TOPOLOGICAL EXTENSION PROPERTIES

BY

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ABSTRACT

The fact that a topological space is compact and Hausdorff if and only if it can be embedded as a closed subset of some topological product of the closed unit interval I, has been known since the late 1930s when the Stone-Čech compactification βX of an arbitrary completely regular, Hausdorff space X was constructed by M. Stone and E. Čech. This led to the more general notion of E-compactness introduced by Engelking and Mrówka. If E is a topological space, then a space X is said to be E-compact if X can be embedded as a closed subset of some topological product of copies of E. A space X is called E-completely regular if X can be embedded as a subspace of some topological product of copies of E. Engelking and Mrówka constructed the maximal E-compact extension $\beta_E X$, for any E-completely regular space X. In the case E=I, the classes of E-completely regular and E-compact spaces are precisely the classes of completely regular, Hausdorff and compact, Hausdorff spaces respectively. In addition, in this case $\beta_E X = \beta X$. Going somewhat further, Herrlich introduced the class of Γ-compact spaces for a given class of spaces Γ, and obtained a maximal Γ-extension of a Γ-completely regular space. Topological extension properties have most recently been investigated in great detail by R.G. Woods. His work in that area has provided much of the motivation for this study.

In this work we are concerned with extension properties which consist only of completely regular, Hausdorff spaces. Such properties relate directly to the class of compact, Hausdorff spaces. In particular, if Φ is a topological extension property and X is a completely regular, Hausdorff space, then there is an extension ΦX such that $X \subseteq \Phi X \subseteq \beta X$. Furthermore, ΦX satisfies many of the conditions satisfied by βX. One of the objects of this work is to examine the relationship between ΦX and βX.

The first chapter investigates pseudocompactness properties related to extension properties. This chapter culminates in a characterization of N-compact spaces which gives a partial answer to the question of which realcompact, zero-dimensional spaces are N-compact.

In the second chapter we focus on one particular extension property, the class of ω_0-bounded spaces. We obtain partial results toward characterizing the pseudocompactness property related to the class of ω_0-bounded spaces.

A well known theorem of Glicksberg states that for a pair of infinite spaces X and Y, $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact. In Chapter 3 this equality is examined for arbitrary extension properties. It is shown that for a large class of extension properties, the Glicksberg theorem remains true with β replaced by Φ.

In Chapter 4 we apply a number of the results of Chapter 3 in order to obtain a characterization of the pseudocompactness of a zero-dimensional topological product.

In the final chapter, which is essentially unrelated to the first four chapters, we obtain lattice theoretic characterizations of those lattices that are isomorphic to the lattice of zero-sets of certain topological spaces.

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Sam Broverman, 1976.

CHAPTER 0

PRELIMINARIES

Our topological notation is the standard notation established in the literature. For background material on compactifications the reader is referred to the Gillman and Jerison text [17]. All spaces will be assumed to be completely regular and Hausdorff unless otherwise mentioned. By a map we will mean a continuous function.

If f is a continuous real-valued function on a space X, the set $\{x \in X: f(x)=0\}$ is called the <u>zero-set</u> of f and $Z(X)$ denotes the family of all zero-sets of X. The <u>Stone-Čech compactification</u> of a completely regular, Hausdorff space X is denoted by βX, and is characterized as the compactification of X to which all bounded, continuous, real-valued functions on X may be continuously extended. For a space X, the points of $\beta X - X$ are the free ultrafilters on $Z(X)$ (or z-ultrafilters on X) and will be regarded both as free z-ultrafilters on X and as points of $\beta X - X$ without specific mention (the reader is referred to Chapter 6 of [17] where βX is constructed by means of z-ultrafilters on X).

A space is called zero-dimensional (0-dimensional) if it has a basis of closed-and-open (henceforth called clopen) sets. The class of 0-dimensional spaces will be denoted by ζ_0. The maximal 0-dimensional compactification of a 0-dimensional space X is denoted by $\beta_0 X$, and is characterized as the 0-dimensional compactification of X to which all continuous {0,1}-valued functions on X may be continuously extended (where {0,1} denotes the two-point discrete space). The reader is referred to [1] for an investigation of $\beta_0 X$. For a space X, B(X) denotes the family of all clopen subsets of X. If X is 0-dimensional, the points of $\beta_0 X-X$ are the free ultrafilters of B(X) (or clopen ultrafilters on X) and will be regarded both as free clopen ultrafilters and points of $\beta_0 X-X$ without specific mention. The reader may be more familiar with $\beta_0 X$ as the Stone space of the Boolean algebra B(X).

If X is a space, E(X) will denote the projective cover of X (the reader is referred to [18] and [38] for an investigation of E(X)). E(X) is an extremally disconnected space such that there is a perfect, irreducible map $k:E(X)\to X$ (a map is perfect if it is closed, continuous, onto and preimages of points are compact, and a map is irreducible if no proper closed subspace of the domain is mapped onto the range). If X is compact then E(X) is the Stone space of the Boolean algebra of regular closed subsets of X, and the map $k:E(X)\to X$ takes an ultrafilter of regular closed sets to its intersection. Then if X is an arbitrary completely regular, Hausdorff space, $E(X)=k^{\leftarrow}(X)$ where $k:E(\beta X)\to\beta X$ as above.

The set (and discrete space) of natural numbers is denoted by N, the space of real numbers by R and the closed unit interval by I. The cardinality of a set S is denoted by $|S|$. We employ the standard conventions from set theory that an ordinal is thought of as its set of predecessors, and a cardinal is an initial ordinal. An <u>ordinal</u>

space is an ordinal with the order topology. If α is an ordinal ω_α denotes the α-th uncountable cardinal. If m is a cardinal number then m^+ denotes its succesor. If τ is a family of sets, then $\cup\tau=\cup\{U:U\in\tau\}$ and $\cap\tau=\cap\{U:U\in\tau\}$.

The main objects of study in this thesis are topological extension properties. We now define them and state some basic facts about them.

0.1 Definition: A topological extension property Φ is a class of spaces which satisfies the following conditions:

i) if $X\in\Phi$ and X is homeomorphic to Y then $Y\in\Phi$,

ii) if $X\in\Phi$ and C is a closed subset of X, then $C\in\Phi$,

iii) if $\{X_\alpha:\alpha\in A\}\subseteq\Phi$ then $\Pi\{X_\alpha:\alpha\in A\}\in\Phi$ (Π is cartesian product),

iv) $I\in\Phi$.

A 0-dimensional topological extension property Φ is a class of 0-dimensional spaces satisfying i),ii), and iii) above and containing the space $\{0,1\}$ (it is clear that if Φ contains a space with more than one point, then the last condition follows from ii)).

The reader may be more familiar with extension properties and 0-dimensional extension properties as epi-reflective left-fitting subcategories of the category of completely regular, Hausdorff spaces or 0-dimensional, Hausdorff spaces respectively. Chapter 10 of [40] gives an introduction to extension properties from a categorical point of view.

If $f: X \to Y$ is a continuous map, then f admits a continuous extension $\beta(f): \beta X \to \beta Y$ such that $\beta(f)|X = f$. This is shown in Theorem 6.4 of [17]. A similar proof to the proof of this fact shows that if X and Y are 0-dimensional spaces and $f: X \to Y$, then f admits a continuous extension $\beta_0(f): \beta_0 X \to \beta_0 Y$. These maps will be used often and will be referred to as Stone extensions of the map f.

In [32], the concept of E-compactness is discussed. If E is a (not necessarily completely regular and Hausdorff) space, then a space X is called E-compact if X can be embedded as a closed subset of E^m for some cardinal m. If E is completely regular and Hausdorff, and I can be embedded as a closed subset of E^m, then the class of E-compact spaces forms an extension property. The classes of I-compact spaces, R-compact spaces and $\{0,1\}$-compact spaces are equal to the extension properties consisting of the compact Hausdorff, realcompact, and compact 0-dimensional spaces respectively. In [16] and [43], examples are given of extension properties not contained in the class of E-compact spaces for any space E.

Our basic references for extension properties are [23], [24], [32] and [43]. The following result can be found in [23].

0.2 Theorem: Let Γ be a class of (0-dimensional) spaces. Let $K\Gamma$ ($K_0\Gamma$) denote all (0-dimensional) spaces which are homeomorphic to a closed subset of a topological product of members of $\Gamma \cup \{I\}$ ($\Gamma \cup \{\{0,1\}\}$). Then $K\Gamma$ ($K_0\Gamma$) is the smallest (0-dimensional) extension property containing Γ.

The following is Theorem 1 of [24].

0.3 Theorem: Let Φ be a (0-dimensional) extension property and X a (0-dimensional) space. Then there exists a space ΦX such that

i) $X \subseteq \Phi X \subseteq \beta X$ ($X \subseteq \Phi X \subseteq \beta_0 X$),

ii) $\Phi X \in \Phi$,

iii) if $f: X \to Y$ is a continuous map and $Y \in \Phi$ then $(\beta(f))(\Phi X) \subseteq Y$ ($(\beta_0(f))(\Phi X) \subseteq Y$) where $\beta(f)$ denotes the Stone extension of the map f (i.e. $\beta(f): \beta X \to \beta Y$) and similarly for $\beta_0(f)$.

It follows that ΦX must be unique up to homeomorphism with respect to conditions i) - iii). That is, if ΦX and ΨX were extensions of X both satisfying i) - iii) then there would exist a homeomorphism from ΦX onto ΨX which keeps X pointwise fixed. If Φ is an extension property, we will refer to ΦX as the maximal Φ-extension (or just Φ-extension) of X. The class of all (0-dimensional) extension properties will be denoted by $\mathcal{H}$ ($\mathcal{H}_0$).

0.4 Theorem: If $\Phi, \Psi \in \mathcal{H}(\mathcal{H}_0)$ and $\Phi \subseteq \Psi$ and X is a (0-dimensional) space, then $\Psi X \subseteq \Phi X$.

Proof: Let $T = \Phi(\Psi X)$. Let $f: X \to Y$ be a map where $Y \in \Phi$. Since $Y \in \Psi$, $(\beta(f))(\Psi X) \subseteq Y$. Thus $\beta(f)|\Psi X: \Psi X \to Y$. Since $Y \in \Phi$, $(\beta(f))(\Phi(\Psi X)) \subseteq Y$. Thus $(\beta(f))(T) \subseteq Y$ and T satisfies i), ii) and iii) of Theorem 0.3 above. By the uniqueness of ΦX we have that $T = \Phi X$. Thus $\Psi X \subseteq T = \Phi X$. ¤

The proof of the following lemma is left to the reader.

0.5 Lemma: Let Γ be a class of (0-dimensional) spaces and suppose that $\Phi=K\Gamma$ $(K_0\Gamma)$. Let X be a (0-dimensional) space and $p\in\beta X-X$ $(\beta_0 X-X)$. Then every continuous map f: $X\to Y$ where $Y\in\Phi$, admits a continuous extension to the point p if and only if every continuous f: $X\to C$ where $C\in\Gamma$, admits a continuous extension to p.

The following is Theorem 4.1 of [32].

0.6 Theorem: Let $\Phi\in\mathcal{H}(\mathcal{H}_0)$ and let X be a (0-dimensional) space. Then a point $p\in\beta X-X$ $(\beta_0 X-X)$ is in ΦX if and only if given a space $Y\in\Phi$ and a continuous map f: $X\to Y$, f admits a continuous extension to the point p.

The following is an immediate corollary to 0.5 and 0.6.

0.7 Corollary: Let $\Phi\in\mathcal{H}(\mathcal{H}_0)$ such that $\Phi=K\Gamma$ $(K_0\Gamma)$. Suppose $X\subseteqq T\subseteqq\beta X(\beta_0 X)$ and every map f: $X\to Y$ where $Y\in\Gamma$ admits a continuous extension to T. Then $T\subseteqq\Phi X$.

The following two results are found in Proposition 2 of [24].

0.8 Theorem: Let $\Phi\in\mathcal{H}$ $(\mathcal{H}_0)$ and let X be a (0-dimensional) space. Then $\Phi X=\cap\{T\subseteqq\beta X\ (\beta_0 X): X\subseteqq T \text{ and } T\in\Phi\}$.

0.9 Theorem: Let $\Phi\in\mathcal{H}$ $(\mathcal{H}_0)$ and let X be a (0-dimensional) space. If f: $X\to Y$ is a perfect map and $Y\in\Phi$, then $X\in\Phi$.

0.10 Definition: Let $\Phi\in\mathcal{H}$ $(\mathcal{H}_0)$. Then Φ' is the class of all (0-dimensional) spaces which satisfy the equality $\Phi X=\beta X$ $(\Phi X=\beta_0 X)$ and is called the pseudocompactness property of Φ. The class of all (0-dimensional) pseudocompactness properties will be denoted by $\mathcal{H}'$ ($\mathcal{H}_0'$).

For example, if Φ is the class of realcompact spaces, Φ' is the class of pseudocompact spaces. The following is Theorem 2.1 of [43].

0.11 Theorem: Let $\Phi, \Psi \in \mathcal{H}$ ($\mathcal{H}_0$). Then

i) if $X \in \Phi'$ and X is dense in T then $T \in \Phi'$.

ii) $X \in \Phi \cap \Phi'$ if and only if X is compact.

iii) if $X \in \Phi'$ and $f: X \to Y$ is continuous and onto, then $Y \in \Phi'$.

iv) if $\Phi \subseteq \Psi$ then $\Psi' \subseteq \Phi'$.

Finally, we note once again that a point of βX ($\beta_0 X$) will also be regarded as an ultrafilter of zero-sets (clopen sets) on X. If $p \in \beta X$ ($\beta_0 X$), we will use without explicit mention the fact that a zero-set (clopen set) Z is a member of the z-ultrafilter (clopen ultrafilter) p if and only if $p \in cl_{\beta X}(Z)$ ($p \in cl_{\beta_0 X}(Z)$).

CHAPTER 1

PSEUDOCOMPACTNESS PROPERTIES

In this chapter we develop the theory of pseudocompactness properties which was first introduced by R.G. Woods in [43]. Several of the results of this chapter will be used in subsequent chapters. However, the theory is developed in this chapter to lead to a characterization of those realcompact, locally compact, 0-dimensional spaces that are N-compact (i.e. homeomorphic to a closed subset of some power of the space N) in terms of a density condition on the outgrowth $(\beta X-X)$ of the Stone-Čech compactification. In addition, we answer in the negative, by means of counterexample, two questions raised in [43]. Specifically, in 2.10 of [43] it is shown that for $\Phi \in \mathcal{H}_0$, either Φ is equal to the class of compact 0-dimensional spaces or Φ' does not properly contain the class of pseudocompact 0-dimensional spaces. We show that the corresponding statement for $\Phi \in \mathcal{H}$ is false. Also, an extension property Φ is exhibited with a space $X \in \Phi'$ such that $E(X) \notin \Phi'$ (in fact $E(X) \in \Phi$). This contrasts the situation in which Φ is the class of realcompact spaces and Φ' is the class of "ordinary" pseudocompact spaces.

As pointed out in Proposition 3.13 of [43], there is a partial ordering by inclusion on $\mathcal{H}$ and $\mathcal{H}_0$ which behaves very much like a complete lattice ordering (the difference being that neither $\mathcal{H}$ nor $\mathcal{H}_0$ is a set).

1.1 <u>Definition</u>: Let A be a set and let $\{\Phi_\alpha : \alpha \in A\} \subseteq \mathcal{H}$ $(\mathcal{H}_0)$. Define $\Lambda\{\Phi_\alpha : \alpha \in A\} = \cap\{\Phi_\alpha : \alpha \in A\}$. Also define $V\{\Phi_\alpha : \alpha \in A\} = K(\cup\{\Phi_\alpha : \alpha \in A\})$ $(K_0(\cup\{\Phi_\alpha : \alpha \in A\}))$.

Given a space X there is a close connection between the join (V) operation on a set of extension properties and the corresponding extension of X.

1.2 <u>Proposition</u>: Let X be a space, A a set, and $\{\Phi_\alpha : \alpha \in A\} \subseteq \mathcal{H}$. Let $\Psi = V\{\Phi_\alpha : \alpha \in A\}$. Then $\Psi X = \cap\{\Phi_\alpha X : \alpha \in A\}$.

Proof: Since $\Phi_\alpha \subseteq \Psi$ for all $\alpha \in A$, it follows from 0.4 that $\Psi X \subseteq \cap\{\Phi_\alpha X : \alpha \in A\}$. Since $\Psi = K(\cup\{\Phi_\alpha : \alpha \in A\})$ by definition, by 0.5 and 0.6 it is sufficient to show that for every $\alpha \in A$ and $Y \in \Phi_\alpha$ and continuous map $f: X \to Y$, $(\beta(f))(\cap\{\Phi_\alpha X : \alpha \in A\}) \subseteq Y$ in order to show that $\cap\{\Phi_\alpha X : \alpha \in A\} \subseteq \Psi X$. But since $(\beta(f))(\Phi_\alpha X) \subseteq \Phi_\alpha Y = Y$ and $\cap\{\Phi_\alpha X : \alpha \in A\} \subseteq \Phi_\alpha X$, we have $(\beta(f))(\cap\{\Phi_\alpha X : \alpha \in A\}) \subseteq Y$ and hence $\Psi X = \cap\{\Phi_\alpha X : \alpha \in A\}$.¤

The problem of how to formulate the extension of X corresponding to the meet (Λ) of extension properties in terms of the individual extensions, is left open to the reader. However, as the following example shows, if $\Phi = \Psi \Lambda \Theta$, Φ, Ψ, $\Theta \in \mathcal{H}$, then ΦX is not necessarily the union of ΨX and ΘX.

1.3 <u>Example</u>: If m is an infinite cardinal number, a space X is called m-bounded if every subset of X of cardinality at most m is contained in a compact subset of X. In [41], it is shown that for a given infinite cardinal m, the class of m-bounded spaces is an extension property and that the m-bounded extension of a space X is the set of points in βX that are in the closure of some subset of X of cardinality at most m (in Chapter 2 we examine m-bounded spaces in detail). If we let Ψ be the class of ω_0-bounded spaces and Θ the class of realcompact spaces, then $\Psi\Lambda\Theta$ is the class of compact spaces (as an ω_0-bounded space is countably compact hence pseudocompact, and by 5H2 of [17] , a pseudocompact, realcompact space is compact). But if X is the discrete space of cardinality ω_1, then $X \in \Theta$ and hence $\Theta X = X$. Also $\Psi X \subsetneqq \beta X$ (as there are points in βX not in the closure of a countable subset of X, these are the uniform ultrafilters on X). Thus $\Theta X \cup \Psi X = X \cup \Psi X = \Psi X \subsetneqq \beta X = \Phi X$ (clearly if Φ is the class of compact spaces $\Phi X = \beta X$ for any space X). This concludes the example.

Proposition 1.2 above can easily be seen to hold with $\mathcal{H}$ replaced by $\mathcal{H}_0$. The close relationship between $\mathcal{H}$ and $\mathcal{H}_0$ leads to interesting and useful results. Let X be a 0-dimensional space and $i: X \to \beta_0 X$ be the identity map. Then there is a map $\beta(i):\beta X \to \beta_0 X$ the Stone extension of i. In the following theorem, a point $p \in \beta X$ ($\beta_0 X$) will also be regarded as an ultrafilter of zero-sets (clopen sets) on X as mentioned in Chapter 0.

1.4 <u>Theorem</u>: Let X be 0-dimensional and $\beta(i): \beta X \to \beta_0 X$ as defined above. Then

i) if $p \in \beta X$, then$(\beta(i))(p) = p \cap B(X)$; and

ii) any map from βX to a 0-dimensional space factors through $\beta_0 X$ by means of $\beta(i)$.

Proof: i) Let $U \in p \cap B(X)$. Then $p \in cl_{\beta X}(U)$. Hence $(\beta(i))(p) \in cl_{\beta_0 X}((\beta(i))(U)) = cl_{\beta_0 X}(U)$. Thus $U \in (\beta(i))(p)$. Conversely, if $U \in (\beta(i))(p)$, then $(\beta(i))(p) \in cl_{\beta_0 X}(U)$. Thus $p \in cl_{\beta X}(U)$. For if $p \in cl_{\beta X}(X-U)$ then $(\beta(i))(p) \in cl_{\beta_0 X}((\beta(i))(X-U)) = cl_{\beta_0 X}(X-U)$. But this is impossible since U and $X-U$ have disjoint closures in $\beta_0 X$. Note that i) implies that if $q \in \beta_0 X$ then $(\beta(i))^{\leftarrow}(q) = \{p \in \beta X : p \cap B(X) = q\}$.

ii) Let $f:\beta X \to Y$ where Y is 0-dimensional. Then $f = \beta(f|X)$. However, there is also an extension $\beta_0(f|X)$: $\beta_0 X \to Y$. Since $(f|X)\cdot i = f|X$, and both $(\beta_0(f|X))\cdot(\beta(i))$ and $\beta(f|X) = f$ agree on X, we must have $f = (\beta_0(f|X))\cdot(\beta(i))$. In particular, any such map f is constant on all preimages of points of $\beta_0 X$ under the map $\beta(i)$.¤

1.4 ii) above can be found in [1] in a somewhat different form. If X is compact then X^* is the quotient space of X by a map k which collapses the connected components of X. It is shown in [1] that any map from X to a 0-dimensional space factors through k. Noticing that if X is 0-dimensional that $(\beta X)^*$ is $\beta_0 X$ and k is $\beta(i)$, we have 1.4 ii).

1.5 <u>Definition:</u> Let $\Phi \in \mathcal{H}_0$. Then $(I \times \Phi) = K\Phi$.

Since $\Phi \in \mathcal{H}_0$, it is clear that $X \in K\Phi$ if and only if X is homeomorphic to a closed subset of $I^m \times P_0$, where m is some cardinal number and $P_0 \in \Phi$ (hence the notation $I \times \Phi$).

1.6 Theorem: Let $\Phi \in \mathcal{H}_0$ and let X be a 0-dimensional space. Then $(I \times \Phi)X = (\beta(i))^{\leftarrow}(\Phi X)$.

Proof: Let $f: X \to P$ where $P \in \Phi$. Then, as in Theorem 1.4 ii), $\beta(f) = (\beta_0(f)) \cdot (\beta(i))$. Since $(\beta_0(f))(\Phi X) \subseteq P$, we have that $(\beta(f))((\beta(i))^{\leftarrow}(\Phi X)) = (\beta_0(f))(\Phi X) \subseteq P$. Clearly every map from X to I extends to $(\beta(i))^{\leftarrow}(\Phi X)$ (in fact to all of βX), thus by 0.5 and 0.6 $(\beta(i))^{\leftarrow}(\Phi X) \subseteq (I \times \Phi)X$. In addition, since $\beta(i)$ is a perfect map and $\Phi X \in \Phi \subseteq I \times \Phi$, $(\beta(i))^{\leftarrow}(\Phi X) \in I \times \Phi$. Thus, $(I \times \Phi)X \subseteq (\beta(i))^{\leftarrow}(\Phi X)$ by 0.8. Hence $(I \times \Phi)X = (\beta(i))^{\leftarrow}(\Phi X)$.¤

1.7 Definition: The operation o: $\mathcal{H} \to \mathcal{H}_0$ is defined by $o(\Phi) = \Phi \cap \zeta_0$. For notational convenience let $o(\Phi) = \Phi_0$.

The following theorem shows that the operation "o" defined above is onto (i.e. if $\Psi \in \mathcal{H}_0$ then there is a $\Phi \in \mathcal{H}$ such that $\Psi = o(\Phi) = \Phi_0$).

1.8 Theorem: Let $\Psi \in \mathcal{H}_0$. Then $(I \times \Psi)_0$ $(= o(I \times \Psi)) = \Psi$.

Proof: Since $\Psi \subseteq I \times \Psi$, we must have that $\Psi \subseteq (I \times \Psi) \cap \zeta_0 = (I \times \Psi)_0$. Let $X \in (I \times \Psi)_0$. Thus $X \in I \times \Psi$, and $(I \times \Psi)X = X = (\beta(i))^{\leftarrow}(\Psi X)$ by Theorem 1.6, since X is 0-dimensional. Thus, $\Psi X = X$ and hence, $X \in \Psi$.¤

It is clear that if $\Phi \in \mathcal{H}$ then $\Phi \supseteq I \times \Phi_0$. If Φ is the class of spaces such that $X \in \Phi$ if and only if the connected components of X are compact, then it is easy to verify that $\Phi \in \mathcal{H}$. Since if $\Psi \in \mathcal{H}_0$, every member of $I \times \Psi$ has compact components, an obvious conjecture is that $\Phi = I \times \Psi$ for some $\Psi \in \mathcal{H}_0$. We show this conjecture is false.

1.9 Lemma: A space X is 0-dimensional if and only if the connected component in βX of any $x \in X$ is $\{x\}$.

Proof: Let X be 0-dimensional and $x \in X$. Recall that in a compact space the connected component of a point is equal to its quasi-component (i.e. the intersection of all clopen sets containing the point - see Theorem 16.15 of [17]). Let $p \in \beta X$, $p \neq x$. Then there is a neighborhood U in βX, of x such that $p \notin cl_{\beta X}(U)$. Since $X \cap U$ is a neighborhood of x in X, and X is 0-dimensional, there is a clopen set V in X such that $x \in V \subseteq X \cap U$. But then $cl_{\beta X}(V) \subseteq cl_{\beta X}(U)$, hence $p \notin cl_{\beta X}(V)$. Since $x \in cl_{\beta X}(V)$ and $cl_{\beta X}(V)$ is clopen, p is not in the quasi-component of x in βX. Thus the connected component of x in βX is $\{x\}$. This proves the necessity.

To see the sufficiency, let $x \in X$, $x \in U$, U open in X. Then $U = X \cap W$ where W is open in βX. Since $\{x\}$ is the intersection of the clopen sets in βX containing x, and $\beta X - W$ is compact, there must be a clopen set V in βX such that $x \in V \subseteq W$. Thus $x \in V \cap X \subseteq W \cap X = U$. and since $V \cap X$ is clopen in X, X must be 0-dimensional.¤

Let X be a totally disconnected, non-0-dimensional space such that any two points of X can be separated by clopen sets (e.g. the space 16L of [17]). Since X is not 0-dimensional, and no two points of X are in the same quasi-component of βX (as they can be separated by clopen sets), by Lemma 1.9 there are points $x \in X$, $p \in \beta X - X$ such that x and p are the same connected component of βX. But clearly $X \in \Phi$ as X is totally disconnected. But if $f: X \to Y$ is any map where $Y \in \Phi_0 = \zeta_0$ (recall that Φ is the class of spaces whose connected components are compact), then $(\beta(f))(p) = (\beta(f))(x)$ as the image of a connected set must lie inside a connected set, and by Lemma 1.9, if Y is 0-dimensional, the only connected sets containing points of Y are those points themselves. Thus, by 0.6 $X \notin I \times \Phi_0$. Since $I \times \Phi_0 \supseteq I \times \Psi$ for any $\Psi \in \mathcal{H}_0$, this shows that $\Phi \neq I \times \Psi$ for any $\Psi \in \mathcal{H}_0$. The problem of characterizing those members of $\mathcal{H}$ which are equal to $I \times \Psi$ for some $\Psi \in \mathcal{H}_0$ is left open to the reader.

By an atom in $\mathcal{H}(\mathcal{H}_0)$ we will mean an element Φ such that if $\Psi \subseteq \Phi$ then either $\Psi = \Phi$ or Ψ is equal to the class of compact (0-dimensional) spaces (the class of compact (0-dimensional) spaces is the smallest in $\mathcal{H}$ $(\mathcal{H}_0)$ with respect to the order by inclusion - hence the notation). The following theorem relates the atoms in $\mathcal{H}$ to those in $\mathcal{H}_0$.

1.10 <u>Theorem</u>: Φ is an atom in $\mathcal{H}_0$ if and only if $I \times \Phi$ is an atom in $\mathcal{H}$.

Proof: Necessity. Suppose Φ is an atom of $\mathcal{H}_0$. Let $\Psi \subseteq I \times \Phi$. Then $\Psi_0 \subseteq (I \times \Phi)_0 = \Phi$ (by Theorem 1.8). Since Φ is an atom, we must have $\Psi_0 = \Phi$ or Ψ_0 is the class of compact 0-dimensional spaces. Suppose $I \times \Phi$ is not an atom, i.e. there is a $\Psi \in \mathcal{H}$ which properly contains the class of compact spaces and is properly contained in $I \times \Phi$. Let $X \in \Psi$ such that X is non-compact. Then $E(X) \in \Psi_0$ and $E(X)$ is non-compact. But $\Psi_0 \subsetneqq \Phi$ (for if $\Psi_0 = \Phi$ then $I \times \Phi = I \times \Psi_0 \subseteq \Psi$ contrary to the assumption that $\Psi \subsetneqq I \times \Phi$). Hence Ψ_0 is equal to the class of compact 0-dimensional spaces as Φ is an atom. This contradicts the existence of the non-compact space $E(X) \in \Psi_0$. Thus $I \times \Phi$ must be an atom.

Sufficiency. Suppose $I \times \Phi$ is an atom in $\mathcal{H}$. Let $\Psi \subsetneqq \Phi$, where $\Psi \in \mathcal{H}_0$. Then $I \times \Psi \subsetneqq I \times \Phi$ (for by Theorem 1.8, if $I \times \Psi = I \times \Phi$, then $\Psi = (I \times \Psi)_0 = (I \times \Phi)_0 = \Phi$). Hence $I \times \Psi$ is the class of all compact spaces as $I \times \Phi$ is an atom. But then $\Psi = (I \times \Psi)_0$ is the class of all compact 0-dimensional spaces. Hence Φ is an atom.¤

Note that the first part of the above proof also shows that if $\Phi \in \mathcal{H}_0$, then $I \times \Phi$ is the smallest member of $\mathcal{H}$ whose intersection with ζ_0 is equal to Φ.

Recall that if $\Phi \in \mathcal{H}$ ($\mathcal{H}_0$), the pseudocompactness property associated with Φ is denoted by Φ' and consists of all (0-dimensional) spaces X for which the equality $\Phi X = \beta X$ ($\beta_0 X$) holds.

1.11 <u>Theorem</u>: Let $\Phi \in \mathcal{H}$ ($\mathcal{H}_0$) and suppose $\Phi = K\Gamma$ ($K_0\Gamma$) for some class Γ of (0-dimensional) spaces. Then $X \in \Phi'$ if and only if given $Y \in \Gamma$ and $f: X \to Y$, then $\mathrm{cl}_Y(f(X))$ is compact.

Proof: Neccesity. Let $X \in \Phi'$. By 0.11, $f(X) \in \Phi'$ and hence $\mathrm{cl}_Y(f(X)) \in \Phi'$. But $Y \in \Gamma \subseteqq \Phi$, thus $\mathrm{cl}_Y(f(X)) \in \Phi \cap \Phi'$. Again, by 0.11, $\mathrm{cl}_Y(f(X))$ must be compact.

Sufficiency. Suppose $X \notin \Phi'$. Then $\Phi X \subsetneqq \beta X$ ($\beta_0 X$). Let $p \in \beta X - \Phi X$ ($\beta_0 X - \Phi X$). By 0.5 and 0.6 there is a $C \in \Gamma$ and $f: X \to C$ such that $(\beta(f))(p) \in \beta C - C$ ($(\beta_0(f))(p) \in \beta_0 C - C$). Thus $\mathrm{cl}_C(f(X))$ is not compact.¤

The following definitions and results are quoted directly from section 3 of [43] (specifically, they are 3.6 - 3.9).

1.12 Definition: i) Let Ψ be a class of spaces. Then $\overline{\Psi} = \{X$: every subspace of X which is in Ψ has compact closure in $X\}$.

ii) If $\Phi, \Psi \in \mathcal{H}$ $(\mathcal{H}_0)$ then Φ and Ψ are said to be copseudocompact if $\Phi' = \Psi'$. The copseudocompactness class of $\Phi \in \mathcal{H}$ $(\mathcal{H}_0)$ is the class of all $\Psi \in \mathcal{H}$ $(\mathcal{H}_0)$ which are copseudocompact with Φ.

1.13 Theorem: Let Ψ be a class of spaces such that any continuous image of any member of Ψ is again in Ψ. Then $\overline{\Psi} \in \mathcal{H}$.

1.14 Lemma: Let Φ, Ψ be classes of spaces such that $\overline{\Phi}, \overline{\Psi} \in \mathcal{H}$. If $\Phi \subseteq \Psi$, then $\overline{\Psi} \subseteq \overline{\Phi}$.

1.15 Theorem: Let $\Phi \in \mathcal{H}$. Then $\overline{(\Phi')} \in \mathcal{H}$, $\overline{(\Phi')}$ is copseudocompact with Φ, and $\overline{(\Phi')}$ contains any other member of $\mathcal{H}$ copseudocompact with Φ (i.e. $\overline{(\Phi')}$ is the largest member of its copseudocompactness class).

If Ψ is a class of 0-dimensional spaces, then $\overline{\overline{\Psi}}$ may be defined in the obvious analogous way to 1.12 and it can be quickly seen that 1.13 - 1.15 remain valid in a 0-dimensional context.

1.16 Lemma: Let Ψ be a class of spaces closed under the formation of continuous images. Then $(\overline{\Psi})'$ is the smallest pseudocompactness property containing the class Ψ.

Proof: Suppose $\Phi \in \mathcal{H}$ and $\Psi \subseteq \Phi'$. Then $\overline{(\Phi')} \subseteq \overline{\Psi}$ by Lemma 1.14. By Theorem 0.11, $(\overline{\Psi})' \subseteq (\overline{(\Phi')})'$. However, $(\overline{(\Phi')})' = \Phi'$ by Theorem 1.15. Hence, $(\overline{\Psi})' \subseteq \Phi'$. ¤

There is a natural "function" from $\mathcal{H}$ to $\mathcal{H}'$ ($\mathcal{H}_0$ to $\mathcal{H}_0'$) namely the one which carries an extension property Φ to its pseudocompactness property Φ'. $\mathcal{H}'$ ($\mathcal{H}_0'$) can be partially ordered by inclusion. The following theorem shows the close relationship between the partial orders of $\mathcal{H}$ ($\mathcal{H}_0$) and $\mathcal{H}'$ ($\mathcal{H}_0'$).

1.17 <u>Theorem</u>: Let $\{\Phi_\alpha:\alpha \in A\}$ be a subclass of $\mathcal{H}$.

i) $(\vee\{\Phi_\alpha:\alpha \in A\})' = \cap\{\Phi'_\alpha:\alpha \in A\}$; and

ii) $(\wedge\{\Phi_\alpha:\alpha \in A\})' = (\overline{(\cup\{\Phi'_\alpha:\alpha \in A\})})'$ if $\Phi_\alpha = \overline{(\Phi'_\alpha)}$ for all $\alpha \in A$.

Proof: i) By Defn. 1.1, $\vee\{\Phi_\alpha:\alpha \in A\} = K(\cup\{\Phi_\alpha:\alpha \in A\})$. Thus by Theorem 1.11 $X \in (\vee\{\Phi_\alpha:\alpha \in A\})'$ if and only if given $\alpha \in A$, $Y \in \Phi_\alpha$ and $f: X \to Y$, then $cl_Y(f(X))$ is compact; i.e. $X \in (\vee\{\Phi_\alpha:\alpha \in A\})'$ if and only if for each $\alpha \in A$, $X \in \Phi'_\alpha$.

ii) For convenience let $\cup\{\Phi'_\alpha:\alpha \in A\} = \Psi$. Clearly $\Psi \subseteq (\wedge\{\Phi_\alpha:\alpha\in A\})'$ (for $\wedge\{\Phi_\alpha:\alpha \in A\} \subseteq \Phi_\alpha$ for all $\alpha \in A$, hence by 0.11 $\Phi'_\alpha \subseteq (\wedge\{\Phi_\alpha:\alpha\in A\})'$ for all $\alpha \in A$). Since $\Phi'_\alpha \in \mathcal{H}'$ for all $\alpha \in A$, we have that Φ'_α is closed under the formation of continous images by 0.11. Thus $\cup\{\Phi'_\alpha:\alpha\in A\}$ is closed under the formation of continuous images. By Lemma 1.16, $(\overline{\Psi})'$ is the smallest pseudocompactness property containing Ψ. Thus $(\overline{\Psi})' \subseteq (\wedge\{\Phi_\alpha:\alpha \in A\})'$. Suppose $X \in \overline{\Psi}$. Since $\Phi'_\alpha \subseteq \Psi$ for $\alpha\in A$, $\overline{\Psi} \subseteq \overline{(\Phi'_\alpha)} = \Phi_\alpha$ by hypothesis. Thus $X \in \wedge\{\Phi_\alpha:\alpha \in A\}$ and $\overline{\Psi} \subseteq \wedge\{\Phi_\alpha:\alpha\in A\}$, hence $(\wedge\{\Phi_\alpha:\alpha \in A\})' \subseteq (\overline{\Psi})'$. ¤

The question of whether or not the hypothesis $\Phi_\alpha = \overline{(\Phi'_\alpha)}$ is necessary in 1.17 ii) has not been answered by the author, and is left open to the reader.

The maximum member of a copseudocompactness class seems to play a central role in the theory of pseudocompactness properties. The following result shows how to obtain all such maximal members.

1.18 Definition: Let $\mathcal{E}$ be a class of spaces. Let $\Phi_{\mathcal{E}} = \{X$: if $C \in \mathcal{E}$ and $f: C \to X$ then $cl_X(f(C))$ is compact$\}$.

This definition is a generalization of the notion of $(E,\beta E)$-compactness introduced in [20]. Given a space E, a space X is called $(E,\beta E)$-compact if the image of any map from E to X has compact closure in X. Thus $\Phi_{\mathcal{E}}$ is equal to the class of spaces that are $(E,\beta E)$-compact for all $E \in \mathcal{E}$. It is easy to verify that $\Phi_{\mathcal{E}} \in \mathcal{H}$ for any class of spaces $\mathcal{E}$.

1.19 Theorem: i) $\overline{((\Phi_{\mathcal{E}})')} = \Phi_{\mathcal{E}}$ for any class of spaces $\mathcal{E}$;
ii) if $\Phi = \overline{(\Phi')}$ then $\Phi = \Phi_{\mathcal{E}}$ where $\mathcal{E} = \Phi'$.

Proof: i) Let $\Theta = \overline{((\Phi_{\mathcal{E}})')}$. By Theorem 1.15 $\Phi_{\mathcal{E}} \subseteq \Theta$. Let $X \in \Theta$ and let $C \in \mathcal{E}$ and $f: C \to X$. By Theorem 1.11 it is clear that $\mathcal{E} \subseteq (\Phi_{\mathcal{E}})'$. Thus $C \in (\Phi_{\mathcal{E}})' = \Theta'$ and hence $cl_X(f(C)) \in (\Phi_{\mathcal{E}})'$ and is compact by 0.11 as $cl_X(f(C)) \in \Theta$ (being a closed subset of X). Thus $X \in \Phi_{\mathcal{E}}$.

ii) Suppose $\Phi = \overline{(\Phi')}$. If $X \in \Phi$, $C \in \Phi'$ and $f: C \to X$ then $cl_X(f(C))$ is compact by Theorem 1.11. Let $\mathcal{E} = \Phi'$. Then this shows that $\Phi \subseteq \Phi_{\mathcal{E}}$. Let $X \in \Phi_{\mathcal{E}}$ and A a subset of X such that $A \in \Phi'$. Then the embedding map $i: A \to X$ is continuous. Since $A \in \Phi' = \mathcal{E}$, $cl_X(i(A)) = cl_X(A)$ is compact. Thus $X \in \overline{(\Phi')} = \Phi$. ¤

1.16 - 1.19 have obvious analogues in $\mathcal{H}_0$ as the reader may easily verify. Corresponding to the study of maximal members of a copseudocompactness class is the study of extension properties which are the smallest members of their copseudocompactness classes. The following theorem relates to this investigation.

1.20 Theorem: Let $\{E_\alpha : \alpha \in A\}$ be a family of spaces. Suppose that each E_{α_0} satisfies the following condition: if $p \in \beta E_{\alpha_0} - E_{\alpha_0}$ then $(E_{\alpha_0} \cup \{(\beta(f))(p) : f : E_{\alpha_0} \to E_{\alpha_0}\}) \in \cap\{ (K\{E_\alpha\})' : \alpha \in A\}$. Then $K\{E_\alpha : \alpha \in A\}$ is the smallest member of its copseudocompactness class.

Proof: Let $\Phi = K\{E_\alpha : \alpha \in A\}$ and suppose $\Psi \in \mathcal{H}$ such that $\Phi' = \Psi'$. It is enough to show that $E_\alpha \in \Psi$ for all $\alpha \in A$. Suppose $E_\alpha \notin \Psi$ for some $\alpha \in A$. Then there is a $p \in \Psi E_\alpha - E_\alpha$. Let $T = (E_\alpha \cup \{(\beta(f))(p) : f : E_\alpha \to E_\alpha\})$. Then $T \in \cap\{(K\{E_\alpha\})' : \alpha \in A\} = (K\{E_\alpha : \alpha \in A\})'$ by Theorem 1.17 i). Thus $T \in \Phi' = \Psi'$. We show that $E_\alpha \in \Psi'$. To show this we show that for every $f : E_\alpha \to Y$ and $Y \in \Psi$, $cl_Y(f(E_\alpha))$ is compact and then invoke Theorem 1.11.

Let $f : E_\alpha \to Y$ where $Y \in \Psi$. Since $p \in \Psi E_\alpha$, we must have $(\beta(f))(p) \in Y$. But then $(\beta(f))(T) \subseteq Y$. For let $q \in T$. Then there is a map $g : E_\alpha \to E_\alpha$ such that $(\beta(g))(p) = q$. Now, $f \cdot g : E_\alpha \to Y$, hence $(\beta(f \cdot g))(p) \in Y$. But $\beta(f \cdot g) = \beta(f) \cdot \beta(g)$ and hence $(\beta(f \cdot g))(p) = (\beta(f))((\beta(g))(p)) = (\beta(f))(q)$. Thus $(\beta(f))(q) \in Y$ and hence $(\beta(f))(T) \subseteq Y$. Since $T \in \Psi'$, $(\beta(f))(T) \in \Psi'$ by 0.11. Thus $cl_Y((\beta(f))(T)) \in \Psi \cap \Psi'$ and must be compact by 0.11. Since E_α is dense in T we must have that $cl_Y(f(E_\alpha))$ is compact and hence $E_\alpha \in \Psi' = \Phi'$. But $E_\alpha \in \Phi$ and therefore must be compact.

Since Ψ contains all compact spaces, we must have that $E_\alpha \in \Psi$ which contradicts our initial assumption. Thus $E_\alpha \in \Psi$ for all $\alpha \in A$ and hence $\Phi \subseteq \Psi$.¤

Note that the above theorem holds if β is replaced by β_0, K by K_0, $\{E_\alpha : \alpha \in A\} \subseteq \zeta_0$ and $\Psi \in \mathcal{K}_0$. An identical proof applies.

1.21 Definition: A space X is almost compact if $|\beta X - X| \leq 1$.

Clearly any almost compact space satisfies the hypothesis of Theorem 1.20. Thus the following corollary is immediate from Theorem 1.20.

1.22 Corollary: Let $\{E_\alpha : \alpha \in A\}$ be a family of almost compact spaces. Then $K\{E_\alpha : \alpha \in A\}$ is the smallest member of its copseudocompactness class.

An ordinal space (i.e. an ordinal with the order topology) is almost compact if and only if its cofinality is uncountable (this is shown in 5.12 of [17] for the ordinal space ω_1, however the same proof applies to any ordinal with uncountable cofinality). Thus the condition of almost compactness on an ordinal space is equivalent to countable compactness (in fact ω_0-boundedness). Thus any family of ω_0-bounded ordinal spaces generates (by the operations K or K_0) an extension property minimal in its copseudocompactness class.

1.23 Definition: A space X is called weakly homogeneous if for every $p \in \beta X - X$, the set $\{q \in \beta X - X : q = (\beta(f))(p)$ for some $f: X \to X\}$ is dense in $\beta X - X$.

This is the density condition related to the result on N-compact spaces mentioned at the outset of this chapter. It is somewhat weaker than the condition of almost homogeneity defined in [2], which requires that f be a homeomorphism of X onto X. It is not difficult to prove that N is almost homogeneous hence weakly homogeneous. We can go somewhat further.

1.24 <u>Theorem</u>: If X is $I \times N$-compact (i.e. $X \in K\{I \times N\}$) then X is weakly homogeneous.

Proof: Let X be $I \times N$-compact and let $p \in \beta X - X$. By 0.6 there is a map $f: X \to N$ such that $(\beta(f))(p) \notin N$. Let U be a non-empty open subset of $\beta X - X$. Then there is an open set V in βX such that $(cl_{\beta X}(V)) - X$ is non-empty and is contained in U. Now, $cl_{\beta X}(V) = cl_{\beta X}(cl_X(V \cap X))$. Since $(cl_{\beta X}(V)) - X$ is non-empty $cl_X(V \cap X)$ is non-compact. But X is $I \times N$-compact, hence realcompact, so $cl_X(V \cap X)$ is realcompact and non-compact, hence is not pseudocompact. Thus by Corollary 1.21 of [17], $cl_X(V \cap X)$ contains $A = \{a_i : i \in N\}$, a closed, C-embedded copy of N. Since A is closed in $cl_X(V \cap X)$, we must have that $(cl_{\beta X}(A)) - A$ is non-empty and is contained in $(cl_{\beta X}(V)) - X \subseteq U$. Let $h: X \to X$ be defined as follows: $h(x) = a_i$ if and only if $f(x) = i$. Clearly h is continuous as f is continuous. Also $(\beta(h))(p) \notin A$ as $(\beta(f))(p) \notin N$, hence $(\beta(h))(p) \in U$. Thus X is weakly homogeneous.¤

It is not true that every $I \times N$-compact space is almost homogeneous. A strongly 0-dimensional, realcompact space is N-compact as shown in [23] (a space X is strongly 0-dimensional if βX is

0-dimensional) hence is $I \times N$-compact. Let D be the discrete space of cardinality ω_1. Then D is realcompact and strongly 0-dimensional, hence is N-compact (D is shown to be realcompact in Theorem 12.2 of [17], and since D is extremally disconnected, βD must also be extremally disconnected by 6M1 of [17], hence βD is 0-dimensional). Let $p \in \beta D - D$ be such that p is not in the closure of any countable subset of D, and let $U = (cl_{\beta D}(V)) - D$ where V is a countably infinite subset of D. Clearly if $h: D \to D$ is any homeomorphism, then $(\beta(h))(p) \notin U$. Thus D is not almost homogeneous, but D is weakly homogeneous by Theorem 1.24. However, the reader may verify that R is almost homogeneous, in a straightforward manner.

The following result is Theorem 3.1 of [14]. It will play a central role in the proofs of some subsequent theorems of this chapter.

1.25 Theorem: Let X be a locally compact, realcompact space. Then a subset $D \subseteqq \beta X - X$ is dense in $\beta X - X$ if and only if $X \cup D$ is pseudocompact.

1.26 Corollary: K{R} is the smallest member of its copseudocompactness class.

Proof: If $\Phi = K\{R\}$ then $\Phi X = \upsilon X$, the Hewitt realcompactification of X for any space X. Thus, by 8A4 of [17], $(K\{R\})'$ is the class of pseudocompact spaces. Noting that R is realcompact and locally compact, the result follows immediately from Theorems 1.20 and 1.25 and the fact, mentioned above, that R is weakly homogeneous.¤

1.27 <u>Lemma</u>: Let X be a 0-dimensional, realcompact and non-compact space. Then $(K_0\{X\})'$ is the class of 0-dimensional pseudocompact spaces.

Proof: If A is pseudocompact and $f\colon A \to X$ is continuous, then $cl_X(f(A))$ is pseudocompact, and hence must be compact as X is realcompact. Conversely, if every map from A to X is such that its image in X has compact closure and A is 0-dimensional, then A must be pseudocompact. For if A is not pseudocompact then as is shown in[33], there is a map $f\colon X \to N$ which is onto. Since X is not pseudocompact, X contains a C-embedded, closed copy $B = \{b_i : i \in N\}$, of N. We can map A onto B by g where $g(x) = b_i$ if and only if $f(x) = i$. Since B is closed in X, $cl_X(g(A)) = cl_X(B) = B$ is not compact contrary to the assumption that the image of every map from A to X has compact closure. Thus A must be pseudocompact. By invoking Theorem 1.11 we obtain the desired result.¤

1.28 <u>Corollary</u>: Let X be a realcompact, locally compact, weakly homogeneous, 0-dimensional, non-compact space. Then $K_0\{X\}$ is the smallest member of its copseudocompactness class.

Proof: We show that X satisfies the 0-dimensional version of Theorem 1.20. Let $p \in \beta_0 X - X$. We must show that $A = X \cup \{(\beta_0(f))(p) : f\colon X \to X\}$ is pseudocompact. Let $q \in \beta X - X$ such that $(\beta(i))(q) = p$ where $i\colon X \to \beta_0 X$ is the identity. Let $B = X \cup \{(\beta(f))(q) : f\colon X \to X\}$. Then $A = (\beta(i))(B)$. But B is pseudocompact by Theorem 1.25. Thus A is pseudocompact.¤

1.29 Theorem: Let X be locally compact. The following two conditions on X are equivalent.

i) X is realcompact, 0-dimensional and weakly homogeneous,

ii) X is N-compact.

Proof: ii) $\Rightarrow$ i). Since N is realcompact and 0-dimensional, X must also be realcompact and 0-dimensional. By Theorem 1.24, X is weakly homogeneous.

i) $\Rightarrow$ ii). If X is compact then X is N-compact (a compact 0-dimensional space is $\{0,1\}$-compact). Let us assume that X is non-compact. By Corollary 1.28, $K_0\{X\}$ is the smallest member of its copseudocompactness class as is $K_0\{N\}$. However, by Lemma 1.27, $(K_0\{X\})' = (K_0\{N\})'$. Thus $K_0\{X\} = K_0\{N\}$ and hence X is N-compact.¤

1.30 Corollary: Let X be a 0-dimensional space. Suppose that the only weakly homogeneous, proper subspaces of βX which contain X are almost compact. Then X is pseudocompact.

Proof: If X is not pseudocompact then $X \notin (K_0\{N\})'$ (by Lemma 1.27, as N is 0-dimensional, realcompact and non-compact). Hence, by Theorem 1.11 $X \notin (K\{I \times N\})'$. Thus $(K\{I \times N\})X \subsetneq \beta X$. But this space is realcompact and non-compact hence is not pseudocompact, thus it is not almost compact (in 6J of [17] it is shown that an almost compact space is pseudocompact). However, since it is $I \times N$-compact, it must be weakly homogeneous contrary to our initial hypothesis. Thus X must be pseudocompact.¤

1.31 Corollary: Let X be a realcompact, locally compact, weakly homogeneous space. The following statements are equivalent.

i) X is copseudocompact with R (i.e. $(K\{X\})' = (K\{R\})'$),

ii) $K\{X\} = K\{R\}$.

Proof: i) $\Rightarrow$ ii). Since both X and R are locally compact, weakly homogeneous and realcompact, they both satisfy the hypothesis of Theorem 1.20. Thus both $K\{X\}$ and $K\{R\}$ are the smallest members of their copseudocompactness classes. Since by hypothesis their copseudocompactness classes are equal, we must have $K\{X\} = K\{R\}$.

ii) $\Rightarrow$ i). This implication is trivial.¤

We now present the examples mentioned at the beginning of this chapter.

1.32 Example: Let Ψ be the class of connected spaces. Since Ψ is closed under the formation of continuous images, $\overline{\Psi} \in \mathcal{H}$. Let $\Phi = \overline{\Psi}$. Then $X \in \Phi$ if and only if all connected components of X are compact. Thus, as mentioned earlier, all totally disconnected (hence 0-dimensional) spaces are in Φ. By Theorem 1.11, it is clear that $\Psi \subseteqq \Phi'$. Since R is connected, $R \in \Phi'$. But $E(R)$, the projective cover of R, is extremally disconnected and hence is 0-dimensional. Thus $E(R) \in \Phi - \Phi'$, and the property of being Φ-pseudocompact is not preserved in passing to the projective cover. This contrasts the fact (proved as Proposition 2.5 of [42]) that $E(X)$ is pseudocompact if and only if X is pseudocompact.

1.33 Example: In Corollary 2.10 of [43], it is shown that for $\Phi \in \mathcal{H}_0$ either Φ is the class of compact 0-dimensional spaces or Φ' does not properly contain the class of pseudocompact 0-dimensional spaces. Let $\Phi = K\{I \times N\}$. Then $\Phi \in \mathcal{H}$ and Φ is not equal to the class of compact spaces. Since $I \times N$ is realcompact, $\Phi \subseteqq K\{R\}$. Thus $(K\{R\})' \subseteqq \Phi'$, i.e. every pseudocompact space is in Φ'. In addition, by Theorem 1.11, Φ' contains all connected spaces (as the image of a continuous map from a connected space to $I \times N$ has compact closure in $I \times N$). Hence Φ' properly contains the class of pseudocompact spaces. This answers in the negative the question raised after Corollary 2.10 of [43] as to whether or not 2.10 remains true for $\Phi \in \mathcal{H}$.

An extension ΦX of a space X lying inside βX ($\beta_0 X$) can be viewed as a certain set of ultrafilters of zero-sets (clopen sets) in X. Using a generalization of the notion of stable families of sets defined in [30], we characterize ΦX in terms of Φ-stable z-ultrafilters (clopen ultrafilters) on X for a certain class of extension properties.

1.34 Definition: Let $\Phi \in \mathcal{H}$ ($\mathcal{H}_0$) such that $\Phi = K\Gamma$ ($K_0\Gamma$). A z-ultrafilter (clopen ultrafilter) A on a space X is called Φ_Γ- stable if given a map $f: X \to Y$ where $Y \in \Gamma$, there is an $F \in A$ such that $cl_Y(f(F))$ is compact.

1.35 Theorem: Suppose $\Phi \in \mathcal{K}$ such that $\Phi = K\Gamma$ where every member of Γ is locally compact. Then $X \in \Phi$ if and only if every Φ_Γ-stable z-ultrafilter on X is fixed.

Proof: Necessity. Suppose $X \in \Phi$ and A is a Φ_Γ-stable z-ultrafilter on X. Since A is a z-ultrafilter on X, there must be exactly one point $p \in \beta X$ such that A converges to p. Suppose A is free, i.e. $p \in \beta X - X$. By Theorem 0.6 there is a $C \in \Gamma$ and a map $f: X \to C$ such that $(\beta(f))(p) \in \beta C - C$. But A is Φ_Γ-stable and $C \in \Gamma$, hence there is an $F \in A$ such that $cl_C(f(F))$ is compact. Thus $(\beta(f))(cl_{\beta X}(F)) \subseteq cl_{\beta C}(f(F)) = cl_C(f(F))$, hence $(\beta(f))(p) \in C$. This contradiction shows that $p \in X$ and thus A is fixed.

Sufficiency. Suppose $X \notin \Phi$. Suppose $p \in \Phi X - X$ and let $A_p = \{Z \in Z(X): p \in cl_{\beta X}(Z)\}$. Then as shown in Chapter 6 of [17], A_p is the unique z-ultrafilter on X converging to p. Let $f: X \to C$ where $C \in \Gamma$. Since $C \in \Phi$, $(\beta(f))(p) \in C$. A compact zero-set neighborhood U of $(\beta(f))(p)$ in C can be chosen by the local compactness of C. But then $f^{\leftarrow}(U) \in A_p$ and $cl_C(f(f^{\leftarrow}(U)))$ is compact. Thus A_p is Φ_Γ-stable but not fixed. This contradiction proves the theorem.¤

1.36 Theorem: Suppose $\Phi \in \mathcal{K}$ and Φ satisfies the same hypothesis as in Theorem 1.35 above. Then for any space X, $\Phi X = \{p \in \beta X$: p is a Φ_Γ-stable z-ultrafilter on X$\}$.

Proof: Let $T = \{p \in \beta X$: p is a Φ_Γ-stable z-ultrafilter on X$\}$. Let $p \in T$ and suppose $f: X \to C$ where $C \in \Gamma$. Then there is an $F \in p$ such that $cl_C(f(F))$ is compact. Hence $(\beta(f))(p) \in C$. Thus by

Lemma 0.5 and Theorem 0.6 $p \in \Phi X$. Hence $T \subseteq \Phi X$. Suppose $q \in \beta X - T$. Then q is not Φ_Γ-stable. Hence there is a $C \in \Gamma$ and a map $f: X \to C$ such that $cl_C(f(F))$ is not compact for any $F \in q$. Thus $(\beta(f))(q) \in \beta C - C$ by the local compactness of C. By Theorem 0.6 $q \notin \Phi X$. Hence $\Phi X \subseteq T$ and the theorem is proved.¤

Identical proofs of Theorems 1.35 and 1.36 yield analagous results for $\Phi \in \mathcal{H}_0$ with β replaced by β_0 and z-ultrafilters replaced by clopen ultrafilters.

The author has not been able to determine whether or not all extension properties in $\mathcal{H}$ ($\mathcal{H}_0$) are equal to $K\Gamma$ ($K_0\Gamma$) for some class of locally compact spaces Γ. This question is left open to the reader.

CHAPTER 2

m-BOUNDED TOPOLOGICAL SPACES

If m is an infinite cardinal number, then a space X is called m-bounded if every subset of X of cardinality at most m has compact closure in X. This class of spaces was first introduced in [22]. In [41] it was shown that for a given infinite cardinal m, the class of m-bounded spaces is an extension property and the m-bounded extension of a space X consists of those points of βX which are in the closure of a subset of X of cardinality at most m. In this chapter we answer some questions raised in [43]. Specifically, in 5.5 of [43] it is asked whether or not $K_0(\{\omega_\alpha: cf(\omega_\alpha) > \omega_0\})$ is equal to the class of all 0-dimensional ω_0-bounded spaces (where ω_α has the topology induced by the well-order and cf denotes cofinality). We show that the answer to this question is no by means of a counterexample. We also give an example of a space X which is in the pseudocompactness property of ω_0-boundedness, and a compact space K such that $X \times K$ is not in the pseudocompactness property of ω_0-boundedness, answering question 5.1 of [43] in the negative. This contrasts the fact that the product of a pseudocompact space (i.e. a space in the pseudocompactness property of the extension property of realcompactness) with a compact space is again pseudocompact. We give a characterization in terms of clopen ultrafilters, of those spaces that are in the extension property generated by the class of Σ-spaces and show that this class is properly contained in the class of ω_0- bounded spaces. We obtain various necessary conditions for a space to be in the pseudocompactness property of the extension property consisting of m-bounded spaces. The results indicate that a complete internal characterization of the spaces in that class will depend on the set theoretic axioms assumed. Finally, the class of strongly-m-bounded spaces is introduced and a characterization of the strongly-m-bounded extension of an extremally disconnected, locally compact space is given.

Recall that ω_α denotes the α-th uncountable cardinal. As an ordinal, ω_α may assume the topology induced by its total order, and in that case will be called an ordinal space. Endowed with the order topology, an ordinal space is 0-dimensional, locally compact and normal. It is clear that if m is an infinite cardinal, then ω_α is m-bounded if and only if $cf(\omega_\alpha) > m$. Furthermore, the proof of 5.12(c) in [17] which shows that $\beta\omega_1 = \omega_1+1$ (the one-point compactification of ω_1) can also be seen to show that $\beta\omega_\alpha = \omega_\alpha+1$ if $cf(\omega_\alpha) > \omega_0$ (thus, since $\omega_\alpha+1$ is 0-dimensional, ω_α is strongly 0-dimensional in this case). Let $\Gamma = \{\omega_\alpha : cf(\omega_\alpha)>\omega_0\}$. In 5.5 of [43] it is asked whether or not $K_0\Gamma$ is the class of all 0-dimensional, ω_0-bounded spaces. In Theorem 4 of [5], it is shown that if $cf(\omega_\alpha) = \omega_t$ then $\omega_\alpha \in K_0\{\omega_t\}$. In view of the fact that $cf(\omega_\alpha)$ is a regular cardinal for any ω_α (a cardinal m is called regular if cf(m)=m), if $\Gamma_1 = \{\omega_\alpha : \omega_\alpha>\omega_0$ and ω_α is regular$\}$, then $K_0\Gamma = K_0\Gamma_1$. We now offer a characterization in terms of clopen ultrafilters, of those spaces in $K_0\Gamma$. In the following theorem, let $(a,b) = \{x \in \omega_\alpha : a<x<b\}$ for a and b in ω_α, and similarly let $[a,b]$ denote a closed interval in ω_α.

2.1 Theorem: Let X be a 0-dimensional space. The following statements are equivalent.

i) $X \in K_0\Gamma = K_0\Gamma_1$,

ii) for every $p \in \beta_0 X - X$, there is an ordinal ω_α with $cf(\omega_\alpha) = \omega_\alpha > \omega_0$ and a family $\mathcal{T} = \{U_i : i < \omega_\alpha\} \subseteqq p$ such that $\cap\mathcal{T}$ is empty and $U_j \subseteqq \cap\{U_i : i < j\}$ for all $j < \omega_\alpha$.

Proof: i) $\Rightarrow$ ii). The comments above have shown the validity of the equality in statement i). Suppose $X \in K_0\Gamma$ and $p \in \beta_0 X - X$. By Theorem 0.6 there is an ordinal $\omega_\alpha \in \Gamma_1$ (i.e. $cf(\omega_\alpha) = \omega_\alpha > \omega_0$) and a continuous map $f: X \to \omega_\alpha$ such that $(\beta_0(f))(p) = \omega_\alpha$ (ω_α also denotes the point at infinity in $\omega_\alpha + 1$, the one-point and Stone-Čech compactification of ω_α). Let $U_i = f^{\leftarrow}((i, \omega_\alpha))$ for all $i < \omega_\alpha$. Then $U_i \in p$; for U_i is clearly clopen and if $(X - U_i) \in p$ then $p \in cl_{\beta_0 X}(X - U_i)$, hence $(\beta_0(f))(p) \in cl_{\omega_\alpha}(f(X - U_i)) \subseteqq [0, i]$ which is contrary to hypothesis. Let $\mathcal{T} = \{U_i : i < \omega_\alpha\}$. If $x \in X$ then $f(x) < \omega_\alpha$ and hence there is an $i < \omega_\alpha$ such that $f(x) < i$; therefore $x \notin U_i$. Thus $\cap\mathcal{T}$ is empty. Clearly $\mathcal{T}$ satisfies condition ii).

ii) $\Rightarrow$ i). Let $p \in \beta_0 X - X$. By hypothesis, there is an $\omega_\alpha \in \Gamma_1$ and a family $\mathcal{T} = \{U_i : i < \omega_\alpha\} \subseteqq p$ such that $\cap\mathcal{T}$ is empty and $U_j \subseteqq \cap\{U_i : i < j\}$ for all $j < \omega_\alpha$. Let $f: X \to \omega_\alpha$ be defined as follows: $f(x) = \min\{i < \omega_\alpha : x \notin U_i\}$. Then $f^{\leftarrow}((i, j)) = U_i - \cap\{U_t : t < j\}$ if $i < j < \omega_\alpha$. Since U_i is clopen for all $i < \omega_\alpha$, this set is open and hence f is continuous. Since $f^{\leftarrow}((i, \omega_\alpha)) = U_i \in p$ for every $i < \omega_\alpha$, and $p \in cl_{\beta_0 X}(U_i)$ we must have $(\beta_0(f))(p) \in cl_{\omega_\alpha + 1}(i, \omega_\alpha) = (i, \omega_\alpha]$ for every $i < \omega_\alpha$. Thus $(\beta_0(f))(p) = \omega_\alpha \in (\omega_\alpha + 1) - \omega_\alpha = \beta\omega_\alpha - \omega_\alpha$. Hence by Theorem 0.6 $X \in K_0\Gamma$. ¤

In order to answer question 5.5 of [43] we introduce the class of Σ-spaces, which has been extensively investigated in [39].

2.2 Definition: Let $\{0,1\}$ denote the two-point discrete space and let α be an infinite cardinal number. The space Σ_α is defined as follows: $\Sigma_\alpha = \{ p \in \{0,1\}^\alpha : |\{i \in \alpha : \pi_i(p) = 1\}| < \alpha\}$ (where $\{0,1\}^\alpha$ denotes the topological product of α copies of $\{0,1\}$, and π_i denotes the i-th projection map from $\{0,1\}^\alpha$ to $\{0,1\}$).

It is shown in [39] that $\beta(\Sigma_\alpha) = \{0,1\}^\alpha$ for any uncountable cardinal α (hence Σ_α is strongly 0-dimensional). The following lemma fixes the position of Σ-spaces as m-bounded spaces.

2.3 Lemma: Let m and α be infinite cardinal numbers. Then Σ_α is m-bounded if and only if $m < cf(\alpha)$.

Proof: Necessity. Suppose Σ_α is m-bounded and $m \geq cf(\alpha)$. Let $\{\alpha_i : i < m\}$ be a family of cardinals such that $\alpha_i < \alpha_j$ if $i < j$, $\alpha_i < \alpha$, and $\Sigma\{\alpha_i : i < m\} = \alpha$ (here Σ denotes the sum of cardinals). Let $\{A_i : i < m\}$ be a family of subsets of α such that $|A_i| = \alpha_i$ for all $i < m$, $A_i \subseteqq A_j$ if $i < j$, and $\cup\{A_i : i < m\} = \alpha$. Define points p_i in Σ_α for $i < m$ as follows: for $t \in \alpha$ and $i < m$, let $\pi_t(p_i) = 1$ if $t \in A_i$, $\pi_t(p_i) = 0$ if $t \notin A_i$. Then $\{p_i : i < m\} \subseteqq \Sigma_\alpha$ and $|\{p_i : i < m\}| = m$. Thus $cl_{\{0,1\}^\alpha}(\{p_i : i < m\})$ is compact and contained in Σ_α (as Σ_α is m-bounded). But the point p, defined by $\pi_t(p) = 1$ for all $t \in \alpha$ is such that $p \in (cl_{\{0,1\}^\alpha}(\{p_i : i < m\})) - \Sigma_\alpha$. This contradiction shows that $m < cf(\alpha)$.

Sufficiency. Suppose $m < cf(\alpha)$ and $\{p_i : i < m\} \subseteq \Sigma_\alpha$. Let $A_i = \{t \in \alpha : \pi_t(p_i) = 1\}$ for each $i < m$. Then $|A_i| < \alpha$ as $p_i \in \Sigma_\alpha$. Thus $|\cup\{A_i : i < m\}| < \alpha$ as $m < cf(\alpha)$. Let $A = \cup\{A_i : i < m\}$. Then $A \subseteq \alpha$. Thus

$cl_{\{0,1\}^\alpha}(\{p_i : i < m\}) \subseteq (\,\Pi\{\,\{0,1\}_t : t \in A\}\,) \times (\,\Pi\{\,\{0\}_t : t \in \alpha - A\}) \subseteq \Sigma_\alpha$.

As the set in the middle is obviously compact, Σ_α is m-bounded.¤

2.4 Definition: If m is an infinite cardinal number, then a space X is called strongly-m-bounded if every union of at most m compact subsets of X has compact closure in X.

Later in this chapter we show that for a given cardinal m, the class of strongly-m-bounded spaces is an extension property. It is easy to see that every member of the class Γ above is strongly-ω_0-bounded (in fact ω_α is strongly-m-bounded if and only $cf(\omega_\alpha) > m$). Thus, every member of $K_0\Gamma$ is strongly-ω_0-bounded. However, Σ_{ω_1} has the following dense, σ-compact subset A: $A = \cup\{A_i : i \in N\}$ where $A_i = \{p \in \{0,1\}^{\omega_1} : |\{t : \pi_t(p) = 1\}| \leq i\}$. Since Σ_{ω_1} is not compact, it cannot be strongly-ω_0-bounded as it has a dense σ-compact subset. Thus Σ_{ω_1} is ω_0-bounded (by Lemma 2.3, as $cf(\omega_1) > \omega_0$) but $\Sigma_{\omega_1} \notin K_0\Gamma$ as it is not strongly-ω_0-bounded. We can show a somewhat more startling result; we show that $\Sigma_\alpha \in (K_0\Gamma)'$ (the pseudocompactness property) for any infinite cardinal α.

2.5 Definition: The cellularity of a space X, denoted c(X), is defined as follows: $c(X) = \sup\{\,m : m$ is a cardinal number and

X has a pairwise disjoint family of open sets of cardinality $m\}$.

The cardinal function $c(X)$, is investigated along with various other cardinal functions of topological spaces, in [9] and [29]. In particular, it is shown in both [9] and [29] that if $\Pi\{X_\alpha : \alpha \in A\}$ is a product space, then $c(\Pi\{X_\alpha : \alpha \in A\}) = \sup\{c(\Pi\{X_\alpha : \alpha \in F\}) : F$ is a finite subset of $A\}$ (i.e. the cellularity of a product space is the supremum of the cellularities of its finite subproducts). Thus $c(\{0,1\}^\alpha) = \omega_0$ (as each finite subproduct is a finite space and hence has cellularity less that ω_0). Clearly if X is dense in Y, then $c(X) = c(Y)$. Hence $c(\Sigma_\alpha) = \omega_0$ for every infinite cardinal α. We use this fact to show that $\Sigma_\alpha \in (K_0\Gamma)'$ by means of Theorem 1.11 of Chapter 1. Let $f\colon \Sigma_\alpha \to \omega_t$, where $\omega_t \in \Gamma_1$. Suppose $f(\Sigma_\alpha)$ is cofinal in ω_t. Then $|f(\Sigma_\alpha)| = \omega_t$ because ω_t has regular cardinality. Now, $f(\Sigma_\alpha)$ is a well-ordered set, as a subset of the well-ordered set ω_t, and with the order topology, $f(\Sigma_\alpha)$ has ω_t isolated points (for every δ in a well-ordered space, $\delta+1$ is an isolated point). However, by 304 of [17], the topology on $f(\Sigma_\alpha)$ inherited as a subspace of ω_t contains the topology on $f(\Sigma_\alpha)$ induced by the well-order on $f(\Sigma_\alpha)$. Thus, as a subspace of ω_t, $f(\Sigma_\alpha)$ contains $\omega_t > \omega_0$ isolated points. The preimages under f of these isolated points form a pairwise disjoint family of $\omega_t > \omega_0$ open sets in Σ_α. This contradicts the fact that $c(\Sigma_\alpha) = \omega_0$. Thus $f(\Sigma_\alpha)$ is not cofinal in ω_t, and hence $\mathrm{cl}_{\omega_t}(f(\Sigma_\alpha))$ is compact. Thus $\Sigma_\alpha \in (K_0\Gamma)'$ by Theorem 1.11. Since Σ_{ω_1} is ω_0-bounded and non-compact, and $\Sigma_{\omega_1} \in (K_0\Gamma)'$, we must have that $\Sigma_{\omega_1} \notin K_0\Gamma$ (for by

Theorem 0.11, $X \in \Phi \cap \Phi'$ if and only if X is compact).

Let us denote by ∇ the class of all spaces Σ_α for which $cf(\alpha) > \omega_0$. Then every member of $K_0\nabla$ is ω_0-bounded by Lemma 2.3, and $\nabla \subseteq (K_0\Gamma)'$. The following result shows that every member of Γ is in $K_0\nabla$ and hence $K_0\Gamma \subsetneqq K_0\nabla$. Recall that $\beta(\Sigma_\alpha) = \{0,1\}^\alpha$ for $\alpha \geq \omega_1$, as shown in [39]. Since $\{0,1\}^\alpha$ is 0-dimensional, this shows that Σ_α is strongly 0-dimensional, i.e. $\beta(\Sigma_\alpha) = \beta_0(\Sigma_\alpha)$.

2.6 Theorem: Let X be a 0-dimensional space. The following are equivalent.

i) $X \in K_0\nabla$,

ii) for every $p \in \beta_0 X - X$ there is a cardinal m with $cf(m) > \omega_0$ and a family $T = \{U_i : i < m\} \subseteq p$ such that for every $x \in X$, $|\{i < m : x \in U_i\}| < m$.

Proof: i) $\Rightarrow$ ii). Let $X \in K_0\nabla$ and $p \in \beta_0 X - X$. By Theorem 0.6, there is a cardinal m with $cf(m) > \omega_0$ and a continuous map $f: X \to \Sigma_m$ such that $(\beta_0(f))(p) \in \beta_0(\Sigma_m) - \Sigma_m$. Thus, if $q = (\beta_0(f))(p)$, $|\{i \in m : \pi_i(q) = 1\}| = m$. Let $A = \{i \in m : \pi_i(q) = 1\}$. For each $i \in A$, let $U_i = f^{\leftarrow}(\pi_i^{\leftarrow}(\{1\}))$. Since $\{1\}$ is a clopen subset of $\{0,1\}$, U_i is a clopen subset of X. Also, $U_i \in p$, for if $X - U_i \in p$, then $p \in cl_{\beta_0 X}(X - U_i)$ and hence $(\beta_0(f))(p) = q \in cl_{\{0,1\}^m}(f(X - U_i)) \subseteq cl_{\{0,1\}^m}(\pi_i^{\leftarrow}(\{0\}) \cap \Sigma_m) \subseteq \pi_i^{\leftarrow}(\{0\})$, and hence $\pi_i(q) = 0$ which is false since $i \in A$. Let $T = \{U_i : i \in A\}$ and let $x \in X$. Clearly $x \in U_i$ if and only if $\pi_i(f(x)) = 1$. Since $f(x) \in \Sigma_m$, $|\{i \in A : x \in U_i\}| < m$. Hence $T \subseteq p$, and the family T satisfies condition ii) as $|T| = |A| = m$.

ii) $\Rightarrow$ i). Let X satisfy condition ii). Let $p \in \beta_0 X - X$. Then there is a cardinal m with $cf(m) > \omega_0$, and a family $\mathcal{T} = \{U_i : i < m\} \subseteq p$ as in condition ii). Define $f: X \to \Sigma_m$ as follows: for $i \in m$ (note that since m is a cardinal number, $i < m$ if and only if $i \in m$) let $\pi_i(f(x)) = 1$ if $x \in U_i$, and $\pi_i(f(x)) = 0$ if $x \notin U_i$. Then f is continuous, for $f^{\leftarrow}(\pi_i^{\leftarrow}(\{1\})) = U_i$, which is clopen for all $i \in m$. In addition, $\pi_i((\beta_0(f))(p)) = 1$ for all $i \in m$ (as $p \in cl_{\beta_0 X}(U_i)$) hence $(\beta_0(f))(p) \in \beta_0 \Sigma_m - \Sigma_m$. Thus, by Theorem 0.6, $X \in K_0 \nabla$.¤

If $\omega_\alpha \in \Gamma_1$, and we let $\mathcal{T} = \{(i,\omega_\alpha) : i \in \omega_\alpha\}$, then $\mathcal{T}$ satisfies condition ii) in Theorem 2.6, as $cf(\omega_\alpha) > \omega_0$. Therefore, $\omega_\alpha \in K_0\{\Sigma_{\omega_\alpha}\} \subseteq K_0 \nabla$. It is actually true that if $\omega_\alpha \in \Gamma_1$, then ω_α can be embedded as a closed subset of Σ_{ω_α}. Thus $K_0 \Gamma_1 \subsetneq K_0 \nabla$. It would be very satiafying if $K_0 \nabla$ could be shown to be the class of all 0-dimensional ω_0-bounded spaces. However, as the following example shows, it is not.

2.7 <u>Example</u>: Let X be the following subspace of the space $Y = \Pi\{(\omega_1+1)_i : i \in \omega_2\}$: $X = \{p \in Y : |\{i \in \omega_2 : \pi_i(p) = \omega_1\}| \le \omega_1\}$. (Note again that ω_1+1 denotes $\beta\omega_1$, or the one-point compactification of the ordinal space ω_1, and ω_1 also denotes the point at infinity in ω_1+1, i.e. by $\pi_i(p) = \omega_1$, we mean that $\pi_i(p)$ is equal to the point at infinity in ω_1+1).

A straightforward verification (similar to the proof of Lemma 2.3) shows that X is ω_0-bounded. Clearly both X and Y are

0-dimensional. We show that X does not satisfy condition ii) of Theorem 2.6, and hence $X \notin K_0\nabla$.

First we note that $\Pi\{(\omega_1)_i : i \in \omega_2\} \subseteq X \subseteq Y$, and this first space is ω_0-bounded (and, in particular, is pseudocompact). Theorem 1 of [19] shows that the Stone-Čech compactification of a product space is equal to the product of the Stone-Čech compactifications of the factor spaces if and only if the product space is pseudocompact. Thus $\beta(\Pi\{(\omega_1)_i : i \in \omega_2\}) = Y$. Hence, $\beta X = \beta_0 X = Y$. Let $p \in \beta_0 X - X$ be defined as follows: $\pi_i(p) = \omega_1$ for all $i \in \omega_2$. We show that p does not satisfy condition ii) of Theorem 2.6.

Suppose $\mathcal{T} = \{U_j : j < \omega_1\} \subseteq p$ (each U_j is a clopen subset of X and p is also regarded as a clopen ultrafilter on X). We may assume that for each $j < \omega_1$, $U_j = V_j \cap X$, where V_j is a basic, canonical clopen subset of the space Y (i.e. $V_j = \Pi\{\pi_i(V_j) : i \in \omega_2\}$ for each $j < \omega_1$, and $\pi_i(V_j) = \omega_1 + 1$ for all but finitely many $i \in \omega_2$). Thus, for each $j < \omega_1$, $V_j = \mathrm{cl}_Y(U_j)$, and $p \in V_j$. Hence, for each $j < \omega_1$ and $i \in \omega_2$, $\omega_1 \in \pi_i(V_j)$ (as $p \in V_j$ and $\omega_1 = \pi_i(p)$ for each $i \in \omega_2$). For each $j < \omega_1$, let $B_j = \{k \in \omega_2 : \pi_k(V_j) \neq \omega_1\}$. Then B_j is finite for each $j < \omega_1$, for V_j is a canonical clopen subset of Y. Let $T = \cup\{B_j : j < \omega_1\}$. Then $|T| \leq \omega_1$ since each B_j is a finite set. We now show that there is a point q such that $q \in X$ and q is also in every member of $\mathcal{T}$.

Define q as follows: $\pi_k(q) = \omega_1$ for all $k \in T$, and $\pi_k(q) = 1$ otherwise. Then $q \in X \cap (\cap\tau)$, as $q \in V_j$ for all $j < \omega_1$, and $U_j = X \cap V_j$. Thus the cardinal ω_1 cannot play the role of m in ii) of Theorem 2.6. Suppose that $\tau = \{U_j : j < \omega_t\}$ where $\omega_t > \omega_1$. In Theorem 13 of [9] it is shown that a product of 2^m spaces, each having a dense subset of cardinality m, will also have a dense subset of cardinality m. Noting that $\omega_2 \le 2^{\omega_1}$, and that X contains $\Pi\{(\omega_1)_i : i \in \omega_2\}$ as a dense subset, X must have a dense subset, D, of cardinality ω_1 by the theorem just quoted. It is also clear that Y hase a base of cardinality ω_2 hence we may assume that $\omega_t = \omega_2$. For every $d \in D$, let $\tau_d = \{U_j : d \in U_j\}$. Then $\tau = \cup\{\tau_d : d \in D\}$. Hence there is a $d_0 \in D$ such that $|\tau_{d_0}| = \omega_2$. Thus ω_2 cannot play the role of m in condition ii) of Theorem 2.6, and hence $X \notin K_0\nabla$.

We now give the example mentioned at the beginning of this chapter which concerns the pseudocompactness property of the class of ω_0-bounded spaces. If υ denotes the class of realcompact spaces then $\upsilon \in \mathcal{K}$, and υX is the Hewitt realcompactification of X. In 8A4 of [17] it is shown that υ', the pseudocompactness property of υ, is precisely the class of pseudocompact spaces. It is also well-known (see 9.14 of [17])that the product of a pseudocompact space with a compact space is again pseudocompact. Woods, in 5.1 of [43] asks the following question: If $\Phi \in \mathcal{K}$, $X \in \Phi'$ and K is a compact space, is it true that $X \times K \in \Phi'$? In Chapter 3 of this thesis, we show that the answer is yes for a large class of extension properties. However, the following example shows that if Φ is the class of ω_0-bounded spaces, then the answer is no.

2.8 Example: It is shown in [41] that if m is an infinite cardinal number, the maximal m-bounded extension of a space X consists of those points in βX which are in the closure of a subset of X of cardinality at most m. Thus, if X has a dense subset of cardinality m, then X is in the pseudocompactness property of the extension property of m-bounded spaces. In particular, N is in the pseudocompactness property of the class of ω_0-bounded spaces. Let us denote by Φ the class of ω_0-bounded spaces. Then $N \in \Phi'$. Recall that in showing that $\Sigma_{\omega_1} \notin K_0\Gamma$ earlier in this chapter, we exhibited a σ-compact, dense subset of Σ_{ω_1}, namely $A = \cup\{A_i : i \in N\}$ where $A_i = \{p \in \{0,1\}^{\omega_1} : |\{j : \pi_j(p) = 1\}| \leq i\}$. Clearly $|A_i| = \omega_1$ for each $i \in N$ (as there are ω_1 finite subsets of ω_1), and each A_i is compact. Thus $A \notin \Phi'$, for if $A \in \Phi'$ then by 0.11, $\Sigma_{\omega_1} \in \Phi'$ as A is dense in Σ_{ω_1}. But then $\Sigma_{\omega_1} \in \Phi \cap \Phi'$, as Σ_{ω_1} is ω_0-bounded. This implies, again by Theorem 0.11, that Σ_{ω_1} is compact, which is false. Hence, $A \notin \Phi'$. Let D denote the discrete space of cardinality ω_1, and for each $i \in N$, let $f_i : \beta D \to A_i$ be a continuous, onto map. Let $f = \cup\{f_i : i \in N\}$. Then $f; N \times \beta D \to \cup\{A_i : i \in N\} = A$ and f is a continuous, onto map. Thus, by Theorem 0.11 iii), $N \times \beta D \notin \Phi'$ although $N \in \Phi'$ and βD is compact.

We now turn to the pseudocompactness peoperty of the extension property of m-bounded spaces. Let Φ_m denote the class of m-bounded spaces. Since $\Phi_m X$ consists of those points in βX which are in the closure of a subset of X of cardinality at most m, a space X is in Φ'_m if and only if every point of βX is in the closure of a subset of X of cardinality at most m. Obviously, if X has a dense subset of cardinality m, then $\Phi_m X = \beta X$, hence $X \in \Phi'_m$. If D denotes the discrete space of cardinality ω_1 and $X = D \cup \{p \in \beta D - D : p \text{ is a uniform ultrafilter on } D\}$, then $X \in \Phi'_{\omega_0}$, but X is not separable. Thus, the condition that the density of X be equal to m (the density of a space X, denoted by d(X), is the least cardinal of a dense subset of X) is not necessary for $X \in \Phi'_m$.

2.9 Definition: i) Let m be an infinite cardinal. A space X has weak covering number m if every open cover of X has a subfamily of cardinality at most m, whose union is dense in X.

ii) X is called pseudo-m^+-compact if every locally finite family of open sets in X has cardinality at most m.

2.10 Theorem: Let m be an infinite cardinal. Suppose that X is a space such that every point of βX is in the closure of a subset of X with weak covering number m. Then X is pseudo-m^+-compact.

Proof: Suppose that X is not pseudo-m^+-compact. Then X possesses a locally finite family $\mathcal{T} = \{U_i : i < m^+\}$ of non-empty open sets. We may assume, by Lemma 8 of [9], that $\mathcal{T}$ is also a pairwise disjoint family.

Let D denote the discrete space of cardinality m^+. Let $p \in \beta D - D$ be a uniform ultrafilter on D (i.e. an ultrafilter all of whose members have cardinality m^+ - Zorn's Lemma implies the existence of such ultrafilters). Choose $x_i \in U_i$ for all $i < m^+$. Let $S = \{x_i : i < m^+\}$. Then S is a closed, discrete, C-embedded subset of X (the discreteness of S follows from the fact that τ is a pairwise disjoint family, S is closed because a locally finite family of sets is closure preserving, and by 3L1 of [17], S is C-embedded). Let $f\colon D \to S$ be a bijection, hence a homeomorphism. Then $\mathrm{cl}_{\beta X}(S) = \beta S$ as S is C-embedded in X, and $\beta(f)\colon \beta D \to \beta S$ is a homeomorphism. In particular, $q = (\beta(f))(p) \in \beta S - S \subseteqq \beta X - X$. We show that q is not in the closure of any subset of X with weak covering number m. This contradiction will prove the theorem.

Suppose A is a subset of X with weak covering number m. We show that A must be pseudo-m^+-compact. Suppose A has a pairwise disjoint, locally finite family $\Omega = \{W_k : k < m^+\}$ of open sets. Let V_k be a non-empty open set such that $\mathrm{cl}_A(V_k) \subseteqq W_k$ for each $k < m^+$. Let $Z = A - \mathrm{cl}_A(\cup\{V_k : k < m^+\})$. Then, since a locally finite family is closure preserving, $\mathrm{cl}_A(\cup\{V_k : k < m^+\}) \subseteqq \cup\{W_k : k < m^+\}$ and hence $\{Z\} \cup \{W_k : k < m^+\}$ is an open cover of A. Clearly the union of no subfamily of cardinality at most m of this cover, will have dense union in A. Hence A does not have weak covering number m, contrary to hypothesis. Thus, A must be pseudo-m^+-compact.

Let $B = \{i < m^+ : A \cap U_i \neq \phi\}$. Then $|B| \leq m$, as A is pseudo-m^+-compact. For each $i < m^+$ let $f_i\colon X \to I$ be defined such that $f_i(x_i) = 1$ and $f_i(X - U_i) = \{0\}$ (this is possible as X is completely regular and Hausdorff). Let $T \subseteqq S$; then $\Sigma\{f_i : i \in T\}$ is a continuous function by the local finiteness of $\mathcal{T}$. Let us denote this function by g_T. Then $g_T(T) = \{1\}$ and $g_T(X - \cup\{U_j : j \in T\}) = \{0\}$. Let $T = S-B$. Then g_T completely separates $M = \{x_i : i \in S-B\}$ and $X - \cup\{U_i : i \in S-B\}$. But the latter set contains A. Thus $(cl_{\beta X}(M)) \cap (cl_{\beta X}(A)) = \phi$. But $q \in cl_{\beta X}(S) = (cl_{\beta X}(M)) \cup (cl_{\beta X}(S-M))$. Since p is a uniform ultrafilter and $|S-M| = |B| \leq m$, $q \notin cl_{\beta X}(S-M)$. Thus $q \in cl_{\beta X}(M)$, and hence $q \notin cl_{\beta X}(A)$.¤

It is clear that any space of cardinality not greater than m, has weak covering number m. Thus, the next corollary follows immediately from Theorem 2.10.

2.11 <u>Corollary</u>: If $X \in \Phi'_m$ then X is pseudo-m^+-compact.

It is well known that a space is pseudo-ω_0-compact (i.e. every locally finite family of open sets is finite) if and only if the space is pseudocompact. Also, an m-bounded space is pseudocompact for any $m \geq \omega_0$. Thus, every m-bounded space is pseudo-ω_0-compact, hence pseudo-m^+-compact for any $m \geq \omega_0$. So if X is m-bounded and non-compact, then X is pseudo-m^+-compact but $X \notin \Phi'_m$. Therefore the converse to Corollary 2.11 is false.

The remaining results on Φ'_m which we are about to present, are related to various set theoretic assumptions which we now explain. The continuum hypothesis which is denoted [CH], is well known and states that $\omega_1 = 2^{\omega_0} = c$. The generalized continuum hypothesis is denoted [GCH] and states that $m^+ = 2^m$ for all infinite cardinals m. Martin's Axiom is denoted [MA] and states th compact, Hausdorff space X with $c(X) = \omega_0$ cannot be the union of less than $c = 2^{\omega_0}$ nowhere dense subsets. The reader is referred to [31] and [34] for an investigation of Martin's Axiom. It is well known that [CH] and [GCH] are consistent with ZFC (the usual Zermelo - Fraenkel axioms of set theory along with the axiom of choice). It is shown in [31] that [MA] and [MA + $\neg$CH] (i.e. the assumption of Martin's Axiom and the negation of the continuum hypothesis) are also consistent with ZFC.

The following is a slight generalization of a Theorem in [28], in which it is shown that assuming $2^{\omega_0} < 2^{\omega_1}$, any separable, normal space has no uncountable discrete subset.

2.12 Theorem: Suppose m is an infinite cardinal number such that $2^m < 2^{(m^+)}$. Let X be a normal space such that $X \in \Phi'_m$. Then any closed discrete subset of X has cardinality at most m.

Proof: Suppose X contains a closed, discrete subset S of cardinality m^+. Then S is C-embedded in X as X is normal. Thus $cl_{\beta X}(S) = \beta S$ and $\beta S - S \subseteqq \beta X - X$. As in Theorem 2.10, let $p \in \beta S - S$ be a uniform ultrafilter on S. We show that p is not in the closure of any subset of X which has cardinality at most m.

Let A be a subset of X such that $|A| = m$. Then $|(cl_X(A)) \cap S| \leq m$. For if $|(cl_X(A)) \cap S| = m^+$, then $(cl_X(A)) \cap S$ is a discrete, C-embedded subset of $cl_X(A)$ of cardinality m^+, hence $|C(cl_X(A))| \geq 2^{(m^+)}$ (where $C(X)$ denotes the ring of continuous functions of a space X). But $cl_X(A)$ has a dense subset of cardinality m (namely A), thus $|C(cl_X(A))| \leq 2^m$. This contradiction shows that $|(cl_X(A)) \cap S| \leq m$.

Since X is normal, $cl_X(A)$ and $S - cl_X(A)$ are completely separated in X (every subset of S is closed in X). Hence $(cl_{\beta X}(A)) \cap (cl_{\beta X}(S - cl_X(A))) = \phi$. But $p \in cl_{\beta X}(S - cl_X(A))$ since p is a uniform ultrafilter. Thus $p \notin cl_X(A)$.¤

Under the assumption of [GCH], $2^m = m^+ < 2^{(m^+)}$ for any infinite cardinal number m, hence Theorem 2.12 is valid under that assumption. However, under the assumption of [MA + ¬CH] there is a separable, normal space X with a closed discrete subset of cardinality ω_1. This space is called the ω_1-Cantor tree (the reader is referred to Chapter 3 of [34] for a detailed investigation of this space). Thus $X \in \Phi'_{\omega_0}$ as X is separable, but X does not satisfy Theorem 2.12 (i.e. X shows that Theorem 2.12 is false under the new set-theoretic assumption of [MA + ¬CH]). The proof of Theorem 2.12 depended heavily on the assumption that $2^m < 2^{(m^+)}$. However, as is shown in Chapter 4 of [34], $2^{\omega_0} = 2^\kappa$ for all $\omega_0 \leq \kappa < c$ under the assumption of [MA]. Since $\omega_1 < c$ if we assume [MA + ¬CH], we have $2^{\omega_0} = 2^{\omega_1}$. However, we do have the following corollary to Theorem 2.12.

2.13 <u>Corollary</u>: Let X be a normal space such that $X \in \Phi'_m$. Then every closed discrete subset of X has cardinality less than 2^m (in fact, if $n = \min\{\alpha: 2^\alpha > 2^m\}$, then every closed discrete subset of X has cardinality less than n).

The proof of Corollary 2.13 is identical to the proof of Theorem 2.12.

We now show that the converse to Theorem 2.12 is false under the assumption of [CH] with $m = \omega_0$. A Souslin line S is a totally ordered space which is non-separable, but satisfies $C(S) = \omega_0$. It has been shown (see [27] or [36]) that the existence of a Souslin line is independent of the usual axioms of set theory (ZFC) and is consistent with [CH]. Theorem 2.8 of [29] shows that every discrete subset of S is countable, and S has a dense subset of cardinality ω_1. By 301 of [17] and the fact that $\omega_1^{\omega_0} = \omega_1$ (since $\omega_1 = c$ under [CH]), S has precisely ω_1 open sets. By (1) of Chapter 3 of [34], we may assume that every countable subset of S is nowhere dense. By 309 of [17], S is normal (as S is a totally ordered space). S is perfectly normal (i.e. every closed subset of S is a G_δ), hence every closed subset of S is a zero-set. In addition, S is Lindelöf and non-compact, hence S is not pseudocompact. Thus, S satisfies the hypotheses of Theorem 2.3 of [14] and therefore there exists a free z-ultrafilter A_p, on S such that every dense open set of S contains a member of A_p. Thus A_p cannot contain any nowhere dense closed subset of S. In particular, if B is a countable subset of S, $cl_S(B) \notin A_p$. Thus, the point $p \in \beta S - S$ to which the z-ultrafilter A_p converges, is not in the closure of any countable subset of S and hence $S \notin \Phi'_{\omega_0}$. This shows that the converse to Theorem 2.12 with $m = \omega_0$ is false.

Strongly-m-bounded spaces were defined earlier in this chapter as spaces in which any union of at most m compact subsets has compact closure. We show now that for a given cardinal m, the class Ψ_m of strongly m-bounded spaces is in $\mathcal{K}$. Let $\mathcal{E}_m$ denote the class of all spaces that are unions of at most m compact subspaces. Then $\mathcal{E}_m$ is closed under the formation of continuous images. Thus, by Theorem 1.13 of Chapter 1, $\overline{(\mathcal{E}_m)} \in \mathcal{K}$. Since $\overline{(\mathcal{E}_m)} = \Psi_m$, $\Psi_m \in \mathcal{K}$.

Recall from Chapter 1 that if $\Phi \in \mathcal{K}$, then Φ_0 consists of those spaces in Φ that are 0-dimensional, and $\Phi_0 \in \mathcal{K}_0$. Since every member of Γ_i (ordinal spaces with regular cardinality) is in $(\Psi_{\omega_0})_0$, $K_0\Gamma_i \subseteqq (\Psi_{\omega_0})_0$. The author has not been able to determine whether or not these two classes are equal, and this question is left open to the reader. We finish this chapter with a number of results concerning strongly-m-bounded spaces.

2.14 <u>Theorem</u>: Let X be a space and let m = sup{α:X is strongly-α-bounded}. If m is singular, or if $m < \omega_m$, then X is strongly-m-bounded.

Proof: m is singular. Let $\mathcal{T} = \{T_i : i < m\}$ be a family of compact subsets of X. Since m is singular, cf(m) < m, hence there is an $n < m$ and a family $\{s_j : j < n\}$ of cardinals such that $s_j < m$ for all $j < n$ and such that $\Sigma\{s_j : j < n\} = m$. Let $\mathcal{T}_j = \{T_i : i < s_j\}$ for all $j < n$. Then $\cup\mathcal{T}_j$ has compact closure in X for every $j < n$ as $|\mathcal{T}_j| = s_j < m$. But then $\cup\{\cup\mathcal{T}_j : j < n\}$ has compact closure in X as $n < m$. Thus X is strongly-m-bounded as $\mathcal{T} = \cup\{\cup\mathcal{T}_j : j < n\}$.

m is regular and $m < \omega_m$. Let $m = \omega_i$. Then i is a non-limit ordinal (if i is a limit ordinal, then $cf(m) \leq i$ hence, since m is regular, $m = cf(m) \le i$ and so $\omega_m \le \omega_i = m$ which is contrary to the hypothesis that $m < \omega_m$). Thus $i = j+1$ and $m = \omega_i = \omega(j+1) = (\omega_j)^+$. Clearly X is strongly-m-bounded.¤

On page 166 of [9], it is remarked that the usual axioms of set theory are consistent with the assumption that no uncountable regular limit cardinals exist. Under this hypothesis it is clear that $m < \omega_m$ for all regular cardinals m. Thus, under this set-theoretic hypothesis, the hypotheses in Theorem 2.14 are always satisfied. However, if we assume the existence of an uncountable regular limit cardinal m, then if $m = \omega_\alpha$(as a totally ordered space), ω_α is not strongly-m-bounded, although $m = \omega_\alpha = \sup\{t:\omega_\alpha$ is strongly-t-bounded$\}$.

2.15 <u>Theorem</u>: Suppose X is a locally compact, non-compact, strongly-m-bounded space for a given infinite cardinal m. Then $c(X) \ge m^+$.

Proof: Let $p_1 \in X$, and let U_1 be a compact neighborhood of p_1. Suppose we have chosen p_i, U_i for $i < \alpha < m^+$ such that U_i is a compact neighborhood of p_i and $U_i \cap cl_X(\cup\{U_j:j < i\}) = \phi$ for all $i < \alpha$. Then $cl_X(\cup\{U_i:i < \alpha\})$ is compact as $|\alpha| \le m$. Let $p_\alpha \in X - cl_X(\cup\{U_i:i < \alpha\})$ (this set is non-empty as X is non-compact), and let U_α be a compact neighborhood of p_α such that $U_\alpha \cap cl_X(\cup\{U_i:i < \alpha\}) = \phi$. By transfinite induction, we get a family $\{U_i:i < m^+\}$, of pairwise disjoint, compact, non-empty neighborhoods. Hence, $c(X) \ge m^+$.¤

Earlier in this chapter we showed that $\nabla \subseteq (K_0\Gamma)'$ by showing that $c(\Sigma_\alpha) = \omega_0$ for all $\Sigma_\alpha \in \nabla$. The above theorem shows that no locally compact, strongly-ω_0-bounded, non-compact space X has $c(X) = \omega_0$. This seems to strengthen the conjecture that $K_0\Gamma$ is the class of all strongly-ω_0-bounded, 0-dimensional spaces.

2.16 Theorem: Let m be an infinite cardinal. Let X be a locally compact, 0-dimensional and m-disconnected space (a space is m-disconnected if the union of no more than m clopen subsets of the space has clopen closure in the space). Let Ξ denote the family of all families of no more than m compact clopen subsets of X. Then $\Psi_m X = \cup\{cl_{\beta X}(\cup\tau):\tau \in \Xi\}$.

Proof: Let $T = \cup\{cl_{\beta X}(\cup\tau):\tau \in \Xi\}$. Clearly $T \subseteq \Psi_m X$. Let $\{A_i:i < m\}$ be a family of compact subsets of T. Let $i < m$ and $p \in A_i$. Then $p \in T$ and hence there is a family $\tau_p = \{V_j:j < m\}$ in Ξ such that $p \in cl_{\beta X}(\cup\tau_p)$. Since X is m-disconnected, $cl_{\beta X}(\cup\tau_p)$ is clopen in βX, hence is a neighborhood of p in βX. As A_i is compact, there is a finite set $\{p_1,\ldots,p_n\} \subseteq A_i$ such that $A_i \subseteq \cup\{cl_{\beta X}(\cup\tau_{p_j}):j = 1,\ldots n\}$. Let $\tau_i = \cup\{\tau_{p_j}:j = 1,\ldots n\}$. Then $A_i \subseteq cl_{\beta X}(\cup\tau_i)$ and $\tau_i \in \Xi$. Now, let $\tau = \cup\{\tau_i:i < m\}$. Then $\tau \in \Xi$, and $cl_{\beta X}(\cup\{A_i:i < m\}) \subseteq cl_{\beta X}(\cup\tau) \subseteq T$, and hence T is strongly-m-bounded. By the local compactness of X, $X \subseteq T$ (every point of X has a compact, clopen neighborhood). Thus, by Theorem 0.8 of Chapter 0, $T \supseteq \Psi_m X$, hence $T = \Psi_m X$.¤

Noting that an extremally disconnected space is m-disconnected for all infinite cardinals m, and a basically disconnected space is ω_0- disconnected, the next two corollaries follow immediately from Theorem 2.16.

2.17 Corollary: Let X be a locally compact, extremally disconnected space and let m be an infinite cardinal. Then $\Psi_m X$ is as in Theorem 2.16.

2.18 Corollary: Let X be a locally compact, basically disconnected space. Let Ξ denote the family of all countable families of compact, clopen subsets of X. Then
$\Psi_{\omega_0} X = \cup\{cl_{\beta X}(\cup\mathcal{T}) : \mathcal{T} \in \Xi\}$.

CHAPTER 3

THE TOPOLOGICAL EXTENSION OF A PRODUCT

In this chapter we are concerned with the problem of determining those conditions on an extension property Φ and spaces X and Y which imply or are equivalent to the equality $\Phi(X \times Y) = \Phi X \times \Phi Y$ (this equality means that there is a homeomorphism between the two spaces which fixes $X \times Y$ pointwise). A considerable amount of work has been done on this problem in the cases where Φ is the class of compact spaces or realcompact spaces. The case in which Φ is the class of compact spaces was solved in [19], where it was shown that the Stone-Čech compactification of a product space is equal to the product of the Stone-Čech compactifications of the factor spaces if and only if the product space is pseudocompact. The reader is referred to [8], [25] and [26] for partial solutions to the case where Φ is the class of realcompact spaces. In this chapter we show that for a 0-dimensional product space $\Pi\{X_\alpha : \alpha \in A\}$, we have the equality $\beta_0(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\beta_0 X_\alpha : \alpha \in A\}$ if and only if the product space is pseudocompact. We show that for any $\Phi \in \mathcal{H}$ and spaces X and Y, $\Phi(X \times Y) = \Phi X \times \Phi Y$ if and only if $X \times Y$ is C*-embedded in $\Phi(X \times Y)$. In addition, we show that if $\Phi \in \mathcal{H}$ such that every member of Φ is pseudocompact, then $\Phi(X \times Y) = \Phi X \times \Phi Y$ if and only if $X \times Y$ is pseudocompact. We then give an example (assuming the continuum hypothesis) to show that this statement is false if $X \times Y$ is replaced by an infinite product space. This contrasts the fact, mentioned above, that β distributes over a product space if and only if the product is pseudocompact. This fact was proved as Theorem 1 of [19]. This result plays a crucial role in much of the work of this chapter. We state it now without proof.

3.1 <u>Theorem</u>: Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces, and suppose $\Pi\{X_\alpha : \alpha \in (A - \{\alpha_0\})\}$ is infinite for every $\alpha_0 \in A$. Then the following statements are equivalent.

i) The topological product $\Pi\{X_\alpha : \alpha \in A\}$ is pseudocompact.

ii) $\beta(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\beta X_\alpha : \alpha \in A\}$.

The proof of the following lemma is straightforward and is left to the reader.

3.2 <u>Lemma</u>: Let $\{X_\alpha : \alpha \in A\}$, $\{Y_\alpha : \alpha \in A\}$ be families of spaces, and for each $\alpha \in A$ let $f_\alpha : X_\alpha \to Y_\alpha$ be a perfect map. Let f denote the map from $\Pi\{X_\alpha : \alpha \in A\}$ to $\Pi\{Y_\alpha : \alpha \in A\}$ which takes the point $((x_\alpha)_\alpha)$ to the point $((f(x_\alpha))_\alpha)$ (i.e. f is the product of the f_α's). Then f is a perfect map.

Recall that if $\Psi \in \mathcal{H}_0$, then $I \times \Psi \in \mathcal{H}$ and $I \times \Psi$ consists of those spaces that can be embedded as closed subsets of $I^m \times P$, where m is a cardinal and $P \in \Psi$. The following result links the $I \times \Psi$ extension of a 0-dimensional product with its Ψ-extension. Note that the equality $\Phi(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\Phi X_\alpha : \alpha \in A\}$ means that $\Pi\{\Phi X_\alpha : \alpha \in A\}$ is the unique maximal Φ-extension of $\Pi\{X_\alpha : \alpha \in A\}$, i.e. every map from $\Pi\{X_\alpha:\alpha \in A\}$ to a space in Φ admits a continuous extension to $\Pi\{\Phi X_\alpha:\alpha \in A\}$, or equivalently, there is a homeomorphism from $\Phi(\Pi\{X_\alpha:\alpha \in A\})$ to $\Pi\{\Phi X_\alpha:\alpha \in A\}$ which fixes $\Pi\{X_\alpha:\alpha \in A\}$ pointwise.

3.3 Lemma: Let $\{X_\alpha:\alpha \in A\}$ be a family of 0-dimensional spaces. Let $i_\alpha: X_\alpha \to \beta_0 X_\alpha$ be the identity embedding and $\beta(i_\alpha): \beta X_\alpha \to \beta_0 X_\alpha$ the Stone extension of i_α. Let $k: \Pi\{\beta X_\alpha:\alpha \in A\} \to \Pi\{\beta_0 X_\alpha:\alpha \in A\}$ be defined by $k((x_\alpha)_\alpha) = ((\beta(i_\alpha))(x_\alpha))_\alpha$ (i.e. k is the product of the maps $\beta(i_\alpha)$ for α in A; note that $(x_\alpha)_\alpha$ denotes a point whose α-th coordinate is x_α). Let $f: \Pi\{\beta X_\alpha:\alpha \in A\} \to Y$ be a continuous map where Y is 0-dimensional. Then f is constant on all sets of the form $k^{\leftarrow}(\{p\})$ where $p \in \Pi\{\beta_0 X_\alpha:\alpha \in A\}$.

Proof: Note that if X is 0-dimensional, $i: X \to \beta_0 X$ is the identity embedding and $\beta(i): \beta X \to \beta_0 X$ is the Stone extension of i, and if $f: \beta X \to Y$ where Y is 0-dimensional, then $f|X: X \to Y$ and hence admits an extension $\beta_0(f|X): \beta_0 X \to \beta_0 Y$. By Theorem 1.4 of Chapter 1, $\beta_0(f|X) \cdot \beta(i) = f$, thus f must be constant on sets of the form $\beta(i)^{\leftarrow}(\{p\})$ where $p \in \beta_0 X$. So the statement of the Lemma is true for a family consisting of one 0-dimensional space.

Suppose X and Y are 0-dimensional spaces. Let $i: X \to \beta_0 X$ and $j: Y \to \beta_0 Y$ be the identity embeddings. Let $f: \beta X \times \beta Y \to Z$ where Z is a 0-dimensional space. Let $p = (p_X,p_Y) \in \beta_0 X \times \beta_0 Y$ and $a,b \in \beta X \times \beta Y$ where $a = (a_X,a_Y)$, $b = (b_X,b_Y)$ and suppose that $(\beta(i) \times \beta(j))(a) = (\beta(i) \times \beta(j))(b) = p$; i.e. $(\beta(i))(a_X) = (\beta(i))(b_X) = p_X$ and similarly for Y. Since the Lemma is true for a family of one 0-dimensional space, and noting that $\beta X \times \{a_Y\}$ is homeomorphic to X, we must have that $f(a) = f((a_X,a_Y)) = f((b_X,a_Y))$. Similarly, as $\{b_X\} \times \beta Y$ is homeomorphic to βY, we must have that $f((b_X,a_Y)) = f((b_X,b_Y)) = f(b)$. Thus $f(a) = f(b)$ and the Lemma is true for a family of two 0-dimensional spaces. A simple induction shows that the Lemma is valid for any finite family of 0-dimensional spaces.

Let $f\colon \Pi\{\beta X_\alpha : \alpha \in A\} \to Y$ where Y is 0-dimensional. Let $q = (q_\alpha)_\alpha$, $r = (r_\alpha)_\alpha \in \Pi\{\beta X_\alpha : \alpha \in A\}$, and $p = (p_\alpha)_\alpha \in \Pi\{\beta_0 X_\alpha : \alpha \in A\}$ be such that $k(q) = k(r) = p$, i.e. $(\beta(i_\alpha))(q_\alpha) = (\beta(i_\alpha))(r_\alpha) = p_\alpha$ for all α in A. Let $F = \{\alpha_1,\ldots,\alpha_n\} \subseteqq A$ be a finite subset of A. By what we have just shown above, $f((q_\alpha)_\alpha) = f((r_{\alpha_1},\ldots,r_{\alpha_n},q_\alpha)_{\alpha \in A-F})$. However, since f is continuous, and r is the limit in $\Pi\{\beta X_\alpha : \alpha \in A\}$ of the net of points $\{(r_{\alpha_1},\ldots,r_{\alpha_n},q_\alpha)_{\alpha \in A-F} : F = \{\alpha_1,\ldots,\alpha_n\} \subseteqq A \text{ is finite}\}$, we have that $f(r) = f(q)$.¤

3.4 Theorem: Let $\Psi \in \mathcal{H}_0$ and let $\{X_\alpha : \alpha \in A\}$ be a family of 0-dimensional spaces. If $(I \times \Psi)(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{(I \times \Psi)X_\alpha : \alpha \in A\}$ then $\Psi(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\Psi X_\alpha : \alpha \in A\}$.

Proof: Recall from Theorem 1.6 of Chapter 1 that $(I \times \Psi)X_\alpha = (\beta(i_\alpha))^{\leftarrow}(\Psi X_\alpha)$ for every $\alpha \in A$. Since $\beta(i_\alpha)\colon \beta X_\alpha \to \beta_0 X_\alpha$ is a perfect map and $(I \times \Psi)X_\alpha$ is a total preimage of a subset of $\beta_0 X_\alpha$, $\beta(i_\alpha)|(I \times \Psi)X_\alpha$ is a perfect map. In order to prove the theorem, it is enough to show that any map from $\Pi\{X_\alpha : \alpha \in A\}$ to Y where $Y \in \Psi$, admits a continuous extension to $\Pi\{\Psi X_\alpha : \alpha \in A\}$(by the uniqueness of the extension).

The map k of Lemma 3.3 is a perfect map by Lemma 3.2. Hence, $k|\Pi\{(I \times \Psi)X_\alpha : \alpha \in A\}$ is a perfect map, being the restriction of k to a total preimage. Let us denote this restriction by h. Let $f: \Pi\{X_\alpha : \alpha \in A\} \to Y$ where $Y \in \Psi$. By hypothesis, there is a map $g: \Pi\{(I \times \Psi)X_\alpha : \alpha \in A\} \to Y$ which is a continuous extension of f. By Lemma 3.3, g must be constant on sets of the form $h^{\leftarrow}(\{p\})$, where $p \in \Pi\{\Psi X_\alpha : \alpha \in A\}$ (as Y is 0-dimensional because $\Psi \in \mathcal{K}_0$). Thus, we can define a map $\overline{f}: \Pi\{\Psi X_\alpha : \alpha \in A\} \to Y$ by $\overline{f}(p) = g(h^{\leftarrow}(\{p\}))$. The function $\overline{f}$ is well-defined, for, as we have said, g is constant on fibers of h. Furthermore, since h is a perfect map, and $\overline{f} \cdot h = g$ is a continuous map, we must have that $\overline{f}$ is a continuous map (since h is perfect, it is a closed map and hence is a quotient map). Clearly $\overline{f}$ is an extension of the map f. This completes the proof.¤

The following result can be found as Lemma 1.2 of [15], and is stated without proof.

3.5 <u>Lemma</u>: Let X and Y be infinite spaces. If the topological product $X \times Y$ is not pseudocompact, then there exists a locally finite sequence $\{U_n \times V_n : n \in N\}$ of non-empty, canonical open subsets of $X \times Y$ such that the sequences $\{U_n : n \in N\}$ and $\{V_n : n \in N\}$ are pairwise disjoint families in X and Y respectively.

3.6 <u>Theorem</u>: The following conditions on infinite, 0-dimensional spaces X and Y are equivalent.

i) $X \times Y$ is pseudocompact.

ii) $\beta_0(X \times Y) = \beta_0 X \times \beta_0 Y$.

Proof: i) $\Rightarrow$ ii). Let Ψ be the class of compact 0-dimensional spaces. Then $I \times \Psi$ is the class of all compact spaces. By Theorem 3.1, $\beta(X \times Y) = \beta X \times \beta Y$. Since $\Psi X = \beta_0 X$ and $(I \times \Psi)X = \beta X$ we invoke Theorem 3.4 to get the result.

ii) $\Rightarrow$ i). Suppose $X \times Y$ is not pseudocompact. By Lemma 3.5 there is a locally finite sequence $\{U_n \times V_n : n \in N\}$ of canonical open subsets of $X \times Y$ such that $\{U_n : n \in N\}$ and $\{V_n : n \in N\}$ are pairwise disjoint sequences in X and Y respectively. Since X and Y are 0-dimensional, U_n and V_n can be taken to be clopen for all n. Let $U = \cup\{U_n \times V_n : n \in N\}$. Clearly U is open. U is also closed since $\{U_n \times V_n : n \in N\}$ is a locally finite sequence (hence closure preserving) and each $U_n \times V_n$ is clopen.

$\beta_0 X \times \beta_0 Y$ is compact, hence pseudocompact, so the sequence $\{cl_{\beta_0 X}(U_n) \times cl_{\beta_0 Y}(V_n) : n \in N\}$ must have a cluster point $(p,q) \in \beta_0 X \times \beta_0 Y$. Let $V = (X \times Y) - U$. Suppose $A \times B$ is a neighborhood of (p,q) in $\beta_0 X \times \beta_0 Y$. Then there are elements $n_1, n_2 \in N$ such that $n_1 \neq n_2$ and $A \cap U_{n_i} \neq \phi$, $B \cap V_{n_i} \neq \phi$ for $i = 1,2$ (as (p,q) is a cluster point of the sequence of canonical open sets). Let $x_i \in A \cap U_{n_i}$, $y_i \in B \cap V_{n_i}$ for $i = 1,2$. Then $(x_1,y_2) \in A \times B$. But $(x_1,y_2) \notin U$ (otherwise there is an $n \in N$ such that $(x_1,y_2) \in U_n \times V_n$, and hence $x_1 \in U_n \cap U_{n_1}$, $y_2 \in V_n \cap V_{n_2}$ so that $n_1 = n = n_2$ which is false). Thus $(x_1,y_2) \in V$ and hence $(p,q) \in cl_{\beta_0 X \times \beta_0 Y}(V)$.

Since U is clopen in $X \times Y$, the function $f: X \times Y \to \{0,1\}$

defined by $f(U) = \{0\}$, $f(V) = \{1\}$ is continuous. But $(p,q) \in (cl_{\beta_0 X \times \beta_0 Y}(U)) \cap (cl_{\beta_0 X \times \beta_0 Y}(V))$ hence f cannot extend continuously to (p,q). Thus $\beta_0(X \times Y) \neq \beta_0 X \times \beta_0 Y$ contradicting ii). Therefore $X \times Y$ is pseudocompact.¤

3.7 Theorem: Let $\{X_\alpha : \alpha \in A\}$ be a family of 0-dimensional spaces such that $\Pi\{X_\alpha : \alpha \in A - \{\alpha_0\}\}$ is infinite for every $\alpha_0 \in A$. Then the following are equivalent.

i) $\Pi\{X_\alpha : \alpha \in A\}$ is pseudocompact.

ii) $\beta_0(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\beta_0 X_\alpha : \alpha \in A\}$.

Proof: i) ⇒ ii). Letting Ψ be the class of all compact 0-dimensional spaces, the result follows from Theorems 3.1 and 3.4.

ii) ⇒ i). In view of Theorem 3.6 above, the proof of this implication is identical to that given in Theorem 1 of [19].¤

The notion of E-compactness introduced in [13] is closely related to that of an extension property. If E is a topological space (not necessarily completely regular and Hausdorff) then a space X is called E-compact if X can be embedded as a closed subset of a power of the space E, and X is called E-completely regular if X can be embedded as a subspace of some power of the space E. Every E-completely regular space X has a maximal E-compactification (i.e. E-compact extension) $\beta_E X$ which is an E-compact space in which X is densely embedded such that any continuous map from X to E admits a continuous extension to $\beta_E X$. In general, classes of

E-compactness are not extension properties (for instance if E is not completely regular and Hausdorff), and not all extension properties are classes of E-compactness for some space E (it is shown in Theorem 4.7 of [43] that the class of ω_0-bounded spaces is not contained in any class of E-compactness). However, a simple modification of the proofs of Theorems 3.4 and 3.7 yields the following theorem.

3.8 Theorem: Let E be a compact, Hausdorff space. Suppose $\{X_\alpha : \alpha \in A\}$ is a family of E-completely regular spaces such that $\Pi\{X_\alpha : \alpha \in (A - \{\alpha_0\})\}$ is infinite for every $\alpha_0 \in A$. Then condition i) implies condition ii).

i) $\Pi\{X_\alpha : \alpha \in A\}$ is pseudocompact.

ii) $\beta_E(\Pi\{X_\alpha : \alpha \in A\}) = \Pi\{\beta_E X_\alpha : \alpha \in A\}$.

These two conditions are shown to be equivalent for E = I, the closed unit interval, in Theorem 3.1 (Theorem 1 of [19]), and Theorem 3.7 above shows the two conditions to be equivalent for E = {0,1}, the two-point discrete space. However, the author has not been able to prove that ii) implies i) in the more general case where E is an arbitrary compact, Hausdorff space. This problem is left open to the reader.

Recall from Theorem 0.4 that if $\Phi, \Psi \in \mathcal{K}$, such that $\Phi \subseteq \Psi$, then $X \subseteq \Psi X \subseteq \Phi X \subseteq \beta X$ for any space X. The following result is mentioned without proof in [10], and has some interesting consequences.

3.9 Proposition: Suppose Φ, $\Psi \in \mathcal{K}$, such that $\Phi \subseteq \Psi$. If, for two spaces X and Y, $\Phi(X \times Y) = \Phi X \times \Phi Y$, then $\Psi(X \times Y) = \Psi X \times \Psi Y$.

Proof: Let $f:(X \times Y) \to Z$ be a continuous map where $Z \in \Psi$. Let $f_y = f|(X \times \{y\})$ for each $y \in Y$. Since $\Phi Z \in \Phi$, $(\beta(f))(\Phi X \times \Phi Y) \subseteq \Phi Z$ by hypothesis. Since $X \times \{y\}$ is homeomorphic to X for every y in Y, $(\beta(f_y))(\Psi X \times \{y\}) \subseteq Z$ as $Z \in \Psi$. It is clear by the continuity of the maps that $\beta(f)|(\Psi X \times \{y\}) = \beta(f_y)|(\Psi X \times \{y\})$ for all y in Y. Thus, $(\beta(f))(\Psi X \times Y) \subseteq Z$. By a similar argument for points in ΨX, it is apparent that $(\beta(f))(\Psi X \times \Psi Y) \subseteq Z$. Thus every map $f: (X \times Y) \to Z$ where $Z \in \Psi$, admits a continuous extension to $\Psi X \times \Psi Y$, i.e. $\Psi(X \times Y) = \Psi X \times \Psi Y$.¤

It is obvious that a similar proof to the one above would apply if $\mathcal{K}$ were replaced by $\mathcal{K}_0$. One immediate result of Proposition 3.9 is that if $\Phi \in \mathcal{K}$, and $X \times Y$ is pseudocompact, then $\Phi(X \times Y) = \Phi X \times \Phi Y$ by virtue of the fact that $\beta(X \times Y) = \beta X \times \beta Y$ (Theorem 3.1) and the observation that the class of compact spaces is contained in Φ for any $\Phi \in \mathcal{K}$.

We know that βX is characterized as that compactification of X in which X is C*-embedded (and for 0-dimensional X, $\beta_0 X$ is that 0-dimensional compactification of X in which X is {0,1}-embedded, i.e. every {0,1}-valued continuous map on X can be extended to $\beta_0 X$). Thus $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is C*-embedded in $\beta X \times \beta Y$. We now show that the C*-embedding of $X \times Y$ in $\Phi X \times \Phi Y$ characterizes the distributivity of Φ over the product $X \times Y$.

3.10 Theorem: Let $\Phi \in \mathcal{K}$, and let X and Y be spaces. Then $\Phi(X \times Y) = \Phi X \times \Phi Y$ if and only if $X \times Y$ is C*-embedded in $\Phi X \times \Phi Y$.

Proof: Necessity. Let $f:(X \times Y) \to R$ be a bounded continuous map. Then $\mathrm{cl}_R(f(X \times Y))$ is compact, hence is in Φ. Since $\Phi(X \times Y) = \Phi X \times \Phi Y$, we have that $(\beta(f))(\Phi X \times \Phi Y) \subseteq \mathrm{cl}_R(f(X \times Y))$. Thus $X \times Y$ is C*-embedded in $\Phi X \times \Phi Y$ (alternatively, since $Z \subseteq \Phi Z \subseteq \beta Z$ for any space Z by Theorem 0.3, we have that $X \times Y \subseteq \Phi(X \times Y) = \Phi X \times \Phi Y \subseteq \beta(X \times Y)$, hence $X \times Y$ must be C*-embedded in $\Phi X \times \Phi Y$ as $X \times Y$ is C*-embedded in $\beta(X \times Y)$).

Sufficiency. Since $X \times Y$ is C*-embedded in $\Phi X \times \Phi Y$, we must have that $X \times Y \subseteq \Phi X \times \Phi Y \subseteq \beta(X \times Y)$. Now, by a proof identical to that of Proposition 3.9, it can be shown that if $f:(X \times Y) \to Z$ where $Z \in \Phi$, then $(\beta(f))(\Phi X \times \Phi Y) \subseteq Z$ and hence $\Phi(X \times Y) = \Phi X \times \Phi Y$.¤

Note that Theorem 3.10 remains valid if $\Phi \in \mathcal{K}_0$, X and Y are 0-dimensional, and C*-embedding is replaced by $\{0,1\}$-embedding.

If $\Phi \in \mathcal{K}$, then $X \subseteq \Phi X \subseteq \beta X$ for any space X. Hence, $\beta(\Phi X) = \beta X$ for any $\Phi \in \mathcal{K}$ and any space X.

3.11 Theorem: Let $\Phi \in \mathcal{K}$ such that every member of Φ is pseudocompact. Let X and Y be any infinite spaces. Then the following are equivalent.

i) $X \times Y$ is pseudocompact.

ii) $\Phi(X \times Y) = \Phi X \times \Phi Y$.

Proof: i) $\Rightarrow$ ii). If $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$. Hence, by Proposition 3.9, $\Phi(X \times Y) = \Phi X \times \Phi Y$ (alternatively, since

$X \times Y \subseteqq \Phi X \times \Phi Y \subseteqq \beta X \times \beta Y$, and $X \times Y$ is C*-embedded in $\beta X \times \beta Y$, $X \times Y$ must be C*-embedded in $\Phi X \times \Phi Y$, and by Theorem 3.10 we obtain the desired result).

ii) $\Rightarrow$ i). If $\Phi(X \times Y) = \Phi X \times \Phi Y$, then by Theorem 3.10, $X \times Y$ is C*-embedded in $\Phi X \times \Phi Y$. By hypothesis, $\Phi X \times \Phi Y$ is pseudocompact. Thus, by Theorem 3.1, $\beta(\Phi X \times \Phi Y) = \beta(\Phi X) \times \beta(\Phi Y) = \beta X \times \beta Y$ (as noted above, $\beta(\Phi X) = \beta X$ for all X). Therefore, $\Phi X \times \Phi Y$ is C*-embedded in $\beta X \times \beta Y$, so that $X \times Y$ must be C*-embedded in $\beta X \times \beta Y$. By Theorem 3.10, this is equivalent to $\beta(X \times Y) = \beta X \times \beta Y$. Thus, by Theorem 3.1, $X \times Y$ is pseudocompact.¤

As the following result shows, if not every member of Φ is pseudocompact, then Theorem 3.11 cannot hold.

3.12 <u>Proposition</u>: Let $\Phi \in \mathcal{K}$. The following are equivalent.

i) For any infinite spaces X and Y, $\Phi(X \times Y) = \Phi X \times \Phi Y$ if and only $X \times Y$ is pseudocompact.

ii) Every member of Φ is pseudocompact.

Proof: i) $\Rightarrow$ ii). Suppose $X \in \Phi$ and X is not pseudocompact. Then X contains a closed, C-embedded copy of N (see Corollary 1.20 of [17]), and in particular, $N \in \Phi$. Then $\Phi(N \times N) = N \times N = \Phi N \times \Phi N$, but $N \times N$ is not pseudocompact which contradicts i).

ii) $\Rightarrow$ i). This is the statement of Theorem 3.11.¤

In view of Theorem 3.6, it is clear that Theorem 3.11 and Proposition 3.12 remain valid if $\mathcal{K}$ is replaced by $\mathcal{K}_0$, and X and Y are assumed to be 0-dimensional spaces. It is also easy to see by means of a simple induction, that Proposition 3.9 and Theorems 3.10 and 3.11 remain valid for any finite product of spaces. We now give an example (assuming the continuum hypothesis) that shows that 3.9, 3.10 and 3.11 are false in the case of an infinite product space.

3.13 Example: Under the assumption of the continuum hypothesis, it has been shown in Theorem 4.2 of [35] that $\beta N - N$ has 2^c P-points (c denotes the cardinality of the continuum, and a point x of a space X is a P-point of X if every G_δ set containing x is a neighborhood of x). If we denote by A the set of P-points in $\beta N - N$, then for every $p \in A$, $\beta N - \{p\}$ is a pseudocompact, locally compact space (as $\beta N - \{p\}$ is almost compact - see 6J of [17]). Thus, by Theorem 4 of [19] (which states that a product of pseudocompact, locally compact spaces is pseudocompact) $\Pi\{(\beta N - \{p\}) : p \in A\}$ is a pseudocompact space. Let Φ be the class of ω_0-bounded spaces. Recall from Chapter 2 that ΦX consists of the set of points in βX which are in the closure of a countable subset of X. Thus, if X is separable, $X \in \Phi'$. Note that since $\Pi\{(\beta N - \{p\}) : p \in A\}$ is pseudocompact, $\beta(\Pi\{(\beta N - \{p\}) : p \in A\}) = \Pi\{\beta(\beta N - \{p\}) : p \in A\} = \Pi\{\beta N : p \in A\}$ by Theorem 3.1. Since $\beta N - \{p\}$ is separable, $\Phi(\beta N - \{p\}) = \beta(\beta N - \{p\}) = \beta N$. Thus $\Pi\{\Phi(\beta N - \{p\}) : p \in A\} = \Pi\{\beta N : p \in A\}$. We show that the point $(x_p)_{p \in A} \in \Pi\{\beta N : p \in A\}$ defined by $x_p = p$, is not in the closure of any countable subset of $\Pi\{(\beta N - \{p\}) : p \in A\}$. Hence, $\Phi(\Pi\{(\beta N - \{p\}) : p \in A\}) \subsetneq \Pi\{\beta N : p \in A\} = \Pi\{\Phi(\beta N - \{p\}) : p \in A\}$. Since every ω_0-bounded space is countably compact, hence pseudocompact, this will show that 3.9, 3.10 and 3.11 are false if the finite product is replaced by an infinite product. We shall require some preliminary remarks.

In the space βN, the points of $\beta N-N$ are free ultrafilters of subsets of N. In what follows, a free ultrafilter $\mathcal{E}$, of subsets of N, will be regarded both as an ultrafilter of subsets of N and as a point of $\beta N-N$.

Let $\{x_n : n \in N\}$ be a sequence in a space X, and let $\mathcal{E}$ be a free ultrafilter on N. Following [3], we call a point x in X a $\mathcal{E}$-limit of the sequence $\{x_n : n \in N\}$ if given a neighborhood U of x, the set $\{n : x_n \in U\}$ is a member of the ultrafilter $\mathcal{E}$. It is straightforward to check that if $f: X \to Y$ is a continuous map and x is the $\mathcal{E}$-limit in X of $\{x_n : n \in N\}$, then $f(x)$ is the $\mathcal{E}$-limit in Y of the sequence $\{f(x_n) : n \in N\}$(we speak of x being <u>the</u> $\mathcal{E}$-limit of a sequence, because, as is easily verified, in a Hausdorff space, $\mathcal{E}$-limits are unique when they exist). Let X be a space and $D = \{x_n : n \in N\}$ a sequence in X such that $x \in cl_X(D) - D$. Then there exists a free ultrafilter $\mathcal{E}$, on N such that x is the $\mathcal{E}$-limit of $\{x_n : n \in N\}$ (we can take the sets $B_U = \{n \in N : x_n \in U\}$, where U is a neighborhood of x, as the base for a free ultrafilter on N). Suppose that $\mathcal{E}$ is a free ultrafilter on N. Then there are at most $c = 2^{\omega_0}$ points x in $\beta N-N$ such that x is the $\mathcal{E}$-limit of a sequence of points in N. This is true because there are c sequences of elements of N, and $\mathcal{E}$-limits are unique in the Hausdorff space βN.

Now, suppose that the point $(x_p)_{p\in A} \in \Pi\{\beta N:p \in A\}$, defined by $x_p = p$, is in the closure of some countable subset of $\Pi\{(\beta N-\{p\}):p \in A\}$. Let this countable set be $\{x_n:n \in N\}$. Then there exists a free ultrafilter $\mathcal{E}$, on N, such that $(x_p)_{p \in A}$ is the $\mathcal{E}$-limit of $\{x_n:n \in N\}$. Let π_p denote the p-th projection map from $\Pi\{\beta N:p \in A\}$ to βN. Then for each $p \in A$ we must have that $\pi_p(x)$ is the $\mathcal{E}$-limit in βN of the sequence $\{\pi_p(x_n):n \in N\}$ (where $x = (x_p)_{p \in A}$). But we know that there are at most c points in $\beta N-N$ which are the $\mathcal{E}$-limit of a sequence in N. Since A has cardinality 2^c, we can find a point $p_0 \in A$ such that p_0 is not the $\mathcal{E}$-limit of any sequence in N. Since p_0 is a P-point of $\beta N-N$, and $\{\pi_{p_0}(x_n):n \in N\}$ is a countable set not containing p_0, there exists a neighborhood U of p_0 in βN such that

$U \cap (\beta N-N) \cap \{\pi_{p_0}(x_n):n \in N\} = \phi$. Let us define a sequence $\{y_n:n \in N\}$ in N by

$y_n = 1$ if $\pi_{p_0}(x_n) \in \beta N-N$,

$y_n = \pi_{p_0}(x_n)$ otherwise.

Let V be a neighborhood of p_0 in βN. Then $V \cap U \cap (\beta N-\{1\})$ is a neighborhood in βN of p_0. Since p_0 is the $\mathcal{E}$-limit of $\{\pi_{p_0}(x_n):n \in N\}$, the set $S = \{n:\pi_{p_0}(x_n) \in V \cap U \cap (\beta N-\{1\})\} \in \mathcal{E}$. But $S \subseteq \{n:y_n \in V\}$, hence $\{n:y_n \in V\} \in \mathcal{E}$. Thus p_0 is the $\mathcal{E}$-limit of $\{y_n:n \in N\}$, a sequence in N. This contradicts the fact that p_0 is not the $\mathcal{E}$-limit of any sequence in N. Thus $(x_p)_{p \in A}$ is not in the closure of any countable subset of $\Pi\{(\beta N-\{p\}):p \in A\}$, and the example is completed.

Note that if $\Pi\{X_\alpha:\alpha \in B\}$ is pseudocompact and $|B| \leq c$, then $\Phi(\Pi\{X_\alpha:\alpha \in B\}) = \Pi\{\Phi X_\alpha:\alpha \in B\}$ where Φ is the class of ω_0-bounded spaces. This is true because any product of at most c separable spaces is separable.

The proof of the following theorem gives a method which can often be useful in showing that an extension property distributes over a pseudocompact product space. Recall that Σ_{ω_1} denotes the set of points in $\{0,1\}^{\omega_1}$ at most countably many of whose coordinates are 1. Let $\Phi = K_0\{\Sigma_{\omega_1}\}$.

3.14 Theorem: Let $\{X_\alpha:\alpha \in A\}$ be a family of 0-dimensional spaces such that $\Pi\{X_\alpha:\alpha \in (A - \{\alpha_0\})\}$ is infinite for every $\alpha_0 \in A$. Then the following are equivalent.

i) $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact.

ii) $\Phi(\Pi\{X_\alpha:\alpha \in A\}) = \Pi\{\Phi X_\alpha:\alpha \in A\}$.

Proof: i) $\Rightarrow$ ii). By Theorem 3.7 $\beta_0(\Pi\{X_\alpha:\alpha \in A\}) = \Pi\{\beta_0 X_\alpha:\alpha \in A\}$. Let $T = \Pi\{\Phi X_\alpha:\alpha \in A\} \subseteq \Pi\{\beta_0 X_\alpha:\alpha \in A\}$. By Theorem 0.8, $\Phi(\Pi\{X_\alpha:\alpha \in A\}) \subseteq T$. Suppose $p = (p_\alpha)_{\alpha \in A} \in (T - \Phi(\Pi\{X_\alpha:\alpha \in A\}))$. By Theorem 2.6 of Chapter 2, there is a family $\{U_i:i < \omega_1\}$ of canonical clopen neighborhoods of p such that $|\{i < \omega_1 :x \in U_i\}| \leq \omega_0$ for every $x \in \Pi\{X_\alpha:\alpha \in A\}$. Since $p \in T$, $p_\alpha \in \Phi X_\alpha$ for all $\alpha \in A$. We now employ a result about regular uncountable cardinals that can be found as Lemma 10 of [9]. If m is a regular uncountable cardinal and $\{S_\lambda:\lambda < m\}$ is a collection of (not necessarily distinct) finite sets, then there is a (possibly empty) set F and subset Ω of m such that $|\Omega| = m$ and $S_k \cap S_j = F$ whenever k and j are distinct members of Ω. Since for each $i < \omega_1$ U_i is a canonical clopen subset of $\Pi\{\beta_0 X_\alpha : \alpha \in A\}$, $U_i = \Pi\{U_i^\alpha : \alpha \in A\}$ where $U_i^\alpha = \beta_0 X_\alpha$ for all but finitely many α. Let S_i denote the finite subset of A for which $U_i^\alpha \neq \beta_0 X_\alpha$. Since ω_1 is regular and uncountable, there is a subset $\Omega \subseteqq \omega_1$ and a finite set $F \subseteqq A$ such that $|\Omega| = \omega_1$, and $S_j \cap S_k = F$ for all $j \neq k$, $j,k \in \Omega$. Let $F = \{\alpha_1,\ldots,\alpha_n\}$. Since $p_\alpha \in \Phi X_\alpha$ for each $\alpha \in A$, and $|\{U_i^\alpha : i \in \Omega\}| = \omega_1$ for each $\alpha \in A$, by Theorem 2.6 of Chapter 2 there must be a point $x_1 \in X_{\alpha_1}$ and a set $\Omega_1 \subseteqq \Omega$ such that $|\Omega_1| = \omega_1$ and $x_1 \in (\cap\{U_i^{\alpha_1} : i \in \Omega_1\}) \cap X_{\alpha_1}$. By induction we get subsets $\Omega_n \subseteqq \Omega_{n-1} \subseteqq \cdots \subseteqq \Omega_1 \subseteqq \Omega$ and points x_t for $t = 1,\ldots n$ such that $|\Omega_t| = \omega_1$ and $x_t \in (\cap\{U_i^{\alpha_t} : i \in \Omega_t\}) \cap X_{\alpha_t}$. If $\alpha \in A-F$, choose $x_\alpha \in U_i^\alpha \cap X_\alpha$ (there is at most one $i \in \Omega_n$ such that $U_i^\alpha \neq \beta_0 X_\alpha$ by the condition that $S_i \cap S_j = F$ for $i \neq j$, $i,j \in \Omega$). But then the point $(x_1,\ldots,x_n,x_\alpha)_{\alpha \in A-F}$ is in the set $(\Pi\{X_\alpha : \alpha \in A\}) \cap (\cap\{U_i : i \in \Omega_n\})$. This contradicts the assumption that a point of $\Pi\{X_\alpha : \alpha \in A\}$ is in at most ω_0 of the U_i's. Thus $T \in \Phi(\Pi\{X_\alpha : \alpha \in A\})$.

ii) $\Rightarrow$ i). Since every member of Φ is pseudocompact, this implication follows from Theorem 3.7.¤

Essentially, in the proof of Theorem 3.14, we were able to reduce the infinite product case to a finite product, $\Pi\{X_\alpha : \alpha \in F\}$, by means of Lemma 10 of [9]. By the nature of the proof, the theorem remains valid if ω_1 is replaced by any other regular uncountable cardinal.

Recall from Chapter 2 that it was shown that if Φ is the class of ω_0-bounded spaces, then $N \in \Phi'$ but $N \times \beta D \notin \Phi'$, where D denotes the discrete space of cardinality ω_1. This answered the question, asked in [43], of whether or not given $\Phi \in \mathcal{H}$, $X \in \Phi'$ and K compact, is it then true that $X \times K \in \Phi'$.

However, suppose $\Phi \in \mathcal{H}$ and $R \in \Phi$. Then $K\{R\} \subseteq \Phi$, hence $\Phi' \subseteq (K\{R\})'$ by Theorem 0.11. Thus every member of Φ' is is pseudocompact. Let $X \in \Phi'$ and let K be compact. Then $X \times K$ is pseudocompact (the product of a compact space with a pseudocompact space is pseudocompact - see 9.14 of [17]). Thus, by Theorem 3.1, $\beta(X \times K) = \beta X \times \beta K = \beta X \times K$. Then by Proposition 3.9, $\Phi(X \times K) = \Phi X \times \Phi K = \Phi X \times K$. But $X \in \Phi'$, thus $\Phi X = \beta X$. Hence $\Phi(X \times K) = \Phi X \times K = \beta X \times K = \beta(X \times K)$ and so $(X \times K) \in \Phi'$. If $\Phi \in \mathcal{H}_0$ and $N \in \Phi$, then we can invoke Theorem 3.6 to obtain the corresponding result for 0-dimensional spaces X and K.

CHAPTER 4

THE STRUCTURE OF CONTINUOUS {0,1}-VALUED FUNCTIONS ON A TOPOLOGICAL PRODUCT

In this chapter we examine the question of which continuous {0,1}-valued functions on a product space $\Pi\{X_\alpha:\alpha \in A\}$, admit continuous extensions to $\Pi\{\beta X_\alpha:\alpha \in A\}$. This problem is equivalent to determining which clopen subsets of a product have clopen closure in the product the the Stone-Čech compactifications of the factor spaces. In solving this problem, we are led to a characterization of pseudocompactness in 0-dimensional topological products. We generalize some of the results of [11] and [12], and in particular, get a characterization of the pseudocompactness of a 0-dimensional product space $X \times Y$ in terms of a condition on the the spaces X and Y and a condition on the projection maps.

If X is compact, let X^* denote the component space of X, i.e. X^* is the quotient space of X formed by collapsing connected components of X. Since in a compact space the connected component of a point is equal to its quasi-component (the intersection of all clopen sets containing the point - see Theorem 16.15 of [17]), X^* becomes a compact, Hausdorff, 0-dimensional space. Let $q: X \to X^*$ be the quotient map. Suppose $f:X \to Y$ where Y is compact and 0-dimensional. It is easy to see that f must be constant on the connected components of X, and hence there is a map $f^*:X^* \to Y$ such that $f^* \cdot q = f$. Since q is a quotient map, f^* is continuous. If X is a 0-dimensional space, then $(\beta X)^* = \beta_0 X$, and the map q is equal to $\beta(i)$ where $i: X \to \beta_0 X$ is the identity embedding. The following Lemma is well-known (see [6], Part I, p. 110 for a proof).

4.1 Lemma: Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces. Then the connected component of a point $(x_\alpha)_{\alpha \in A} \in \Pi\{X_\alpha : \alpha \in A\}$ is equal to $\Pi\{C_\alpha : \alpha \in A\}$ where C_α is the connected component of x_α in X_α for each α in A.

4.2 Theorem: Let $\{K_\alpha : \alpha \in A\}$ be a family of compact spaces. Then the map which takes the component of the point $(x_\alpha)_{\alpha \in A}$ in $\Pi\{K_\alpha : \alpha \in A\}$ to the product of the components of the x_α's is a homeomorphism from $(\Pi\{K_\alpha : \alpha \in A\})^*$ onto $\Pi\{K_\alpha^* : \alpha \in A\}$.

Proof: Let $k: \Pi\{K_\alpha : \alpha \in A\} \to (\Pi\{K_\alpha : \alpha \in A\})^*$ and $k_\alpha: K_\alpha \to K_\alpha^*$ be the quotient maps. Let h denote the following map: $h((x_\alpha)_{\alpha \in A}) = (k_\alpha(x_\alpha))_{\alpha \in A}$, i.e. h is the product of the k_α's. Thus h is a continuous map from $\Pi\{K_\alpha : \alpha \in A\}$ to $\Pi\{K_\alpha^* : \alpha \in A\}$. As noted above, there is a map $g: (\Pi\{K_\alpha : \alpha \in A\})^* \to \Pi\{K_\alpha^* : \alpha \in A\}$ such that $g \cdot k = h$. Clearly g is the map described in the statement of the theorem. Furthermore, g is continuous, as k is a quotient map, g is onto (because h is onto), and g is one-to-one by Lemma 4.1.¤

4.3 Definition: Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces, and let $f: \Pi\{X_\alpha : \alpha \in A\} \to \{0,1\}$. Then f is called finitely decomposable if there exists a finite subset $F = \{\alpha_1, \dots, \alpha_n\} \subseteq A$, and finite decompositions (into pairwise disjoint clopen sets) T_i of X_{α_i}

for $i = 1,\ldots n$ such that f is constant on each set of the form $\Pi\{C_i : i = 1,\ldots n\} \times \Pi\{X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\}$ where $C_i \in \mathcal{T}_i$ for $i = 1,\ldots n$.

4.4 Theorem: Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces. Suppose $f\colon \Pi\{X_\alpha : \alpha \in A\} \to \{0,1\}$ is a continuous map. Then f admits a continuous extension to $\Pi\{\beta X_\alpha : \alpha \in A\}$ if and only if f is finitely decomposable.

Proof: Sufficiency. Suppose f is finitely decomposable. Then there is a finite subset $F = \{\alpha_1,\ldots,\alpha_n\} \subseteq A$ and finite decompositions $\mathcal{T}_i$ of X_{α_i} such that f is constant on each set of the form $\Pi\{C_i : i = 1,\ldots n\} \times \Pi\{X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\}$, where $C_i \in \mathcal{T}_i$. Since each C_i is clopen, the set $\Pi\{cl_{\beta X_{\alpha_i}}(C_i) : i = 1,\ldots,n\} \times \Pi\{\beta X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\}$ is clopen and is the closure in $\Pi\{\beta X_\alpha : \alpha \in A\}$ of the set mentioned above. Since $\mathcal{T}_i$ is finite for $i = 1,\ldots,n$, it follows that the sets of the form $\Pi\{cl_{\beta X_{\alpha_i}}(C_i) : i = 1,\ldots,n\} \times \Pi\{\beta X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\}$ form a finite decomposition of $\Pi\{\beta X_\alpha : \alpha \in A\}$ into clopen sets. If we define $f^*\colon \Pi\{\beta X_\alpha : \alpha \in A\} \to \{0,1\}$ by
$f^*(\Pi\{cl_{\beta X_{\alpha_i}}(C_i) : i = 1,\ldots,n\} \times \Pi\{\beta X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\}) = f(\Pi\{C_i : i = 1,\ldots,n\} \times \Pi\{X_\alpha : \alpha \neq \alpha_i, i = 1,\ldots,n\})$, then f^* is continuous and is an extension of f.

Necessity. Suppose $f^*\colon \Pi\{\beta X_\alpha : \alpha \in A\} \to \{0,1\}$ is a continuous extension of $f\colon \Pi\{X_\alpha : \alpha \in A\} \to \{0,1\}$. Let $k_\alpha\colon \beta X_\alpha \to (\beta X_\alpha)^*$ be the quotient maps for all $\alpha \in A$, and let $h\colon \Pi\{\beta X_\alpha : \alpha \in A\} \to \Pi\{(\beta X_\alpha)^* : \alpha \in A\}$ be the product of the k_α's. By Theorem 4.2, f^* factors through

$\Pi\{(\beta X_\alpha)^*:\alpha \in A\}$, i.e. there is a map $g: \Pi\{(\beta X_\alpha)^*:\alpha \in A\} \to \{0,1\}$ such that $g \cdot h = f^*$. Since $\Pi\{(\beta X_\alpha)^*:\alpha \in A\}$ is compact and 0-dimensional, it is easy to see that there is a finite subset of A, $F = \{\alpha_1,\ldots,\alpha_n\}$ such that $g^{\leftarrow}(\{0\})$ and $g^{\leftarrow}(\{1\})$ are sets of the form $U = \cup\{(\Pi\{U_{ij}:j=1,\ldots,n\} \times \Pi\{(\beta X_\alpha)^*:\alpha\neq\alpha_t,t=1,\ldots,n\}):i=1,\ldots,k\}$ where k and n are positive integers, and U_{ij} is a clopen subset of $(\beta X_{\alpha_j})^*$ for $j=1,\ldots,n$ and $i=1,\ldots k$ (this is because the canonical clopen sets form a base for the open sets). Let $k_1,k_2 \in N$ and

$U = g^{\leftarrow}(\{0\}) = \cup\{(\Pi\{U_{ij}:j=1,..n\} \times \Pi\{(\beta X_\alpha)^*:\alpha \in A-F\}):i=1,..,k_1\}$,

$V = g^{\leftarrow}(\{1\}) = \cup\{(\Pi\{V_{ij}:j=1,..n\} \times \Pi\{(\beta X_\alpha)^*:\alpha \in A-F\}):i=1,..,k_2\}$. Fix $j\leq n$. Let $T_j = \{U_{ij}:i=1,..,k_1\} \cup \{V_{ij}:i=1,..,k_2\}$. Clearly $\cup T_j = (\beta X_{\alpha_j})^*$ for $j=1,..,n$. Let $1 \leq j \leq n$ and suppose $|T_j| = n_j \in N$. Then $T_j = \{H_s:s=1,..,n_j\}$ where each H_s is clopen in $(\beta X_{\alpha_j})^*$. Let $H_s^0 = H_s$ and $H_s^1 = (\beta X_{\alpha_j})^* - H_s$ for $s=1,..,n_j$. For every $r: n_j \to \{0,1\}$, let $H_r = \cap\{H_s^{r(s)}:s=1,..,n_j\}$ and let $T_j^* = \{H_r: r: n_j \to \{0,1\}\}$. By the construction of T_j^* it is clear that if $T \in T_j$ and $H_r \cap T \neq \phi$, then $H_r \subseteq T$.

Suppose $C_j \in T_j^*$ for $j=1,..,n$. Then g is constant on $C = \Pi\{C_j:j=1,..,n\} \times \Pi\{(\beta X_\alpha)^*:\alpha \in A-F\}$. To verify this, suppose $C \cap U \neq \phi$. Then there is an $i_1 \in N$ such that $1 \leq i_1 \leq k_1$, and $C \cap (\Pi\{U_{i_1 j}:j=1,..,n\} \times \Pi\{(\beta X_\alpha)^*:\alpha \in A-F\}) \neq \phi$. Thus, $C_j \cap U_{i_1 j} \neq \phi$ for $j=1,..,n$. Since for each $j \leq n$ C_j is an H_r in T_j^*, $C_j \subseteq U_{i_1 j}$ for $j=1,..,n$ and hence $C \subseteq U$. A similar argument holds if $C \cap V \neq \phi$. Thus g is constant on C.

If we define $\Theta_j = \{k_{\alpha_j}^{\leftarrow}(W) \cap X_{\alpha_j} : W \in \mathcal{T}_j^*\}$ for $j=1,..,n$, then it is easy to see that f is finitely decomposable with respect to the decompositions Θ_j of X_{α_j} for $j=1,..,n$.¤

It is clear that no special property of βX was used in Theorem 4.4 other than the fact that clopen subsets of X have clopen closures in βX. Thus, if γX_α, $\alpha \in A$ are compactifications of X_α such that clopen subsets of X_α have clopen closures in γX_α, then Theorem 4.4 remains valid with βX_α replaced by γX_α (in fact, the necessity remains valid with βX_α replaced by any compactification of X_α).

We now use Theorem 4.4 and Theorem 3.7 of Chapter 3 to obtain a characterization of pseudocompactness in a 0-dimensional product space. Recall that if X is 0-dimensional, then $\beta_0 X$ can be characterized as the 0-dimensional compactification of X to which every continuous $\{0,1\}$-valued function on X can be continuously extended. Thus, any clopen subset of X has clopen closure in $\beta_0 X$. Hence, assuming all X_α are 0-dimensional in Theorem 4.4, we may replace βX_α by $\beta_0 X_\alpha$.

4.5 <u>Definition</u>: Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces and let $f: \Pi\{X_\alpha : \alpha \in A\} \to Y$ where Y is any space. Then f is said to depend on finitely many coordinates if there exists a finite subset $F \subseteq A$, and a map $g: \Pi\{X_\alpha : \alpha \in F\} \to Y$ such that $f = g \cdot \pi_F$, where π_F is the projection map from $\Pi\{X_\alpha : \alpha \in A\}$ to $\Pi\{X_\alpha : \alpha \in F\}$.

The following theorem gives various conditions on a 0-dimensional product space which are equivalent to the space being pseudocompact (i.e. every real-valued continuous function is bounded).

4.6 Theorem: Let $\{X_\alpha:\alpha \in A\}$ be a family of 0-dimensional spaces such that $\Pi\{X_\alpha:\alpha \in (A-\{\alpha_0\})\}$ is infinite for all $\alpha_0 \in A$. The following are equivalent.

i) $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact,

ii) Every countable subproduct of $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact,

iii) $\beta(\Pi\{X_\alpha:\alpha \in A\}) = \Pi\{\beta X_\alpha:\alpha \in A\}$,

iv) $\beta_0(\Pi\{X_\alpha:\alpha \in A\}) = \Pi\{\beta_0 X_\alpha:\alpha \in A\}$,

v) Every continuous $\{0,1\}$-valued function on $\Pi\{X_\alpha:\alpha \in A\}$ is finitely decomposable,

vi) Every finite subproduct of $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact, and every continuous $\{0,1\}$-valued function on $\Pi\{X_\alpha:\alpha \in A\}$ depends on finitely many coordinates.

Proof: Conditions i), ii), and iii) are shown to be equivalent for any family of (not necessarily 0-dimensional) spaces in [19], and are included to provide a contrast to iv), v) and vi). Conditions i) and iv) are shown to be equivalent in Theorem 3.7 of Chapter 3. Since statement iv) means precisely that every continuous $\{0,1\}$-valued function on $\Pi\{X_\alpha:\alpha \in A\}$ admits a continuous extension to $\Pi\{\beta_0 X_\alpha:\alpha \in A\}$, by Theorem 4.4 and the remarks following it, we get the equivalence of iv) and v).

i) $\Rightarrow$ vi). Since pseudocompactness is preserved by continuous maps, every subproduct of $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact (being the image of $\Pi\{X_\alpha:\alpha \in A\}$ under a projection map) if $\Pi\{X_\alpha:\alpha \in A\}$ is pseudocompact. It is clear that any finitely decomposable map from $\Pi\{X_\alpha:\alpha \in A\}$ to $\{0,1\}$ depends on finitely many coordinates. Thus, by the equivalence of i) and v) which has already been shown, it follows that every continuous $\{0,1\}$-valued function on $\Pi\{X_\alpha:\alpha \in A\}$ depends on finitely many coordinates.

vi) $\Rightarrow$ iv). Let $f: \Pi\{X_\alpha:\alpha \in A\} \to \{0,1\}$ be a continuous map. Then there is a finite set $F \subseteqq A$ and a continuous map $g: \Pi\{X_\alpha:\alpha \in F\} \to \{0,1\}$ such that $f = g \cdot \pi_F$. But, $\Pi\{X_\alpha:\alpha \in F\}$ is pseudocompact by hypothesis. Thus, by the equivalence of i) and iv), g admits a continuous extension g^* to $\Pi\{\beta_0 X_\alpha:\alpha \in F\}$ (we do not need the hypothesis that $\Pi\{X_\alpha:\alpha \in F - \{\alpha_0\}\}$ is infinite for every $\alpha_0 \in F$, for if one such product is finite then g obviously admits an extension g^*). Let $\overline{\pi}_F: \Pi\{\beta_0 X_\alpha:\alpha \in A\} \to \Pi\{\beta_0 X_\alpha:\alpha \in F\}$ be the projection map. Then, if we let $f^* = g^* \cdot \overline{\pi}_F$, $f^*: \Pi\{\beta_0 X_\alpha:\alpha \in A\} \to \{0,1\}$ and is an extension of f. Hence $\beta_0(\Pi\{X_\alpha:\alpha \in A\}) = \Pi\{\beta_0 X_\alpha:\alpha \in A\}$.∎

If we consider any finite family of 0-dimensional spaces X_i, $i=1,\ldots,n$ then every continuous function on $\Pi\{X_i:i=1,\ldots,n\}$ depends on finitely many coordinates. However, if X_i is a non-pseudocompact space for each $i=1,\ldots,n$, then $\Pi\{X_i:i=1,\ldots,n\}$ satisfies the second part of condition vi) but not the first. Furthermore, in [7] an example is given of a 0-dimensional, non-pseudocompact product space, all of whose finite subproducts are pseudocompact. Evidently, this space satisfies the first part of condition vi) but not the second. Hence, neither of the two parts of condition vi) may be removed.

We now obtain some 0-dimensional analogues of results from [11] and [12].

4.7 Definition: A space X is said to be {0,1}-embedded in a space Y if every continuous {0,1}-valued function on X admits a continuous extension to Y. A pair of spaces (X,Y) is called a {0,1}-pair if $X \times Y$ is {0,1}-embedded in both $\beta X \times Y$ and $X \times \beta Y$. If (X,Y) is a {0,1}-pair such that $X \times Y$ is not {0,1}-embedded in $\beta X \times \beta Y$, then (X,Y) is called a proper {0,1}-pair. A map $f: X \to Y$ is called a clopen map if the image under f of a clopen subset of X is clopen in Y.

The following theorem is the "0-dimensional analogue" of Theorem 2.1 of [12]. The proof requires only minor modifications of the original proof, to which the reader is referred.

4.8 Theorem: Let X and Y be 0-dimensional spaces. If $X \times Y$ is {0,1}-embedded in $\beta X \times Y$, then either X is pseudocompact, or Y is a P-space (a P-space is a space in which every G_δ is open).

An investigation of the proof of Theorem 3.6 of Chapter 3 shows that for 0-dimensional spaces X and Y, $X \times Y$ is {0,1}-embedded in $\beta X \times Y$ if and only if $X \times Y$ is {0,1}-embedded in

$\beta_0 X \times Y$. Thus, "$\beta X \times Y$" may be replaced by "$\beta_0 X \times Y$" in Theorem 4.8. Theorem 4.8 yields the following analogue to Theorem 2.2 of [12].

4.9 Theorem: If (X,Y) is a proper $\{0,1\}$-pair and X and Y are 0-dimensional spaces, then both X and Y are P-spaces.

Proof: By hypothesis $X \times Y$ is $\{0,1\}$-embedded in $\beta X \times Y$, hence also in $\beta_0 X \times Y$ by the remarks above. If Y is pseudocompact, then $\beta_0 X \times Y$ is pseudocompact, and then, by Theorem 3.6 of Chapter 3, $\beta_0(\beta_0 X \times Y) = \beta_0 X \times \beta_0 Y$, i.e. $\beta_0(X \times Y) = \beta_0 X \times \beta_0 Y$ (as $X \times Y$ would then be $\{0,1\}$-embedded in $\beta_0 X \times \beta_0 Y$) which is contrary to the hypothesis that (X,Y) is a proper $\{0,1\}$-pair. Thus, Y is not pseudocompact. Hence, by Theorem 4.8, X is a P-space. Similarly, Y is a P-space.¤

Theorems 2.1 and 2.2 of [12] now follow from Theorems 4.8 and 4.9, in the case where X and Y are 0-dimensional spaces. The following are parallels to Theorems 3.1 of [12] and 4.3 of [11].

4.10 Theorem: The following conditions on a product space $X \times Y$ are equivalent:

i) The projection map $\pi_Y: X \times Y \to Y$ is a clopen map,

ii) $X \times Y$ is $\{0,1\}$-embedded in $\beta X \times Y$.

Proof: i) $\Rightarrow$ ii). Trivial modifications of the proof of Theorem 3.1 of [12] yield this implication.

ii) $\Rightarrow$ i). Let U be a clopen subset of $X \times Y$. By hypothesis, there is a clopen set $V \subseteq \beta X \times Y$ such that $V \cap (X \times Y) = U$. Let π_Y denote the projection map from $X \times Y$ to Y and $\bar{\pi}_Y$ the projection map from $\beta X \times Y$ to Y. Since βX is compact, $\overline{\pi}_Y$ is a closed map, hence (as all projection maps are open maps) is a clopen map. Thus, $\overline{\pi}_Y(V)$ is clopen in Y. Clearly $\pi_Y(U) \subseteq \overline{\pi}_Y(V)$. If $y \notin \pi_Y(U)$ then $(X \times \{y\}) \cap U = \phi$, hence $(\beta X \times \{y\}) \cap V = \phi$ as $X \times \{y\}$ is dense in $\beta X \times \{y\}$. Thus $y \notin \overline{\pi}_Y(V)$. Therefore $\pi_Y(U) = \overline{\pi}_Y(V)$ and hence π_Y is a clopen map.¤

4.11 Theorem: Let X and Y be 0-dimensional spaces. The following are equivalent.

i) $X \times Y$ is pseudocompact,

ii) X and Y are both pseudocompact spaces and π_X and (or) π_Y are clopen maps.

Proof: i) $\Rightarrow$ ii). Since $X \times Y$ is pseudocompact, $\beta(X \times Y) = \beta X \times \beta Y$ by Theorem 3.1 of Chapter 3, and hence $X \times Y$ is a $\{0,1\}$-pair. Thus by Theorem 4.10, both π_X and π_Y are clopen maps. Since both X and Y are continuous images of $X \times Y$, both X and Y are pseudocompact (note that this implication does not make use of the 0-dimensionality of X and Y).

ii) $\Rightarrow$ i). Since π_X is clopen, $X \times Y$ is $\{0,1\}$-embedded in $X \times \beta Y$ by Theorem 4.10. Thus, by the remarks preceding 4.9, $X \times Y$ is $\{0,1\}$-embedded in $X \times \beta_0 Y$. Since X is pseudocompact and $\beta_0 Y$ is compact, $X \times \beta_0 Y$ is pseudocompact. Thus, by Theorem 3.6, $\beta_0(X \times \beta_0 Y) = \beta_0 X \times \beta_0 Y$. But $X \times Y$ is $\{0,1\}$-embedded in $X \times \beta_0 Y$, hence $X \times Y$ is $\{0,1\}$-embedded in $\beta_0 X \times \beta_0 Y$, i.e. $\beta_0(X \times Y) = \beta_0 X \times \beta_0 Y$. By Theorem 3.6, $X \times Y$ is pseudocompact. Thus, the proof does not require the hypothesis that π_Y be clopen, although this must be the case by Theorem 4.10.¤

CHAPTER 5

LATTICES OF ZERO-SETS

In this chapter we give characterizations of those lattices that are lattice isomorphic to the lattice of zero-sets of a compact Hausdorff, Lindelöf, realcompact normal, or realcompact space. T.P. Speed has independently, and at about the same time as the author, obtained characterizations of those lattices that are zero-set lattices of compact Hausdorff or realcompact spaces. The characterization obtained by Speed of the zero-set lattice of a realcompact space differs significantly from the one given here. His work may be found in [37].

The basic references for lattice theory that we use are [4] and [21]. We use without further comment the lattice theoretic terminology therein. The lattice operations of "meet" and "join" will be denoted by "$\wedge$" and "$\vee$" respectively.

5.1 Definition: Let L be a lattice with a 0 element. Then an ultrafilter on L is a non-empty subset $M \subseteq L$ such that:

i) if $a, b \in M$ then $a \wedge b \in M$,

ii) if $a \in M$ and $a \leq b$, then $b \in M$,

iii) $0 \notin M$,

iv) M is maximal with respect to i), ii) and iii).

Let L be a bounded distributive lattice (a lattice is bounded if it has both a 0 and a 1 element). Let $M(L)$ denote the set of ultrafilters on L. Then the Stone topology on M(L) is the topology generated by the following base for the closed sets: $\{A(z):z \in L\}$ where $A(z) = \{M \in M(L):z \in M\}$. With this topology, M(L) is a compact (not necessarily Hausdorff) space and is also called the Stone space of L (see section 11 of [21]). This construction plays a central role in the characterization theorems.

We now introduce several lattice conditions which will occur frequently. The symbols on the left will be used to denote the conditions. Let L denote a bounded lattice.

ℓ_1) If $s, t \in L$ and $s \wedge t = 0$, then there exist elements $a, b \in L$ such that $s \wedge a = b \wedge t = 0$ and $a \vee b = 1$ (this is condition 2.1 of [37], called normality).

ℓ_2) If $s > t$ then there is an $a \in L$ such that $a \wedge s > 0$, and $a \wedge t = 0$.

ℓ_3) If $\{z_i:i \in N\} \subseteq L$ then $\wedge\{z_i:i \in N\} \in L$.

ℓ_4) If $z \in L$ then there are sequences $\{z_i:i \in N\}$, $\{w_i:i \in N\} \subseteq L$ such that $z = \wedge\{z_i:i \in N\}$, $z_{i+1} \leq z_i$, $z_{i+1} \wedge w_i = 0$, and $z_i \vee w_i = 1$ for all $i \in N$ (this is condition 2.2 of [37]).

The following is Lemma 2.4 of [37] to which the reader is referred for a proof.

5.2 Lemma: Let L be a bounded lattice. If L satisfies condition ℓ_4 then L satisfies ℓ_2.

By saying that an ultrafilter M on a lattice L is closed under countable meets, we mean that if $\{z_i:i \in N\} \subseteq M$ and $\wedge\{z_i:i \in N\}$ exists in L then $\wedge\{z_i:i \in N\} \in M$. Such ultrafilters are also referred to as real ultrafilters.

5.3 Lemma: Let L be a bounded distributive lattice satisfying ℓ_3. Suppose that every ultrafilter on L is closed under countable meets. If L satisfies condition ℓ_4 then L satisfies ℓ_1.

Proof: Let $s \wedge t = 0$. By ℓ_4 we get sequences $\{p_i\}$, $\{q_i\}$, $\{r_i\}$, $\{u_i\} \subseteq L$ such that $s = \wedge\{p_i\}$, $p_i \vee q_i = 1$, $s \wedge q_i = 0$, $t = \wedge\{r_i\}$, $r_i \vee u_i = 1$ and $t \wedge u_i = 0$ (since $s \leq p_i$ for each i and $p_{i+1} \wedge q_i = 0$ we have that $s \wedge q_i = 0$, similarly for t and u_i). Without loss of generality, $p_1 \wedge r_1 = 0$ (if $p_n \wedge r_n > 0$ for all $n \in N$, then $\{p_n, r_n\}_{n \in N} \subseteq M$ for some ultrafilter M on L, hence, by hypothesis, $(\wedge\{p_i : i \in N\}) \wedge (\wedge\{r_i : i \in N\}) \in M$, i.e. $0 = s \wedge t \in M$ which is not possible). Then $q_1 \vee u_1 = 1$, for $1 = q_1 \vee p_1 \leq q_1 \vee u_1$ ($p_1 \leq u_1$ since $p_1 \wedge r_1 = 0$, thus $p_1 \wedge u_1 = (p_1 \wedge r_1) \vee (p_1 \wedge u_1) = p_1 \wedge (r_1 \vee u_1) = p_1 \wedge 1 = p_1$), and $q_1 \wedge s = 0$, $u_1 \wedge t = 0$. Thus L satisfies ℓ_1.¤

The following theorem was obtained independently as Theorem 4.1 of [37]. The methods developed in the proof will be employed in later characterization theorems.

5.4 Theorem: Let L be a bounded distributive lattice. Then $L \cong Z(X)$ for some compact Hausdorff space X if and only if L satisfies conditions ℓ_3 and ℓ_4 and every ultrafilter on L is real (i.e. every ultrafilter on L is closed under countable meets).

Proof: Necessity. Suppose X is a compact Hausdorff space. It is shown in 1.14(a) of [17] that Z(X) satisfies ℓ_3 for any space X. In addition, if $Z \in Z(X)$ then $Z = f^{\leftarrow}(\{0\})$ for some positive continuous function $f: X \to R$. Let $s_n = f^{\leftarrow}([0, \frac{1}{n}])$, $t_n = f^{\leftarrow}([\frac{2}{2n+1}, \infty))$. Then clearly the sequences $\{s_n\}$ and $\{t_n\}$ satisfy condition ℓ_4 for Z (we do not use the compactness of X as the lattice of zero-sets for any space will satisfy this condition). Finally, every ultrafilter of zero-sets on a compact space intersects in a single point, hence any ultrafilter must be closed under countable meets (a zero-set will be in the ultrafilter if and only if it contains the point of intersection).

Sufficiency. Let M(L) be the set of ultrafilters on L with the Stone topology. Let $\{A(z): z \in J\}$ be a family of basic closed sets in M(L) with f.i.p. (where A(z) is as defined above, and $J \subseteq L$). Then J has the finite meet property (i.e. if $F \subseteq J$ and F is finite then $\wedge F > 0$). Thus there is an ultrafilter M on L such that $J \subseteq M$ (any subset of L with f.m.p. is contained in an ultrafilter by Zorn's Lemma). Clearly $M \in \cap\{A(z): z \in J\}$. Thus M(L) is compact.

To see that M(L) is Hausdorff, let $M_1 \neq M_2$, $M_1, M_2 \in M(L)$. By the maximality of M_1 and M_2 there are elements $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 \wedge m_2 = 0$. By Lemma 5.3, L satisfies ℓ_1, hence there are a, $b \in L$ such that $a \wedge m_1 = b \wedge m_2 = 0$ and $a \vee b = 1$. Then A(a) and A(b) are basic closed sets such that $A(a) \cup A(b) = M(L)$ (every ultrafilter is prime, hence either a or b is in M for every M in M(L)) and $M_1 \notin A(a)$, $M_2 \notin A(b)$. Thus $M(L) - A(a)$ and $M(L) - A(b)$ are disjoint neighborhoods of M_1 and M_2.

Let C(M(L)) denote the closed sets of M(L). Let $\psi: L \to C(M(L))$ be defined by $\psi(z) = A(z)$. Clearly ψ is well-defined, and $\psi(L)$ forms a base for the closed sets of $M(L)$. $\mathcal{C}(M(L))$ is a bounded distributive lattice under the operations of union and intersection. Clearly ψ is a lattice homomorphism which preserves 0 and 1. Furthermore, ψ is one-to-one. For, by Lemma 5.2, L satisfies ℓ_2. Thus, if $s \neq t$ then without loss of generality, $s \vee t > s$. By ℓ_2, there is an element $a \in L$ such that $a \wedge (s \vee t) > 0$ but $a \wedge s = 0$. Thus $a \wedge (s \vee t) = (a \wedge s) \vee (a \wedge t) = 0 \vee (a \wedge t) = a \wedge t > 0$. Then there must be an $M \in M(L)$ such that $a \wedge t \in M$. But $s \notin M$ as $a \wedge s = 0$. Thus $\psi(s) \neq \psi(t)$, and ψ is one-to-one.

We now show that ψ is actually a homomorphism into $Z(M(L))$. Let $z \in L$. By condition ℓ_4 there are sequences $\{z_i\}_{i \in N}$ and $\{w_i\}_{i \in N}$ in L such that $z \leq z_{i+1} \leq z_i$, $z_{i+1} \wedge w_i = 0$, $w_i \vee z_i = 1$ and $z = \wedge\{z_i\}_{i \in N}$. Hence, $\psi(z) \subseteqq \psi(z_{i+1}) \subseteqq \psi(z_i)$, $\psi(z_{i+1}) \cap \psi(w_i) = \psi(z_{i+1} \wedge w_i) = \psi(0) = \phi$, and $M(L) = \psi(1) = \psi(w_i \vee z_i) = \psi(w_i) \cup \psi(z_i)$. Thus $\psi(z) \subseteqq \psi(z_{i+1}) \subseteqq M(L) - \psi(w_i) \subseteqq \psi(z_i)$. But $\psi(z) = \psi(\wedge\{z_i\}_{i \in N}) = \cap\{\psi(z_i)\}_{i \in N}$ (for clearly $\psi(z) \subseteqq \psi(z_i)$ for all $i \in N$, and if $M \in M(L)$ such that $M \in \cap\{\psi(z_i)\}_{i \in N}$ then $\{z_i\}_{i \in N} \subseteqq M$ and hence since L staisfies ℓ_3 and every ultrafilter on L is real, $z = \wedge\{z_i\}_{i \in N} \in M$, i.e. $M \in \psi(z)$). Thus, $\psi(z) = \cap\{\psi(z_i)\}_{i \in N} = \cap\{M(L) - \psi(w_i)\}_{i \in N}$, and $\psi(z)$ is a closed G_δ set in the compact Hausdorff space $M(L)$. Therefore, by 3.11 of [17], $\psi(z) \in Z(M(L))$.

Finally, we show that ψ maps L onto $Z(M(L))$. Let $Z \in Z(M(L))$. Then Z is a G_δ and hence $Z = \cap\{U_i\}_{i \in N}$, where U_i is open in $M(L)$ for all i in N. Since $M(L)$ is compact and $\psi(L)$ forms a base for the closed sets of $M(L)$, for every $i \in N$ there is a $z_i \in L$ such that $Z \subseteqq \psi(z_i) \subseteqq U_i$. Then $Z = \cap\{\psi(z_i)\}_{i\in N} = \psi(\wedge\{z_i\}_{i\in N})$ and ψ is onto. Therefore ψ is a lattice isomorphism between L and $Z(M(L))$.¤

In view of 8D1 of [17] which shows that $Z(X)$ and $Z(\upsilon X)$ are always lattice isomrphic (where υX denotes the Hewitt realcompactification of X) and the facts that a compact space is pseudocompact and $\upsilon X = \beta X$ for any pseudocompact space X, we see that "compact" may be replaced by "pseudocompact" in the statement of Theorem 5.4.

Recall that a space X is called Lindelöf if every family of closed sets with the countable intersection property (c.i.p.) has non-empty intersection. A filter F on a lattice L (i.e. F satisfies conditions i), ii), and iii) in Definition 5.1, but not necessarily iv)) is said to have the countable meet property (c.m.p.) if $\{z_i\}_{i\in N} \subseteqq F$ implies that $\wedge\{z_i\}_{i\in N} \in F$.

5.5 Theorem: Let L be a bounded distributive lattice. Then $L = Z(X)$ for some Lindelöf space X if and only if L satisfies conditions ℓ_1, ℓ_3, ℓ_4 and every filter on L with the c.m.p. is contained in a real ultrafilter.

Proof: Necessity. As was shown in Theorem 5.4, $Z(X)$ satisfies $\ell_1 - \ell_4$ for any space X. If X is Lindelöf and F is a filter on $Z(X)$ with c.m.p. then $\cap F \neq \phi$. Thus F must be contained in the ultrafilter on $Z(X)$ consisting of all zero-sets containing any one particular point of $\cap F$. Such an ultrafilter is always real.

Sufficiency. Let $M\upsilon(L)$ denote the subspace of $M(L)$ consisting of the real ultrafilters on L. As in Theorem 5.4, $M(L)$ is compact and Hausdorff (ℓ_1 implies $M(L)$ is Hausdorff), thus $M\upsilon(L)$ is completely regular and Hausdorff. Recall from Theorem 5.4 the map $\psi: L \to C(M(L))$ defined by $\psi(z) = \{M \in M(L): z \in M\}$. Let $\tau: L \to C(M\upsilon(L))$ be defined by $\tau(z) = \psi(z) \cap M\upsilon(L)$. Then $\tau(L)$ forms a base for the closed sets of $M\upsilon(L)$. Let $\{C_i\}_{i\in I} \subseteq C(M\upsilon(L))$ be a family of closed sets with the countable intersection property. Then for each $i \in I$ there is a subset $J_i \subseteq L$ such that $C_i = \cap\{\tau(z)\}_{z\in J_i}$ (as $\{\tau(z)\}_{z\in L}$ forms a base for $C(M\upsilon(L))$). But since $\{C_i\}_{i\in I}$ has c.i.p., $\cup\{J_i: i \in I\}$ must have c.m.p. and hence is contained in some $M \in M\upsilon(L)$ by hypothesis. Clearly $M \in \cap\{C_i\}_{i\in I}$ and hence, $M\upsilon(L)$ is Lindelöf. Thus, by 3D4 of [17], which shows that every regular Lindelöf space is normal, $M\upsilon(L)$ is normal.

By Lemma 5.2, L satisfies ℓ_2. Hence, since every non-0 element of L is contained in a real ultrafilter, τ is a one-to-one homomorphism into $C(M\upsilon(L))$(the proof is identical to that of Theorem 5.4 which showed that ψ is one-to-one). Since $M\upsilon(L)$ consists only of real ultrafilters, if $\{z_i\}_{i\in N} \subseteq L$ then $\tau(\wedge\{z_i\}_{i\in N}) = \cap\{\tau(z_i)\}_{i\in N}$. Thus, we may show that $\tau(z)$ is a closed G_δ set in $M\upsilon(L)$ just as it was shown in Theorem 5.4 that $\psi(z)$ is a closed G_δ in $M(L)$. Hence, since $M\upsilon(L)$ is normal, by 3D3 of [17], which shows that every closed G_δ in a normal space is a zero-set, $\tau(z) \in Z(M\upsilon(L))$ for all $z \in L$. Thus $\tau: L \to Z(M\upsilon(L))$.

To see that τ maps onto $Z(M\upsilon(L))$, let Z be a zero-set in $M\upsilon(L)$. Then there is a subset $B \subseteq L$ such that $Z = \cap\{\tau(b)\}_{b\in B}$. Also, Z is a G_δ set, so $Z = \cap\{U_i\}_{i\in N}$ where each U_i is open in $M\upsilon(L)$. Since $M\upsilon(L)$ is Lindelöf, for each $i \in N$ there is a countable set $B_i \subseteq B$ such that $Z \subseteq \cap\{\tau(b)\}_{b\in B_i} \subseteq U_i$. Let $B' = \cup\{B_i : i \in N\}$. Then $Z \subseteq \cap\{\tau(b)\}_{b\in B'} \subseteq \cap\{U_i\}_{i\in N} = Z$. Since B' is countable, $Z = \cap\{\tau(b)\}_{b\in B'} = \tau(\wedge\{b\}_{b\in B'})$. Thus, τ is onto and is an isomorphism from L onto $Z(M\upsilon(L))$.¤

Note that since L is isomrphic to $Z(M\upsilon(L))$, that $M(L)$ and $M(Z(M\upsilon(L)))$ are homeomorphic. But $M(Z(M\upsilon(L))) = \beta(M\upsilon(L))$ (in Chapter 6 of [17] it is shown that $M(Z(X)) = \beta X$ for any completely regular, Hausdorff space X). Thus $M(L)$ is homeomorphic to $\beta(M\upsilon(L))$. It can be seen directly that $\beta(M\upsilon(L)) = M(L)$ once it is observed that Theorem 5.5 implies that disjoint zero-sets in $M\upsilon(L)$ have disjoint closures in $M(L)$.

A space X is called realcompact if every ultrafilter of zero-sets on X closed under countable intersection (i.e. a real z-ultrafilter) is fixed (i.e. has non-empty intersection).

5.6 <u>Theorem</u>: Let L be a bounded distributive lattice. Then $L = Z(X)$ for some normal realcompact space X if and only if L satisfies ℓ_1, ℓ_3, ℓ_4 and the following conditions:

i) If $z \neq 0$ and $z \in L$, then there is a real ultrafilter M on L such that $z \in M$.

ii) If $S = A \cup B \subseteq L$ such that S is not contained in any real ultrafilter on L then there are elements $a, b \in L$ such that $a \vee b = 1$ and neither of the sets $A \cup \{a\}$ or $B \cup \{b\}$ is contained in any real ultrafilter on L.

Proof: Necessity. Let X be realcompact and normal. As in 5.5 and 5.4, $Z(X)$ satisfies ℓ_1, ℓ_3, and ℓ_4. Suppose $Z \in Z(X)$, and $Z \neq \phi$. Let $p \in Z$. Then $\{W \in Z(X) : p \in W\}$ is a real z-ultrafilter on X which contains Z. Thus $Z(X)$ satisfies condition i) (no use was made of the hypothesis that X is realcompact normal). Note that in a realcompact space, a subset $S \subseteq Z(X)$ is contained in a real z-ultrafilter on X if and only if $\cap S \neq \phi$ (a realcompact space is one in which a z-ultrafilter is real if and only if it has non-empty intersection). Suppose $S \subseteq Z(X)$, $S = A \cup B$ and $\cap S = \phi$. Then $(\cap A) \cap (\cap B) = \phi$, hence, by the normality of X there are zero-sets $a, b \in Z(X)$ such that $(\cap A) \cap a = \phi$, $(\cap B) \cap b = \phi$ and $a \cup b = X$ (since we may separate A and B by cozero sets). Thus $Z(X)$ satisfies condition ii).

Sufficiency. Let $M\upsilon(L)$ denote the subspace of $M(L)$ consisting of all real ultrafilters on L. Again, $M(L)$ is a compact, Hausdorff space, hence, $M\upsilon(L)$ is completely regular and Hausdorff. Let $C(M\upsilon(L))$ and $\tau : L \to C(M\upsilon(L))$ be as in Theorem 5.5 Then, as in Theorem 5.5, since L satisfies condition i) and $M\upsilon(L)$ consists only of real ultrafilters, τ is a one-to-one homomorphism.

We show that $M\upsilon(L)$ is normal. Let C_1, C_2 be disjoint closed subsets of $M\upsilon(L)$. Then there sets $A_1, A_2 \subseteq L$ such that

$C_i = \cap\{\tau(z)\}_{z\in A_i}$ for $i = 1,2$. Since $C_1 \cap C_2 = \phi$, there can be no real ultrafilter on L containing $A_1 \cup A_2$. By condition ii), there are elements $a_1, a_2 \in L$ such that $a_1 V a_2 = 1$ and neither $A_1 \cup \{a_1\}$ nor $A_2 \cup \{a_2\}$ are contained in any real ultrafilter on L. Thus $C_1 \subseteq M\upsilon(L) - \tau(a_1)$, $C_2 \subseteq M\upsilon(L) - \tau(a_2)$ which are disjoint open sets in $M\upsilon(L)$. Thus $M\upsilon(L)$ is normal.

Let $z \in L$. Then there are sequences $\{z_i\}_{i\in N}$ $\{w_i\}_{i\in N}$ contained in L which satisfy condition ℓ_4. As in Theorem 5.5, $\tau(z) = \tau(\Lambda\{z_i\}_{i\in N}) = \cap\{\tau(z_i)\}_{i\in N}$, and $\tau(z) \subseteq M\upsilon(L) - \tau(w_i) \subseteq \tau(z_i)$. Thus $\tau(z)$ is a closed G_δ in a normal space, hence is a zero-set. Therefore, $\tau: L \to Z(M\upsilon(L))$ is a one-to-one homomorphism.

We show that τ is onto. Let $Z \in Z(M\upsilon(L))$. Then $Z = \cap\{U_i\}_{i\in N}$ where U_i is open in $M\upsilon(L)$ for each $i \in N$. Let $i \in N$. Then Z and $M\upsilon(L) - U_i$ are disjoint closed sets. There are sets $A, B \subseteq L$ such that $Z = \cap\{\tau(z)\}_{z\in A}$, $M\upsilon(L) - U_i = \cap\{\tau(z)\}_{z\in B}$. By condition ii), there are elements $a, b \in L$ such that $aVb = 1$, and neither $A \cup \{a\}$ nor $B \cup \{b\}$ are contained in a real ultrafilter on L. Thus, $\tau(b) \cap (M\upsilon(L) - U_i) = \phi$, and hence $\tau(b) \subseteq U_i$. Also, $\tau(a) \cap Z = \phi$, hence $Z \cap \tau(b) = Z \cap (\tau(b) \cup \tau(a)) = Z \cap (\tau(aVb)) = Z \cap M\upsilon(L) = Z$, i.e. $Z \subseteq \tau(b) \subseteq U_i$. Thus, for every $i \in N$ there is an $x_i \in L$ such that $Z \subseteq \tau(x_i) \subseteq U_i$. Then $Z = \cap\{\tau(x_i)\}_{i\in N} = \tau(\Lambda\{x_i\}_{i\in N})$ and τ is onto.

We can now show that $M\upsilon(L)$ is realcompact. Let F be a real ultrafilter on $Z(M\upsilon(L))$. Then $\tau^{\leftarrow}(F)$ is a real ultrafilter on L, as τ is an isomorphism. Hence, $\tau^{\leftarrow}(F) = M \in M\upsilon(L)$. Clearly

$\cap\mathcal{F} = \{M\}$. Thus, $\mathcal{F}$ is fixed and $M\upsilon(L)$ is realcompact.¤

In view of the fact that for any space X, $Z(X)$ satisfies conditions ℓ_1, ℓ_3, ℓ_4, and condition i) of Theorem 5.6, and $M\upsilon(Z(X))$ is precisely υX (see Chapter 8 of [17]), the following two corollaries are immediate from the last two theorems. These corollaries may also be proved directly without any difficulty.

5.7 Corollary: Let X be a space. Then υX is Lindelöf if and only if every family of zero-sets of X with the countable intersection property is contained in a real z-ultrafilter.

5.8 Corollary: Let X be a space. Then υX is normal if and only if $Z(X)$ satisfies condition ii) of Theorem 5.6

5.9 Theorem: Let L be a bounded distributive lattice. Then $L = Z(X)$ for some realcompact space X if and only if L satisfies conditions ℓ_1, ℓ_3 and the following three conditions:

i) Every element of L is contained in a real ultrafilter on L.

ii) If $z \in L$, then there are subsets $\{z_r : r \in Q \cap (0,1)\}$, $\{w_r : r \in Q \cap (0,1)\} \subseteq L$ (where $Q \cap (0,1)$ is the set of rational numbers in the open unit interval) such that $z_r \le z_s$, $w_r \ge w_s$ for all $r \le s$, $z_r \vee w_r = 1$ for all r, and $z_r \wedge w_s = 0$ for all $r < s$ where $r,s \in Q \cap (0,1)$, and $z = \wedge\{z_r : r \in Q \cap (0,1)\}$.

iii) $\beta(M\upsilon(L)) = M(L)$.

Proof: Necessity. Let X be a space. If $Z \in Z(X)$ then $Z = f^{\leftarrow}(\{0\})$ for some continuous $f: X \to I$. Let $Z_r = f^{\leftarrow}([0,r])$, $W_r = f^{\leftarrow}([r,1])$

for $r \in Q \cap (0,1)$. Then $\{Z_r : r \in Q \cap (0,1)\}$ and $\{W_r : r \in Q \cap (0,1)\}$ can easily be seen to satisfy condition ii). Since $M\upsilon(Z(X)) = \upsilon X$ (see Chapter 8 of [17]) and $M(Z(X)) = \beta X$, we have that $\beta(M\upsilon(Z(X))) = \beta(\upsilon X) = \beta X$, hence condition iii). The other conditions are satisfied as shown in the previous theorems.

Sufficiency. Suppose L satisfies the above conditions. Then the function $\tau: L \to C(M\upsilon(L))$ defined by $\tau(z) = \{M \in M\upsilon(L) : z \in M\}$ is a one-to-one homomorphism onto a base for the closed sets of $M\upsilon(L)$, just as shown in Theorems 5.6 and 5.5 (condition ii) implies ℓ_4).

We show that $\tau(z) \in Z(M\upsilon(L))$ for $z \in L$. Given $z \in L$ there are sets $\{z_r : r \in Q \cap (0,1)\}$, $\{w_r : r \in Q \cap (0,1)\} \subseteqq L$ as in condition ii). Then $z \wedge w_r = 0$ for all $r \in Q \cap (0,1)$, hence, $\tau(z) \cap \tau(w_r) = \phi$ and thus $\tau(z) \subseteqq (M\upsilon(L) - \tau(w_r)) \subseteqq \tau(z_r)$ (as $z_r \vee w_r = 1$). Thus, $\tau(z) \subseteqq \cap\{(M\upsilon(L) - \tau(w_r)) : r \in Q \in (0,1)\} \subseteqq \cap\{\tau(z_r) : r \in Q \in (0,1)\} = \tau(\wedge\{z_r : r \in Q \in (0,1)\}) = \tau(z)$. Thus we have open sets $U_r = M\upsilon(L) - \tau(w_r)$ for $r \in Q \cap (0,1)$ such that if $r \leq s$ then $U_r \subseteqq U_s$ (as $w_r \geq w_s$, hence $\tau(w_r) \supseteqq \tau(w_s)$). Also, if $r < s$, then $z_r \wedge w_s = 0$, thus $U_r = M\upsilon(L) - \tau(w_r) \subseteqq \tau(z_r) \subseteqq M\upsilon(L) - \tau(w_s) = U_s$. Hence $cl_{M\upsilon(L)}(U_r) \subseteqq U_s$. In addition, $\tau(z) = \cap\{U_r : r \in Q \cap (0,1)\}$. Let $g: M\upsilon(L) \to R$ be defined by

$g(x) = \inf\{r \in Q \cap (0,1) : x \in U_r\}$ if $x \in \cup\{U_r : r \in Q \cap (0,1)\}$, and

$g(x) = 1$ if $x \in M\upsilon(L) - \cup\{U_r : r \in Q \cap (0,1)\}$.

Then, as in Lemma 3.12 of [17], g is a continuous map. Furthermore, $g^{\leftarrow}(\{0\}) = \cap\{U_r : r \in Q \in (0,1)\} = \tau(z)$. Thus, $\tau(z) \in Z(M\upsilon(L))$.

To see that τ maps onto $Z(M\upsilon(L))$, let $Z \in Z(M\upsilon(L))$. Then there is a map $f: M\upsilon(L) \to [0,1]$ such that $Z = f^{\leftarrow}(\{0\})$. For $n \in N$, let $V_n = f^{\leftarrow}([\frac{1}{n},1]) \in Z(M\upsilon(L))$. Then $Z \cap V_n = \phi$ for all $n \in N$, and $Z = \cap\{(M\upsilon(L) - V_n):n \in N\}$. But $\beta(M\upsilon(L)) = M(L)$, so Z and V_n have disjoint closures in $M(L)$ for all $n \in N$. Thus, there is a $z_n \in L$ such that $cl_{M(L)}(Z) \subseteqq \psi(z_n) \subseteqq M(L) - cl_{M(L)}(V_n)$ (where $\psi: L \to C(M(L))$ is as in Theorem 5.4). This is true because $cl_{M(L)}(Z) = \cap\{\psi(z_\alpha):\alpha \in A\}$ for some family $\{z_\alpha:\alpha \in A\}$. But then $cl_{M(L)}(V_n) \subseteqq M(L) - cl_{M(L)}(Z) = M(L) - \cap\{\psi(z_\alpha):\alpha \in A\} = \cup\{(M(L) - \psi(z_\alpha)):\alpha \in A\}$. Since $cl_{M(L)}(V_n)$ is compact, there is a finite subset $F \subseteqq A$ such that $cl_{M(L)}(V_n) \subseteqq \cup\{ M(L) - \psi(z_\alpha):\alpha \in F\} = M(L) - \cap\{\psi(z_\alpha):\alpha \in F\} = M(L) - \psi(\wedge\{z_\alpha :\alpha \in F\})$. Thus $cl_{M(L)}(Z) = \cap\{\psi(z_\alpha):\alpha \in A\} \subseteqq \cap\{\psi(z_\alpha):\alpha \in F\} \subseteqq M(L) - cl_{M(L)}(V_n)$. Hence, $Z \subseteqq \tau(\wedge\{z_\alpha:\alpha \in F\}) \subseteqq M\upsilon(L) - V_n$. Let $z_n = \wedge\{z_\alpha:\alpha \in F\}$. Then $Z \subseteqq \tau(z_n) \subseteqq M\upsilon(L) - V_n$ for all $n \in N$. Thus, $Z = \cap\{\tau(z_n):n \in N\} = \tau(\wedge\{z_n:n \in N\})$. Hence, τ is onto and is an isomorphism.¤

In view of the fact that $Z(X)$ is isomorphic to $Z(\upsilon X)$ for any space X (see 8D1 of [17]), Theorem 5.9 characterizes the lattice of zero-sets of any completely regular and Hausdorff space.

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