is homocyclic, that is, a direct product of cyclic subgroups all of the same order.

We define two A - invariant subgroups of P :

$$C_P(A) = \{g \in P \mid g^{\alpha} = g \text{ for all } \alpha \in A\}$$

$$[P, A] = \langle g^{-1} g^{\alpha} \mid g \in P, \alpha \in A \rangle$$

LEMMA 2.4 [6 , p. 177]

$$P = C_P(A) \times [P, A]$$

We define a characteristic, and hence A - invariant elementary abelian subgroup of P :

$$\Omega(P) = \{g \in P \mid g^p = 1\}$$

LEMMA 2.5

If P is A - basic then $\Omega(P)$ is also A - basic.

Proof. Let P be A - basic. Then P is homocyclic of exponent p^n, for some integer n. If P has exponent p then $P = \Omega(P)$. Thus we may assume P is not elementary abelian. Suppose $\Omega(P)$ is not A - basic. Then

$$\Omega(P) = M \times N$$

where M and N are nontrivial and A - invariant. It can be assumed that N is A -basic.

Set $\overline{P} = P/N$. Since P is homocyclic and not elementary abelian, N is contained in $\Phi(P)$.

This implies that a minimal set of generators of P has the same number of elements as a minimal set of generators of $\overline{P}$. To state this in simple form, we say that P and $\overline{P}$ have the same number of generators. The generators of P are of order p^n and N is nontrivial, so some of the generators of $\overline{P}$ will be of order p^{n-1}. Clearly P and $\Omega(P)$ have the same number of generators. But M is nontrivial, so N has less generators than P. Therefore $\overline{P}$ will have some generators of order p^n. Thus $\overline{P}$ is not homocyclic.

Since N is A - invariant, $\overline{P}$ is acted on by A. But $\overline{P}$ cannot be A - basic since it is not homocyclic. Any direct factor of $\overline{P}$ which is A - basic will be of exponent either p^n or p^{n-1}. Thus we have

$$\overline{P} = \overline{K} \times \overline{L}$$

$\overline{K}$ is A - invariant of exponent p^n

$\overline{L}$ is A - invariant of exponent p^{n-1}

Let K , L be the inverse images in P of $\overline{K}$ and $\overline{L}$. Since K is A - invariant and $\Phi(K)$ char K, it follows that $\Phi(K)$ is A - invariant. Thus $\Phi(K) \cap N$ is also A - invariant. Since N is A - basic and elementary abelian, we must have either $\Phi(K) \cap N = 1$ or $N \leq \Phi(K)$. But if $N \leq \Phi(K)$ then K and $\overline{K}$ have the same number of generators, so $\overline{K}$ cannot be homocyclic of exponent p^n. Therefore $\Phi(K) \cap N = 1$ and N is a direct factor of K. By lemma 1, $K = N \times F$ where F is A - invariant.

Now $P = F L$ and $|P| = |F||L|$. Thus $F \cap L = 1$. We conclude that $P = F \times L$ since P is abelian. Since F and L are A - invariant, P is not A - basic, which is a contradiction.

LEMMA 2.6

If P is homocyclic then $P/\Phi(P) \underset{A}{\cong} \Omega(P)$

Proof. Let P have exponent p^n. We can define a homomorphism $\theta : P \longrightarrow \Omega(P)$ by the equation

$$g^{\theta} = g^{p^{n-1}} \qquad \text{for all } g \in P$$

This is clearly an A - homomorphism of P onto $\Omega(P)$. The kernel consists of all elements of order not greater than p^{n-1}, that is, the Frattini subgroup $\Phi(P)$. The result now follows from the basic homomorphism theorems for groups with operators.

π - primary and π - secondary Minimal Covers of $L(G)$.

Let π be a set of primes. Throughout this section G will be a solvable π - restricted group. Set $X = L(G)$ and let Y be a fixed minimal cover of X in G.

Let p be some prime in π. Since X is nilpotent it has a unique Sylow p - subgroup. Let M denote this subgroup. We will write $M = \mathrm{Syl}_p(X)$ to indicate that it is unique.

We have M char X char G. Therefore M char G. Similarly M' char G. Since p belongs to π, G is p - restricted.

LEMMA 2.7

Let P be a Sylow p - subgroup of G. Then M' is contained in $Z(P)$. Hence P is contained in $C_G(M')$.

Proof. Since M is normal in G, M is contained in P. This implies M' is contained in P'. But G is p - restricted so P has class two at most. Hence P' is contained in $Z(P)$ and M' is also.

Since M' is normal in G, we can say that G induces a group of automorphisms on M' by conjugation, or by our previous terminology, we can say that G has an action on M' which is defined by conjugation in G. An element g of G induces the identity automorphism on M' if and only if g is contained in $C_G(M')$. Let $A_G(M')$ denote the group of automorphisms induced on M' by G. This is the image of G in the homomorphism of G into $Aut(M')$ which defines the action of G on M'. Now $C_G(M')$ is a normal subgroup of G since M' is normal in G. Thus we have the isomorphism

$$A_G(M') \cong G/C_G(M')$$

By lemma 7, P is contained in $C_G(M')$, hence $A_G(M')$ is a p' - group.

LEMMA 2.8

M' is a characteristic abelian p - subgroup of G, and G induces a p' - group of automorphisms on M'.

Since M' is contained in Z(M) and X is nilpotent, it follows that M' is contained in Z(X). Thus X is contained in $C_G(M')$. It then follows from the fact that $G = XY$ that Y induces the same group of automorphisms on M' as G does.

COROLLARY 2.9

Y induces a p' - group of automorphisms on M'. Furthermore if N is a subgroup of M', then N is normal in G if and only if N is Y - invariant.

We are now in a position to give a new definition of the concept of a p - mapping. Let G be a solvable p - restricted group. Set $X = L(G)$ and let Y be a minimal cover of X in G. We define a p - mapping from X to Y to be a triple (θ , N , K/L) such that

(1) N is a normal p - subgroup of G which is contained in X' but not contained in Y, and N is Y - basic

(2) K/L is a complemented p - chief factor of Y

(3) $\theta : N \longrightarrow K/L$ is a Y - homomorphism of N onto K/L.

Generally we will simply say that

$$\Theta : N \xrightarrow[Y]{} K/L$$

is a p - mapping.

To establish the equivalence of the two definitions we show that the conditions on N are the same :

(a) N is a normal p - subgroup of G which is contained in X', and $N/\Phi(N)$ is a chief factor of G

(b) N is a normal p - subgroup of G which is contained in X', and N is Y - basic

In both cases Y induces a p' - group of automorphisms on N, and N is an abelian p - group.

Assume (a) holds. By corollary 2, N is a direct product of Y - basic subgroups. Each Y - basic direct factor of N is normal in G, by corollary 9. Since $N/\Phi(N)$ is a chief factor of G, N must be Y - basic, so (b) holds.

Assume (b) holds. Since $\Phi(N)$ is characteristic in N, it is normal in G. By lemma 5, $\Omega(N)$ is Y - basic, so from lemma 6 we get that $N/\Phi(N)$ is also Y - basic. Since $N/\Phi(N)$ is elementary abelian, it must be a chief factor of G. Thus (a) holds.

Let G be a Π - restricted solvable group. As in chapter 1, a Π - mapping is defined to be a p - mapping for some prime p in Π. We say that Y is a <u>Π - primary</u> minimal cover of X if there do not exist any Π - mappings from X to Y. If there do exist Π - mappings, then