FINITE DIFFERENCE METHODS

FOR

THE NUMERICAL INTEGRATION OF $Y''=F(X,Y)$

A Thesis

Presented to

The Faculty of Graduate Studies and Research

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In Partial Fulfillment

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Master of Science

in

Computer Science

by

Mamunda A. V. Nair

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ABSTRACT

In this thesis, we consider the problem of obtaining approximate solutions of the second order differential equation,

$$y'' = f(x,y), \quad a \le x \le b \tag{1}$$

where $f(x,y)$ is a continuous function of x and y and $f(x,y)$ satisfies Lipschitz condition with respect to y.

First, we consider the numerical solution of (1) subject to the initial conditions

$$y(a) = \eta, \quad y'(a) = \eta_1 \tag{2}$$

by multi-step methods of the predictor-corrector type. For the predictor, we use an explicit finite difference formula of the type

$$\sum_{i=0}^{k} \alpha_i \, y_{n+i} = h^2 \sum_{i=0}^{k-1} \tilde{\beta}_i \, f_{n+i} \quad (n=0,1,2,\ldots) \tag{3}$$

and as corrector, the implicit finite difference formula

$$\sum_{i=0}^{k} \alpha_i \, y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i \, f_{n+i} \quad (n=0,1,2,\ldots). \tag{4}$$

Special cases of (3) and (4) for $k=2,3$ and 4 are considered. For $k=4$, α's, $\tilde{\beta}$'s, β's and the truncation errors for 3 different families of one parameter formulae are derived. Many numerical illustrations are given. Experimental results show that one of the families, designated as I-3 turns out to be the best (i.e., the resulting $|\text{error}|$ is the least). The fourth order Runge-Kutta method is discussed briefly and numerical results based on this method are given for comparison. Graphs showing $\log_2 (1/2h)$ versus $\log_{10} |\text{error}|$ are shown for some of the problems.

Next, we consider the differential equation (1) subject to the

boundary conditions

$$y(a) = A \text{ and } y(b) = B \tag{5}$$

or the more general boundary conditions

$$y'(a) - dy(a) = A \text{ and } y'(b) + ey(b) = B \ . \tag{6}$$

We consider boundary value problems where $f(x,y)$ is continuous and bounded and $f_y(x,y) \geq 0$. Both linear and non-linear cases are discussed. We solve them by using finite difference methods. Methods to set up finite difference schemes with k=2 and k=4 are shown. Numerical results based on the three families mentioned before and other formulae are presented.

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CHAPTER I

INTRODUCTION

1. General

The general solution of the nth order differential equation (D.E.)

$$y^{(n)}(x) = F[x, y(x), y'(x), \ldots, y^{(n-1)}(x)]$$

for a real function $y(x)$ normally depends on n parameters. Here, $y^{(m)}(x)$ denotes the value of the mth derivative of $y(x)$ at the point x. The parameters are determined in an initial value problem by prescribing the values

$$y^{(m)}(x) = A_m \quad (m=0,1,\ldots,n-1)$$

at a fixed point x=a. If the conditions are specified at more than one point, the problem is called a boundary value problem. The boundary conditions usually have the form

$$\Phi_i\,[y(x),\ y'(x), \ldots,\ y^{(n-1)}(x)] = 0 \quad (i=0,1,\ldots,n-1)$$

at the boundary points x=a and x=b. The functions F and Φ_i may be linear or non-linear.

Many problems in science and engineering can be formulated as one or more differential equations. In mechanical engineering or in astronomy, for example, a large number of problems are associated with force and motion. When a scientist deals with these problems, he uses a mixture of observed mathematical variables and one or more of hypothesized variables to build a model. Often, these models are expressed in terms of differential equations. But only a few, relatively speaking, of these models are simple enough to be solved analytically.

Hence one resorts to a numerical method. Sometimes, even when an analytical method is feasible, the resulting expression may be too complex for the tabulation of a function, so that one may prefer a numerical technique.

Differential equations may be of first or higher orders. Systems of equations of higher order can always be treated numerically by reducing them to a larger system of first order equations. For example, it is possible to integrate a D.E. of second order

$$y'' = f(x,y,y') \tag{1.1}$$

by reducing it to a system

$$y' = z, z' = f(x,y,z)$$

This situation is somewhat different if the equation to be integrated is of the form

$$y'' = f(x,y) \tag{1.2}$$

i.e., if no derivative appears on the right hand side of the D.E. In solving these problems, there is no need to calculate the first derivatives. These types of problems frequently arise in applications. For example, a simple pendulam satisfies the D.E.

$$\theta'' + \sin\theta = 0$$

Let the amplitude of the swing be $\theta(0) = \pi/4$. Then we have the additional condition $\theta'(0) = 0$. The period is obtained by integrating this initial value problem which is of the type (1.2).

As a second illustration [1], consider the bending of a strut with flexural rigidity $E(t)$ and axial compressive load P by a distributed transverse load $p(t)$, t being the coordinate along the axis of the strut. The bending moment distribution $M(t)$ then satisfies the D.E.

$$\frac{d^2M}{dt^2} + \frac{P}{E(t)} M = -p(t)$$

Assuming that the transverse load is a constant p, the flexural rigidity is a variable and that 2ℓ is the length when bent

$$E(t) = \frac{E_0}{1+(\frac{t}{\ell})^2}$$

If we take $P = \frac{E_0}{\ell^2}$ and introduce the variables

$$x = \frac{t}{\ell} \text{ and } y = \frac{M}{\ell^2 p},$$

the D.E. becomes

$$y'' + (1+x^2)y = -1$$

We take M=0 at each end, so that the boundary conditions are $y(-1) = y(1)=0$. This then becomes a linear boundary value problem of second order.

There occur cases where $f(x,y)$ in (1.2) is non-linear in y. In such cases, we have a non-linear boundary value problem.

2. Initial Value Problems

Consider the second order initial value problem

$$y'' = f(x,y),\ y(a) = \eta,\ y'(a) = \eta_1 \tag{1.3}$$

where $f(x,y)$ is a continuous function of x and y and $f(x,y)$ satisfies the Lipschitz condition with respect to y. We seek a solution in the range $a \leq x \leq b$, where a and b are real constants. We wish to integrate (1.3) using multi-step methods. Let $\{x_n\}$ be a sequence of pointsdefined by $x_n = a+nh$ $(n=0,1,2...)$. The approximate solutions will be obtained not on the continuous interval $a \leq x \leq b$, but at discrete points x_n, with $n = (b-a)/h$, h being the step length. Let y_n be an approximation to the theoretical solution at x_n, i.e., to $y(x_n)$ and $f_n = f(x_n,y_n)$. A general

linear multi-step formula may then be written as

$$\sum_{i=0}^{k} \alpha_i \, y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i \, f_{n+i} \tag{1.4}$$

Given y_{n+i} $(i=0,1,\ldots,k-1)$, y_{n+k} is to be estimated. If $\beta_k \neq 0$ in (1.4), we have an implicit formula. However, if $\beta_k=0$, we have an explicit formula. For an explicit formula, equation (1.4) gives y_{n+k} in terms of y_{n+i}, f_{n+i}, $i=0, 1,\ldots, k-1$ and α's and β's.

If we normalise by putting $\alpha_k=1$ and use the implicit formula, we have

$$y_{n+k} + \sum_{i=0}^{k-1} \alpha_i \, y_{n+i} = h^2 \, \beta_k f(x_{n+k}, y_{n+k}) + h^2 \sum_{i=0}^{k-1} \beta_i \, f_{n+i}$$

By using an iterative scheme, we have

$$y_{n+k}^{(t+1)} + \sum_{i=0}^{k-1} \alpha_i \, y_{n+i} = h^2 \, \beta_k \, f(x_{n+k}, y_{n+k}^{(t)}) + h^2 \sum_{i=0}^{k-1} \beta_i \, f_{n+i}$$

$$(t = 0,1,2,\ldots)$$

where $y_{n+k}^{(0)}$ is obtained by first using the explicit formula. The iteration is continued until

$$\left| y_{n+k}^{(t+1)} - y_{n+k}^{(t)} \right| < \varepsilon$$

where ε is a preassigned small +ve quantity.

The explicit formula is the predictor and the implicit formula is the corrector. But for the predictor we need starting vales. These may be obtained by using some other appropriate method, say the Runge-Kutta method [16].

Since the corrector is used iteratively to obtain the solution, it is primarily responsible for the accuracy of the method. We shall discuss in the next chapter three separate families of corrector formulae.

3. Boundary Value Problems

A boundary value problem is said to be of class M [4], if it is of the form (1.2) with the boundary conditions

$$y(a) = A, \; y(b) = B \tag{1.5}$$

where A, B are arbitrary constants, a and b are arbitrary finite constants and f(x,y) is a continuous function of the variables with $\frac{\partial f}{\partial y}(x,y)$ continuous, bounded and non-negative in this strip S defined by $a \leq x \leq b$ and $-\infty < y < \infty$. A boundary value problem of class M has a unique solution [4]. We shall discuss methods for obtaining approximate solutions of boundary value problems of class M in this thesis.

We shall also treat the case when (1.2) has the more general boundary conditions

$$\begin{aligned} y'(a) - dy(a) &= A, \\ y'(b) + ey(b) &= B \end{aligned} \tag{1.6}$$

with $d \geq 0$, $e \geq 0$ and $d + e > 0$.

The boundary value problem (1.2) - (1.6) also has a unique solution [4].

Among the methods available for the numerical solution of boundary value problems, the two prominent methods are the shooting method and the finite difference method.

The Shooting Method

The name comes from the situation in the two-point boundary value problem for a second order D.E. with the initial and final values of the solution prescribed. Varying the initial slope gives rise to a set of profiles which suggest the trajectory of a projectile 'shot' from the initial point. The initial slope is sought which

results in the trajectory 'hitting' the target, the final value.

Consider the boundary value problem (1.2) - (1.5) assumed to have a unique solution. We make the initial guess α_1 for the quantity $y'(a)$ at the initial point. The D.E. now becomes an initial value problem and may be solved easily. As a result we get a value B_1 for the end point, which in general will be different from B, the correct value. Repeating the process with a new slope α_2, we get another value B_2 at the end point. From these values at the end point, the correct initial slope is sought by interpolation and the problem is solved as an initial value problem.

Finite Difference Method

In this method, the D.E. is replaced by an appropriate finite difference equation. The solution of the problem is sought at discrete values of the independent variable. The effect is to replace the original problem by the problem of solving a finite number of algebraic equations. If the original D.E. is linear, the finite difference equations will be linear. If, on the other hand, the original D.E. is non-linear, the resulting finite difference equations will be non-linear.

A Finite Difference Scheme

In this scheme, we set up a finite number of grid points x_1, x_2, ..., x_{N+1} where

$$x_n = a+n\,h,\ n=0,\ 1,\ldots,\ N+1$$

$$h = (b-a)/(N+1)$$

and $$x_0 = a,\ x_{N+1} = b\ .$$

N is an appropriate positive integer.

If we denote the true solution of the boundary value problem at x_n by $y\ (x_n)$, a method is designed to obtain the numbers y_n, which approxi-

mate closely the values of $y(x_n)$. A convenient way to obtain such a scheme is to have the values y_n satisfy a difference equation (∇.E.) of the form

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i y''_{n+i}, \quad n=0,1,2,\ldots, N-k+1 \tag{1.7}$$

where $y''_i = f(x_i, y_i)$

$\alpha_k \neq 0$

$|\alpha_0| + |\beta_0| \neq 0$

Here, k is the order of the ∇.E. The equation may be normalised by choosing $\alpha_k = 1$. Equation (1.7) leads to N-k+2 equations involving $y_1, y_2, \ldots, y_N$ in a linear or non-linear form. y_0 and y_{N+1} are determined by the boundary conditions.

If we denote by T_{n+k}, the truncation error, the true solution of the boundary value problems will satisfy a ∇.E. of the form

$$\sum_{i=0}^{k} \alpha_i y(x_{n+i}) = h^2 \sum_{i=0}^{k} \beta_i y''(x_{n+i}) + T_{n+k} \tag{1.8}$$

Defining the discretisation error e_n by

$$e_n = y(x_n) - y_n \tag{1.9}$$

we have from (1.8) and (1.9)

$$\sum_{i=0}^{k} \alpha_i e_{n+i} = h^2 \sum_{i=0}^{k} \beta_i [y''(x_{n+i}) - y''_{n+i}] + T_{n+k} \tag{1.10}$$

$$= h^2 \sum_{i=0}^{k} \beta_i e_{n+i} g_{n+i} + T_{n+k}$$

where $g_{n+i} = \frac{\partial f}{\partial y}(x_{n+i}, \xi)$ ξ being an appropriate value between $y(x_{n+i})$ and y_{n+i}.

Assuming that y(x) has continuous derivatives of sufficiently high

orders, we associate with (1.7) the operator

$$L\,[y(x);h] = \sum_{i=0}^{k} \alpha_i\, y(x+ih) - h^2 \sum_{i=0}^{k} \beta_i\, y''(x+ih) \tag{1.11}$$

Expanding (1.11) using Taylor's Theorem, we have

$$L\,[y(x;h] = \sum_{n=0}^{\infty} C_n\, h^n\, y^{(n)}\,(x) \tag{1.12}$$

where

$$C_0 = \Sigma\alpha_i,\ C_1 = \Sigma i\alpha_i,\ C_2 = \frac{1}{2}\,\Sigma i^2\alpha_i - \Sigma\,\beta_i \tag{1.13}$$

and $$C_q = \frac{1}{q!}\,\Sigma i^q\alpha_i - \frac{1}{(q-2)!}\,\Sigma i^{q-2}\,\beta_i,\ q=3,4.,,,$$

the summations being from 0 to k.

Now we define the degree of the ∇.E. (1.7) as the unique integer p such that

$$C_q = 0\,(q=0,1,\ldots,\ p+1);\ C_{p+2} \neq 0$$

Then we have

$$L\,[y(x);\ h] = C_{p+2}\ h^{p+2}\ y^{(p+2)}(x) + 0(h^{p+3}) \tag{1.14}$$

Therefore, the truncation error is given by

$$T_{n+k} = C_{p+2}\ h^{p+2}\ \ y^{(p+2)}(x_n) + 0(h^{p+3}) \tag{1.15}$$

However, for a number of difference operators we can write

$$L\,[y(x);h] = C_{p+2}\ h^{p+2}\ y^{(p+2)}(\S) \tag{1.16}$$

where § is a suitable number in the interval (x,x+kh). Henrici [4] refers to (1.16) as the generalised mean value therom.

Let the polynomials [4] associated with the ∇.E. (1.7) be

$$\alpha(x) = \Sigma\alpha_i\ x^i \tag{1.17}$$

$$\beta(x) = \Sigma\beta_i\ x^i$$

the summations again from 0 to k.

If the ∇.E. has a positive degree $p \geq 1$, we have

$$C_0 = C_1 = C_2 = 0$$

In terms of $\alpha(x)$ and $\beta(x)$, we then have

$$\alpha(1) = 0 \tag{1.18}$$

$$\alpha'(1) = 0$$

$$\alpha''(1) = 2\beta(1)$$

The first two conditions of (1.18) imply that x=1 is a root of $\alpha(x)=0$ of at least multiplicity 2. Hence we can write $\alpha(x)$ in the form

$$\alpha(x) = (x-1)^2 \gamma(x) \tag{1.19}$$

where $\gamma(x)$ is a monic polynomial of degree k-2.

CHAPTER II

DEVELOPMENT OF FAMILIES OF ONE PARAMETER FORMULAS

1. Hull and Newberry have discussed in a paper [6], the integration procedures with respect to initial value problems of the first order. They have derived three families of one parameter formulas. Here, we shall consider integration procedures with respect to D.E.'s of the second order.

2. Using Taylor's Theorem, (1.8) can be expanded as

$$\alpha_0 y_n + \alpha_1 \left[\Sigma \frac{h^j y_n^{(j)}}{j!}\right] + \alpha_2 \left[\Sigma \frac{(2h)^j}{j!} y_n^{(j)}\right]$$

$$+ \cdots\cdots + \alpha_k \left[\Sigma \frac{(kh)^j}{j!} y_n^{(j)}\right]$$

$$= h^2 \{\beta_0 y_n'' + \beta_1 \left[\Sigma \frac{h^j}{j!} y_n^{(j+2)}\right] + \beta_2 \left[\Sigma \frac{(2h)^j}{j!} y_n^{(j+2)}\right]$$

$$+ \cdots\cdots + \beta_k \left[\Sigma \frac{(kh)^j}{j!} y_n^{(j+2)}\right]\}$$

All the summations are from 0 to ∞. (2.1)

Equating the coefficients of the powers of $h^2, h^3, \ldots, h^{k+2}$ we obtain

$$\begin{aligned}
&\frac{1}{2!} \Sigma i^2 \alpha_i = \Sigma \beta_i \\
&\frac{1}{3!} \Sigma i^3 \alpha_i = \Sigma i\beta_i \\
&\frac{1}{4!} \Sigma i^4 \alpha_i = \frac{1}{2!} \Sigma i^2 \beta_i \\
&\vdots \qquad\qquad \vdots \\
&\frac{1}{(k+2)!} \quad i^{k+2} \alpha_i = \frac{1}{k!} \Sigma i^k \beta_i
\end{aligned} \tag{2.2}$$

This system of k+1 equations can be written in matrix form

as follows:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & (k-1) & k \\ 0 & 1 & 2^2 & \cdots & (k-1)^2 & k^2 \\ - & - & - & \cdots & - & - \\ - & - & - & \cdots & - & - \\ 0 & 1 & 2^k & \cdots & (k-1)^k & k^k \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ - \\ - \\ \beta_k \end{bmatrix} = \begin{bmatrix} \frac{1}{1.2} \Sigma\, i^2 \alpha_i \\ \frac{1}{2.3} \Sigma\, i^3 \alpha_i \\ \frac{1}{3.4} \Sigma\, i^4 \alpha_i \\ - \\ - \\ \frac{1}{(k+1)(k+2)} \Sigma\, i^{k+2} \alpha_i \end{bmatrix} \tag{2.3}$$

All the summations are from $i=0$ to $i=k$.

If we denote the square matrix by M, the vector of β's by B and the vector on the right hand side of (2.3) by A, we can write

$$MB = A$$

The above matrix is a Vandermonde matrix and hence it is non-singular and the coefficients β's are uniquely determined. The β's can be obtained either by using the inverse of the matrix (Parker, [9]) or by using Gaussian elimination.

3. The Truncation Error

If we denote the truncation error by T_{n+k}, we get an expression

$$T_{n+k} = \frac{R\, h^{k+3}}{(k+1)!} y^{(k+3)}(x_n) + O(h^{k+4}) \tag{2.4}$$

where R is given by the equation

$$\frac{1}{(k+1)(k+2)} \sum_{i=0}^{k} i^{k+3} \alpha_i = \sum_{i=0}^{k} i^{k+1} \beta_i + R \tag{2.5}$$

which is obtained by equating the coefficients of h^{k+3} in the expansion of (2.1).

The last k equations of (2.2) and the equation (2.5) form a system

of k+1 equations with $\beta_1, \beta_2, \ldots, \beta_k$, R as the unknowns and these can be written in a matrix form

$$\begin{bmatrix} 0 & k & (k-1) & \ldots & 2 & 1 \\ 0 & k^2 & (k-1)^2 & \ldots & 2^2 & 1 \\ 0 & k^3 & (k-1)^3 & \ldots & 2^3 & 1 \\ - & - & - & \ldots & - & - \\ - & - & - & \ldots & - & - \\ 0 & k^k & (k-1)^k & \ldots & 2^k & 1 \\ 1 & k^{k+1} & (k-1)^{k+1} & \ldots & 2^{k+1} & 1 \end{bmatrix} \cdot \begin{bmatrix} R \\ \beta_k \\ \beta_{k-1} \\ - \\ - \\ \beta_2 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2.3} \Sigma\, i^3 \alpha_i \\ \frac{1}{3.4} \Sigma\, i^4 \alpha_i \\ \frac{1}{4.5} \Sigma\, i^5 \alpha_i \\ - \\ - \\ \frac{1}{(k+1(k+1)} \Sigma\, i^{k+2} \alpha_i \\ \frac{1}{(k+2)(k+3)} \Sigma\, i^{k+3} \alpha_i \end{bmatrix} \tag{2.6}$$

the summations again being from i=0 to i=k.

In solving these equations, it is convenient to introduce the following polynomial in x:

$$D(x) = \begin{vmatrix} x & k & (k-1) & \ldots & 1 \\ x^2 & k^2 & (k-1)^2 & \ldots & 1 \\ - & - & - & \ldots & - \\ - & - & - & \ldots & - \\ x^{k+1} & k^{k+1} & (k-1)^{k+1} & \ldots & 1 \end{vmatrix} \tag{2.7}$$

Using Cramer's Rule an expression for R can be obtained which is the quotient of two determinants.

The denominator is

$$\frac{1}{(k+1)!} \frac{d^{k+1}}{dx^{k+1}} D(x)$$

while the numerator is the polynomial D(x) with the power x^j is replaced by $\frac{1}{(j+1)(j+2)} \Sigma\, i^{j+2} \alpha_i$ for j=1, 2,..., k+1.

After cancelling common factors from the numerator and denominator, we have

$$R = x(x-1)(x-2)\ldots(x-k) = \pi(x)$$

$$= x^{k+1} + b_k x^k + \ldots + b_1 x, \text{ where the power } x^j \text{ is replaced by } \frac{1}{(j+1)(j+2)} \Sigma\, i^{j+2}\alpha_i,$$

$$= \frac{1}{(k+2)(k+3)} \Sigma\, i^{k+3}\alpha_i + \frac{b_k}{(k+1)(k+2)} \Sigma\, i^{k+2}\alpha_i + \ldots + \frac{b_1}{2.3} \Sigma\, i^3\alpha_i$$

$$= \Sigma\, \alpha_i \int_0^i \int_0^t \pi(x)\, dx\, dt$$

$$= \Sigma\, (\alpha_i + \alpha_{i+1} + \ldots + \alpha_k) \int_{i-1}^{i} \int_0^t \pi(x)\, dx\, dt \tag{2.8}$$

$$= \sum_{i=1}^{k} \bar{\alpha}_{i-1} L_i$$

where $L_i = \int_{i-1}^{i} \int_0^t \pi(x)\, dx\, dt$

and $\bar{\alpha}_i = \sum_{i=1+1}^{k} \alpha_j$.

4. A Difference Scheme of Order k=4

If we consider a difference equation of order k=4, from (1.19) we have

$$\alpha(x) = (x-1)^2\, \gamma(x) \tag{2.9}$$

where $\gamma(x)$ is a monic polynomial of degree 2.

In order to study the behaviour of the propagated error, let us consider the initial value problem

$$y'' = \lambda y,\ y(a) = \eta,\ y'(a) = \eta_1$$

If we use a corrector formula of type (1.4), we get an error

equation

$$\Sigma\ \alpha_i\ e_{n+i} = h^2\ \lambda\ \Sigma\ \beta_i\ e_{n+i} + T_{n+k}$$

where e_i is as given in (1.10).

i.e., $\Sigma(\alpha_i - h^2\lambda\beta_i)\ e_{n+i} = T,$ (2.10)

assuming T_{n+k} is a constant equal to T.

The general solution of (2.10) is given by the relation

$$e_n = A_1 s_1^{\ n} + A_2 s_2^{\ n} + A_3 s_3^{\ n} + A_4 s_4^{\ n} - \frac{T}{h^2\lambda\Sigma\beta_i}$$

where s_i (i=1,2,3,4) are roots of the equation

$$\Sigma\ (\alpha_i - h^2\lambda\beta_i)\ s^i = 0.$$

Now $s_1 \simeq 1+\sqrt{\lambda}\ h \simeq e^{\sqrt{\lambda}\ h}$

and $s_2 \simeq 1-\sqrt{\lambda}\ h \simeq e^{-\sqrt{\lambda}\ h}$

The other roots s_3 and s_4 are extraneous and have been introduced because a second order D.E. is replaced by a fourth order ∇.E.

For stable methods, we need not know s_3 and s_4 so long as they have magnitudes less than unity.

Hence $s_i^{\ n} \to o$ for i=3,4.

Therefore, $e_n \simeq A_1 s_1^{\ n} + A_2 s_2^{\ n} - \dfrac{T}{h^2\lambda\Sigma\beta_i}$

$$e_0 = A_1 + A_2 - \frac{T}{h^2\lambda\Sigma\beta_i}$$

and $e_1 = A_1 s_1 + A_2 s_2 - \dfrac{T}{h^2\lambda\Sigma\beta_i}$

If we put $e_0 + \dfrac{T}{h^2\lambda\Sigma\beta_i} = E_0$

and $$e_1 + \frac{T}{h^2\lambda\Sigma\beta_i} = E_1,$$

we have

$$A_1 + A_2 = E_0$$

$$A_1 s_1 + A_2 s_2 = E_1$$

Solving for A_1 and A_2, we have

$$A_1 = \frac{E_1 - E_0 s_2}{s_1 - s_2}$$

and $$A_2 = \frac{E_0 s_1 - E_1}{s_1 - s_2}$$

Hence $$e_n = \frac{(E_1 - E_0 s_2)}{s_1 - s_2} s_1^n + \frac{(E_0 s_1 - E_1)}{s_1 - s_2} s_2^n - \frac{T}{h^2\lambda\Sigma\beta_i}$$

Now $$e_0 = 0 \quad , \quad e_1 \simeq 0$$

$$\frac{E_1 - E_0 s_2}{s_1 - s_2} \simeq \frac{T}{2h^2\lambda\Sigma\beta_i}$$

$$\frac{E_0 s_1 - E_1}{s_1 - s_2} \simeq \frac{T}{2h^2\lambda\Sigma\beta_i}$$

Hence $$e_n \simeq \frac{T}{h^2\lambda\Sigma\beta_i} \left(\frac{s_1^n + s_2^n}{2} - 1 \right)$$

$$\simeq \frac{T}{h^2\lambda\Sigma\beta_i} \left[\frac{e^{\sqrt{\lambda}(x_n - x_o)} + e^{-\sqrt{\lambda}(x_n - x_0)}}{2} - 1 \right]$$

since $s_1^n \simeq e^{\sqrt{\lambda} nh} = e^{\sqrt{\lambda}\,(x_n - x_0)}$ and

$s_2^n \simeq e^{-\sqrt{\lambda} nh} = e^{-\sqrt{\lambda}\,(x_n - x_0)}$

$$\simeq \frac{T}{h^2\lambda\Sigma\beta_i} \{ \cosh (\sqrt{\lambda}\,(x_n - x_0)) - 1 \}$$

$$\simeq \frac{2T}{h^2 \lambda \Sigma \beta_i} \quad \sinh^2 \frac{\sqrt{\lambda}\,(x_n - x_0)}{2}$$

This expression suggests that e_n will be small if (a) T is small (b) $\Sigma\ \beta_i$ is large. For T to be small, we need $T=O(h^p)$, p the maximum possible. From (1.13) we have the expressions

$$\Sigma\ i\alpha_i = 0$$

and

$$\Sigma\ \beta_i = \frac{1}{2}\ \Sigma\ i^2\ \alpha_i$$

If r_3 and r_4 are the extraneous roots, we have

$$\alpha(x) = \Sigma\alpha_i\ x^i = (x-1)^2\ (x-r_3)\ (x-r_4)$$

$$\alpha''(1) = \Sigma\ i^2\alpha_i = 2 \prod_{i=3}^{4} (1-r_i)$$

Hence, $$\Sigma\beta_i = \prod_{i=3}^{4} (1-r_i)$$

= (product of the distances of the extraneous roots from unity)

To minimize the estimated error we want $\Sigma\beta_i$ to be as large as possible. But for stability r's must be in or near the unit circle.

Analogous to the three families considered by Hull and Newberry we shall also consider here three different one parameter family of formulas.

For the first family, we will take $r_2 = r_3 = -c$. Equation (2.9) now becomes

$$\alpha(x) = (x-1)^2(x+c)^2 \tag{2.11}$$

5. <u>'Westward' Family</u>

The family of one parameter formulas we will derive based on equation (2.11), will be called 'Westward', analogous to the naming

convention used by Hull and Newberry[6].

The α's are given by the vector (2.12)

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$
$$= (c^2, 2c-2c^2, c^2-4c+1, 2c-2, 1)$$

For k=4, using the inverse of the resulting Vandermonde matrix, we have

$$\begin{bmatrix} 1 & -50/24 & 35/24 & -10/24 & 1/24 \\ 0 & 96/24 & -104/24 & 36/24 & -4/24 \\ 0 & -72/24 & 114/24 & -48/24 & 6/24 \\ 0 & 32/24 & -56/24 & 28/24 & -4/24 \\ 0 & -6/24 & 11/24 & -6/24 & -1/24 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{1.2} \Sigma i^2 \alpha_i \\ \frac{1}{2.3} \Sigma i^3 \alpha_i \\ \frac{1}{3.4} \Sigma i^4 \alpha_i \\ \frac{1}{4.5} \Sigma i^5 \alpha_i \\ \frac{1}{5.6} \Sigma i^6 \alpha_i \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} \quad (2.13)$$

On solving, β's are given by the vector

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \quad (2.14)$$
$$= [Q(19c^2-2c-1), Q(204c^2+48c+4), Q(14c^2+388c+14),$$
$$Q(4c^2+48c+204), Q(19-2c-c^2)]$$

where Q=1/240.

The expression for R is given by

$$R = \sum_{i=0}^{4} \bar{\alpha}_{i-1} \int_{i-1}^{i} \int_{0}^{t} \{x(x-1)(x-2)(x-3)(x-4)\}\, dx\, dt$$

$$= (-c^2)\ \frac{107}{84} + (c^2-2c)\ \frac{149}{84} + (2c-1)\ \frac{149}{84} + 1 \cdot \frac{107}{84}$$

$$= (c^2-1)\ /2$$

Using (2.4), the truncation error now becomes

$$T.E. = \begin{cases} \frac{1}{240}(c^2-1)\, h^7\, y^{(7)}, & c \neq \pm 1 \\ \frac{-1}{240}\, h^8\, y^{(8)}, & c=-1 \\ \frac{-2}{945}\, h^8\, y^{(8)}, & c=+1 \end{cases} \tag{2.15}$$

The difference equation now is

$$c^2 y_n + (2c-2c^2)y_{n+1} + (c^2-4c+1)y_{n+2} + (2c-2)y_{n+3} + y_{n+4}$$

$$- h^2 \sum_{i=0}^{4} \beta_i\, y''_{n+i} = 0 \tag{2.16}$$

where c is an arbitrary real constant and the β's are given by (2.14).

6. <u>'East-West' Family</u>

In this family, we take the two extraneous roots at $-c$ and $+c$ respectively, so that (2.9) now becomes

$$\alpha\ (x) = (x-1)^2 (x+c)(x-c)$$

The α's are given by the vector

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$= (-c^2, 2c^2, 1-c^2, -2, 1) \tag{2.17}$$

From the system of equations (2.13), β's are obtained as

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \tag{2.18}$$

$$= [Q(-1 -19\ c^2),\ Q(4- 204\ \ c^2),\ Q(14 - 14c^2),$$
$$Q(204 - 4c^2),\ Q(c^2 + 19)]$$

where $Q = 1/240$.

The ∇.E., therefore, is

$$-c^2 y_n + 2c^2 y_{n+1} + (1-c^2)\ y_{n+2} - 2y_{n+3} + y_{n+4} \tag{2.19}$$

$$- h^2 \sum_{i=0}^{4} \beta_i\, y''_{n+i} = 0$$

where the β's are given by (2.18).

The truncation error associated with this ∇.E. is

$$T.E. = -\frac{1}{240}(c^2+1)\, h^7\, y^{(7)}$$

7. <u>'Radial' Family</u>

If k is the order of the ∇.E; for this family, we have the relation

$$\alpha(x) = (x-1)^2 \prod (x-c\, e^{(2\pi i\, m)/(k-1)}), \quad m=1,2,\ldots,k-2$$

For k=4, this then becomes

$$\alpha(x) = (x-1)^2(x^2+cx+c^2)$$

From this equation we have the α's given by

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \tag{2.20}$$

$$= (c^2, c-2c^2, c^2-2c+1, c-2, 1)$$

The β's are obtained as before, as

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \tag{2.21}$$

$$= [Q(19c^2-c-1),\ 4Q(51c^2+6c+1),\ 2Q(7c^2+97c+7),\ 4Q(c^2+6c+51),\ Q(19-c-c^2)]$$

where $Q = 1/240$.

The ∇.E. then becomes

$$c^2 y_n + (c-2c^2)y_{n+1} + (c^2-2c+1)y_{n+2} + (c-2)y_{n+3} + y_{n+4} - h^2 \sum_{i=0}^{4} \beta_i\, y''_{n+i} = 0 \tag{2.22}$$

where β's are given by (2.21).

The truncation error associated with this ∇.E. is given by

$$T.E. = \begin{cases} \frac{1}{240}(c^2-1)\, h^7\, y^{(7)}, & c \neq \pm 1 \\ -\frac{221}{60480}\, h^8\, y^{(8)}, & c = -1 \\ -\frac{158}{60480}\, h^8\, y^{(8)}, & c = +1 \end{cases} \tag{2.23}$$

8. Another Family

Another one parameter family of formulas of order k=4 developed by Usmani [12], has α's and β's given by

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \tag{2.24}$$

$$= (1, c-2, 2-2c, c-2, 1)$$

and

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \tag{2.25}$$

$$= [Q(18-c), 8Q(3c+26), 2Q(97c+14), 8Q(3c+26), Q(18-c)]$$

where $Q = 1/240$.

Here, the ∇.E. is

$$y_n + (c-2)y_{n+1} + (2-2c)y_{n+2} + (c-2)y_{n+3} + y_{n+4} \tag{2.26}$$

$$- h^2 \sum_{i=0}^{4} \beta_i \, y''_{n+i} = 0$$

where β's are given by (2.25) and has a truncation error

$$\text{T.E.} = \begin{cases} \dfrac{31c-190}{60,480} \; h^8 y^{(8)}, & c \neq \dfrac{190}{31} \\[2ex] \dfrac{-79}{585,900} \; h^{10} \, y^{(10)}, & c = \dfrac{190}{31} \end{cases} \tag{2.27}$$

with $2 < c \leq 14/3$.

CHAPTER III

INITIAL VALUE PROBLEMS

1. In this chapter we shall consider the problem of obtaining approximate solution of the initial value problem (1.3). We use multi-step methods of the predictor-corrector type. For corrector, we use the ∇.E. given by (1.4). When $\alpha_k \neq 0$ and $|\alpha_0|+|\beta_0| > o$, the order of k of the ∇.E. is uniquely determined. For predictor, we use (1.4) with $\beta_k = 0$.

2. Existence and Uniqueness of Solution

Let f(x,y) satisfy the conditions stated earlier. If the ∇.E. (1.4) has a unique solution $\{y_n\}$ where $(x_n \varepsilon [a,b])$ for arbitrarily chosen initial values $y_0, y_1, \cdots, y_k$, then we have to show that the relation (1.4) considered as an equation for y_{n+k} has a unique solution for arbitrary values of $y_n, y_{n+1}, \cdots, y_{n+k-1}$. This is the case if $\beta_k=0$, because the relation (1.4) represents y_{n+k} explicitly as a function of $y_n, y_{n+1}, \cdots, y_{n+k-1}$. Thus, in the case of the predictor, we have a ∇.E. of this type. If, on the other hand y_{n+k} occurs [as an argument in $f_{n+k} = f(x_{n+k}, y_{n+k})$] also on the right hand side, (1.4) represents an equation for y_{n+k}, which conceivably might have several solutions or no solution at all. If a unique solution exists, we have to find it.

If we write (1.4) in the form

$$y = F(y) \tag{3.1}$$

where $y = y_{n+k}$,

$$F(y) = h^2 \frac{\beta_k}{\alpha_k} y''(x_{n+k}, y)+C \tag{3.2}$$

where $C = \frac{1}{\alpha_k} \{ h^2[\beta_{k-1} y''_{n+k-1} + \ldots + \beta_0 y''_n] - \alpha_{k-1} y_{n+k-1} - \ldots - \alpha_0 y_n \}$

An iterative procedure takes the form

$$y^{(\nu+1)} = F(y^{(\nu)}), \quad \nu = 0, 1, 2, \ldots \tag{3.3}$$

where $y^{(0)}$ is a suitable first approximation. The following theorem enables us to prove (for sufficiently small values of h) not only the convergence of the sequence $\{y^{(\nu)}\}$ to a solution of (1.4), but also the uniqueness of the solution.

Theorem 3.1

Let the function F(y) be defined for $-\infty < y < \infty$ and let there exist a constant K such that $0 \leq K < 1$ and $|F(y^*) - F(y)| \leq K|y^* - y|$ (3.4)

for arbitrary values of y^* and y. Then the following statements hold:

(i) Equation (3.1) has a unique solution y.

(ii) For arbitrary $y^{(0)}$ the sequence defined by (3.3) converges to y.

(iii) For $\nu = 1, 2, \ldots$ there hold the estimates

$$|y - y^{(\nu)}| \leq \frac{K}{1-K} |y^{(\nu)} - y^{(\nu-1)}| \leq \frac{K^\nu}{1-K} |y^{(1)} - y^{(0)}| \tag{3.5}$$

If F(y) is defined by (3.2) and if f(x,y) satisfies a Lipschitz condition with respect to y with Lipschitz constant L, then condition (3.4) is satisfied with

$$K = \left| \frac{h^2 \beta_k}{\alpha_k} \right| . L \tag{3.6}$$

and this is less than 1 for all sufficiently small values of h.

Proof

If y and y^* are two solutions of (3.1), then

$$y = F(y), \quad y^* = F(y^*)$$

Subtracting the first relation from the second and using (3.4), we have

$$|y^*-y| \leq K|y^*-y|$$

which by virtue of $|K| < 1$ is impossible unless $y^*=y$.

Thus, (3.1) can have at most one solution. In order to prove the existence of a solution, subtract from (3.3) the relation $y^{(\nu)} = F(y^{(\nu-1)})$. Then we have

$$|y^{(\nu+1)} - y^{(\nu)}| = |F(y^{(\nu)}) - F(y^{(\nu-1)}|$$

Using (3.4), there follows

$$|y^{(\nu+1)} - y^{(\nu)}| \leq K|y^{(\nu)} - y^{(\nu-1)}|$$

Using this estimate repeatedly, there exist the relations

$$|y^{(\nu+\mu)} - y^{(\nu+\mu-1)}| \leq K^{\mu}|y^{(\nu)} - y^{(\nu-1)}|$$

and $$|y^{(\nu+1)} - y^{(\nu)}| \leq K^{\nu}|y^{(1)} - y^{(o)}|, \quad (\nu,\mu=1,2,\ldots)$$

As a consequence, for any positive integer μ

$$\begin{aligned} |y^{(\nu+\mu)}-y^{(\nu)}| &\leq |y^{(\nu+\mu)}-y^{(\nu+\mu-1)}|+\ldots+|y^{(\nu+1)}-y^{(\nu)}| \\ &\leq (K^{\mu}+K^{\mu-1}+\ldots+K)|y^{(\nu)}-y^{(\nu-1)}| \\ &\leq \frac{K^{\mu}}{1-K}|y^{(1)}-y^{(0)}| \end{aligned} \tag{3.7}$$

Given any $\varepsilon>0$, there exists an integer ν_0 such that

$$\frac{K^{\nu}}{1-K}|y^{(1)}-y^{(0)}| < \varepsilon$$

for all $\nu>\nu_0$. The sequence $\{y^{(\nu)}\}$ is thus shown to satisfy the Cauchy criterion for convergence and thus has a finite limit y. Letting $\nu\to\infty$ in (3.3) and using the fact that F(y) is continuous by virtue of its satisfying a Lipschitz condition, we get

$$y = \lim_{\nu\to\infty} y^{(\nu)} = \lim_{\nu\to\infty} F(y^{(\nu)}) = F(\lim_{\nu\to\infty} y^{(\nu)}) = F(y)$$

The limit y is thus recognised as being a solution, and in view

of the uniqueness already established, the only solution of (3.1). The estimate (3.5) follows by letting $\mu\to\infty$ in (3.7) while keeping ν fixed.

3. Convergence of a ∇.E. of Order k

Although (1.4) represents an implicit equation for y_{n+k} if $\beta_k \neq o$, it follows by theorem 3.1 that if the function $f(x,y)$ satisfies a Lipschitz condition with Lipschitz constant L,(1.4) has a unique solution y_{n+k} for all values of h satisfying

$$h < \left| \frac{\alpha_k}{\beta_k L} \right|^{1/2} \tag{3.8}$$

For all values of h satisfying (3.8), the values $y_m (m=k,k+1,\ldots)$ may thus be regarded as uniquely determined functions of the starting values $y_0, y_1, \ldots, y_{k-1}$, which in turn are functions of h:

$$y_\mu = \eta_\mu(h), \quad \mu = 0,1,\ldots,k-1.$$

We expect that the values y_n thus generated tend to the value of the exact solution at the point x as $h\to 0$ provided that the starting values are properly chosen.

Definition of Convergence

The linear multi-step defined by (1.4) is called convergent if the following statement is true for all functions $f(x,y)$ satisfying the conditions given earlier and all constants η and η_1:

If $y(x)$ denote the solution of the initial value problem (1.3), then

$$\lim_{\substack{h\to 0 \\ n\to\infty}} y_n = y(x_n) \tag{3.9}$$

such that $nh \to (x_n - a)$ and holds for all $x_n \in [a,b]$ and for all sequences $\{y_n\}$ defined by (1.4) with starting values $y_\mu = \eta_\mu(h)$ satisfying the two

conditions

$$\lim_{h\to o} \eta_\mu(h) = \eta, \quad \mu=o,1,\ldots,k-1 \tag{3.10}$$

and

$$\lim_{h\to o} \frac{\eta_\mu(h)-\eta_0(h)}{\mu\, h} = \eta_1, \quad \mu=1,2,\ldots,k-1 . \tag{3.11}$$

The sufficient, but not necessary condition, in order that (3.10) and (3.11) are satisfied, is that the starting values are exact, i.e., $y_\mu=y(x_\mu)$.

Convergence: Condition of Stability

Let polynomials (1.17) are associated with the ∇.E. (1.4).

Theorem 3.2

A necessary condition for the convergence of the linear multi-step method defined by (1.4) is that the modulus of no root of the polynomial $\alpha(x)$ exceed 1 and that the multiplicity of the roots of modulus 1 be at most 2.

For a proof of the above theorem, see Henrici ([4], p. 301).

The condition thus imposed on the roots of $\alpha(x)$ is the condition of stability and this condition guarantees that small initial disturbances are not unduly amplified.

Convergence: Condition of Consistency

The condition of consistency ensures that the ∇.E. is locally a good approximation to the D.E.

Theorem 3.3

The degree of a convergent linear multi-step method is at least 1.

For proof of this theorem, see Henrici ([4], p. 301-303).

Now we shall discuss some stable and consistent formulas (both explicit and implicit) for k=2, 3 and 4. The explicit ∇.E. will be

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k-1} \beta_i \tilde{y}''_{n+i} \qquad (n=0,1,2,\ldots) \tag{3.12}$$

4. <u>∇.E. of Order k=2</u>

The explicit ∇.E. of order k=2 is of the form

$$y_n - 2y_{n+1} + y_{n+2} = h^2 y''_{n+1} \tag{3.13}$$

This formula has a truncation error given by

$$\text{T.E.} = \frac{1}{12} h^4 y^{(4)}$$

The implicit ∇.E. of order k=2 turns out to be

$$y_n - 2y_{n+1} + y_{n+2} = \frac{h^2}{12} (y''_n + 10\, y''_{n+1} + y''_{n+2}) \tag{3.14}$$

which has a truncation error

$$\text{T.E.} = -\frac{1}{240} h^6 y^{(6)}$$

Equation (3.13) can be used as a predictor and (3.14) as a corrector.

5. <u>∇.E. of Order k=3</u>

When k=3, a one parameter family of ∇.E.'s can be obtained involving a real arbitrary constant c.

Consider an equation of the form

$$\alpha(x) = (x-1)^2 (x+c)$$

The α's are then given by the vector

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$$
$$= (c, 1-2c, c-2, 1)$$

Using a system of equations similar to (2.13) when k=3, the β's are obtained as

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$$

$$= [c/12,\ (10\,c+1)/12,\ (c+10)/12,\ 1/12]$$

The $\tilde{\beta}$'s are given by the vector

$$\tilde{\beta} = (\tilde{\beta}_0,\ \tilde{\beta}_1,\ \tilde{\beta}_2)$$

$$= [(c+1)/12,\ (10c-2)/12,\ (c+13)/12]$$

The explicit ∇.E. of order k=3 is then written as

$$c\,y_n + (1-2c)y_{n+1} + (c-2)y_{n+2} + y_{n+3}$$

$$- \frac{h^2}{12}\,[\,(c+1)y''_n + (10c-2)y''_{n+1} + (c+13)y''_{n+2}\,] = 0 \qquad (3.15)$$

The truncation error associated with this ∇.E. is

$$\text{T.E.} = \frac{1}{12}\,h^5\,y^{(5)}$$

The implicit ∇.E. of order k=3 is given by

$$c\,y_n + (1-2c)y_{n+1} + (c-2)y_{n+2} + y_{n+3}$$

$$- \frac{h^2}{12}\,[\,c\,y''_n + (10c+1)y''_{n+1} + (c+10)y''_{n+2} + y''_{n+3}\,] = 0 \qquad (3.16)$$

which has a truncation error

$$\text{T.E.} = \begin{cases} -\dfrac{(c+1)}{240}\,h^6\,y^{(6)}, & c \neq -1 \\[2ex] -\dfrac{1}{240}\,h^7\,y^{(7)}, & c = -1 \end{cases}$$

6. <u>∇.E.'s of order k=4</u>

(i) First we shall consider the 'Westward' family. For the explicit ∇.E., the α's are given by (2.12) and the $\tilde{\beta}$'s are given by the vector

$$\tilde{\beta} = (\tilde{\beta}_0,\ \tilde{\beta}_1,\ \tilde{\beta}_2,\ \tilde{\beta}_3) \qquad (3.17)$$

$$= [\ Q(c^2-1),\ Q(10c^2+2c+4),\ Q(c^2+20c-5),\ Q(2c-14)\ ]$$

where Q=1/12.

The ∇.E. for the predictor can then be written as

$$c^2 y_n + (2c-2c^2)y_{n+1} + (c^2-4c+1)y_{n+2} + (2c-2)y_{n+3} + y_{n+4} \qquad (3.18)$$

$$- h^2 \sum_{i=0}^{3} \tilde{\beta}_i \, y''_{n+i} = 0$$

where $\tilde{\beta}$'s are given by (3.17) and this equation has a truncation error associated with it given by

$$\text{T.E.} = \frac{1}{240} h^6 \, (19-2c-c^2)y^{(6)}$$

The implicit ∇.E. for the corrector will be the equation (2.16).

(ii) For the 'East-West' family, the α's are given by (2.17) and the $\tilde{\beta}$'s are given by

$$\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) \qquad (3.19)$$

$$= [\, Q(-1-c^2),\ Q(4-10c^2),\ Q(-5-c^2),\ 14\,Q \,]$$

where $Q = 1/12$.

The explicit ∇.E. then becomes

$$-c^2 y_n + 2c^2 y_{n+1} + (1-c^2)y_{n+2} - 2y_{n+3} + y_{n+4} \qquad (3.20)$$

$$- h^2 \sum_{i=0}^{3} \tilde{\beta}_i \, y''_{n+i} = 0$$

where $\tilde{\beta}$'s are given by (3.19) and the truncation error associated with this equation is

$$\text{T.E.} = \frac{1}{240} (c^2+19) \, h^6 \, y^{(6)}$$

For the corrector, the ∇.E. will be (2.19).

(iii) For the third family, i.e., 'Radial', the α's are given by (2.20) and the $\tilde{\beta}$ vector is

$$\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) \qquad (3.21)$$

$$= [\, Q(c^2-1),\ Q(10c^2+c+4),\ Q(c^2+10c-5),\ Q(c+14) \,]$$

where $Q=1/12$.

The ∇.E. for the predictor is

$$c^2 y_n + (c-2c^2)y_{n+1} + (c^2-2c+1)y_{n+2} + (c-2)y_{n+3} + y_{n+4} \tag{3.22}$$

$$- h^2 \sum_{i=0}^{3} \tilde{\beta}_i \, y''_{n+i} = 0$$

with $\tilde{\beta}$'s given by (3.21) and this equation has a truncation error

$$\text{T.E.} = \frac{1}{240} (19-c-c^2) \, h^{(6)} \, y^{(6)}$$

The implicit ∇.E. is given by (2.22).

(iv) For the fourth family we outlined in chapter 2, the α's are given by (2.24) and the $\tilde{\beta}$'s by the vector

$$\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) \tag{3.23}$$

$$= [\ 0, \frac{1}{12}(c+14), \frac{1}{12}(10c-4), \frac{1}{12}(c+14)\ ]$$

The explicit ∇.E. then is given by

$$y_n + (c+2)y_{n+1} + (2-2c)y_{n+2} + (c-2)y_{n+3} + y_{n+4} \tag{3.24}$$

$$- h^2 \sum_{i=0}^{3} \tilde{\beta}_i \, y''_{n+i} = 0$$

with $\tilde{\beta}$'s given by (3.23). The truncation error associated with this ∇.E. is

$$\text{T.E.} = \frac{1}{240} (18-c) \, h^{(6)} \, y^{(6)}.$$

For the corrector we have the ∇.E. (2.26).

7. Error Bound for the 'Westward' family of Order k=4

The error equation associated with ∇.E. (2.16) is

$$\sum_{i=0}^{4} \alpha_i \, e_{n+i} = h^2 \sum_{i=0}^{4} \beta_i \, g_{n+i} \, e_{n+i} + T_{n+4} \tag{3.25}$$

where $\quad g_i e_i = f\ (x_i, y(x_i)\ ) - f\ (x_i, y_i)$

so that g_i is $\frac{\partial f}{\partial y}$ evaluated at a point between $(x_i, y(x_i))$ and (x_i, y_i).

For the derivation of a bound on $|e_n|$ we will need the following lemmas:

Let $\alpha(\zeta)$ and $\beta(\zeta)$ be the polynomials given by (1.17). Let $1, 1, \zeta_1, \zeta_2$ be the roots of $\alpha(\zeta)$. Then $|\zeta_i| \leq 1$ for $i \geq 3$. Also, $\alpha(1) = \alpha'(1) = 0$, $\alpha''(1) = 2\,\beta(1)$.

Lemma 3.1

$$\text{If } \frac{1}{\alpha_4 + \alpha_3\zeta + \alpha_2\zeta^2 + \alpha_1\zeta^3 + \alpha_0\zeta^4} = \gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \ldots, \tag{3.26}$$

then there exist two constants Γ and γ (non-negative) such that

$$|\gamma_\ell| \leq \ell\Gamma + \gamma, \quad \ell = 0,1,2,\ldots \tag{3.27}$$

For a proof of this lemma, see Henrici [4, p. 312].

In the case of the 'Westward' family, (3.27) is true when $\Gamma=1$ and $\gamma=1$ for $c=1$.

Lemma 3.2 (i)

If $\gamma_\ell (\ell=0,1,2,\ldots)$ are given by (3.27), then

$$\alpha_4\gamma_e + \alpha_3\gamma_{\ell-1} + \alpha_2\gamma_{\ell-2} + \alpha_1\gamma_{\ell-3} + \alpha_0\gamma_{\ell-4} = \begin{cases} 1, & \ell=0 \\ 0, & \ell>0 \end{cases}$$

Proof

The proof follows from the relation

$$1 = (\alpha_4 + \alpha_3\zeta + \alpha_2\zeta^2 + \alpha_1\zeta^3 + \alpha_0\zeta^4)(\gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \ldots).$$

We assume $\gamma_\ell = 0$ for $\ell < 0$.

Lemma 3.2(ii)

If $A = |\alpha_0| + |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|$ with $\alpha_4 = 1$, then $\gamma A \geq 1$.

Proof

From lemma (3.1), $\alpha_4\,\gamma_0 = 1$ or $\gamma_0 = 1$ since $\alpha_4 = 1$.

Also $A \geq 1$. Using relation (3.27) for $\ell=0$, we have

$$\gamma_0 = 1 \leq 0.\Gamma + \gamma$$

which implies that

$$\gamma \geq 1. \quad \text{Therefore, } A\,\gamma \geq 1.$$

Lemma 3.3

Let $\{x_n\}$ be a sequence of numbers $(n = 0, 1, \ldots, N)$ that satisfy the following inequality

$$|x_n| \leq S_1 \sum_{m=0}^{n-1} |x_m| + S_2$$

where S_1 and S_2 are certain non-negative constants independent of n and

$$|x_n| \leq S_2 \quad (n=0,1,\ldots,M), \; M<N$$

Then $$|x_n| \leq S_2\,(1+S_1)^n, \quad (n=0,1,\ldots,N)$$

For a proof of this lemma, see Usmani [15].

Theorem 3.4

Let $\{e_n\}$ satisfy the error equation (3.25) and let

$$E = \max_{t=0,1,2,3} (|e_t|)$$

Then $$|e_n| \leq K^* \exp\{\, n\, h^2 L^* \}\,, \tag{3.28}$$

where

$$K^* = [4AE(n\Gamma+\gamma) + h^2 C_{p+2}\, M_{p+2}\, (\frac{n^2}{2}\Gamma + n\gamma)]/(1-h^2\beta_4 L), \tag{3.29}$$

$$L^* = [BL(N\Gamma+\gamma)]/(1-h^2\beta_4 L)$$

$$A = |\alpha_0|+|\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_4|, \quad B=|\beta_0|+|\beta_1|+|\beta_2|+|\beta_3|+|\beta_4|,$$

$$M_i = \max \left|\frac{d^i y}{dx^i}\right| \text{ for } x \in [x_0, x_n], \text{ the range of integration}$$

and C_i are given by (1.13).

Proof

If n is replaced by $n-\ell-4$ in the error equation (3.25), we have

$$\alpha_0 e_{n-\ell-4} + \alpha_1 e_{n-\ell-3} + \ldots + \alpha_4 e_{n-\ell}$$
$$= h^2 [\beta_0 g_{n-\ell-4} e_{n-\ell-4} + \beta_1 g_{n-\ell-3} e_{n-\ell-3} + \ldots + \beta_4 g_{n-\ell} e_{n-\ell} + T_{n-\ell} \quad (3.30)$$

Multiply (3.30) by γ_ℓ, $(\ell = 0,1,\ldots,n-4)$. On adding the terms on the left hand side, we have

$$\text{Sum on l.h.s.} = \gamma_0 \alpha_4 e_n + \sum_{t=0}^{3} \nabla_t e_t \quad (3.31)$$

where

$$\nabla_i = \sum_{j=0}^{i} \gamma_{n-4-j} \alpha_{i-j} \quad (3.32)$$

Other terms on the left hand side vanish in view of Lemma (2.2(i)).

$$\text{Sum on r.h.s.} = h^2 \gamma_0 \beta_4 g_n e_n + h^2 \sum_{t=0}^{n-1} \Delta_t g_t e_t + \sum_{\ell=0}^{n-4} \gamma_\ell T_{n-\ell}, \quad (3.33)$$

where $$\Delta_i = \sum_{j=0}^{i} \gamma_{n-4-j} \beta_{i-j} \quad (3.34)$$

From (3.31) and (3.33), we have

$$e_n = - \sum_{t=0}^{3} \nabla_t e_t + h^2 \beta_4 g_n e_n + h^2 \sum_{t=0}^{n-1} \Delta_t g_t e_t + \sum_{\ell=0}^{n-4} \gamma_\ell T_{n-\ell}$$

i.e., $$|e_n| = \sum_{t=0}^{3} |\nabla_t e_t| + h^2 \beta_4 L |e_n| + h^2 L \sum_{t=0}^{n-1} |\Delta_t e_t| + \sum_{\ell=0}^{n-4} |\gamma_\ell| \cdot |T_{n-\ell}|$$

where L is the Lipschitz constant.

$$(1-h^2\beta_4 L)|e_n| \leq \sum_{t=0}^{3} |\nabla_t e_t| + h^2 L \sum_{t=0}^{n-1} |\Delta_t e_t| + \sum_{\ell=0}^{n-4} |\gamma_\ell| \cdot |T_{n-\ell}| \quad (3.35)$$

Now, $$|T_i| \leq h^{p+2} C_{p+2} M_{p+2}$$

From (3.32), we have

$$|\nabla_i| \leq A.\ \text{max. of } |\gamma_\ell| \text{ used}$$

$$\leq A.\ |\gamma_{n-4}|$$

$$\leq A.\ [(n-4)\Gamma + \gamma], \text{ using (3.27)}$$

or $$|\nabla_i| < A.\ (n\Gamma + \gamma)$$

Similarly,

$$|\Delta_i| < B.\ (n\Gamma + \gamma)$$

also,

$$\sum_{i=0}^{n-4} \gamma_i < \frac{n^2}{2}\Gamma + n\gamma$$

From (3.33), it follows

$$(1-h^2\beta_4 L)\ |e_n| \leq 4A(n\Gamma+\gamma)E + h^2 LB(n\Gamma+\gamma) \sum_{\ell=0}^{n-1} |e_\ell| + h^{p+2} C_{p+2} M_{p+2} \sum_{\ell=0}^{n-4} |\gamma_\ell|$$

$$\leq h^2 BL(n\Gamma+\gamma)\sum_{\ell=0}^{n-1} |e_\ell| + 4AE(n\Gamma+\gamma) + h^{p+2} C_{p+2} M_{p+2} \left(\frac{n^2}{2}\Gamma + n\gamma\right) \tag{3.36}$$

or $$|e_n| \leq L^* h^2 \sum_{\ell=0}^{n-1} |e_\ell| + K^*$$

Since

$A\gamma \geq 1$, by using lemma 3.2(ii) $E < K^*$

Using lemma (3.3), from equation (3.36) we have

$$|e_n| \leq K^* (1+h^2 L^*)^n$$

$$\leq K^* (e^{h^2 L^*})^n, \text{ using the inequality } 1+h^2 L^* < e^{h^2 L^*}$$

$$\leq K^* \exp\{nh^2 L^*\}$$

This completes the proof.

If we set $\Gamma^* = \dfrac{\Gamma}{1-h^2\beta_4 L}$ and $a^* = a - \dfrac{h\gamma}{\Gamma}$,

we have

$$|e_n| \leq K^* \exp\{BL\Gamma^*(x_n - a^*)^2\}$$

where $K^* = [4AE(n\Gamma+\gamma)+\Gamma\frac{h^p}{2} C_{p+2}M_{p+2}(x_n-a^*)^2] / (1-h^2\beta_4 L)$

Clearly $|e_n| \to 0$ as $E,h \to 0$.

Also $|| e_n || = O(h^p)$

Similar error bounds may be obtained for the other families too.

Note: For a vector $\nu = (\nu_i)$, $|\nu| = (|\nu_i|)$, $||\nu|| = \max_i |\nu_i|$

8. Runge-Kutta Method

A commonly used fourth order Runge-Kutta formula for obtaining the numerical solution of an initial value problem of the first order, viz;

$$y' = f(x,y), \; y(a) = \eta$$

is given by

$$y_{n+1} = y_n + \frac{1}{6}(k_0+2k_1+2k_2+k_3) + O(h^5) \tag{3.37}$$

where

$$\begin{aligned} k_0 &= h\, f\,(x_n, y_n) \\ k_1 &= h\, f\,(x_n+\tfrac{1}{2}h, y_n+\tfrac{1}{2}k_0) \\ k_2 &= h\, f\,(x_n+\tfrac{1}{2}h, y_n+\tfrac{1}{2}k_1) \\ k_3 &= h\, f\,(x_n+h, y_n+k_2) \end{aligned} \tag{3.38}$$

where h is the step size.

These formulas can be generalised [5] to the treatment of simultaneous equations of the form

$$\frac{dy}{dx} = f\ (x,y,z)$$

$$\frac{dz}{dx} = g\ (x,y,z) \tag{3.39}$$

where y and z are prescribed at x=a. Formula (3.37) generalises as follows:

$$y_{n+1} = y_n + \frac{1}{6}\ (k_0+2k_1+2k_2+k_3) + O(h^5) \tag{3.40}$$

$$z_{n+1} = z_n + \frac{1}{6}\ (m_0+2m_1+2m_2+m_3) + O(h^5)$$

where

$$k_0 = h\ f(x_n,y_n,z_n)$$

$$k_1 = h\ f(x_n+\frac{1}{2}\ h, y_n+\frac{1}{2}\ k_0, z_n+\frac{1}{2}\ m_0) \tag{3.41}$$

$$k_2 = h\ f(x_n+\frac{1}{2}h,\ y_n+\frac{1}{2}k_1,\ z_n+\frac{1}{2}m_1)$$

$$k_3 = h\ f(x_n+h,\ y_n+k_2, z_n+m_2)$$

and

$$m_0 = h\ g(x_n,y_n,z_n)$$

$$m_1 = h\ g(x_n+\frac{1}{2}h,\ y_n+\frac{1}{2}k_0, z_n+\frac{1}{2}m_0) \tag{3.42}$$

$$m_2 = h\ g(x_n+\frac{1}{2}h,\ y_n+\frac{1}{2}k_1,\ z_n+\frac{1}{2}m_1)$$

$$m_3 = h\ g(x_n+h,\ y_n+k_2,\ z_n+m_2)$$

When f=z, the system of equations (3.39) is equivalent to

$y'' = g(x,y,y')$ with $y(a)$ and $y'(a)$ given.

With $z=y'$, (3.41) becomes

$$k_0 = h\ y'_n$$

$$k_1 = h\ y'_n + \frac{h}{2}\ m_0$$

$$k_2 = h\, y'_n + \frac{h}{2} m_1$$

$$k_3 = h\, y'_n + h\, m_2$$

Equations (3.40) and (3.42) then reduce to

$$y_{n+1} = y_n + hy'_n + \frac{h}{6}(m_0 + m_1 + m_2) + O(h^5) \tag{3.43}$$

$$y'_{n+1} = y'_n + \frac{1}{6}(m_0 + 2m_1 + 2m_2 + m_3) + O(h^5)$$

where

$$m_0 = h\, g(x_n, y_n, y'_n)$$

$$m_1 = h\, g(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hy'_n,\ y'_n + \tfrac{1}{2} m_0)$$

$$m_2 = h\, g(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\, y'_n + \tfrac{1}{4}h\, m_0, y'_n + \tfrac{1}{2} m_1) \tag{3.44}$$

$$m_3 = h\, g(x_n + h,\ y_n + hy'_n + \tfrac{1}{2}hm_1,\ y'_n + m_2)$$

Many variations and generalisations of these formulas are available. Formulas (3.43) and (3.44) are clearly simplified when the D.E. is independent of y'.

9. Various V.E.'s Employed in Experiments

In the last chapter we shall compare numerical results based on the various formulas outlined in this chapter. For reference purposes, we shall designate the predictor-connector combination given by the pair of equations (3.13) and (3.14) as I-1. Similarly we shall designate the pair of equations given by (3.15) and (3.16) as I-2, the pair of equations given by (3.18) and (2.16) as I-3, the pair of equations given by (3.20) and (2.19) as I-4, the pair of equations given by (3.22) and (2.22) as I-5 and the pair of equations given by (3.24) and (2.26) as I-6. Also, we shall designate the method of solution using the Runge-Kutta method as I-7.

CHAPTER IV

LINEAR BOUNDARY VALUE PROBLEMS

1. We shall consider here, the linear case of the boundary value problems of class M given by (1.2)-(1.5). When k=2, if the ∇.E. (1.7) has a positive degree, from (1.19) we have

$$\alpha(x) = (x-1)^2.$$

Therefore, $\alpha_0=1$, $\alpha_1= -2$ and $\alpha_2=1$.

The ∇.E. then, is necessarily of the form

$$y_{n-1}-2y_n+y_{n+1}-h^2(\beta_0 y''_{n-1}+\beta_1 y''_n+\beta_2 y''_{n+1}) = 0 \tag{4.1}$$

with $\beta_0+\beta_1+\beta_2 = 1$.

Frequently used difference equations for the numerical integration of D.E.'s of the form (1.2)-(1.5) are

$$y_{n-1}-2y_n+y_{n+1}-h^2 y''_n = 0 \quad (p=2) \tag{4.2}$$

and

$$y_{n-1}-2y_n+y_{n+1}-\frac{h^2}{12}(y''_{n-1}+10y''_n+y''_{n+1}) = 0 \quad (p=4) \tag{4.3}$$

The ∇.E. (4.3) is the well known Numerov formula.

2. Setting up a Finite Difference Scheme with k=2.

Since y_0 and y_{N+1} are determined by the boundary conditions, the unknowns in this case are $y_1, y_2, \cdots, y_N$. We need as many equations for the determination of the unknowns as there are unknowns. If the order of the ∇.E. is greater then 2, new unknowns such as y_{-1} and y_{N+2} are introduced for which there are no equations. This difficulty can be overcome by modifying the difference equation near the boundaries. However, when k=2, the smallest possible value of k, this problem does not arise.

Using the difference equation (4.1) in approximating the differential equation (1.2)-(1.5) at the discrete points n=1, 2,...,N of (4.1), we get the system

$$Jy+h^2Bf(y) = d \tag{4.4}$$

where

$$J = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & - & - & - & & \\ & & & - & - & - & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta_1 & \beta_2 & & & & & \\ \beta_0 & \beta_1 & \beta_2 & & & & \\ & \beta_0 & \beta_1 & \beta_2 & & & \\ & & & - & - & & \\ & & & - & - & - & \\ & & & & \beta_0 & \beta_1 & \beta_2 \\ & & & & & \beta_0 & \beta_1 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_N \end{bmatrix}, \qquad f(y) = \begin{bmatrix} f(x_1,y_1) \\ f(x_2,y_2) \\ \cdot \\ \cdot \\ f(x_N,y_N) \end{bmatrix}, \qquad d = \begin{bmatrix} A-\beta_0 h^2 f(a,A) \\ 0 \\ \cdot \\ \cdot \\ B-\beta_2 h^2 f(b,B) \end{bmatrix}$$

Since we are concerned with the linear case of boundary value problems of class M, $f(x,y)$ is of the form

$$f(x,y) = g(x)y + s(x) \tag{4.5}$$

Hence $f(x,y)$ can be written in the form

$$f(y) = Gy + s$$

where

$$G = \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_N \end{bmatrix} \text{ and } s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$$

Here, $g_i = g(x_i)$ and $s_i = s(x_i)$.

The system of equations (4.4) now reduces to

$$A\,y = \bar{b} \tag{4.6}$$

where $A = J+h^2DG$ and $\bar{b} = d-h^2Ds$.

Matrices A, J and D share the property of having non-zero elements only on the main diagonal and the two diagonals adjacent to it and hence are tridiagonal. Solutions of systems involving non-singular tridiagonal matrices are not difficult.

3. <u>Setting up a Finite Difference Scheme of Higher Orders</u>

Equation (2.16) of order k=4 may be used to approximate the solution of D.E. (1.2)-(1.5) . This equation may be considered as being centred on x_{n+2} and forms an equation for y_{n+2}. Two additional points are, therefore, involved on either side. Hence it can not be used as an equation for y_1 or y_N and two special equations are needed at these two points.

Now, consider equation (3.16) obtained with k=3. Writing $\bar{\beta}$ for the vector β in (3.16) and putting n=0, we have

$$- cy_o + (2c-1)y_1+(2-c)y_2-y_3+h^2 \sum_{i=0}^{3} \bar{\beta}_i y''_i = 0 \tag{4.7}$$

Replacing h by -h in (3.16),

$$-cy_n+(2c-1)y_{n-1}+(2-c)y_{n-2}-y_{n-3}+h^2\sum_{i=0}^{3} \bar{\beta}_{3-i}\, y''_{n-3+i} = 0 \tag{4.8}$$

From (4.8), we have

$$-y_{n-2}+(2-c)y_{n-1}+(2c-1)y_n-cy_{n+1}+h^2 \sum_{i=0}^{3} \bar{\beta}_{3-i} y''_{n-2+i} = 0 \tag{4.9}$$

Equation (4.7) can be used at one boundary. With n=N, (4.9) can be used at the other boundary. Equation (2.16) will be used at the interior points. Now the linear boundary value problem of class M can be replaced by

$$J(c)y + h^2D(c)f(y) = d \tag{4.10}$$

where

$$J(c)=-\begin{bmatrix} \bar{\bar{\alpha}}_1 & \bar{\alpha}_2 & \bar{\alpha}_3 & & & & & & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & & \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & \\ & - & - & - & - & & & & \\ & & - & - & - & - & - & & \\ & & & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \\ & & & & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \\ & & & & & \bar{\alpha}_3 & \bar{\alpha}_2 & \bar{\alpha}_1 & \end{bmatrix}$$

$$D(c) = \begin{bmatrix} \bar{\beta}_1 & \bar{\beta}_2 & \bar{\beta}_3 & & & & & \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & & & & \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & & & \\ & - & - & - & - & - & & \\ & & - & - & - & - & - & \\ & & & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ & & & & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ & & & & & \bar{\beta}_3 & \bar{\beta}_2 & \bar{\beta}_1 \end{bmatrix}$$

$$d = \begin{bmatrix} \bar{\alpha}_0 A - h^2 \bar{\beta}_0 f(a,A) \\ \alpha_0 A - h^2 \beta_0 f(a,A) \\ - \\ - \\ \alpha_4 B - h^2 \beta_4 f(b,B) \\ \bar{\alpha}_0 B - h^2 \bar{\beta}_0 f(b,B) \end{bmatrix}$$

y and f(y) are as defined before.

Since $f(x,y) = g(x)y + s(x)$

$f(y) = Gy + s$

where G and s are also as defined before.

The system of equations (4.10) now reduces to

$$A(c)y = \bar{b} \tag{4.11}$$

where $A(c) = J(c) + h^2 D(c)G$

and $\bar{b} = d - h^2 D(c)s$

Matrices A(c), J(c), and D(c) have the common feature of having

non-zero elements only on the main diagonal and two diagonals each on either side adjacent to the main diagonal.

4. Computational Procedure

If by $A = (a_{n,m})$ we denote the five band matrix $A(c)$, we have

$$a_{n,n-2} = -\alpha_0 + h^2\beta_0 g_{n-2}$$

$$a_{n,n-1} = -\alpha_1 + h^2\beta_1 g_{n-1}$$

$$a_{n,n} = -\alpha_2 + h^2\beta_2 g_n$$

$$a_{n,n+1} = -\alpha_3 + h^2\beta_3 g_{n+1}$$

$$a_{n,n+2} = -\alpha_4 + h^2\beta_4 g_{n+2}$$

for $n=1,2,\ldots,N$, $a_{n,m}=0$ for $|n-m| > 2$ and $1 \leq m \leq N$.

At the boundaries, α's and β's will be replaced by the appropriate $\bar{\alpha}$'s and $\bar{\beta}$'s.

Linear systems involving non-singular five band matrices can be solved by an adaptation of the Gaussian algorithm. If A can be resolved into two non-singular matrices $L=(\ell_{m,n})$ and $U=(u_{m,n})$ of the form

$$L = \begin{bmatrix} 1 & & & & & & \\ \ell_{21} & 1 & & & & & \\ \ell_{31} & \ell_{32} & 1 & & & & \\ - & - & - & - & & & \\ & - & - & - & & & \\ & & & \ell_{N-1,N-3} & \ell_{N-1,N-2} & 1 & \\ & & & & \ell_{N,N-2} & \ell_{N,N-1} & 1 \end{bmatrix}$$

$\ell_{m,m} = 1$, $\ell_{m,n} = 0$ for $n > m$ or $n < m-2$ and

$$u = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & & & & \\ & u_{2,2} & u_{2,3} & u_{2,4} & & & \\ & & - & - & - & & \\ & & & - & - & - & \\ & & & u_{N-2,N-2} & u_{N-2,N-1} & u_{N-2,N} \\ & & & & u_{N-1,N-1} & u_{N-1,N} \\ & & & & & u_{N,N} \end{bmatrix}$$

$u_{m,n} = 0$ for $n < m$ or $n > m+2$

so that $A = L\,U$. (4.12)

In order to solve (4.11), we first have to determine the vector z such that

$$Lz = \bar{b} \tag{4.13}$$

and then the vector y such that

$$Uy = z \tag{4.14}$$

Now $y = U^{-1}z = U^{-1}L^{-1}\bar{b} = A^{-1}\,\bar{b}$.

y thus satisfies (4.11).

The relation (4.12) is equivalent to the relations

$$u_{1,1} = a_{1,1},\ u_{1,2} = a_{1,2}$$

$$\ell_{2,1}\,u_{1,1} = a_{2,2}$$

$$\ell_{2,1}\,u_{1,2} + u_{2,2} = a_{2,2}$$

$$\ell_{2,1}\,u_{1,3} + u_{2,3} = a_{2,3}$$

$$\left.\begin{aligned}
\ell_{n,n-2}\, u_{n-2,n-2} &= a_{n,n-2} \\
\ell_{n,n-2}\, u_{n-2,n-1} + \ell_{n,n-1}\, u_{n-1,n-1} &= a_{n,n-1} \\
\ell_{n,n-2}\, u_{n-2,n} + \ell_{n,n-1}\, u_{n-1,n} + u_{n,n} &= a_{n,n} \\
\ell_{n,n-1}\, u_{n-1,n+1} + u_{n,n+1} &= a_{n,n+1}
\end{aligned}\right] \quad n = 3,4\ldots,N$$

$$u_{n,n+2} = a_{n,n+2}, \quad n=1,2,\ldots,N-2 \qquad (4.15)$$

From then relations ℓ's and u's can be obtained recursively as follows:

$$u_{1,1} = a_{1,1} \quad , \quad u_{1,2} = a_{1,2}$$

$$\ell_{2,1} = a_{2,1} \,/\, u_{1,1}$$

$$u_{2,2} = a_{2,2} - \ell_{2,1}\, u_{1,2}$$

$$u_{2,3} = a_{2,3} - \ell_{2,1}\, a_{1,3}$$

In addition to (4.15), we have

$$\left.\begin{aligned}
\ell_{n,n-2} &= a_{n,n-2} \,/\, u_{n-2,n-2} \\
\ell_{n,n-1} &= (a_{n,n-1} - \ell_{n,n-2}\, u_{n-2,n-1}) \,/\, u_{n-1,n-1} \\
u_{n,n} &= a_{n,n} - \ell_{n,n-2}\, u_{n-2,n} - \ell_{n,n-1}\, u_{n-1,n} \\
u_{n,n+1} &= a_{n,n+1} - \ell_{n,n-1}\, u_{n-1,n+1}
\end{aligned}\right] \quad \text{for } n=3,4,\ldots,N$$

The vector z can be determined from the relation

$$z_n = b_n - \ell_{n,n-1}\, z_{n-1} - \ell_{n,n-2}\, z_{n-2}, \quad n=1,2,\ldots,N$$

with $z_\gamma \equiv 0$ for $\gamma \leq 0$

If (4.14) is written out in components, we have

$$u_{n-2,n-2}\, y_{n-2} + u_{n-2,n-1}\, y_{n-1} + u_{n-2,n}\, y_n = z_{n-2} \quad (n=3,4\ldots,N+2)$$

with $y_\gamma \equiv 0$ for $\gamma \geq N+1$

These relations can be rearranged so as to furnish the components of y, starting with the last component. We get

$$y_n = (z_n - u_{n,n+1}\, y_{n+1} - u_{n,n+2}\, y_{n+2}) \,/\, u_{n,n}$$

$$(n = N, N-1, \ldots, 1) \text{ with } y_\gamma \equiv 0 \text{ for } \gamma \geq N+1.$$

5. Linear Boundary Value Problems with Mixed Boundary Conditions

We shall consider here the numerical approximation of the solution of the boundary value problem (1.2) - (1.6), by introducing a finite set $\{ x_i \}$ of grid points

$$x_i = a + (i-1)h, \quad i=1,2,\ldots,N$$

where $h = (b-a) \,/\, (N-1)$.

A scheme is then designed, as before, for the determination of the sequence y_n which approximates $y(x_n)$.

∇.E.'s with k=2

The sequence of real numbers y_n satisfies the ∇.E.'s.

$$\text{i)} \quad \left(1+hd+\frac{5}{12}h^2 g_1+\frac{h^3}{12}(dg_1+g'_1)\right) y_1+\left(-1+\frac{h^2}{12}g_2\right) y_2$$

$$= -hA - \frac{h^2}{12}(5\, s_1+s_2) - \frac{h^3}{12}(A\, g_1+s'_1)$$

$$\text{ii)} \quad \left(-1+\frac{h^2}{12}g_{n-1}\right) y_{n-1} + \left(2+\frac{10}{12}h^2 g_n\right) y_n+\left(-1+\frac{h^2}{12}g_{n+1}\right) y_{n+1} \tag{4.16}$$

$$= -\frac{h^2}{12}(s_{n-1} + 10s_n + s_{n+1}), \quad n=2,3,\ldots,N-1$$

$$\text{iii)} \quad \left(-1+\frac{h^2}{12}g_{N-1}\right) y_{N-1} + \left(1+h\,e + \frac{5}{12}h^2 g_N + \frac{h^3}{12}(e\, g_N-g'_N)\right) y_N$$

$$= hB - \frac{h^2}{12}(s_{N-1} + 5\, s_N) + \frac{h^3}{12}(B\, g_N + s'_N)$$

where $g_i = g(x_i)$, $s_i = s(x_i)$, $g'_i = \left(\frac{dg(x)}{dx}\right)_{x=x_i}$, $s'_i = \left(\frac{ds(x)}{dx}\right)_{x=x_i}$

The equation (4.16(ii)) is based on Numerov's formula. For the derivation of (4.16(i) and (iii)), see Usmani[21].

∇.E.'s with k=4

For k=4, we introduce the following ∇.E.'s:

(i) $$[1+2hd + \frac{2h^3}{15}(dg_1 + g'_1) + \frac{13}{15}h^2 g_1]\, y_1 + \frac{h^2}{15}\, 16\, g_2 y_2$$
$$+(-1+\frac{h^2}{15} g_3) y_3 = -2hA - \frac{h^2}{15}(13\, s_1 + 16 s_2 + s_3) - \frac{2h^3}{15}(Ag_1 + s'_1)$$

(ii) $$[-c + \frac{h^2}{12} c\, g_1]\, y_1 + [-1(1-2c) + \frac{h^2}{12}(10c+1) g_2]\, y_2 + [-(c-2) + \frac{h^2}{12}(c+10) g_3]\, y_3$$
$$+ (-1+\frac{h^2}{12} g_4)\, y_4 = -\frac{h^2}{12}[c\, s_1 + (10c+1) s_2 + (c+10) s_3 + s_4]$$

(iii) $$[-c^2 + \frac{h^2}{240}(19c^2-2c-1)\, g_{n-2}]\, y_{n-2} + [-(2c-2c^2) + \frac{h^2}{240}(204c^2+48c+4) g_{n-1}] y_{n-1}$$
$$+[-(c^2-4c+1) + \frac{h^2}{240}(14c^2+388c+14) g_n]\, y_n + [-2c+2) + \frac{h^2}{240}(4c^2+48c+204) g_{n+1}] y_{n+1}$$
$$+[-1+\frac{h^2}{240}(19-2c-c^2) g_{n+2}] y_{n+2} = -\frac{h^2}{240}[(19c^2-2c-1)\, s_{n-2}$$
$$+ (204c^2+48c+4) s_{n-1} + (14c^2+388c+14) s_n + (4c^2+48c+204) s_{n+1}$$
$$+ (19-2c-c^2) s_{n+2}],\quad n=3,4,\ldots,N-2 \qquad (4.17)$$

(iv) $$[\, -1+\frac{h^2}{12} g_{N-3}\,]\, y_{N-3} + [\, -(c-2) + \frac{h^2}{12}(c+10) g_{N-2}\,] y_{N-2}$$
$$+ [-1(1-2c) + \frac{h^2}{12}(10c+1) g_{N-1}]\, y_{N-1} + [-c + \frac{h^2}{12} c\, g_N]\, y_N$$
$$= -\frac{h^2}{12}[\, s_{N-3} + (c+10) s_{N-2} + (10c+1)\, s_{N-1} + c.s_N]$$

(v) $$[\, -1+\frac{h^2}{15} g_{N-2}\,]\, y_{N-2} + \frac{h^2}{15}.16\, g_{N-1} y_{N-1} + [1+2he - \frac{2h^3}{15}(g'_N - eg_N) + \frac{h^2}{15}.13 g_N] y_N$$
$$= 2h\, B - \frac{h^2}{15}(s_{N-2} + 16 s_{N-1} + 13 s_N) + 2\,\frac{h^3}{15}(B\, g_N + s'_N)$$

Formula (4.17 (iii)) is based on (2.16) and formulas (4.17(ii)

and (iv)) are based on (4.7) and (4.9) respectively. In order to derive (4.17(i) and (v)) we develop

$$-y_1+y_3 = 2\ hy'_1+\frac{h^2}{15}\ (13y''_1+16y''_2+y''_3)+\frac{2}{15}\ h^3y'''_1-\frac{2}{1575}\ h^7y^{(7)}(\xi_1),$$

$$x_1<\xi_1<x_3 \tag{4.18}$$

and

$$-y_{N-2}+y_N = 2\ hy'_N-\frac{h^2}{15}\ (y''_{N-2}+16y''_{N-1}+13y''_N) + \frac{2}{15}\ h^3y'''_N - \frac{2}{1575}$$

$$h^7y^{(7)}\ (\xi_N), x_{N-2}<\xi_N<x_N \tag{4.19}$$

For (4.18) we assume that y_1, y_2 and y_3 satisfy an equation of the form

$$a_0y_1+a_1y_2+a_2y_3 = \lambda h^2y'_1+h^2(b\ y_1''+b_1y_2''+b_2y_3'') \tag{4.20}$$

$$+\ c_0h^3y_1''' + T_1$$

We use the Taylor series method to determine the coefficients a_0, a_1, a_2, λ, b_0, b_1, b_2, c_0 and the local truncation error T_1. Then in (4.18) we substitute

$$y'' = g\ y + s \text{ and}$$

$$y''' = g'\ y + g\ y' + s'$$

for $x = x_1$, x_2 and x_3, use the boundary condition $y'(a)-d(a)=A$ and neglect the truncation error T_1 and we obtain (4.17(i)). In a similar manner using (4.19), (4.17(v)) is derived.

6. Various ∇.E.'s Used in Experiments

Equations (2.19), (2.22) or (2.26) may be used instead of (2.16) at the interior points. Numerical results based on these various ∇.E.'s are presented in the last chapter.

For reference purposes, the methods based on (4.2) and (4.3)

are referred to as B-1 and B-2 respectively. Methods using equations (2.16), (2.19), (2.22) and (2.26) at the interior points are referred to as B-3, B-4, B-5 and B-6 respectively. Equations (4.7) and (4.9) are used at the boundaries in all cases with k=4. In case of the more general boundary conditions, we shall refer to the method using (4.16) as BG-2 and the method using (4.17) as BG-3.

CHAPTER V

NON-LINEAR BOUNDARY VALUE PROBLEMS

1. If the function $f(x,y)$ is non-linear, we cannot expect to solve the system of equations (4.10) by the usual algebraic methods. A solution may be obtained by using an iterative procedure.

A system of non-linear equations in N unknowns y_n can be expressed as

$$\phi_n(y_1,y_2, \quad , y_N)=0, \quad n=1,2,\ldots,N \tag{5.1}$$

or in the vector form

$$\phi(y)=0 \tag{5.2}$$

where $\phi=(\phi_n)$ and $y=(y_n)$ are N-dimensional vectors. The solution to the system may be obtained by Newton-Raphson's method ([4], [17]).

Let y_0 be an approximation of y satisfying (5.2). We generate a sequence of vectors y_m by

$$y_{m+1}=y_m-\left[\frac{\partial\phi(y_m)}{\partial y}\right]^{-1}\phi(y_m), \quad m=0,1,2, \tag{5.3}$$

The matrix $\frac{\partial\phi}{\partial y} = (\lambda_{m,n})$ is the Jacobian of the system (5.2) written

$$\lambda_{m,n}=\frac{\partial\phi_m}{\partial y_n} \text{ for all } m,n=1,2,\ldots,N. \tag{5.4}$$

The iterative scheme defined by (5.3) converges quadratically for sufficiently small values of h. Thus the correct number of significant figures is doubled in the numerical value of y_m at each iteration.

The initial vector may be chosen so that $y_0=(Q(x_n))$, $n=1,2,\ldots,N$; where the function $Q(x)$ satisfies the linear D. E.

$$Q''(x)=f(0,0)+x\,\frac{\partial f}{\partial x}(0,0)+y\,\frac{\partial f}{\partial y}(0,0).$$

2. Finite Difference Scheme

Now, for a given y, the residual vector r(y) is defined as

$$r(y)=Jy+h^2Bf(y)-d \tag{5.5}$$

with J, B and d as defined in chapter 4.

If the vector $y^{(0)}$ is an approximation to the solution, $(y^{(0)})$ is small. We shall solve the linearised system

$$r(y^{(0)})+[J+h^2BF(y^{(0)})]\Delta y^{(0)}=0 \tag{5.6}$$

where

$$F(y) = \begin{bmatrix} f_y(_1,y_1) & & & \\ & f_y(x_1,y_2) & & \\ & & \ddots & \\ & & & f_y(x_N,y_N) \end{bmatrix}$$

If we put $A(y^{(0)})=J+h^2BF(y^{(0)})$, the solution is given by

$$\Delta y^{(0)}=-[A(y^{(0)})]^{-1}\, r(y^{(0)}) \tag{5.7}$$

The vector $y^{(1)}=y^{(0)}+\Delta y^{(0)}$ will then be a better approximation to the exact solution. The residual vector $r(y^{(1)})$ will be smaller and the process can be repeated with $y^{(1)}$ taking the place of $y^{(0)}$ and so on until the result converges to a sufficiently accurate number.

∇.E.'s of Order k=2

∇.E.'s used frequently for obtaining approximate solution of the problem (1.2)-(1.5) are of order k=2, namely, (4.2) or (4.3). As in the linear case, here also the solution is greatly facilitated by the fact that $A(y^{(0)})$ is tridiagonal.

If $A(y^{(0)}) = (a_{m,n})$, we have

$$a_{n,n-1} = -1+h^2\beta_0 \frac{\partial f(x_{n-1},y_{n-1}^{(0)})}{\partial y}, \quad (n=2,3,\ldots,N)$$

$$a_{n,n} = 2+h^2\beta_1 \frac{\partial f(x_n,y_n^{(0)})}{\partial y}, \quad (n=1,2,\ldots,N) \tag{5.8}$$

$$a_{n,n+1} = -1+h^2\beta_2 \frac{\partial f(x_{n+1},y_{n+1}^{(0)})}{\partial y}, (n=1,2,\ldots,N-1)$$

and all the other elements are zero.

The method described in chapter 4 for the linear boundary value problem is applicable here too. The only additional work required is the evaluation of the residual vector $r(y^{(m)})$ and the derivation of the partial derivatives

$$\frac{\partial f(x_n,y_n)}{\partial y}, \quad (n=1,2,\ldots,N) \tag{5.9}$$

For more details, see [4].

V.E.'s of order k=4

When k=4, $A(y^{(0)})$ is a five band matrix similar to the case when the D.E. was linear. The elements of this matrix will be

$$\begin{aligned}
a_{n,n-2} &= -\alpha_0+h^2\beta_0 \frac{\partial f(x_{n-2},y_{n-2}^{(0)})}{\partial y} \\
a_{n,n-1} &= -\alpha_1+h^2\beta_1 \frac{\partial f(x_{n-1},y_{n-1}^{(0)})}{\partial y} \\
a_{n,n} &= -\alpha_2 +h^2\beta_2 \frac{\partial f(x_n,y_n^{(0)})}{\partial y} \\
a_{n,n+1} &= -\alpha_3+h^2\beta_3 \frac{\partial f(x_{n+1},y_{n+1}^{(0)})}{\partial y} \\
a_{n,n+2} &= -\alpha_4+h^2\beta_4 \frac{\partial f(x_{n+2},y_{n+2}^{(0)})}{\partial y}
\end{aligned} \tag{5.10}$$

for $n=1,2,\ldots,N$; $a_{n,m}=0$ for $|n-m| > 2$ and $1 \leq m \leq N$. At the boundaries, the α's and β's will be replaced by the appropriate $\bar{\alpha}$'s and $\bar{\beta}$'s.

In the next chapter, we present numerical results based on all

the three families of formulae derived earlier. For comparison, results based on a family of formulae given by Usmani [14], are also presented.

3. Non-linear Boundary Value Problems with Mixed Boundary Conditions

When the boundary conditions are of the type (1.6), we have formulae similar to those discussed in the last chapter.

For k=2, the formulae are

(i) $$[1+hd(1+\frac{h^2}{12}q_1)]y_1-y_2+\frac{h^2}{12}(5f_1+f_2)+hA(1+\frac{h^2}{12}q_1)+\frac{h^3}{12}p_1=0$$

(ii) $$-y_{n-1}+2y_n-y_{n+1}+\frac{h^2}{12}(f_{n-1}+10f_n+f_{n+1})=0,\ n=2,3,\ldots,N-1 \qquad (5.11)$$

(iii) $$-y_{n-1}+[1+he(1+\frac{h^2}{12}q_N]y_N+\frac{h^2}{12}(f_{N-1}+5f_N)-hB(1+\frac{h^2}{12}q_N)-\frac{h^3}{12}p_N=0$$

where $p_m = \dfrac{\partial f(x_m,y_m)}{\partial x}$, $q_m = \dfrac{\partial f(x_m,y_m)}{\partial y}$

When k=4, analogous to the case when f(x,y) is linear, at the boundaries we have the formulae

$$[1+2hd(1+\frac{h^2}{12}q_1)]y_1-y_3+\frac{h^2}{15}(13f_1+16f_2+f_3)+2hA(1+\frac{h^2}{15}q_1)$$
$$+\frac{2}{15}h^3p_1 = 0$$

$$-y_{N-2}+[1+2he(1+\frac{h^2}{15}q_N)]y_N+\frac{h^2}{15}(f_{N-2}+16f_{N-1}+13f_N)-2hB(1+\frac{h^2}{15}q_N)$$
$$-\frac{2}{15}h^3p_N = 0.$$

As before, we use (2.16), (4.7) and (4.9) at the interior points.

Numerical results based on the mixed boundary conditions are also given in the next chapter.

CHAPTER VI

NUMERICAL RESULTS AND CONCLUDING REMARKS

1. All the computations were performed either on an IBM 360/65 or on an IBM 370/158 computer at the University of Manitoba computer centre. The programs were written in FORTRAN IV. All calculations were done using double precision arithmetic in order to keep the round-off errors to a minimum.

A number of D.E.'s were chosen to obtain the numerical results. The true solutions of these equations were used to check on the accuracy of the numerical solutions.

2. Initial Value Problems

We present here, the numerical results based on the following D.E.'s:

$$x^2y''-2y=-x,\ y(2)=0, y'(2)=0.2105 \tag{6.1}$$

with $y(x)=\frac{1}{38}(19x-5x^2-\frac{36}{x})$ (collatz [1], p. 178)

$$y''=\frac{1}{2}(x+y+1)^3, y(0)=0, y'(0)=-0.5 \tag{6.2}$$

with $y(x)=\frac{2}{2-x}-x-1$ (Ciarlet, Schultz and Varga [10], p. 425).

During the experiments with the one parameter formulas, the value of 'c' was varied from 0.1 to 1.3 in increments of 0.1. It was observed from the experiments that the optimum choice of 'c', (optimum in the sense that the resulting |error| was least) depends on 'h' to a great extent. Sometimes, the behaviour of this dependence varied from problem

to problem. Generally, we get better results when the value of 'c' approaches 1.

In Table 1(a) we present the results of solving the problem (6.1) over the range (2,x) with x=3,6,9,12 and 15. At the starting points we used the exact values. In this table, results using I-1, I-2 and I-3 are shown. We obtained similar results using I-4, I-5 and I-6. For comparison, in Table 1(b) we present the results using I-7 (Runge-Kutta method). We write 0.nnn E-xx for 0.nnn x 10^{-xx}. This convention is followed throughout this thesis.

Figure 1 is a graph showing the variation of $\log_{10} |\text{error}|$ versus $\log_2 (1/2h)$ for problem (6.1) at x=3 using methods I-1 and I-3. Figure 2 is a similar graph for the problem (62.) at x=1.

From the results it emerges that methods using k=4 gave better results than the others. Among the methods with k=4, method I-3 gave best results. In chapter 3, we have shown that $||e||$ when using I-3 is of the order h^p. The value of p is 5. Results in Table 1(a) confirms this.

TABLE 1(a)

EXPERIMENT WITH PROBLEM (6.1) Viz. $x^2y''-2y=-x$, $y(2)=0$, $y'(2)=0.2105$
USING I-1, I-2 and I-3

Method	x	h	|Error|
I-1	3.0	1/8	0.903 E-06
		1/16	0.685 E-07
	6.0	1/8	0.695 E-05
		1/16	0.485 E-06
	9.0	1/8	0.165 E-04
		1/16	0.115 E-05
	12.0	1/18	0.296 E-04
		1/16	0.207 E-05
	15.0	1/8	0.465 E-04
		1/16	0.324 E-05
I-2	3.0	1/8	0.667 E-06
		1/16	0.566 E-07
	6.0	1/8	0.564 E-05
		1/16	0.435 E-06
	9.0	1/8	0.134 E-04
		1/16	0.103 E-05
	12.0	1/8	0.242 E-04
		1/16	0.185 E-05
	15.0	1/8	0.380 E-04
		1/16	0.291 E-05
I-3	3.0	1/8	0.121 E-07
		1/16	0.250 E-09
	6.0	1/8	0.874 E-07
		1/16	0.168 E-08
	9.0	1/8	0.197 E-06
		1/16	0.284 E-08
	12.0	1/8	0.345 E-06
		1/16	0.682 E-08
	15.0	1/8	0.532 E-06
		1/16	0.106 E-07

The value of 'c' for I-2 was set at 0.9. For I-3, the value of 'c' was at 0.99.

TABLE 1(b)

EXPERIMENT WITH PROBLEM (6.1) Viz. $x^2y''-2y=-x, y(2)=0, y'(2)=0.2105$ USING I-7

Method	x	h	|Error|
I-7	3.0	1/8	0.469 E-06
		1/16	0.291 E-07
	6.0	1/8	0.166 E-05
		1/16	0.103 E-06
	9.0	1/8	0.362 E-05
		1/16	0.225 E-06
	12.0	1/8	0.639 E-05
		1/16	0.397 E-06
	15.0	1/8	0.995 E-05
		1/16	0.619 E-06

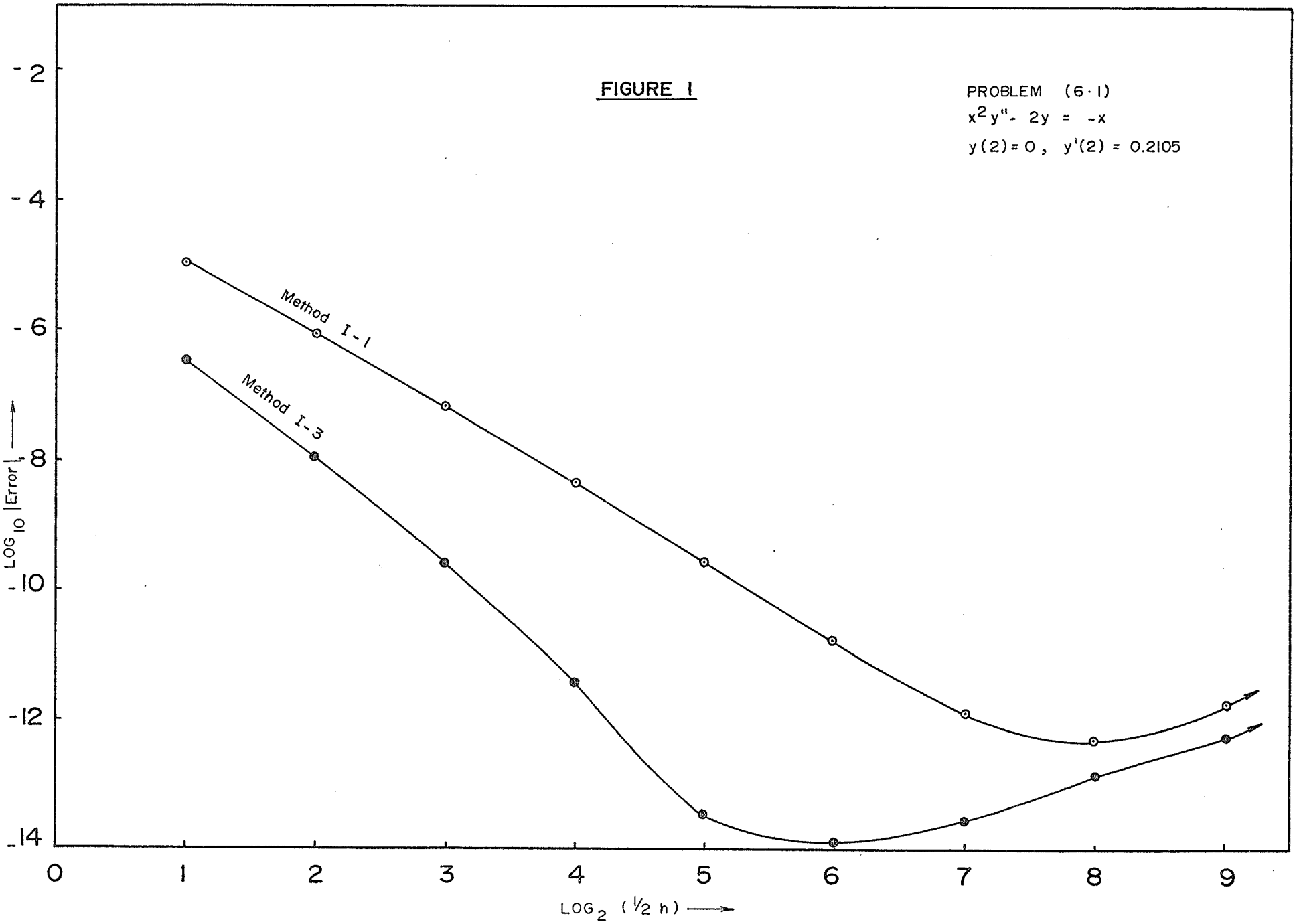
FIGURE I
PROBLEM (6·1)
$x^2y'' - 2y = -x$
$y(2) = 0, \quad y'(2) = 0.2105$
Method I-1
Method I-3
$-\log_{10}|\text{Error}| \longrightarrow$
$\log_2(\tfrac{1}{2}h) \longrightarrow$
-2
-4
-6
-8
-10
-12
-14
0
1
2
3
4
5
6
7
8
9

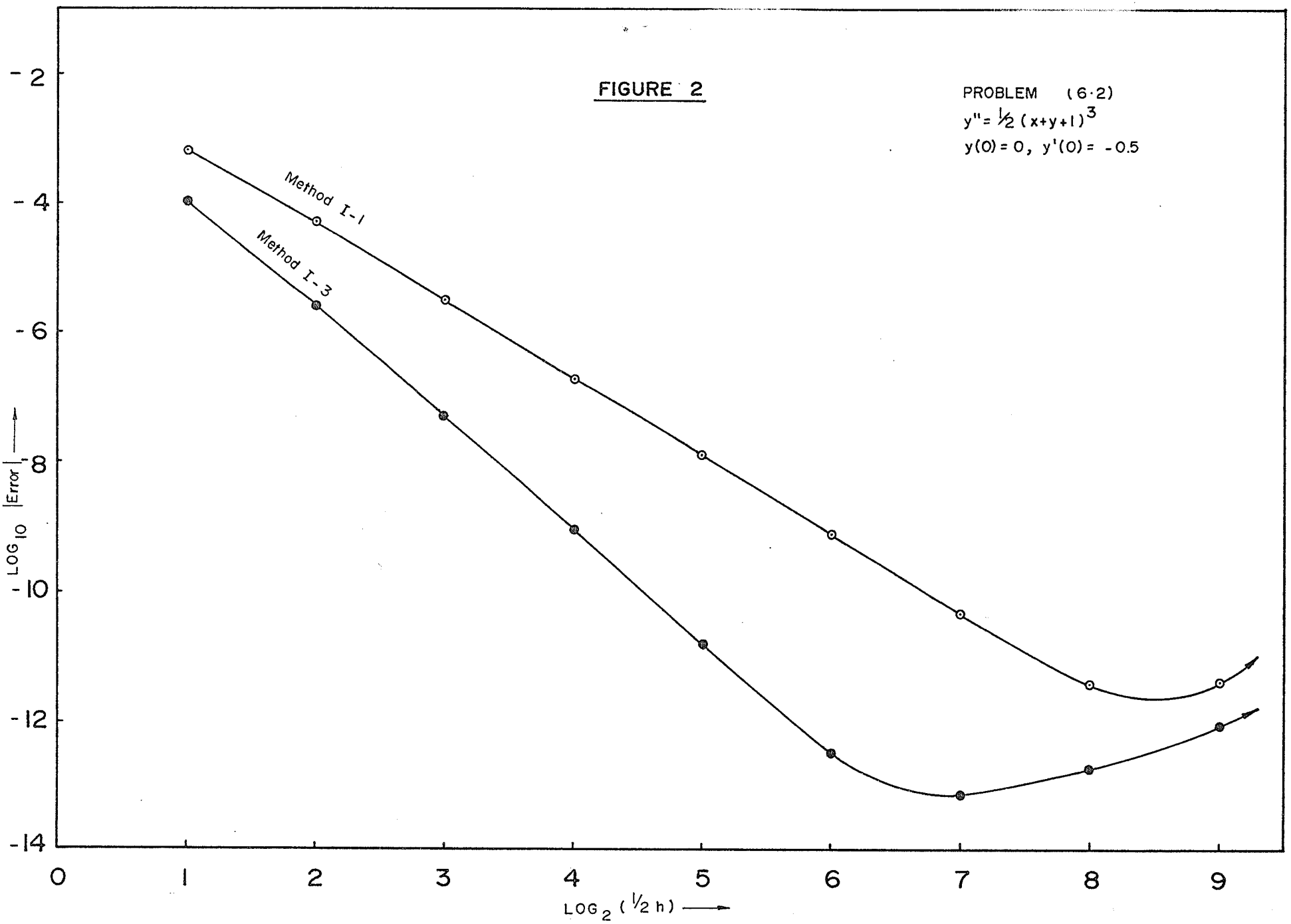
FIGURE 2
PROBLEM (6·2)
y'' = ½ (x+y+1)³
y(0) = 0, y'(0) = -0.5
Method I-1
Method I-3
-2
-4
-6
-8
-10
-12
-14
- LOG10 |Error| ⟶
0
1
2
3
4
5
6
7
8
9
LOG2 (½ h) ⟶

3. Linear Boundary Value Problems of Class M

In this section, we present results based on B-1, B-2 (order k=2) and B-3, B-4, B-5 and B-6 (order k=4). We solved a number of D.E.'s using these methods. Here, we give results in solving the following equations:

$$x^2y''-2y=-x,\ y(2)=y(3)=0 \tag{6.3}$$

with $y(x)=\frac{1}{38}(19x-5x^2-\frac{36}{x})$ (Collatz [1]).

$$x^2y''-3y/4=x^2,\ y(1)=-2.2,\ y(2)=3.2 \tag{6.4}$$

with $y(x)=0.8x^2+(x^2-4)/\sqrt{x}$

and

$$y''-y=x^2-1,\ y(0)=0,\ y(1)=1 \tag{6.5}$$

with $y(x)=(2\sinh x/\sinh 1)-x^2$

Problem (6.5) is a slightly modified form of the one from Roberts and Shipman ([17], p. 202).

From the experiments it emerges that the optimum choice of 'c', optimum in the sense that the |error| was smallest in case of the one-parameter families depends on 'h'. Also this dependence varies from problem to problem. Usmani [12] has discussed the choice of 'c' in the case of method B-6. For B-3, B-4 and B-5 it was observed from experiments that we get better results if the value of 'c' is set to 1 at the boundaries.

Table 2 shows numerical results obtained in solving problem (6.3) using B-3 for c = .1(.1)1.3 varying 'c' at the interior points for h = 1/8 and h = 1/16. We get better results when the value of 'c' approaches 1. Table 3 shows the results for c=0.75(0.01)1.00 for h = 1/8, 1/16.

In Tables 4(a) and 4(b) we present the results in solving problem

(6.3) using the various methods. Tables 5(a) and 5(b) show the results in solving problem (6.4). The value of 'c' for methods B-3, B-4 and B-5 was 0.99. The value of 'c' for method B-6 was 3. These tables show that the methods B-3 and B-5 compare very favourably with the other methods. Method B-3 emerges as the best method (i.e., the |error| is the smallest).

Figure 3 is a graph showing the variation of $\log_{10}$ error versus $\log_2(1/2h)$ for problem (6.3) using methods B-2 and B-3. Figures 4 and 5 are similar graphs for problems (6.4) and (6.5) respectively. These curves show that as h decreases both methods become more accurate till they hit the sound-off region.

We also solved the D.E.

$$y''-y=-4xe^x \tag{6.6}$$

having the more general boundary conditions of the form (1.6), Viz

$$y'(0)-y(0)=1 \text{ and } y'(1)+y(1)=-e=-2.7183$$

with $y(x)=x(1-x)e^x$.

In Table 6 we present the results using the methods, designated as BG-2 and BG-3. Clearly, BG-3 turns out to be superior.

TABLE 2

EXPERIMENT WITH PROBLEM (6.3) Viz. $x^2y''-2y=-x, y(2)=0, y(4)=-0.4321$ USING B-3.

	MAXIMUM ABSOLUTE ERROR	
c	h=1/8	h=1/16
0.1	0.369 E 00	0.405 E 00
0.2	0.665 E-02	0.429 E 00
0.3	0.380 E-04	0.442 E 00
0.4	0.874 E-06	0.609 E-02
0.5	0.334 E-07	0.921 E-05
0.6	0.165 E-07	0.434 E-07
0.7	0.103 E-07	0.346 E-09
0.8	0.458 E-08	0.250 E-09
0.9	0.532 E-08	0.734 E-10
1.0	0.877 E-08	0.119 E-09
1.1	0.130 E-07	0.281 E-09
1.2	0.170 E-07	0.448 E-09
1.3	0.197 E-07	0.494 E-08

TABLE 3

EXPERIMENT WITH PROBLEM (6.3) Viz. $x^2y''-2y=-x, y(2)=0, y(4)=-0.4321$ USING B-3

	MAXIMUM ABSOLUTE ERROR	
c	h=1/8	h=1/16
0.75	0.740 E-08	0.344 E-09
0.76	0.683 E-08	0.327 E-09
0.77	0.626 E-08	0.309 E-09
0.78	0.569 E-08	0.290 E-09
0.79	0.513 E-08	0.270 E-09
0.80	0.458 E-08	0.250 E-09
0.81	0.403 E-08	0.230 E-09
0.82	0.352 E-08	0.212 E-09
0.83	0.375 E-08	0.195 E-09
0.84	0.398 E-08	0.177 E-09
0.85	0.421 E-08	0.159 E-09
0.86	0.444 E-08	0.142 E-09
0.87	0.466 E-08	0.125 E-09
0.88	0.488 E-08	0.107 E-09
0.89	0.510 E-08	0.903 E-10
0.90	0.532 E-08	0.734 E-10
0.91	0.554 E-08	0.569 E-10
0.92	0.575 E-08	0.428 E-10
0.93	0.596 E-08	0.472 E-10
0.94	0.617 E-08	0.515 E-10
0.95	0.645 E-08	0.558 E-10
0.96	0.692 E-08	0.619 E-10
0.97	0.739 E-08	0.744 E-10
0.98	0.786 E-08	0.891 E-10
0.99	0.831 E-08	0.105 E-09
1.00	0.877 E-08	0.119 E-09

TABLE 4(a)

EXPERIMENT WITH PROBLEM (6.3) Viz. $x^2y''-2y=-x$, $y(2)=0$, $y(b)=\frac{1}{38}(19b-5b^2-\frac{36}{b})$ WITH $y(x)=\frac{1}{38}(19x-5x^2-\frac{36}{x})$.

		MAXIMUM ABSOLUTE ERROR IN METHOD		
b	h	B-1	B-2	B-3
3	1/4	0.159 E-03	0.260 E-05	-
	1/8	0.413 E-04	0.174 E-06	0.710 E-08
	1/16	0.104 E-04	0.110 E-07	0.820 E-10
	1/32	0.261 E-05	0.685 E-09	0.874 E-12
4	1/4	0.318 E-03	0.448 E-05	0.547 E-06
	1/8	0.803 E-04	0.290 E-06	0.831 E-08
	1/16	0.202 E-04	0.182 E-07	0.105 E-09
	1/32	0.505 E-05	0.114 E-08	0.115 E-11
6	1/4	0.462 E-03	0.582 E-05	0.575 E-06
	1/8	0.117 E-03	0.370 E-06	0.920 E-08
	1/16	0.293 E-04	0.233 E-07	0.119 E-09
	1/32	0.732 E-05	0.146 E-08	0.128 E-11
10	1/4	0.536 E-03	0.634 E-05	0.606 E-06
	1/8	0.135 E-03	0.402 E-06	0.966 E-08
	1/16	0.339 E-04	0.253 E-07	0.125 E-09
	1/32	0.849 E-05	0.158 E-08	0.162 E-11
18	1/4	0.558 E-03	0.647 E-05	0.613 E-06
	1/8	0.141 E-03	0.411 E-06	0.978 E-08
	1/16	0.353 E-04	0.258 E-07	0.126 E-09
	1/32	0.883 E-05	0.161 E-08	0.173 E-10

TABLE 4(b)

EXPERIMENT WITH PROBLEM (6.3) Viz. $x^2y''-2y=-x$, $y(2)=0$, $y(b)=\frac{1}{38}(19b-5b^2-\frac{36}{b})$.

b	h	MAXIMUM ABSOLUTE ERROR IN METHOD B-4	B-5	B-6
3	1/8	0.527 E-07	0.752 E-08	0.524 E-07
	1/16	0.609 E-08	0.129 E-09	0.108 E-08
	1/32	0.462 E-09	0.121 E-11	0.196 E-10
4	1/4	0.284 E-05	0.550 E-06	0.213 E-05
	1/8	0.415 E-06	0.123 E-07	0.541 E-07
	1/16	0.363 E-07	0.130 E-09	0.110 E-08
	1/32	0.238 E-08	0.197 E-11	0.198 E-10
6	1/4	0.170 E-04	0.782 E-06	0.220 E-05
	1/8	0.186 E-05	0.101 E-07	0.550 E-07
	1/16	0.140 E-06	0.192 E-09	0.111 E-08
	1/32	0.758 E-08	0.196 E-11	0.199 E-10
10	1/4	0.626 E-04	0.601 E-06	0.222 E-05
	1/8	0.597 E-05	0.147 E-07	0.553 E-07
	1/16	0.382 E-06	0.152 E-09	0.111 E-08
	1/32	0.494 E-08	0.166 E-10	0.199 E-10
18	1/4	0.190 E-03	0.879 E-06	0.223 E-05
	1/8	0.153 E-04	0.104 E-07	0.553 E-07
	1/16	0.715 E-06	0.339 E-09	0.111 E-08
	1/32	0.828 E-06	0.983 E-10	0.199 E-10

TABLE 5(a)

EXPERIMENT WITH PROBLEM (6.4) Viz. $x^2y''-\frac{3}{4}y=x^2, y(1)=-2.2, y(b)=.8b^2+(b^2-4)/\sqrt{b}$ WITH $y(x)=.8x^2+(x^2-4)/\sqrt{x}$.

		MAXIMUM ABSOLUTE ERROR IN METHOD		
b	h	B-1	B-2	B-3
2	1/8	0.859 E-03	0.101 E-04	0.105 E-05
	1/16	0.217 E-03	0.645 E-06	0.156 E-07
	1/32	0.543 E-04	0.405 E-07	0.194 E-09
3	1/4	0.531 E-02	0.215 E-03	0.566 E-04
	1/8	0.138 E-02	0.141 E-04	0.117 E-05
	1/16	0.349 E-03	0.894 E-06	0.181 E-07
	1/32	0.874 E-04	0.561 E-07	0.230 E-09
5	1/4	0.691 E-02	0.249 E-03	0.572 E-04
	1/8	0.178 E-02	0.165 E-04	0.127 E-05
	1/16	0.448 E-03	0.104 E-05	0.195 E-07
	1/32	0.112 E-03	0.654 E-07	0.250 E-09
9	1/4	0.772 E-02	0.267 E-03	0.588 E-04
	1/8	0.198 E-02	0.175 E-04	0.130 E-05
	1/16	0.498 E-03	0.111 E-05	0.202 E-07
	1/32	0.125 E-03	0.694 E-07	0.261 E-09
17	1/4	0.798 E-02	0.273 E-03	0.599 E-04
	1/8	0.205 E-02	0.178 E-04	0.131 E-05
	1/16	0.515 E-03	0.113 E-05	0.205 E-07
	1/32	0.129 E-03	0.710 E-07	0.269 E-09

TABLE 5(b)

EXPERIMENT WITH PROBLEM (6.4) Viz. $x^2y''-\frac{3}{4}y=x^2$, $y(1)=-2.2$, $y(b)=.8b^2+(b^2-4)/\sqrt{b}$.

		MAXIMUM ABSOLUTE ERROR IN METHOD		
b	h	B-4	B-5	B-6
2	1/8	0.579 E-05	0.108 E-05	0.436 E-05
	1/16	0.825 E-06	0.231 E-07	0.107 E-06
	1/32	0.712 E-07	0.245 E-09	0.213 E-08
3	1/4	0.184 E-03	0.518 E-04	0.139 E-03
	1/8	0.380 E-04	0.162 E-05	0.455 E-05
	1/16	0.406 E-05	0.205 E-07	0.109 E-06
	1/32	0.301 E-06	0.373 E-09	0.215 E-08
5	1/4	0.101 E-02	0.743 E-04	0.144 E-03
	1/8	0.156 E-03	0.131 E-05	0.464 E-05
	1/16	0.144 E-04	0.303 E-07	0.110 E-06
	1/32	0.914 E-06	0.313 E-09	0.216 E-08
9	1/4	0.384 E-02	0.589 E-04	0.146 E-03
	1/8	0.514 E-03	0.192 E-05	0.467 E-05
	1/16	0.406 E-04	0.222 E-07	0.110 E-06
	1/32	0.207 E-05	0.332 E-09	0.217 E-08
17	1/4	0.125 E-01	0.791 E-04	0.147 E-03
	1/8	0.143 E-02	0.133 E-05	0.468 E-05
	1/16	0.878 E-04	0.315 E-07	0.111 E-06
	1/32	0.128 E-04	0.199 E-08	0.217 E-08

TABLE 6

EXPERIMENT WITH PROBLEM (6.6) Viz. $y''-y=-4xe^{x}$ WITH $y'(0)-y(0)=1$ AND $y'(1)+y(1)=-e$.

METHOD	h	MAXIMUM ABSOLUTE ERROR IN METHOD
BG-2	1/8	0.232 E-04
	1/16	0.146 E-05
	1/32	0.913 E-07
	1/64	0.571 E-08
BG-3	1/8	0.752 E-07
	1/16	0.920 E-09
	1/32	0.211 E-10
	1/64	0.645 E-12

Value of 'c' for BG-3 was set at 0.99.

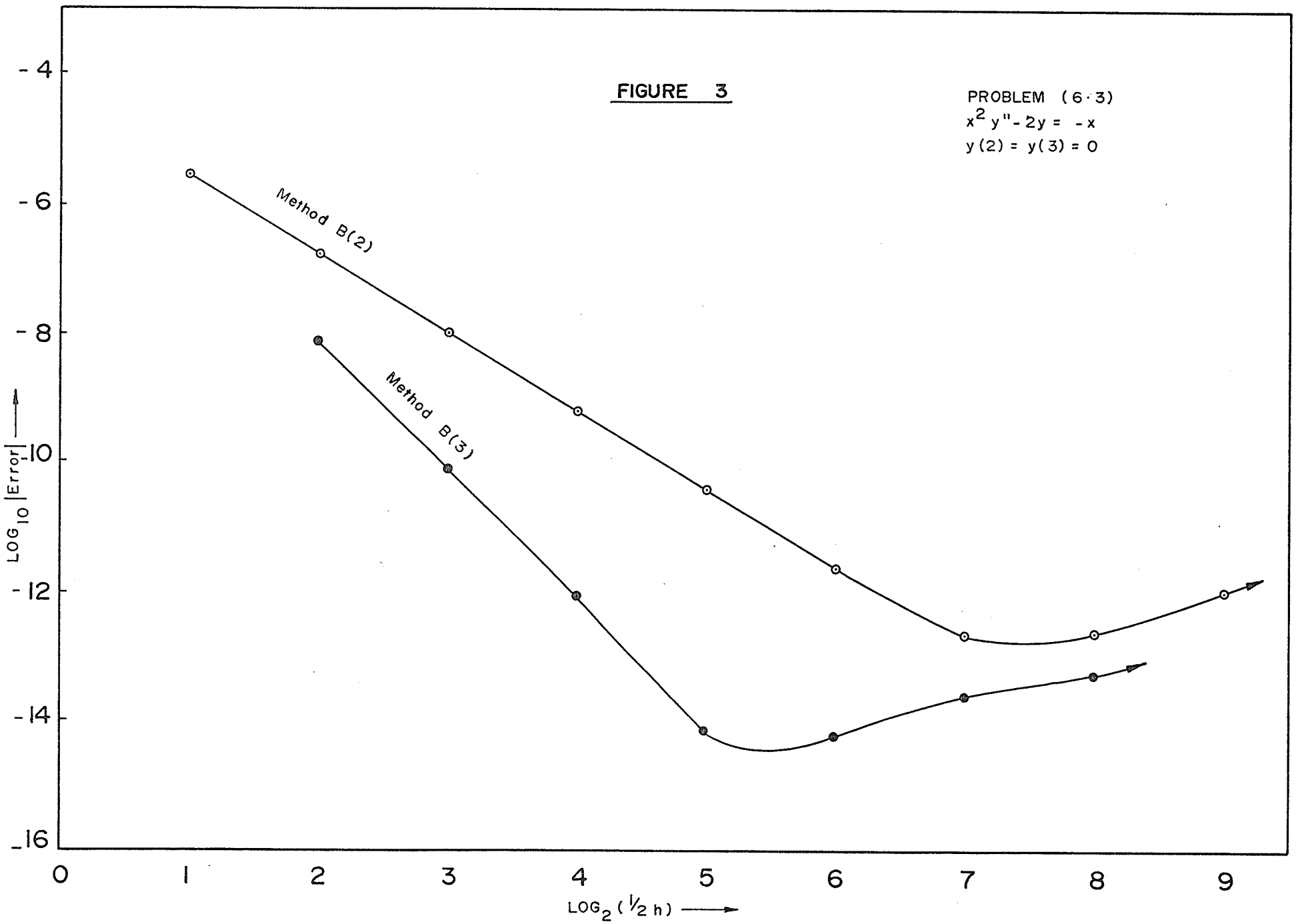
FIGURE 3
PROBLEM (6·3)
$x^2 y'' - 2y = -x$
$y(2) = y(3) = 0$
Method B(2)
Method B(3)
-4
-6
-8
-10
-12
-14
-16
$\text{LOG}_{10}|\text{Error}|$
0
1
2
3
4
5
6
7
8
9
$\text{LOG}_2(\frac{1}{2}h)$

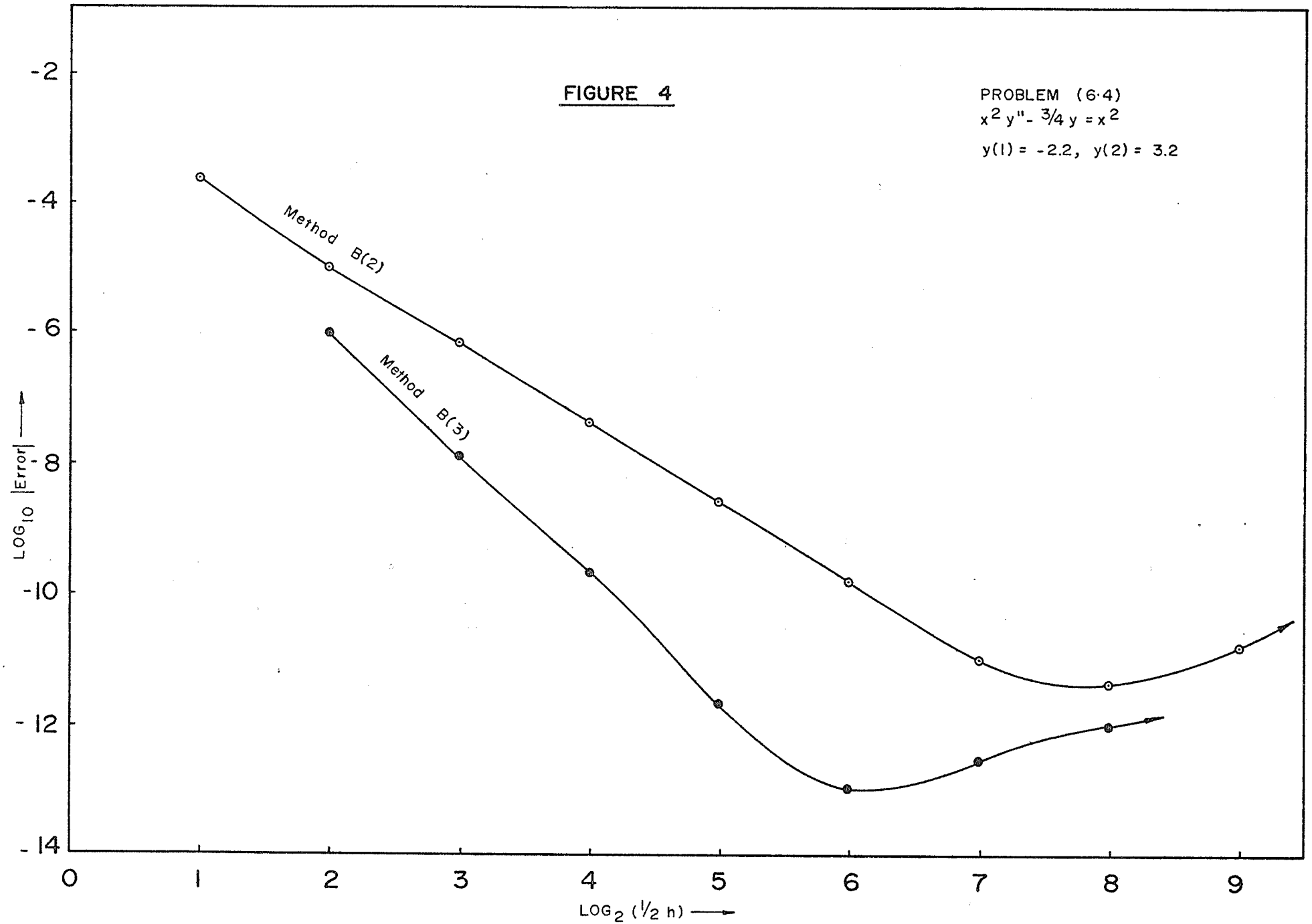
FIGURE 4
PROBLEM (6·4)
$x^2 y'' - 3/4\, y = x^2$
$y(1) = -2.2,\ y(2) = 3.2$
Method B(2)
Method B(3)
-2
-4
-6
-8
-10
-12
-14
0
1
2
3
4
5
6
7
8
9
$\mathrm{LOG}_{10}\ |\mathrm{Error}|$
$\mathrm{LOG}_2\ (1/2\ h)$

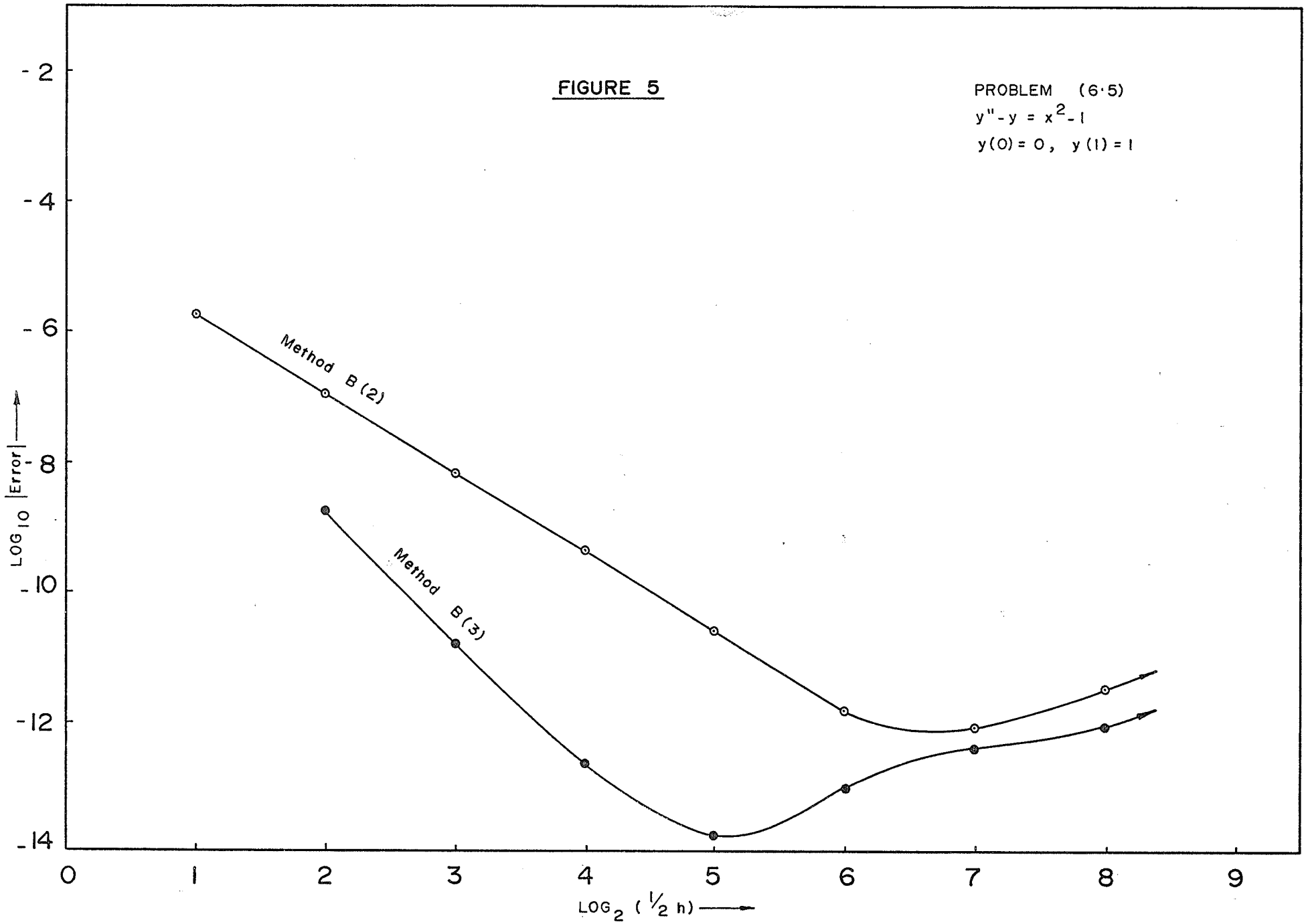
FIGURE 5
PROBLEM (6·5)
y'' - y = x^2 - 1
y(0) = 0, y(1) = 1
Method B(2)
Method B(3)
LOG10 |Error| ⟶
LOG2 (½ h) ⟶
-2
-4
-6
-8
-10
-12
-14
0
1
2
3
4
5
6
7
8
9

4. Non-linear Problems

We shall present in this section numerical results using iterative procedures, based on methods referred to earlier as B-2, B-3, B-4, B-5 and B-6. We shall solve non-linear boundary value problems of class M.

Let us consider the D.E.

$$y''=3y^2/2 \tag{6.7}$$

having the boundary conditions $Y(0)=4$ and $y(1)=1$

with $y(x)=4/(1+x)^2$ (Collatz [1], p. 145).

Let the initial approximations $y^{(0)}$ be defined by

$$y_n^{(0)}=Q(x_n), \quad (n=1,2,\ldots,N)$$

where $Q(x)$ satisfy the linear boundary value problem

$$Q''(x) = 0, \; Q(0)=4, \; Q(b)=4/(1+b)^2, b>0$$

with $Q(x)=4-4(b+4)x/(1+b)^2$.

The criterion for stopping iterations is that the remainder as defined by (5.5) be such that

$$||r(y^{(m)})|| < 1.0E-10, \quad (m=0,1,\ldots)$$

The D.E. (6.7) is solved over the range [0,b] using Newton's method.

We solved the problem over various ranges with b=1,2 and 4. The results are shown in Table 7.

We also solved the following non-linear D.E.:

$$y'' = 0.5(x+y+1)^3 \tag{6.8}$$

with the boundary conditions $y(0)=y(1)=0$, and the analytical solution

$$y(x) = [2/(2-x)] - x - 1.$$

The initial approximations $y_n^{(0)} = Q(x_n)$ are obtained by using

the linear boundary value problem

$$Q''(x) = 0.5(x+1)^3, \quad Q(0) = Q(1) = 0,$$

$$\text{with } Q(x) = [-1-31x+(1+x)^5]/40.$$

The problem was solved for different values of h. The results are presented in Table 8.

From the Tables 7 and 8 it is clear that the numerical results obtained by using B-3 and B-5 compare very favourably with the results obtained by using the other methods. Overall, B-3 emerges as the best method.

We also solved the problems (6.7) and (6.8) with mixed boundary conditions. In Table 9, we present the results in solving the problem (6.7) subject to the boundary conditions $y'(0)-2y(0)=-16$ and $y'(1)+y(1)=0$. The problem was solved in the range (0,1) for different values of h using BG-3. Similarly, in Table 10, the solutions of problem (6.8) with boundary conditions $y'(0)-y(0)=-0.5$ and $y'(1)+y(1)=1$ are given. Here again, the problem was solved in the range (0,1) for different values of h.

TABLE 7

EXPERIMENT WITH PROBLEM (6.7) Viz. $y''=\frac{3}{2}y^2$, $y(0)=4$, $y(b)=4/(1+b)^2$ WITH $y(x)=4/(1+x)^2$.

h=0.1

METHOD	b	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
B-2	1	0	0.206 E 00	
		1	0.760 E-02	0.430 E-01
		2	0.271 E-04	0.106 E-03
		3	0.351 E-09	0.641 E-04
		4	0.111 E-15	0.641 E-04
	2	0	0.219 E 00	
		1	0.195 E-01	0.252 E 00
		2	0.858 E-03	0.128 E-01
		3	0.243 E-05	0.601 E-04
		4	0.196 E-10	0.679 E-04
	4	0	0.229 E 00	
		1	0.321 E-01	0.661 E 00
		2	0.425 E-02	0.141 E 00
		3	0.257 E-03	0.104 E-01
		4	0.161 E-05	0.646 E-04
		5	0.697 E-10	0.682 E-04
B-3	1	0	0.623 E 00	
		1	0.263 E-01	0.430 E-01
		2	0.925 E-04	0.152 E-03
		3	0.119 E-08	0.657 E-05
		4	0.334 E-15	0.657 E-05
	2	0	0.717 E 00	
		1	0.697 E-01	0.252 E 00
		2	0.305 E-02	0.128 E-01
		3	0.861 E-05	0.358 E-04
		4	0.694 E-10	0.623 E-05
	4	0	0.783 E 00	
		1	0.115 E 00	0.661 E 00
		2	0.153 E-01	0.141 E 00
		3	0.923 E-03	0.104 E-01
		4	0.578 E-05	0.679 E-04
		5	0.250 E-09	0.604 E-05
		6	0.275 E-15	0.604 E-05

(Cont'd)

TABLE 7 (Cont'd)

METHOD	b	NO. OF ITERATIONS	||r(Y)||	||e||
B-4	1	0	0.319 E-01	
		1	0.532 E-02	0.430 E-01
		2	0.175 E-04	0.196 E-03
		3	0.229 E-09	0.466 E-04
		4	0.549 E-15	0.466 E-04
	2	0	0.199 E-01	
		1	0.110 E-01	0.252 E 00
		2	0.378 E-03	0.129 E-01
		3	0.102 E-05	0.158 E-03
		4	0.824 E-11	0.122 E-03
	4	0	0.215 E-01	
		1	0.151 E-01	0.661 E 00
		2	0.140 E-02	0.141 E 00
		3	0.764 E-04	0.106 E-01
		4	0.480 E-06	0.215 E-03
		5	0.210 E-10	0.163 E-03
B-5	1	0	0.467 E 00	
		1	0.195 E-01	0.430 E-01
		2	0.681 E-04	0.153 E-03
		3	0.875 E-09	0.909 E-05
		4	0.453 E-15	0.909 E-05
	2	0	0.537 E 00	
		1	0.521 E-01	0.252 E 00
		2	0.228 E-02	0.128 E-01
		3	0.643 E-05	0.370 E-04
		4	0.518 E-10	0.389 E-05
	4	0	0.588 E 00	
		1	0.864 E-01	0.661 E 00
		2	0.115 E-01	0.141 E 00
		3	0.692 E-03	0.104 E-01
		4	0.433 E-05	0.681 E-04
		5	0.187 E-09	0.902 E-05
		6	0.402 E-15	0.902 E-05
B-6	1	0	0.113 E 01	
		1	0.482 E-01	0.430 E-01
		2	0.171 E-03	0.146 E-03
		3	0.220 E-08	0.304 E-04
		4	0.344 E-14	0.304 E-04

(Cont'd)

TABLE 7 (Cont'd)

METHOD	b	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
B-6	2	0	0.130 E 01	
		1	0.126 E 00	0.252 E 00
		2	0.553 E-02	0.128 E-01
		3	0.156 E-04	0.330 E-04
		4	0.126 E-09	0.306 E-04
		5	0.344 E-14	0.306 E-04
	4	0	0.141 E 01	
		1	0.208 E 00	0.661 E 00
		2	0.276 E-01	0.141 E 00
		3	0.166 E-02	0.104 E-01
		4	0.104 E-04	0.674 E-04
		5	0.451 E-09	0.306 E-04
		6	0.250 E-14	0.306 E-04

The value of 'c' for methods B-3, B-4 and B-5 was set at 0.9. The value of 'λ' in the case of B-6[14] was 2.5.

TABLE 8

EXPERIMENT WITH PROBLEM (6.8) Viz. $y''=0.5(x+y+1)^3$, $y(0)=y(1)=0$, WITH $y(x)=[2/(2-x)]-x-1$.

METHOD	h	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
B-2	1/6	0	0.217 E-01	
		1	0.196 E-03	0.403 E-03
		2	0.107 E-07	0.509 E-04
		3	0.529 E-16	0.509 E-04
	1/8	0	0.123 E-01	
		1	0.102 E-03	0.435 E-03
		2	0.612 E-08	0.164 E-04
		3	0.416 E-16	0.164 E-04
	1/10	0	0.802 E-02	
		1	0.663 E-04	0.443 E-03
		2	0.407 E-08	0.681 E-05
		3	0.382 E-16	0.683 E-05
B-3	1/6	0	0.718 E-01	
		1	0.548 E-03	0.440 E-03
		2	0.341 E-07	0.124 E-04
		3	0.156 E-15	0.124 E-04
	1/8	0	0.420 E-01	
		1	0.336 E-03	0.447 E-03
		2	0.207 E-07	0.241 E-05
		3	0.971 E-16	0.242 E-05
	1/10	0	0.280 E-01	
		1	0.227 E-03	0.448 E-03
		2	0.138 E-07	0.675 E-06
		3	0.538 E-16	0.697 E-06
B-4	1/6	0	0.135 E-01	
		1	0.142 E-03	0.463 E-03
		2	0.915 E-08	0.240 E-04
		3	0.376 E-16	0.240 E-04
	1/8	0	0.625 E-02	
		1	0.674 E-04	0.462 E-03
		2	0.439 E-08	0.156 E-04
		3	0.348 E-16	0.156 E-04

(Cont'd)

TABLE 8 (Cont'd)

METHOD	h	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
B-4	1/10	0	0.346 E-02	
		1	0.370 E-04	0.457 E-03
		2	0.241 E-08	0.101 E-04
		3	0.321 E-16	0.100 E-04
B-5	1/6	0	0.522 E-01	
		1	0.386 E-03	0.157 E-02
		2	0.176 E-07	0.163 E-02
		3	0.492 E-15	0.163 E-02
	1/8	0	0.312 E-01	
		1	0.245 E-03	0.445 E-03
		2	0.151 E-07	0.309 E-05
		3	0.625 E-16	0.312 E-05
	1/10	0	0.208 E-01	
		1	0.168 E-03	0.448 E-03
		2	0.101 E-07	0.806 E-06
		3	0.590 E-16	0.832 E-06
B-6	1/6	0	0.134 E 00	
		1	0.101 E-02	0.434 E-03
		2	0.641 E-07	0.329 E-04
		3	0.208 E-15	0.329 E-04
	1/8	0	0.763 E-01	
		1	0.629 E-03	0.445 E-03
		2	0.380 E-07	0.801 E-05
		3	0.194 E-15	0.802 E-05
	1/10	0	0.511 E-01	
		1	0.418 E-03	0.447 E-03
		2	0.255 E-07	0.256 E-05
		3	0.208 E-15	0.257 E-05

The value of 'c' for B-3, B-4, and B-5 was set at 0.9. For B-6, the value of 'λ' was 2.5.

TABLE 9

EXPERIMENT WITH PROBLEM (6.7) Viz. $y''=1.5y^2$ WITH $y'(0)-2y(0)=-16$ AND $y'(1)+y(1)=0$ USING METHOD BG-3.

h	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
1/8	0	0.891 E 00	
	1	0.380 E-01	0.591 E-01
	2	0.274 E-03	0.564 E-03
	3	0.253 E-07	0.410 E-04
	4	0.236 E-15	0.410 E-04
1/16	0	0.451 E 00	
	1	0.102 E-01	0.591 E-01
	2	0.713 E-04	0.558 E-03
	3	0.651 E-08	0.167 E-05
	4	0.668 E-15	0.164 E-05
1/32	0	0.267 E 00	
	1	0.259 E-02	0.591 E-01
	2	0.181 E-04	0.558 E-03
	3	0.164 E-08	0.721 E-07
	4	0.444 E-15	0.496 E-07
1/64	0	0.145 E 00	
	1	0.651 E-03	0.591 E-01
	2	0.453 E-05	0.558 E-03
	3	0.411 E-09	0.577 E-07
	4	0.458 E-15	0.117 E-08

The value of 'c' was set at 0.9.

TABLE 10

EXPERIMENT WITH PROBLEM (6.8) Viz. $y''=0.5(x+y+1)^3$, WITH $y'(0)-y(0)=-0.5$ AND $y'(1)+y(1)=1$ USING METHOD BG-3.

h	NO. OF ITERATIONS	$\|\|r(Y)\|\|$	$\|\|e\|\|$
1/8	0	0.437 E-01	
	1	0.342 E-03	0.879 E-03
	2	0.869 E-07	0.214 E-05
	3	0.937 E-14	0.239 E-05
1/16	0	0.262 E-01	
	1	0.935 E-04	0.882 E-03
	2	0.225 E-07	0.246 E-06
	3	0.241 E-14	0.127 E-06
1/32	0	0.138 E-01	
	1	0.240 E-04	0.883 E-03
	2	0.571 E-08	0.285 E-06
	3	0.626 E-15	0.118 E-07
1/64	0	0.700 E-02	
	1	0.603 E-05	0.883 E-03
	2	0.143 E-08	0.288 E-06
	3	0.173 E-15	0.559 E-08

The value of 'c' was set at 0.9.

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