THE UNIVERSITY OF MANITOBA

CONSTRUCTIONS OF EXTENDED TRIPLE SYSTEMS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS AND ASTRONOMY

WINNIPEG, MANITOBA R3T 2N2

April 1976

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ABSTRACT

An extended triple system is a pair (S, W) where S is a finite set and W is a collection of unordered triples from S, where each triple may have repeated elements, such that every pair of elements of S, not necessarily distinct, belongs to exactly one triple of W. Algebraically, an extended triple system on an n-set S is equivalent to a quasigroup on S satisfying the laws $x(xy) = y$ and $(yx)x = y$. If all elements of the quasigroup are idempotent, then the system is equivalent to a Steiner triple system. We define $\{n; b\}$ to be the class of all extended triple systems on n elements with b idempotent elements. If $\{n; b\}$ is non-empty, we shall say $\{n; b\}$ exists. It is known that necessary conditions for the existence of $\{n; a\}$, $0 \leq a \leq n$, are:

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \leq \frac{n}{2}$;

(4) if $a = n - 1$, then $n = 2$.

This thesis is concerned mainly with the sufficiency of the known necessary conditions for the existence of $\{n; a\}$. A direct method of construction is used to show that if the necessary conditions are satisfied, then $\{n; a\}$ exists. In addition, we shall give recursive methods of construction including the use of the direct and singular direct product of quasigroups. It is shown that, with a few obvious exceptions, there exist at least two non-isomorphic systems in each class $\{n; a\}$.

ACKNOWLEDGEMENTS

I am greatly indebted to Professor N.S. Mendelsohn, who granted me the privilege of pursuing this research under his supervision. His patience, encouragement and many helpful suggestions throughout the various stages of this work have been greatly appreciated.

Thanks are also due to Mr. Barry Wolk for calling my attention to the work of Th. Skolem, and to Professor C.C. Lindner for making other useful references readily available.

The author is particularly grateful to Professors D.M. Johnson and M. Doob for their valuable comments and help in checking the original version of this manuscript, and to Mrs. M. McTavish, who patiently and skilfully performed the formidable task of typing this thesis.

Finally, I wish to express my great indebtedness to the University of Manitoba for financial assistance in the form of a Graduate Fellowship during this study.

CHAPTER 1

Introduction

A Steiner triple system, briefly STS, is a pair (S, T) where S is a finite set and T a collection of 3-subsets of S, called triples, such that every pair of distinct elements of S is contained in exactly one triple of T . The number $|S|$ is called the order of (S, T) . It is known (see, e.g., [5]) that there is an STS of order n, briefly STS(n), if and only if $n \equiv 1$ or $3 \pmod 6$.

Algebraically the set of all Steiner triple systems may be looked upon as a variety of quasigroups satisfying the three identities

$$\text{(i)} \quad x^2 = x \quad \text{(idempotent)}$$

$$\left.\begin{array}{ll}\text{(ii)} & x(xy) = y \\ \text{(iii)} & (yx)x = y\end{array}\right\} \text{(totally symmetric)}$$

Quasigroups satisfying (i), (ii), (iii) are called Steiner quasigroups. Quasigroups satisfying (ii) and (iii) are commutative, and are called totally symmetric quasigroups.

An extended triple system is a pair (S, W) where S is a finite set and W is a collection of non-ordered triples from S, where each triple may have repeated elements, such that every pair of elements of S, not necessarily distinct, belongs to exactly one triple of W .

The class of all extended triple systems is co-extensive with the variety of quasigroups satisfying only the totally symmetric identities $x(xy) = y$ and $(yx)x = y$.

Quasigroups satisfying the totally symmetric identities are not necessarily idempotent. We shall denote by $\{n; b\}$ the class of all extended triple systems on n elements which have b idempotents. Clearly, $\{n; n\}$ is the class of all STS(n) and this class is non-empty if and only if $n \equiv 1$ or $3 \pmod 6$.

We say $\{n; a\}$ exists if there exist systems with parameters n and a. If (S, W) belongs to $\{n; a\}$, we simply write $W \in \{n; a\}$.

Johnson and Mendelsohn [8] gave the following conditions which are necessary for the existence of $\{n; a\}$, $0 \le a \le n$:

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \le \frac{n}{2}$;

(4) if $a = n-1$, then $n = 2$.

It was also conjectured in [8] that the necessary conditions for the existence of $\{n; a\}$ given above were sufficient.

Let (S, W) and (S^*, W^*) be two extended triple systems. If $S \subseteq S^*$ and $W \subseteq W^*$, we shall say that (S, W) is a subsystem of (S^*, W^*) and that (S^*, W^*) contains (S, W). If $W \cap W^* = \phi$, then we say that (S, W) and (S^*, W^*) are disjoint. If there is a bijection $\alpha : S \longrightarrow S^*$ such that $(W)\alpha = W^*$, then we say that (S, W) and (S^*, W^*) are isomorphic (or equivalent).

There are essentially two inequivalent systems based on the set $S = \{1, 2, 3\}$. We shall denote these by

$\mathfrak{T}_0 = \{(1, 1, 2), (2, 2, 3), (3, 3, 1)\} \in \{3; 0\}$,

$\mathfrak{T}_3 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3)\} \in \{3; 3\}$.

There are two inequivalent systems on $S = \{1, 2, 3, 4\}$. We denote these by

$\mathcal{K}_4 = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (2, 3, 4)\} \in \{4; 1\}$,

$\mathcal{K}_4^* = \{(1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2), (1, 3, 4)\} \in \{4; 1\}$.

Algebraically $\mathcal{K}_4$ is associated with the "four-group". Up to isomorphism, there is only one system on $S = \{1, 2, 3, 4, 5\}$. We denote this system by

$\mathcal{V}_5 = \{(1,1,1), (2,2,3), (3,3,4), (4,4,5), (5,5,2), (1,2,4), (1,3,5)\} \in \{5; 1\}$.

A system $W \in \{n; a\}$ which contains the maximum possible number $\left[\frac{a}{3}\right]$ mutually disjoint copies of $\mathcal{T}_3$ is called a <u>consistent</u> system. ($[x]$ denotes the largest integer not exceeding x.)

For convenience and completeness we state

<u>Theorem 1.1</u> (Johnson and Mendelsohn [8]). Suppose $\{n; a\}$ exists with $0 \le a \le n$. Then

(1) if $n \equiv 0 (\mathrm{mod}\ 3)$, then $a \equiv 0 (\mathrm{mod}\ 3)$;

(2) if $n \not\equiv 0 (\mathrm{mod}\ 3)$, then $a \equiv 1 (\mathrm{mod}\ 3)$;

(3) if n is even, then $a \le \frac{n}{2}$;

(4) if $a = n - 1$, then $n = 2$.

Chapter 2 of this thesis deals mainly with some of the essential properties of $\{n; a\}$. Some basic constructions are given, and examples of consistent systems, for small values of n, are included at the end. These examples provide enough initial cases for recursive constructions which follow. In Chapter 3 we use direct construction methods to show that the necessary conditions for the existence of $\{n; a\}$ given in Theorem 1.1 are also sufficient. Small embeddings of extended triple systems are considered in Chapter 4, and a recursive method of construction is developed. An algebraic approach is taken in

Chapter 5. In this chapter we construct consistent systems using Direct Products and Singular Direct Products. In Chapter 6 we comment on the existence of non-isomorphic extended triple systems, and some examples of non-isomorphic systems are given. Apart from a few exceptions, it is shown that there exist at least two non-isomorphic systems in each class $\{n; a\}$.

CHAPTER 2

Properties of $\{n; a\}$ and some basic constructions

1. Preliminaries.

A partial Steiner triple system is a pair (P, T) where P is a finite set and T a collection of 3-subsets of P, called triples, such that every pair of distinct elements of P is contained in at most one triple of T. Unlike Steiner triple systems, there is no cardinality restriction on P. The number $|P|$ is called the order of the partial Steiner triple system (P, T). A partial Steiner triple system (P, T) is called maximal if $|T'| \leq |T|$ for every partial Steiner triple system (P, T′). It is known [7, 19] that a maximal partial Steiner triple system of order n exists for every positive integer n. In particular, if $n \equiv 1$ or 3(mod 6), it is clear that a maximal partial Steiner triple system of order n is indeed an STS(n). In the following section we shall be concerned mainly with maximal partial Steiner triple systems.

2. Some useful connections.

It is already clear that $\{n; n\}$ exists if and only if there exists an STS(n). The purpose of this section is to establish further connections between a certain class of extended triple systems and maximal partial Steiner triple systems. We shall utilize some properties of $\{n; a\}$ determined in [8].

If there is a system $W \in \{n; a\}$, the triples of W are essentially of three types: (1) (i, i, i), (2) (j, j, k), (3) (i, j, k), where i, j, k are pairwise distinct. Johnson and Mendelsohn [8] proved the following

Lemma 2.1. Suppose there is a system $W \in \{n; a\}$.

(i) If n is even, then amongst the type (2) triples (j, j, k) each idempotent appears an odd number of times as the single element k , and each non-idempotent appears an even number of times.

(ii) If n is odd, then amongst the type (2) triples (j, j, k) each non-idempotent must appear at least once as the element k . In particular, if the non-idempotents are $1, 2, 3, \cdots, b$, then in the triples $(1, 1, \alpha_1), (2, 2, \alpha_2), (3, 3, \alpha_3), \cdots, (b, b, \alpha_b)$ the elements $\alpha_1, \alpha_2, \cdots, \alpha_b$ are a permutation of $1, 2, \cdots, b$ with $\alpha_i \neq i$.

(iii) The number of type (3) triples (i, j, k) is

$$\frac{n(n-1)}{6} - \frac{(n-a)}{3} .$$

J. Schönheim [19] proved in essence

Lemma 2.2. Let (P, T) be a maximal partial Steiner triple system of order n . Then

$$|T| = \begin{cases} 6k^2 - 2k & \text{if} \quad n = 6k, \\ 6k^2 + k & \text{if} \quad n = 6k+1, \\ 6k^2 + 2k & \text{if} \quad n = 6k+2, \\ 6k^2 + 5k + 1 & \text{if} \quad n = 6k+3, \\ 6k^2 + 6k + 1 & \text{if} \quad n = 6k+4, \\ 6k^2 + 9k + 2 & \text{if} \quad n = 6k+5\ . \end{cases}$$

We are now in a position to prove the following theorems.

Theorem 2.3. There is a system $W \in \{n; \frac{n}{2}\}$ if and only if there is an STS$(n+1)$.

Proof. Let (S, T) be an STS$(n+1)$ where $S = \{1, 2, \cdots, n+1\}$. We shall eliminate from T the collection of all triples containing a particular element, say $n+1$. Let $T^{(n+1)}$ be the collection of all triples of T containing the element $n+1$ and assume, without loss of generality, $T^{(n+1)} = \{(n+1, 1, 2), (n+1, 3, 4), \cdots, (n+1, n-1, n)\}$. Let $T^* = \{(1, 1, 2), (2, 2, 2), (3, 3, 4), (4, 4, 4), \cdots, (n-1, n-1, n), (n, n, n)\}$.

Put $W = (T - T^{(n+1)}) \cup T^*$. Then $W \in \{n; \frac{n}{2}\}$, based on $S - \{n+1\}$.

Assume there is a system $W \in \{n; \frac{n}{2}\}$ based on the set $S^* = \{1, 2, \cdots, n\}$. Introduce a new element ∞ and put $S = S^* \cup \{\infty\}$. We now derive from W an STS$(n+1)$.

By Theorem 1.1, $n \equiv 0$ or $2 \pmod 6$. So, by Lemma 2.1, we find that amongst the type (2) triples (j, j, k) in W, the element k must be idempotent. So, up to permuting the elements of S^*, W must contain the triples $J = \{(1, 1, 2), (2, 2, 2), (3, 3, 4), (4, 4, 4), \cdots, (n-1, n-1, n), (n, n, n)\}$.

Let $T_\infty = \{(1, 2, \infty), (3, 4, \infty), \cdots, (n-1, n, \infty)\}$. Put $T = (W - J) \cup T_\infty$. Then it is easily seen that (S, T) is an $STS(n+1)$, which completes the proof.

<u>Remark 1.</u> In our construction of $W \in \{n; \frac{n}{2}\}$ there are several possibilities for the collection T^*, because in each of the $\frac{n}{2}$ disjoint pairs $(1, 2), (3, 4), \cdots, (n-1, n)$ any one of the two elements may be idempotent in W. This is a crucial point, since our choice of T^* may very well determine the internal structure of the system W.

For example, let (S, T) be the $STS(7)$ where $S = \{1, 2, 3, 4, 5, 6, 7\}$, and

$T = \{(1, 2, 4), (1, 3, 7), (1, 5, 6), (2, 3, 5), (2, 6, 7), (3, 4, 6), (4, 5, 7)\}$.

$T^{(7)} = \{(1, 3, 7), (2, 6, 7), (4, 5, 7)\}$.

Let W_1 and W_2 be the following two systems derived from (S, T) by deleting $T^{(7)}$ from the collection T:

$$W_1 = \{(1, 1, 1), (3, 3, 1), (2, 2, 2), (6, 6, 2), (5, 5, 5), (4, 4, 5), (1, 2, 4), (1, 5, 6), (2, 3, 5), (3, 4, 6)\} \in \{6; 3\}.$$

$$W_2 = \{(1, 1, 1), (3, 3, 1), (2, 2, 2), (6, 6, 2), (4, 4, 4), (5, 5, 4), (1, 2, 4), (1, 5, 6), (2, 3, 5), (3, 4, 6)\} \in \{6; 3\}.$$

It is easy to check that W_1 contains no copy of $\mathfrak{T}_3$, while W_2 contains a copy of $\mathfrak{T}_3$ on the set $\{1, 2, 4\}$. This remark is not restricted to the construction given in Theorem 2.3. It is worth noting that the collection $\{(i, i, i), (j, j, i)\}$ may always be replaced with the collection $\{(i, i, j), (j, j, j)\}$ within any system. This procedure is treated more generally in the next section.

Theorem 2.4. There is a system $W \in \{6n+4; 3n+1\}$ if and only if there is a maximal partial Steiner triple system of order $6n+4$.

Proof. Let (P, T) be a maximal partial Steiner triple system where $P = \{1, 2, \cdots, 6n+4\}$. It follows from Lemma 2.2 that $|T| = 6n^2+6n+1$ and so the triples of T give rise to a total of $18n^2+18n+3$ unordered pairs. As a result, there are $\binom{6n+4}{2} - (18n^2+n+3) = 3n+3$ pairs of elements from P not contained in any triple of T . The number of pairs containing a particular element in (P, T) is an even number, and each element is contained in an odd number $6n+3$ pairs, counting all pairs from P . Consequently, one particular element must appear in 3 of these $3n+3$ excluded pairs, while all other elements appear in one pair. Up to permuting the elements of P , we may assume that the pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(5, 6)$, $(7, 8)$, $\cdots$, $(6n+3, 6n+4)$ do not appear in any triple of T , but every other pair appears in exactly one triple of T .

$$T^* = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (5, 5, 5), (6, 6, 5), (7, 7, 7), (8, 8, 7), \cdots, (6n+3, 6n+3, 6n+3), (6n+4, 6n+4, 6n+3)\}$$

Let T^* be as above. Put $W = T \cup T^*$. It is easily verified that $W \in \{6n+4; 3n+1\}$ based on P . Conversely, let us assume that there is a system $W \in \{6n+4; 3n+1\}$ based on some set $P = \{1, 2, \cdots, 6n+4\}$. Let T be the collection of all type (3) triples (i, j, k) in W . By Lemma 2.1, $|T| = \frac{(6n+4)(6n+3)}{6} - (n+1) = 6n^2+6n+1$. It is clear that (P, T) is indeed a maximal partial Steiner triple system of order $6n+4$. This completes the proof of the theorem.

<u>Theorem 2.5.</u> There is a system $W \in \{6n+5; 6n+1\}$ if and only if there is a maximal partial Steiner triple system of order $6n+5$.

<u>Proof.</u> Let (P, T) be a maximal partial Steiner triple system, where $P = \{1, 2, \cdots, 6n+5\}$. By Lemma 2.2, $|T| = 6n^2 + 9n+2$. Thus the triples of T give rise to $18n^2+27n+6$ unordered pairs. There must be $\binom{6n+5}{2} - (18n^2+27n+6) = 4$ pairs of elements of P not contained in any triple of T . Each element is contained in an even number $6n+4$ pairs, counting all pairs from P . In (P, T) the number of pairs containing a particular element is always even. So we may assume, without loss of generality, that the pairs $(1, 2), (2, 3), (3, 4), (4, 1)$ do not appear in any triple of T , but that every other pair appears in exactly one triple of T .

Let $T^* = \{(1, 1, 2), (2, 2, 3), (3, 3, 4), (4, 4, 1), (5, 5, 5), (6,6,6), \cdots, (6n+5, 6n+5, 6n+5)\}$.

Put $W = T \cup T^*$. Clearly $W \in \{6n+5; 6n+1\}$ based on the set P .

Conversely, let us assume that $W \in \{6n+5; 6n+1\}$ based on some set $P = \{1, 2, \cdots, 6n+5\}$. Let T be the collection of all type (3) triples (i, j, k) of W . Then it is easily verified that $|T| = 6n^2+9n+2$ and (P, T) is a maximal partial Steiner triple system of order $6n+5$. This completes the proof.

<u>Theorem 2.6.</u> There is a system $W \in \{6n+4; 3n+1\}$ if and only if there is a system $W^* \in \{6n+5; 6n+1\}$.

<u>Proof.</u> Suppose there is a system $W \in \{6n+4; 3n+1\}$ based on the set $S = \{1, 2, \cdots, 6n+4\}$. Then by Lemma 2.1, each of the $3n+1$

idempotents must appear an odd number of times as the element k in type (2) triples (j, j, k) . Up to a permutation on S , it is clear that W must contain one of the two collections of triples:

$A = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (5, 5, 6), (6, 6, 6),$
$(7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}$,

or

$B = \{(1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2), (5, 5, 6), (6, 6, 6),$
$(7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}$.

Let us introduce a new element ∞ , and put $S^* = S \cup \{\infty\}$. We shall construct $W^* \in \{6n+5;\ 6n+1\}$ based on S^* as follows.

If W contains the collection A, we let

$A^* = \{(1, 1, 2), (2, 2, \infty), (\infty, \infty, 3), (3, 3, 1), (1, 4, \infty), (4, 4, 4),$
$(5,5,5), \cdots, (6n+4, 6n+4, 6n+4), (5,6,\infty), (7,8,\infty), \cdots,$
$(6n+3, 6n+4, \infty)\}$.

Then put $W^* = (W - A) \cup A^*$. It is easily verified that $W^* \in \{6n+5;\ 6n+1\}$, based on S^* .

If W contains the collection B, we let

$B^* = \{(1, 1, \infty), (\infty, \infty, 3), (3, 3, 2), (2, 2, 1), (2, 4, \infty), (4, 4, 4),$
$(5,5,5), \cdots, (6n+4, 6n+4, 6n+4), (5,6,\infty), (7,8,\infty), \cdots,$
$(6n+3, 6n+4, \infty)\}$.

Put $W^* = (W - B) \cup B^*$. Then $W^* \in \{6n+5;\ 6n+1\}$. Conversely, we assume there is a system $W^* \in \{6n+5;\ 6n+1\}$ based on the set $S^* = \{0, 1, 2, \cdots, 6n+4\}$. Again by Lemma 2.1, we find that, up to a permutation on S , W^* must contain the collection of triples

$T^* = \{(0, 0, 1), (1, 1, 2), (2, 2, 3), (3, 3, 0), (4, 4, 4), (5, 5, 5), \cdots,$
$(6n+4, 6n+4, 6n+4)\}$.

We shall construct $W \in \{6n+4;\ 3n+1\}$ by eliminating some triples from W^*, but we must be careful which triples we delete. If we delete all triples containing one of the elements 0, 1, 2, 3, this procedure will always guarantee us the desired result. Otherwise, we may be forced into a situation where the end result is not always a system in $\{6n+4;\ 3n+1\}$. To this end we let T_0^* be the collection of all triples in W^* containing the element 0 and assume, without any loss of generality,

$$T_0^* = \{(0, 0, 1), (3, 3, 0), (0, 2, 4), (0, 5, 6), (0, 7, 8), \cdots, (0, 6n+3, 6n+4)\}.$$

Let

$$A^* = \{(2, 2, 2), (1, 1, 2), (3, 3, 2), (4, 4, 2), (5, 5, 6), (6, 6, 6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}.$$

Put $W = (W^* - (T^* \cup T_0^*)) \cup A^*$. It can be checked that $W \in \{6n+4;\ 3n+1\}$ based on $S^* - \{0\}$. This completes the proof of the theorem.

Theorem 2.7. There is a system $W \in \{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$ if and only if there is a system $W^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathcal{V}_5$.

Proof. Assume there is a system $W \in \{6n+4;\ 3n+1\}$ based on the set $S = \{1, 2, \cdots, 6n+4\}$ and containing a copy of $\mathcal{K}_4$ based on the set $\{1, 2, 3, 4\}$. We may assume without loss of generality that W contains the collection

$$A = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4,4,1), (2,3,4), (5,5,6), (6,6,6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}.$$

Introduce a new element ∞ and put $S^* = S \cup \{\infty\}$.

Let

$A^* = \{(1, 1, 2), (2, 2, \infty), (\infty, \infty, 3), (3, 3, 1), (2, 3, 4), (1, 4, \infty),$
$(4,4,4), (5,5,5), \cdots, (6n+4, 6n+4, 6n+4), (5,6,\infty), (7,8,\infty), \cdots,$
$(6n+3, 6n+4, \infty)\}$.

Put $W^* = (W - A) \cup A^*$. Then $W^* \in \{6n+5;\ 6n+1\}$ and W^* contains a copy of $\mathcal{V}_5$ based on the set $\{1, 2, 3, 4, \infty\}$. Conversely, assume there is a system $W^* \in \{6n+5;\ 6n+1\}$ based on the set $S^* = \{0, 1, 2, \cdots, 6n+4\}$ and containing a copy of $\mathcal{V}_5$ based on the set $\{0, 1, 2, 3, 4\}$. We may assume that W^* contains the following two collections of triples:

$T^* = \{(0, 0, 1), (1, 1, 2), (2, 2, 3), (3, 3, 0), (0, 2, 4), (1, 3, 4),$
$(4,4,4), (5,5,5), \cdots, (6n+4, 6n+4, 6n+4)\}$, and

$T_0^* = \{(0, 0, 1), (3, 3, 0), (0, 2, 4), (0, 5, 6), (0, 7, 8), \cdots,$
$(0, 6n+3, 6n+4)\}$.

Let

$A^* = \{(2, 2, 2), (1, 1, 2), (3, 3, 2), (4, 4, 2), (1, 3, 4), (5, 5, 6),$
$(6,6,6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4),$
$(6n+4, 6n+4, 6n+4)\}$.

Put $W = (W^* - (T^* \cup T_0^*)) \cup A^*$. Then $W \in \{6n+4;\ 3n+1\}$ and W contains a copy of $\mathcal{K}_4$ based on the set $\{1, 2, 3, 4\}$.

3. The replacement property.

So far we have seen how extended triple systems may be derived from Steiner and similar triple systems. Perhaps more important is the fact that extended triple systems may be derived from each other. We outline here a technique that will prove very effective in most of our constructions that appear in subsequent chapters.

<u>Lemma 2.8.</u> Suppose there is a system $W \in \{n; a\}$ containing a subsystem $Q \in \{q; r\}$. Suppose there is another system $Q^* \in \{q; s\}$. Then there is a system $W^* \in \{n; a-r+s\}$ containing a copy of Q^* .

<u>Proof.</u> We may assume, without loss of generality, that the system Q^* is based on the same set of elements as Q ; for otherwise we can relabel the elements to achieve this. We put $W^* = (W - Q) \cup Q^*$. It is readily verified that $W^* \in \{n; a-r+s\}$.

<u>Remark 2.</u> The fact that we can remove subsystems, and appropriately replace them, will prove quite crucial in most of our constructions. It is worth noting that even if Q and Q^* are isomorphic in Lemma 2.8, then W and W^* need not be isomorphic. For example, in $\mathcal{K}_4$ we may replace the sybsystem $Q = \{(1, 1, 1), (2, 2, 1)\}$ with $Q^* = \{(1, 1, 2), (2, 2, 2)\}$ thus obtaining a copy of $\mathcal{K}_4^*$.

<u>Theorem 2.9.</u> Suppose there is a system $W \in \{n; a\}$ containing t mutually disjoint copies of $\mathfrak{J}_3$. Then there is a system $W_k \in \{n; a-3k\}$ containing at least k mutually disjoint copies of $\mathfrak{J}_0$ and $t-k$ mutually disjoint copies of $\mathfrak{J}_3$, where $k = 1, 2, \cdots, t$.

<u>Proof.</u> The proof follows directly by applying Lemma 2.8. We obtain W_1 from W by removing one of the t disjoint copies of $\mathfrak{J}_3$ and replacing it with the appropriate copy of $\mathfrak{J}_0$. For $k=1, 2, \cdots, t-1$, we obtain W_{k+1} from W_k by the same procedure.

Corollary 2.10. If there exists a consistent system $W \in \{n; a\}$, then there exists a consistent system $W_k \in \{n; a-3k\}$ where $k = 1, 2, \cdots, \left[\frac{a}{3}\right]$.

Example 2.11. Let $W \in \{9; 9\}$ be given by

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9), (2, 6, 7),
(3, 4, 9), (3, 5, 7), (3, 6, 8).

We obtain $W_1 \in \{9; 6\}$:

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 8), (8, 8, 9), (9, 9, 7), (1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8),
(2, 5, 9), (2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

$W_2 \in \{9; 3\}$:

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 5), (5, 5, 6), (6, 6, 4), (7, 7, 8), (8, 8, 9), (9, 9, 7), (1, 4, 7),
(1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9), (2, 6, 7), (3, 4, 9), (3, 5, 7),
(3, 6, 8).

$W_3 \in \{9; 0\}$:

(1, 1, 2), (2, 2, 3), (3, 3, 1), (4, 4, 5), (5, 5, 6), (6, 6, 4), (7, 7, 8),
(8, 8, 9), (9, 9, 7), (1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9),
(2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

For an application of Lemma 2.8, let us consider the following theorem and example.

<u>Theorem 2.12.</u> Let $n \equiv 2$ or $4 \pmod 6$, $n \geq 4$. Then there exists a system $W \in \{n; 1\}$ which contains at least one copy of $\mathcal{K}_4$.

<u>Proof.</u> Let (S, T) be any $STS(n-1)$. We add to S a new element ∞, and put $S^* = S \cup \{\infty\}$. Let $T^* = \{(x, x, \infty) \mid x \in S^*\}$ and $W = T \cup T^*$. Then $W \in \{n; 1\}$, based on S^*. Since $n \geq 4$, T contains at least one triple (a, b, c). Hence, W contains the collection $K = \{(\infty, \infty, \infty), (a, a, \infty), (b, b, \infty), (c, c, \infty), (a, b, c)\}$, which is a copy of $\mathcal{K}_4$.

<u>Remark 3.</u> The system W constructed in the proof of Theorem 2.12 is associated algebraically with a totally symmetric loop (see for example [2]). It is clear that W does not contain a copy of $\mathcal{K}_4^*$, but contains as many copies of $\mathcal{K}_4$ as there are triples in T. The copies of $\mathcal{K}_4$ pairwise intersect in (∞, ∞, ∞).

<u>Corollary 2.13.</u> Let $n \equiv 2$ or $4 \pmod 6$, $n \geq 4$. Then there exists a system $W^* \in \{n; 1\}$ containing a unique copy of $\mathcal{K}_4^*$.

<u>Proof.</u> Let $W \in \{n; 1\}$ be as constructed in the proof of Theorem 2.12. Let Q be a copy of $\mathcal{K}_4$ in W. Let Q^* be a copy of $\mathcal{K}_4^*$, based on the same set as Q. Put $W^* = (W - Q) \cup Q^*$. A straightforward verification shows $W^* \in \{n; 1\}$ contains Q^* as its only copy of $\mathcal{K}_4^*$.

<u>Example 2.14.</u> This example illustrates Theorem 2.12 and Corollary 2.13 for the cases $n = 8, 10$.

(i)

$W \in \{8;\ 1\}$

$(\underline{\infty, \infty, \infty})$, $(\underline{1, 1, \infty})$, $(\underline{2, 2, \infty})$, $(\underline{3, 3, \infty})$, $(\underline{1, 2, 3})$,
$(4, 4, \infty)$, $(5, 5, \infty)$, $(6, 6, \infty)$, $(7, 7, \infty)$, $(1, 4, 7)$, $(1, 5, 6)$, $(2, 4, 5)$,
$(2, 6, 7)$, $(3, 4, 6)$, $(3, 5, 7)$.

$W^* \in \{8;\ 1\}$

$(\infty, \infty, \infty)^*$, $(1, 1, \infty)^*$, $(2, 2, 1)^*$, $(3, 3, 1)^*$, $(2, 3, \infty)^*$,
$(4, 4, \infty)$, $(5, 5, \infty)$, $(6, 6, \infty)$, $(7, 7, \infty)$, $(1, 4, 7)$, $(1, 5, 6)$, $(2, 4, 5)$,
$(2, 6, 7)$, $(3, 4, 6)$, $(3, 5, 7)$.

(ii)

$W \in \{10;\ 1\}$

$(\underline{\infty, \infty, \infty})$, $(\underline{1, 1, \infty})$, $(\underline{2, 2, \infty})$, $(\underline{3, 3, \infty})$, $(\underline{1, 2, 3})$,
$(4, 4, \infty)$, $(5, 5, \infty)$, $(6, 6, \infty)$, $(7, 7, \infty)$, $(8, 8, \infty)$, $(9, 9, \infty)$, $(1, 4, 7)$,
$(1, 5, 9)$, $(1, 6, 8)$, $(2, 4, 9)$, $(2, 5, 8)$, $(2, 6, 7)$, $(3, 4, 8)$, $(3, 5, 7)$,
$(3, 6, 9)$, $(4, 5, 6)$, $(7, 8, 9)$.

$W^* \in \{10;\ 1\}$

$(\infty, \infty, \infty)^*$, $(1, 1, \infty)^*$, $(2, 2, 1)^*$, $(3, 3, 1)^*$, $(2, 3, \infty)^*$,
$(4, 4, \infty)$, $(5, 5, \infty)$, $(6, 6, \infty)$, $(7, 7, \infty)$, $(8, 8, \infty)$, $(9, 9, \infty)$, $(1, 4, 7)$,
$(1, 5, 9)$, $(1, 6, 8)$, $(2, 4, 9)$, $(2, 5, 8)$, $(2, 6, 7)$, $(3, 4, 8)$, $(3, 5, 7)$
$(3, 6, 9)$, $(4, 5, 6)$, $(7, 8, 9)$.

Remark 4. In (i) and (ii) of Example 2.14 the systems W and W^* are clearly non-isomorphic. Algebraically the system $W \in \{8;\ 1\}$ is a group, but the system $W \in \{10;\ 1\}$ cannot be associated with any group.

Before we proceed to list some examples of systems which are consistent, we give some simple examples to illustrate that there are systems which are not consistent.

Example 2.15. In each case the system W given below is easily checked to be inconsistent:

(i) $W \in \{7; 7\}$

(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5),
(6, 6, 6), (7, 7, 7), (1, 2, 4), (1, 3, 7), (1, 5, 6),
(2, 3, 5), (2, 6, 7), (3, 4, 6), (4, 5, 7).

(ii) $W \in \{11; 7\}$

(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6),
(7, 7, 7), (8, 8, 9), (9, 9, 10), (10, 10, 11), (11, 11, 8),
(1, 2, 3), (1, 4, 8), (1, 5, 11), (1, 6, 10), (1, 7, 9), (2, 4, 10),
(2, 5, 9), (2, 6, 11), (2, 7, 8), (3, 4, 5), (3, 6, 7), (3, 8, 10),
(3, 9, 11), (4, 6, 9), (4, 7, 11), (5, 6, 8), (5, 7, 10).

Observe $W \in \{11; 7\}$ contains a copy of $\mathcal{V}_5$ on $\{3, 8, 9, 10, 11\}$.

(iii) $W \in \{12; 6\}$

(1, 1, 1), (2, 2, 2), (5, 5, 5), (6, 6, 6), (9, 9, 9),
(10, 10, 10), (7, 7, 1), (4, 4, 2), (12, 12, 5), (8, 8, 6),
(11, 11, 9), (3, 3, 10), (1, 2, 3), (1, 4, 11), (1, 5, 10),
(1, 6, 9), (1, 8, 12), (2, 5, 9), (2, 6, 12), (2, 7, 11),
(2, 8, 10), (3, 4, 8), (3, 5, 7), (3, 6, 11), (3, 9, 12),
(4, 7, 12), (4, 9, 10), (5, 8, 11), (6, 7, 10), (7, 8, 9), (10, 11, 12).

Remark 5. There are examples of consistent systems in $\{11; 7\}$ and $\{12; 6\}$, but there can be no consistent $W \in \{7; 7\}$ since the copies of $\mathfrak{T}_3$ in such a W pairwise intersect.

4. Illustrative Examples: Consistent systems.

Theorem 2.9 will be one of our most effective tools used in future constructions. We list here some examples, for small values of n, of systems W to which the theorem applies. The techniques outlined in the previous sections have been utilized in constructing these examples. The parameters n, a, t are as in the statement of Theorem 2.9. The examples provide a source for future reference.

$W \in \{8; 4\}$ $\quad n = 8, a = 4, t = 1$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(7, 7, 7), (6, 6, 1), (5, 5, 2), (4, 4, 3), (8, 8, 7),
(1, 4, 7), (1, 5, 8), (2, 4, 8), (2, 6, 7), (3, 5, 7),
(3, 6, 8), (4, 5, 6).

$W \in \{9; 9\}$ $\quad n = 9, a = 9, t = 3$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9),
(2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

Observe $W \in \{9; 9\}$ and $W \in \{8; 4\}$ are derived from the same STS(9).

$W \in \{10; 4\}$ $n = 10$, $a = 4$, $t = 1$.

(a) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(6, 6, 6), (7, 7, 6), (5, 5, 1), (4, 4, 2), (10, 10, 3),
(8, 8, 10), (9, 9, 10), (1, 4, 9), (1, 6, 10), (1, 7, 8),
(2, 5, 10), (2, 6, 8), (2, 7, 9), (3, 4, 8), (3, 5, 7),
(3, 6, 9), (4, 5, 6), (4, 7, 10), (5, 8, 9).

(b) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (8, 8, 4), (5, 5, 1), (6, 6, 2), (7, 7, 3),
(9, 9, 1), (10, 10, 1), (1, 4, 6), (1, 7, 8), (2, 4, 5),
(2, 7, 9), (2, 8, 10), (3, 4, 9), (3, 5, 8), (3, 6, 10),
(4, 7, 10), (5, 6, 7), (5, 9, 10), (6, 8, 9) .

(c) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (8, 8, 4), (5, 5, 1), (6, 6, 2), (7, 7, 3),
(9, 9, 5), (10, 10, 5), (1, 4, 6), (1, 7, 8), (1, 9, 10),
(2, 4, 5), (2, 7, 9), (2, 8, 10), (3, 4, 9), (3, 5, 8),
(3, 6, 10), (4, 7, 10), (5, 6, 7), (6, 8, 9).

We observe the following properties concerning the three systems (a), (b), (c).

(i) (a) contains no copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

(ii) (b) contains a copy of $\mathcal{K}_4$ on {1, 5, 9, 10} .

(iii) (c) contains a copy of $\mathcal{K}_4^*$ on {1, 5, 9, 10}, and (c) is derived from (b) by an application of Lemma 2.8.

(iv) (a) is derived from $W \in \{11; 7\}$ which follows.

$W \in \{11; 7\}$ $n = 11$, $a = 7$, $t = 2$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 10), (9, 9, 11), (10, 10, 9), (11, 11, 8),
(1, 4, 9), (1, 5, 11), (1, 6, 10), (1, 7, 8), (2, 4, 11),
(2, 5, 10), (2, 6, 8), (2, 7, 9), (3, 4, 8), (3, 5, 7),
(3, 6, 9), (3, 10, 11), (4, 7, 10), (5, 8, 9), (6, 7, 11) .

$W \in \{12; 6\}$ $n = 12$, $a = 6$, $t = 2$.

(3, 3, 3), (6, 6, 6), (11, 11, 11), (3, 6, 11),
(4, 4, 4), (7, 7, 7), (12, 12, 12), (4, 7, 12),
(10, 10, 3), (8, 8, 6), (9, 9, 11), (2, 2, 4), (1, 1, 7),
(5, 5, 12), (1, 2, 3), (4, 5, 6), (7, 8, 9), (10, 11, 12),
(1, 4, 11), (1, 5, 10), (1, 6, 9), (1, 8, 12), (2, 5, 9),
(2, 6, 12), (2, 7, 11), (2, 8, 10), (3, 4, 8), (3, 5, 7),
(3, 9, 12), (4, 9, 10), (5, 8, 11), (6, 7, 10).

$W \in \{13; 13\}$ $n = 13$, $a = 13$, $t = 4$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(10, 10, 10), (11, 11, 11), (12, 12, 12), (10, 11, 12),
(13, 13, 13), (1, 4, 11), (1, 5, 10), (1, 6, 9), (1, 7, 13),
(1, 8, 12), (2, 4, 13), (2, 5, 9), (2, 6, 12), (2, 7, 11),
(2, 8, 10), (3, 4, 8), (3, 5, 7), (3, 6, 11), (3, 9, 12),
(3, 10, 13), (4, 7, 12), (4, 9, 10), (5, 8, 11), (5, 12, 13),
(6, 7, 10), (6, 8, 13), (9, 11, 13) .

Note $W \in \{13;\ 13\}$ and $W \in \{12;\ 6\}$ come from the same STS(13) .

$W \in \{14;\ 7\}$ $n = 14,\ a = 7,\ t = 2$.

(1, 1, 1), (5, 5, 5), (9, 9, 9), (1, 5, 9),

(2, 2, 2), (7, 7, 7), (14, 14, 14), (2, 7, 14),

(8, 8, 1), (11, 11, 5), (12, 12, 9), (10, 10, 2), (4, 4, 7), (13, 13, 14),

(3, 3, 3), (6, 6, 3), (1, 2, 3), (1, 4, 14), (1, 6, 12), (1, 7, 10),

(1, 11, 13), (2, 4, 12), (2, 5, 13), (2, 6, 8), (2, 9, 11), (3, 4, 11),

(3, 5, 7), (3, 8, 13), (3, 9, 10), (3, 12, 14), (4, 5, 6), (4, 8, 10),

(4, 9, 13), (5, 8, 12), (5, 10, 14), (6, 7, 11), (6, 9, 14), (6, 10, 13),

(7, 8, 9), (7, 12, 13), (8, 11, 14), (10, 11, 12).

$W \in \{15;\ 15\}$ $n = 15,\ a = 15,\ t = 5$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),

(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),

(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),

(10, 10, 10), (11, 11, 11), (12, 12, 12), (10, 11, 12),

(13, 13, 13), (14, 14, 14), (15, 15, 15), (13, 14, 15),

(1, 4, 14), (1, 5, 9), (1, 6, 12), (1, 7, 10), (1, 8, 15), (1, 11, 13),

(2, 4, 12), (2, 5, 13), (2, 6, 8), (2, 7, 14), (2, 9, 11), (2, 10, 15),

(3, 4, 11), (3, 5, 7), (3, 6, 15), (3, 8, 13), (3, 9, 10), (3, 12, 14),

(4, 8, 10), (4, 9, 13), (4, 7, 15), (5, 11, 15), (5, 8, 12), (5, 10, 14),

(6, 7, 11), (6, 9, 14), (6, 10, 13), (7, 12, 13), (8, 11, 14), (9, 12, 15).

$W \in \{15;\ 15\}$ and $W \in \{14;\ 7\}$ are derived from the same STS(15) .

W ∈ {16; 7}   n = 16, a = 7, t = 2 .

(1, 1, 1), (6, 6, 6), (8, 8, 8), (1, 6, 8),
(2, 2, 2), (4, 4, 4), (12, 12, 12), (2, 4, 12),
(15, 15, 15), (13, 13, 15), (14, 14, 15), (16, 16, 15), (7, 7, 1), (3, 3, 6),
(5, 5, 8), (11, 11, 2), (10, 10, 4), (9, 9, 12), (1, 2, 3), (1, 4, 13),
(1, 5, 9), (1, 10, 14), (1, 11, 15), (1, 12, 16), (2, 5, 13), (2, 6, 14),
(2, 7, 15), (2, 8, 16), (2, 9, 10), (3, 4, 11), (3, 5, 14), (3, 7, 16),
(3, 8, 10), (3, 9, 15), (3, 12, 13), (4, 5, 6), (4, 7, 14), (4, 8, 15),
(4, 9, 16), (5, 7, 12), (5, 10, 15), (5, 11, 16), (6, 7, 11), (6, 9, 13),
(6, 10, 16), (6, 12, 15), (7, 8, 9), (7, 10, 13), (8, 12, 14), (8, 13, 16),
(9, 11, 14), (10, 11, 12), (13, 14, 16).

W ∈ {17; 13}   n = 17, a = 13, t = 4 .

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(10, 10, 10), (11, 11, 11,), (12, 12, 12), (10, 11, 12),
(13, 13, 13), (14, 14, 15), (15, 15, 16), (16, 16, 17), (17, 17, 14),
(1, 4, 13), (1, 5, 9), (1, 6, 8), (1, 7, 17), (1, 10, 14), (1, 11, 15),
(1, 12, 16), (2, 4, 12), (2, 5, 13), (2, 6, 14), (2, 7, 15), (2, 8, 16),
(2, 9, 10), (2, 11, 17), (3, 4, 11), (3, 5, 14), (3, 6, 17), (3, 7, 16),
(3, 8, 10), (3, 9, 15), (3, 12, 13), (4, 7, 14), (4, 8, 15), (4, 9, 16),
(4, 10, 17), (5, 7, 12), (5, 8, 17), (5, 10, 15), (5, 11, 16), (6, 7, 11),
(6, 9, 13), (6, 10, 16), (6, 12, 15), (7, 10, 13), (8, 12, 14), (8, 13, 16),
(9, 11, 14), (9, 12, 17), (13, 14, 16), (13, 15, 17).

Concerning the systems $W \in \{16;\ 7\}$, $W \in \{17;\ 13\}$ note the following:

(i) $W \in \{16;\ 7\}$ is derived from $W \in \{17;\ 13\}$ by deleting all triples containing the number 17 from the system $W \in \{17;\ 13\}$.

(ii) $W \in \{16;\ 7\}$ contains a copy of $\mathcal{K}_4$ on $\{13, 14, 15, 16\}$.

(iii) $W \in \{17;\ 13\}$ contains a copy of $\mathcal{V}_5$ on $\{13, 14, 15, 16, 17\}$.