FINITE AND INFINITE MATRICES AND SOME APPLICATIONS

by

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ABSTRACT

In this thesis, properties of finite as well as infinite matrices are studied. For the finite matrices, primary emphasis is on the class of Λ-d.d. matrices. The results derived are then applied to various branches of applied mathematics such as error analysis for boundary value problems, convergence of iterative methods for linear algebraic systems, and stability of the solutions of linear differential systems. For the infinite matrices, we consider the existence,uniqueness and finite approximations of bounded solutions of infinite algebraic and differential systems. A numerical method for finding the conformal mapping which maps the doubly connected region between a general curvilinear polygon and a circle onto a circular annulus is also derived. In the above, known results are generalized and some new results are established .

TABLE OF CONTENTS

		page
	INTRODUCTION	
I.	PRELIMINARIES CONCERNING FINITE MATRICES	
	1. Irreducible and Diagonally Dominant Matrices	1
	2. M-Matrices	3
	3. Iterative Methods for Algebraic Systems	6
	4. Stability Conditions for Linear Differential systems	10
	5. Boundary Value Problems	11
II.	Λ-d.d. MATRICES	
	1. Introduction	14
	2. Some Results for d.d. L-matrices	14
	3. Λ-d.d. Matrices	17
	4. Bounds for Determinants of Λ-d.d. Matrices	23
	5. Characterization of the Inverses of Λ-d.d. Matrices	25
	6. An Application	29
III.	CONVERGENCE OF ITERATIVE METHODS	
	1. Introduction	31
	2. A Convergence Theorem for J-method	31
	3. Convergence of the General Iterative Methods	34
	4. Bounds for the Spectral radii for Iteration Matrices	37
IV.	STABILITY CONDITIONS FOR LINEAR DIFFERENTIAL SYSTEMS	
	1. Introduction	43
	2. Systems with M-matrices	46
	3. Comparison Theorems	48
	4. Systems with Λ-d.d. Matrices	53
	5. Boundedness of Solutions	56
	6. Concluding Remarks	58

V. LINEAR ALGEBRAIC EQUATIONS IN INFINITE MATRICES

1. Introduction 59
2. Preliminaries 64
3. Main Results 68
4. Examples 72

VI. LINEAR DIFFERENTIAL EQUATIONS IN INFINITE MATRICES

1. Introduction 74
2. Existence and Uniqueness Theorems 78
3. Approximation and Behavior of Solutions 83
4. Inhomogeneous Case 88

VII. INFINITE LINEAR EQUATIONS IN A MAPPING PROBLEM

1. Introduction 99
2. Conformal Mapping Considerations 101
3. The Region 101
4. The Method 103
5. Convergence Considerations 107
6. Numerical Results and Conclusions 107

INTRODUCTION

In recent years, the theory of strictly diagonally dominant, irreducibly diagonally dominant, and diagonally dominant M-matrices have been very rapidly developed [10 25,26,27] . It is interesting to note that, whenever a result is proved for one of the above classes the same can be established for the other two. It is thus quite natural to ask whether there is a class of matrices which contains and shares some of the nice properties common to all of these three classes. Indeed, the class of Λ-d.d. matrices which we will study in Chapter II of this thesis fulfils this goal. In Chapter II, non-singularity of the Λ-d.d. matrices is established. Bounds for the determinants and the entries of the inverses of Λ-d.d. matrices are also given. These bounds generalize those bounds given by Oeder [25] , Ostrowski [26] , and Taussky [41] .The results obtained in Chapter II also simplify discussion of error analysis in numerical solution of boundary value problems for $y''=f(x,y)$ when the resulting matrix is not an M-matrix.

Chapter III is devoted to give a sufficient condition for the convergence of a general iterative method for linear algebraic systems. Bounds for the spectral radii of the iteration matrices are also given. The theorems obtained extend the results of James [16] to the case where the matrix of the system is neither strictly diagonally dominant nor irreducibly diagonally dominant.

Given a complex matrix $A=(a_{ij})$, let $\tilde{A}=(\tilde{a}_{ij})$ be defined by $\tilde{a}_{ii}=|a_{ii}|$; $\tilde{a}_{ij}=-|a_{ij}|$, $i\neq j$. Fan [10] shows that if a complex matrix A and an M-matrix B satisfy $\tilde{A}\geq B$, then $|A^{-1}|\leq B^{-1}$. In Chapter IV , we use Fan's condition ($\tilde{A}\geq B$) to make a comparison of the solutions of two linear differential systems. The result is then used to establish the stability conditions for the systems .

In the finite matrix theory, determinants play a fundamental role ; but their value is lost in the theory of infinite matrices. Hence it might not be expected that we could have a simple extension of the theory of finite matrices to the theory of infinite matrices. Due to existence, convergence and other difficulties, the treatment of infinite matrices differs radically from that of finite matrices . Consequently, the types of problem solved by aid of infinite matrices are of a completely different character from those solved by use of finite matrices. For analysis of infinite matrices and references to earlier literature on the related topics, we refer to Cooke [8] .

In their recent papers, Shivakumar and Wong [38] , Shivakumar [39], and McClure and Wong [21] give, under the assumption that the infinite matrix A is strictly diagonally

Chapter I

PRELIMINARIES CONCERNING FINITE MATRICES

1. Irreducible and Diagonally Dominant Matrices

Let $A=(a_{ij})$ be an nxn matrix and $N=\{1,2,\ldots,n\}$. We define σ_i and Λ_A as follows :

(1.1) $$\sigma_i |a_{ii}| = \sum_{j\neq i}^{n} |a_{ij}| \quad , \quad i=1,2,\ldots,n ,$$

(1.2) $$\Lambda_A = \{i \in N \mid \sigma_i < 1\} \quad .$$

Definition 1.1

A matrix A is diagonally dominant (d.d.) if and only if

(1.3) $$\sigma_i \leq 1 \quad , \quad \text{for all } i\in N .$$

A d.d. matrix with $\Lambda_A \neq \emptyset$ is called weakly diagonally dominant (w.d.d.). If $\Lambda_A=N$, then A is said to be strictly diagonally dominant (s.d.d.) .

Definition 1.2

A matrix $A=(a_{ij})$ is irreducible if and only if, given any two integers $i,j\in N$, there exists a sequence of non-zero entries of A of the form $a_{ii_1}, a_{i_1i_2}, \ldots, a_{i_sj}$. A is irreducibly diagonally dominant (i.d.d.) if and only if it is irreducible and w.d.d. .

Theorem 1.3 [26,27,41,43]

If A is i.d.d., then none of the diagonal entries of A vanish and the determinant, det A , is non-zero .

dominant, sets of sufficient conditions for the existence, uniqueness, and finite approximation of a bounded solution to the linear algebraic equations involving the infinite matrix A . In Chapter V , some of our results improve the existing ones .

Linear differential equations involving infinite matrices are considered in Chapter VI . We establish the existence, uniqueness and approximation theorems by a 'classical analysis' approach, i.e., by using uniform convergence properties and changes of orders of summation to justify the extension of finite dimensional solutions based on the infinite series for the exponential of the matrix involved. Shaw [36,37] also used this approach, however , he required some additional conditions .

Solution of a large number of problems in modern technology hinges critically on the possibility of conformal transformation of a doubly connected region onto a circular annulus [15,20,24,46] . In Chapter VII , we discuss the conformal mapping of the doubly connected region between a curvilinear polygon of n 'sides' (n=2,3,...) and a circle onto an annulus. The problem of mapping is shown to be equivalent to the solution of an infinite linear algebraic system. The method is illustrated by numerical examples.

<u>Theorem 1.4</u> [26,27,41,43]

If A is s.d.d., then $\det A \neq 0$.

Theorems 1.3 and 1.4 have been improved in many ways. For example, each of the following is known to be a sufficient condition for non-vanishing of determinant of A:

(1.4) $\sigma_i \sigma_j < 1$, $i \neq j$; $i,j=1,2,\ldots,n$. (Taussky [41])

(1.5) $k_i m_i < 1$, $i=1,2,\ldots,n$,

where

$$m_i |a_{ii}| = \max_{j \neq i} |a_{ij}|$$

and k_i are positive numbers satisfying

$$\sum_{j=1}^{n} (1+k_j)^{-1} \leq 1 .$$ (Ostrowski [48])

(1.6) $\sigma_i^{\alpha} \gamma_i^{1-\alpha} < 1$, $i=1,2,\ldots,n$; $0 \leq \alpha \leq 1$,

where

$$\gamma_i |a_{ii}| = \sum_{j \neq i}^{n} |a_{ji}| .$$ (Ostrowski [27])

For bounds for the determinants of d.d. matrices, we refer to [2,6,25,26,27,32,41]. We mention the following lower bounds for $|\det A|$, where the matrix $A=(a_{ij})$ is a s.d.d. matrix :

(1.7) $M_1 = |a_{nn}| \prod_{i=1}^{n-1} |a_{ii}| (1-\bar{\delta}_i)$, (Taussky [41]) and

$$(1.8) \qquad M_2 = |a_{11}| \prod_{i=2}^{n} |a_{ii}| (1-\underline{\delta}_i) \quad , \quad \text{(Oeder [25])}$$

where

$$\bar{\delta}_i \; |a_{ii}| = \sum_{j=i+1}^{n} |a_{ij}| \quad , \; i=1,2,\ldots,n-1 \; ,$$

$$\underline{\delta}_i \; |a_{ii}| = \sum_{j=1}^{i-1} |a_{ij}| \quad , \; i=2,3,\ldots,n \; .$$

For the s.d.d. matrix A , Ostrowski[26]gives a characterization of the inverse $A^{-1}=(b_{ij})$ of A , namely

$$(1.9) \qquad |b_{ij}| \le \sigma_i \; |b_{jj}| \quad , \; i\neq j \; ; \; i,j=1,2,\ldots,n \; .$$

Let A_{ij} be the cofactor of the (i,j)-entry of A .Then (1.9) if and only if

$$(1.10) \qquad |A_{ij}| \le \sigma_j \; |A_{ii}| \quad , \; i\neq j \; ; \; i,j=1,2,\ldots,n \; .$$

2.M-Matrices

A real matrix $A=(a_{ij})$ is an L-matrix if

$$a_{ii} > 0 \quad , \quad i=1,2,\ldots,n \; ,$$

and

$$a_{ij} \le 0 \quad , i\neq j \; ; \; i,j=1,2,\ldots,n \; .$$

Given any complex matrix $A=(a_{ij})$ with non-zero diagonal entries, we define

$$(2.1) \qquad \tilde{a}_{ij} = \begin{cases} |a_{ij}| , & \text{if } i=j , \\ -|a_{ij}| , & \text{if } i \neq j . \end{cases}$$

Then $\tilde{A}=(\tilde{a}_{ij})$ is an L-matrix . We shall call $\tilde{A}$ the L-matrix induced by A .(Let $A=(a_{ij})$,$B=(b_{ij})$.We say $A \geq B$ if $a_{ij} \geq b_{ij}$,$\forall i,j$).

Definition 2.1

An L-matrix $A=(a_{ij})$ is an M-matrix if and only if A is non-singular and $A^{-1} \geq 0$.

The following are equivalent conditions for M-matrices [10,43].

Theorem 2.2

An L-matrix A is an M-matrix if and only if the determinant of each of the principal submatrices of A is positive .

Theorem 2.3

An L-matrix A is an M-matrix if and only if for any vector v , $Av \geq 0$ implies $v \geq 0$.

A symmetric real matrix A is positive definite if and only if for all vectors $v \neq 0$, we have $v'Av > 0$, where v' is the transpose of v . It is well known that [45,p.29] a symmetric real matrix is positive definite if and only if the determinant of each of the principal submatrices of A is positive . From Theorem 2.2 , we have the following

Theorem 2.4

Every positive definite L-matrix is an M-matrix.

Every symmetric M-matrix is positive definite .

Ky Fan [10] has given the following remarkable result concerning bounds for determinants and inverse entries of M-matrices :

Theorem 2.5

If a complex matrix $A=(a_{ij})$ and an M-matrix $B=(b_{ij})$ satisfy

$$b_{ii} \leq |a_{ii}| \ , \quad i=1,2,\ldots,n \ ,$$

and

$$|a_{ij}| \leq |b_{ij}| \ , \quad i \neq j \ ; \ i,j=1,2,\ldots,n \ ,$$

then

$$|\det A| \geq \det B$$

and

$$\left|\frac{A_{ij}}{\det A}\right| \leq \left|\frac{B_{ij}}{\det B}\right| \ , \quad i,j=1,2,\ldots,n \ ,$$

where A_{ij} and B_{ij} are cofactors in A and B respectively .

As immediate consequences of Theorem 2.5 , we have the following .

Corollary 2.6

Let A , B be M-matrices such that $A \geq B$ .Then $A^{-1} \leq B^{-1}$.

Corollary 2.7

Let A be a complex matrix , If $\tilde{A}$ is an M-matrix, then

(2.2) $$|\det A| \geq \det \tilde{A} > 0$$

and

(2.3) $$\left|\frac{A_{ij}}{\det A}\right| \leq \frac{\tilde{A}_{ij}}{\det \tilde{A}} \quad , \quad i,j=1,2,\ldots,n \; .$$

3. Iterative Methods for Algebraic Systems

Consider the system

(3.1) $$Ax = b \quad ,$$

where $A=(a_{ij})$ is an nxn matrix . Cramer's Rule states that if $\det A \neq 0$, then the system (3.1) has a unique solution given by

(3.2) $$x_j = \frac{\Delta_j}{\det A} \quad , \quad j=1,2,\ldots,n \; ,$$

where Δ_j is the determinant of the matrix obtained from A by replacing the j^{th} column of A by b .

We assume throughout that $\det A \neq 0$ and the diagonal entries of A are non-zero . Write A as

(3.3) $$A = D(I+L+U) \quad ,$$

where I is the identity matrix and D=diag A , and L ,U are strictly lower and upper triangular matrices respectively. We can then write (3.1) as one of the following :

(3.4) $$x = -(L+U)x + D^{-1}b \quad ,$$

(3.5) $$x = -(I+L)^{-1}Ux + (I+L)^{-1}D^{-1}b \quad ,$$

(3.6) $$x = \{(1-\omega)I - \omega L - \omega U\}x + \omega D^{-1}b \quad ,$$

$$(3.7) \qquad x = (I+\omega L)^{-1}\{(1-\omega)I - \omega U\}\, x + \omega(I+\omega L)^{-1}D^{-1}b \quad ,$$

where ω is any real number. Equations (3.4),(3.5),(3.6) and (3.7) give respectively the following iterative methods :

The Jacobi method (J) :

$$(3.8) \qquad x^{(m+1)} = -(L+U)x^{(m)} + D^{-1}b \quad ,$$

The Gauss-Seidel method (GS) :

$$(3.9) \qquad x^{(m+1)} = -(I+L)^{-1}Ux^{(m)} + (I+L)^{-1}D^{-1}b \quad ,$$

The Simultaneous Overrelaxation method (JOR) :

$$(3.10) \qquad x^{(m+1)} = \{(1-\omega)I - \omega L - \omega U\}\, x^{(m)} + \omega D^{-1}b \quad ,$$

The Successive Overrelaxation method (SOR) :

$$(3.11) \qquad x^{(m+1)} = (I+\omega L)^{-1}\{(1-\omega)I-\omega U\}\, x^{(m)} + \omega(I+\omega L)^{-1}D^{-1}b \;,$$

for $m \geq 0$,where $x^{(0)}$ is initial estimate of the unique solution x of (3.1) .

The general iterative method for (3.1) has the form

$$(3.12) \qquad x^{(m+1)} = Mx^{(m)} + g \quad , \quad m \geq 0 \quad ,$$

where $x^{(0)}$ is the initial guess solution of (3.1) . The matrix M is called the iteration matrix for the iterative method.

To the iterative method (3.12) , we associate the error vectors $e^{(m)}$ defined by

$$(3.13) \qquad e^{(m)} = x^{(m)} - x \;, \quad m \geq 0$$

where x is the unique vector solution of (3.1) . The iterative method converges if and only if $e^{(m)} \to 0$ as $m \to \infty$.

From (3.12) , since x is the solution of (3.1) , we have $x=Mx+g$, and thus

$$(3.14) \qquad e^{(m+1)} = x^{(m+1)} - x$$

$$= (Mx^{(m)}+g) - (Mx+g)$$

$$= M(x^{(m)} - x) = Me^{(m)} \quad ,$$

on using (3.12) and (3.13) . Hence

$$(3.15) \qquad e^{(m)} = M^m e^{(0)} \quad ,$$

and the iterative method (3.12) converges for all $e^{(0)}$ if and only if $M^m \to 0$ as $m \to \infty$.

Definition 3.1

The spectral radius of a matrix A is defined as

$$(3.16) \qquad \rho(A) = \max_{\lambda \varepsilon S_A} |\lambda| \quad ,$$

where S_A is the set of all eigenvalues of A .

It is well known [47,p.35] that $A^m \to 0$ as $m \to \infty$ if and only if $\rho(A) < 1$. Hence we have

Theorem 3.2

The iterative method (3.12) converges if and only if $\rho(M) < 1$.

Consider the iterative method (3.12) where the iteration matrix M is given by

$$(3.17) \qquad M(\alpha,\Omega)=(I+\alpha\Omega L)^{-1}\{(I-\Omega)-(1-\alpha)\Omega L-\Omega U\} \quad ,$$

where $\Omega=\text{diag}(\omega_1,\omega_2,\ldots,\omega_n)$ is a diagonal relaxation matrix and α is a real number . Note that $M(0,I)$,$M(1,I)$,$M(o,\omega I)$ and $M(1,\omega I)$ are respectively the J,GS,JOR and SOR iteration matrices.

Denoting by ℓ_i and u_i the sums of the absolute values of the entries in the i^{th} row of the triangular matrices $L=(\ell_{ij})$ and $U=(u_{ij})$,James [16] has proved the following theorems .

Theorem 3.3

If the matrix A of (3.1) is strictly diagonally dominant, then the iterative method defined in (3.17) converges for $0 \leq \alpha \leq 1$ subject to the sufficient conditions

$$0 < \omega_i < \frac{2}{1+\ell_i+u_i} \quad , \quad i=1,2,\ldots,n.$$

Theorem 3.4

If the matrix A of (3.1) is irreducibly diagonally dominant,then the iterative method defined in (3.17) converges for $\alpha=0$, $\frac{1}{2} < \alpha \leq 1$ subject to the sufficient conditions

$$0 < \omega_i \leq \frac{2}{1+\ell_i+u_i} \quad ,i=1,2,\ldots,n \ ,$$

provided the strict inequality holds at least for one row for which $\ell_i+u_i < 1$.

In [16] , bounds on the spectral radius $\rho(M)$ of the iteration matrix M are also given :

$$(3.18) \qquad \rho(M) \le \max_i \frac{|1-\omega_i|+|\omega_i|(|1-\alpha|\ell_i+u_i)}{1-|\omega_i\alpha|\ell_i} \quad ,$$

provided $|\omega_i\alpha|\ell_i < 1$ for all i ; and

$$(3.19) \qquad \rho(M) \ge \min_i \frac{|1-\omega_i|-|\omega_i|(|1-\alpha|\ell_i+u_i)}{1+|\omega_i\alpha|\ell_i} \quad .$$

4.Stability conditions for Linear Differential System

Recently, Kahane [17] considered the system of n linear differential equations

$$(4.1) \qquad \dot{y}(t) = A(t)y(t) \quad , \quad y(0)=c \quad , \; t \ge 0 \; .$$

and proved that the solution of (4.1) tends to zero as $t \to \infty$ if the matrix $A(t)=(a_{ij}(t))$ is strictly column diagonally dominant with negative diagonal entries,i.e.,

$$(4.2) \qquad -a_{ii}(t) > \sum_{j\neq i}^{n} |a_{ji}(t)| \quad , \; i=1,2,\ldots,n, \quad (t \ge 0) \; .$$

Since many systems of the form (4.1) can be transformed by certain transformations,which do not alter the character of the solutions as $t \to \infty$, into systems with constant coefficients (e.g.,the system with periodic coefficients A(t) can always be transformed to a system with constant coefficients [12,p.119]) , we shall mainly be concerned with the system

$$(4.3) \qquad \dot{x}(t) = Ax(t) \quad , \quad x(0)=c \quad , \quad t \ge 0 \quad ,$$

where A is a constant matrix . For earlier literature concerning the stability of the solution of (4.3) we refer to [4,12] .

It is well known [12] that a necessary and sufficient condition that the solution of the system (4.3) , regardless of the value c , approach zero as $t \to \infty$,is that all the eigenvalues of A have negative real parts . The following theorem thus provides a condition for the stability of the system with matrix A satisfying the hypothesis.

Theorem 4.1[12,p.74]

The real parts of all the eigenvalues of a real matrix $A=(a_{ij})$ with $a_{ij} \geq 0$, $i \neq j$, are negative if and only if

$$(4.4) \qquad a_{11}<0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, (-1)^n \det A>0 .$$

5. Boundary Value Problems

A boundary value problem is said to be of class M if it is of the form [14,p.347]

$$(5.1) \qquad y'' = f(x,y) \ , \ y(a)=y_a \ , \ y(b)=y_b \ , \ -\infty<a<b<\infty \ ,$$

where y_a and y_b are arbitrary constants and the function $f(x,y)$ is defined and continuous in the strip

$$(5.2) \qquad S = \{(x,y) \mid x\varepsilon[a,b] \ , \ -\infty<y<\infty\} .$$

Furthermore, the partial derivative $\frac{\partial f}{\partial y}$ is assumed to be continuous, bounded and non-negative in S . It is well known that the boundary value problem of class M has a unique solution .

For the numerical approximation of the solution $y(x)$ of the problem (5.1) , we set up a finite set of grid points

(5.3) $x_i = a+ih \quad , \quad h=\frac{b-a}{n+1} \quad , \quad i=0,1,\ldots,n+1 \ ,$

n being an appropriate positive integer . A scheme is then designed for the determination of the numbers y_i which approximate closely to $y(x_i)$. A convenient way to obtain such a scheme is to demand that y_i satisfy a system of difference equations of the form

(5.4) $$\sum_{j=0}^{k} \alpha_j y_{i+j} = h^2 \sum_{j=0}^{k} \beta_j y''_{i+j} \quad , \quad i=0,1,\ldots,n-k+1 \quad ,$$

where

(5.5) $$y''_j = f(x_j,y_j) \quad , \quad \alpha_k \neq 0 \ , \ |\alpha_0|+|\beta_0| \neq 0 \quad .$$

We normalize (5.4) by choosing $\alpha_k=1$. The positive integer k is called the order of the difference equations (5.4) .

System (5.4) is a system of n-k+2 equations involving the n unknowns $y_1,y_2,\ldots,y_n$; y_0 and y_{n+1} being determined by the boundary conditions . If k> 2 , then we have less than n equations in n unknowns . This difficulty is overcome by suitably modifying the difference equations near the boundaries (ss e.g.[42]) .

If we now approximate the differential equation (5.1) by using difference equations (5.4) , we get a system of n equations in n unknowns . If f(x,y) is linear in y , then the system will be of the form

(5.6) $$AY = b \ ,$$

where A is a constant matrix of band structure , and Y is the unknown vector with entries y_i .We may assume that the true solutions $y(x_i)$ satisfy

(5.7) $$\sum_{j=0}^{k} \alpha_j y(x_{i+j}) = h^2 \sum_{j=0}^{k} \beta_j y''(x_{i+j}) + t_{i+1} \quad ,i=0,1,..,n-1.$$

The discretization error $E=(e_i)$,$e_i=y(x_i) - y_i$,and the truncation error $T=(t_i)$ are then related by

(5.8) $$AE = T .$$

If A is non-singular , we have

(5.9) $$E = A^{-1}T .$$

Any further analysis of the discretization error E now depends on the properties of A^{-1} .

Chapter II

Λ-d.d. MATRICES

1. Introduction

s.d.d. , i.d.d. and d.d.M-matrices play an important role in numerical solutions of certain boundary value problems [14,42] ,convergence of iterative methods for algebraic systems [43,47] , existence and uniqueness of bounded solutions for infinite algebraic systems [38,39] , and many other branchs of applied mathematics . In this chapter , we will study the properties of a class of matrices, namely , 'Λ-d.d. matrices' , which includes s.d.d. , i.d.d. and d.d.M-matrices as subclasses .

2. Some Results for d.d.L-matrices

We recall that an L-matrix is a real matrix with positive diagonal entries and non-positive off diagonal entries . For the d.d.L-matrices , we establish the following:

Theorem 2.1 [D]

Let $A=(a_{ij})$ be a d.d.L-matrix . Let $A_{ij}^{(h)}$ be the cofactor of the entry a_{ij} in

$$(2.1) \qquad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1h} \\ \vdots & & & \\ a_{h1} & a_{h2} & \cdots & a_{hh} \end{pmatrix} , \quad h=2,3,\ldots,n.$$

Then

$$(2.2) \qquad 0 \leq A_{ij}^{(h)} \leq A_{ii}^{(h)} \quad , \quad i\neq j \; ;i,j=1,2,..,n;h=2,3,..,n.$$

Proof:

The theorem is valid for h=2 . Assume the validity of the theorem for h<n .Since A is d.d.L-matrix , we have

(2.3) $$-a_{jn} \leq \sum_{i=1}^{n-1} a_{ji} .$$

The hypothesis of induction shows that the $A_{jk}^{(n-1)}$ are non-negative . Multiplying both sides of (2.3) by $A_{jk}^{(n-1)}$ and summing for $j=1,2,\ldots,n-1$,we obtain

$$\sum_{j=1}^{n-1} (-a_{jn}) A_{jk}^{(n-1)} \leq \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} a_{ji} A_{jk}^{(n-1)}$$

$$= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ji} A_{jk}^{(n-1)}$$

$$= \sum_{j=1}^{n-1} a_{jk} A_{jk}^{(n-1)}$$

$$= A_{nn}^{(n)} .$$

We can also verify that

$$A_{nk}^{(n)} = \sum_{j=1}^{n-1} (-a_{jn}) A_{jk}^{(n-1)} \geq 0 .$$

Hence

$$0 \leq A_{nk}^{(n)} \leq A_{nn}^{(n)} .$$

This proves that the cofactors in the n^{th} row are non-negative. For general index i , we can interchange the i^{th} , n^{th} rows and also i^{th} , n^{th} column . This completes the proof .

Corollary 2.2 [D]

Let $A=(a_{ij})$ be a d.d. L-matrix . Let $A[i,a]$ be the matrix obtained from A by replacing the entry a_{ii} by an arbitrary non-positive number a .Then

(2.4) $$\det A[i,a] \leq 0 .$$

Proof:

We have

$$\det A[i,a] = aA_{ii} + \sum_{j\neq i}^{n} a_{ij}A_{ij} \leq 0$$

since A_{ij} are non-negative by Theorem2.1 , and a, a_{ij}, $i\neq j$; are non-positive .This completes the proof .

<u>Theorem 2.3</u> [D]

Let $A=(a_{ij})$ be a d.d. L-matrix . Then

(2.5) $$0 \leq a_{nn}\prod_{i=1}^{n-1} a_{ii}(1-\bar{\delta}_i) \leq \det A \leq \prod_{i=1}^{n} a_{ii} ,$$

where $\bar{\delta}_i$ are given by

$$\bar{\delta}_i \; a_{ii} = \sum_{j=i+1}^{n} |a_{ij}| , \quad i=1,2,\ldots,n-1 .$$

<u>Proof</u>:

Observe that

$$\det A = a_{11}(1-\bar{\delta}_1)\det A(1) + \det A\{1\} ,$$

where A(i,j,..,k) denotes the matrix obtained from A with i-,j-,...,k- rows and columns deleted , and A{i} is the matrix A with entry a_{ii} replaced by $\sum_{j=i+1}^{n} |a_{ij}|$. From

Theorem 2.1 , we have $\det A(1) \geq 0$ and $\det A\{1\} \geq 0$, hence

$$\det A \geq a_{11}(1-\bar{\delta}_1)\det A(1) \geq 0 .$$

Similarly , we have

$$\det A(1) \geq a_{22}(1-\bar{\delta}_2)\det A(1,2) \geq 0 ,$$

$$\vdots$$

$$\det A(1,2,\ldots,n-1) \geq a_{nn} \geq 0 .$$

Hence

$$\det A \geq a_{nn} \prod_{i=1}^{n-1} a_{ii}(1-\bar{\delta}_i) \geq 0 .$$

On the other hand,

$$\det A = a_{11}\det A(1) + \det A[1,0] \quad ,$$

where $A[1,0]$ is defined as in Corollary 2.2 . From the above corollary , we have $\det A[1,0] \leq 0$, hence

$$\det A \leq a_{11}\det A(1) \quad .$$

Similarly , we have

$$\det A(1) \leq a_{22}\det A(1,2) \quad ,$$

$$\vdots$$

$$\det A(1,2,\ldots,n-1) \leq a_{nn} \quad .$$

Hence

$$\det A \leq \prod_{i=1}^{n} a_{ii} \quad .$$

This completes the proof .

3. Λ-d.d. Matrices

Given an nxn matrix $A=(a_{ij})$, let σ_i and Λ_A be defined as in (I1.1) and (I1.2) .

Definition 3.1

An nxn matrix $A=(a_{ij})$ is an Λ-d.d. matrix if and only if it is w.d.d. , i.e.,

$$(3.1) \qquad \begin{cases} 0 \leq \sigma_i \leq 1 \quad , \quad i=1,2,\ldots,n \ , \\ \Lambda_A \neq \emptyset \quad ; \end{cases}$$

and satisfying

(H) for each $j \notin \Lambda_A$ there is a sequence of non-zero entries of A of the form $a_{ji_1}, a_{i_1 i_2}, \ldots, a_{i_s i}$ with $i \in \Lambda_A$.

Clearly , the class of Λ-d.d. matrices contains the class of s.d.d. and the class of i.d.d. matrices .At the end of this section , we will show that every d.d. M-matrix is Λ-d.d. . We now establish the non-singularity of Λ-d.d. matrices .

Theorem 3.2 [A]

Every Λ-d.d. matrix A is non-singular .

To prove Theorem 3.2 , we need only to show that the L-matrix $\tilde{A}$ induced by A is an M-matrix (Corollary I2.7).Thus we shall show

Theorem 3.3 [A]

If $A=(a_{ij})$ is an Λ-d.d. matrix , then the L-matrix $\tilde{A}$ induced by A is an M-matrix .

We need the following lemma and results to prove the above theorem .

Lemma 3.4 [A]

Let $A=(a_{ij})$ be an Λ-d.d. matrix . Then for any non-empty subset L of $N=\{1,2,...,n\}$ such that $L \cap \Lambda_A = \emptyset$,there is a non-zero entry a_{ij} with $i \epsilon L$ and $j \notin L$.

Proof:

Let L be a non-empty subset of N such that $L \cap \Lambda_A = \emptyset$. Choose $i_1 \epsilon L$, then $i_1 \notin \Lambda_A$ and ,hence, there is a sequence of non-zero entries of A of the form $a_{i_1 i_2}, a_{i_2 i_3}, ..., a_{i_{s-1} i_s}$ for some $i_s \epsilon \Lambda_A$. Let r be the first integer such that $i_r \notin L$ and note that $2 \le r \le s$ since $i_1 \epsilon L$ and $i_s \notin L$ .Then $a_{i_{r-1} i_r} \neq 0$ with $i_{r-1} \epsilon L$ and $i_r \notin L$ .This proves Lemma 3.4 .

Corollary 3.5 [A]

Let $A=(a_{ij})$ be an Λ-d.d. matrix . If $\Lambda_A=\{i_1,i_2,..,i_k\}$, then there is a permutation $(i_1i_2...i_n)$ such that ,for each $j=k+1,...,n$, $a_{i_ji_\ell}\neq 0$ for some $\ell<j$.

Proof:

If $\Lambda_A=N$, then there is nothing to prove .Suppose $\Lambda_A\neq N$,then $L_1=N-\Lambda_A$ is non-empty and $L_1\cap\Lambda_A=\emptyset$ .Hence , by Lemma 3.4 , there are an $i_{k+1}\varepsilon L_1$ and $j\notin L_1$ such that $a_{i_{k+1}j}\neq 0$. Since $j\notin L_1$,we have $j\varepsilon N-L_1=\Lambda_A$ .Hence $j=i_\ell$ for some $1\leq\ell\leq k$.

Let $L_2=L_1-\{i_{k+1}\}$. If $L_2=\emptyset$,then the proof is completed. Suppose $L_2\neq\emptyset$ .Obviously $L_2\cap\Lambda_A=\emptyset$ .Again ,by Lemma 3.4 ,there are $i_{k+2}\varepsilon L_2$ and $j\notin L_2$ such that $a_{i_{k+2}j}\neq 0$. Since $j\notin L_2$,we have $j\varepsilon\Lambda_A\cup\{i_{k+1}\}$. Hence $j=i_\ell$ for some $1\leq\ell\leq k+1$ .The corollary is proved by repeating the above process until $L_p=L_{p-1}-\{i_{k+p-1}\}=\emptyset$.

Corollary 3.6 [A]

Let $A=(a_{ij})$ be an Λ-d.d. matrix .If $\Lambda_A=\{i_1,i_2,..,i_k\}$, then there is a permutation $(i_1i_2...i_n)$ such that

$$(3.2)\qquad \delta_j\left|a_{i_ji_j}\right|=\sum_{\ell=j+1}^{n}\left|a_{i_ji_\ell}\right| \quad ,0\leq\delta_j<1\ ,j=1,2,...,n-1.$$

Proof:

From Corollary 3.5 , there is a permutation $(i_1i_2..i_n)$ of N such that , for each $j=k+1,...,n$, $a_{i_ji_\ell}\neq 0$ for some $\ell<j$. Notice that if $i_j\varepsilon\Lambda_A$,then