THE UNIVERSITY OF MANITOBA

CONTRIBUTIONS TO COMBINATORIAL LATTICE THEORY

by

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0. PROLOGUE

Combinatorics today connotes much more than puzzle-solving or code-cracking or, even, that tenuous description of it as the study of "arrangement" and "enumeration". Indeed, current mathematical literature suggests that the arsenal of mathematical techniques is now equipped with what is heuristically called a "combinatorial" argument. In this context the adjective "combinatorial" has come more to refer to the "flavour" or "style" of the argument rather than to its particular formalism.

But this is not at all surprising since the term "combinatorial" is in its popular usage an aesthetic description, a description based on a mathematician's artistic judgement.

In this spirit, we shall not give an explicit formulation of *combinatorial lattice theory*; we shall, instead, let the contents speak as a heuristic illustration of it. This may be weak philosophy; it should not hurt the mathematics.

1. ARITHMETIC OF LATTICE THEORY

The unique factorization of any positive integer into primes (up to permutation of the components) is for good reason called the *fundamental theorem of arithmetic*. In lattice theory there is an important analogue of this theorem in which the role of prime numbers is played by *irreducible* elements. To clarify this analogy we shall need some terminology.

An element x in a lattice L is *join-reducible* (*meet-reducible*) *in* L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z$ $(x = y \wedge z)$; x is *join-irreducible* (*meet-irreducible*) *in* L if it is not join-reducible (meet-reducible) in L; x is *doubly irreducible in* L if it is both join- and meet-irreducible in L. Let J(L), M(L), and Irr(L) denote the set of all join-irreducible elements in L, meet-irreducible elements in L, and doubly irreducible elements in L, respectively, and $\ell(L)$ the *length* of L, that is, the order of a maximum-sized chain in L minus one. The join (meet) $\bigvee A$ $(\bigwedge A)$ of a subset A of L which exists in L is *irredundant* if, for every non-empty proper subset A' of A whose join (meet) exists in L, $\bigvee A' < \bigvee A$ $(\bigwedge A' > \bigwedge A)$. Finally, for $x, y \in L$, x *covers* y ($x \succ y$ or $y \prec x$) *in* L if $x > y$ and $x \geq z > y$ implies $x = z$, for every $z \in L$.

The theme of *irreducible* elements in lattice theory is really the underlying one in all our work here; in fact, this is, up to the present, the basic theme of combinatorial lattice theory.

The analogue of the fundamental theorem of arithmetic is contained in the following simple and elementary result.

THEOREM 0. *Every element in a lattice of finite length can be represented as a finite irredundant join (meet) of join-irreducible (meet-irreducible) elements.*

This "factorization" result is really part of the folklore of lattice theory. Over the past four decades the theme of irreducible elements in combinatorial lattice theory has been developed primarily in the remarkable work of R. P. DILWORTH (cf. [8]-[14]).

Arithmetic of Distributive Lattices. Within lattice theory the best understood class of lattices is the class of *distributive* lattices and, from the standpoint of a "factorization" result, this is not at all surprising.

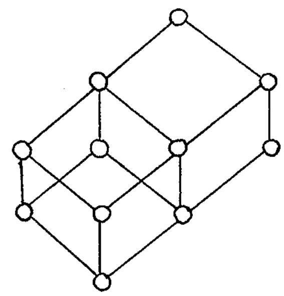
THEOREM 1 (G. BIRKHOFF [6]). *Every element in a finite distributive lattice has a unique irredundant representation as a join (meet) of join-irreducible (meet-irreducible) elements.*

The extent to which a converse of Theorem 1 holds was established by R. P. DILWORTH [8].

THEOREM 2 (R. P. DILWORTH [8], cf. [13], [14]). *Let* L *be a lattice of finite length. Every element of* L *has a unique finite irredundant representation as a join (meet) of join-irreducible (meet-irreducible) elements if and only if, for every* $a \in L$, *the interval* *sublattice* $[\bigwedge(b \in L|a \succ b), a]$ ($[a, \bigvee(b \in L|b \succ a)]$) *is distributive.*

If, in a lattice L of finite length, both irredundant join and meet representations by irreducibles are unique, then L is distributive; however, if only one of these conditions holds, then the lattice is not even modular [8] (see, for example, the lattice of Figure 1).

Figure 1



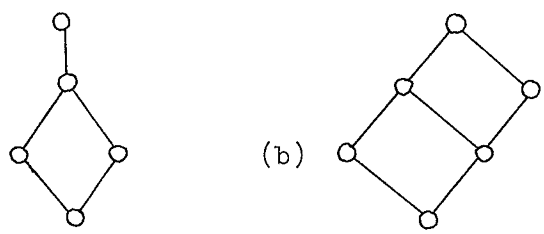
A non-distributive lattice in which every element has a unique irredundant representation as a join of join-irreducibles.

In a finite distributive lattice L every join-irreducible element a satisfies the following property (G. BIRKHOFF [6]): *if* $b, c \in L$ *and* $a \leq b \vee c$, *then* $a \leq b$ *or* $a \leq c$. In fact, this property characterizes finite distributive lattices. Furthermore, it is well known that a finite distributive lattice L satisfies the Jordan-Dedekind chain condition and that $|J(L)| = \ell(L) - 1 = |M(L)|$. Recently, G. MARKOWSKY [34] showed that these conditions also characterize finite distributive lattices in the class of finite lattices.

Though the partially ordered subsets J(L) and M(L) of a finite distributive lattice L (with universal bounds 0, 1) behave dually with respect to each other as far as "factorization" of elements in L is concerned, it turns out, that, as partially ordered sets, $J(L) - \{0\}$ is isomorphic to $M(L) - \{1\}$. The 5-element modular non-distributive lattice M_5 shows, however, that this property does not characterize distributivity.

For subsets A, B of a partially ordered set a *matching from A into* B is an injection f: A → B such that $x \leq f(x)$ for every $x \in A$. A matching from J(L) into M(L) always exists for a finite distributive lattice L. The lattice of Figure 2(a) illustrates that the property of *matching* the irreducible elements of a lattice does not make sense for the deleted partially ordered sets J(L) - {0}, M(L) - {1} and that, in general, the 0 is not *matched* to the 1. It is, consequently, not surprising that there may be no matching f: J(L) → M(L) in a distributive lattice L which, when restricted to J(L) - {0}, induces an isomorphism onto M(L) - {1} (see, for example, Figure 2(b)).

Figure 2 (a) (b)



Again, M_5 shows that even a bijective matching f: J(L) → M(L) does not characterize distributivity.

Arithmetic of Modular Lattices. Of course, most of the "factorization" results available for distributive lattices are lost once we go over to the theory of modular lattices. Furthermore, a survey of the arithmetical theory of modular lattices tends to substantiate the view that modularity is not only far less understood than distributivity, but is also far more complicated.

Still, a classical "factorization" result can be recovered.

THEOREM 3 (A. KUROSCH [32] and O. ORE [35]). *Let* L *be a modular lattice. If* $a = x_1 \vee x_2 \vee \dots \vee x_m = y_1 \vee y_2 \vee \dots \vee y_n$ *are two irredundant join representations of an element* a *in* L *by join-irreducible elements then for each* x_i *there exists a* y_j *such that* $a = x_1 \vee x_2 \vee \dots \vee x_{i-1} \vee y_j \vee x_{i+1} \vee \dots \vee x_m$ *is also an irredundant join representation of* a.

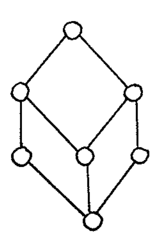
The property expressed in this theorem is called the *replacement property* for modular lattices. In this connection there is an analogue of Theorem 2 which follows from a more general result of P. CRAWLEY [7] (cf. [8], [13], [14]).

THEOREM 4 (P. CRAWLEY [7]). *A lattice* L *of finite length has the replacement property if and only if, for every* $a, b \in L$,
$a \succ \bigwedge(c \in L \mid a \succ c \geq a \wedge b)$ *implies* $a \vee b \succ \bigwedge(c \in L \mid a \vee b \succ c \geq b)$.

The condition in Theorem 4 may alternatively be stated as follows. If $a, b \in L$ and $[a, a \wedge b]$ contains a unique element covered by a, then $[b, a \vee b]$ contains a unique element covered by $a \vee b$.

Even a lower semimodular lattice satisfying this modified version of upper semimodularity need not be modular (see Figure 3); these two conditions do, however, imply local modularity, that is, $[\bigwedge(c \in L \mid a \succ c), a]$ is a modular sublattice for every $a \in L$ (cf. [8]).

Figure 3



A non-modular, lower semimodular lattice satisfying the condition of Theorem 4.

It is a simple matter to check that, in a finite modular lattice L, the partially ordered subsets J(L) - {0} and M(L) - {1} need not be isomorphic as in the distributive case. In the middle 30's it was, however, conjectured that *in a finite modular lattice the number of join-irreducible elements is equal to the number of meet-irreducible elements.* Two decades later R. P. DILWORTH proved the following remarkable combinatorial result which, as a special case, settled this conjecture in the affirmative.

THEOREM 5 (R. P. DILWORTH [11], [19]). *In a finite modular lattice the number of elements covered by precisely* k *elements is equal to the number of elements covering precisely* k *elements.*

The critical steps of Dilworth's proof of Theorem 5 depend upon properties of the generalized Möbius function; consequently, the proof is rather intricate. Recently, B. GANTER and the author [19] succeeded in supplying a simple proof based on the following lemma.

LEMMA 6 ([19]). *In a finite complemented modular lattice the number of* k*-element subsets of atoms whose join is the unit is equal to the number of* k*-element subsets of coatoms whose meet is the zero.*

The proof of Lemma 6 presented in [19] uses the classical coordinatization theorem for projective geometries. R. P. DILWORTH has since pointed out that even the coordinatization theorem can be avoided by an elementary lattice-theoretic argument.

PROBLEM 1. *Does every modular lattice* L *of finite length admit a matching from* J(L) *into* M(L)*?*

D. KELLY and the author have shown that, at least in the special case in which the breadth is ≤ 2, the answer is in the affirmative.

2. STRUCTURE OF SUBLATTICES OF A LATTICE

By now it is clear that the sets $J(L)$ and $M(L)$ play a central role in the arithmetic of a lattice L of finite length and particularly, in the case that L is distributive. The *quotient set* $Q(L) = \{b/a \mid a \in J(L),\ b \in M(L),\ a \leq b\}$ introduced in [40] plays a somewhat analogous role in the study of the sublattices of a lattice L of finite length. Indeed, the basic result in this direction which is due to B. WOLK and the author [40] is the following.

THEOREM 1 ([40]). *If* S *is a sublattice of a lattice* L *of finite length then* $S = L - \bigcup([a, b] \mid b/a \in A)$, *for some* $A \subseteq Q(L)$.

In view of Theorem 1 it is natural to classify sublattices of a lattice L of finite length in terms of subsets of $Q(L)$. For $A \subseteq Q(L)$, let us define $C\ell(A) = \{y/x \in Q(L) \mid [x, y] \subseteq \bigcup([a, b] \mid b/a \in A)\}$ and $C\ell(Q(L)) = \{C\ell(A) \mid A \subseteq Q(L)\}$. $C\ell$ is a closure operator on $Q(L)$ so that $C\ell(Q(L))$ is a lattice with respect to set inclusion [40]. We denote by $\mathrm{Sub}(L)$ the lattice of all sublattices of a lattice L.

Sublattices of Distributive Lattices. The familiar modular lattice M_5 illustrates that in general the converse of Theorem 1 need not hold; nonetheless, for distributive lattices the converse does hold, and even characterizes distributivity.

THEOREM 2 ([40]). *For a lattice* L *of finite length the following conditions are equivalent:*

(*i*) L *is distributive;*

(*ii*) $L - \bigcup([a, b] \mid b/a \in A)$ *is a sublattice of* L *for every* $A \subseteq Q(L)$;

(*iii*) *for every* $S \subseteq L$, S *is a sublattice of* L *if and only if* $S = L - \bigcup([a, b] \mid b/a \in A)$ *for some* $A \subseteq Q(L)$;

(*iv*) *the mapping* $\psi(S) = C\ell(A)$, *where* $S = L - \bigcup([a, b] \mid b/a \in A)$, $A \subseteq Q(L)$, *is an isomorphism between* Sub(L) *and the dual of* $C\ell(Q(L))$.

COROLLARY 3 ([39]). *If* L *is a finite distributive lattice and* K *is a maximal proper sublattice then there exists* $b/a \in Q(L)$ *such that* (*i*) $L - M = [a, b]$, (*ii*) $(a, b] \subseteq L - J(L)$ *and,* (*iii*) $[a, b) \subseteq L - M(L)$.

A problem that has attracted considerable attention in recent years (cf. [16], [29], [30], [47], [49], [50]) concerns topologies on finite sets. Because topological spaces on an n-element set correspond to sublattices preserving the zero and unit of the Boolean lattice $\underset{\sim}{2}^n$ on n atoms such a problem has a clear-cut lattice-theoretic formulation.

PROBLEM 2. *Characterize the lattice of* {0, 1}*-preserving sublattices of* $\underset{\sim}{2}^n$. *What is the order of this lattice?*

Corollary 3 yields $2^{n(n-1)}$ as an upper bound to the order of this lattice (cf. [29], [30], [39]). This bound is, however, far from best possible; for example, for n = 3 the actual order is 29.

A less trivial result related to this problem states that *the maximum size of a non-discrete topology on an* n*-element set is* $\frac{3}{4}\cdot 2^n$. This was first established by H. SHARP [47] using graph-theoretic and linear algebraic tools and by D. STEVEN [50] using essentially topological ones (cf. R. P. STANLEY [49]). Applying Corollary 3 again, we get a rather elegant lattice-theoretic proof.

COROLLARY 4 ([39]). *If* L *is a finite Boolean lattice with* $|L| \geq 4$ *and* M *is a maximal proper sublattice of* L *then (i)* $|M| = \frac{3}{4}|L|$ *and (ii)* $\ell(M) = \ell(L)$.

Since every finite distributive lattice is embeddable in a finite Boolean lattice further information about the orders of {0, 1}-preserving sublattices of a Boolean lattice essentially reduces to the general question of studying sublattices of finite distributive lattices.

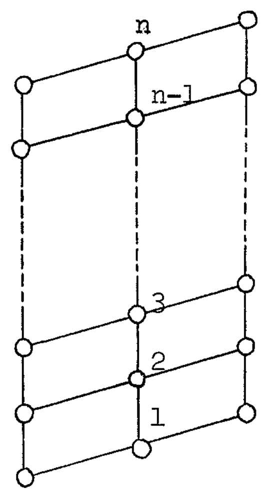
Once again, applying Corollary 3 and enumerating sufficiently many joins and meets of subsets of J(L) and M(L), respectively, of a finite distributive lattice L, we can establish the following analogue of Corollary 4.

THEOREM 5 ([39]). *If* L *is a finite distributive lattice with* $|L| \geq 3$ *and* M *is a maximal proper sublattice of* L *then (i)* $|M| \geq \frac{2}{3}|L|$ *and (ii)* $\ell(M) \geq \ell(L) - 1$.

Moreover, these inequalities are best possible in the sense that for every integer $n \geq 1$ there is a distributive lattice L_n

with a maximal proper sublattice K_n such that $|L_n| = 3n$, $|K_n| = 2n$, and $\ell(K_n) = \ell(L_n) - 1$, (see Figure 4).

Figure 4



L_n, $n \geq 1$

$K_n = L_n - \{1, 2, \ldots, n\}$

Applying Theorem 5 in a topological setting yields the following.

COROLLARY 6. *Every topological space with* n *open sets,* $n \geq 3$, *contains a proper subspace with at least* $\frac{2}{3}$ n *open sets.*

Since the partially ordered subset of join-irreducible elements, J(L), of a finite distributive lattice L determines L, obtaining more information about the sublattice structure of L would seem to hinge on our knowledge of the relationship between the partially ordered subset of join-irreducible elements, J(S), of a sublattice S of L and J(L).

Because there is a categorical duality between finite distributive lattices and finite partially ordered sets, a subset S of a finite distributive lattice L is a $\{0, 1\}$-preserving sublattice of L if and only if there is a monotone map from $J(L)$ onto $J(S)$. From another standpoint it is easy to verify that, for a sublattice S of a finite distributive lattice L, $J(S) = A \dot{\cup} B$ where $A \subseteq J(L)$, $B \subseteq L - J(L)$, and, if $a, b \in A$ and $a \wedge b \in J(L)$ then $a \wedge b \in A$.

In the next theorem we consider this connection from yet another standpoint. For $A \subseteq L$, we define the *covering neighbourhood* of A by $\mathrm{cov}(A) = \{x \in L \mid x \succ a,\ x \prec a, \text{ or } x = a \text{ for some } a \in A\}$.

THEOREM 7 ([40]). *Let* L *be a finite distributive lattice and let* $K = L - [a, b]$, $(b/a \in Q(L),\ a \neq b)$, *be a maximal proper sublattice of* L. *Then (i)* $\mathrm{cov}([a, b])$ *is a sublattice of* L *isomorphic to the direct product of* $[a, b]$ *with a* 3*-element chain and,* *(ii)* $J(K) = (J(L) - \{a\}) \cup \{c\}$, *where* $a \prec c \in K$.

The proof of Theorem 7 depends on an essential application of Theorem 4. Somewhat unexpected is the information this theorem supplies concerning orders of sublattices. The first corollary below is a consequence of (i) of the theorem while the second follows from (ii).

COROLLARY 8 ([40]). *Every distributive lattice of order* $n \geq 3$ *which contains a maximal proper sublattice of order* m *also contains sublattices of orders* $n-m$, $2(n-m)$, *and* $3(n-m)$.

COROLLARY 9 ([40]). *Every finite distributive lattice* L *contains a* *maximal proper sublattice* K *such that either* $|K| = |L| - 1$ *or* $|K| \geq 2\ell(L)$.

The estimate in Corollary 9 is best possible in the sense that if, for every positive integer n, B_n is the linear sum of n copies of the Boolean latiice $\underset{\sim}{2}^3$, then the maximum order of a maximal proper sublattice of B_n is precisely $2\ell(B_n)$.

Sublattices of Modular Lattices. We have seen that the arithmetic of modular lattices is considerably more involved and less understood than that of distributive lattices. As expected this dichotomy carries over to the study of the structure of sublattices of a lattice.

A term useful in the study of sublattices of an arbitrary lattice L is the *boundary* bd(A) of a subset A of L, defined by bd(A) = cov(A) - A. It is easy to show: *if* S *is a sublattice of a lattice* L *of finite length and if* $A \subseteq L$ *such that* $\mathrm{bd}(A) \subseteq S$, *then* $S \cup A$ *is a sublattice of* L ([44]).

A subset A of L is *connected* if, for every a, b ε A, there is a sequence $a = x_0, x_1, \ldots, x_n = b$ of elements in A such that either $x_i \prec x_{i-1}$ or $x_i \succ x_{i-1}$ for every i = 1, 2,...,n. Again, it is easy to verify: *if* M *is a maximal proper sublattice of a lattice* L *of finite length then* L - M *is a connected subset of* L ([44]).

It will be helpful to quantify the notion of a boundary as follows: *for* $A \subseteq L$ *and* a ε A, $A_*(a) = |\{x \in L - A | x \prec a\}|$ *and* $A^*(a) = |\{x \in L - A | x \succ a\}|$. Obviously, if L - A is a sublattice of L then $A_*(a) \leq 1$ and $A^*(a) \leq 1$ for every $a \in A$.

PROPOSITION 10 ([44]). *Let* A *be a subset of a modular lattice* L *of finite length satisfying the conditions: (i)* A *is convex; (ii)* $A_*(a) \leq 1$ *and* $A^*(a) \leq 1$ *for every* $a \in A$. *Then* L - A *is a sublattice of* L.

In this way we can generate maximal proper sublattices of modular lattices of finite length.

THEOREM 11 ([44]). *Let* A *be a subset of a modular lattice* L *of finite length satisfying the conditions: (i)* A *is convex; (ii)* A *is a connected subset of* L*; (iii)* $A_*(a) = 1 = A^*(a)$ *for every* $a \in A$. *Then* L - A *is a maximal proper sublattice of* L.

For finite distributive lattices we have already seen (cf., Corollary 3 and Theorem 7) that the conditions (i), (ii), and (iii) of Theorem 14 characterize maximal proper sublattices; whether this extends to arbitrary modular lattices of finite length seems much more difficult to settle.

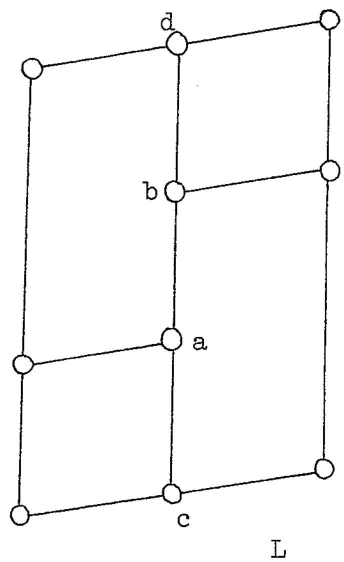
PROBLEM 3. *Characterize maximal proper sublattices of a modular lattice of finite length. In particular, do the conditions (i), (ii), and (iii) characterize such sublattices?*

Unfortunately, as to properties of L - M, where M is a maximal proper sublattice of a modular lattice L of finite length, very little, apart from the next result, is available.

PROPOSITION 12 ([44]). *If* M *is a maximal proper sublattice of a modular lattice* L *of finite length and if* $b \succ a$, *where* $a, b \in L - M$, *then either* b *is join-reducible or* a *is meet-reducible.*

The lattice L of Figure 5 illustrates the necessity of modularity in Proposition 12.

Figure 5



L

$K = L - \{a, b, c, d\}$

Slightly more information about sublattices of modular lattices than that provided by Theorem 1 can be obtained from the following.

PROPOSITION 13 ([44]). *If* S *is a sublattice of a modular lattice* L *of finite length then for every* $x \in L - S$ *there exists* $b/a \in Q(L)$ *such that (i)* $x \in [a, b] \subseteq L - S$, *(ii)* $(a, x) \subseteq L - J(L)$, *and (iii)* $(x, b) \subseteq L - M(L)$.

Both Propositions 12 and 13 rely on the following property of irreducible elements in a modular lattice L of finite length: *for* $a, b, c \in L$ *if* $a \in J(L)$ *and* $b < a \leq b \vee c$ *then* $a \leq c$.

Let M be a maximal proper sublattice of a lattice L. If L is a finite Boolean lattice then Corollary 4 shows that $\frac{|M|}{|L|} = \frac{3}{4}$ and, if L is finite distributive then by Theorem 7, $\frac{|M|}{|L|} \geq \frac{2}{3}$. However, if L is modular there is, in general, no non-zero constant k such that $\frac{|M|}{|L|} \geq k$. In fact, B. WOLK has pointed out that if P_n denotes the lattice of subspaces of a projective plane of order n then a maximal proper sublattice M of P_n satisfies either $|M| = 2n + 4$ or $|M| = 2n + 6$ so that $\lim_{n\to\infty} \frac{|M|}{|P_n|} = 0$. (Figure 6 illustrates the two possible maximal proper sublattices of P_n.)

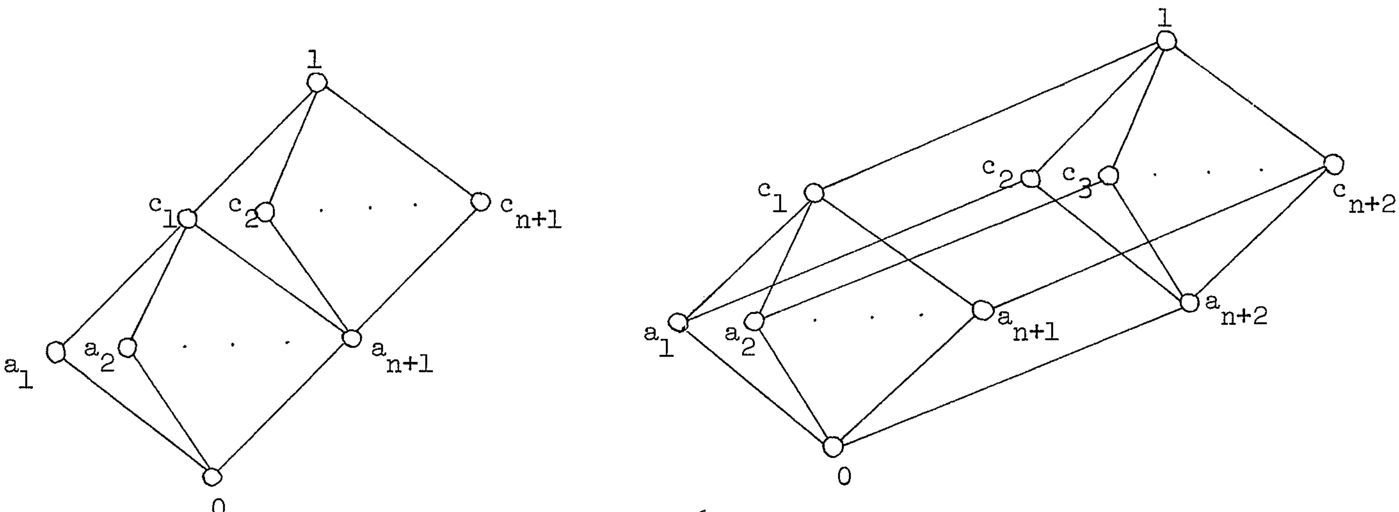


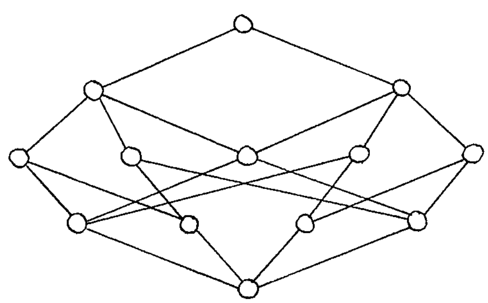
Figure 6

At least for the lattice of subspaces of a projective plane the conjecture in Problem 3 does hold.

Lattice of Sublattices. The problem (Problem 2) of characterizing the lattice of $\{0, 1\}$-preserving sublattices of $\underset{\sim}{2}^n$ is, as we have seen, a very difficult one: it is no wonder that the problem of characterizing the lattice of sublattices of an arbitrary lattice has so far remained almost beyond reach (cf. G. GRÄTZER [20, p.66]).

D. SACHS [45] has characterized the lattice of *Boolean* sublattices of a Boolean lattice (cf. G. GRÄTZER, K. M. KOH, and M. MAKKAI [21]). In general, however, for a finite lattice L, Sub(L) seems to have no familiar property apart from the immediate observation that every element is a join of atoms. Figure 7 illustrates Sub($\underset{\sim}{2}^2$).

Figure 7



Sub($\underset{\sim}{2}^2$)

K. M. KOH has observed that, for a lattice L of finite length, Sub(L) *is upper semimodular if and only if* L *is a chain.* H. LAKSER [33], on the other hand, has shown that Sub(L) *is lower semimodular if and only if* L *contains no sublattice isomorphic to the direct product of a 2-element chain with a 3-element chain.*

Carrying on the work of L. N. ŠEVRIN [46] for semilattices, N. D. FILIPPOV [17] investigated the problem of determining the relationship between lattices L and K such that Sub(L) ≅ Sub(K).

THEOREM 14 (N. D. FILIPPOV [17], cf. [38]). *Let* L *and* K *be lattices such that* Sub(L) ≅ Sub(K). *If* L *is modular (distributive, Boolean) then* K *is modular (distributive, Boolean).*

In fact, Filippov's proof relies on a lengthy and detailed investigation which culminates in proving that *if* L *is modular then* K *is obtained from* L *by a permutation of the linear components of* L, *some possibly dualized.* Recently, the author [38] gave a simple and elementary proof based essentially on the observation that the comparability relation ($x \geq y$ or $x \leq y$) in a lattice L is determined by the elements of rank 2 in Sub(L).

PROBLEM 4 (G. GRÄTZER [20, p.66]). *Characterize the comparability relation for lattices.*

PROBLEM 5 (G. GRÄTZER [20, p.66]). *For which equational classes* K *of lattices does* $L \in K$, Sub(L) ≅ Sub(L') *imply that* $L' \in K$?

Finite Sublattices Generated by Order-Isomorphic Subsets. Classes of lattices characterized by the noncontainment of sublattices isomorphic to certain prescribed lattices arise frequently in lattice theory: for example, the classes of modular and distributive lattices. Indeed many such characterizations are part of the folklore of lattice theory. A related question concerns classes of lattices characterized by the noncontainment of subsets order-isomorphic to certain prescribed partially ordered sets: for example, the classes of *dismantlable* lattices (see Section 3) and *planar* lattices (see Section 4). If L is a fixed finite lattice and if an arbitrary lattice K contains no subset order-isomorphic to L then certainly K contains no sublattice isomorphic to L. Far less trivial is the problem of determining those finite lattices L with the property that if an arbitrary lattice K contains no sublattice isomorphic to L then K contains no subset order-isomorphic to L. Equivalently, which finite lattices L satisfy the condition: $\underset{\sim}{A}(L)$: *every lattice which contains an order-isomorphic copy of* L *also contains a lattice-isomorphic copy of* L?

The next result, which is due to W. POGUNTKE and the author [37], provides a complete solution.

THEOREM 15 ([37]). *For a finite lattice* L *the following conditions are equivalent:*

(*i*) $\underset{\sim}{A}(L)$ *holds;*

(*ii*) L *is distributive and every element of* L *is either join-irreducible or meet-irreducible;*

(*iii*) L *is a linear sum of components each of which is either a single element,* $\underset{\sim}{2}^3$, *or a product of two chains one of which has precisely two elements.*

The equivalence of (ii) and (iii) was already established by

F. GALVIN and B. JÓNSSON [18]. The class of all finite lattices satisfying condition (iii) of Theorem 15 has already arisen in several apparently different contexts (cf. [3]). One of these is in connection with distributive sublattices of free lattices. The following corollary, also first proven in [18], is immediate.

COROLLARY 16 (F. GALVIN and B. JÓNSSON [18], cf. [37]). *A finite lattice is a distributive sublattice of a free lattice if and only if it satisfies condition (iii) of Theorem* 19.

Evidently, the class of all finite lattices L satisfying Λ(L) is rather small. It would seem reasonable, therefore, to generalize this condition in the following direction. Let $\underset{\sim}{K}$ be an arbitrary class of lattices and let L be a finite lattice. We define $\Lambda_{\underset{\sim}{K}}$(L): *every* M ε $\underset{\sim}{K}$ *which contains an order-isomorphic copy of* L *also contains a lattice-isomorphic copy of* L. Of course, when $\underset{\sim}{K}$ is the class of all lattices this is just Λ(L).

If $\underset{\sim}{K}$ is closed with respect to the formation of sublattices then $\Lambda_{\underset{\sim}{K}}$(L) holds if and only if whenever M ε $\underset{\sim}{K}$ and M contains an order-isomorphic copy L' of L then the sublattice in M generated by L' already contains a lattice-isomorphic copy of L. Furthermore, if $\underset{\sim}{K}$ also contains all finite distributive lattices and $\Lambda_{\underset{\sim}{K}}$(L) holds then L is distributive.

If, for example, $\underset{\sim}{B}$ is just the class of all finite Boolean lattices then it is easy to check that $\Lambda_{\underset{\sim}{B}}$(L) holds for every finite distributive lattice L.

PROBLEM 6 ([37]). *Determine the class* $\underset{\sim}{K}$ *of all lattices such that* $\underset{\sim}{\Lambda}_{\underset{\sim}{K}}(L)$ *holds for every finite distributive lattice. Does this class, for example, contain all finite direct products of chains?*

In another direction, if $\underset{\sim}{D}$ is the class of all distributive lattices then $\underset{\sim}{\Lambda}_{\underset{\sim}{D}}(L)$ holds for every finite Boolean lattice L [37].

PROBLEM 7 ([37]). *Characterize those finite lattices* L *for which* $\underset{\sim}{\Lambda}_{\underset{\sim}{D}}(L)$ *holds. Are these, for example, just the finite lattices projective in* $\underset{\sim}{D}$ (cf. [5], [22])*?*

3. DISMANTLABLE LATTICES

The various "factorization" results heralded by Theorem 1.0 have demonstrated the usefulness of studying the join-irreducible (and dually, meet-irreducible) elements of a lattice. Similarly, the analysis of sublattices of a lattice made possible by Theorem 2.1 has illustrated the usefulness of studying "matched" pairs of irreducible elements. We shall now investigate those properties of lattices determined by their doubly irreducible elements.

By way of preliminaries let us observe that three of the numerical invariants of a lattice L, namely, $|L|$, $\ell(L)$, and $|\mathrm{Irr}(L)|$, can be related by a simple inequality.

THEOREM 1 ([41]). *Every lattice* L *of finite length satisfies the inequality* $|L| \geq 2(\ell(L) + 1) - |\mathrm{Irr}(L)|$.

Among all lattices L of finite length such that $\mathrm{Irr}(L) = \phi$ this inequality is best possible in the sense that for every integer $n \geq 3$ there is a lattice L_n such that $\mathrm{Irr}(L_n) = \phi$, $\ell(L_n) = n$, and $|L_n| = 2(\ell(L_n) + 1)$, (see Figure 8).

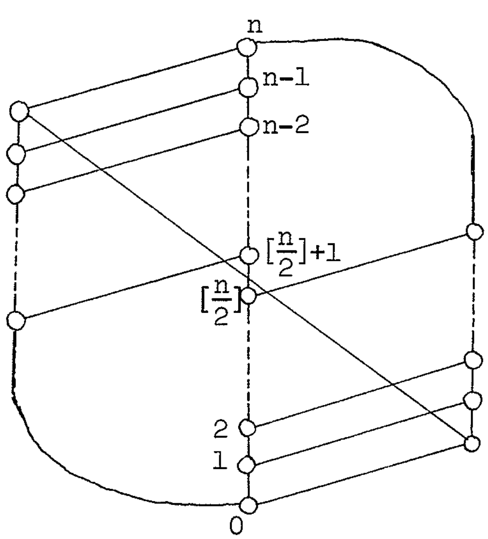


Figure 8

$L_n \quad (n \geq 3)$

Indeed, as D. KELLY has pointed out, for positive integers n, m such that $n \geq m + 3$ the inequality of Theorem 1 is best possible in the sense that there is a lattice $L_{n,m}$ such that $|\mathrm{Irr}(L_{n,m})| = m$, $\ell(L_{n,m}) = n$ and $|L_{n,m}| = 2(\ell(L_{n,m}) + 1) - |\mathrm{Irr}(L_{n,m})|$. $L_{n,m}$ can be taken to be the linear sum of L_{n-m} with an m-element chain.

Once we observe that $L - A$ is a sublattice of L for every $A \subseteq \mathrm{Irr}(L)$ the following corollary is immediate.

COROLLARY 2 ([41]). *If* n *is a positive integer and* L *is a lattice of finite length satisfying* $|L| \leq 2(\ell(L) + 1) - n$ *then there is a chain* $S_n \subset S_{n-1} \subset \ldots \subset S_0 = L$ *of sublattices of* L *such that* $|S_i| = |S_{i-1}| - 1$ *for every* $i = 1, 2, \ldots, n$.

Continuing the theme of sublattices there is yet another inequality similar to that of Theorem 1.

THEOREM 3 ([41]). *Any lattice* L *such that* $\ell(\mathrm{Sub}(L))$ *is finite satisfies* $\ell(\mathrm{Sub}(L)) = |\mathrm{Irr}(L)| + \ell(\mathrm{Sub}(L - \mathrm{Irr}(L)))$.

Several numerical invariants associated with a finite lattice L have been investigated over the years. For example, E. SPERNER [48] showed

$$w(\underset{\sim}{2}^n) \leq \binom{n}{[\frac{n}{2}]},$$

where w(L) denotes the *width* of L. A simple consequence of this inequality is

$$w(L) \leq \binom{|J(L)|}{\left\lceil \frac{|J(L)|}{2} \right\rceil},$$

for every finite lattice L. Let b(L) denote the *breadth* of L and $\delta(L)$ its *dimension* (that is, the least number m of chains $C_1, C_2, \ldots, C_m$ such that L is order-isomorphic to a subset of $C_1 \times C_2 \times \ldots \times C_m$). It is trivial that, for a finite lattice L,

$$b(L) \leq w(L).$$

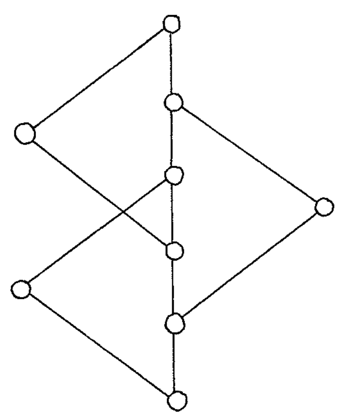
On the other hand, K. A. BAKER [2] has shown that, for any finite lattice L,

$$b(L) \leq \delta(L)$$

and equality holds whenever L is distributive (cf. R. P. DILWORTH [10]). One direction of study suggested by G. GRÄTZER is the following. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be numerical invariants which can be defined for any finite lattice L (such as order, length, width, breadth, dimension, etc.) and let us call an n-tuple of integers $\langle a_1, a_2, \ldots, a_n \rangle$ *representable* with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$ if there exists a finite lattice L such that $\alpha_i(L) = a_i$ for every $i = 1, 2, \ldots, n$. Which n-tuples of integers are representable with respect to $\alpha_1, \alpha_2, \ldots, \alpha_n$? For example, a reasonable starting point would be to take pairs or triples of such invariants.

Dismantlable Lattices. In [4], K. A. BAKER, P. C. FISHBURN, and F. S. ROBERTS showed that every finite planar lattice has a doubly irreducible element. Since, plainly, any sublattice of a planar lattice is planar, it follows that every sublattice of a planar lattice has a doubly irreducible element; that is, every finite planar lattice can be completely "dismantled" by removing one element at a time leaving a sublattice at each stage. In [41] the author introduced the concept of a finite *dismantlable* lattice as a lattice L of order n such that there is a chain $L_1 \subset L_2 \subset \ldots \subset L_n = L$ of sublattices of L satisfying $|L_i| = i$ for every $i = 1, 2, \ldots, n$. As we have just indicated, every finite planar lattice is dismantlable, but not conversely (see, for example, Figure 9).

Figure 9



A dismantlable non-planar lattice

Repeated application of Corollary 2 shows that every lattice with seven or fewer elements is dismantlable. Of course, for every integer $n \geq 8$ there is a lattice of order n which is not dismantlable (for example, the linear sum of the Boolean lattice $\underset{\sim}{2}^3$ with a chain of order n - 8). A simple modification of a proof of G. HAVAS and M. WARD [23] shows that, given an integer n, any large enough lattice ($|L| \geq n^{3^n}$ will do [23], cf. [20, p.67]) contains a dismantlable sublattice with precisely n elements.

THEOREM 4 ([41]). *For a finite lattice* L *the following conditions are equivalent:*

(i) L *is dismantlable;*

(ii) $\ell(\mathrm{Sub}(L)) = |L|$*;*

(iii) $\mathrm{Irr}(S) \neq \phi$ *for every sublattice* S *of* L*;*

(iv) *for every chain* C *in* L *there is a positive integer* n *and a chain* $C = S_0 \subset S_1 \subset \ldots \subset S_n = L$ *of sublattices of* L *such that* $|S_i| = |S_{i-1}| + 1$ *for every* $i = 1, 2, \ldots, n$.

COROLLARY 5 ([41]). *Every sublattice and epimorphic image of a dismantlable lattice is dismantlable.*

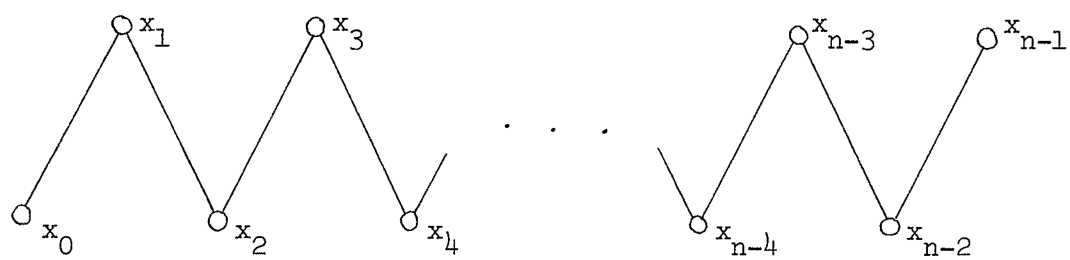
The concept of a *dismantlable* lattice can be extended in the natural way to infinite lattices. (i), (iii) and (iv) of Theorem 4 as well as Corollary 5 carry over [25].

As D. KELLY and the author [25] discovered, dismantlable lattices are very closely related to *fences* and *crowns*. A (lower) *fence* is a partially ordered set $\{x_i | 0 \leq i < n\}$ for which the comparabilities that hold are precisely:

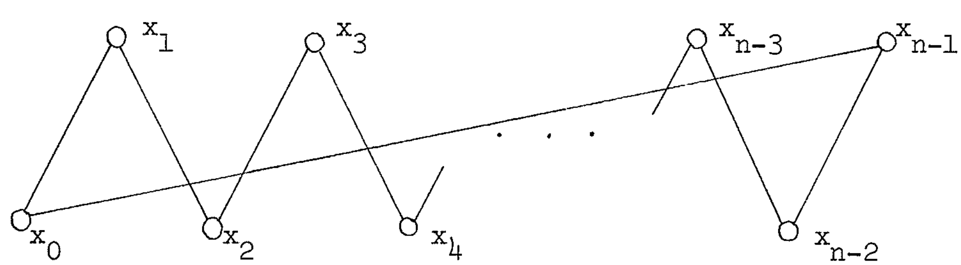
(*) $x_i < x_{i+1}$ (i even), $x_i > x_{i+1}$ (i odd). For finite even

$n \geq 6$, a *crown* is a partially ordered set $\{x_i | 0 \leq i < n\}$ for which $x_0 < x_{n-1}$ and (*) are precisely the comparabilities that hold (see Figure 10).

Figure 10



A lower fence



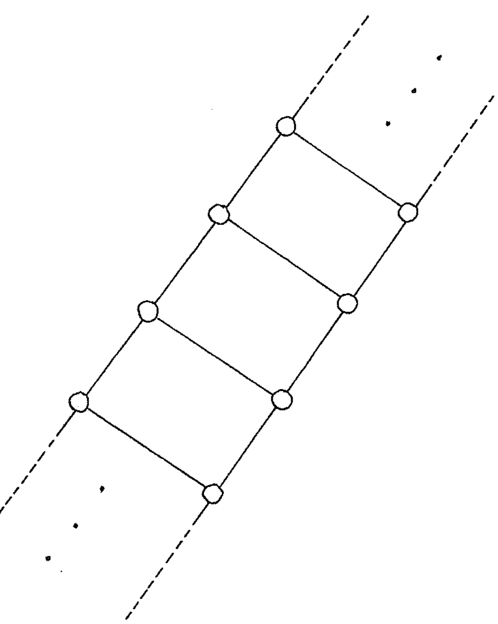
A crown of order n

The concept of a *fence* turns out to be the natural link between that of a *crown* and that of a *dismantlable* lattice. (Indeed, observe that the removal of one element, or two comparable elements from a crown leaves a fence.) The main result proved in [25] is the following characterization of dismantlable lattices in terms of crowns.

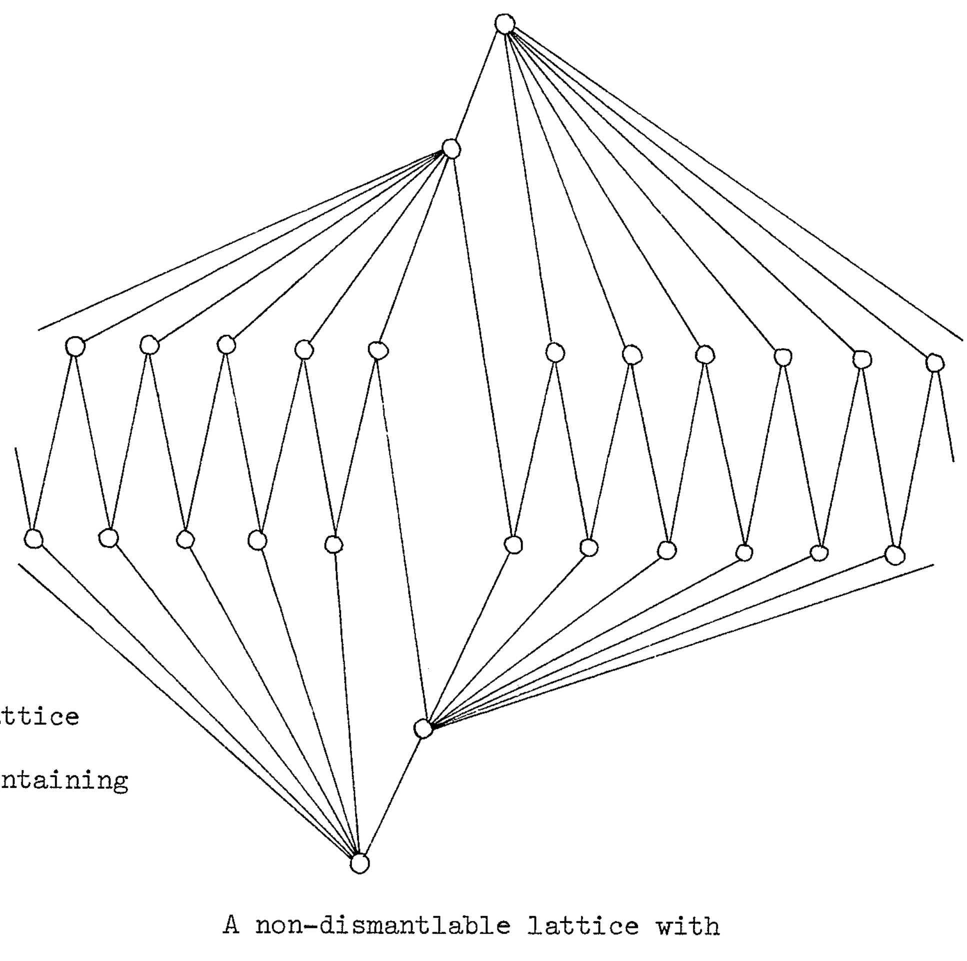
THEOREM 6 ([25]). *A lattice which contains no infinite chains and no infinite fences is dismantlable if and only if it contains no crowns.*

Lattices showing the necessity of both hypotheses are illustrated in Figure 11.

Figure 11



A non-dismantlable lattice of infinite length containing no crowns



A non-dismantlable lattice with infinite fences containing no crowns.

Let L be a lattice containing a crown $\{x_i \mid 0 \le i < 6\}$. Then L also contains the crown $\{(x_0 \vee x_2) \wedge (x_0 \vee x_4), x_0 \vee x_2,$ $(x_0 \vee x_2) \wedge (x_2 \vee x_4), x_2 \vee x_4, (x_2 \vee x_4) \wedge (x_0 \vee x_4), x_0 \vee x_4\}$ which generates a sublattice of L containing no doubly irreducible elements. Carrying out this construction for an arbitrary crown establishes the "necessity" of the condition in Theorem 6. The proof of the "sufficiency", though elementary, is rather long and technical. In outline the proof depends upon the construction of a sequence $((F_n, Q_n))_n$ of pairs such that, for $n \ge 1$, Q_n is a nonempty convex subset of L, $(Q_0 = L)$, and F_n is a "maximal" fence in Q_{n-1} (see Figure 12). In fact, for $n \ge 1$, one of the endpoints e_n of F_n is distinguished with its two predecessors f_n, g_n in F_n and Q_n is defined by

$$Q_n = \{x \in L \mid x \ge f_n \text{ and } x \| g_n\} \quad \text{if} \quad g_n > f_n \quad \text{and} \quad f_n < e_n, \quad \text{or}$$

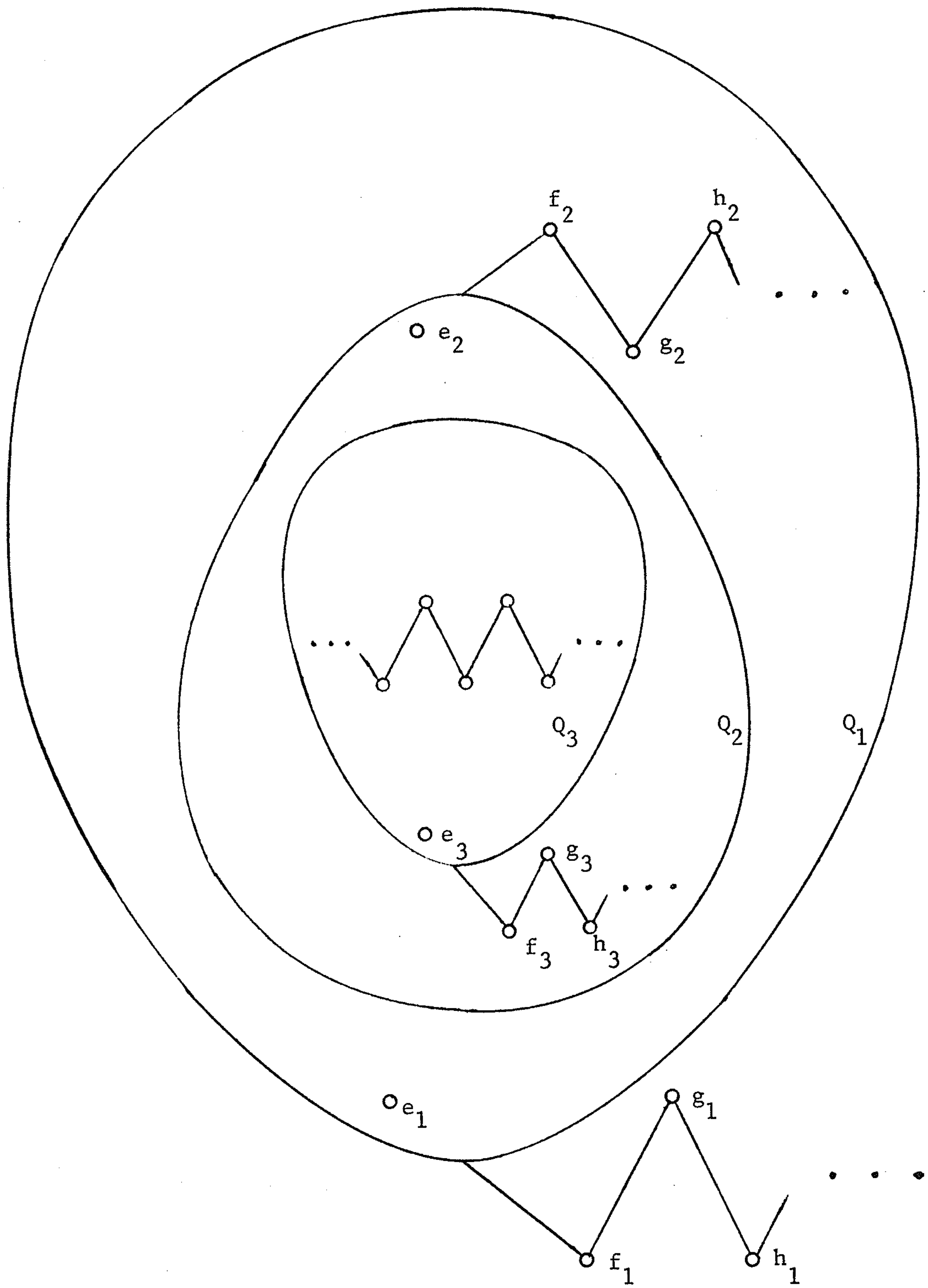
$$Q_n = \{x \in L \mid x \le f_n \text{ and } x \| g_n\} \quad \text{if} \quad g_n < f_n \quad \text{and} \quad f_n > e_n.$$

It is then shown that for $n \ge 1$, $|F_n| \ge 5$, under the hypotheses that $\mathrm{Irr}(L) = \phi$ and L contains no crowns. From this it follows that $(Q_n)_n$ is "nested", that is, for $n \ge 1$, $Q_n \subseteq Q_{n-1}$. But then $\bigcup_n Q_n$ must contain an infinite chain, contradicting the hypothesis.

A simple application of the construction used in the proof of Theorem 6 yields the following.

THEOREM 7 ([25]). *A dismantlable lattice which is not a chain and which contains no infinite chains and no infinite fences contains*

Figure 12



The construction scheme for the proof of Theorem 6 *at least two incomparable doubly irreducible elements.*

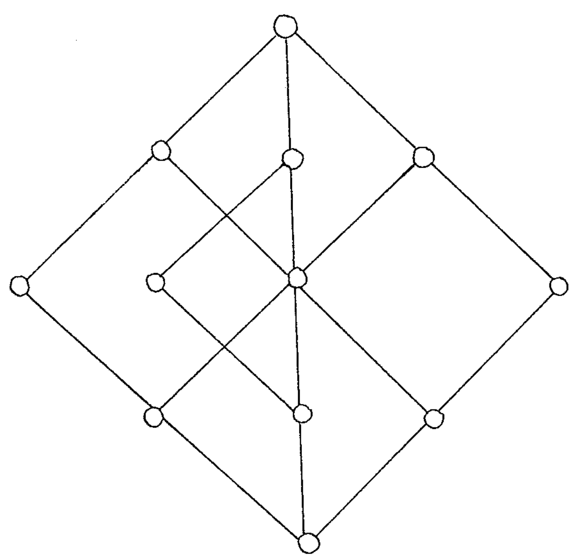
Modular and distributive dismantlable lattices have a very particular structure.

THEOREM 8 ([25]). *A modular lattice of finite length is dismantlable if and only if its breadth is* ≤ 2.

The crown of order 8 with universal bounds 0 and 1 adjoined shows that modularity is essential in Theorem 8. Modular lattices of finite length with breadth ≤ 2 have been studied by C. HERRMANN in [24] where they are called *quasiplanar*. Although not every such lattice is planar (see Figure 13) we have the following.

COROLLARY 9 ([25]). *A finite distributive lattice is dismantlable if and only if it is planar.*

Figure 13



A non-planar dismantlable modular lattice

4. PLANAR LATTICES

Just as for a graph, it is customary and convenient to represent a finite lattice by a diagram in the plane. Unlike graphs, the diagram of a finite lattice L must somehow account for the partial ordering of L. This is accomplished in the obvious way: small circles, corresponding to elements of L, are arranged in such a way that, for a, b ε L, the circle representing a is higher than the circle representing b whenever a > b and a straight line segment is drawn to connect the two circles whenever a covers b. L is *planar* if it has a diagram in which straight line segments do not intersect.

It should be mentioned at the outset that the connecting links for covering pairs in the diagram of a finite lattice could be taken more generally as continuous functions with respect to the y-axis; planarity could then be defined in the analogous way. However, using the fact that, in either case, a finite planar lattice is dismantlable, a simple induction establishes the equivalence of both approaches; that is, a finite lattice which has a planar diagram using curves of non-zero slope also has a planar diagram using only straight line segments.

Although a few characterizations of planarity are available (cf. [4]) the most useful to date is that due essentially to J. ZILBER [6, p.32, ex. 7(c)] who showed that *a finite lattice* L *with partial ordering* $\leq$ *is planar if and only if there exists a partial ordering* $\leq'$ *on* L *such that, for* a, b ε L, a $\leq$ b *or* b $\leq$ a, *if and only if* a *is incomparable with* b *with respect to* $\leq'$. Intuitively $\leq'$ orders the elements of L from left to right.

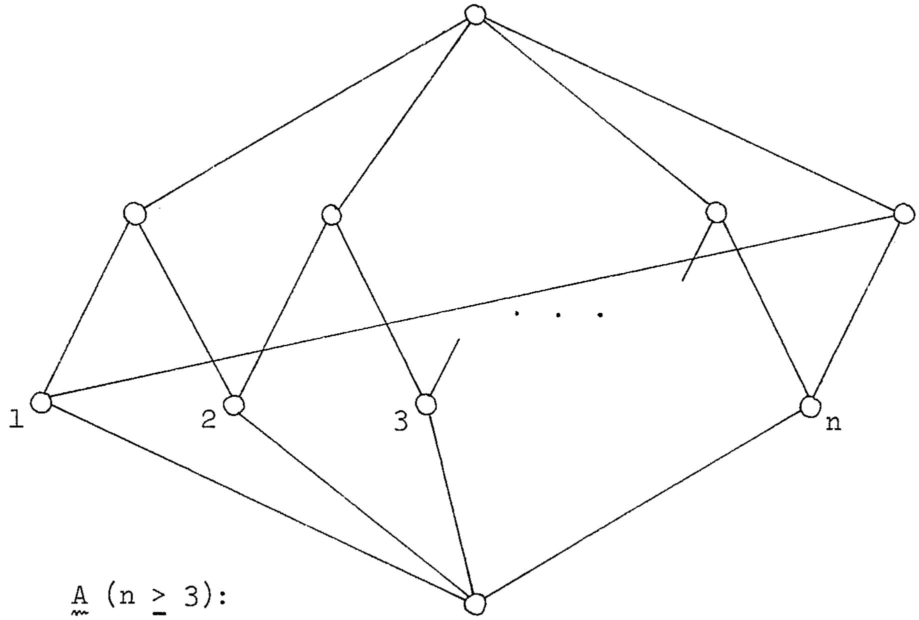
A term useful in the study of planar lattices is that of the *dimension* of a lattice L (see Section 3) which was first introduced by B. DUSHNIK and E. W. MILLER [15]. The characterization of planarity for lattices due to J. ZILBER combined with a result of B. DUSHNIK and E. W. MILLER yields: *a finite lattice is planar if and only if it has dimension* ≤ 2 (cf. [4], [26]).

Using a graph-theoretical approach C. PLATT [36] has shown that *a finite lattice* L *is planar if and only if the covering graph of* L *together with a directed edge from* 1 *to* 0 *is a planar graph.* Taking the graph-theoretical analogue one step further, it is tempting, in view of the famous characterization of planarity for graphs due to K. KURATOWSKI [31], to hope for a characterization of planarity for lattices by exhibiting a list of excluded lattices. Since planar lattices are contained in the much wider class of dismantlable lattices Theorem 3.6 shows that if such a list were to exist it could not be finite. The following characterization of finite planar lattices is due to D. KELLY and the author [26].

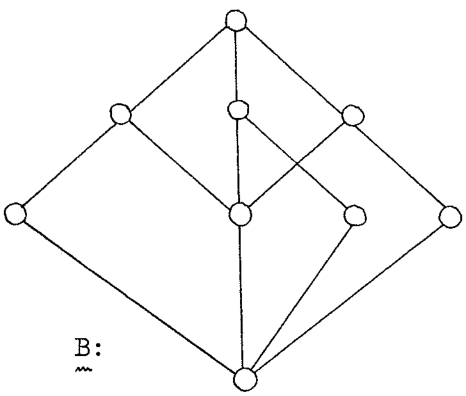
THEOREM 1 ([26]). *A finite lattice is planar if and only if it contains no subset order-isomorphic (or dually order-isomorphic) to any of the lattices* $\underset{\sim}{A}$ - $\underset{\sim}{I}$ *illustrated in Figure* 14.

Though this characterization of planarity is the obvious analogue for lattices of the characterization of planarity for graphs, the

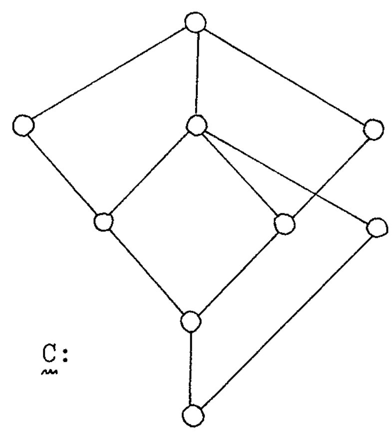
Figure 14



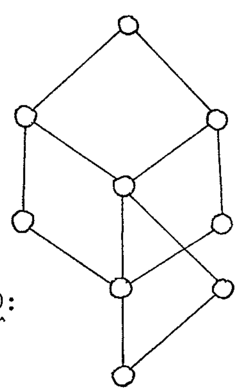
$\underset{\sim}{A}$ $(n \geq 3)$:



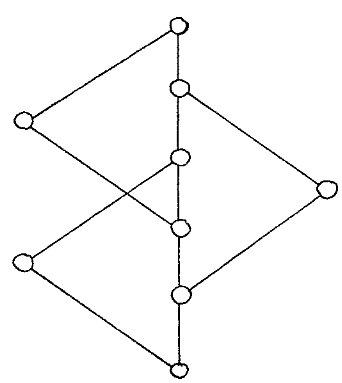
$\underset{\sim}{B}$:



$\underset{\sim}{C}$:

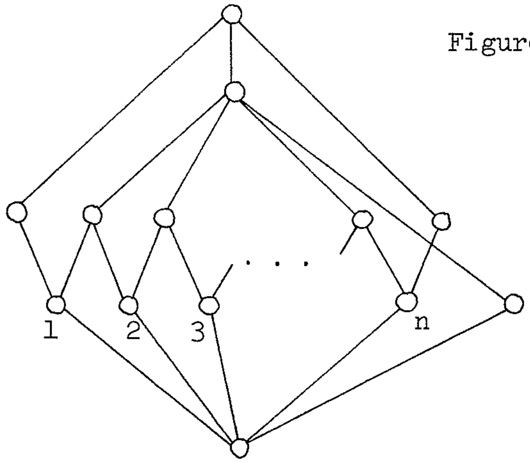


$\underset{\sim}{D}$:

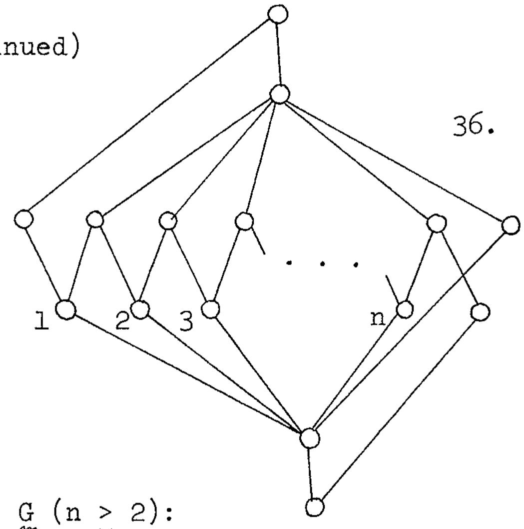


$\underset{\sim}{E}$:

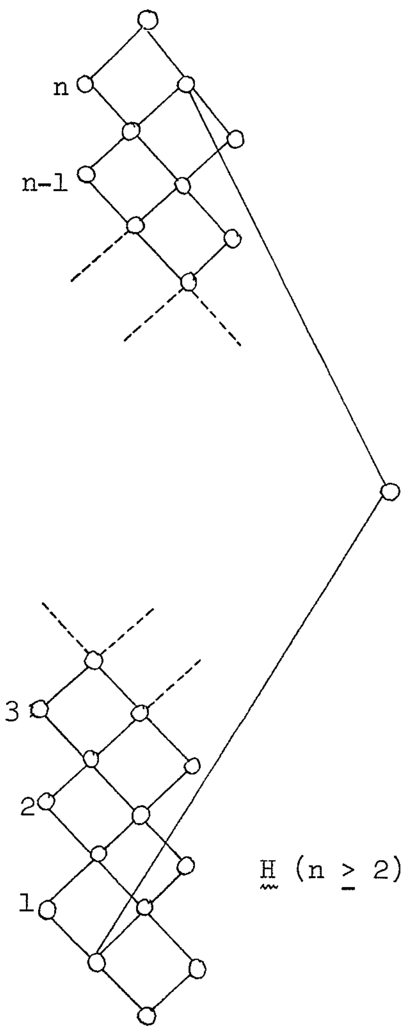
Figure 14 (continued)



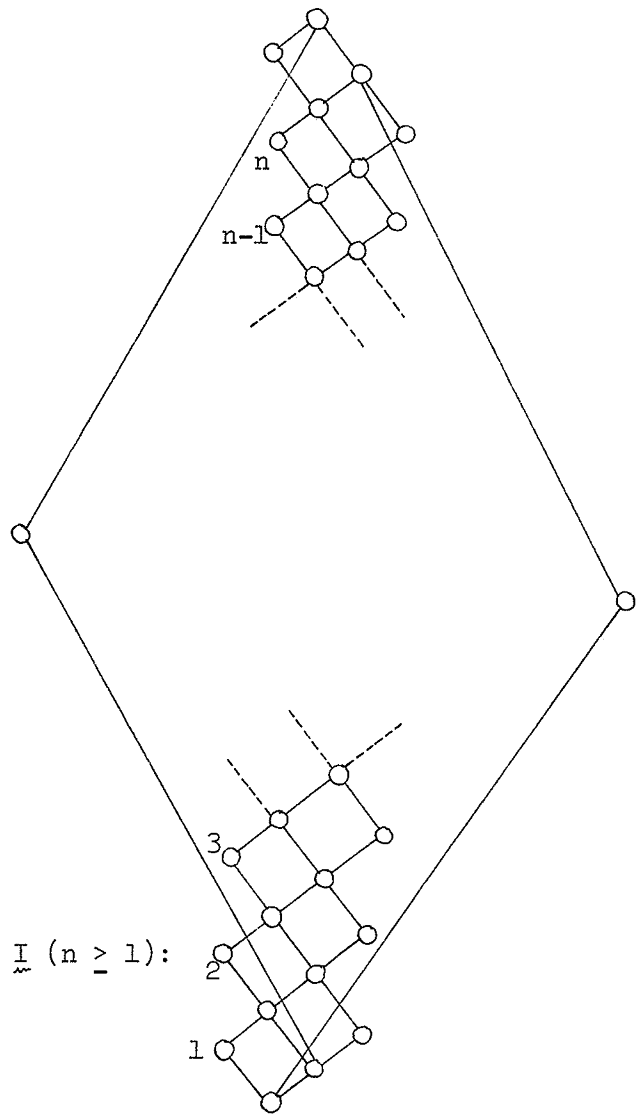
F (n ≥ 2):



G (n ≥ 2):



H (n ≥ 2):



I (n ≥ 1):

techniques required bear little resemblance to those needed to prove K. KURATOWSKI's theorem.

Indeed, by virtue of Theorem 3.6 it suffices to characterize finite planar lattices among finite dismantlable lattices. To prove that a finite non-planar lattice contains a subset order-isomorphic (or dually order-isomorphic) to one of $\underset{\sim}{A} - \underset{\sim}{I}$ it suffices to consider a finite non-planar dismantlable lattice L in which the removal of a doubly irreducible element c leaves a planar lattice. If $b \succ c \succ a$ in L then in every planar embedding of $L - \{c\}$, a *is not visible from* b, that is any polygonal curve from a to b crosses a line of the planar embedding of $L - \{c\}$. The proof involves the investigation of the (a, b)-*components* of $L - \{c\}$ (that is, the maximal connected subsets of the open interval (a, b)) and their *dangling points* (that is, elements $x \in L - \{c\}$, $x > a$, $x \| b$ or $x < b$, $x \| a$).

Any planar embedding $e_2(L - \{c\})$ of $L - \{c\}$ is obtained (up to "left-right ordering") from a given planar embedding $e_1(L - \{c\})$ by a sequence of "permutations" and "reflections" of certain (u, v)-components, $u, v \in L - \{c\}$, $u < v$, which have no dangling points. It, therefore, becomes feasible to systematically analyze all planar embeddings of $L - \{c\}$ to locate the forbidden subsets of L.

By specializing the proof of Theorem 1 to modularity we get the characterization of finite modular planar lattices due to R. WILLE [51].

COROLLARY 2 (R. WILLE [51], cf. [26]). *A finite modular lattice is planar if and only if it contains no subset order-isomorphic (or dually order-isomorphic) to any of the lattices* $\underset{\sim}{A}$ (n = 3), $\underset{\sim}{B}$, *or* $\underset{\sim}{C}$ *illustrated in Figure* 14.

For arbitrary lattices we get the following.

COROLLARY 3 ([26]). *A lattice has dimension* ≤ 2 *if and only if it contains no subset order-isomorphic (or dually order-isomorphic) to any of the lattices* $\underset{\sim}{A}$ - $\underset{\sim}{I}$ *illustrated in Figure* 14.

We have seen in Theorem 3.7 that a finite dismantlable lattice which is not a chain contains at least two incomparable doubly irreducible elements. Modifying some of the techniques used in the proof of Theorem 1 this result can be sharpened.

THEOREM 4 ([26]). *A finite dismantlable non-planar lattice contains at least three pairwise incomparable doubly irreducible elements.*

The class of dismantlable lattices is considerably larger than the class of planar lattices. D. KELLY and the author have shown for example, that for any integer n, there is a finite dismantlable lattice with dimension n.

PROBLEM 8. *In a finite dismantlable lattice* L *is the number of pairwise incomparable doubly irreducible elements* $\geq$ *the dimension of* L?

5. SPECTRUM OF A LATTICE

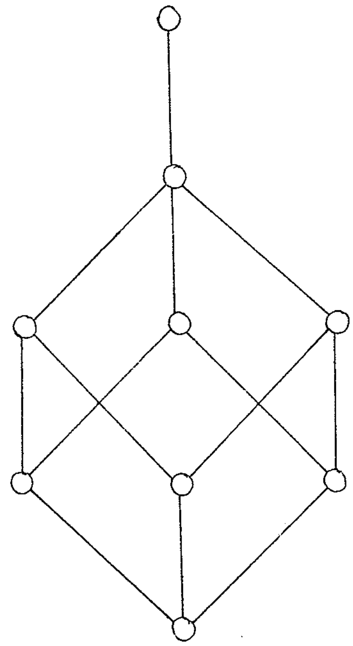
We have already seen that information about the orders of maximal proper sublattices of a distributive lattice (Theorem 2.5) can be useful in determining the structure of sublattices (Theorem 2.7); this, in turn, can provide further information about orders of sublattices (Corollaries 8 and 9). Our concern now shall be to approach the general question of describing the orders of sublattices of a lattice. What makes matters particularly difficult is the absence, for lattices, of a result as powerful as Lagrange's Theorem for groups.

A useful term arising naturally in this connection is the *spectrum* sp(L) of a lattice L; that is, the set of all integers n such that L has an n-element sublattice. The spectrum of L is *complete* if $\mathrm{sp}(L) = \{n \mid 0 \leq n \leq |L|\}$. Of course, every dismantlable lattice has a complete spectrum (though the converse is false (see, for example, Figure 15)). Therefore, from the standpoint of the spectrum of a lattice, finite dismantlable lattices would on the surface appear to be quite uninteresting. Nevertheless, the notion of a dismantlable lattice will, in the sequel, be a recurring theme.

The outstanding problem in this area is the following.

PROBLEM 9 (G. BIRKHOFF [6, p.19]). *Given* n, *what is the smallest integer* $\psi(n)$ *such that every lattice with at least* $\psi(n)$ *elements contains a sublattice with exactly* n *elements?*

Figure 15



A non-dismantlable lattice with complete spectrum

G. HAVAS and M. WARD [23] have shown that this function exists; indeed, that $\psi(n) \leq n^{3^n}$ for every positive integer n. However, apart from this fact very little else seems to be known about the general behaviour of this function.

As to values of $\psi(n)$ for small n very little is known beyond the fact that $\psi(n) = n$ for $n \leq 6$. Over the years, since FR. KLEIN-BARMEN's proof for the modular case, this has become a part of the folklore of the problem. One way to prove this fact is by means of the next result, which serves also to suggest techniques for the evaluation of $\psi(n)$ for larger n.

THEOREM 1 ([43]). *Let* $k = 1$ *or* 2 *and let* C *be a maximal chain in a lattice* L *of finite length. If* $|L| \geq |C| + k$ *then* C *is contained in a dismantlable sublattice of* L *with precisely* $|C| + k$ *elements.*

Since the sublattices so constructed are in a certain obvious sense "narrow", they turn out to be dismantlable. (Actually, a sublattice containing a maximal chain and only one or two additional elements is dismantlable by virtue of Theorem 3.1.)

Classifying lattices according to length and applying Theorem 1 gives

COROLLARY 2 ([43]). *$\psi(n) = n$ for $n \leq 6$.*

To evaluate $\psi(7)$ and $\psi(8)$ some further technical information is needed.

THEOREM 3 ([43]). *If L is a lattice with n atoms, m coatoms, $n \leq m$, $\ell(L) = 3$ and $\mathrm{Irr}(L) = \phi$ then $3 \leq n \leq m \leq \binom{n}{2}$. Conversely, given integers n, m such that $3 \leq n \leq m \leq \binom{n}{2}$ there is a lattice L with n atoms, m coatoms, $\ell(L) = 3$ and $\mathrm{Irr}(L) = \phi$.*

Of course, an obvious application of Theorem 3 is to establish that ψ can have no fixed points beyond $n = 8$.

COROLLARY 4 ([43]). *$\psi(n) \geq n + 2$ for $n = 7$ and for all $n \geq 9$.*

Together with Theorems 3.1 and 1, Theorem 3 also yields the following corollaries.

COROLLARIES 5 ([43]). *There is a unique smallest lattice without doubly irreducible elements; namely, $\underset{\sim}{2}^3$.*

COROLLARY 6 ([43]). *Every lattice with precisely 9 elements has a complete spectrum. In fact, if $\ell(L) \leq 6$ then every maximal chain in L is contained in a 7-element sublattice of L.*

Returning finally to the evaluation of $\psi(7)$ and $\psi(8)$ the following obvious analogue of Theorem 1 is needed.

THEOREM 7 ([43]). *Every maximal chain* C *in a lattice* L *of finite length satisfying* $|L| \geq |C| + 5$ *is contained in a dismantlable sublattice with precisely* $|C| + 3$ *elements.*

COROLLARY 8 ([43]). $\psi(7) = 9$ *and* $\psi(8) = 8$.

Perhaps somewhat unexpected is the fact that ψ is not in general an increasing function. Furthermore, since, as we shall now see, $\psi(n)$ becomes quite large for certain periodic values of n, the possibility of explicitly determining it for $n \geq 9$ with existing techniques would seem to be rather dim indeed.

In fact, an immediate consequence of Corollary 2.4 is the following estimate.

COROLLARY 9 ([43]). $\psi(3 \cdot 2^{n-2} + 1) \geq 2^n + 1$ *for every* $n \geq 3$.

By taking the linear sum of $\underset{\sim}{2}^n$ with a suitable-sized chain Corollary 9 in turn gives

COROLLARY 10 ([43]). $\psi(3 \cdot 2^{n-2} + k) \geq 2^n + k$ *for all* $k \leq 2^{n-2} - 1$ *and* $n \geq 3$.

On the other hand, we have seen (Section 2) that the order of a maximal proper sublattice of the lattice of subspaces of a projective plane of order n is either 2n + 4 or 2n + 6 and that it has no sublattice of order $2n + 5$ (cf. Figure 6).

COROLLARY 11 ([43]). *$\psi(2n + 5) \geq 2n^2 + 2n + 5$ for all prime power n.*

Taking again the linear sum of the lattice of subspaces of a projective plane of order n with a suitable chain we get the analogue of Corollary 10.

COROLLARY 12 ([43]). *$\psi(2n + k) \geq 2n^2 + 2n + k$ for all k such that $6 < k \leq 2n^2 - 7$ and all prime power n.*

Finally, Corollary 6 suggests the problem of determining bounds on the size of a lattice in order that it have a complete spectrum. In this direction we have the following result.

THEOREM 13 ([42]). *Every modular lattice L of finite length satisfying $|L| \leq \frac{1}{3}(5\ell(L) + 7)$ has a complete spectrum.*

The proof is based on Theorem 3.1 together with the next result which bears an obvious resemblance to Theorems 1 and 7.

THEOREM 14 ([42]). *Every maximal chain C in an upper semimodular lattice L is contained in a dismantlable sublattice of L with precisely $|C| + |C - M(L)|$ elements.*

The order of the dismantlable sublattice prescribed by Theorem 14 is in general maximum. In fact, if for every positive integer n, B_n is the linear sum of n copies of $\underset{\approx}{2}^3$ and C is any maximal chain in B_n, then $|C| = 4n$, $|C - M(L)| = 2n$ and $6n$ is, indeed, the maximum order of a dismantlable sublattice of B_n.

On the other hand, though it is not known whether the bound of Theorem 13 is best possible, it cannot exceed $2(\ell(L) + 1)$. (It is also unknown whether modularity is needed.) The Boolean lattice $\underset{\sim}{2}^3$ does not have a 7-element sublattice; thus, the inequality is best possible for lattices of length 3. Furthermore, the lattice obtained by adjoining a new unit to $\underset{\sim}{2}^3$ (see Figure 15) does have a complete spectrum. The inequality of Theorem 3.1 is, as we have seen (see Figure 8), best possible among all lattices L of finite length such that $\mathrm{Irr}(L) = \phi$. The modular lattices B_n $(n \geq 1)$ described above also satisfy the conditions: $|B_n| = 2(\ell(B_n) + 1)$; $\mathrm{Irr}(B_n) = \phi$. Therefore, $|B_n| - 1 \notin \mathrm{sp}(B_n)$. In particular, for every integer $n \geq 1$, the spectrum of B_n is not complete.

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PROJECTIVE IMAGES OF MODULAR (DISTRIBUTIVE, COMPLEMENTED) LATTICES ARE MODULAR (DISTRIBUTIVE, COMPLEMENTED)

I. RIVAL

A lattice L' is a *projective image* of another lattice L if there is an isomorphism between the lattice of sublattices of L and that of L', that is, $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$. By way of a lengthy and edtailed investigation N. D. Filippov [1] found necessary and sufficient conditions on lattices L, L' in order that L' be a projective image of L. In terms of familiar classes of lattices the main result of [1] is that of the title. We give a simple direct proof.

Let Ψ be an isomorphism of $\mathrm{Sub}(L)$ onto $\mathrm{Sub}(L')$. Restricted to the atoms of Sub (L), Ψ induces a bijection φ from L to L' defined by $\varphi(x)=y$, where $\Psi(\{x\})=\{y\}$.

We show first that if L is modular (distributive) then L' is modular (distributive). It suffices to prove that $\mathrm{Sub}(L) \cong \mathrm{Sub}(N_5)$ $(\mathrm{Sub}(L) \cong \mathrm{Sub}(M_5))$ implies that $L \cong N_5$ $(L \cong M_5)$, where N_5 is the five-element non-modular lattice (M_5 is the five-element modular non-distributive lattice). The comparability relation ($x \leqslant y$ or $x \geqslant y$) in a lattice is determined by its set of two-element sublattices, that is, the set of elements of height two in its lattice of sublattices. But since N_5 and M_5 are as lattices determined up to isomorphism by the comparability relation we are done.

Now let L be a complemented lattice. We show that L' is complemented. Since x is comparable to y in L if and only if $\varphi(x)$ is comparable to $\varphi(y)$ in L', the set of universal bounds $\{0, 1\}$ of L must correspond under Ψ to the set $\{0', 1'\}$ of universal bounds of L', that is, $\Psi(\{0, 1\}) = \{0', 1'\}$. If y is a complement of x in L then $\varphi(y)$ is a complement of $\varphi(x)$ in L'. Indeed,

$$\begin{aligned}\{0', \varphi(x), 1'\} \vee \{0', \varphi(y), 1'\} &= \Psi(\{0, x, 1\}) \vee \Psi(\{0, y, 1\}) = \Psi(\{0, x, 1\} \vee \{0, y, 1\}) \\ &= \Psi(\{0, x, y, 1\}) = \{0', \varphi(x), \varphi(y), 1'\}.\end{aligned}$$

Thus, $\{0', \varphi(x), \varphi(y), 1'\}$ is a sublattice of L' and $\varphi(x)$ is incomparable with $\varphi(y)$.

In particular, if L' is a projective image of a Boolean lattice L then L' is Boolean.

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MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES

IVAN RIVAL

ABSTRACT. A best possible estimate is established for the size and length of maximal proper sublattices of finite distributive lattices.

.. **Introduction.** Papers of H. Sharp [3] and D. Steven [4] have estab-ıed the following result:

ΓHEOREM 1. *If L is a finite Boolean lattice with $|L| \geqq 4$ and M is a ximal proper sublattice of L then* (i) $|M| = \frac{3}{4}|L|$ *and* (ii) $l(M) = l(L)$, *ere $l(L)$ denotes the length of L.*

Γhe purpose of this note is to prove an analogous result for finite dis-ıutive lattices, as well as to provide a simple alternative proof of eorem 1.

ΓHEOREM 2. *If L is a finite distributive lattice with $|L| \geqq 3$ and M is a ximal proper sublattice of L then* (i) $|M| \geqq \frac{2}{3}|L|$ *and* (ii) $l(M) \geqq l(L) - 1$.

Furthermore, these inequalities are best possible in the sense that for ry integer $n \geqq 1$ there is a distributive lattice L_n with a maximal proper lattice M_n such that $|L_n| = 3n$, $|M_n| = 2n$ and $l(M_n) = l(L_n) - 1$ (see ure 1).

. **Preliminaries.** Let $J(L) = \{x \in L | x$ join-irreducible$\}$, $M(L) = \{x \in L | x$ et-irreducible$\}$ and for all further notation and terminology refer to [2]. call that in lattices of finite length every element can be expressed as the ı of all join-irreducibles contained in it and dually.

The basic result we need is that characterizing maximal proper sub-ices of an arbitrary finite distributive lattice [1]. In what follows we vide a new simple proof of this result.

.EMMA 1. *If L is a lattice of finite length and S is a proper sublattice ı there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such that $S \cap [a, b] = \varnothing$.*

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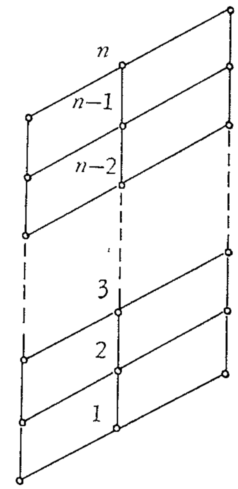

$L_n,\ n \geq 1$

$M_n = L_n - \{1, 2, \cdots, n\}$

FIGURE 1

PROOF. Since S is a proper sublattice there exists $a \in J(L)-S$. Let $\{b \in M(L) | a \leqq b\}$; clearly $B \neq \varnothing$. If for every $b \in B$ there were sc $x_b \in S \cap [a, b]$ then $\bigwedge (x_b | b \in B) \in S$ since S is a complete sublattice. the other hand, $a \leqq \bigwedge (x_b | b \in B) \leqq \bigwedge B = a$, that is, $\bigwedge (x_b | b \in B) = a$ which is a contradiction. Thus, we can conclude that there is some b such that $S \cap [a, b] = \varnothing$.

LEMMA 2. *If L is a distributive lattice, $a \in J(L)$, $b \in M(L)$, and a then $L-[a, b]$ is a sublattice of L.*

PROOF. Let us assume that there exist $x, y \in L-[a, b]$ such $x \vee y \in [a, b]$. Since $a \in J(L)$ and L is distributive we get that $a \leqq x$ or a so that either $x \in [a, b]$ or $y \in [a, b]$, contradicting our choice.

We can now give the basic characterization [**1**].

THEOREM 3. *If L is a finite distributive lattice and M is a maximal pro sublattice, then there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such that* (i) $L-$ $[a, b]$, (ii) $(a, b] \subseteq L-J(L)$ *and* (iii) $[a, b) \subseteq L-M(L)$.

PROOF. (i) By Lemma 1 there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such $M \cap [a, b] = \varnothing$, that is, $M \subseteq L-[a, b] \subset L$. By Lemma 2, $L-[a, b]$ sublattice so that by the maximality of M, $M = L-[a, b]$.

(ii) If there were some $c \in (a, b]$ such that $c \in J(L)$, then in view of ·mma 2, $M=L-[a, b]\subseteq L-[c, b]\subset L$ is a chain of sublattices con- ıdicting the maximality of M. (iii) follows dually.

We are now ready to prove Theorems 1 and 2.

3. **The Boolean case.** Suppose M is a maximal proper sublattice of ǝ Boolean lattice $\mathbf{2}^n$. Without loss of generality we may take $n\geqq 3$ in ıich case it must be that $0, 1 \in M$. By Theorem 3 there exist $a \in J(L)$, $\in M(L)$ such that $L-M=[a, b]$. But then a must be an atom and b a atom, so that $[a, b]\cong \mathbf{2}^{n-2}$, and this, of course, implies that $|M|=\frac{3}{4}|\mathbf{2}^n|$. ıally, if a' is the complement of a, then $[0, a']\cup[a', 1]\subseteq M$, since 0, $\in M$ and $L-M$ is a convex sublattice of L. Since all maximal chains have ǝ same order, $l(M)=l(L)$ which completes the proof of Theorem 1.

4. **The distributive case.** The proof of Theorem 2 amounts essentially an enumeration of sufficiently many joins and meets of subsets of $J(L)$ d $M(L)$, respectively.

We can certainly assume that $|L-M|\geqq 2$ so that by the maximality of , $0, 1 \in M$ and $L-M\subseteq L-(J(L)\cap M(L))$. By Theorem 3 there exist $\in J(L)$, $b \in M(L)$, $a\leqq b$, such that $L-M=[a, b]$.

Now, L is finite, so in particular, for every $x \in (a, b]$ there exists $A\subseteq$ $L)$ such that $x=\bigvee A$. If $A\subseteq M$ then $\bigvee A \in M$ since M is a finite sub- tice. Therefore, $A\cap[a, b]\neq\varnothing$, and, in fact, by Theorem 3, $J(L)\cap$ $b]=\{a\}$, so that $a \in A$. This shows that for every $x \in (a, b]$ we can ect a subset $f(x)$ in $J(L)\cap M$ such that $x=a\vee\bigvee f(x)$, where $\bigvee f(x) \in M$; fact, $\bigvee f(x) \in M-\{0, 1\}$. Moreover, it is immediate that if $x, y \in (a, b]$, $f(x)=\bigvee f(y)$ if and only if $x=y$. Thus, we can conclude that

$$\{\bigvee f(x) \mid x \in (a, b]\} \subseteq M - \{0, 1\}$$

d

$$|\{\bigvee f(x) \mid x \in (a, b]\}| = |(a, b]| = |L - M| - 1.$$

Dually, we get a choice function g from $[a, b)$ into the set of subsets of (L) such that $y=b\wedge\bigwedge g(y)$, for every $y \in [a, b)$, $\{\bigwedge g(y)|y \in [a, b)\}\subseteq$ $-\{0, 1\}$, and $|\{\bigwedge g(y)|y \in [a, b)\}|=|L-M|-1$.

We now claim that $\{\bigvee f(x)|x \in (a, b]\}\cap\{\bigwedge g(y)|y \in [a, b)\}=\varnothing$. In ɔt, suppose there were suitable $x, y \in [a, b]$ such that $\bigvee f(x)=\bigwedge g(y)$. ǝarly, $b\geqq x=a\vee\bigvee f(x)\geqq\bigvee f(x)$ so that $\bigvee f(x)=b\wedge\bigvee f(x)=b\wedge\bigwedge g(y)=$ which is impossible since $\bigvee f(x) \in M$ while $y \in L-M$.

It now follows that $|L|\geqq|L-M|+|\{\bigvee f(x)|x \in (a, b]\}|+|\{\bigwedge g(y)|y \in$ $b)\}|+|\{0, 1\}|=3|L-M|$, from which (i) is an immediate consequence.

To establish (ii) take some $c \in L$ such that $b\succ c\geqq\bigvee f(b)$ (b covers c). $\in [a, b]$ then $c\geqq\bigvee f(b)$ and $c\geqq a$ together imply that $b\succ c\geqq a\wedge\bigvee f(b)=$ which is impossible. Thus, $c \in M$ and by the convexity of $L-M$,

$[0, c] \subseteq M$. Finally, taking a maximal chain in $[0, c]$, adjoining to it maximal chain in $(b, 1]$ and recalling that in distributive lattices maximal chains have the same order, we can conclude that $l(M) \geqq l(L) -$ completing the proof of Theorem 2.

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MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES (II)

Ivan Rival

1. Introduction. Let L be a lattice, $J(L) = \{x \in L \mid x \text{ join-irreducible in } L\}$ and $M(L) = \{x \in L \mid x \text{ meet-irreducible in } L\}$. As is well-known the sets $J(L)$ and $M(L)$ play a central role in the arithmetic of a lattice L of finite length and particularly, in the case that L is distributive. We show (Proposition 1) that the "quotient set" $Q(L) = \{b/a \mid a \in J(L), b \in M(L), a \leq b\}$ plays a somewhat analogous role in the study of the sublattices of a lattice L of finite length. If L is a finite distributive lattice, its quotient set $Q(L)$ in a natural way determines (Theorem 1) the lattice $\mathrm{Sub}(L)$ of all sublattices of L.

By examining (Theorem 2) the connection between $J(K)$ and $J(L)$, where K is a maximal proper sublattice of a finite distributive lattice L, we can derive some useful information about the orders of sublattices of finite distributive lattices; namely, <u>every finite distributive lattice of order</u> $n \geq 3$ <u>which contains a maximal proper sublattice of order</u> m <u>also contains sublattices of orders</u> $n-m$, $2(n-m)$, <u>and</u> $3(n-m)$; and, <u>every finite distributive lattice</u> L <u>contains a maximal proper sublattice</u> K <u>such that either</u> $|K| = |L| - 1$ <u>or</u> $|K| \geq 2\ell(L)$, where $\ell(L)$ denotes the length of L.

The author wishes to thank Barry Wolk for suggesting the proof presented here for Proposition 1. For all terminology not explained here we refer to G. Birkhoff [1].

2. A Connection Between $Q(L)$ and $Sub(L)$. Proposition 1 below serves to underline a basic connection between $Q(L)$ and the sublattices of a lattice L of finite length, a connection which, specialized to finite distributive lattices, has been the motivation for the results presented in this paper.

Proposition 1, in fact, is a generalization of Lemma 1 [2]. We shall throughout adopt the abbreviation $\bigcup_A[a,b]$ for $\bigcup_{b/a\in A}[a,b]$, where $A \subseteq Q(L)$.

Proposition 1. *If S is a sublattice of a lattice L of finite length then* $S = L - \bigcup_A[a,b]$, *for some* $A \subseteq Q(L)$.

Proof. We must show that for every $x \in L - S$ there is some $b/a \in Q(L)$ such that $x \in [a,b] \subseteq L - S$. Let us suppose that this does not hold for some $x \in L - S$. Let $A = \{a \in J(L) \mid a \leq x\}$ and $B = \{b \in M(L) \mid x \leq b\}$; clearly, $A \neq \emptyset \neq B$ and $\bigvee A = x = \bigwedge B$. But then by our assumption, for every $a \in A$ and for every $b \in B$ there exists $y_a^b \in S \cap [a,b]$. Since L is of finite length it is complete; therefore, $\bigvee_{a\in A} \bigwedge_{b\in B} y_a^b \in S$. On the other hand,

$$x = \bigvee A = \bigvee_{a\in A} \bigwedge_{b\in B} a \leq \bigvee_{a\in A} \bigwedge_{b\in B} y_a^b \leq \bigvee_{a\in A} \bigwedge_{b\in B} b = \bigwedge B = x,$$

that is, $\bigvee_{a\in A} \bigwedge_{b\in B} y_a^b = x \in L - S$, which is a contradiction.

In view of Proposition 1 it is natural to classify sublattices of a lattice L of finite length in terms of subsets of $Q(L)$. Indeed, for $A \subseteq Q(L)$ we define $Cl(A) = \{y/x \in Q(L) \mid [x,y] \subseteq \bigcup_A [a,b]\}$ and $Cl(Q(L)) = \{Cl(A) \mid A \subseteq Q(L)\}$.

The following lemma is straightforward.

Lemma 1. *Let* L *be a lattice of finite length and* $A, B \subseteq Q(L)$. *Then*

(i) $\bigcup_A [a,b] = \bigcup_{Cl(A)} [x,y]$ *and*,

(ii) $\bigcup_{Cl(A)} [x,y] \subseteq \bigcup_{Cl(B)} [u,v]$ *if and only if* $Cl(A) \subseteq Cl(B)$.

The next lemma is an easy consequence of Lemma 1.

Lemma 2. *Let* L *be a lattice of finite length. Then*

(i) Cl *is a closure operator on* $Q(L)$ *and*,

(ii) $Cl(Q(L))$ *is a lattice with respect to set inclusion*.

Theorem 1. *For a lattice* L *of finite length the following conditions are equivalent*:

(i) L *is distributive*;

(ii) $L - \bigcup_A [a,b]$ *is a sublattice of* L *for every* $A \subseteq Q(L)$;

(iii) *for every* $S \subseteq L$, S *is a sublattice of* L *if and only if* $S = L - \bigcup_A [a,b]$ *for some* $A \subseteq Q(L)$;

(iv) *the mapping* $\varphi(S) = Cl(A)$, *where* $S = L - \bigcup_A [a,b]$, $A \subseteq Q(L)$, *is an isomorphism between* $Sub(L)$ *and the dual of* $Cl(Q(L))$.

Proof. That (i) implies (ii) follows from the fact that join-irreducible elements in a distributive lattice are *join-prime*, that is, if $a \in J(L)$ and $a \leq b \vee c$ then $a \leq b$ or $a \leq c$. Applying Proposition 1 we get that (ii) implies (iii). On the other hand, Proposition 1 together with Lemma 1(ii) shows that φ is well-defined, one-one, isotone, and

and that, in fact, φ^{-1} is isotone. From (iii) we have that φ is onto, so that φ is, indeed, an isomorphism; thus, (iii) implies (iv). It remains only to show that (iv) implies (i).

Let M_5 and N_5 be the two five-element non-distributive lattices labelled as in Figure 1. Suppose that L satisfies (iv) but L is non-distributive. Then L contains as a sublattice a copy of M_5 or N_5. Let d be a join-irreducible in L such that $d \leq a$ but $d \nleq b$ and $d \nleq c$, and e a meet-irreducible in L such that $e \geq b \vee c$. By the surjectivity of φ, $L - \bigcup_{Cl(\{e/d\})}[x,y]$ is a sublattice of L. In view of Lemma 1(i) $L - \bigcup_{Cl(\{e/d\})}[x,y] = L - [d,e]$. But $b \vee c \in [d,e]$ although $b, c \in L - [d,e]$, which is a contradiction. Thus, (iv) implies (i), completing the proof.

3. Maximal Proper Sublattices of Finite Distributive Lattices. We define a partial ordering on $Q(L)$ as follows:

$$b/a \leq d/c \text{ if and only if } [a,b] \subseteq [c,d].$$

If b/a is minimal with respect to this ordering then $Cl(\{b/a\}) = \{b/a\}$ so that by Theorem 1, $L - [a,b]$ is a maximal proper sublattice of L in the case that L is finite distributive. Note that if $b/a \in Q(L)$ and $a \neq b$ then b/a is minimal if and only if $[a,b) \subseteq L - M(L)$ and $(a,b] \subseteq L - J(L)$, (cf. [2, Theorem 3]).

For $x, y \in L$, x <u>covers</u> y ($x \succ y$ or $y \prec x$) <u>in</u> L if $x > y$ and $x \geq z > y$ implies $x = z$, for every $z \in L$. For $A \subseteq L$ we define $cov(A) = \{x \in L \mid x \succ a \text{ or } x \prec a \text{ or } x = a, \text{ for some } a \in A\}$. Observe that $a \in L - J(L)$ ($a \in L - M(L)$) if and only if there exist $b, c \in cov(\{a\})$ such that $a = b \vee c$ ($a = b \wedge c$).

Theorem 2. <u>Let</u> L <u>be a finite distributive lattice and</u> $K = L - [a,b]$ $(b/a \in Q(L),\ a \neq b)$ <u>be a maximal proper sublattice of</u> L. <u>Then</u> (i) $\mathrm{cov}([a,b])$ <u>is a sublattice of</u> L <u>isomorphic to the direct product of</u> $[a,b]$ <u>with a three-element chain, and</u> (ii) $J(K) = (J(L) - \{a\}) \cup \{c\}$, <u>where</u> $a \prec c \in L - [a,b]$.

Proof. Set

$$A = \{y \in L \mid y \prec x \text{ for some } x \in [a,b]\}$$

$$B = \{y \in L \mid y \succ x \text{ for some } x \in [a,b]\}$$

$$A' = \{x \in [a,b] \mid x \succ y \text{ for some } y \in A\}$$

$$B' = \{x \in [a,b] \mid x \prec y \text{ for some } y \in B\}.$$

To establish (i) it suffices to show that $A \cong [a,b] \cong B$. Since b/a is minimal in $Q(L)$ and $a \neq b$, $a \neq 0$ and $b \neq 1$; thus, $a \in A'$ and $b \in B'$. Furthermore, since $L - [a,b]$ is a sublattice of L, every element in A' covers precisely one element in A and every element in B' is covered by precisely one element in B.

Suppose now that c_1', c_2' are distinct minimal elements in B' with covers $c_1, c_2 \in B$. Since $[a,b]$ is a sublattice of L, $c_1 \neq c_2$; since c_1' is incomparable with c_2', c_1 is incomparable c_2; and since $L-[a,b]$ is a sublattice of L, $c_1, c_2 > c_1 \wedge c_2 \in L - [a,b]$. Now, if $c_2' = c_2' \vee (c_1 \wedge c_2)$ then $c_2' \wedge c_2' < c_1 \wedge c_2 < c_2'$, so that $c_1 \wedge c_2 \in [a,b]$. Therefore, $c_2' < c_2' \vee (c_1 \wedge c_2) \leq c_2$ and, since $c_2' \prec c_2$, we have that $c_2' \prec c_2' \vee (c_1 \wedge c_2) = c_2$ which, by transposition implies that $c_2' \wedge (c_1 \wedge c_2) \prec c_1 \wedge c_2$. But $c_1' \wedge c_2' \leq c_2' \wedge (c_1 \wedge c_2) < c_2'$ so that $c_2' \wedge (c_1 \wedge c_2) \in B'$, contradicting the minimality of c_2'. Thus, B' has a unique minimal element c' with precisely one cover c in B; dually, A' has a unique maximal element d' covering precisely one element d in A. Now, if f is the unique cover of b and e the unique element covered by a then by transposition we have that $A = [e,d] \cong [a,d'] = A'$ and $B = [c,f] \cong [c',b] = B'$. From this it follows that $\mathrm{cov}([a,b]) = A \cup [a,b] \cup B$ is a sublattice of L and that in fact, b/a is minimal in $Q(\mathrm{cov}([a,b]))$. In this case $A \cup B$ is a maximal proper sublattice of $\mathrm{cov}([a,b])$ so that by Theorem 2 [2], $|A \cup B| \geq \frac{2}{3}|\mathrm{cov}([a,b])|$. Now, if $d < b$ or $a < c$ then $|\mathrm{cov}([a,b])| = |A| + |[a,b] + |B| < 3\,|[a,b]|$. But $[a,b] = \mathrm{cov}([a,b]) - (A \cup B)$ so that $|A \cup B| < \frac{2}{3}|\mathrm{cov}([a,b])|$, which is a contradiction. Thus, $a = c'$ and $b = d'$ so that $A \cong [a,b] \cong B$, form which (i) follows.

To show (ii) observe first that $J(L) - \{a\} \subsetneq J(K)$ and $J(K) \cap A \subsetneq J(L) - \{a\}$. It suffices then to show that $J(K) \cap B = \{c\}$.

Let $x \in B - \{c\}$. Choose some $y \in B$ such that $x \succ y$. Then there exist $x_1\ y_1 \in [a,b]$ and $x_2, y_2 \in A$ such that $x \succ x_1 \succ x_2$ and $y \succ y_1 \succ y_2$. By transposition $x_1 \succ y_1$, $x_2 \succ y_2$, and $x_1 \wedge y = y_1$. If $x_2 < y$ then $y_1 = x_1 \wedge y \geq x_2 \succ y_2$, and since $y_1 \succ y_2$ we have that $y_1 = x_2$, which is impossible. Thus, x_2 is incomparable with y and, in fact, x covers x_2 in K, and since x also covers y in K, we get that x is join-reducible in K.

It remains only to show that $c \in J(K)$. We may without loss of generality assume that c covers two distinct incomparable elements $c_1, c_2 \in L$, both incomparable with a. But a is join-irreducible in L, that is, it covers only e. By transposition we get that $\{a, c_1, c_2, e, c\}$ is a sublattice of L isomorphic to the five-element modular, non-distributive lattice M_5 which, of course, is a contradiction. The proof of the theorem is now complete.

The following corollary is an immediate consequence of Theorem 2(i).

Corollary 1. Every distributive lattice of order $n \geq 3$ which contains a maximal proper sublattice of order m also contains sublattices of orders $n-m$, $2(n-m)$, and $3(n-m)$.

Corollary 2. Every finite distributive lattice L contains a maximal proper sublattice K such that either $|K| = |L| - 1$ or $|K| \geq 2\ell(L)$.

Proof. We may without loss of generality assume that $\mathrm{Irr}(L) = \emptyset$. Recall that for finite distributive lattices $|J(L)| = \ell(L) + 1 = |M(L)|$. Furthermore, the inequality $|L| \geq |J(L)| + |M(L)| - |\mathrm{Irr}(L)|$ holds in every lattice L of finite length, so that if L is distributive we have that $|L| \geq 2(\ell(L) + 1) - |\mathrm{Irr}(L)|$. (This latter inequality, incidentally, holds in every lattice of finite length, cf. [3, Theorem 1].)

If $J(K) = J(L) - \{a\}$ then $M(K) = M(L) - \{b\}$, and since $J(L) \cap M(L) = \mathrm{Irr}(L) = \emptyset$ we also have that $\mathrm{Irr}(K) = \emptyset$. In this case

$$|K| \geq 2(\ell(K) + 1) - |\mathrm{Irr}(K)| = 2|J(L) - \{a\}| = 2\ell(L) .$$

Otherwise, $J(K) \neq J(L) - \{a\}$. By Theorem 2(ii) and its dual there exist $c, d \in L$ such that $J(K) = (J(L) - \{a\}) \cup \{c\}$, $c \notin J(L)$, and $M(K) = (M(L) - \{b\}) \cup \{d\}$, $d \notin M(L)$. Observe that $(J(L) - \{a\}) \cap (M(L) - \{b\}) \subseteq J(L) \cap (M(L) = \mathrm{Irr}(L) = \phi$. Therefore, $\mathrm{Irr}(K) \subseteq \{c,d\}$, so that in this case

$$|K| \geq 2(\ell(K) + 1) - |\mathrm{Irr}(K)| \geq 2|(J(L) - \{a\}) \cup \{c\}| + 1) - 2 = 2\ell(L).$$

The estimate on the order of maximal proper sublattices of finite distributive lattices prescribed in Corollary 2 is best possible in the sense that, if for every positive integer n, L_n is the ordinal sum of n copies of the Boolean lattice $\underset{\sim}{2}^3$, then the <u>maximum</u> order of a maximal proper sublattice of L_n is $2\ell(L_n)$.

References

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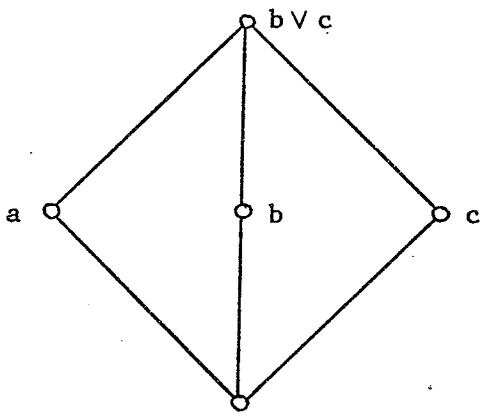
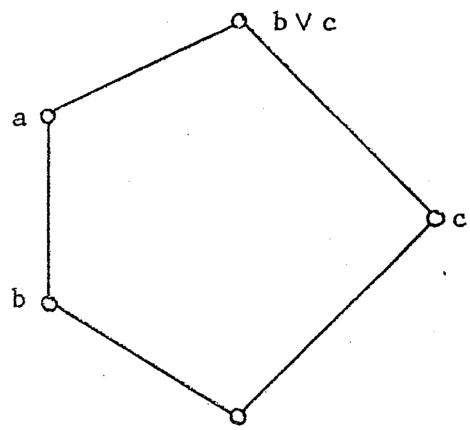


M_5



N_5

Figure 1.

Corrections "Maximal Sublattices of Distributive Lattices (II)"

Line	reads	should read
7-8	... $d \nleq b$ and $d \nleq c$,	... $d \nleq b \wedge c$
9	surjectivity of φ	surjectivity of φ^{-1} ...
19	that if $b/a \in Q(L)$ and $a \neq b$ then b/a is minimal ...	that if $b/a \in Q(L)$ then b/a is minimal ...
7	$A = \{y \in L \mid y \prec x$ for some $x \in [a,b]\}$	$A = \{y \in K \mid y \prec x$ for some $x \in [a,b]\}$
8	$B = \{y \in L \mid y \succ x$ for some $x \in [a,b]\}$	$B = \{y \in K \mid y \succ x$ for some $x \in [a,b]\}$
18	... c_1 is incomparable c_2;	... c_1 is incomparable with c_2;
20	... $c_2' \wedge c_2'$ ...	... $c_1' \wedge c_2'$ ...
14	... form	... from ....

LATTICES WITH DOUBLY IRREDUCIBLE ELEMENTS

by

IVAN RIVAL

Introduction. An element x in a lattice L is join-reducible (meet reducible) in L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z$ $(x = y \wedge z)$; x is join-irreducible (meet-irreducible) in L if it is not join-reducible (meet-reducible) in L; x is doubly irreducible in L if it is both join- and meet-irreducible in L. Let $J(L)$, $M(L)$, and $Irr(L)$ denote the set of all join-irreducible elements in L, meet-irreducible elements in L, and doubly irreducible elements in L, respectively, and $\ell(L)$ the length of L, that is, the order of a maximum-sized chain in L minus one.

In this paper we investigate some combinatorial properties of lattices in terms of their doubly irreducible elements. First, we show (Theorem 1) that any lattice L of finite length satisfies $|L| \geq 2(\ell(L) + 1) - |Irr(L)|$, an inequality which, among all lattices L of finite length such that $Irr(L) = \phi$, is best possible. This inequality is in turn useful in the computation (Corollary 1) of orders of sublattices of "small" lattices.

Next, we examine and characterize (Theorem 2) dismantlable lattices, that is, lattices which can be completely "dismantled" by removing one element at a time leaving a sublattice at each stage. All finite planar lattices are dismantlable [1]; furthermore, given a positive integer n, any large enough lattice ($|L| \geq n^{3^n}$ will do [3][2, p. 67]) contains a dismantlable sublattice with precisely n elements.

Finally, if $Sub(L)$ denotes the lattice of all sublattices of a lattice L, we show that every lattice L such that $\ell(\mathrm{Sub}(L))$ is finite satisfies $\ell(\mathrm{Sub}(L)) = |\mathrm{Irr}(L)| + \ell(\mathrm{Sub}(L - \mathrm{Irr}(L)))$.

n Inequality. Let C be a chain of maximum order in a lattice L of 'inite length and $x_1 < x_2 < \dots < x_n$ a labelling of C. Since every element in a lattice of finite length can be represented as a oin of all the join-irreducibles that it contains, there is a one - one choice function f from C into J(L) defined as follows: $f(x_1) \leq x_1$; $f(x_i) \leq x_i$ and $f(x_i) \not\leq x_{i-1}$ for every $i = 2,3,\dots n$. Thus, $|J(L)| \geq |C|$; ually, we have that $|M(L)| \geq |C|$. Combining these inequalities with he fact that $|L| \geq |J(L)| + |M(L)| - |\mathrm{Irr}(L)|$ establishes

THEOREM 1. Every lattice L of finite length satisfies the inequality

$$|L| \geq 2(\ell(L) + 1) - |\mathrm{Irr}(L)|.$$

Among all lattices L of finite length such that $\mathrm{Irr}(L) = \emptyset$ this inequality is best possible in the sense that for every integer $n \geq 3$ there is a lattice L_n such that $\mathrm{Irr}(L_n) = \emptyset$, $\ell(L_n) = n$ and $|L_n| = 2(\ell(L) + 1)$ (see Figure 1).

Once we observe that $L - A$ is a sublattice of L for every $A \subsetneq \mathrm{Irr}(L)$ the following corollary is immediate.

COROLLARY 1. If n is a positive integer and L is a lattice of finite length satisfying $|L| \leq 2(\ell(L) + 1) - n$, then there is a chain $S_n \subset S_{n-1} \subset \cdots \subset S_0 = L$ of sublattices of L such that $|S_i| = |S_{i-1}| - 1$ for every $i = 1,2,\cdots,n$.

Dismantlable Lattices. With every finite lattice L we can associate a family of sublattices defined as follows: $L_0 = L$; $L_i = L_{i-1} - \mathrm{Irr}(L_{i-1})$ for $i = 1,2,\cdots$. (Note that $\mathrm{Irr}(L_i) \cap \mathrm{Irr}(L_j) = \emptyset$ if $i \neq j$.) In this way we obtain a descending chain $L = L_0 \supset L_1 \supset \cdots$ of sublattices of L which, since L is finite, must end; that is, there is a smallest integer n such that either $L_n = \emptyset$ or $\mathrm{Irr}(L_n) = \emptyset$. A finite lattice L is dismantlable if there is an integer n such that $L_n = \emptyset$ (or equivalently, $L = \bigcup_{i=0}^{n} \mathrm{Irr}(L_i)$).

It was shown in [1] that every finite planar lattice has a doubly irreducible element. Since, plainly, any sublattice of a planar lattice is planar, it follows that every finite planar lattice is dismantlable. On the other hand, the lattice of Figure 2 illustrates that not every dismantlable lattice is planar.

If $|L| \leq 5$ it is easy to verify that L is dismantlable. Now suppose that $|L| = 6$. If $\ell(L) \leq 2$ then certainly L is dismantlable; if $\ell(L) \geq 3$ they by Corollary 1, L has a 5-element sublattice (which is dismantlable) so that L is dismantlable. If $|L| = 7$ a similar argument shows that L is dismantlable. However, for every integer $n \geq 8$ there is a lattice of order n which is not dismantlable (for example, the ordinal sum of the Boolean lattice $\underset{\sim}{2}^3$ with a chain of order $n - 8$).

G. Havas and M. Ward [3] have shown that any lattice L such that $|L| \geq n^{3^n}$ contains a sublattice of order n. In fact, their proof shows that if $|L| \geq n^{3^n}$ then L contains a dismantlable sublattice of order n (cf. [2, p.67]).

THEOREM 2. For a finite lattice L the following conditions are equivalent:

(i) L is dismantlable.

(ii) $\ell(\mathrm{Sub}(L)) = |L|$.

(iii) $\mathrm{Irr}(S) \neq \phi$ for every sublattice S of L.

(iv) For every chain C in L there is a positive integer n and a chain $C = S_0 \subset S_1 \subset \cdots \subset S_n = L$ of sublattices of L such that $|S_i| = |S_{i-1}| + 1$ for every $i = 1,2,\cdots,n$.

We shall need the following lemma.

LEMMA 1. Let C be a maximal chain in a lattice L of finite length and S a subset of L disjoint from $\mathrm{Irr}(L) \cap C$. Then S is a sublattice of $L - (\mathrm{Irr}(L) \cap C)$ containing $C - (\mathrm{Irr}(L) \cap C)$ if and only if $S \cup (\mathrm{Irr}(L) \cap C)$ is a sublattice of L containing C.

Proof. The "if" part is obvious. Let S be a sublattice of $L - (\mathrm{Irr}(L) \cap C)$ containing $C - (\mathrm{Irr}(L) \cap C)$. It suffices to show that for every $x \in \mathrm{Irr}(L) \cap C$ and $y \in S$ such that x is incomparable with y, $x \vee y, x \wedge y \in S \cup (\mathrm{Irr}(L) \cap C)$. Now take $x = x_0 \prec x_1 \prec \cdots \prec x_r = x \vee y$ to be a covering chain between x and $x \vee y$ (x_i covers x_{i-1} for every $i = 1,2,\cdots,r$). Since x is doubly irreducible in L, x_1 is its unique cover and since C is a maximal chain, $x_1 \in C$. If x_1 is not doubly irreducible in L then $x_1 \in C - (\mathrm{Irr}(L) \cap C)$, otherwise $x_2 \in C$. Iterating, there exists a positive integer $i \leq r$ such that $x_i \in C - (\mathrm{Irr}(L) \cap C)$. Thus, $x \vee y \leq x_i \vee y \leq x \vee y$ and since $x_i, y \in S$ we have that $x \vee y = x_i \vee y \in S$. A dual argument shows that $x \wedge y \in S$.

Proof of Theorem 2. That each of (ii), (iii), and (iv) implies (i) is obvious, as is (i) implies (ii).

(i) implies (iii): Let S be an arbitrary sublattice of a dismantlable lattice L. We show that $\mathrm{Irr}(S) \neq \emptyset$. Let m be the smallest integer such that $S \cap (\bigcup_{i=0}^{m} \mathrm{Irr}(L_i)) \neq \emptyset$. If x is join-reducible in S then there exist $y, z \in S$ both distinct from x such that $x = y \vee z$. Now if $y \in \mathrm{Irr}(L_i)$ and $z \in \mathrm{Irr}(L_j)$, for $i, j \geq m$, then $y, z \in L_m$, which is impossible since $x \in \mathrm{Irr}(L_m)$. Otherwise, either $i < m$ or $j < m$, which, however, contradicts the minimality of m. In any case then, x must be join-irreducible in S and dually, x must be meet-irreducible in S, that is, $x \in \mathrm{Irr}(L)$.

(i) implies (iv): Let C be a chain in a dismantlable lattice L. Without loss of generality we may take C to be a maximal chain in L. We proceed by induction on $|L|$. By assumption $\mathrm{Irr}(L) \neq \emptyset$.

If $\mathrm{Irr}(L) \cap C = \emptyset$ and $x \in \mathrm{Irr}(L)$ then clearly $L - \{x\}$ is a dismantlablesublattice of L containing C. Applying the inductive hypothesis to $L - \{x\}$ we are done.

If $\mathrm{Irr}(L) \cap C \neq \emptyset$ then $L - (\mathrm{Irr}(L) \cap C)$ is a dismantlable sublattice of L. Now take B a maximal chain in $L - (\mathrm{Irr}(L) \cap C)$ containing $C - (\mathrm{Irr}(L) \cap C)$. Applying the inductive hypothesis we get a chain $B = S'_0 \subset S'_1 \subset \cdots \subset S'_m = L - (\mathrm{Irr}(L) \cap C)$ of sublattices of L such that $|S'_i| = |S'_{i-1}| + 1$ for every $i = 1,2,\cdots,m$. Now let $B - C = \{b_1, b_2, \cdots, b_k\}$ ($B - C$ may be empty) and define a chain of subsets of L as follows: $S_0 = C$; $S_j = C \cup \{b_1, b_2, \cdots, b_j\}$ for every $j = 1,2,\cdots,k$; $S_{k+i} = S_k \cup S'_i$ for every $i = 1,2,\cdots,m$. Finally, in view of Lemma 1, $S_0, S_1, \cdots, S_{k+m}$ are all sublattices of L. The proof of the theorem is now complete.

COROLLARY 2. Every sublattice and epimorphic image of a dismantlable lattice is dismantlable.

Proof. The first part follows at once from Theorem 2(iii).

That epimorphic images of a dismantlable lattice are dismantlable we prove in the more convenient terminology of congruence relations. Let L be dismantlable and $\underline{\Theta}$ be a congruence relation on L. We show that the quotient $L/\underline{\Theta}$ is dismantlable. Since every sublattice of $L/\underline{\Theta}$ is of the form $S/\underline{\Theta}_S$, where S is a sublattice of L and $\underline{\Theta}_S$ is the restriction of Θ to S, it suffices by Theorem 2(iii) to prove that $\mathrm{Irr}(S/\underline{\Theta}_S) \neq \phi$ for every sublattice S of L. This we do by induction on $|S|$.

Let S be a sublattice of L. By the first part S is dismantlable so in particular there is an $x \in \mathrm{Irr}(S)$. Again $S - \{x\}$ is a sublattice of L and therefore, by the inductive hypothesis $\mathrm{Irr}(S-\{x\}/\underline{\Theta}_{S-\{x\}}) \neq \phi$. If the congruence class $[x]\underline{\Theta}_S$ has at least two elements then $S/\underline{\Theta}_S \cong S - \{x\}/\underline{\Theta}_{S-\{x\}}$ and we are done. Otherwise $[x]\underline{\Theta}_S = \{x\}$.

If $[x]\underline{\Theta}_S = [y]\underline{\Theta}_S \vee [z]\underline{\Theta}_S$, where $y, z \in S$, then $x \equiv y \vee z \ (\underline{\Theta}_S)$ which implies that $x = y \vee z$. But $x \in \mathrm{Irr}(S)$ so that $x = y$ or $x = z$, that is, $[x]\underline{\Theta}_S = [y]\underline{\Theta}_S$ or $[x]\underline{\Theta}_S = [z]\underline{\Theta}_S$. Thus, $[x]\underline{\Theta}_S$ is join-irreducible in $S/\underline{\Theta}_S$, and by a dual argument, $[x]\underline{\Theta}_S$ is meet-irreducible in $S/\underline{\Theta}_S$ as well. Thus, $\mathrm{Irr}(S/\underline{\Theta}_S) \neq \phi$ and the induction is complete.

REMARK. If L is a dismantlable lattice then there is a positive integer n such that $L = \bigcup_{i=0}^{n} \mathrm{Irr}(L_i)$ and in fact, $\ell(\mathrm{Sub}(L)) = |\bigcup_{i=0}^{n} \mathrm{Irr}(L_i)|$. An analogous result holds in a more general context.

<u>Any lattice L such that $\ell(\mathrm{Sub}(L))$ is finite satisfies $\ell(\mathrm{Sub}(L)) =$
$= |\mathrm{Irr}(L)| + \ell(\mathrm{Sub}(L - \mathrm{Irr}(L)))$.</u>

We show by induction on $\ell(\mathrm{Sub}(L))$ that if $\mathrm{Irr}(L) \neq \emptyset$ then $\ell(\mathrm{Sub}(L)) = 1 + \ell(\mathrm{Sub}(L - \{x\}))$ for every $x \in \mathrm{Irr}(L)$. Observe that

(1) $\ell(\mathrm{Sub}(L)) = 1 + \max(\ell(\mathrm{Sub}(M)) \mid M$ maximal proper sublattice of $L)$.

Suppose that the maximum in (1) is attained by some maximal proper sublattice M which is not of the form $L - \{x\}$ where $x \in \mathrm{Irr}(L)$. Since M is maximal $\mathrm{Irr}(L) \subseteq M$. In particular, $\mathrm{Irr}(L) \subseteq \mathrm{Irr}(M)$ and $\mathrm{Irr}(M) \neq \emptyset$. By the inductive hypothesis

(2) $\ell(\mathrm{Sub}(M)) = 1 + \ell(\mathrm{Sub}(M - \{x\}))$ for every $x \in \mathrm{Irr}(M)$.

Now if x is an arbitrary doubly irreducible element in L, $M - \{x\} \subset L - \{x\}$, so that

(3) $\ell(\mathrm{Sub}\ (M - \{x\})) \leq \ell(\mathrm{Sub}\ (L - \{x\})) - 1$.

Combining (2) and (3), and bearing in mind the choice of M in (1) we get that $\ell(\mathrm{Sub}\ (M)) = \ell(\mathrm{Sub}(L - \{x\}))$ and we are done.

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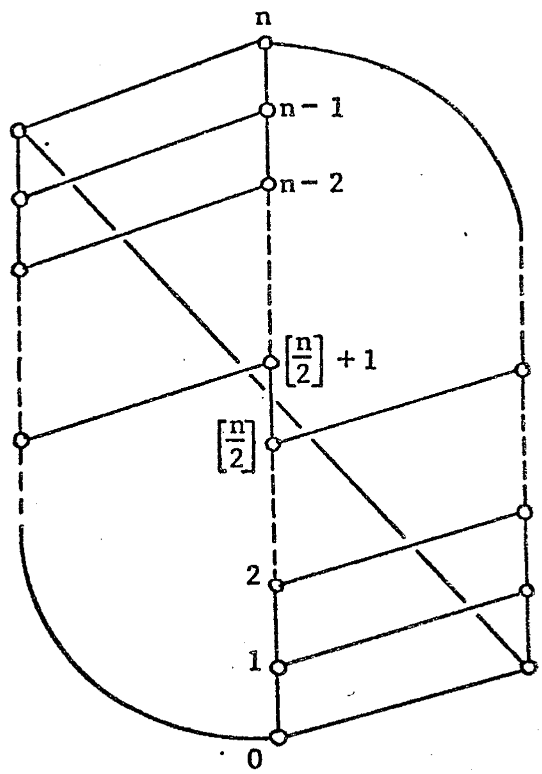

L_n $(n \geq 3)$

Figure 1

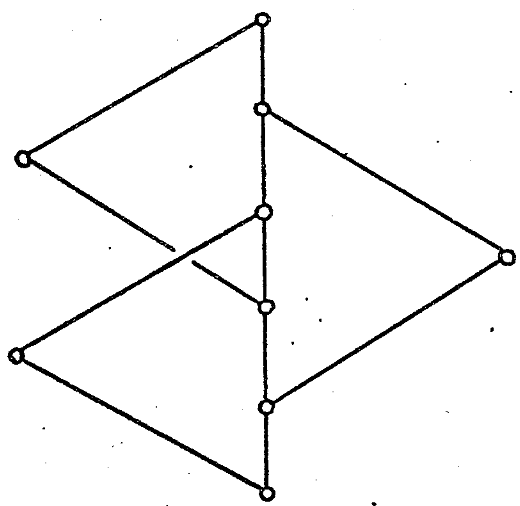

Figure 2

Finite Modular Lattices with Sublattices of all Orders

IVAN RIVAL

The spectrum $sp(L)$ of a lattice L is the set of all integers n such that L has an n-element sublattice; the spectrum of L is complete if $sp(L) = \{n | 0 \leq n \leq |L|\}$. Let $\ell(L)$ denote the length of L, that is, the order of a maximum-sized chain in L minus one.

The purpose of this note is to prove the

THEOREM. *Every modular lattice L of finite length satisfying $|L| \leq \frac{1}{3}(5\ell(L) + 7)$ has a complete spectrum.*

1. PRELIMINARIES. Let $J(L) = \{x \in L | x$ join-irreducible in $L\}$, $M(L) = \{x \in L | x$ meet-irreducible in $L\}$, and $Irr(L) = J(L) \cap M(L)$, that is, the set of all doubly irreducible elements in L. For $x, y \in L$, x covers y ($x \succ y$ or $y \prec x$) in L if $x > y$ and $x \geq z > y$ implies $x = z$, for every $z \in L$. A lattice L of finite length is upper (lower) semimodular if $x \succ x \wedge y$ ($x \vee y \succ x$) implies $x \vee y \succ y$ ($y \succ x \wedge y$), for every $x, y \in L$; L is modular if it is both upper and lower semimodular (cf. G. Birkhoff [1]). A finite lattice L of order n is dismantlable [2] if there is a chain $\phi = S_0 \subset S_1 \subset \cdots \subset S_n = L$ of sublattices of L such that $|S_i| = |S_{i-1}| + 1$, for every $i = 1, 2, \cdots, n$. Clearly, every dismantlable lattice has a complete spectrum; the converse is, however, false (see, for example, the lattice illustrated in Figure 1).

We shall use the following lemmas in the proof of the theorem. The first was already established in [2, Theorem 1], but since its proof is short we include it here for completeness.

LEMMA 1. <u>Every lattice L of finite length satisfies the inequality</u> $|L| \geq 2(\ell(L) + 1) - |\mathrm{Irr}(L)|$.

<u>Proof</u>. Let $x_1 < x_2 < \cdots < x_n$ be a labelling of a chain C of maximum order in L . Since every element in L is representable as a join of all the join-irreducibles that it contains, there is a one-one choice function f from C into $J(L)$ defined as follows: $f(x_1) \leq x_1$; $f(x_i) \leq x_i$ and $f(x_i) \nleq x_{i-1}$, for every $i = 2, 3, \ldots, n$. Thus, $|J(L)| \geq |C|$; dually, we have that $|M(L)| \geq |C|$. Combining these inequalities with the fact that $|L| \geq |J(L)| + |M(L)| - |\mathrm{Irr}(L)|$ establishes the result.

The next lemma is an immediate consequence of Lemma 1.

LEMMA 2. <u>If in a lattice L of finite length</u> $|L| < 2(\ell(L) + 1)$ <u>then</u> $\mathrm{Irr}(L) \neq \emptyset$.

LEMMA 3. <u>Every maximal chain C in an upper semimodular lattice L is contained in a dismantlable sublattice of L with precisely</u> $|C| + |C - M(L)|$ <u>elements</u>.

<u>Proof</u>. Let $x_1 < x_2 < \cdots < x_n$ be a labelling of $C - M(L)$. Note that each x_i has a cover in $L - C$. We now give an inductive procedure for selecting a subset $\{y_i | i = 1, 2, \cdots, n\}$ of these covers: let $y_1 \in L - C$ be an arbitrary cover of x_1; for $i > 1$, if $x_i \vee y_{i-1} \in C$ take $y_i \in L - C$ to be an arbitrary cover of x_i and, if $x_i \vee y_{i-1} \in L - C$ let $y_i = x_i \vee y_{i-1}$. Set $S = C \cup \{y_i | i = 1, 2, \cdots, n\}$; clearly, $|S| = |C| + |C - M(L)|$. Since L is upper semimodular, $y_i \succ x_i$, for every $i = 1, 2, \cdots, n$. Thus, S is closed with respect to $\wedge$. To show that S is closed with respect to $\vee$ we shall need the following fact:

(1) <u>if</u> $i > j$ <u>and</u> $x_i \vee y_j \in L - C$ <u>then</u> $y_j < y_{j+1} < \cdots < y_i$ <u>is a maximal chain in</u> S <u>between</u> y_j <u>and</u> y_i.

We prove (1) by induction on $i - j$. If $i = j + 1$ we have it by the construction above. If $i > j + 1$ then $x_{i-1} \vee y_j \in L - C$ (otherwise, $x_{i-1} \vee y_j \in C$ and by upper semimodularity $x_{i-1} \prec x_{i-1} \vee y_j \leq x_i$ which implies $y_j \leq x_i$). Thus, by the inductive hypothesis, $y_j < y_{j+1} < \cdots < y_{i-1}$ is a maximal chain in S between y_j and y_{i-1}. If $x_i \vee y_{i-1} \in C$ then by upper semimodularity $x_i \prec x_i \vee y_{i-1}$ and $x_i \prec x_i \vee y_j \leq x_i \vee y_{i-1}$ which implies that $x_i \vee y_j = x_i \vee y_{i-1} \in C$, contradicting our assumption. Thus, $x_i \vee y_{i-1} \in L - C$, so that $y_i = x_i \vee y_{i-1} > y_{i-1}$, proving (1).

Suppose now that x_i is incomparable with y_j for some i and j, and that $x_i \vee y_j \in L - C$. Then $i > j$ and by (1) $y_j < y_i$ Since $y_i \succ x_i$, $x_i \vee y_j = y_i$. This shows that $x_i \vee y_j \in S$, for all $i, j = 1, 2, \cdots, n$. If $x \in C$ and x is incomparable with some y_j, and $x_m = \bigwedge(x_i | x_i \geq x)$, then $x \vee y_j = x_m \vee y_j$. Thus, $x \vee y_j \in S$, for every $x \in C$ and for every $j = 1, 2, \cdots, n$. If y_i is incomparable with y_j, for some i and j, $i > j$, then x_i is incomparable with y_j. By (1), $x_i \vee y_j \in C$; by upper semimodularity $x_i \vee y_j \succ x_i$ and $y_i \vee y_j = y_i \vee (x_i \vee y_j) \succ y_i$. But in view of the preceding observations $y_i \vee (x_i \vee y_j) \in S$, that is, $y_i \vee y_j \in S$. Thus, we have shown that S is closed with respect to $\vee$.

It remains yet to show that S is dismantlable. By our construction $y_1 \in J(S)$. If $y_1 \notin M(S)$ then y_1 has at least two covers in S, one of which must be some y_i, and since $y_i \succ x_i$, $x_i \vee y_1 = y_i \in L - C$. Applying (1) we get that $i = 2$. We conclude that there is only one y_i covering y_1 in S, namely, y_2. But then there must be some $x \in C$ such that $x \succ y_1$ in S. Clearly, $x > x_2$ so that $x \geq x_2 \vee y_1 = y_2 \succ y_1$ in S, which implies that $x = y_2$, a contradiction. Therefore, $y_1 \in \mathrm{Irr}(S)$. Iterating this argument we have that $y_i \in \mathrm{Irr}(S - \{y_1, y_2, \cdots, y_{i-1}\})$, for every $i = 2, 3, \cdots, n$ and since C is itself dismantlable, we get that S is dismantlable. The proof of the lemma is now complete.

The order of the dismantlable sublattice prescribed by Lemma 3 is in general maximum. In fact, if for every positive integer n, L_n is the ordinal sum of n copies of the Boolean lattice $\underset{\sim}{2}^3$ and C is any maximal chain in L_n, then $|C| = 4n$, $|C - M(L)| = 2n$ and it is easy to verify that $6n$ is, indeed, the maximum order of a dismantlable sublattice of L_n.

2. PROOF OF THE THEOREM. The idea of the proof is to construct a dismantlable sublattice of L of order at least $|L| - |\mathrm{Irr}(L)| - 1$ which would do since $\{n \mid |L| - |\mathrm{Irr}(L)| \leq n \leq |L|\} \subseteq \mathrm{sp}(L)$.

We proceed by induction on $|L|$. Since $\frac{1}{3}(5\ell(L) + 7) < 2(\ell(L) + 1)$ we have by Lemma 2 that $\mathrm{Irr}(L) \neq \emptyset$. If there exists some $x \in \mathrm{Irr}(L)$ and a maximal chain C in L such that $x \notin C$ then $\ell(L - \{x\} = \ell(L)$ (all maximal chains in a modular lattice of finite length have the same order) from which it follows that $|L - \{x\}| \leq \frac{1}{3}(5\ell(L - \{x\}) + 7)$. Once we apply the inductive hypothesis to the sublattice $L - \{x\}$ we are done. Thus, without loss of generality, we may assume that

(1) $\mathrm{Irr}(L) \subseteq C$, <u>for every maximal chain in</u> L.

From (1) we get that

(2) _every_ $x \in \mathrm{Irr}(L)$ _is comparable with every_ $y \in L$,
which, in turn, shows that

(3) _if_ S _is a sublattice of_ L and $A \subseteq \mathrm{Irr}(L)$ _then_ $S \cup A$ _is a sublattice of_ L.

Let $L' = L - \mathrm{Irr}(L)$. Without loss of generality $L' \neq \emptyset$. If there exists some $x \in J(L') - J(L)$ then x must cover precisely two elements $a, b \in L$, one of which must be doubly irreducible in L. But a is incomparable with b so that in view of (2) this cannot occur. Therefore, $J(L') \subseteq J(L)$, and dually, $M(L') \subseteq M(L)$. In particular, $\mathrm{Irr}(L') = J(L') \cap M(L') \subseteq J(L) \cap M(L) = \mathrm{Irr}(L)$. This shows that

(4) $\mathrm{Irr}(L') = \emptyset$.

Applying (1) again we get that

(5) $\ell(L') = \ell(L) - |\mathrm{Irr}(L)|$.

Let C be a maximal chain in L'. By virtue of (4) either $|C - M(L')| \geq \dfrac{|C|}{2}$ or $|C - J(L')| \geq \dfrac{|C|}{2}$. Using Lemma 3, its dual, and (5) we get a dismantlable sublattice S of L' satisfying $|S| \geq \frac{3}{2}|C| = \frac{3}{2}(\ell(L) - |\mathrm{Irr}(L)| + 1)$. By (3) and since S is dismantlable, we have that $\{n \mid 0 \leq n \leq |S| + |\mathrm{Irr}(L)|\} \subseteq \mathrm{sp}(L)$. On the other hand, $\{n \mid |L| - |\mathrm{Irr}(L)| \leq n \leq |L|\} \subseteq \mathrm{sp}(L)$. It suffices then to show that

$$\tfrac{3}{2}(\ell(L) - |\mathrm{Irr}(L)| + 1) + |\mathrm{Irr}(L)| \geq |L| - |\mathrm{Irr}(L)| - 1 ,$$

or, equivalently, that

(6) $\frac{3}{2}(\ell(L)+1)+\frac{1}{2}|\mathrm{Irr}(L)|+1 \geq |L|$.

By Lemma 1 and the assumption we have that

$$\frac{1}{3}(5\ell(L)+7) = 2(\ell(L)+1) - (\frac{1}{3}\ell(L)-\frac{1}{3}) \geq |L| \geq 2(\ell(L)+1) - |\mathrm{Irr}(L)|,$$

so that $|\mathrm{Irr}(L)| \geq \frac{1}{3}\ell(L)-\frac{1}{3}$.

But then

$$\frac{3}{2}(\ell(L)+1)+\frac{1}{2}|\mathrm{Irr}(L)|+1 \geq \frac{1}{3}(5\ell(L)+7) \geq |L|$$

establishing (6) and completing the proof.

The Boolean lattice $\underset{\sim}{2}^3$ does not have a 7-element sublattice; thus, the inequality of the Theorem is best possible for lattices of length 3 . On the other hand, the lattice obtained by adjoining a new unit to $\underset{\sim}{2}^3$ (as illustrated in Figure 1) does have a complete spectrum. Furthermore, it was shown in [2] that the inequality of Lemma 1 is best possible among all lattices L of finite length such that $\mathrm{Irr}(L) = \emptyset$. The modular lattices L_n $(n \geq 1)$ described in section 1 also satisfy the conditions: $|L_n| = 2(\ell(L_n)+1)$; $\mathrm{Irr}(L_n) = \emptyset$. Therefore, $|L_n| - 1 \notin \mathrm{sp}(L_n)$. In particular, for every integer $n \geq 1$, the spectrum of L_n is not complete.

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[redacted]
IVAN RIVAL
Figure 1 .

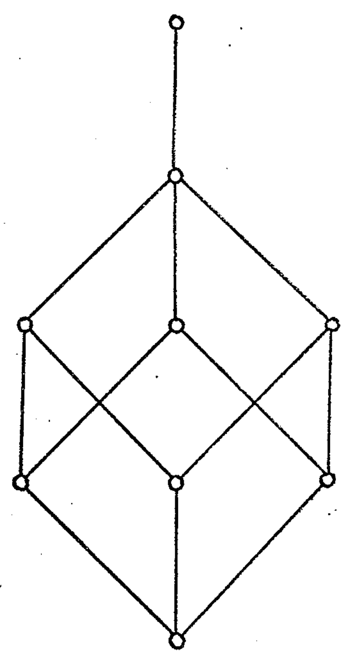

Figure 1

A lattice with a complete spectrum but which is not dismantlable.

LATTICES WITH SUBLATTICES OF A GIVEN ORDER $N \leq 9$.

Ivan Rival

1. INTRODUCTION.

G. Birkhoff [2, p. 19] poses the following question: *Given* n, *what is the smallest integer* $\psi(n)$ *such that every lattice with order* $r \geq \psi(n)$ *elements contains a sublattice of exactly* n *elements?* G. Havas and M. Ward [3] have shown that this function exists, indeed, that $\psi(n) \leq n^{3^n}$ *for every positive integer* n. However, apart from this fact very little seems to be known about the general behaviour of this function. As to values of $\psi(n)$ for small n very little beyond the fact that $\psi(n) = n$ *for* $n \leq 6$ is known which, over the years since Fr. Klein-Barmen's proof [5] for the modular case, has become a part of the folklore of the problem. As a consequence of more general combinatorial considerations on lattices of finite length we shall in this paper prove: $\psi(n) = n$ *for* $n \leq 6$; $\psi(7) = 9$; $\psi(8) = 8$, *that is,* ψ *is not in general an increasing function*; $\psi(9) \geq 17$; *and* ψ *has no further fixed points, that is,* $\psi(n) \geq n+2$ *for* $n \geq 10$.

A useful term naturally arising in connection with this problem is that of the *spectrum* $sp(L)$ of a lattice L defined as the set of all integers n such that L has an n-element sublattice; the spectrum of L is *complete* if $sp(L) = \{n \mid 0 \leq n \leq |L|\}$. The author has shown elsewhere [7], that *every modular lattice* L *of finite length satisfying* $|L| \leq \frac{1}{3}(5\,\ell(L) + 7)$ *has a complete spectrum, where* $\ell(L)$ *denotes the* length of L. A finite lattice L of order n is said to be dismantlable if there is a chain $\phi = S_0 \subset S_1 \subset \cdots \subset S_n = L$ of sublattices of L such that $|S_i| = |S_{i-1}| + 1$, for every $i = 1, 2, \cdots, n$. Every finite planar lattice and every lattice with seven or fewer elements is dismantlable (cf. [1], [4], [6]). Furthermore, it is obvious that every dismantlable lattice has a complete spectrum. Although not every eight-element lattice is dismantlable or, for that matter, even has a complete spectrum (for example, the Boolean lattice $\underset{\sim}{2}^3$) we shall show that every nine-element lattice does in fact have a complete spectrum, even though it need not be dismantlable (see Figure 1a).

2. PRELIMINARIES.

An element x in a lattice L is join-reducible (meet-reducible) in L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z$ $(x = y \wedge z)$; we denote by $L(\vee)$ $(L(\wedge))$ the set of all join-reducible elements in L (the set of all meet-reducible elements in L). Let $Irr(L) = L - (L(\vee) \cup L(\wedge))$, the set of all doubly irreducible elements in L. For $x, y \in L$, x covers y $(x \succ y$ or $y \prec x)$ in L if $x > y$ and $x \geq z > y$ implies $x = z$, for every $z \in L$. The length $\ell(L)$ of a lattice L is the order of a maximum-sized chain in L minus one. For all further notation and terminology we refer the reader to [2].

The proofs of the following elementary results have already appeared elsewhere [6].

LEMMA 2.1. *If in a lattice* L *of finite length* $|L| < 2(\ell(L) + 1)$ *then* $\mathrm{Irr}(L) \neq \emptyset$.

Once we observe that $L - \{x\}$ is a sublattice of L for every $x \in \mathrm{Irr}(L)$, it follows from Lemma 2.1, by classifying lattices according to their length, that

LEMMA 2.2. *Every lattice with seven or fewer elements is dismantlable.*

LEMMA 2.3. *A lattice* L *is dismantlable if and only if for every chain* C *in* L *there is a positive integer* n *and a chain* $C = S_0 \subset S_1 \subset \cdots \subset S_n = L$ *of sublattices of* L *such that* $|S_i| = |S_{i-1}| + 1$ *for every* $i = 1, 2, \ldots, n$.

The next lemma will be useful in the extension of given sublattices of a lattice of finite length to yet larger ones.

LEMMA 2.4. *Let* C *be a maximal chain in a lattice* L *of finite length and* S *a subset of* L *disjoint from* $\mathrm{Irr}(L) \cap C$. *Then* S *is a sublattice of* $L - (\mathrm{Irr}(L) \cap C)$ *containing* $C - (\mathrm{Irr}(L) \cap C)$ *if and only if* $S \cup (\mathrm{Irr}(L) \cap C)$ *is a sublattice of* L *containing* C.

3. $\psi(n) = n$ FOR $n \leq 6$.

The approach we take here is that of extending arbitrary maximal chains in a lattice of finite length to sublattices of a prescribed order. Since the sublattices so constructed will in a certain obvious sense be "narrow", they turn out to be dismantlable.

PROPOSITION 3.1. _Let_ $k = 1$ _or_ 2 _and let_ C _be a maximal chain in a lattice_ L _of finite length_. _If_ $|L| \geq |C| + k$ _then_ C _is contained in a dismantlable sublattice of_ L _with precisely_ $|C| + k$ _elements_.

Proof. Once a sublattice containing a maximal chain and only one or two additional elements is constructed it is dismantlable by virtue of Lemma 2.1 and the fact that every lattice of length two is dismantlable.

(i) Let $k = 1$. Since L is not a chain $L(\wedge) \cap C \neq \phi$. Let $a = \bigvee(L(\wedge) \cap C)$, $b = \bigwedge(x \in C \mid x > y$ for some $y \in L - C$ such that $y > a)$, and B be a maximal chain in $L - ((a, b) \cap C)$ containing $C - (a, b)$. Then $B - C \neq \phi$ and $(a, b) \cap C \subseteq \mathrm{Irr}([a, b])$. By Lemma 2.4 $([a, b] \cap C) \cup A$ is a sublattice of $[a, b]$ for every $A \subseteq B - C$. Therefore, $C \cup A$ is a sublattice of L for every $A \subseteq B - C$. Once we choose A such that $|A| = 1$ we are done.

(ii) Let $k = 2$. In view of the construction in (i) we may, without loss of generality, assume that $B - C$ consists of precisely one element c, say, and that $C \cup \{c\}$ is a sublattice of L. By symmetry, if there were a maximal chain B' in $L - ((a, b) \cap C)$ containing $C - (a, b)$ distinct from B, then we could assume that $B' - C$ consists of precisely one element c' and, that $C \cup \{c'\}$ is a sublattice of L. But then $C \cup \{c, c'\}$ would be a sublattice of L and since $c \neq c'$ we would be done. Therefore, we may assume that $[a, b] = ([a, b] \cap C) \cup \{c\}$.

By our construction if $c \in L(\wedge)$ then $c = \bigvee(L(\wedge) \cap B)$. Applying the construction in (i), with B substituted for C and c for a, we get an element c' such that $B \cup \{c'\}$ is a sublattice of L, in which case $C \cup \{c, c'\}$ is a sublattice of L of the prescribed size. Therefore, we may assume that $c \notin L(\wedge)$.

If there were an element $c' \in L - C$ distinct from c such that $c' > a$ then $C \cup \{c, c'\}$ would do.

This finally leaves the case $[a, 1] = ([a, 1] \cap C) \cup \{c\}$. Since $k = 2$, $a' = \bigvee((L(\wedge) \cap C) - \{a\})$ exists. If we now choose $c' \in L - C$ such that $c' > a'$ then $C \cup \{c, c'\}$ is a sublattice of L, and this completes the proof.

Using Proposition 3.1 it is a simple matter to determine $\psi(n)$ for $n \leq 6$. For example, if $n = 6$ and L is an arbitrary lattice with 6 or more elements then L certainly has a 6-element sublattice, as long as $\ell(L) = 2$ or $\ell(L) \geq 5$. If $\ell(L) = 4, 3$ then L has a 6-element sublattice by applying Proposition 3.1 for $k = 1, 2$, respectively.

THEOREM 3.2. $\psi(n) = n$ for $n \leq 6$.

4. LATTICES L SUCH THAT $\ell(L) = 3$ AND $\mathrm{Irr}(L) = \emptyset$.

To establish lower bounds for $\psi(n)$ it suffices to construct lattices which have no n-element sublattices. The most obvious lattices without a complete spectrum are those without doubly irreducible elements, and these abound even at length three.

PROPOSITION 4.1. <u>If</u> L <u>is a lattice with</u> n <u>atoms,</u> m <u>coatoms,</u> $n \le m$, $\ell(L) = 3$ <u>and</u> $\mathrm{Irr}(L) = \emptyset$ <u>then</u> $3 \le n \le m \le \binom{n}{2}$. <u>Conversely, given integers</u> n, m <u>such that</u> $3 \le n \le m \le \binom{n}{2}$ <u>there is a lattice</u> L <u>with</u> n <u>atoms,</u> m <u>coatoms,</u> $\ell(L) = 3$ <u>and</u> $\mathrm{Irr}(L) = \emptyset$.

Proof. If L is a lattice satisfying the hypotheses of the first statement then every coatom is join-reducible and, therefore, the join of a subset of pairs of atoms. Since there are at most $\binom{n}{2}$ distinct such pairs we have that $m \le \binom{n}{2}$. On the other hand $n \le m$ by hypothesis so that $n \le \binom{n}{2}$ which in turn implies that $3 \le n$.

Suppose now that n, m are integers such that $3 \le n \le m \le \binom{n}{2}$. Let $A = \{a_1, a_2, \dots, a_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ be disjoint sets each with n distinct elements. We define a partial ordering on $A \cup C$ as follows:

$$a_1, a_n < c_n \ ; \ a_i, a_{i+1} < c_i \quad \text{for} \quad i = 1, 2, \dots, n-1 .$$

If we now adjoin a 0 and a 1 to $A \cup C$ we get a lattice L' with atoms A, coatoms C, $\ell(L') = 3$ and $\mathrm{Irr}(L') = \emptyset$. Finally, we choose $m-n$ pairs $\{a_i, a_j\}$ of atoms in L' such that $a_i \vee a_j = 1$ (there are $\binom{n}{2} - n$ such pairs in L') and simply enlarge L' by adjoining a distinct coatom c_{ij} to L' for each of these $m-n$ pairs $\{a_i, a_j\}$ and then extend the partial ordering of L' by $a_i, a_j < c_{ij}$. The resultant lattice L so obtained satisfies the prescribed conditions.

COROLLARY 4.2. <u>There is a unique smallest lattice without doubly irreducible elements; namely,</u> $\underset{\sim}{2}^3$.

Proof. Clearly, $\mathrm{Irr}(\underset{\sim}{2}^3) = \emptyset$. On the other hand, let L be a lattice of minimum order such that $\mathrm{Irr}(L) = \emptyset$. Certainly, $\ell(L) \geq 3$. If $\ell(L) \geq 4$ then by Lemma 2.1 $|L| \geq 10$; thus, $\ell(L) = 3$. Finally, by Proposition 4.1 L must have precisely 3 atoms, 3 coatoms, and, distinct pairs of atoms must join in L to distinct coatoms and vice versa.

The next corollary will be of use in the evaluation of $\psi(7)$ and $\psi(8)$.

COROLLARY 4.3. <u>Every lattice</u> L <u>with precisely</u> 9 <u>elements has a complete spectrum</u>. <u>In fact, if</u> $\ell(L) \leq 6$ <u>then every maximal chain in</u> L <u>is contained in a</u> 7-<u>element sublattice of</u> L.

Proof. Let L be a lattice such that $|L| = 9$. Without loss of generality $\ell(L) \geq 3$.

We show first that $\mathrm{Irr}(L) \neq \emptyset$ or, what is the same, that $8 \in sp(L)$. By Lemma 2.1 we may assume that $\ell(L) = 3$, but then Proposition 4.1 implies that $\mathrm{Irr}(L) \neq \emptyset$.

In view of Lemma 2.2 it suffices now to establish the second statement. By Proposition 3.1 we may assume that $\ell(L) = 3$. Now if $a \in \mathrm{Irr}(L)$ then either $L - \{a\} \cong \underset{\sim}{2}^3$ (by Corollary 4.2) or L is dismantlable (by Lemma 2.2). In the latter case we are done by Lemma 2.3. On the other hand, if $L - \{a\} \cong \underset{\sim}{2}^3$ it is straightforward to verify that L must be isomorphic to one of the three lattices illustrated in Figure 1b, 1c or 1d, and these lattices do, indeed, satisfy the assertion. The proof is now complete.

Of course, the most obvious application of Proposition 4.1 is to establish that ψ can have no fixed points beyond $n = 8$.

THEOREM 4.4. $\psi(n) \geq n+2$ *for* $n = 7$ *and for all* $n \geq 9$.

5. $\psi(7) = 9$, $\psi(8) = 8$ AND $\psi(9) \geq 17$.

The next result is the obvious analogue of Proposition 3.1 needed to evaluate $\psi(7)$.

PROPOSITION 5.1. *Every maximal chain* C *in a lattice* L *of finite length satisfying* $|L| \geq |C| + 5$ *is contained in a dismantlable sublattice with precisely* $|C| + 3$ *elements*.

Proof. Let L be a lattice of finite length and C an arbitrary maximal chain in L such that $|L| \geq |C| + 5$. We induct on $|L|$. Let us observe at the outset that once a sublattice of L containing C with $|C| + 3$ elements is constructed, it is dismantlable by repeated application of Lemma 2.1.

Suppose now that $Irr(L) \neq \emptyset$. We distinguish two cases.

(i) $Irr(L) \cap C \neq \emptyset$. Let B be a maximal chain in $L - Irr(L)$ containing $C - Irr(L)$. If $B - C = \emptyset$ we can apply the inductive hypothesis to the maximal chain B in $L - Irr(L)$ to get a sublattice of $L - Irr(L)$ containing B with $|B| + 3$ elements. We then apply Lemma 2.4 to get the assertion. If $|B - C| = 2,1$ we apply Proposition 3.1 for $n = 1, 2$, respectively, and then again apply Lemma 2.4. If $|B - C| \geq 3$ choose $A \subseteq B - C$ such that $|A| = 3$. By Lemma 2.4 $C \cup A$ is a sublattice of L .

(ii) $\mathrm{Irr}(L) \cap C = \phi$. Let $a \in \mathrm{Irr}(L)$. Since $a \in L - C$ we can apply the inductive hypothesis to $L - \{a\}$, as long as $|L| > |C| + 5$. Otherwise, $|L - \{a\}| = |C| + 4$. If there exists $b \in \mathrm{Irr}(L - \{a\}) - C$ then $L - \{a, b\}$ is a sublattice of L with the prescribed conditions. If $\mathrm{Irr}(L - \{a\}) \cap C \neq \phi$ we simply apply the reasoning of (i). Therefore, we may assume that $\mathrm{Irr}(L - \{a\}) = \phi$. But then, by Lemma 2.1, $|C| = 4$ so that $|L| = 9$, in which case we can apply Corollary 4.3.

Therefore, we may without loss of generality assume that $\mathrm{Irr}(L) = \phi$.

We shall now show that if C is any maximal chain in L such that $|L| \geq |C| + 5$ and c is a coatom in C then without loss of generality $|[0, c]| = |[0, c] \cap C| + 1$.

Let $|[0, c]| = |[0, c] \cap C| + i$. If $i = 5$ we apply the inductive hypothesis to $[0, c]$ to get a sublattice S' containing $[0, c] \cap C$ with $|[0, c] \cap C| + 3$ elements; setting $S = S' \cup \{1\}$ we are done; if $i = 4$ we apply Proposition 3.1 for $k = 2$ and again adjoin the 1 ; if $i = 3$ we are already done; if $i = 2$ it suffices to adjoin the 1 together with any coatom y in L $(1 \in L(\vee))$ distinct from c to $[0, c]$.

By duality, we may assume that if C is any maximal chain in L such that $|L| \geq |C| + 5$ and a is an atom in C then $|[a, 1]| = |[a, 1] \cap C| + 1$.

Since $\mathrm{Irr}(L) = \phi$ it now follows that if C is a maximal chain in L such that $|L| \geq |C| + 5$ we may in fact assume that $|C| = 4$.

Let $C = \{0 \prec a \prec c \prec 1\}$ be such a maximal chain and $a', c' \in L - C$ such that $0 \prec a' \prec c$ and $a \prec c' \prec 1$. Now $\mathrm{Irr}(L) = \emptyset$ so that $a' \in L(\wedge)$. Let $c'' \in L - C$ such that $a' \prec c''$. Then $c'' \prec 1$. If $c'' \wedge c' = 0$ we are done. Otherwise, $0 \prec c'' \wedge c' \prec c'$ and we get a copy of $\underset{\sim}{2}^3$ as a sublattice. By hypothesis there must be at least one other element in L not contained in this copy of $\underset{\sim}{2}^3$ and, indeed, it must be incomparable with every atom and coatom in $\underset{\sim}{2}^3$. In this way we get a 9-element sublattice of L of length 3. Applying Corollary 4.3 completes the proof.

Let L be a lattice such that $|L| \geq 9$. If $\ell(L) \geq 4$ we can apply Proposition 3.1 to get a 7-element sublattice of L. If $\ell(L) = 3$ Proposition 5.1 yields a 7-element sublattice.

THEOREM 5.2. $\psi(7) = 9$.

THEOREM 5.3. $\psi(8) = 8$.

Proof. We show by induction on $|L|$ that if L is a lattice of finite length such that $|L| \geq 8$ then L has an 8-element sublattice. Without loss of generality we may assume that $\ell(L) \geq 3$ and, by the inductive hypothesis, that $\mathrm{Irr}(L) = \emptyset$. Therefore, by Corollary 4.3 $|L| \geq 10$. If $\ell(L) \geq 4$ then in view of Propositions 3.1 and 5.1 L has an 8-element sublattice; thus, we may assume that $\ell(L) = 3$. It follows that every maximal chain in L has order 4.

We now show that without loss of generality every maximal chain $C = \{0 \prec a \prec c \prec 1\}$ in L satisfies $|[0, c]| = 4 = |[a, 1]|$. By duality it suffices to show that $|[0, c]| = 4$. We show then that if $|[0, c]| \geq 5$ we can construct an 8-element sublattice S of L. By the inductive hypothesis we may assume that $|[0, c]| \leq 7$. If $|[0, c]| = 7$ set $S = [0, c] \cup \{1\}$; if $|[0, c]| = 6$ set $S = [0, c] \cup \{1, c'\}$, where c' is a coatom in L distinct from c $(1 \in L(\vee))$. This leaves only the case in which $[0, c]$ consists of 5 distinct elements $0, a, a', a'', c$ such that $0 \prec a \prec c$ and $0 \prec a', a'' \prec c$. Since $Irr(L) = \phi$ there exist $c' \neq c''$ both distinct from c such that $a' \prec c'$ and $a'' \prec c''$. Clearly $c' \wedge c'' \neq a$. If $c' \wedge c'' = 0$ set $S = [0, c] \cup \{c', c'', 1\}$; if $0 \prec c' \wedge c''$ set $S = ([0, c] - \{a\}) \cup \{c', c'', c' \wedge c'', 1\}$.

Finally, if $C = \{0 \prec a \prec c \prec 1\}$ is a maximal chain in L and $[0, c] = \{0, a, a', c\}$, $[a, 1] = \{a, c, c', 1\}$ it suffices to take the only other cover c'' of a' (besides c) and the only other element a'' covered by c' (besides a) and set $S = C \cup \{a', a'', c', c''\}$ which is an 8-element sublattice of L. The proof is now complete.

To get a lower bound for $\psi(9)$ we examine the spectrum of the lattice $\mathcal{P}$ of all subspaces of the projective plane of order 2 illustrated in Figure 2. Once we verify that all maximal proper sublattices of $\mathcal{P}$ are isomorphic either to $\mathcal{P} - \{a_1, a_2, a_4, c_1, c_2, c_4\}$ or to $\mathcal{P} - \{a_1, a_2, a_6, a_7, c_1, c_2, c_4, c_7\}$ we get that $sp(\mathcal{P}) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 16\}$.

THEOREM 5.4. $\psi(9) \geq 17$.

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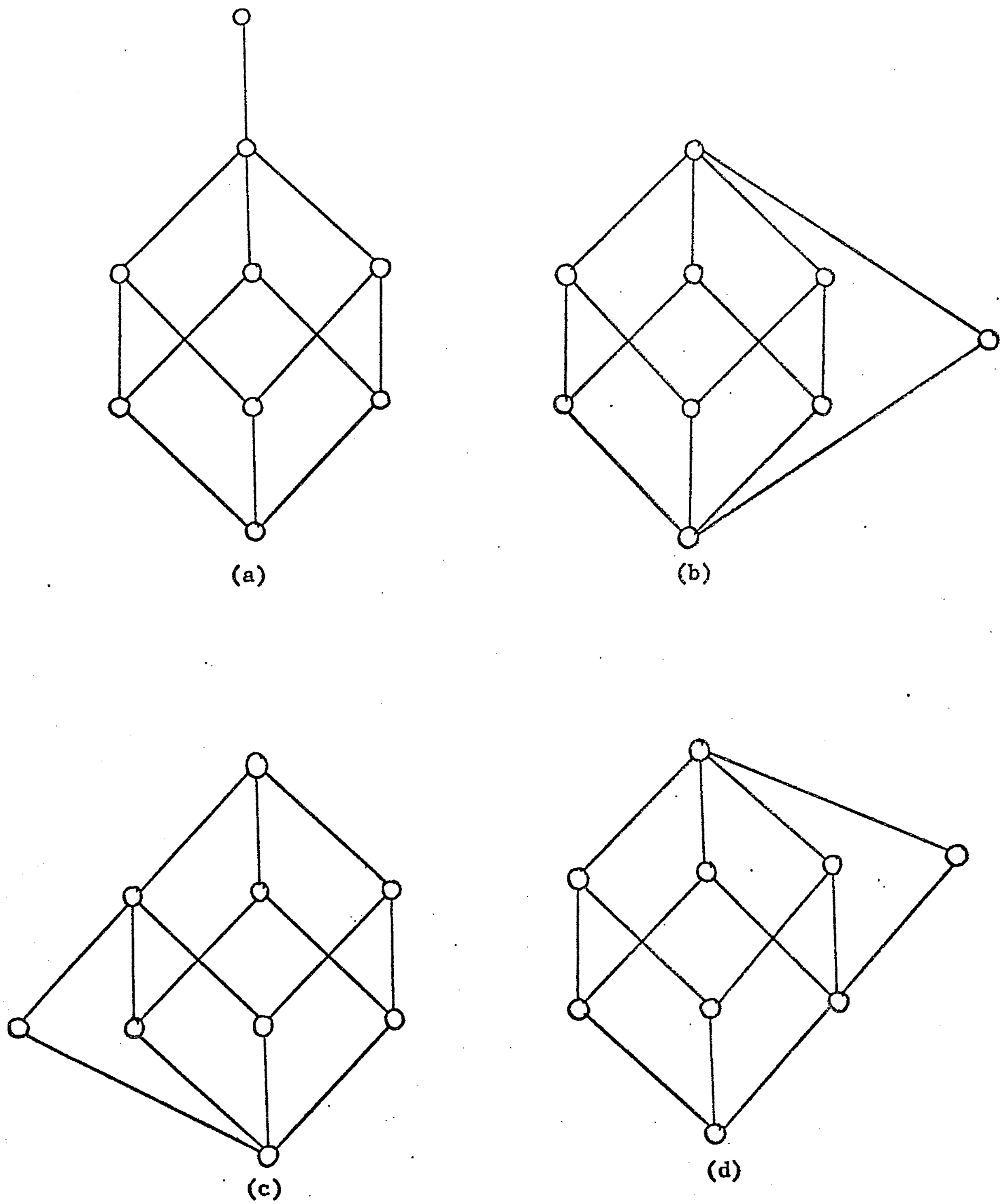
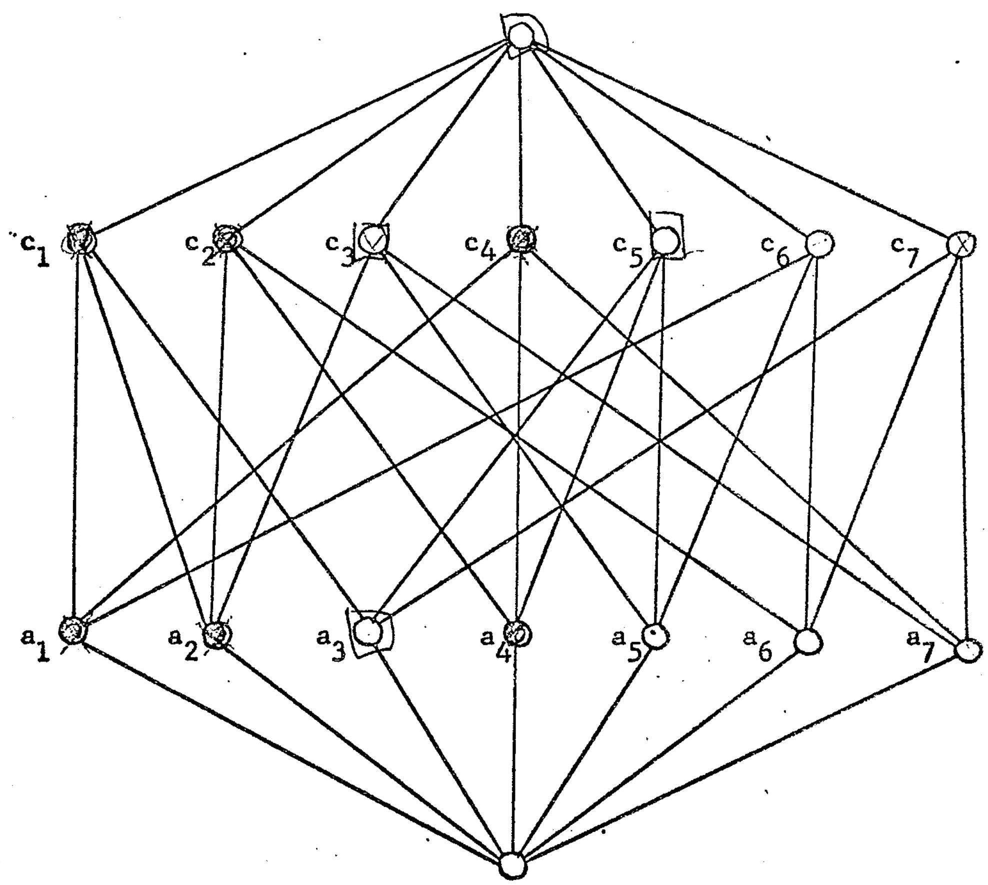


IVAN RIVAL
Figure 1



IVAN RIVAL
Figure 2

;ure 2

The lattice of all subspaces of the projective plane of order 2

SUBLATTICES OF MODULAR LATTICES OF FINITE LENGTH

by

Ivan Rival

It is well-known that the join-irreducible elements $J(L)$ and the meet-irreducible elements $M(L)$ of a lattice L of finite length play a central role in its arithmetic and, especially, in the case that L is distributive. In [2] it was shown that the _quotient set_ $Q(L) = \{b/a \mid a \in J(L),\ b \in M(L),\ a \le b\}$ plays a somewhat analogous role in the study of the sublattices of L. Indeed, _in a lattice_ L _of finite length if_ S _is a sublattice of_ L _then_ $S = L - \bigcup_{b/a \in A} [a, b]$ _for some_ $A \subseteq Q(L)$. Furthermore, the converse actually characterizes finite distributive lattices [2].

On the other hand, the arithmetical theory of a modular lattice of finite length in terms of its join-irreducible and meet-irreducible elements is far more involved than it is for a finite distributive lattice; consequently, it is not unexpected that the study of the structure of sublattices of a modular lattice of finite length is also more involved than it is for a finite distributive lattice. The purpose of this paper is to introduce and investigate some new concepts useful in the general study of sublattices of a lattice of finite length and particularly, in the case that the lattice is modular.

The Boundary of a Subset of a Lattice. An element x in a lattice L is _join-reducible_ (_meet-reducible_) _in_ L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z$ $(x = y \wedge z)$; x is _join-irreducible_ (*meet-irreducible*) *in* L if it is not join-reducible (meet-reducible) in L . Let J(L) and M(L) denote the sets of join-irreducible and meet-irreducible elements in L , respectively. For $x, y \in L$, x is *incomparable* with $y (x \parallel y)$ if $x \nleq y$ and $y \nleq x$; x *covers* $y (x \succ y$ or $y \prec x)$ *in* L if $x > y$ and $x \geq z > y$ implies $x = z$ for every $z \in L$. A subset A of L is *connected* if, for every $a, b \in A$, there is a sequence $a = x_0, x_1, \ldots, x_n = b$ of elements in A such that either $x_i \succ x_{i=1}$ or $x_i \prec x_{i-1}$ for every $i = 1, 2, \ldots, n$; thus, every subset of a finite lattice can be partitioned into components, that is, maximal connected subsets.

PROPOSITION 1. *Let* L *be a lattice of finite length and let* M *be a maximal proper sublattice of* L . *Then* L − M *is a connected subset of* L .

Proof. Let us suppose that L − M is not a connected subset of L , that is, L − M is partitioned into components A_i , $i \in I$, and $|I| > 1$. In view of the maximality of M there exist $x_i \in A_i$, $x_j \in M \cup A_i$ such that $x_i \parallel x_j$ and, say, $x_i \vee x_j \notin M \cup A_i$, that is, $x_i \vee x_j \in A_k$ for some $k \neq i$. Let C_i, C_j be maximal chains from x_i , x_j , respectively, to $x_i \vee x_j$; then $C_i \subseteq L - M$ or $C_j \subseteq L - M$. Indeed, if $x_j \in M$ and $y_i \in C_i$ then $x_i \vee x_j \leq y_i \vee x_j \leq x_i \vee x_j$ implies that $y_i \vee x_j \in L - M$ so that $y_i \in L - M$; if $x_j \in A_i$ and there exists $y_i \in C_i \cap M$ then again $y_i \vee x_j = x_i \vee x_j$ so that by the preceeding argument $C_j \subseteq L - M$. In either case $A_i \cup A_k$ is a connected subset of L − M , contradicting our assumption.

$\square$

In [2] we have considered the usefulness of the <u>covering neighbourhood</u> $\mathrm{cov}(A)$ of a subset A of L defined by

$$\mathrm{cov}(A) = \{x \in L \mid x \succ a ,\ x \prec a ,\ \text{or}\ \ x = a \ \text{ for some } \ a \in A\} ,$$

particularly, in the study of sublattices of a finite distributive lattice L. Closely related to the covering neighbourhood of $A \subseteq L$ is the <u>boundary</u> $\mathrm{bd}(A)$ of A defined by

$$\mathrm{bd}(A) = \mathrm{cov}(A) - A .$$

PROPOSITION 2. <u>Let</u> L <u>be a lattice of finite length,</u> S <u>a sublattice of</u> L <u>and</u> A <u>a subset of</u> L <u>such that</u> $\mathrm{bd}(A) \subseteq S$. <u>Then</u> $S \cup A$ <u>is a sublattice of</u> L.

Proof. If $x_i, x_j \in A - S$, $x_i \parallel x_j$, and C_i, C_j are maximal chains from x_i, x_j, respectively, to $x_i \vee x_j$, say, then there exist elements $y_i = \bigwedge(C_i - A)$ and $y_j = \bigwedge(C_j - A)$ such that $y_i, y_j \in \mathrm{bd}(A)$ and $x_i \vee x_j \leq y_i \vee y_j \leq x_i \vee x_j$. By hypothesis, $y_i \vee y_i \in S$. The remaining cases are treated similarly. $\square$

A subset B of a lattice L of finite length can be the boundary of several disjoint subsets of L; for example, take $B = \{0, 1\}$ in the 4-element lattice of length 2. On the other hand, a subset of L is determined by its boundary and a system of representatives of its components; that is, if A and A' are subsets of L, A_i, A_i', $i = 1, 2, \cdots, n$, are their respective components, $A_i \cap A_i' \neq \phi$ for all i, and $\mathrm{bd}(A) = \mathrm{bd}(A')$ then $A = A'$. In fact, if $a \in A_i \cap A_i'$ and $x \in A_i$ there is a sequence $a = y_0, y_1, \cdots, y_m = x$ of elements in A_i such that $y_j \succ y_{j-1}$

or $y_i \prec y_{j-1}$ for every $j = 1, 2, \cdots, m$. If $z_{m-1} \in A_i'$ and $z_{m-1} \succ z_m$, say, than $z_m \in A_i'$ since otherwise, $z_m \in bd(A_i') = bd(A_i)$. Thus, $A_i \subseteq A_i'$ and by symmetry, $A_i' \subseteq A_i$.

It will be helpful to quantify the notion of the boundary of a subset of a lattice. Let L be a lattice of finite length, $A \subseteq L$, and $a \in A$. We define

$$A_*(a) = |\{x \in L - A \mid x \prec a\}| \quad \text{and}$$
$$A^*(A) = |\{x \in L - A \mid x \succ a\}| \quad .$$

Clearly, *if* $L - A$ *is a sublattice of* L *then* $A_*(a) \leq 1$ *and* $A^*(a) \leq 1$ *for every* $a \in A$.

Sublattices of Modular Lattices. In the case that L is modular of finite length we can recover a partial converse.

PROPOSITION 3. *Let* L *be a modular lattice of finite length and let* A *be a subset of* L *satisfying the following conditions:* (i) $A_*(a) \leq 1$ *and* $A^*(a) \leq 1$ *for every* $a \in A$; (ii) A *is a convex subset of* L . *Then* $L - A$ *is a sublattice of* L .

Proof. Let us suppose that there exist $x, y \in L - A$ such that $x \wedge y \in A$. In this case we may choose a maximal element $a \in A$ such that there exist $x, y \in L - A$ with $x \wedge y = a$. In view of (i) x and y cannot both cover a so that we may furthermore assume that there exists $z \in A$ such that $y > z \succ a$. Now, if $x \vee z = x \vee y$ then by the modularity of L , $y = z$; thus, $x \vee z < x \vee y$. By virtue of (ii) $x \vee z \in L - A$. But, since $y \succ z$ and $x \vee z \not\geq y$ we have that $(x \vee z) \wedge y = z \in A$ contradicting the maximality of A . □

We are now in a position to describe at least one method of generating maximal proper sublattices of a modular lattice of finite length.

COROLLARY 4. *Let* L *be a modular lattice of finite length and let* A *be a subset of* L *satisfying the following conditions:* (i) $A_*(a) = 1 = A^*(a)$ *for every* $a \in A$; (ii) A *is a convex subset of* L ; (iii) A *is a connected subset of* L . *Then* $L - A$ *is a maximal proper sublattice of* L .

Proof. By Proposition 3, $L - A$ is a sublattice of L . If $L - A$ is not a maximal proper sublattice of L then there exists $\emptyset \neq A' \subset A$ such that $M = (L - A) \cup A'$ is a maximal proper sublattice of L . In view of (iii) there exist $a_1 \in A'$ and $a_2 \in A - A'$ such that either $a_1 \succ a_2$ or $a_2 \succ a_1$. We may suppose that $a_2 \succ a_1$. By (i) there exists $a_3 \in L - A$ such that $a_2 \succ a_3$. Obviously, $a_1 \parallel a_3$, $a_1, a_3 \in M$ and $a_2 = a_1 \vee a_3$ which must then lie in M although $a_2 \in A - A'$. $\square$

For finite distributive lattices we have already seen in [1] that the conditions (i) - (iii) of Corollary 4 characterize maximal proper sublattices; whether this extends to modular lattices of finite length seems much more difficult to settle.

PROBLEM. *Characterize maximal proper sublattices of a modular lattice of finite length. In particular, do the conditions* (i) - (iii) *of Corollary 4 characterize such sublattices?*

Unfortunately, as to properties of $L - M$, where M is a maximal proper sublattice of a modular lattice L , very little apart from the next proposition is available. It will be convenient to keep in mind the following property concerning irreducible elements in a modular lattice L of finite length: *for* $a, b, c \in L$, *if* $a \in M(L)$ *and* $b > a \geq b \wedge c$ *then* $a \geq c$; *if* $a \in J(L)$ *and* $b < a \leq b \vee c$ *then* $a \leq c$.

PROPOSITION 5. *Let* L *be a modular lattice of finite length and let* M *be a maximal proper sublattice of* L. *If* $a, b \in L - M$ *and* $b \succ a$ *in* L *then either* b *is join-reducible in* L *or* a *is meet-reducible in* L.

Proof. Let us suppose that $b \in J(L)$ and $a \in M(L)$ and set $M' = M \cup \{x \in L \mid x \geq b\}$. Therefore, the sublattice in L generated by M' is L. On the other hand, M' is a join-subsemilattice of L; hence, there exist $y \in M - \{x \in L \mid x \geq b\}$ and $z \geq b$ such that $y \wedge z \in L - M'$ and $y \wedge z \ngeq b$. But $a \in M(L)$ and L is modular so that $y \wedge z \nleq a$ and $a \vee (y \wedge z) = z \wedge (a \vee y)$. Furthermore, $z \geq b$ and $a \vee y = b \vee y \geq b$ (since $b \succ a$ and $a \in M(L)$), so that $a \vee (y \wedge z) \geq b$. Finally, since $b \in J(L)$ we have that $y \wedge z \geq b$, contradicting our assumption. $\square$

The lattice of Figure 1 illustrates the necessity of modularity in Proposition 5.

In [2] we have shown that *in a lattice* L *of finite length, if* S *is a sublattice of* L *then* $S = L - \bigcup_{b/a \in A} [a, b]$ *for some* $A \subseteq Q(L)$. Slightly more information can be obtained in the case that L is modular.

PROPOSITION 6. *Let* L *be a modular lattice of finite length and let* S *be a sublattice of* L. *Then, for every* $x \in L - S$ *there exists* $b/a \in Q(L)$ *such that* (i) $x \in [a, b] \subseteq L - S$, (ii) $(a, x) \subseteq L - J(L)$, *and* (iii) $(x, b) \subseteq L - M(L)$.

Proof. In view of the remark above and duality it suffices to show that there exists $b \in M(L)$ such that $x \leq b$ and $(x, b) \subseteq (L - S) \cap (L - M(L))$. We may assume that $x \notin M(L)$ so that there exists an integer n and a subset $\{b_i \mid 1 \leq i \leq n\}$ of $M(L)$ irredundant with respect to $\bigwedge(b_i \mid 1 \leq i \leq n) = x$. If, for every i, there exists $y_i \in [x, b_i] \cap S$ then $x = \bigwedge(y_i \mid 1 \leq i \leq n) \in S$. Hence, there exists $j \in \{1, 2, \cdots, n\}$ such that $[x, b_j] \subseteq L - S$. Let $z = \bigwedge(b_i \mid 1 \leq i \leq n, i \neq j)$. Clearly, $z \wedge b_j = x$ and, since $\{b_i \mid 1 \leq i \leq n\}$ is an irredundant meet representation of x, $z \nleq b_j$. Thus, for every $u \in (x, b_j)$, $z \wedge b_j < u$ which, since $b_j \in M(L)$ and L is modular, implies that $u \in L - M(L)$. Choosing $b = b_j$ completes the proof. $\square$

Let M be a maximal proper sublattice of a finite lattice L. If L is Boolean then $\frac{|M|}{|L|} = \frac{3}{4}$ and, if L is distributive then $\frac{|M|}{|L|} \geq \frac{2}{3}$ (cf. [1]). However, if L is modular there is, in general, no non-zero constant k such that $\frac{|M|}{|L|} \geq k$. In fact, B. Wolk has pointed out (and it is straightforward to verify) that, if P_n denotes the lattice of subspaces of a projective plane of order n, then a maximal proper sublattice M of P_n satisfies either $|M| = 2n + 4$ or $|M| = 2n + 6$ so that $\lim_{n \to \infty} \frac{|M|}{|P_n|} = 0$. Figure 2 illustrates the two possible maximal proper sublattices of P_n. It is also easy to check that at least in this case the conjecture posed in the problem above does hold.

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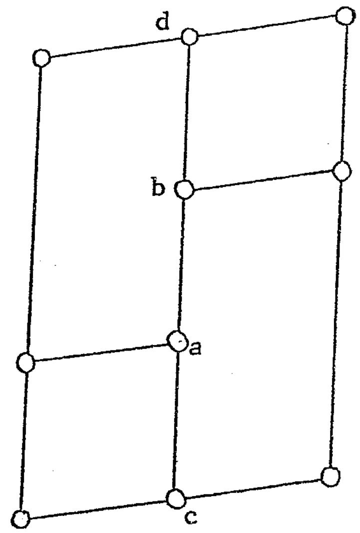

L

$K = L - \{a, b, c, d\}$

Figure 1.

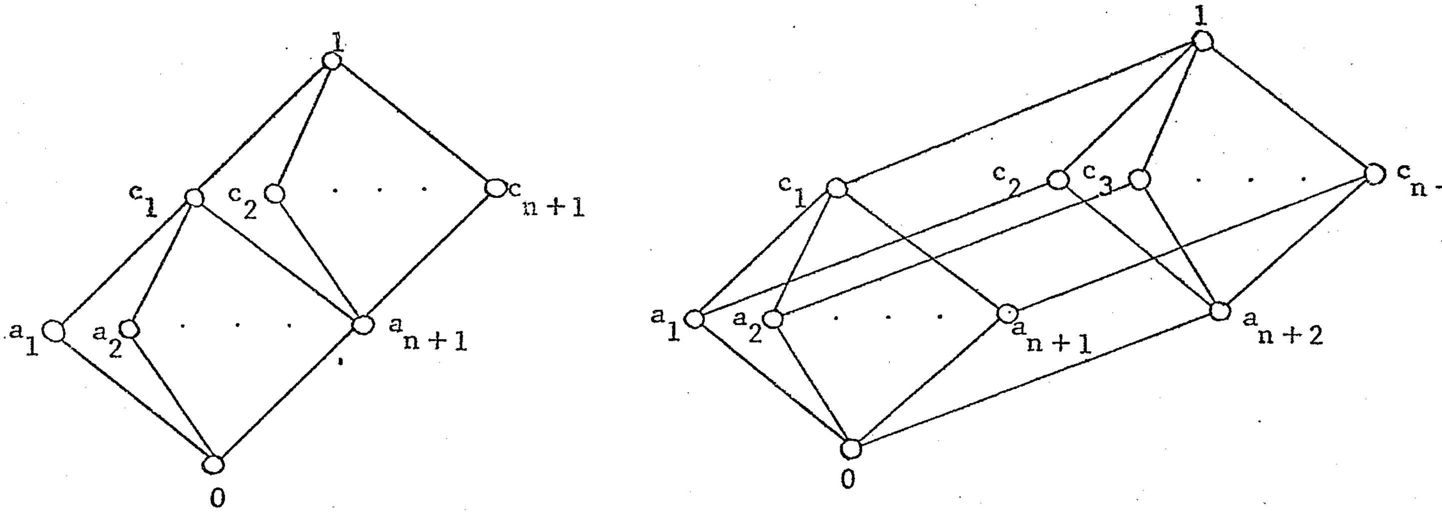

Figure 2.

TECHNISCHE HOCHSCHULE DARMSTADT

DILWORTH'S COVERING THEOREM FOR MODULAR LATTICES: A SIMPLE PROOF

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A SIMPLE PROOF OF A THEOREM OF DILWORTH

by

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In the early 1950's R.P.Dilworth[2] proved the following remarkable combinatorial result on finite modular lattices.

THEOREM. *In a finite modular lattice the number of elements covered by precisely* k *elements is equal to the number of elements covering precisely* k *elements.*

In the special case when $k = 1$, this theorem settled a conjecture of the middle 1930's that *in a finite modular lattice the number of join-irreducibles is equal to the number of meet-irreducibles.*

The critical steps of Dilworth's proof of the theorem depend upon properties of the generalized Möbius function; consequently, the proof is rather detailed. We shall give here a surprisingly simple and elementary proof.

A lemma. Our proof depends on the following basic property of finite complemented modular lattices. For all terminology not explained here we refer to [1].

LEMMA. *In a finite complemented modular lattice the number of* k-*element subsets of atoms whose join is the unit is equal to the number of* k-*element subsets of coatoms whose meet is the zero.*

Proof. For a finite complemented modular lattice M with universal bounds 0,1 , let $a_k(M)$ denote the number of k-element subsets of atoms with join the 1 and $c_k(M)$ its dual. The assertion is trivial if the length of M is 2 or less.

Suppose first that M is indecomposable. If the length of M is 4 or more the result follows by the existence of dual automorphisms in finite projective geometries. Otherwise, there is an integer n such that M is a projective plane of order n and $k \geq 3$. Of the $\binom{n^2+n+1}{k}$ distinct k-element subsets of atoms ("points") the number whose join is not the 1 of M is precisely the number of distinct "collinear" k-element subsets, namely, $\binom{n+1}{k}(n^2+n+1)$ so that $a_k(M) = \binom{n^2+n+1}{k} - \binom{n+1}{k}(n^2+n+1)$. By the principle of duality for projective geometries this is also the number $c_k(M)$.

In general, $M = M_1 \times M_2 \times \ldots \times M_n$, where each of the M_i, with universal bounds 0_i, 1_i, is indecomposable. An atom of M is an n-tuple $(x_1,x_2,\ldots,x_n)$ such that for some i, x_i is an atom of M_i and $x_j = 0_j$ for all $j \neq i$; dually, $(y_1,y_2,\ldots,y_n)$ is a coatom if for some i, y_i is a coatom of M_i and $y_j = 1_j$ for all $j \neq i$. Moreover, a k-element subset of atoms of M whose join is 1, in fact, corresponds to a system of k_i-element subsets of atoms of the M_i's each of whose join is 1_i and such that $k_1 + k_2 + \ldots + k_n = k$. Therefore,

$$a_k(M) = \Sigma_{k_1+k_2+\ldots+k_n=k} \left(\Pi_{i=1}^{n} a_{k_i}(M_i)\right) =$$

$$\Sigma_{k_1+k_2+\ldots+k_n=k} \left(\Pi_{i=1}^{n} c_{k_i}(M_i)\right) = c_k(M),$$

which completes the proof of the lemma.

Proof of the theorem. Let M be a finite modular lattice. An (n+1)-element subset $S = \{p,q_1,q_2,\ldots,q_n\}$ of M is a *covering n-set* if either (i) every q_i covers p or, (ii) p covers every q_i. A covering n-set S is called *lower* if (i) holds and *upper* if (ii) holds; we denote by L the set of all lower covering n-sets of M and by U_n the set of all upper covering n-sets. It is well-known that every covering n-set generates a complemented sublattice of M. For a covering n-set $S = \{p, q_1, q_2, \ldots, q_n\}$ we set $[S] = [p, \bigvee_{i=1}^{n} q_i]$ if S is lower and $[S] = [\bigwedge_{i=1}^{n} q_i, p]$ if S is upper; we shall say that a covering n-set S *spans* the complemented interval [S]. A lower covering n-set (respectively, upper covering n-set) $S = \{p, q_1, q_2, \ldots, q_n\}$ is called *full* if whenever x covers p then $x \in S$ (respectively, if whenever p covers x then $x \in S$).

For any complemented sublattice I of M there is, in view of the lemma, a bijection ψ_I between the spanning lower covering n-sets of I and the spanning upper covering n-sets of I. The map $\psi_n: L_n \longrightarrow U_n$ defined by $\psi_n(S) = \psi_{[S]}(S)$ is, therefore, a bijection. In particular, $|L_n| = |U_n|$ for every positive integer n.

We must show that the number v(k) of full lower covering k-sets is equal to the number w(k) of full upper covering k-sets. If m is the maximum number for which a covering m-set exists in M then $v(m) = |L_m| = |U_m| = w(m)$. Now, let us suppose that the theorem has been established for all integers j such that $k < j \leq m$. Let a_j be the number of lower covering k-sets each of which is contained in a full lower covering j-set and, dually, let c_j be the number of upper covering k-sets each of which is contained in a full upper covering j-set. Obviously, every full covering j-set contains precisely $\binom{j}{k}$ covering k-sets and, in fact, $a_j = \binom{j}{k} v(j) = \binom{j}{k} w(j) = c_j$ for every $j > k$. Finally, since every covering k-set is contained in a full covering j-set for some $j \geq k$, we have that $v(k) = |L_k| - \Sigma_{j>k}\, a_j = |U_k| - \Sigma_{j>k}\, c_j = w(k)$. This completes the proof.

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TECHNISCHE HOCHSCHULE DARMSTADT

FINITE SUBLATTICES GENERATED BY ORDER-ISOMORPHIC SUBSETS

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FINITE SUBLATTICES GENERATED BY ORDER-ISOMORPHIC SUBSETS

by

Werner Poguntke and Ivan Rival

1. Introduction. Classes of lattices characterized by the noncontainment of sublattices isomorphic to certain prescribed lattices arise frequently in lattice theory: for example, the classes of modular and distributive lattices. Indeed, many such characterizations are part of the folklore of lattice theory. A related question concerns classes of lattices characterized by the noncontainment of subsets order-isomorphic to certain prescribed partially ordered sets: for example, dismantlable lattices [5] and modular planar lattices [11] (cf. also [1]). If L is a fixed finite lattice and if an arbitrary lattice K contains no subset order-isomorphic to L then certainly K contains no sublattice isomorphic to L. Far less trivial is the problem of determining those finite lattices L with the property that if an arbitrary lattice K contains no sublattice isomorphic to L then K contains no subset order-isomorphic to L. In particular, the purpose of this note is to characterize finite lattices subject to the condition

Λ(L) : *every lattice which contains an order-isomorphic copy of* L *also contains a lattice-isomorphic copy of* L.

Equivalently, if a lattice K contains a subset L' order-isomorphic to L then the sublattice in K generated by L' already contains a sublattice isomorphic to L. We shall prove the following general combinatorial result:

THEOREM 1. *For a finite lattice* L *the following conditions are equivalent:*

(i) $\underset{\sim}{\wedge}(L)$ *holds;*

(ii) L *is distributive and every element of* L *is either join-irreducible or meet-irreducible;*

(iii) L *is a linear sum of components each of which is either a single element, an eight-element Boolean algebra, or, a product of two chains one of which has precisely two elements.*

The equivalence of (ii) and (iii) has already been established by F. GALVIN and B. JÓNSSON [6], however, since their proof is quite lengthy and computational it seems sensible to include (see section 2) a simple combinatorial proof. The well-known characterization of finite distributive sublattices of free lattices accomplished in [6] is, as we shall see (Corollary 2), a straightforward consequence of Theorem 1. In another direction, we shall investigate (section 4) the connection between the condition $\underset{\sim}{\wedge}$ and a problem related to the lattice of ideals of a lattice.

2. Proof of Theorem 1. If A,B are subsets of a partially ordered set P, we shall write $A < B$ if $a \leq b$ for all $a \in A$ and $b \in B$; P is a *linear sum* of *components* P_i $(i \in I)$ if $P = \bigcup_{i \in I} P_i$ and for any distinct $i,j \in I$, either $P_i < P_j$ or $P_j < P_i$. For all further terminology we refer to [4].

(i) implies (ii): Since, in general, the lattice of order ideals (M-closed subsets) of a finite partially ordered set determines a distributive lattice, it follows that L must be

distributive. On the other hand, if a is an element in a finite lattice $(L, \leq)$ which is both join-reducible and meet-reducible and a_1, a_2 are distinct elements not in L, then it is routine to verify that the set $K := (L - \{a\}) \cup \{a_1, a_2\}$ with the partial ordering $\leq'$ defined by:

for $x, y \in L - \{a\}$, $x \leq' y$ *whenever* $x \leq y$,

$a_1 <' a_2$ *and,*

for $x \in L - \{a\}$, $i = 1,2$, $x <' a_i$ $(x >' a_i)$ *whenever* $x < a$ $(x > a)$,

is a lattice. Furthermore, though $(K, \leq')$ contains an order-isomorphic copy of $(L, \leq)$ it contains no lattice-isomorphic copy of $(L, \leq)$ which, in turn, shows that if $\underset{\sim}{\wedge}(L)$ holds then L contains no element which is both join-reducible and meet-reducible.

(ii) implies (iii): We shall proceed by induction on the length of L. It is an elementary property of an arbitrary finite distributive lattice that, if A is the set of atoms and $|A| = n$, then $[0, \bigvee A] \cong \underset{\sim}{2}^n$, the Boolean algebra on n atoms. This, together with the inductive hypothesis, and the assumption that every element of L is either join-irreducible or meet-irreducible, implies that $n = 2$ or 3.

Let $n = 3$. Then $[0, \bigvee A] \cong \underset{\sim}{2}^3$. We need only show that every other element $x \in L$ satisfies $x > \bigvee A$. Otherwise, if x is incomparable with $\bigvee A$ then $x \wedge \bigvee A = a$, an atom in $[0, \bigvee A]$. Thus, there is an element $y \leq x$ such that $a \prec y$ (y *covers* x) and $y \notin [0, \bigvee A]$. In particular, y is distinct from the two covers $b_1, b_2 \in [0, \bigvee A]$ of a. But then $[a, b_1 \vee b_2 \vee y] \cong \underset{\sim}{2}^3$ which, in turn, implies that b_1 and b_2

are meet-reducible as well as join-reducible.

Let $n = 2$. In fact, let $A = \{a_1, a_2\}$. If both a_1 and a_2 are meet-reducible then there exist three pairwise incomparable elements b_i such that $a_i \prec b_i$ and $a_i \prec a_1 \vee a_2 = b_3$, $i = 1,2$. By assumption b_3 has precisely one cover c and $b_1 \vee b_2 = c$ which, on the other hand, implies that $0 \prec b_1 \wedge b_2$ although $b_1 \wedge b_2 \notin A$. Thus, we may assume that a_1 is meet-irreducible so that $L = \{0, a_1\} \cup [a_2, 1]$. Finally, it suffices, by the inductive hypothesis, to show that a_2 has at most two covers which, however, is a consequence of the fact that $a_1 \vee a_2$ must be meet-irreducible.

(iii) implies (i): Let us first consider the case in which L is an eight-element Boolean algebra and let K be an arbitrary lattice containing an order-isomorphic copy of L labelled as in Figure 1. It is well-known and simple to verify that the sublattice in K generated by $\{a_1 \vee a_2, a_1 \vee a_3, a_2 \vee a_3\}$ is an eight-element Boolean algebra.

Now let L be a direct product of a two-element chain and a finite chain and let K be an arbitrary lattice containing an order-isomorphic copy of L labelled as in Figure 2. We shall show that the subset $\{a_1', a_2', \ldots, a_n', b_1', b_2', \ldots, b_n'\}$ of K defined by $a_i' := a_n \wedge b_i$ and $b_i' := a_i' \vee b_1$ for $1 \leq i \leq n$ forms a sublattice of K isomorphic to L. Clearly, $a_1' \leq a_2' \leq \ldots \leq a_n'$; if $a_i' = a_{i+1}'$ for some $1 \leq i \leq n-1$, then $a_{i+1} \leq a_n' \wedge b_{i+1} = a_n' \wedge b_i \leq b_i$ which is impossible. Thus, $a_1' < a_2' < \ldots < a_n'$. Similarly, $b_1' \leq b_2' \leq \ldots \leq b_n'$; if $b_i' = b_{i+1}'$ for some $1 \leq i \leq n-1$, then $a_{i+1} \leq b_1' \vee (a_n' \wedge b_{i+1}) = b_1' \vee (a_n' \wedge b_i) \leq b_i$ which again is a contradiction, so that $b_1' < b_2' < \ldots < b_n'$. Furthermore, $a_i' \leq b_i'$ for

$1 \leq i \leq n$. If $a_i' = b_i'$ for some i then $b_1 \leq a_i' \vee b_1 = a_n \wedge b_i \leq a_n$ which is a contradiction. Thus, $a_i' < b_i'$ for $1 \leq i \leq n$. Now let $i < k$. $a_k' \vee b_i' = a_k' \vee (a_i' \vee b_1) = b_k'$. On the other hand, $a_k' \wedge b_i' = (a_n \wedge b_k) \wedge ((a_n \wedge b_i) \vee b_1) = a_n \wedge b_i = a_i'$ since $a_n \wedge b_k \geq a_n \wedge b_i$, $(a_n \wedge b_i) \vee b_1 \geq a_n \wedge b_i$, $a_n \geq a_n \wedge b_k$, and $b_i \geq (a_n \wedge b_i) \vee b_1$.

Finally, we consider the case in which L is a finite linear sum $\bigcup_{i \in I} L_i$ where each L_i is either trivial, an eight-element Boolean algebra, or a product of a two-element chain and a finite chain. Let K be an arbitrary lattice containing an order-isomorphic copy $L' = \bigcup_{i \in I} L_i'$ of L and let $[0_i, 1_i]$ denote the universal bounds of each component L_i' in L'. Then $\bigcup_{i \in I} [0_i, 1_i]$ is a sublattice of K and since, for each $i \in I$, the interval sublattice $[0_i, 1_i]$ contains an order-isomorphic copy of L_i, namely L_i', it follows from our considerations above that $[0_i, 1_i]$ contains a sublattice $L_i'' \cong L_i$, so that $\bigcup_{i \in I} L_i''$ is a sublattice of K isomorphic to L. This completes the proof of the theorem.

The class of all finite lattices satisfying condition (iii) of Theorem 1 has already arisen in several apparently different contexts. One of these is in connection with distributive sublattices of free lattices.

COROLLARY 2. (F. GALVIN and B. JÓNSSON [6]) *A finite lattice is a distributive sublattice of a free lattice if and only if it satisfies condition (iii) of Theorem 1.*

Proof. It is well-known that every element in a free lattice is either join-irreducible or meet-irreducible [10]. Thus, the one direction follows by the equivalence of conditions

(ii) and (iii) of Theorem 1.

Let L be a finite lattice satisfying condition (iii) of Theorem 1 and let F be a free lattice on n generators where $n = |L|$. Since L is a homomorphic image of F and L is finite, F contains an order-isomorphic copy of L. In view of the equivalence of conditions (i) and (iii) of Theorem 1 F contains a sublattice isomorphic to L.

3. Remark. Evidently the class of all finite lattices L satisfying $\Lambda(L)$ is rather *small*. It would seem reasonable, therefore, to generalize this condition in the following direction. Let K be an arbitrary class of lattices and let L be a finite lattice. We define

> $\Lambda_K(L)$: *every* $M \in K$ *which contains an order-isomorphic copy of* L *also contains a lattice-isomorphic copy of* L .

Of course, when K is the class of all lattices this is just $\Lambda(L)$.

If K is closed with respect to the formation of sublattices then $\Lambda_K(L)$ holds if and only if whenever $M \in K$ and M contains an order-isomorphic copy L' of L then the sublattice in M generated by L' already contains a lattice-isomorphic copy of L. Furthermore, from the proof of Theorem 1 it follows that if K contains all finite distributive lattices and $\Lambda_K(L)$ holds then L is distributive.

If, for example, B is just the class of all finite Boolean lattices then it is easy to check that $\Lambda_B(L)$ holds for every finite distributive lattice L.

Problem 1: *What is the largest class* $\underset{\sim}{K}$ *of lattices such that* $\underset{\sim}{\Lambda}_{\underset{\sim}{K}}(L)$ *holds for every finite distributive lattice? Does this class, for example, contain all finite direct products of chains?*

In another direction, if $\underset{\sim}{D}$ is the class of all distributive lattices then $\underset{\sim}{\Lambda}_{\underset{\sim}{D}}(L)$ holds for every finite Boolean lattice L. Indeed, let L be the Boolean lattice on n atoms and let L be order-isomorphic to a subset of M, $M \in \underset{\sim}{D}$. Since the sublattice in M generated by the order-isomorphic copy of L is finite we may without loss of generality take M to be finite. Then the dimension (cf. [4,p.99]) of M is at least n; that is, the breadth of M is at least n. But then M contains an element with n covers so that M contains a sublattice isomorphic to L.

Problem 2: *Characterize those finite lattices* L *for which* $\underset{\sim}{\Lambda}_{\underset{\sim}{D}}(L)$ *holds. Are these, for example, just the finite lattices projective in* $\underset{\sim}{D}$ (cf. [2], [3], [9])?

4. Ideal Lattices. Again let $\underset{\sim}{K}$ be an arbitrary class of lattices. We shall briefly investigate the problem of characterizing all finite lattices L subject to the condition: *if* L *is isomorphic to a sublattice of* I(M) *, the lattice of ideals of* M, $M \in \underset{\sim}{K}$ *, then* L *is isomorphic to a sublattice of* M. This is a slightly generalized version of a problem first raised by G. GRÄTZER in [8,p.207] . Such lattices as we shall presently see are rather closely related to lattices satisfying the condition $\underset{\sim}{\Lambda}_{\underset{\sim}{K}}$.

PROPOSITION 3. *Let* L *be a lattice,* S *a finite meet-semilattice and* $\varphi : S \longrightarrow I(L)$ *a meet-embedding. Then there is an order-embedding* $\psi : S \longrightarrow L$ *such that, for* $x,y \in S$, $\psi(x) \in \varphi(y)$ *if and only if* $x \leq y$.

Proof. Let n be the length of S, $H_i := \{x \in S \mid \text{height of } x \text{ in } S \leq i\}$, $R_0 := \{0\}$ and for $i \geq 1$, $R_i := H_i - H_{i-1}$.
We shall define inductively a sequence $\psi_0, \psi_1, \ldots, \psi_n$ of functions each of which satisfies the following conditions:

(i) $\psi_i : H_i \longrightarrow L$ *is an order-embedding;*

(ii) $\psi_i | H_{i-1} = \psi_{i-1}$;

(iii) for $x,y \in H_i$, $\psi_i(x) \in \varphi(y)$ *if and only if* $x \leq y$.

Once done we shall set $\psi := \psi_n$ completing the proof.

Let $x_0 \in \varphi(0)$; we define $\psi_0(0) := x_0$. Suppose now that the functions $\psi_0, \psi_1, \ldots, \psi_k$ $(k < n)$ have been constructed each satisfying the conditions (i),(ii) and (iii). Let $H_{k+1} = \{y_1, y_2, \ldots, y_m\}$ and $1 \leq j \leq m$. For each $i \neq j$ $(1 \leq i \leq m)$ we choose $u_j^i \in \varphi(y_j) - \varphi(y_i)$ and define $x_j' := \bigvee_{i \neq j} u_j^i$. The set $\{x_1', x_2', \ldots, x_m'\}$ is an m-element antichain in L, $x_j' \in \varphi(y_j) - \bigcup_{i \neq j} \varphi(y_i)$ and *if, for every* $j \in \{1,2,\ldots,m\}$, $x_j'' \geq x_j'$ *where* $x_j'' \in \varphi(y_j)$ *then* $\{x_1'', x_2'', \ldots, x_m''\}$ *is again an m-element antichain in* L *such that* $x_j'' \in \varphi(y_j) - \bigcup_{i \neq j} \varphi(y_i)$ *for* $1 \leq j \leq m$.

Our aim now is to carefully select elements $x_j \in \varphi(y_j)$ $(1 \leq j \leq m)$ such that $x_j \geq x_j'$ and the map $\psi_{k+1} : H_{k+1} \longrightarrow L$ defined by

$$\psi_{k+1}(x) := \begin{cases} \psi_k(x) & \text{if } x \in H_k \\ x_j & \text{if } x = y_j \in R_{k+1} \end{cases}$$

is an order-embedding with the desired properties.

In fact, for $j \in \{1,2,\ldots,m\}$, we choose elements $w_j^r \in \varphi(y_j) - \varphi(v_r)$ for every v_r in S covered by y_j $(1 \le r \le l_j)$ and define

$$x_j := x_j' \vee \bigvee_{r=1}^{l_j} (w_j^r \vee \psi_k(v_r)) .$$

Clearly, $x_j \in \varphi(y_j) - (\bigcup_{i \neq j} \varphi(y_i) \cup \bigcup_{r=1}^{l_j} \varphi(v_r))$.

If $\psi_{k+1}(y_j) = \psi_k(x)$ for some $y_j \in R_{k+1}$, $x \in H_k$ then $x_j \in \varphi(y_j \wedge x)$ and $y_j \wedge x < y_j$; that is, $x_j \in \varphi(v_r)$ for some v_r covered by y_j, which is a contradiction. Thus, ψ_{k+1} is injective.

If $\psi_k(x) < x_j$ and $x \nleq y_j$ for some $x \in H_k$ then $\psi_k(x) \in \varphi(y_j) \cap \varphi(x) = \varphi(y_j \wedge x)$. But condition (iii) for ψ_k implies that $x \le y_j \wedge x \le y_j$. This then shows that $\psi_{k+1}: H_{k+1} \longrightarrow L$ as defined above is indeed an order-embedding. By definition ψ_{k+1} satisfies condition (ii). Finally, if $\psi_{k+1}(x) \in \varphi(y)$ but $x \nleq y$ for some $x,y \in H_{k+1}$ then $\psi_{k+1}(x) \in \varphi(x) \cap \varphi(y) = \varphi(x \wedge y)$ and $x \wedge y < x$ which, in turn, implies that $\psi_{k+1}(x) \in \varphi(z)$ for some z covered by x, a contradiction. ψ_{k+1} thus satisfies condition (iii), completing the proof.

COROLLARY 4. *Let* L *be a lattice,* P *a finite partially ordered set and* $\varphi: P \longrightarrow I(L)$ *an order-embedding. Then there is an order-embedding* $\psi: P \longrightarrow L$ *such that, for* $x,y \in P$, $\psi(x) \in \varphi(y)$ *if and only if* $x \le y$.

Proof. It suffices to apply Proposition 3 to the meet-subsemilattice of I(L) generated by $\varphi(P)$.

COROLLARY 5. *Let* $\underset{\sim}{K}$ *be a class of lattices and* L *a finite lattice such that* $\underset{\sim}{\Lambda}_K(L)$ *holds. If* L *is isomorphic to a sublattice of* $I(M), M \in \underset{\sim}{K}$, *then* L *is isomorphic to a sublattice of* M.

Finally, if we restrict ourselves to the class of all lattices then Corollary 5 together with Theorem 1 yields the following result already established by H. GASKILL [7] .

COROLLARY 6 (H. GASKILL [7]). *Let* K *be a lattice and let* L *be a finite distributive lattice in which every element is either join-irreducible or meet-irreducible. If* L *is isomorphic to a sublattice of* I(K) *then* L *is isomorphic to a sublattice of* K .

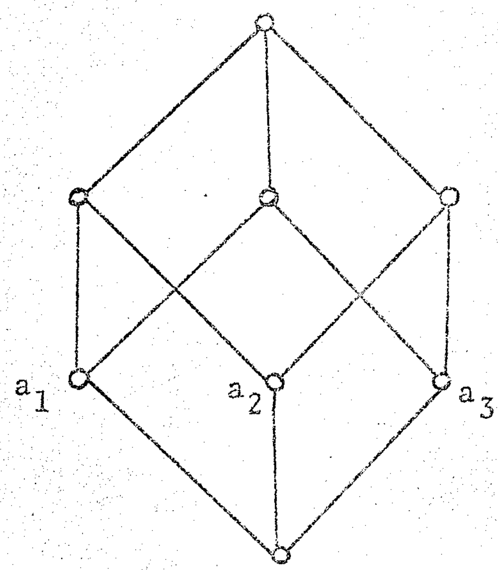

Figure 1

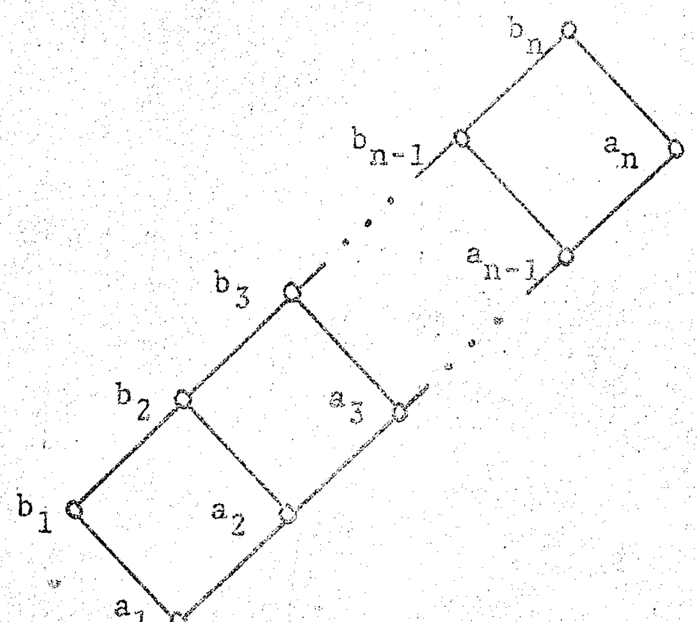

Figure 2

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CORRECTIONS TO

CROWNS, FENCES, AND DISMANTLABLE LATTICES

by

David Kelly and Ivan Rival

Page	Line	Reads	Should Read
2	−1, −3	n (8 times)	m (8 times)
3	−9	...$(x_{n-1} \not\leq x_n)$,	...$(x_{n-2} \not\leq x_n)$,
3	−4	...integer k such that...	...integer $k \geq 2$ such that...
3	−2	$k \geq 3$ ...	$k > 3$ ...
11	11	...is upper, or,	...is upper, and,
13	−7	...If $F_{n>1}$ is lower...	...If F_{n-1} is lower...
18	3	and $x_i \not\geq g_{n-1}$, ...	...and $x_i \geq g_{n-1}$, ...
18	6	$(g_{n-1}, y_1, x_1, a_2, \ldots, a_{m-1}, x_2, y_2)$...	$(y_1, x_1, a_2, \ldots, a_{m-1}, x_2, y_2, g_{n-1})$ ..
	7	...$y_i \not\leq g_{n-1}$, ...	...$y_i \leq g_{n-1}$,...
19	−2	..$(g_{n-1}, y_1, x_1, a_2, \ldots, a_{m-1}, x_2)$...	...$(y_1, x_1, a_2, \ldots, a_{m-1}, x_2, g_{n-1})$...

CROWNS, FENCES, AND DISMANTLABLE LATTICES

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ABSTRACT

An element x of a lattice L is doubly irreducible in L if $x \neq y \vee z$ and $x \neq y \wedge z$ whenever y and z are elements of L distinct from x. A lattice L is dismantlable if every sublattice S of L contains an element that is doubly irreducible in S. Sublattices and homomorphic images of dismantlable lattices are dismantlable. A (lower) fence is a partially ordered set $\{x_i \mid 0 \leq i < n\}$, $(n \leq \omega)$, for which the comparabilities that hold are precisely:

(*) $x_i < x_{i+1}$ (i even), $x_i > x_{i+1}$ (i odd). For finite even $n \geq 6$, a crown is a partially ordered set $\{x_i \mid 0 \leq i < n\}$ for which $x_0 < x_{n-1}$ and (*) are precisely the comparabilities that hold.

Theorem 1. Let L be a lattice which contains no infinite chains and no infinite fences. (a) L is dismantlable if and only if L contains no crowns. (b) If L is dismantlable and is not a chain, then L contains (at least) two incomparable doubly irreducible elements.

Theorem 2. Let L be a modular lattice of finite length. L is dismantlable if and only if L contains no crown of order 6 (i.e., the breadth of L does not exceed 2).

Corollary. A finite distributive lattice is dismantlable if and only if it is planar.

CROWNS, FENCES, AND DISMANTLABLE LATTICES

by

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A finite lattice L of order n is <u>dismantlable</u> [6] if there is a chain $L_1 \subset L_2 \subset \cdots \subset L_n = L$ of sublattices of L such that $|L_i| = i$ for every $i = 1, 2, \cdots, n$. In [1] it was shown that every finite planar lattice is dismantlable. Furthermore, every lattice L with $|L| \leq 7$ is dismantlable [6]; in fact, every large enough lattice contains a dismantlable sublattice with precisely n elements [4]. As well, such lattices are closed under the formation of sublattices and homomorphic images [6]. In section 2, we shall extend the definition of <u>dismantlable</u> to infinite lattices.

For an integer $n \geq 3$ a <u>crown</u> [1] is a partially ordered set $\{x_1, y_1, x_2, y_2, \cdots, x_n, y_n\}$ in which $x_i \leq y_i$, $y_i \geq x_{i+1}$, for $i = 1, 2, \cdots, n-1$, and $x_1 \leq y_n$ are the only comparability relations (see Figure 1).

The main result of this paper is a characterization (Theorem 3.1) of dismantlable lattices in terms of crowns. In fact, we show that <u>every finite lattice is either dismantlable or it contains a crown but not both</u>. For more familiar classes of lattices we prove (Theorem 3.5) that <u>a modular lattice of finite length is dismantlable if and only if it has breadth</u> ≤ 2 (or, <u>equivalently, it contains no crown of order</u> 6). It now follows (Corollary 3.6) that <u>a finite distributive lattice is dismantlable if and only if it is planar</u>.

1. Preliminaries. An element x in a lattice L is join-reducible (meet-reducible) in L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z$ $(x = y \wedge z)$; x is join-irreducible (meet-irreducible) in L if it is not join-reducible (meet-reducible) in L; x is doubly irreducible in L if it is both join-irreducible and meet-irreducible in L. Let $J(L)$, $M(L)$, and $\mathrm{Irr}(L)$ denote the set of all join-irreducible elements in L, meet-irreducible elements in L, and doubly irreducible elements in L, respectively. The length of an n-element chain is $n-1$ and the length of a partially ordered set P is the least upper bound of the lengths of the chains in P. For $x, y \in P$, x is incomparable with y $(x \parallel y)$ if $x \nleq y$ and $y \nleq x$. For all further terminology we refer to [2].

Let us observe that a partially ordered set which contains no infinite chains and in which all maximal chains have the same length is itself of finite length. Furthermore, since in a lattice L with no infinite chains, any join equals a finite join, L is complete and for every $x \in L$, $x = \bigvee(a \in J(L) \mid a \leq x)$.

Lemma 1.1. If x and y are elements in a lattice L with no infinite chains and $x \nleq y$, then there exists $a \in J(L)$ such that $a \leq x$ but $a \nleq y$. $\square$

(Of course, the dual of the preceding lemma holds as well.)

A fence F [1] is a partially ordered set $\{x_1, x_2, \ldots, x_n, \ldots\}$ in which either

(1) $\quad x_1 \leq x_2,\ x_2 \geq x_3,\ \ldots,\ x_{2n-1} \leq x_{2n},\ x_{2n} \geq x_{2n+1},\ \ldots$

or

(1′) $\quad x_1 \geq x_2,\ x_2 \leq x_3,\ \ldots,\ x_{2n-1} \geq x_{2n},\ x_{2n} \leq x_{2n+1},\ \ldots$

are the only comparability relations (denoted by $F = (x_1, x_2, \cdots, x_n, \cdots)$). $F = (x_1, x_2, \cdots, x_n, \cdots)$ is a <u>lower fence</u> (see Figure 2) if (1) holds and an <u>upper fence</u> if (1′) holds. We shall also denote a crown on $\{x_1, y_1, x_2, y_2, \cdots, x_n, y_n\}$ by $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$.

The concept of a fence turns out to be the natural link between that of a crown and that of a dismantlable lattice. (Indeed, observe that the removal of one element, or two comparable elements from a crown leaves a fence.)

We now establish some elementary results about fences in partially ordered sets which contain no crowns.

<u>Lemma 1.2.</u> <u>Let</u> P <u>be a partially ordered set containing no crowns and</u> $F = \{x_1, x_2, \cdots, x_n\}$, $(n \geq 3)$, <u>be a subset of</u> P <u>satisfying</u> (1). <u>Then</u> F <u>is a fence in</u> P <u>for even (odd)</u> n <u>if and only if</u>

$$x_1 \not\geq x_3, \qquad x_{n-2} \not\geq x_n \quad (x_{n-1} \not\leq x_n),$$

$$x_i \not\leq x_{i+3}, \quad \underline{\text{for}}\ i\ \underline{\text{odd and}}\ i \leq n-3,\ \underline{\text{and}}$$

$$x_i \not\geq x_{i+3}, \quad \underline{\text{for}}\ i\ \underline{\text{even and}}\ i \leq n-3\ .$$

<u>Proof.</u> It is enough to consider the case in which n is even. Clearly $x_i \neq x_{i+1}$ for all $i < n$, (for example, if $x_1 = x_2$ then $x_1 \geq x_3$). If F is not a fence in P, choose the least integer k such that either $x_i \leq x_{i+k}$, for some odd i, or $x_i \geq x_{i+k}$, for some even i. By hypothesis, $k \geq 3$ and, in fact, k must be odd. But then $(x_i, x_{i+1}, \cdots, x_{i+k})$ is a crown, which is a contradiction. □

The next two corollaries describe how fences can be constructed by "pasting" together smaller fences.

Corollary 1.3. *If $n \geq 3$ and $(x_1, x_2, \cdots, x_n)$ is a lower fence in a partially ordered set P containing no crowns and $y \in P$, then $(y, x_1, x_2, \cdots, x_n)$ is a fence in P if and only if $y \geq x_1$, $y \not\geq x_2$ and $y \geq x_3$.* $\square$

Corollary 1.4. *Let $(x_1, x_2, \cdots, x_i, y_1, y_2, \cdots, y_j)$ and $(y_1, y_2, \cdots, y_j, z_1, z_2, \cdots, z_k)$ $(j \geq 3)$ be fences in a partially ordered set P containing no crowns. Then $(x_1, x_2, \cdots, x_i, y_1, y_2, \cdots, y_j, z_1, z_2, \cdots, z$ is a fence in P.* $\square$

If $(x_1, x_2, \cdots, x_n)$ is a lower fence in a partially ordered set P and $x_2' \in P$, then $(x_1, x_2', x_3, \cdots, x_n)$ is a lower fence in P whenever $x_1 \leq x_2'$, $x_3 \leq x_2'$, and $x_2' \leq x_2$.

Lemma 1.5. *Let $n \geq 3$ and $(x_1, x_2, \cdots, x_n)$ be a lower fence in a lattice L containing no crowns and $y \in L$. Then $(y, x_1, x_1 \vee x_3, x_3, \cdots, x_n)$ is a fence in L if and only if $y \geq x_1$ and $y \parallel x_1 \vee x_3$.* $\square$

2. Dismantlable Lattices. For finite lattices, the following result was proven in [6, Theorem 2].

Proposition 2.1. *For any lattice L the following two conditions are equivalent:*

(i) *every nonempty sublattice S of L contains an element which is doubly irreducible in S;*

(ii) <u>there is an ordinal</u> α <u>and a family</u> $(L_\gamma \mid 0 \le \gamma \le \alpha)$ <u>of sublattices of</u> L <u>with</u> $L_0 = L$, $L_\alpha = \phi$, <u>and satisfying the following conditions</u>: (a) <u>if</u> $\beta < \alpha$ <u>and</u> $L_\beta \ne \phi$, <u>there exists</u> $x \in L_\beta$ <u>such that</u> $L_{\beta+1} = L_\beta - \{x\}$; <u>if</u> $L_\beta = \phi$, <u>then</u> $L_{\beta+1} = \phi$; <u>and,</u> (b) <u>for a limit ordinal</u> $\beta \le \alpha$, $L_\beta = \bigcap_{\gamma<\beta} L_\gamma$.

<u>Proof.</u> For (i) implies (ii), it is enough to define a family $(L_\gamma \mid 0 \le \gamma \le \alpha)$ by setting x in (ii)(a) to be a doubly irreducible element in L_β.

Suppose now that (ii) holds. Let S be an arbitrary nonempty sublattice of L and let β be the least ordinal such that $S \nsubseteq L_\beta$. We choose some $x \in S - L_\beta$. If β were a limit ordinal, then $x \notin L_\gamma$ for some $\gamma < \beta$ which, however, would contradict the minimality of β; therefore, $\beta = \delta + 1$ for some δ. Since $S \subseteq L_\delta$ and x is doubly irreducible in L_δ, x is also doubly irreducible in S. □

Since Proposition 2.1 is a natural extension of [6, Theorem 2] to arbitrary lattices, we define a lattice to be <u>dismantlable</u> whenever it satisfies either of the equivalent conditions of Proposition 2.1. Obviously every sublattice of a dismantlable lattice is dismantlable. Furthermore, it is possible to extend the proof in [6, Corollary 2] to show that any homomorphic image of an arbitrary dismantlable lattice is dismantlable.

We can now prove a strong version of one direction of our main result (Theorem 3.1).

<u>Proposition 2.2.</u> <u>Every lattice which contains a crown is not dismantlable.</u>

<u>Proof.</u> Let L be a lattice containing a crown $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$, where $n \geq 3$. Then L also contains the crown $(x_1', y_1', x_2', y_2', \cdots, x_n', y_n')$ where $y_i' = x_i \vee x_{i+1}$ and $x_i' = y_{i-1}' \wedge y_i'$, for $1 \leq i \leq n$, (with $x_{n+1} := x_1$ and $y_0' := y_n'$). Then $y_i' = x_i' \vee x_{i+1}'$, for $1 \leq i \leq n$, (with $x_{n+1}' := x_1$), so that every element in the sublattice of L generated by $(x_1', y_1', x_2', y_2', \cdots, x_n', y_n')$ is either join-reducible or meet-reducible. □

REMARK. The argument used in the proof of Proposition 2.2 also shows that a lattice which contains a fence (crown) also contains a fence (crown) of the same order in which for any pair of elements having an upper (lower) bound, the respective bound is actually the join (meet) of the pair in the lattice.

The direct product of a two-element chain and the integers $\mathbb{Z}$ is a (distributive) non-dismantlable lattice containing no crowns. Therefore, only lattices containing no infinite chains will be considered in the sequel.

3. The main result of this paper is

<u>Theorem 3.1.</u> <u>A finite lattice is dismantlable if and only if it contains no crowns.</u>

Actually we shall prove a stronger result:

<u>Theorem 3.2.</u> <u>A lattice which contains no infinite chains and no infinite fences is dismantlable if and only if it contains no crowns.</u>

The necessity is just a special case of Proposition 2.2. Since the property of non-containment of certain partially ordered sets in a lattice is inherited by its sublattices, the proof of Theorem 3.2 finally amounts to proving

Theorem 3.3. Every lattice which contains no infinite chains, no infinite fences, and no crowns must contain at least one doubly irreducible element.

The non-containment of infinite fences in Theorem 3.3 is essential. The lattice on $\{x_i \mid i \in \mathbb{Z}\} \cup \{0, 1\}$ with the partial ordering defined by $0 < x_i < 1$ $(i \in \mathbb{Z})$, $x_i < x_{i+1}$ (i even and $i \in \mathbb{Z}$), $x_i > x_{i+1}$ (i odd and $i \in \mathbb{Z}$), contains no crowns, yet it is not dismantlable. In fact, the lattice $L = F_1 \cup F_2 \cup \{c, d, 0, 1\}$, (see Figure 3) where $F_1 = (a_1, a_2, \cdots, a_n, \cdots)$ is an infinite upper fence, $F_2 = (b_1, b_2, \cdots, b_n, \ldots)$ is an infinite lower fence, $F_1 \cap F_2 = \emptyset$, $d > x$ for every $x \in F_1$, $d > a$, $c < x$ for every $x \in F_2$, $c < d$, and $0, 1$ are the universal bounds is not dismantlable yet, it contains no crowns; note that $\{x_i \mid i \in \mathbb{Z}\}$ as ordered above is not isomorphic to any subset of L.

Furthermore, the non-dismantlable lattice consisting of a crown of order $2n$, $(n \geq 3)$, with universal bounds 0 and 1 adjoined does not contain any crown of order $2m$, for $m \neq n$, so that all crowns must be omitted in Theorems 3.1 to 3.3.

Before proceeding to the proof in Section 4, we prove an analogue of Theorem 3.2 for modular lattices.

The _breadth_ of a lattice L $(b(L))$ is the least positive integer b such that any join $\bigvee_{i=1}^{n} x_i$, $(n > b)$, is always a join of a subset of b of the x_i's.

Lemma 3.4. For a lattice L _the following conditions are equivalent:_

(i) $b(L) \leq 2$;

(ii) L _contains no crown of order_ 6;

(iii) L _contains no sublattice isomorphic to the Boolean lattice_ $\underset{\sim}{2}^3$.

Proof. To prove that (iii) implies (ii) we need only recall the remark following Proposition 2.2; the rest is routine. □

Theorem 3.5. A modular lattice L _of finite length is dismantlable if and only if_ $b(L) \leq 2$.

Proof. The "only if" part follows from Proposition 2.2 and Lemma 3.4.

Suppose that L is a modular lattice of finite length and L contains no crown of order 6. We show, in fact, that every maximal join-irreducible in L is already doubly irreducible in L. If this were not so then there would exist a maximal join-irreducible a with at least two distinct covers x_1, x_2. By the maximality of a, x_1 and x_2 must be join-reducible. Now let $y_1, y_2 \in L$ both distinct from a such that x_1 covers y_1 and x_2 covers y_2. By the lower semimodularity of L, a covers both $a \wedge y_1$ and $a \wedge y_2$, and if c is the unique element which a covers then we must have $y_1 \wedge y_2 = a \wedge y_1 = a \wedge y_2 = c$. Finally, by upper semimodularity $y_1 \vee y_2$ must cover both y_1 and y_2 so that $(y_1, x_1, a, x_2, y_2, y_1 \vee y_2)$ is a crown of order 6 in L which is impossible. Thus, $a \in \mathrm{Irr}(L)$. □

The crown of order 8 with universal bounds 0 and 1 adjoined shows that modularity is essential in Theorem 3.5. Modular lattices of finite length with breadth ≤ 2 have been studied in [5] where they are called <u>quasiplanar</u>. Although not every such lattice is planar (see Figure 4) we have the following

<u>Corollary 3.6.</u> <u>A finite distributive lattice is dismantlable if and only if it is planar.</u>

<u>Proof.</u> It is well-known that in a finite distributive lattice L, $b(L)$ equals the largest integer n such that there exists $x \in L$ and x covers k elements. But then it follows from [3, Theorem 1.2] that L can be embedded into a direct product of $b(L)$ chains. (The special case of this latter statement for $b(L) \leq 2$ is also proven in [7].) □

In Section 5, we shall show that a simple application of the construction used in the proof of Theorem 3.3, yields a proof of

<u>Theorem 3.7.</u> <u>A dismantlable lattice which is not a chain and which contains no infinite chains and no infinite fences, contains at least two incomparable doubly irreducible elements.</u>

For a finite dismantlable lattice L, Theorem 3.7 has a rather simple proof by induction on $|L|$.

Indeed, suppose $b \in Irr(L)$. (We may without loss of generality assume that $0 < b < 1$.) Then there exist unique elements $a, c \in L$ such that b covers a and c covers b. Without loss of generality the dismantlable sublattice $L' = L - \{b\}$ of L is not a chain so that by the inductive hypothesis it contains two incomparable elements x, y both doubly irreducible in L'. If neither x nor y is in $\{a, c\}$ then $x, y \in \mathrm{Irr}(L)$ and we are done. Otherwise, $x \in \{a, c\}$ and $y \notin \{a, c\}$, say, so that $y \in \mathrm{Irr}(L)$, and since $b \in \mathrm{Irr}(L)$, $b \parallel y$.

The next result was established for finite lattices in [6, Theorem 2.].

Corollary 3.8. *A lattice L which contains no infinite chains and no infinite fences is dismantlable if and only if for every chain C in L, there is an ordinal α and a family $(L_\gamma \mid 0 \le \gamma \le \alpha)$ of sublattice of L with $L_0 = L$, $L_\alpha = C$, and satisfying the following conditions: (a) if $\beta < \alpha$ and $L_\beta \supset C$, there exists $x \in L_\beta - C$ such that $L_{\beta+1} = L_\beta - \{x\}$; if $L_\beta = C$, then $L_{\beta+1} = C$; and, (b) for a limit ordinal $\beta \le \alpha$, $L_\beta = \bigcap_{\gamma<\beta} L_\gamma$.*

Proof. The sufficiency is obvious. Let C be a chain in the dismantlable lattice L; we may assume that C is maximal. We show that for every sublattice S of L such that $S \supset C$ there is an element $S - C$ which is doubly irreducible in S. But every such S is dismantlable and is not a chain. Thus, by Theorem 3.7, S contains two incomparable doubly irreducible elements so that one of them must be in $S - C$. □

Let L be the lattice consisting of the infinite fence $(x_1, x_2, \cdots, x_n, \cdots)$ with universal bounds 0 and 1 adjoined. L is dismantlable yet, it contains only one doubly irreducible element. Thus, the hypothesis regarding infinite fences is necessary in Theorem 3.7. This hypothesis is also essential in Corollary 3.8 since for the chain $C = \{x_1, x_2\}$ there is no suitable family $(L_\gamma)_\gamma$ of sublattices that will do.

4. Proof of Theorem 3.3. Throughout this section we shall assume that L is a lattice which contains no infinite chains and no infinite fences, that L contains no crowns, and that $\mathrm{Irr}(L) = \phi$.

Let Q be a convex subset of L and $\mathbf{F} = (y_1, y_2, \cdots, y_n)$, $n \geq 3$, be a fence in Q. We shall call $\mathbf{F}$ maximal in Q whenever

(2) there is no fence $(x_1, x_2, y_2, u, y_4, \cdots, y_n)$ such that $u = y_2 \wedge y_4$ if $\mathbf{F}$ is lower, and, $u = y_2 \vee y_4$ if $\mathbf{F}$ is upper, or,

(2′) there is no fence $(y_1, y_2, \cdots, y_{n-4}, v, y_{n-1}, z_1, z_2)$ such that $v = y_{n-1} \vee y_{n-3}$ if $\mathbf{F}$ is lower (upper) and n is even (odd), and, $v = y_{n-1} \wedge y_{n-3}$ if $\mathbf{F}$ is upper (lower) and n is even (odd).

A fence $\mathbf{F} = (y_1, y_2, \cdots, y_n)$, $n \geq 3$, is left-maximal in Q if (2) holds and right-maximal if (2′) holds.

Lemma 4.1. Every convex subset Q of L which contains a fence of order 3 also contains a left-maximal and a right-maximal fence both of order ≥ 3. Indeed, if Q contains a fence of order 5 it contains a maximal fence of order ≥ 5.

<u>Proof.</u> In view of Corollary 1.4 it suffices to show that Q contains a left-maximal fence of order ≥ 3 if it contains a fence of order 3. If F is a fence in Q of order 3 and Q contains no left-maximal fence then there exists a sequence $(F_m)_m$ of fences in Q such that $F_1 = F$ and, for $m > 1$, F_m is obtained from F_{m-1} as described in (2). But now if, for every $m \geq 1$, y_m is the fourth entry in F_{m+1}, then $(y_1, y_2, \cdots, y_m, \cdots)$ would be an infinite fence in Q, which is impossible. □

Let $Q_0 = L$. For $n \geq 1$, we shall construct a sequence $((F_n, Q_n))_n$ of pairs such that, for $n \geq 1$, Q_n is a nonempty convex subset of L, F_1 is a left-maximal fence in Q_0 and, for $n \geq 2$, F_n is a maximal fence in Q_{n-1} (see Figure 5). For notational ease, we shall always label F_n so that $F_n = (e_n, f_n, g_n, h_n, \cdots)$. Furthermore, by virtue of the remark following Proposition 2.2, we may assume that $g_n = f_n \vee h_n$ $(g_n = f_n \wedge h_n)$ if F_n is upper (lower).

Once F_n has been chosen, Q_n will be defined by

$$(3)\qquad \begin{aligned} Q_n &= \{x \in L \mid x \geq f_n \text{ and } x \parallel g_n\} \text{ if } F_n \text{ is upper} \\ Q_n &= \{x \in L \mid x \leq f_n \text{ and } x \parallel g_n\} \text{ if } F_n \text{ is lower} \end{aligned}$$

It is obvious that Q_n is convex in L. We shall show by induction on $n \geq 1$, that a maximal fence F_n in Q_{n-1} (left-maximal for $n = 1$) can be chosen so that

(i) <u>for every</u> $n \geq 1$, $Q_n \subseteq Q_{n-1}$,

(ii) $|F_1| \geq 3$ <u>and, for</u> $n > 1$, $|F_n| \geq 5$, <u>and</u>

(iii) <u>if</u> F_n <u>is upper (lower)</u>, $x \in Q_n$, $y \leq x$ $(y \geq x)$, <u>and</u> $y \not\leq g_n$ $(y \not\geq g_n)$, <u>then</u> $y \in Q_n$.

For $n = 1$, property (i) is obvious. If b is join-reducible in L and $a \vee c = b$, $a \neq c$, then (a, b, c) is a fence of order 3 in $Q_0 = L$. Thus, by Lemma 4.1, Q_0 must contain a left-maximal fence $F_1 = (e_1, f_1, g_1, h_1, \cdots)$ of order ≥ 3. Property (iii) holds for $n = 1$, precisely because F_1 is left-maximal in $Q_0 = L$.

Let $n \geq 2$. We assume that (F_i, Q_i) have been defined for $i = 1, 2, \cdots, n-1$ and that (i), (ii), and (iii) hold whenever n is replaced by a smaller positive integer.

<u>Lemma 4.2.</u> <u>Let</u> $x \in Q_{n-1}$ <u>and</u> $y \notin Q_{n-1}$.

(a) <u>If</u> $y \geq x$, <u>then</u> $y > g_{n-1}$.

(b) <u>If</u> $y \leq x$, <u>then</u> $y < g_{n-1}$.

<u>Proof.</u> (a) If F_{n-1} is upper, y must be comparable with g_{n-1} and, in fact, $y > g_{n-1}$. If $F_{n>1}$ is lower, $y > g_{n-1}$ by (iii) for $n-1$. □

By Lemma 3.4, every join-reducible (meet-reducible) element x in L has a <u>join</u> (<u>meet</u>) <u>representation</u> $x = b_1 \vee b_2$ $(x = c_1 \wedge c_2)$, where $b_1, b_2 \in J(L)$ $(c_1, c_2 \in M(L))$. A join representation $x = b_1 \vee b_2$ of a join-reducible element x in L is <u>maximal</u> if whenever $x = c_1 \vee c_2$, $c_1, c_2 \in J(L)$, and $c_1 \geq b_1$, $c_2 \geq b_2$ then $c_1 = b_1$ and $c_2 = b_2$. (<u>Minimal</u> meet representations are defined dually.)

The next lemma together with Lemmas 4.1 and 4.2 shows that Q_{n-1} does contain a maximal fence satisfying (ii).

Lemma 4.3. *Let* Q *be a nonempty subset of* L *and* $g \in L - Q$ *such that* $x \| g$ *for all* $x \in Q$. *Furthermore, suppose that whenever* $x \in Q$ *and* $y \notin Q$ *the following conditions are satisfied:*

(a) *if* $y \geq x$, *then* $y > g$;

(b) *if* $y \leq x$, *then* $y < g$.

Then Q *contains a fence of order* 5.

(For this lemma the only assumptions needed on L are that L contains no infinite chains, $\mathrm{Irr}(L) = \emptyset$, and L contains no crown of order 6.)

Proof. It is easy to check that under the hypotheses of the lemma Q is convex. Furthermore, if $x \in Q$ but $x \notin M(L)$ then there exist $y, z \in M(L)$ such that $x = y \wedge z$. At least one of y, z lies in Q since otherwise $y, z > g$ so that $x = y \wedge z \geq g$. Thus, $Q \cap M(L) \neq \emptyset$ and dually, $Q \cap J(L) \neq \emptyset$.

For $a, b, c \in L$, we write a/bc for the ordered set $\{a, b, c\}$ whenever $b, c \in J(L)$, $b \neq c$, $a \in M(L)$, and $b, c < a$. These "quotient pairs" are strictly partially ordered by $a_1/b_1c_1 < a_2/b_2c_2$ if and only if $b_1 < b_2, c_2$ or $c_1 < b_2, c_2$. Quotient pairs bc/a are defined dually. Only certain quotient pairs will be of interest here; in fact, let Δ be the set of all a/bc such that $a, b, c \in Q$, $a = b \vee c$, and if $b < x < a$ or if $c < x < a$ then x is both join- and meet-reducible; and dually, let ∇ be the set of all bc/a such that $a, b, c \in Q$, $a = b \wedge c$, and if $a < x < b$ or if $a < x < c$ then x is both join- and meet-reducible.

Let us assume that Q contains no fence of order 5.

First, we show that either Δ or ∇ is nonempty. Suppose that $\Delta = \phi$. Take b_1 a minimal meet-irreducible in Q and $b_2, b_3 \in J(L)$ a maximal join representation of b_1. If $b_2, b_3 \notin Q$ then $b_2, b_3 < g$ so that $b_1 = b_2 \vee b_3 \leq g$, which is impossible. On the other hand, b_2 and b_3 cannot both be in Q since in that case $b_1/b_2b_3 \in \Delta$, contradicting our assumption. Thus, we may assume that $b_2 \in Q$ but that $b_3 \notin Q$, (i.e., $b_3 < g$).

Now, let us take $b_4, b_5 \in M(L)$ a minimal meet representation of b_2. If one of b_4, b_5 is not in Q, say b_4, then $b_4 > g > b_3$ so that $b_1 = b_2 \vee b_3 \leq b_4$ which, by the minimality of b_4 would imply that $b_4 = b_1 \in Q$. Therefore, we may assume that $b_4, b_5 \in Q$. If $b_1 \wedge b_4$, $b_1 \wedge b_5 > b_2$ we get a fence $(b_4, b_1 \wedge b_4, b_1, b_1 \wedge b_5, b_5)$ of order 5 so that without loss of generality we may take $b_2 = b_1 \wedge b_4$.

Let $b_6, b_7 \in J(L)$ be a maximal join representation of b_4. If $b_6, b_7 \geq b_2$, then $b_4/b_6b_7 \in \Delta$. Thus, we may suppose that $b_6 < b_4$ but that $b_6 \not\geq b_2$. If $b_6 \leq b_1$ then $b_6 \leq b_1 \wedge b_4 = b_2$ which, by the maximality of b_6, gives that $b_6 = b_2$, contradicting $b_6 \not\geq b_2$. Therefore, $b_6 \not\leq b_1$. If $b_6 \notin Q$ we get a crown $(g, b_6, b_4, b_2, b_1, b_3)$. Thus, $b_6 \in Q$.

Now take $b_8 \in M(L)$ minimal with respect to $b_8 > b_6$ and $b_8 \not\geq b_2$. Suppose there exists such a b_8 with $b_8 \not\leq b_2 \vee b_6$. If $b_8 \notin Q$ we get a crown $(b_8, b_6, b_2 \vee b_6, b_2, b_1, b_3)$; thus, $b_8 \in Q$. In this case Q contains a fence $(b_8, b_6, b_2 \vee b_6, b_2, b_1)$ of order 5. Thus, we may assume that every $x \in M(L)$ such that $x > b_6$ but $x \not\geq b_2$ also satisfies $x \leq b_2 \vee b_6$.

Finally, we take $c_1, c_2 \in M(L)$ a minimal meet representation of b_6. Clearly, one of c_1, c_2 must satisfy $x \nleq b_2$, say c_1. Then $c_1 \leq b_2 \vee b_6$. If $c_2 \nleq b_2 \vee b_6$ then $c_2 \geq b_2$; however, in this case, $c_2 \geq b_2 \vee b_6 \geq c_1$, which is impossible. Thus, $c_2 \leq b_2 \vee b_6$ and $c_1c_2/b_6 \in \nabla$. We have then shown that either $\Delta \neq \phi$ or $\nabla \neq \phi$.

Suppose now that $\Delta \neq \phi$ and take c/ab a maximal element in Δ with respect to the strict partial ordering of Δ . (Note that an infinite chain in $(\Delta, <)$ would give an infinite chain in Q.) Take $a_1, a_2, b_1, b_2 \in M(L)$ such that a_1, a_2 is a minimal meet representation of a and, b_1, b_2 is a minimal meet representation of b . If $a_1, a_2 \notin Q$ then $a = a_2 \wedge a_2 \geq g$, which is impossible. Thus, we may assume that $a_1 \in Q$; similarly, we may assume that $b_1 \in Q$. If $a_1 \neq c \neq b_1$ we get a fence (a_1, a, c, b, b_1) in Q of order 5 (for example, if $a_1 \geq b$, then $a_1 \geq a \vee b = c$, contradicting the minimality of a_1). Without loss of generality we may assume then that $a_1 = c \neq b_1$. If $b_2 \notin Q$ then $a_2 \in Q$ (otherwise we get the crown (g, a_2, a, c, b, b_2)). But then (a_2, a, c, b, b_1) is a fence of order 5 in Q (note that $a_2 \neq b_1$ since otherwise $b_1 \geq c$). Therefore, $b_2 \in Q$. Suppose that $b_2 \neq c$. If $b_2 \wedge c, b_1 \wedge c > b$ then $(b_2, b_2 \wedge c, c, b_2 \wedge c, b_2)$ is a fence in Q of order 5. Thus, we may assume that $b_1 \wedge c = b$, (which is a minimal meet representation of b). Furthermore, $a_2 \notin Q$ since otherwise (a_2, a, c, b, b_1) is a fence in Q of order 5.

Let $B = \{x \in J(L) \mid x < b_1 \text{ and } x \nleq c\}$. If there exists $x \in B$ such that $x \nleq b$ then $x \| b$. Take a minimal such x. If $x \notin Q$ we get a crown (a_2, a, c, b, b_1, x); therefore, $x \in Q$, in which case we get a fence (a, c, b, b_1, x) in Q of order 5. Thus, every $x \in B$ satisfies $x \geq b$. Let $b_3, b_4 \in J(L)$ be a maximal join representation of b_1. Obviously, one of b_3, b_4 lies in B, say b_3, so that $b_3 \geq b$. If $b_4 \not\geq b$ then $b_4 \leq c$; thus, $b_4 \leq c \wedge b_1 = b \leq b_3$, which is impossible. Therefore, $b_4 \geq b$ and, in fact, $b_1/b_3b_4 \in \Delta$, and $b_2/b_3b_4 > c/ab$, contradicting the maximality of c/ab. The case $\nabla \neq \phi$ is handled dually. The proof of the lemma is now complete. □

Thus, we are assured of the existence of a maximal fence in Q_{n-1} of order ≥ 5. Let $G = (a_1, a_2, \cdots, a_m)$ be any such fence. Furthermore, we may assume that $a_3 = a_2 \vee a_4$ $(a_2 \wedge a_4)$ if G is upper (lower) and that $a_{m-2} = a_{m-3} \vee a_{m-1}$ $(a_{m-3} \wedge a_{m-1})$ if G is upper and m is odd, or if G is lower and m is even (if G is upper and m is even, or if G is lower and m is odd). There is no loss in generality in assuming that F_{n-1} is upper. It remains now to choose $F_n = (e_n, f_n, g_n, h_n, \cdots)$ in Q_{n-1} and then Q_n as defined by (3) such that (F_n, Q_n) satisfies properties (i), (ii), and (iii). To this end we shall distinguish four cases, and in each case choose $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_1, a_2, \cdots, a_m)$ or $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_m, a_{m-1}, \cdots, a_1)$ so that properties (i) and (iii) hold. Either choice, of course, already satisfies (ii). Furthermore, for either choice, $Q_n \neq \phi$ since $a_1 \in Q_n$ or $a_m \in Q_n$.

Case (a). G is upper and m is odd (see Figure 6). Set $A_1 = \{x \mid x \geq a_2$ and $x \parallel a_3\}$, $A_2 = \{x \mid x \geq a_{m-1}$ and $x \parallel a_{m-2}\}$, $B_1 = \{y \mid y \not\leq a_3$ and $y \leq x$ for some $x \in A_1\}$, and $B_2 = \{y \mid y \not\leq a_{m-2}$ and $y \leq x$ for some $x \in A_2\}$. Either every $x \in A_1$ satisfies $x \not\geq g_{n-1}$ or, every $x \in A_2$ satisfies $x \ngeq g_{n-1}$, (since otherwise, $(g_{n-1}, x_1, a_2, \cdots, a_{m-1}, x_2)$ would be a crown in L, where $x_i \in A_i$ and $x_i \ngeq g_{n-1}$, for $i = 1, 2$). Suppose that every x in A_1 or A_2 satisfies $x \ngeq g_{n-1}$; then, either every $y \in B_1$ satisfies $y \nleq g_{n-1}$ or, every $y \in B_2$ satisfies $y \nleq g_{n-1}$, (since otherwise $(g_{n-1}, y_1, x_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ would be a crown in L, where $y_i \in B_i$, $y_i \leq x_i$ for some $x_i \in A_i$ and $y_i \nleq g_{n-1}$, for $i = 1, 2$). Without loss of generality we may assume that every y in B_1 satisfies $y \nleq g_{n-1}$. But then $A_1 \subseteq Q_{n-1}$ so that by property (iii) for $n-1$, $B_1 \subseteq Q_{n-1}$ and, furthermore, if $y \in B_1$ then $y \geq a_2$ (since otherwise $(y_1, x_1, a_2, \cdots, a_{m-1}, a_m)$, where $y_1 \in B_1$, $y_1 \leq x_1$ for some $x_1 \in A_1$ and $y_1 \ngeq a_2$, would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \cdots, a_m)$). If we now set $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_1, a_2, \cdots, a_m)$ and, therefore, $Q_n = A_1$ we have that the pair (F_n, Q_n) satisfies properties (i), (ii), and (iii).

Therefore, we may, without loss of generality, assume that every $x \in A_2$ satisfies $x \ngeq g_{n-1}$ but that there exists $x_1 \in A_1$ such that $x_1 \geq g_{n-1}$. In this case, we set $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_m, a_{m-1}, \cdots, a_1)$ and therefore, $Q_n = A_2$. Obviously (F_n, Q_n) satisfies (i) and (ii). Furthermore, every $y \in B_2$ satisfies $y \nleq g_{n-1}$ (since otherwise $(x_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ is a crown in L , where $y_2 \in B_2$, $y_2 \leq x_2$ for some $x_2 \in A_2$ and $y_2 \leq g_{n-1}$). To show (iii) we must prove that $B_2 \subseteq A_2$. Again applying (iii) for $n-1$ to $y \in B_2$ we get that $B_2 \subseteq Q_{n-1}$ and if there were $y_2 \in B_2$ such that $y_2 \ngeq a_{m-1}$ and $y_2 \leq x_2$ for some $x_2 \in A_2$, then $(a_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \cdots, a_m)$. Thus, (F_n, Q_n) also satisfies property (iii).

<u>Case (b).</u> G <u>is lower and</u> m <u>is odd.</u> This case is completed by dualizing the argument of case (a).

<u>Case (c).</u> G <u>is upper and</u> m <u>is even</u> (see Figure 7). Define A_1, B_1 as in Case (a) and set $A_2 = \{x \mid x \leq a_{m-1}$ and $x \parallel a_{m-2}\}$ and $B_2 = \{y \mid y \not\leq a_{m-2}$ and $y \geq x$ for some $x \in A_2\}$. (Note that A_2, B_2 here are just the duals of A_2, B_2, respectively, in Case (a).)

Either every $x \in A_1$ satisfies $x \not\leq g_{n-1}$ or, every $x \in A_2$ satisfies $x \not\geq g_{n-1}$, (since otherwise, $(x_1, a_2, a_3, \cdots, a_{m-1}, x_2)$ is a crown in L, where $x_1 \in A_1$, $x_1 \geq g_{n-1}$ and $x_2 \in A_2$, $x_2 \leq g_{n-1}$). Suppose that every x in A_1 satisfies $x \not\leq g_{n-1}$ and every x in A_2 satisfies $x \not\geq g_{n-1}$); then either every $y \in B_1$ satisfies $y \not\geq g_{n-1}$ or, every $y \in B_2$ satisfies $y \not\leq g_{n-1}$ (since otherwise, $(y_1, x_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ is a crown in L, where $y_1 \in B_1$, $y_1 \leq x_1$ for some $x_1 \in A_1$, $y_1 \leq g_{n-1}$, and $y_2 \in B_2$, $y_2 \geq x_2$ for some $x_2 \in A_2$, $y_2 \geq g_{n-2}$). If every y in B_1 satisfies $y \not\geq g_{n-1}$ then the corresponding argument of Case (a) shows that $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_1, a_2, \cdots, a_m)$ and therefore $Q_n = A_1$ satisfy properties (i), (ii), and (iii). If, on the other hand, B_2 satisfies $y \not\leq g_{n-2}$ then we simply dualize the corresponding argument in Case (a) with the fence $(a_1, a_2, \cdots, a_m)$ replaced by the fence $(a_m, a_{m-1}, \cdots, a_1)$.

Suppose now that every $x \in A_1$ satisfies $x \not\leq g_{n-1}$ but that there exists $x_2 \in A_2$ such that $x_2 \leq g_{n-1}$. In this case we set $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_1, a_2, \cdots, a_m)$ and therefore, $Q_n = A_1$. Obviously (F_n, Q_n) satisfies (i) and (ii). Furthermore, every $y \in B_1$ satisfies $y \not\geq g_{n-1}$ (since otherwise $(g_{n-1}, y_1, x_1, a_2, \cdots, a_{m-1}, x_2)$ is a crown in L, where $y_1 \in B_1$, $y_1 \leq x_1$ for some $x_1 \in A_1$, and $y_1 \leq g_{n-1}$).

To show (iii) we must show that $B_1 \subseteq A_1$. Applying (iii) for $n-1$ to $y \in B_1$ we get that $B_1 \subseteq Q_{n-1}$ and if there were $y_1 \in B_1$ such that $y_1 \not\geq a_2$, $y_1 \leq x_1$ for some $x_1 \in A_1$ then $(y_1, x_1, a_2, \cdots, a_m)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \cdots, a_m)$. Thus, also (iii) holds.

If every $x \in A_2$ satisfies $x \not\leq g_{n-1}$ and there exists $x_1 \in A_1$ such that $x_1 \geq g_{n-1}$ we choose $F_n = (e_n, f_n, g_n, h_n, \cdots) = (a_m, a_{m-1}, \cdots, a_1)$ and therefore, $Q_n = A_2$. Clearly, (F_n, Q_n) satisfy (i) and (ii). Again every $y \in B_2$ satisfies $y \not\geq g_{n-1}$ (since otherwise $(g_{n-1}, x_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ is a crown in L, where $y_2 \in B_2$, $y_2 \geq x_2$ for some $x_2 \in A_2$, and $y_2 \geq g_{n-1}$). Thus, $B_2 \subseteq Q_{n-1}$ and if there were $y_2 \in B_2$, $y_2 \geq x_2$ for some $x_2 \in A_2$, and $y_2 \not\leq a_{m-1}$ then $(a_1, a_2, \cdots, a_{m-1}, x_2, y_2)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \cdots, a_m)$. Thus, (F_n, Q_n) also satisfies (iii).

Case (d). G is lower and m is even. This case is completed by replacing the fence G in Case (c) by the fence $(a_m, a_{m-1}, \cdots, a_1)$.

Therefore, we now have a sequence of pairs (F_n, Q_n) for every $n \geq 1$ satisfying properties (i), (ii), and (iii); furthermore, for all $n \geq 1$ $Q_n \neq \phi$, $Q_n \subset Q_{n-1}$ since $f_{n-1} \in Q_{n-1} - Q_n$, and every $x \in Q_n$ is comparable with f_{n-1}.

But then whenever $1 \leq n < m$, $f_n \, \rho_n \, f_m$, where ρ_n is either "<" or ">". Without loss of generality "<" appears an infinite number of times among the ρ_n's which, in turn, gives an infinite increasing chain in L. This is impossible since L has no infinite chains. We conclude that L must contain a doubly irreducible element. $\square$

5. Proof of Theorem 3.7. Let L be a dismantlable lattice which is not a chain and let us assume again that L contains no infinite chains and no infinite fences.

We show first that for some $x \in \mathrm{Irr}(L)$ there exists $y \in L$ such that $x \| y$. Otherwise $\mathrm{Irr}(L)$ is a chain $x_1 < x_2 < \cdots < x_k$ in L and if C is any maximal chain in L then $\mathrm{Irr}(L) \subseteq C$. For each $i = 1, 2, \cdots, k$ take $y_i(z_i)$ to be the unique element covered by (covering) x_i (if $x_1 = 0$ take $y_1 = 0$ and, if $x_k = 1$ take $z_k = 1$). If we set $z_0 = 0$ and $y_{k+1} = 1$ then $L = \mathrm{Irr}(L) \cup \bigcup([z_i, y_{i+1}] \mid 0 \leq i \leq k)$. For some i, $(0 \leq i \leq k)$, $S = [z_i, y_{i+1}]$ is not a chain and $S \cap \mathrm{Irr}(L) = \emptyset$. On the other hand, S is a sublattice of a dismantlable lattice and is therefore dismantlable, i.e., $\mathrm{Irr}(S) \neq \emptyset$. But clearly, $\mathrm{Irr}(S) \subseteq \mathrm{Irr}(L)$ which is a contradiction.

Thus, there exists $x \in \mathrm{Irr}(L)$ and $y \in L$ such that $x \| y$. We now have a fence $G = (y, x \wedge y, x)$ in L and by Lemma 4.1 there exists a left-maximal fence $F_1 = (a_1, a_2, \cdots, a_{m-1}, x)$. We may now define Q_1 as in Section 4. If $Q_1 \cap \mathrm{Irr}(L) = \emptyset$ we can proceed to construct the sequence $((F_n, Q_n))_n$, $(n \geq 2)$, as before which leads to a contradiction. Thus, $Q_1 \cap \mathrm{Irr}(L) \neq \emptyset$. This means there exists $z \in \mathrm{Irr}(L)$ such that $z \| x$. $\square$

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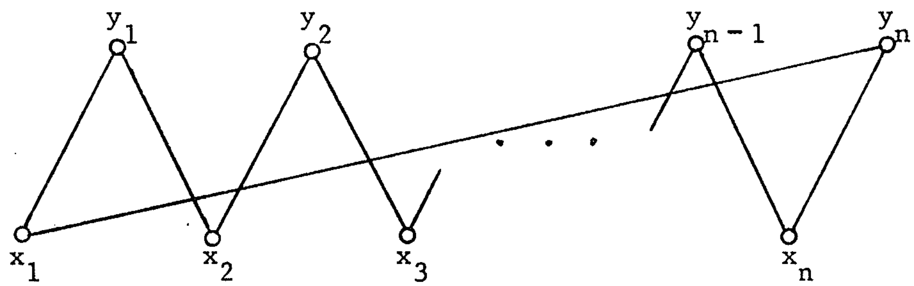


Figure 1. A crown of order 2n

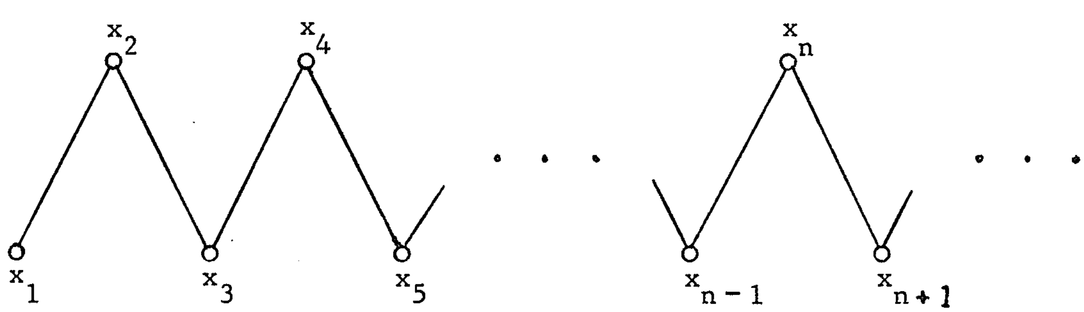


Figure 2. A lower fence $(x_1, x_2, \cdots, x_n, \cdots)$

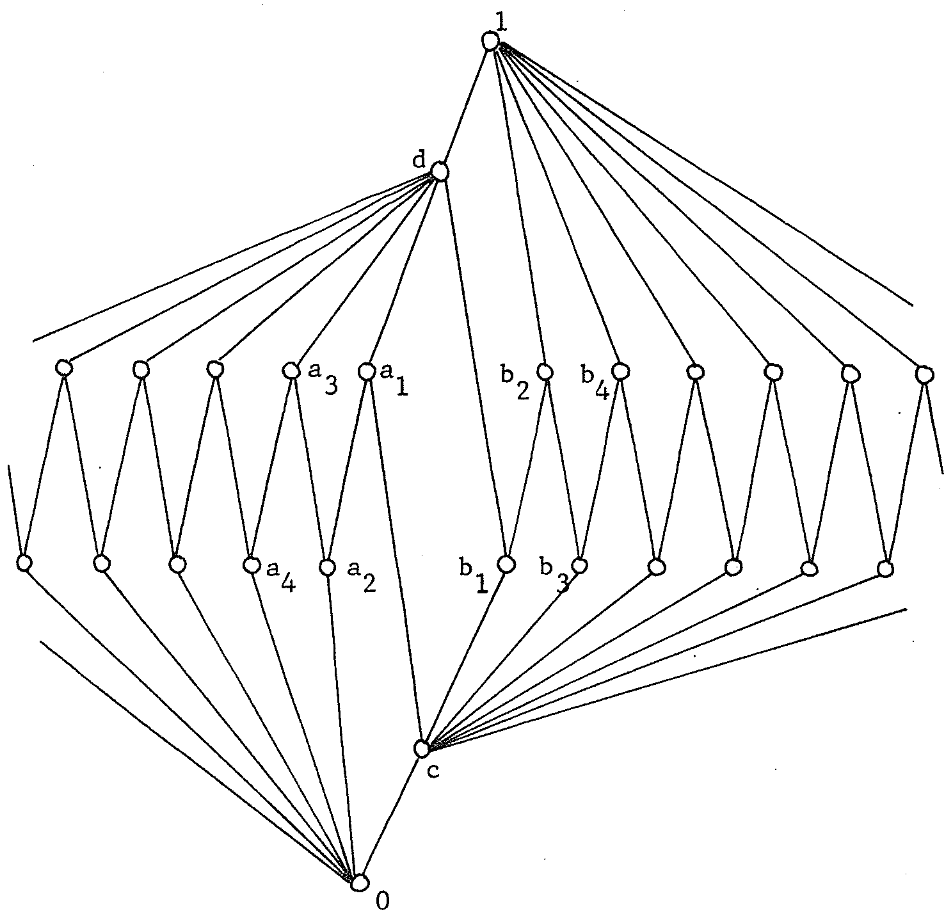


Figure 3. A non-dismantlable lattice with no crowns

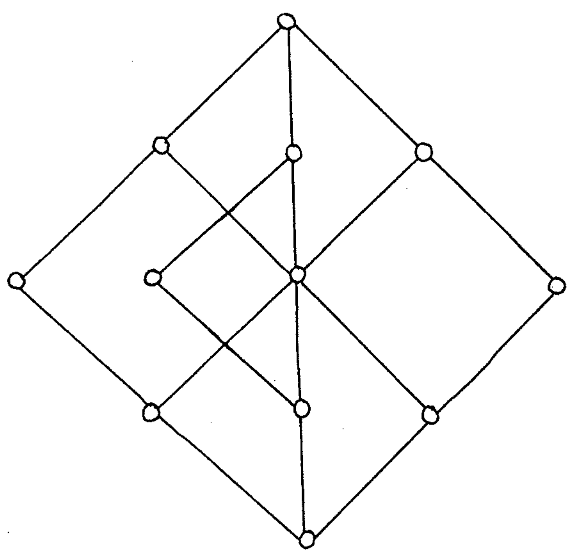


Figure 4. A modular non-planar dismantlable lattice

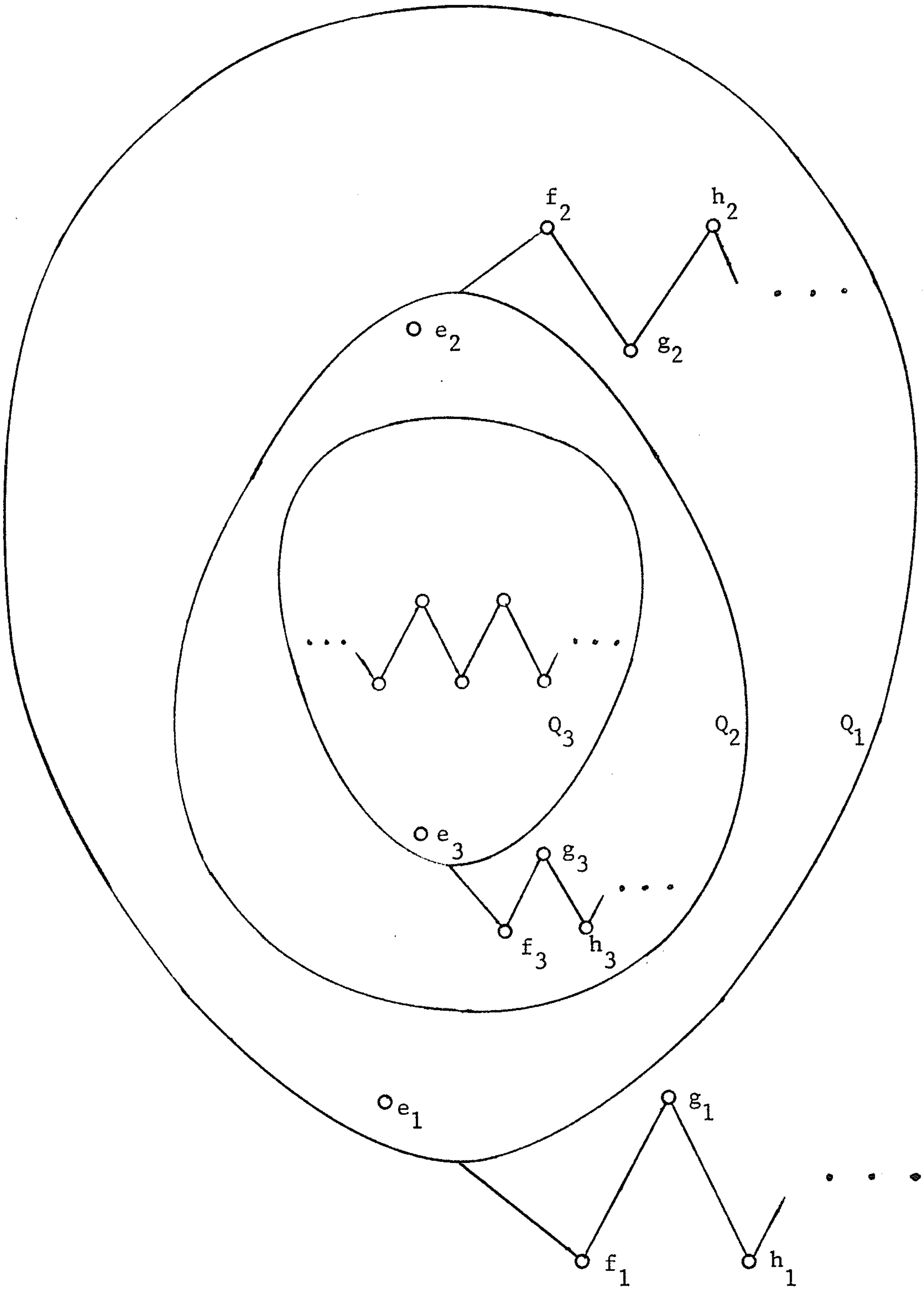


Figure 5. The construction scheme

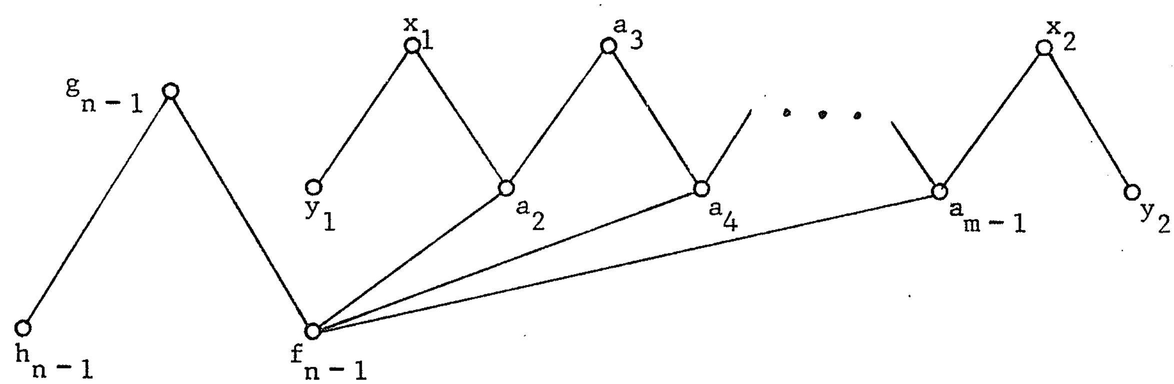


Figure 6. Theorem 3.6. Case (a)

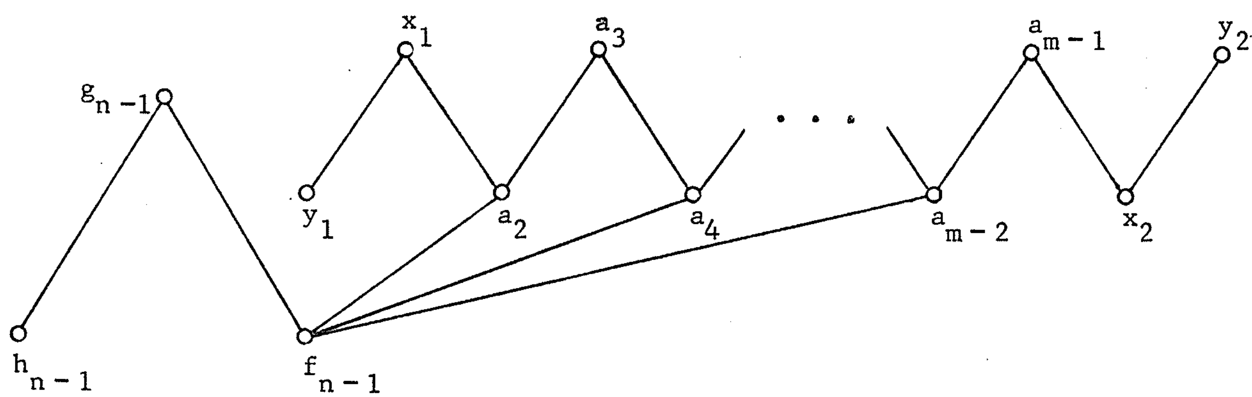


Figure 7. Theorem 3.6. Case (c)

PLANAR LATTICES

by

David Kelly and Ivan Rival

1. Geometry of planar lattices.

2. Dismantlability of planar lattices.

3. Transformations of planar lattice embeddings.

4. Dangles on indecomposable intervals.

5. Proof of Theorem 1.

6. Some results related to Theorem 1.

A finite partially ordered set (poset) P is customarily represented by drawing a small circle for each point, with a lower than b whenever $a < b$ in P, and drawing a straight line segment from a to b whenever a is covered by b in P (see, for example, G. Birkhoff [2, p. 4]). A poset P is planar if such a diagram can be drawn for P in which none of the straight line segments intersect.

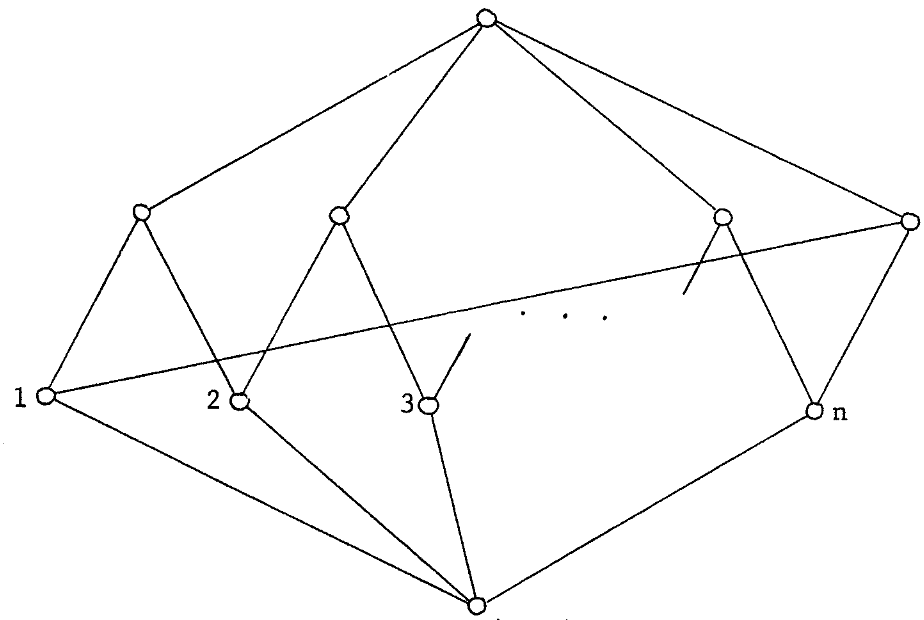
The main result of this paper is the following characterization of planar lattices (answering Problem 9 of G. Grätzer [4, p. 66]). Let $\mathfrak{L} = \{\underset{\sim}{A}_n \mid n \geq 3\} \cup \{\underset{\sim}{B}, \underset{\sim}{B}^d, \underset{\sim}{C}, \underset{\sim}{C}^d, \underset{\sim}{D}, \underset{\sim}{D}^d, \underset{\sim}{E}\} \cup \{\underset{\sim}{F}_n \mid n \geq 2\} \cup \{\underset{\sim}{F}_n^d \mid n \geq 2\} \cup \{\underset{\sim}{G}_n \mid n \geq 2\} \cup \{\underset{\sim}{H}_n \mid n \geq 2\} \cup \{\underset{\sim}{I}_n \mid n \geq 1\}$ be the lattices of Figure 1. (The dual of a poset P is denoted by P^d.)

THEOREM 1. A finite lattice is planar if and only if it does not contain (as a subposet) any lattice in $\mathfrak{L}$. Moreover, $\mathfrak{L}$ is the minimum such list; that is, if $\mathfrak{F}$ is a set of lattices such that the first assertion remains true with $\mathfrak{L}$ replaced by $\mathfrak{F}$, then $\mathfrak{L} \subseteq \mathfrak{F}$.

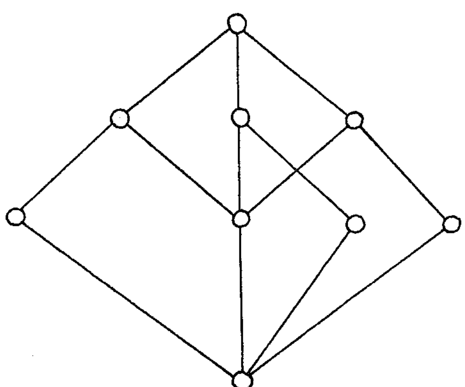
Theorem 1 is analogous in its statement to K. Kuratowski's characterization of planar graphs [7]; however, the corresponding proofs bear little resemblance to each other.

The basic concepts for planar lattices are developed in the first section. Section 2 recalls that planar lattices are dismantlable. Section 3 describes a procedure for obtaining all planar embeddings of a planar finite lattice from one fixed planar embedding. The purpose of Section 4 is to prove a technical lemma that guarantees the existence of particular subposets at various points in the proof of Theorem 1.

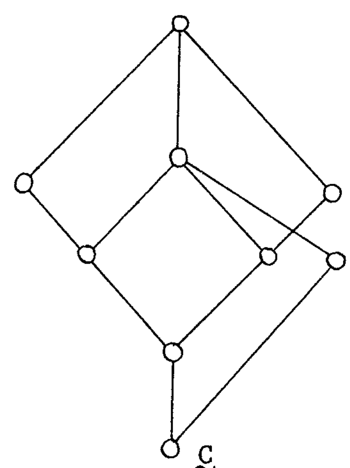
Figure 1.



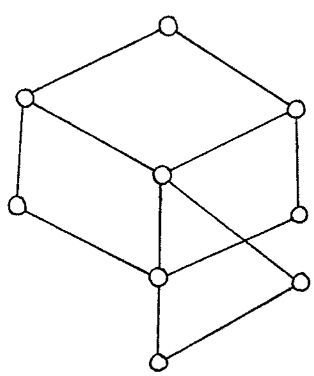
$\underset{\sim}{A}_n$ $(n \geq 3)$



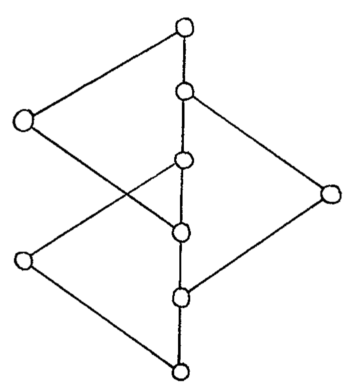
$\underset{\sim}{B}$



$\underset{\sim}{C}$

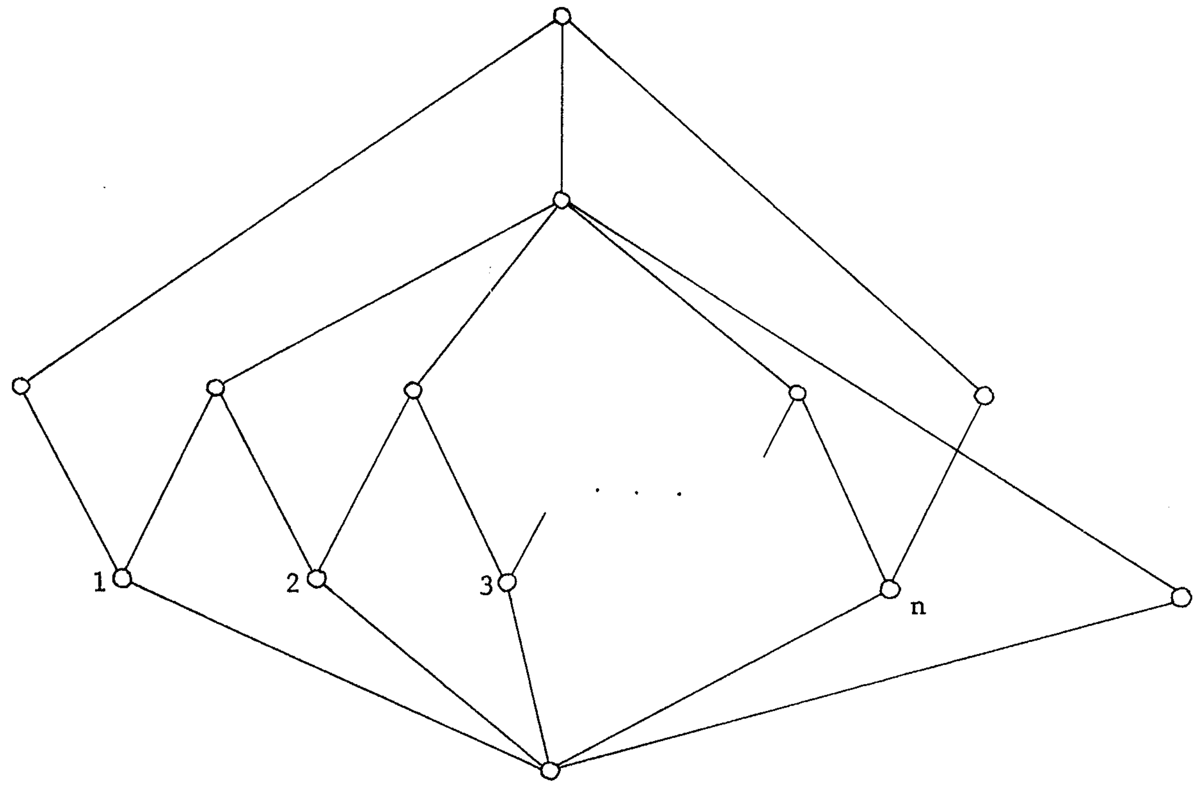


$\underset{\sim}{D}$

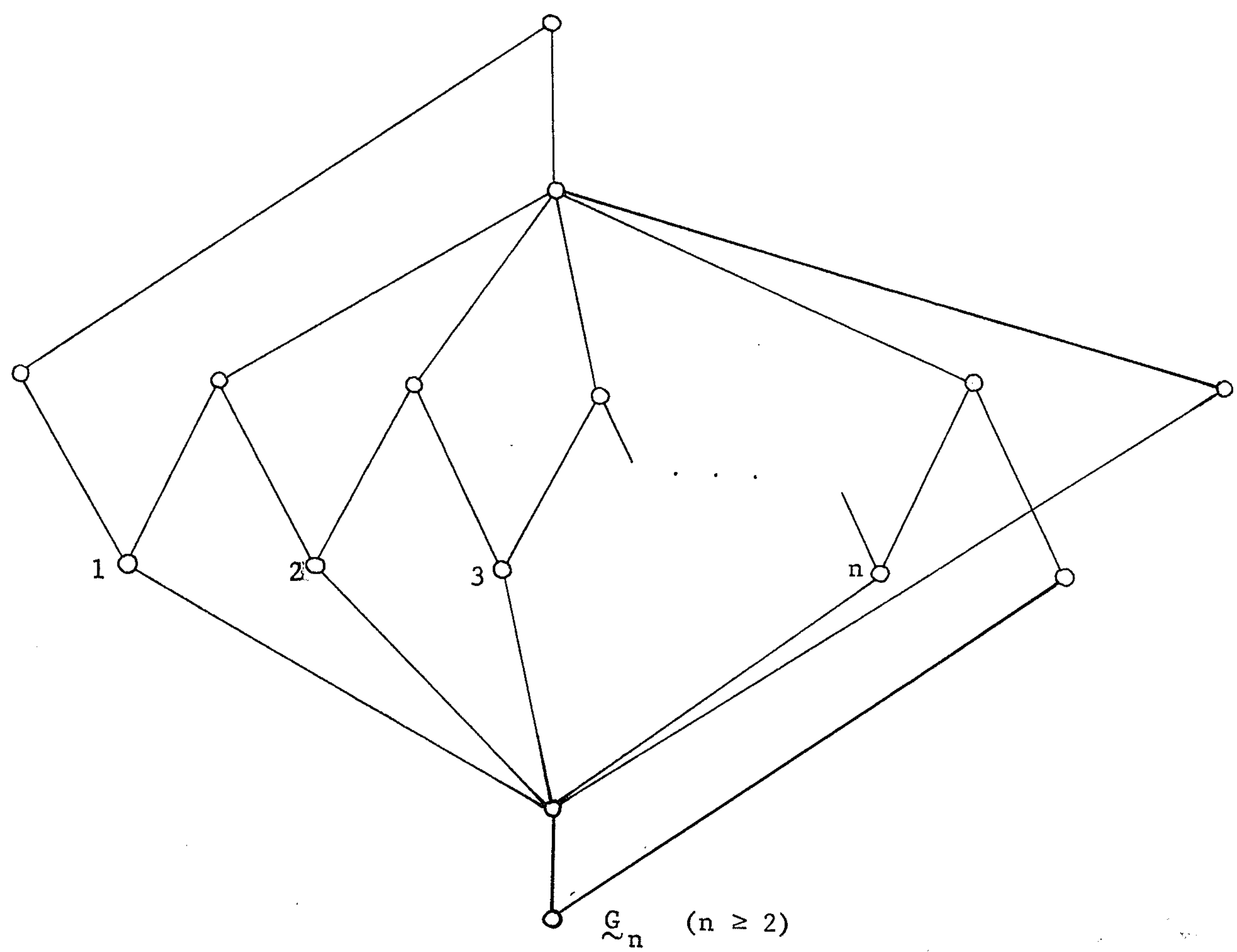


$\underset{\sim}{E}$

Figure 1. (cont'd.)

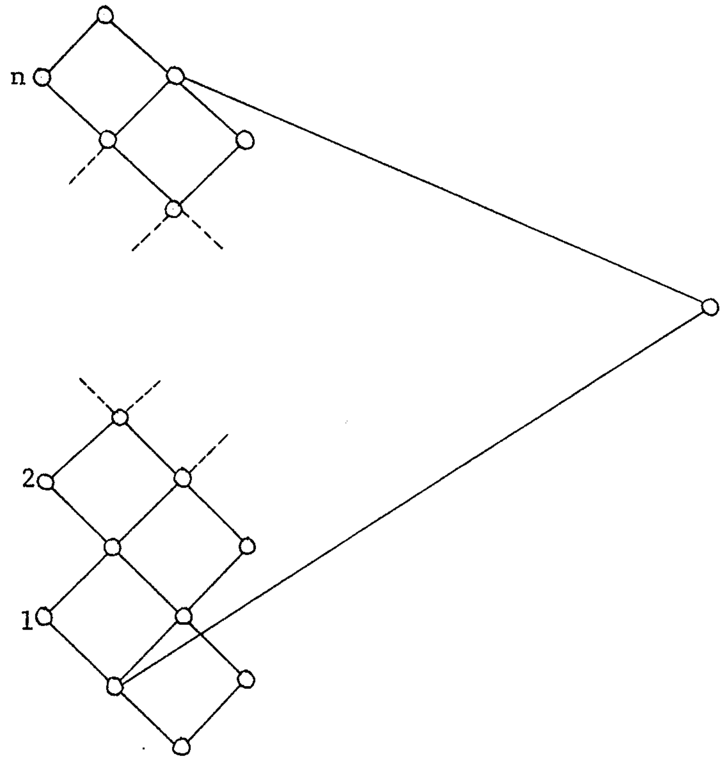


$\underset{\sim}{F}_n$ $(n \geq 2)$

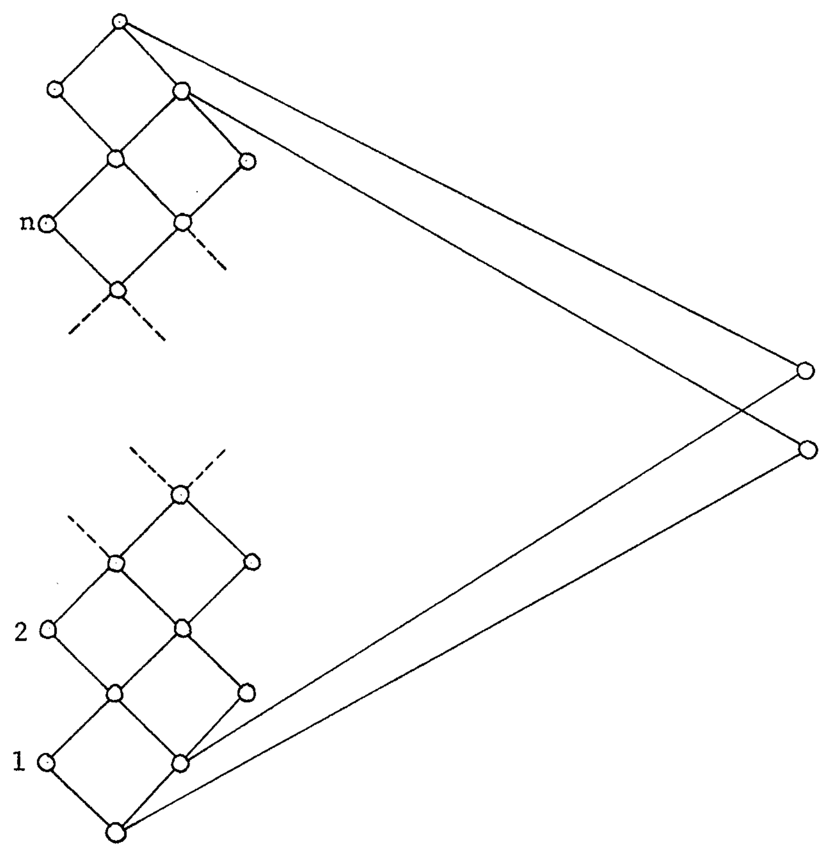


$\underset{\sim}{G}_n$ $(n \geq 2)$

Figure 1. (cont'd.)



$\underset{\sim}{H}_n$ (n ≥ 2)



$\underset{\sim}{I}_n$ (n ≥ 1)

Section 5 consists of the proof of Theorem 1, where essential use is made of our characterization of dismantlable finite lattices [6]. In the final section, Theorem 1 is extended to infinite lattices of dimension ≤ 2 ; in addition, we show that in a dismantlable nonplanar finite lattice, there are at least three doubly irreducible elements which are pairwise incomparable.

1. Geometry of planar lattices.

Let P be a finite partially ordered set (poset). The relation a is covered by b (or b covers a) is denoted by $a \prec b$. π_1 and π_2 are the first and second projections of $\mathbb{R}^2$ onto $\mathbb{R}$. A planar embedding $e(P)$ of P consists of

(P1) an injection $a \mapsto \bar{a}$ from P to $\mathbb{R}^2$ such that $\pi_2(\bar{a}) < \pi_2(\bar{b})$ whenever $a < b$, and

(P2) straight line segments $\overline{ab}$, connecting $\bar{a}$ and $\bar{b}$ whenever $a \prec b$ in P; these segments do not intersect, except possibly at their endpoints.

P is planar if it has a planar embedding.

We shall soon see the need to consider representations of P which allow more general connecting paths than the straight line segments of (P2). A planar representation $e(P)$ of P consists of (P1) and

(P2′) paths $y \mapsto \langle f(y), y \rangle$, denoted by $\overline{ab}$, with endpoints $\bar{a}$ and $\bar{b}$ whenever $a \prec b$ in P, where $f\colon [\pi_2(\bar{a}), \pi_2(\bar{b})] \to \mathbb{R}$ is a continuous function; these paths do not intersect except possibly at their endpoints.

For $a, b \in P$ such that $a \prec b$, a is a lower cover of b (or b is an upper cover of a). An element of P is doubly irreducible in P if it has at most one lower and at most one upper cover in P. A planar representation $e(P)$ of P induces a planar representation of $P - \{c\}$, where c is a doubly irreducible element in P, and $a \prec c \prec b$, by defining $\overline{ab}$ to be $\overline{ac} \cup \overline{cb}$ if $a \prec b$ in $P - \{c\}$, and deleting $\overline{ac} \cup \overline{cb}$ if $a \not\prec b$ in $P - \{c\}$; this induced planar representation of $P - \{c\}$ is denoted by $e(P - \{c\})$. The fact that this induced planar representation $e(P - \{c\})$ need not be a planar embedding of $P - \{c\}$, even if $e(P)$ were a planar embedding, shows the need to consider planar representations.

Actually, it is shown in D. Kelly [5] that the existence of a planar representation of P is equivalent to the planarity of P; Theorem 2.5 will prove this equivalence in case P is a lattice.

For each $a \in P$, a planar representation $e(P)$ of P induces a strict linear ordering on the set $U(a)$ of upper covers of a defined by: for $x, y \in U(a)$, x is to the left of y if and only if $\overline{ax}(m) < \overline{ay}(m)$ where $m = \min\{\pi_2(\bar{x}) \mid x \in U(a)\}$; the ordering of the lower covers of a point is defined dually. Two planar representations of P are similar if, for each $a \in P$, the upper (lower) covers of a have the same ordering with respect to the two representations.

In the remainder of this section, L is assumed to be a finite lattice with a planarrepresentation $e(L)$. All the geometric concepts to be introduced will depend on the choice of the planar representation $e(L)$, although the dependence will not always be stated explicitly. Most such concepts will, however, be invariant with respect to similar planar representations.

A maximal chain from a to b (with $a \le b$ in L) is a sequence $a = x_0, x_1, \cdots, x_n = b$ of points of L with $x_i \prec x_{i+1}$ $(0 \le i \le n-1)$; if a and b are not mentioned, $a = 0$ and $b = 1$ are understood. The function $\varphi : [\pi_2(\bar{a}), \pi_2(\bar{b})] \mapsto \mathbb{R}$ corresponding to such a maximal chain is defined by: $\varphi(y) = f_i(y)$ where f_i is the function on $[\pi_2(\bar{x}_{i-1}), \pi_2(\bar{x}_i)]$ representing $\overline{x_{i-1}\, x_i}$ and y is in this interval. (If $a = b$, φ is $\{\langle\pi_2(\bar{a}), \pi_1(\bar{a})\rangle\}$.)

The only result that we need from analysis is the following immediate consequence of the intermediate value theorem.

LEMMA 1.1. Let $p, q \in \mathbb{R}$, $p < q$, and let φ_1, φ_2 be continuous functions from $[p, q]$ into $\mathbb{R}$. If $\varphi_1(p) \leq \varphi_2(p)$ and $\varphi_1(q) \geq \varphi_2(q)$, then there is $r \in [p, q]$ such that $\varphi_1(r) = \varphi_2(r)$.

If C and D are two maximal chains between a and b, $(a < b)$, in L such that $C \cap D = \{a, b\}$, and φ, ψ are their corresponding functions then, by Lemma 1.1, the infimum of $\{\varphi, \psi\}$ is φ or ψ since any crossing of φ and ψ would correspond to a common element of C and D; in case $\inf\{\varphi, \psi\} = \varphi$, we call C the <u>infimum</u>, and D the <u>supremum</u> of C and D (of course, not to be confused with the join and meet in L).

In general, for maximal chains C and D between a and b $(a < b)$ in L, there are elements $a = x_0 < x_1 < \cdots < x_n = b$ of L and maximal chains C_i and D_i from x_{i-1} to x_i, $(1 \leq i \leq n)$, such that $C = \bigcup_{i=1}^n C_i$, $D = \bigcup_{i=1}^n D_i$ and $C_i \cap D_i = \{x_{i-1}, x_i\}$. Let φ, ψ and φ_i, ψ_i be the functions corresponding to C, D and C_i, D_i, respectively, $1 \leq i \leq n$. Clearly, $\varphi = \bigcup_{i=1}^n \varphi_i$, $\psi = \bigcup_{i=1}^n \psi_i$ and $\inf\{\varphi, \psi\} = \bigcup_{i=1}^n \inf\{\varphi_i, \psi_i\}$; that is, $\inf\{\varphi, \psi\}$ is the function corresponding to the maximal chain $\bigcup_{i=1}^n \inf\{C_i, D_i\}$. The <u>region</u> R defined by C and D is the subposet of L consisting of all elements of L in the area of the plane bounded by φ and ψ; that is, $x \in R$ if and only if $\pi_2(\bar{a}) \leq \pi_2(\bar{x}) \leq \pi_2(\bar{b})$ and $\langle \pi_2(\bar{x}), \pi_1(\bar{x}) \rangle$ lies between $\inf\{\varphi, \psi\}$ and $\sup\{\varphi, \psi\}$. The <u>left</u> (<u>right</u>) <u>boundary</u> of R is $\inf\{C, D\}$ ($\sup\{C, D\}$); the <u>boundary</u> of R is $C \cup D$, and the <u>interior</u> of R is $R - (C \cup D)$.

Correspondingly, the <u>left</u> (<u>right</u>) <u>boundary</u> of L is the infimum (supremum) of all maximal chains in L. The <u>left</u> (<u>right</u>) <u>side</u> of a maximal chain C in L is the region defined by the left (right) boundary of L and C. An element x on the left (right) side of C is also said to be <u>on the left</u> (<u>right</u>) <u>of</u> C. Equivalently, x is on the left of C whenever $\pi_1(\bar{x}) \leq \varphi(\pi_2(\bar{x}))$, where φ is the function corresponding to the maximal chain C of L. Obviously, every element of L is either on the left of or right of C, and it is on both sides precisely when it is an element of C.

LEMMA 1.2. Let $x \leq y$ in L. If x and y are on different sides of a maximal chain C in L, then there is $z \in C$ such that $x \leq z \leq y$.

<u>Proof.</u> Suppose x and y are on the left and right, respectively, of C. Let φ_1 be the function corresponding to a maximal chain between x and y, and let φ_2 be the restriction to $[\pi_2(\bar{x}), \pi_2(\bar{y})]$ of the function corresponding to C, and apply Lemma 1.1. $\square$

LEMMA 1.3. If, in a region R, a and b are the least and greatest elements of the boundary of R, then a and b are the least and greatest elements of R; that is, $R \subseteq [a, b]$.

<u>Proof.</u> Let φ_1 (φ_2) be the function corresponding to the left (right) boundary of R. For an element $x \in R$, let ψ be the function corresponding to a maximal chain between x and 1. Without loss of generality, we may assume that $\varphi_2(\pi_2(\bar{b})) \leq \psi(\pi_2(\bar{b}))$. Since $\psi(\pi_2(\bar{x})) \leq \varphi_2(\pi_2(\bar{x}))$, it follows from Lemma 1.1 that $x \leq b$. $\square$

For a region R and elements $a, b \in R$ as in Lemma 1.3 we call a and b the <u>bounds</u> of R. A sublattice S of L is <u>cover-preserving</u> if $a \prec b$ in S implies $a \prec b$ in L.

PROPOSITION 1.4. A region of L is a cover-preserving sublattice of L.

<u>Proof.</u> Let R be a region of L and suppose that $x, y \in R$ but $x \vee y \notin R$. We may assume that $x \vee y$ is on the right of a maximal chain in L containing the right boundary of R. By Lemma 1.2, there are $x', y' \in C$ such that $x \leq x' < x \vee y$ and $y \leq y' < x \vee y$. Without loss of generality, $x' \leq y'$, and therefore $x, y \leq y' < x \vee y$, a contradiction. Thus, R is a sublattice of L. Furthermore, if $x, y \in R$, $x < y$, $x < z < y$ for some $z \in L$, but $(x, y) \cap R = \emptyset$ then again by Lemma 1.2 there are z_1, z_2 on the right boundary of R, say, such that $x \leq z_1 < z < z_2 \leq y$. Then $x = z_1$ and $y = z_2$ which is impossible since the right boundary of a region is a maximal chain in L. $\square$

It is evident from Proposition 1.4 that, for a region R, the association of each a in R to $\bar{a}$ in $e(L)$, and of each cover $a \prec b$ in R to $\overline{ab}$ in $e(L)$ determines a planar representation of R; this induced representation of R is denoted by $e(R)$.

LEMMA 1.5. A closed interval of L is a region of L.

<u>Proof.</u> Let $a < b$ in L, let $C(D)$ be the infimum (supremum) of all maximal chains from a to b, and let R be the region defined by C and D. Clearly, $[a, b] \subseteq R$, and by Lemma 1.3, $R \subseteq [a, b]$. $\square$

For $x, y \in L$, x is <u>incomparable</u> with y in L $(x \parallel y)$ whenever $x \nleq y$ and $x \ngeq y$. We define the relation λ on L (with respect to $e(L)$) by: $x \lambda y$ if and only if $x \parallel y$ and there are lower covers x' and y' of $x \vee y$ such that $x \leq x'$, $y \leq y'$, and x' is to the left of y' (with respect to $e(L)$).

PROPOSITION 1.6. If $x \lambda y$, then x is on the left of any maximal chain through y. If $x \parallel y$ and x is on the left of some maximal chain through y, then $x \lambda y$.

Proof. First, let us observe that if $x \parallel y$ and x is on the left of a maximal chain C through y, then x is on the left of every maximal chain through y. Indeed, if x were on the right of some maximal chain D through y then x would be in the region defined by $C \cap [y, 1]$ and $D \cap [y, 1]$ or in the region defined by $C \cap [0, y]$ and $D \cap [0, y]$. But then, in view of Lemma 1.3, x and y are comparable.

If $x \lambda y$ then there are lower covers x', y' of $x \vee y$ such that $x \leq x'$, $y \leq y'$ and x' is to the left of y'. Let φ_1 and φ_2 be the functions corresponding to $x' \prec x \vee y$ and C, where C is a maximal chain through $y \leq y' < x \vee y$. Since $\varphi_1(\pi_2(\overline{x \vee y})) = \varphi_2(\pi_2(\overline{x \vee y}))$ and $\varphi_1(m) < \varphi_2(m)$, where $m = \max\{\pi_2(\overline{w}) \mid w \prec z\}$, it follows from Lemma 1.1 that $\pi_1(\overline{x}) = \varphi_1(\pi_2(\overline{x})) < \varphi_2(\pi_2(\overline{x}))$ so that x' is on the left of C. By Lemma 1.3, x is also on the left of C.

Finally, let $x \parallel y$, x', y' be lower covers of $x \vee y$ with $x \leq x'$, $y \leq y'$, and let C be a maximal chain through $y \leq y' < x \vee y$. If x is on the left of some maximal chain through y, then x is on the left of C. Therefore, x' is on the left of C and x' is to the left of y'. $\square$

Clearly, two planar representations of L are similar if and only if they induce the same λ .

It follows from Proposition 1.6 that we get the same relation λ if we define $x \lambda y$ dually in terms of upper covers of $x \wedge y$. In particular, two planar representations of a lattice are similar if and only if the induced orderings on the sets of upper covers are identical.

The next result is due to J. Zilber [2, p. 32, ex. 7(c)].

PROPOSITION 1.7. λ is a strict partial order on L . Moreover, if $x \parallel y$, then $x \lambda y$ or $y \lambda x$.

Proof. The preceeding proposition shows that $x \parallel y$ implies that exactly one of $x \lambda y$ or $y \lambda x$ holds. It only remains to establish the transitivity of λ . Let $x \lambda y$ and $y \lambda z$, and let D be a maximal chain through z ; then y is on the left of D . In view of Proposition 1.4, there is a maximal chain C through y on the left side of D . Therefore, x is on the left of C , and in particular, on the left of D . $\square$

Thus, if x and y are comparable, $x \lambda z$, and $y \parallel z$, then $y \lambda z$.

Since λ is an order relation extending the ordering of the set of upper (or lower) covers of any element of L , it is reasonable to read $x \lambda y$ as x is "to the left of" y .

A connection between elements c and d in a partially ordered set P is a sequence $c = x_0, x_1, \cdots, x_n = d$ of elements of P such that $x_{i-1} \prec x_i$ or $x_i \prec x_{i-1}$ for every $i = 1, 2, \cdots, n$. A fence is a partially ordered set $\{x_1, x_2, \cdots, x_k\}$ in which the comparabilities that hold are precisely $x_1 < x_2, x_2 > x_3, \cdots, x_{2i-1} < x_{2i}, x_{2i} > x_{2i+1}, \cdots$ or $x_1 > x_2, x_2 < x_3, \cdots, x_{2i-1} > x_{2i}, x_{2i} < x_{2i+1}, \cdots$. We usually denote a fence F by $(x_1, x_2, \cdots, x_k)$ and use the terms down and up to indicate which comparability holds between x_1 and x_2 (x_{k-1} and x_k); for example, F is down-up if $x_1 < x_2 > x_3 \cdots > x_{k-1} < x_k$. It is easy to verify that any connection between c and d contains (as a subposet of P) a fence $(x_1, x_2, \cdots, x_k)$ with $x_1 = c$ and $x_k = d$.

Let $a < b$ in L . An $\langle a, b\rangle$-component of L is a connected component of the undirected graph corresponding to the covering relation in (a, b); that is, an $\langle a, b\rangle$-component is a maximal subset of (a, b) in which, between every pair of elements, there is a connection in (a, b). In particular, if x and y are in different $\langle a, b\rangle$-components, then $x \parallel y$. It is also obvious that an $\langle a, b\rangle$-component is a convex subset of L . An $\langle a, b\rangle$-component is proper if whenever $y \leq x$ ($y \geq x$) for some $x \in C$ and $y \in L - C$, then $y \leq a$ ($y \geq b$) .

Let C be an $\langle a, b\rangle$-component of L . Although $C \cup \{a, b\}$ is obviously not an $\langle a, b\rangle$-component, we will, for brevity, call it a bounded $\langle a, b\rangle$-component.

LEMMA 1.8. A bounded component of L is a region of L .

Proof. Let $a < b$ in L and let C be a bounded $\langle a, b\rangle$-component of L . Furthermore, let $D(E)$ be the infimum (supremum) of all maximal chains from a to b contained in C, and let R be the region defined by D and E . Clearly, $C \subseteq R$. If $R \nsubseteq C$, let x be a minimal element of $R - C$ and let

y be a lower cover of x in R. If $y > a$, then $y \in C$ so that $x \in C$. Otherwise, $a \prec x$. Let d and e be upper covers of a in D and E respectively, let F be a maximal chain in R from a to b through x, and let $d = z_0, z_1, \cdots, z_n = e$ be a connection between d and e in (a, b). Let k be the least index i such that z_i is not in the region defined by D and F. Since $z_k \in C$ and $C \subseteq R$, z_k is not on the left side of a maximal chain extending F; thus, by Lemma 1.2, $z_{k-1} \in F$. Then $d = z_0, z_1, \cdots, z_{k-1}, y_1, y_2, \cdots, y_n = x$ is a connection between d and x, where $z_{k-1} \succ y_1 \succ y_2 \succ \cdots \succ y_n = x$ $(n \geq 0)$; therefore $x \in C$, a contradiction. $\square$

Since the intersection of two distinct bounded $\langle a, b\rangle$-components is $\{a, b\}$, the ordering of the functions corresponding to the left boundaries of the bounded $\langle a, b\rangle$-components induces a strict linear ordering $C_1 \lambda C_2 \lambda \cdots \lambda C_n$ on the $\langle a, b\rangle$-components. This ordering can be defined by: $C_i \lambda C_j$ if and only if, for any $x \in C_i$ and $y \in C_j$, $x \lambda y$. The left boundary of $[a, b]$ is clearly the left boundary of $C_1 \cup \{a, b\}$.

Let R be a region with bounds $a < b$. A <u>left up-dangle</u> (<u>down-dangle</u>) <u>on</u> R is an element z such that $z \lambda b$ $(z \lambda a)$ and $z > x$ $(z < x)$ for some $x \in R - \{a, b\}$. <u>Right dangles</u> are defined analogously. The <u>attachment point</u> of an up-dangle (down-dangle) z is the greatest (least) element of R less (greater) than z; clearly, the attachment point is distinct from a and b.

PROPOSITION 1.9. The attachment point of a left dangle on a region R always exists and is on the left boundary of R.

Proof. Let z be a left up-dangle on R, let w be the greatest element on the left boundary C of R that is less than z, and let C_1 be a maximal chain extending C. Let $x \in R$ be such that $z > x$. By Proposition 1.6, z is on the left of C_1; since x in on the right of C_1, there is $y \in C$ such that $x \leq y < z$ (by Lemma 1.2). Then $x \leq w$, and therefore, w is the attachment point of z. □

COROLLARY 1.10. Let the $\langle a, b \rangle$-components of L be $C_1 \lambda\ C_2 \lambda \cdots \lambda\ C_n$, where $a < b$ but $a \not\prec b$ in L, and let C_i' $(1 \leq i \leq n)$ be the corresponding bounded components. The only bounded component that can have a left (right) dangle z is C_1' (C_n'); the corresponding attachment point is $z \wedge b$ $(z \vee a)$ if z is an up-dangle (down-dangle). In particular, all components C_i for $i \neq 1$ or n are proper.

Proof. A left up-dangle z on any bounded $\langle a, b \rangle$-component is a left up-dangle on $[a, b]$, and therefore, has attachment point w on the left boundary of $[a, b]$, which is the left boundary of C_1'; clearly, $w = z \wedge b$. For $i \neq 1$ and any $x \in C_i$, $x \parallel w$ so that z cannot be a dangle on C_i. □

A face is a region of L whose interior is empty and contains no paths of $e(L)$, and whose bounds are the only elements common to both its left and right boundary. It is easy to verify that any two incomparable elements x and y in a face uniquely determine the face. In fact, if $x \lambda y$, the left (right) boundary of any face containing x and y must be the supremum (infimum) of all maximal chains between $x \wedge y$ and $x \vee y$ that pass through x (y). If the $\langle a, b \rangle$-components of L are $C_1 \lambda\ C_2 \lambda \cdots \lambda\ C_n$, then the region defined by the right boundary of $C_i \cup \{a, b\}$ and the left boundary of $C_{i+1} \cup \{a, b\}$ is a face (for $1 \leq i < n$) . Indeed, if there were an element in the interior of this region it would be in (a, b) and therefore, in some $\langle a, b \rangle$-component; if there were only a path in the interior of this region, then elements from different $\langle a, b \rangle$-components would be comparable.

LEMMA 1.11. If $x, z \in [a, b] \subseteq L$ with $x \lambda z$ then there is $y \in [a, b]$ such that x and y are in a common face with $x \lambda y$.

Proof. Since x is not on the right boundary of $[a, b]$, there is a first maximal chain which a horizontal ray from $\bar{x}$ to the right intersects, and an element y on this maximal chain such that $x \lambda y$. Let $C(D)$ be the supremum (infimum) of all maximal chains from $x \wedge y$ to $x \vee y$ that pass through x (y) . The region defined by C and D is a face containing both x and y . $\square$

For $a < b$ in L , b is visible from a (with respect to $e(L)$) if and only if there is a continuous function $\varphi : [\pi_2(\bar{a}), \pi_2(\bar{b})] \to \mathbb{R}$ such that the path $y \mapsto \langle \varphi(y), y \rangle$, $y \in [\pi_2(\bar{a}), \pi_2(\bar{b})]$) , intersects $e(L)$ only at $\bar{a}$ and $\bar{b}$; φ is called a visibility function (for a and b) . The next result will play an important role in the proof of Theorem 1.

THEOREM 1.12. For $a < b$ in L , b is not visible from a if and only if $a \not\prec b$, there is exactly one $\langle a, b \rangle$-component, and $[a, b]$ has both a left and a right dangle.

Proof. Let $p = \pi_2(\bar{a})$ and $q = \pi_2(\bar{b})$, and let us suppose that $[a, b]$ has no right dangle. Let φ_1 be the function corresponding to the right boundary C of $[a, b]$. Define $\psi : [p, q] \to \mathbb{R}$ by $\psi(x) = \varphi_1(x) + (x - p)(q - x)$, and let $\varphi_2 : [p, q] \to \mathbb{R}$ be defined by $\varphi_2(x) = \min\{\psi(x), f(x)\}$, where f is the restriction to $[p, q]$ of the function corresponding to the first maximal chain to the right of C ($f(x) = \infty$ if both a and b are on the right boundary of L). A visibility function is $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$.

If $a \not\prec b$ and there are $\langle a, b\rangle$-components $C_1 \lambda C_2 \lambda \cdots \lambda C_n$ with $n \geq 2$, let φ_1 be the function corresponding to the right boundary of $C_1 \cup \{a, b\}$ and let φ_2 be the function corresponding to the left boundary of $C_2 \cup \{a, b\}$. A visibility function is $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$.

Let us now suppose that $a \not\prec b$ and there is exactly one $\langle a, b\rangle$-component such that $[a, b]$ has both a left and a right dangle. Let $p = \pi_2(\bar{a})$, $q = \pi_2(\bar{b})$, and let φ_1 and φ_2 be the functions corresponding to the left and right boundaries C, D, respectively, of $[a, b]$. Let us, furthermore, assume that there is a visibility function ψ for a and b.

Let z be a left up-dangle on $[a, b]$ with attachment point w, and let δ be the function corresponding to a maximal chain between w and 1 through z. If $\psi(x) < \varphi_1(x)$ for all $x \in (p, q)$, then $\psi(r) < \delta(r)$, where $r = \pi_2(\bar{w})$, and $\delta(q) < \psi(q)$ imply that ψ and δ cross between r and q, which is impossible.

Thus, we may assume that $\varphi_1(x) < \psi(x) < \varphi_2(x)$ for all $x \in (p, q)$. We now proceed as in the proof of Lemma 1.8. Let c and d be upper covers of a in C and D, respectively, and let $c = z_0, z_1, \cdots, z_n = d$ be a connection between c and d in (a, b). Let k be the least index i such that $\langle \pi_2(\overline{z_i}), \pi_1(\overline{z_i}) \rangle$ is not between the functions φ_1 and ψ; then, $\langle \pi_2(\overline{z_{k-1}}), \pi_1(\overline{z_{k-1}}) \rangle$ is between φ_1 and ψ. Let $r = \pi_2(\overline{z_{k-1}})$ and $s = \pi_2(\overline{z_k})$. We will consider only the case that $r < s$; let δ be the function corresponding to $z_{k-1} \prec z_k$. Since $\delta(r) \leq \psi(r)$ and $\delta(s) > \psi(s)$, δ crosses ψ, a contradiction. $\square$

2. Dismantlability of planar lattices.

Let P be a finite partially ordered set. P is <u>dismantlable</u> if $P = \{x_1, x_2, \cdots, x_n\}$ where x_i is doubly irreducible in $\{x_1, x_2, \cdots, x_i\}$ for $1 \le i \le n$. The notion of dismantlability was first introduced for finite lattices in I. Rival [8].

PROPOSITION 2.1. A dismantlable finite bounded poset is a lattice.

<u>Proof.</u> Let c be a doubly irreducible element of a dismantlable finite bounded poset P; then $P - \{c\}$ is a dismantlable bounded poset, and therefore a lattice by induction on $|P|$. It is then immediate that P is a lattice. $\square$

We note that the concepts of "left (right) side of a maximal chain" and "left (right) boundary of a region" extend in the natural way to planar finite bounded posets, and that Lemmas 1.2 and 1.3 remain valid in this context.

The first statement of the next result is due to K.A. Baker, P.C. Fishburn, and F.S. Roberts [1].

PROPOSITION 2.2. A planar finite bounded poset P with $|P| \ge 3$ contains a doubly irreducible element $c \ne 0, 1$ on the left boundary. Moreover, $P - \{c\}$ is planar.

COROLLARY 2.3. A planar finite bounded poset is dismantlable. $\square$

COROLLARY 2.4. A planar finite bounded poset is a lattice. $\square$

Corollary 2.4 appears as an exercise in G. Birkhoff [2, p. 32, ex. 7(a)]. Proposition 2.2 is an immediate consequence of the following theorem which establishes the equivalence (for lattices) between planarity and the existence of a planar representation.

THEOREM 2.5. Every finite bounded poset P with a planar representation $e_1(P)$ has a planar embedding $e_2(P)$ which is similar to $e_1(P)$. Furthermore, if $|P| \geq 3$, then P contains a doubly irreducible element distinct from 0 and 1 on its left boundary.

Proof. We may obviously assume that $|P| \geq 3$. Let B be the left boundary of P, and let c be the maximum element of $B - \{1\}$ which has a unique lower cover a in P. Let $c \prec b$ in B and suppose that c also has an upper cover b_1 distinct from b. In view of the choice of c, b has a lower cover c_1 distinct from c. Now, let $B_0 = B \cap [0, c]$, $B_1 = B \cap [c, 1]$, and C be a maximal chain from c to 1 through b_1. Since $c_1 \parallel c$ we have, by Lemma 1.3, that c_1 is not in the region defined by B_1 and C; thus, c_1 is on the right of $B_0 \cup C$. But b is on the left of $B_0 \cup C$; therefore, by Lemma 1.2, there exists $x \in B_0 \cup C$ such that $c_1 < x < b$ which contradicts $c_1 \prec b$. Thus, c is a doubly irreducible element in P. By induction on $|P|$, there is a planar embedding $e_3(P - \{c\})$ similar to $e_1(P - \{c\})$. If $a \prec b$ in $P - \{c\}$, choose $\bar{c}$ as the midpoint of the line segment $\overline{ab}$ to obtain $e_2(P)$. Otherwise, since a and b are on the left boundary of $e_3(P - \{c\})$, we can adjoin $\bar{c}$ to $e_3(P - \{c\})$ to form a planar embedding $e_2(P)$ by taking $\pi_2(\bar{c}) = \frac{1}{2}\pi_2(\bar{a}) + \frac{1}{2}\pi_2(\bar{b})$, and $\pi_1(\bar{c})$ a sufficiently small real number. $\square$

PROPOSITION 2.6. If d is not on the left boundary B of a planar finite lattice L, there is a doubly irreducible element $c \in B$ which is incomparable with d.

<u>Proof.</u> Let u (v) be the greatest (least) element of B that is $< d$ $(> d)$, let $C = B \cap [u, v]$, and let D be a maximal chain from u to v through d. By Proposition 2.2, there is $c \in C - \{u, v\}$ which is doubly irreducible in the region R defined by C and D; clearly $c \parallel d$. An application of Lemma 1.2 shows that, if $x \in L - R$ and $x > c$, then there is $y \in R$ such that $x > y > c$; therefore, c is doubly irreducible in L. $\square$

3. Transformations of planar lattice embeddings.

The points of a planar embedding of a finite poset can be moved "slightly" without destroying the planarity of the embedding.

LEMMA 3.1. If $e(P)$ is a planar embedding of a finite poset P, there is $\epsilon > 0$ such that if each point $\bar{a}$ of $e(P)$ is replaced by a point $\hat{a}$ with $|\hat{a} - \bar{a}| \leq \epsilon$, then joining each $\hat{a}$ and $\hat{b}$ by a straight line segment whenever $a \prec b$ in P also defines a planar embedding.

Proof. It suffices to take an $\epsilon > 0$ less than the ϵ given by the next lemma and less than $\frac{1}{2}(\pi_2(\bar{b}) - \pi_2(\bar{a}))$ whenever $a \prec b$ in P. $\square$

LEMMA 3.2. If G is a finite planar graph in $\mathbb{R}^2$ with straight line edges, there is $\epsilon > 0$ such that if, for every vertex x of G, a point x' is chosen such that $|x' - x| \leq \epsilon$, and the graph G' consists of the vertices x' and straight line edges connecting x' and y' whenever x and y are connected by an edge in G, then G' is a planar graph.

Proof. For each edge $\overline{xy}$ in G, choose $0 < \epsilon_{xy} < \frac{1}{2}|\overline{xy}|$ such that $\{p \in \mathbb{R}^2 \mid$ there is q on $\overline{xy}$ with $|p - q| \leq \epsilon_{xy}\}$ contains no vertices of G except x and y, and no part of an edge unless it is incident with x or y. It is enough to take $\epsilon = \frac{1}{2}\min\{\epsilon_{xy} \mid \overline{xy}$ in $G\}$. $\square$

Let p and q be points in $\mathbb{R}^2$ with $\pi_2(p) < \pi_2(q)$. A diamond with bottom and top points p and q is the area of $\mathbb{R}^2$ bounded by two paths $\overline{pr} \cup \overline{rq}$ and $\overline{ps} \cup \overline{sq}$ intersecting only at p and q such that $\pi_2(r), \pi_2(s) \in (\pi_2(p), \pi_2(q))$, and each of $\overline{pr}$, $\overline{rq}$, $\overline{ps}$ and $\overline{sq}$

are straight line segments; p and q are also called the extreme points of the diamond. In other words, a diamond corresponds to a planar embedding of $\underset{\sim}{2}^2$.

LEMMA 3.3. Let L be a finite lattice with a planar embedding $e(L)$, and let D be a diamond in $\mathbb{R}^2$ with bottom and top points p and q , respectively. There is a planar embedding $e'(L)$ of L similar to $e(L)$ and contained in D such that $\bar{0} = p$ and $\bar{1} = q$.

Proof. Actually, we shall show slightly more, namely, that all elements on the left boundary of L can be taken on the left boundary D_1 of D . Without loss of generality, $|L| \geq 3$. Let $c \neq 0, 1$ be a doubly irreducible element on the left boundary of L and let $a \prec c \prec b$. By induction on $|L|$, there is a planar embedding $e'(L - \{c\})$ with the desired properties which is similar to $e(L - \{c\})$. If $a \prec b$ in $L - \{c\}$, we can adjoin c to $e'(L - \{c\})$ by choosing $\bar{c}$ on $\overline{ab}$; if $a \not\prec b$ in $L - \{c\}$, we can use Lemma 3.1 to shift any elements on D_1 strictly between a and b slightly to the right, and then suitably place c on D_1 so it can be joined to a and b with straight line segments in D_1 . $\square$

PROPOSITION 3.4. Let $a < b$ but $a \not\prec b$ in a finite lattice L with a planar embedding $e(L)$, and let $C_1 \lambda C_2 \lambda \cdots \lambda C_n$ $(n \geq 1)$ be the proper $\langle a, b\rangle$-components with respect to $e(L)$. If σ is a permutation of $\{1, 2, \cdots, n\}$ and $\tau : \{1, 2, \cdots, n\} \to \{0, 1\}$, then there is a planar embedding $e'(L)$ of L with the "to the left of" relation λ' such that for $x \parallel y$ in L :

$x \lambda' y$ if and only if $x \lambda y$, whenever

(i) x or $y \notin \bigcup_{i=1}^{n} C_i$, or

(ii) for some $1 \leq i, j \leq n$,

$x \in C_i$, $y \in C_j$, and $(i - j)(\sigma(i) - \sigma(j)) > 0$, or

(iii) $x, y \in C_i$ and $\tau(i) = 0$;

$x \lambda' y$ if and only if $y \lambda x$, whenever
for some $1 \leq i, j \leq n$,

(iv) $x \in C_i$, $y \in C_j$, and $(i - j)(\sigma(i) - \sigma(j)) < 0$, or

(v) $x, y \in C_i$ and $\tau(i) = 1$.

REMARK. Under the conditions of the proposition, $C_{\sigma(1)} \lambda' C_{\sigma(2)} \lambda' \cdots \lambda' C_{\sigma(n)}$, and $e'(C_i)$ is similar to $e(C_i)$ if $\tau(i) = 0$, and similar to the reflection of $e(C_i)$ if $\tau(i) = 1$. We say that $e'(L)$ is obtained (up to similarity) by permuting the proper $\langle a, b\rangle$-components of $e(L)$ according to σ , and reflecting them according to τ . We also write $e'(L) \equiv T^{a,b}_{\sigma,\tau}\, e(L)$ (using $\equiv$ to indicate similarity), and call $T^{a,b}_{\sigma,\tau}$ an elementary transformation (with respect to L) .

Proof. As in the argument of Theorem 1.12, there are visibility functions for a and b , one on the left of C_1 and one on the right of C_n . Adding one point to each of these paths gives a planar representation of the lattice $L^* = L \cup \{\alpha, \beta\}$ where $a \prec \alpha$, $\beta \prec b$; let $e^*(L^*)$ be a similar planar embedding. Then $e^*(L)$ is similar to $e(L)$, and $\bar{\alpha}$, $\bar{\beta}$, $\bar{a}$ and $\bar{b}$ form a diamond D in $e^*(L)$ which includes exactly the elements $\{a, b\} \cup \bigcup_{i=1}^{n} C_i$

of L. We now delete all the lines and points in the interior of D, and draw n smaller diamonds inside D with extreme points $\bar{a}$ and $\bar{b}$ which intersect only at $\bar{a}$ and $\bar{b}$. Applying Lemma 3.3 to the i-th inner diamond, we obtain a planar embedding of $C_{\sigma(i)} \cup \{a, b\}$ inside this diamond similar to $e(C_{\sigma(i)} \cup \{a, b\})$ if $\tau(\sigma(i)) = 0$, and to its reflection if $\tau(\sigma(i)) = 1$. Finally, deleting $\bar{\alpha}, \bar{\beta}$ and the edges of D yields a planar embedding $e'(L)$ with the desired properties. □

It is clear that any region R with bounds $a<b$ and without dangles is the union of "consecutive" proper $\langle a, b\rangle$-components $C_k, C_{k+1}, \cdots, C_m$ and the bounds a, b. A new planar embedding is obtained by reversing the order of $C_k, C_{k+1}, \cdots, C_m$ and reflecting each C_i $(k \le i \le \ell)$. We say that this new planar embedding is obtained by <u>reflecting</u> R.

THEOREM 3.5. Let L be a finite planar lattice. If $e(L)$ and $e'(L)$ are two planar embeddings of L, there are elementary transformations $T_1, T_2, \cdots, T_n$ such that

$$e'(L) \equiv T_n \cdots T_2\, T_1\, e(L) .$$

<u>Proof.</u> Let $|L| \ge 3$ and let c be a doubly irreducible element in L such that $a \prec c \prec b$. By induction on $|L|$, there is a sequence $S_1, S_2, \cdots, S_m$ of elementary transformations (with respect to $L - \{c\}$) such that $e'(L - \{c\}) \equiv S_m \cdots S_2\, S_1\, e(L - \{c\})$. We inductively define a sequence $T_1, T_2, \cdots, T_n$ of elementary transformations with respect to L, with a corresponding sequence of planar embeddings $e_0(L), e_1(L), \cdots e_m(L)$, where $e_0(L) = e(L)$ and $e_i(L) \equiv T_i\, e_{i-1}(L)$ for $1 \le i \le m$. If S_i is

$T^{x,y}_{\sigma,\tau}$ and $\{x, y\} \neq \{a, b\}$, then T_i is $T^{x,y}_{\sigma,\tau}$. On the other hand, if S_i is $T^{a,b}_{\sigma,\tau}$, if $\{c\}$ is the j-th proper $\langle a, b\rangle$-component of L with respect $e_{i-1}(L)$, and if there are k proper $\langle a, b\rangle$-components in $L - \{c\}$ then T_i is $T^{a,b}_{\sigma',\tau'}$, where $\sigma' = \alpha\sigma\alpha^{-1} \cup \{\langle j, j\rangle\}$, $\tau' = \alpha\tau\alpha^{-1} \cup \{\langle j, 0\rangle\}$, and $\alpha : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k+1\}$ is defined by $\alpha(t) = t$ for $t < j$, and $\alpha(t) = t+1$ for $t > j$. By induction, it is easy to show that $e_i(L - \{c\}) \equiv S_i \cdots S_2 S_1 e(L - \{c\})$ for $0 \leq i \leq m$. Therefore, $e_m(L - \{c\}) \equiv e'(L - \{c\})$; hence, $e'(L) \equiv T^{a,b}_{\sigma,\tau} e_m(L)$ for a suitable σ and a zero function τ. Finally, if $T_{m+1} = T^{a,b}_{\sigma,\tau}$, then $e'(L) \equiv T_{m+1} \cdots T_2 T_1 e(L)$, which completes the proof. $\square$

4. Dangles on indecomposable intervals.

A _splitting element_ of a poset P is an element comparable with every element of P. P is (_linearly_) _decomposable_ if it contains a splitting element which is not a universal bound of P; otherwise, P is (_linearly_) _indecomposable_. If $0 = d_0 < d_1 < \cdots < d_n = 1$ are all the splitting elements of a nontrivial planar finite lattice L, then $L = \bigcup_{i=1}^{n} [d_{i-1}, d_i]$ and each $[d_{i-1}, d_i]$ is indecomposable. If $x \in L$ is not on the left boundary of L, x is not a splitting element since there exists $y \in L$ such that $y \lambda x$. Thus, the splitting elements of L are precisely those elements common to both boundaries of L.

LEMMA 4.1. If $[u, v]$ is an indecomposable interval in a join-semilattice S and $w \geq u$, $w \in S$, is incomparable with v, then $[u, v \vee w]$ is indecomposable. □

Let S be a join-semilattice. We call a sequence

$$x_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1} \qquad (n \geq 0)$$

of elements of S a _join-extension sequence_ if the following three conditions are satisfied:

(i) $x_0 < x_1$;

(ii) $x_i \parallel y_i \quad (1 \leq i \leq n)$;

(iii) $x_{i+1} = x_i \vee y_i \quad (1 \leq i \leq n)$.

We note that $y_i \parallel y_{i+1}$ and $x_{i+2} = y_i \vee y_{i+1} \quad (1 \leq i \leq n-1)$. A _meet-extension sequence_ is defined dually.

LEMMA 4.2. Let L be an indecomposable planar finite lattice and let $a \neq 0$ be an element on the left boundary of L. Then there is a join-extension sequence $x_0 = 0, x_1 = a, y_1, x_2, y_2, \cdots, x_n, y_n, x_{n+1} = 1$ such that x_i is on the left (right) boundary of L for odd (even) i, and y_i is on the left (right) boundary of L for even (odd) i.

Proof. If $x_i < 1$ has been chosen and x_i is, say, on the left boundary of L, then, since L is indecomposable, x_i is not on the right boundary of L. We can now take x_{i+1} to be the minimum element on the right boundary $> x_i$ and y_i a lower cover of x_{i+1} on the right boundary. □

In a planar finite lattice, whenever there is a dangle on an indecomposable interval with a distinguished element on one boundary of this interval, then, as we show in the next lemma, one of a certain class of posets must occur. This result is applied repeatedly in the proof of Theorem 1.

LEMMA 4.3. Let L be a planar finite lattice, let $[u, v] \subseteq L$ be an indecomposable interval with $a \neq u, v$ on the left boundary of $[u, v]$, and let z be an up-dangle on $[u, v]$ with attachment point $w = z \wedge v$. Then one of the following six cases will occur; in each case, one of the posets of Figure 2 listed for that case will be isomorphic to a subposet of L containing z, a, u, and v.

(i) z is left dangle and $w \geq a$: Poset (a).

(ii) z is left dangle and $w = a$: Poset (a) for $n = 0$.

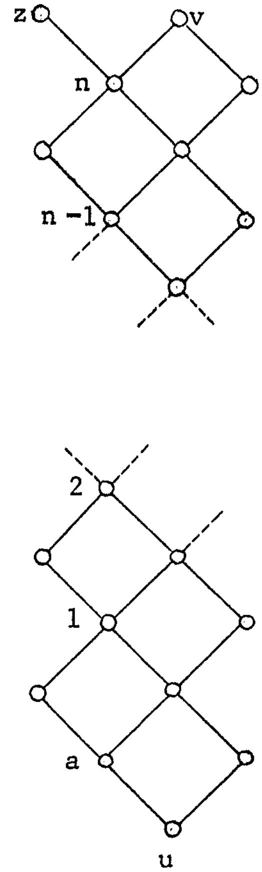
(iii) z is left dangle and $w < a$: Poset (b), Poset (c).

(iv) z is right dangle and $w > a$: Poset (d).

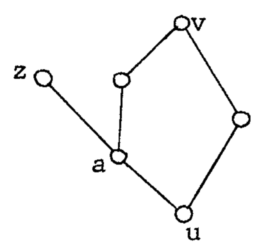
(v) z is right dangle and $w \parallel a$: Poset (e).

(vi) z is right dangle and $w < a$: Poset (f).

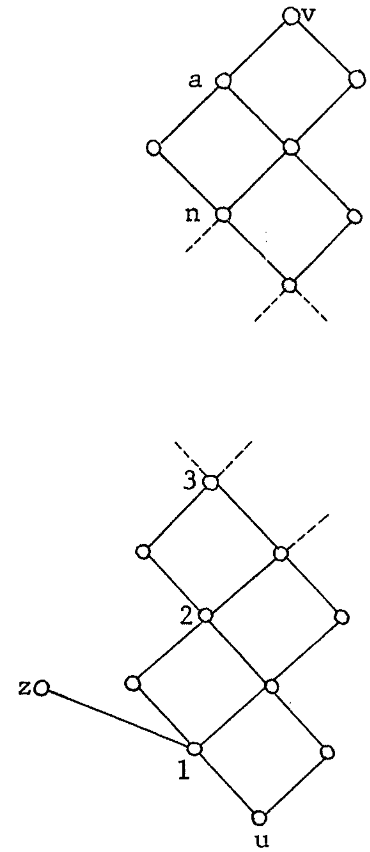
Figure 2.



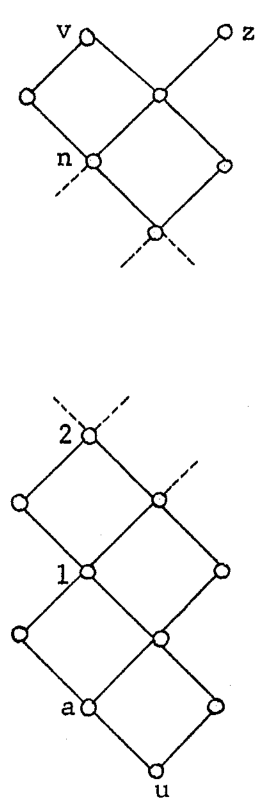
Poset (a)
($n \geq 0$)



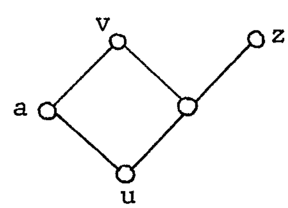
Poset (b)



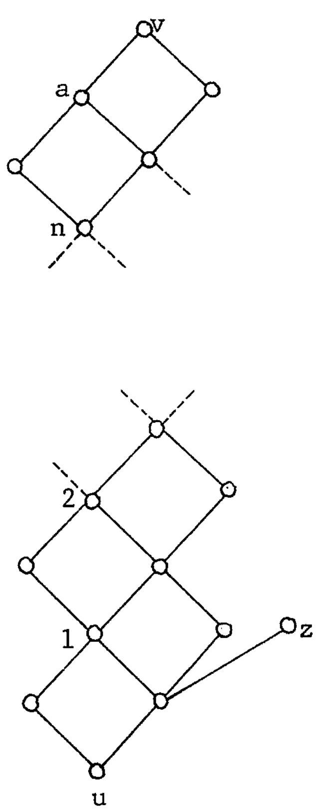
Poset (c)
($n \geq 1$)



Poset (d)
($n \geq 0$)



Poset (e)



Poset (f)
($n \geq 0$)

<u>Proof.</u> One of these cases must occur for an up-dangle z because, if z is a left (right) dangle, then w is on the left (right) boundary by Proposition 1.9, and $w \neq a$ for a right dangle because $[u, v]$ is indecomposable. We now analyse each case separately to determine the posets which appear.

(i) <u>z is left dangle and $w \geq a$</u> : In view of Lemma 4.2, there is a join-extension sequence $x_0 = u, x_1 = a, y_1, x_2, y_2, \cdots, x_n, y_n, x_{n+1} = v$ such that x_i is on the left (right) boundary of $[u, v]$ for odd (even) i, and y_i is on the left (right) boundary of $[u, v]$ for even (odd) i. Since w is on the left boundary of $[u, v]$, there is odd k such that $w \in [x_k, x_{k+2})$ (or $w \in [x_k, x_{k+1})$ in case $x_{k+1} = v$). Then $\{z, u, x_1, y_1, x_2, y_2, \cdots, x_k, y_k, v\}$ is isomorphic to Poset (a). For example, if $w \in [x_k, x_{k+2})$, then $z \geq y_k$ is impossible because it would imply either $y_k \leq y_{k+1}$ if $z \leq y_{k+1}$, or $z \geq y_k \vee y_{k+1} = x_{k+2}$ if $z \geq y_{k+1}$.

(ii) <u>z is left dangle and $w = a$</u> : Trivial.

(iii) <u>z is left dangle and $w < a$</u> : Let $x_0 = v, x_1 = a, y_1, x_2, y_2, \cdots, x_n, y_n, x_{n+1} = u$ be a meet-extension sequence with x_i on the left (right) boundary of $[u, v]$ for odd (even) i, and y_i on the left (right) boundary of $[u, v]$ for even (odd) i. If $w \in (x_2, a)$ with $x_2 = u$ or $w \in (x_3, a)$, then $\{z, u, w, a, y_1, v\}$ is isomorphic to Poset (b). Otherwise, there is odd $k \geq 3$ such that $w \in (x_{k+2}, x_k]$ (or $w \in (x_{k+1}, x_k]$ when $x_{k+1} = u$); then $\{v, x_1, y_1, x_2, y_2, \cdots, x_{k-1}, y_{k-1}, y_k, w, u, z\}$ is isomorphic to Poset (c), since, for example, $w \leq y_k$ and $w \leq y_{k+1}$ would imply $w \leq x_{k+2}$.

(iv) <u>z is right dangle and $w > a$</u> : Let $x_0, x_1, y_1, x_2, y_2, \cdots, x_n, y_n, x_{n+1}$ be the join-extension sequence of (i) . Since w is on the right boundary of $[u, v]$, $w \geq x_2$. If, for even $k \geq 2$, $w \in [x_k, x_{k+2})$, or $w \in [x_k, x_{k+1})$ with $x_{k+1} = v$, then $\{u, x_1, y_1, x_2, y_2, \cdots, x_k, y_k, v, z\}$ is isomorphic to Poset (d).

(v) <u>z is right dangle and $w \parallel a$</u> : Trivial.

(vi) <u>z is right dangle and $w < a$</u> : Let $x_0, x_1, y_1, x_2, y_2, \cdots, x_n, y_n, x_{n+1}$ be the meet-extension sequence of (iii) . There is even $k \geq 2$ such that either $w \in (x_{k+2}, x_k]$, or $w \in (x_{k+1}, x_k]$ when $x_{k+1} = u$. Then $\{v, x_1, y_1, x_2, y_2, \cdots, x_{k-1}, y_{k-1}, y_k, w, u\}$ is isomorphic to Poset (f) . $\square$

5. Proof of Theorem 1.

It is easy to verify that $\underset{\sim}{A}_n$ $(n \geq 3)$ is a lattice. Since $\underset{\sim}{A}_n$ is not dismantlable, it follows from Corollary 2.3 that it cannot be planar. On the other hand, if P is one of the other posets of Figure 1, then there is a doubly irreducible element c in P such that $L = P - \{c\}$ is planar; hence, by Corollary 2.3, P is dismantlable and, consequently, a lattice by Corollary 2.4. In each such poset P, if c is chosen so that the diagram in Figure 1 given for P induces a planar embedding $e(L)$ of the lattice L, then for all $u < v$ in L with $u \not\prec v$, there is at most one proper $\langle u, v\rangle$-component C; either C consists of a doubly irreducible element of L, or $L = C \cup \{u, v, 0, 1\}$. Therefore, by Theorem 3.5, any planar embedding of L is similar to $e(L)$ or its reflection. If $a \prec c \prec b$ in P, then b is not visible from a by Theorem 1.12; by the next lemma, this means that P is nonplanar. Therefore, every poset in Figure 1 is a nonplanar lattice.

LEMMA 5.1. Let L be a planar finite lattice, $a < b$ in L, and $M = L \mathbin{\dot\cup} \{c\}$ be defined by setting $a \prec c \prec b$ in M. M is planar if and only if there is a planar embedding of L in which b is visible from a.

<u>Proof.</u> If $e(M)$ is a planar embedding of M, then b is visible from a in $e(M - \{c\})$ with visibility path $\overline{ac} \cup \overline{cb}$. Now, let $e(L)$ be a planar embedding of L in which b is visible from a. If $a \prec b$, add $\bar{c}$ on the midpoint of $\overline{ab}$ to give a planar embedding of L. Otherwise, add $\bar{c}$ to the visibility path for a and b, forming a planar representation of M, and take a similar planar embedding. $\square$

Let M be a finite lattice which contains a poset P in Figure 1. In D. Kelly [5], it is shown that a finite poset which contains a nonplanar lattice is nonplanar; therefore, M is nonplanar, and the proof of one direction of Theorem 1 is complete. The nonplanarity of M also follows from a consideration of dimension 2 lattices. The dimension of an arbitrary poset is the least number of total orders whose intersection is the partial ordering of the poset (see B. Dushnik and E.W. Miller [3]). As observed in [1], the following characterization of planar lattices is a combination of results of J. Zilber [2, p. 32, ex. 7(c)] and B. Dushnik and E.W. Miller [3, Theorem 3.61].

PROPOSITION 5.2. Let L be a finite lattice. L is planar if and only if dimension(L) ≤ 2 .

Therefore, dimension(M) $\geq$ dimension(P) > 2 so that M is nonplanar.

Now, let M be a nonplanar finite lattice. We will show that M must contain one of the posets (or their duals) of Figure 1. To this end, we suppose that M does not contain any of these posets; it will eventually be shown that this is impossible. At various stages of the proof, certain hypotheses will be shown to be untenable by exhibiting an occurrence in M of one of these posets.

By the characterization of dismantlable lattices established in D. Kelly and I. Rival [6], M is dismantlable since it does not contain $\underset{\sim}{A}_n$ $(n \geq 3)$. If M is dismantled (by removing doubly irreducible elements one at a time), a planar lattice will eventually be obtained; therefore, we can, without loss of generality, assume that $M = L \cup \{c\}$, where L is planar and $a \prec c \prec b$ in M. By Lemma 5.1, b is not visible from a with respect to <u>any</u> planar embedding of L.

Let $e_1(L)$ be a planar embedding of L. In the course of the proof, various planar embeddings of L will be introduced; at any point of the proof, it is to be understood that all statements are with respect to the most recently introduced planar embedding.

By Theorem 1.12, $a \not\prec b$, there is exactly one $\langle a, b\rangle$-component, and there are both left and right dangles on $[a, b]$. We first suppose that there are left and right dangles z_1 and z_2 on $[a, b]$ with attachment points w_1 and w_2, respectively, such that $w_1 \parallel w_2$. Since w_1 and w_2 are in the same $\langle a, b\rangle$-component, there is a fence $F = (w_1 = x_1, x_2, \cdots, x_k = w_2)$ in (a, b) with $k \geq 3$.

Let both z_1 and z_2 be up-dangles. $P = F \cup \{a, c, b, z_1, z_2, 1\}$ is a subposet of M. We show that P contains $\underset{\sim}{C}$ or $\underset{\sim}{F}_n$ for some n as a subposet. If $w_1 \wedge w_2 > a$, then $\underset{\sim}{C} \cong \{a, w_1 \wedge w_2, w_1, w_2, c, b, z_1, z_2, 1\}$. For example, $w_1 < z_2$ would imply that $w_1 \leq z_2 \wedge b = w_2$. We can now assume that $w_1 \wedge w_2 = a$. If F is down-down, then, using the definition of an attachment point and taking into account the incomparabilities that hold in a fence, we have that $\underset{\sim}{F}_n \cong \{a, x_1, x_2, \cdots, x_k, c, b, z_1, z_2, 1\}$ with $n = \frac{1}{2}(k-1)$. If F is down-up, then $\underset{\sim}{F}_n \cong \{a, x_1, x_2, \cdots, x_{k-1}, c, b, z_1, z_2, 1\}$ with $n = \frac{k}{2}$. If F is up-down, then $k \geq 5$ since $w_1 \wedge w_2 = a$, and therefore, $\underset{\sim}{F}_n \cong \{a, x_2, x_3, \cdots, x_{k-1}, c, b, z_1, z_2, 1\}$ with $n = \frac{1}{2}(k-2)$.

By symmetry, we can now assume that z_1 is an up-dangle and z_2 is a down-dangle. We show that $Q = F \cup \{0, z_2, a, c, b, z_1, 1\}$ contains a poset in $\mathcal{L}$. If F is down-up, then $\underset{\sim}{G}_n \cong \{0, z_2, a, x_1, x_2, \cdots, x_k, c, b, z_1, 1\}$ with $n = \frac{k}{2}$. If F is up-up with $k = 3$, then $\underset{\sim}{E} \cong \{0, z_2, a, x_2, x_3, c, b, z_1, 1\}$; if F is up-up with $k \geq 5$, then $\underset{\sim}{G}_n \cong \{0, z_2, a, x_2, x_3, \cdots, x_k, c, b, z_1, 1\}$ with $n = \frac{1}{2}(k-1)$. The remaining case is that F is up-down. If $k = 4$, then $\underset{\sim}{E} \cong \{0, z_2, a, x_2, x_3, c, b, z_1, 1\}$; otherwise, $k \geq 6$ and $\underset{\sim}{G}_n \cong \{0, z_2, a, x_2, x_3, \cdots, x_{k-1}, c, b, z_1, 1\}$ with $n = \frac{1}{2}(k-2)$. Thus, we have shown that w_1 and w_2 must be comparable.

If z_1 (z_2) were an up-dangle (down-dangle) on $[a, b]$ with attachment point w_1 (w_2) such that $w_1 < w_2$, then $\underset{\sim}{E} \cong \{0, z_2, a, w_1, w_2, c, b, z_1, 1\}$; for example, $z_2 < z_1$ would imply that $w_2 = z_2 \vee a \leq z_1 \wedge b = w_1$. If z_1 and z_2 are up-dangles on $[a, b]$ with attachment points $w_1 \leq w_2$ and $z_1 \vee b \parallel z_2 \vee b$, then $\underset{\sim}{D} \cong \{a, w_1, c, b, z_1, z_1 \vee b, z_2, z_2 \vee b, 1\}$; if $w_1 < w_2$ and $z_1 \vee b < z_2 \vee b$, then $\underset{\sim}{E} \cong \{a, c, w_1, w_2, b, z_1, z_1 \vee b, z_2, z_2 \vee b\}$.

Therefore, there is an element $d \in (a, b)$ such that if z is an up-dangle and z' is a down-dangle on $[a, b]$ then $z > d > z'$. Moreover, the set $S = \{[z \wedge b, z \vee b] \mid z \text{ up-dangle on } [a, b]\}$ is a chain (with respect to $\subseteq$) of closed intervals. Let $[r, s]$ be the maximum element of S.

We now show that there is a planar embedding $e_2(L)$, obtained from $e_1(L)$ by elementary transformations $T^{x,y}_{\sigma,\tau}$, $x, y \geq d$, in which b is on the boundary of $[r, s]$. Let $[u, v]$ be the minimum element of S such that b is not on the boundary of $[u, v]$ with respect to $e_1(L)$. We show that there is a planar embedding $e_1'(L)$ obtained from $e_1(L)$ by elementary transformations $T^{x,y}_{\sigma,\tau}$, $x, y \geq d$ in which b is on the boundary of $[u, v]$. Iteration of this procedure will provide the desired planar embedding $e_2(L)$. We can assume that b is on the left boundary of $[u', v']$ with respect to $e_1(L)$ for every $[u', v'] \in S$ such that

$[u', v'] \subset [u, v]$. Let z be an up-dangle on $[a, b]$ such that $u = z \wedge b$ and $v = z \vee b$. If $b \lambda z$ for all such z, then b would be on the left boundary of $[u, v]$. Choose z so that $z \lambda b$. By Lemma 1.11, there is $z_1 \in [u, v]$ such that $z_1 \lambda b$ and z_1 and b are in a common face. Clearly, z_1 is an up-dangle on $[a, b]$, $z_1 \wedge b = u$ and $z_1 \vee b = v$. Let $C_1 \lambda C_2 \lambda \cdots \lambda C_n$ be the $\langle u, v\rangle$-components. If z_1 and b were in the same $\langle u, v\rangle$-component, then, in any connection between z_1 and b in (u, v), there would be two consecutive elements that are on different boundaries of the common face containing z_1 and b. Since this is impossible, $b \notin C_1$.

We first consider the case that $b \notin C_n$. Suppose that both C_1 and C_n are not proper. Let x_1 (x_2) be a left (right) dangle on $[u, v]$ with attachment point y_1 (y_2). If both x_1 and x_2 are up-dangles, then $\underset{\sim}{C} \cong \{a, u, y_1, y_2, c, v, x_1, x_2, 1\}$; if both are down-dangles, then $\underset{\sim}{B} \cong \{0, u, c, b, x_1, y_1, x_2, y_2, v\}$; if x_1 is an up-dangle and x_2 is a down-dangle, then $\underset{\sim}{C}^d \cong \{0, u, c, b, x_2, y_2, v, x_1, 1\}$. Therefore, C_1 (or C_n) is proper. Since b is on the left boundary of the $\langle u, v\rangle$-component B containing b, C_1 (or C_n) can be permuted with (the reflection of) B, giving a planar embedding in which b is on the left (right) boundary of $[u, v]$.

We can now assume that $b \in C_n$. Let D be the maximum indecomposable subinterval of $C_n' = C_n \cup \{u, v\}$ which contains b. Let $p < q$ be the universal bounds of D; obviously, p and q are consecutive splitting elements of C_n'. If there are no dangles on D, then, since b is on the left boundary of D, reflecting D will give a planar embedding of L in which b is on the right boundary of $[u, v]$. We complete the proof of the existence of the planar embedding $e_1'(L)$ by showing that there can be no dangles on D. Let us suppose that there is an up-dangle z on D. If $z < v$, then $z \in C_n$ and q would not be a splitting element of C_n'. Thus, $z \parallel v$, and z is a right up-dangle on $[u, v]$. Let $w = z \wedge v$; since b is not on the right boundary of $[u, v]$, $w \neq b$. If $w > b$, let P be Poset (d) of Figure 2 with a replaced by b. By Lemma 4.3, P is a subposet of L for some $n \geq 0$. Then, $\underset{\sim}{H}_{n+2} \cong (P - \{p, q\}) \cup \{a, c, u, z_1, v, 1\}$. If $w < b$, then $\underset{\sim}{E} \cong \{0, c, p, w, b, z_1, q, z, 1\}$. Therefore, $w \parallel b$; let F be a fence in C_n connecting w and b. In this case, we obtain the same subposets obtained previously for the bounded component $[a, b]$ with left and right dangles having incomparable attachment points. (The poset Q introduced there is isomorphic to $F \cup \{0, c, p, z_1, q, z, 1\}$.)

Now, suppose there is a down-dangle z on D. As above, z must be a right down-dangle on $[u, v]$. Let $w = z \vee u$. If $w < b$, let P be the subposet of L that is isomorphic to the dual of Poset (d) for some $n \geq 0$ with a, u, and v replaced by b, q, and p, respectively. Then, $\underset{\sim}{I}_{n+1} \cong (P - \{p, q\}) \cup \{0, u, c, z_1, v\}$. If $w \parallel b$, we obtain the duals of the subposets that we had for the case of a bounded component $[a, b]$ with two up-dangles and incomparable attachment points. (If P is the poset introduced there, $P^d \cong F \cup \{0, c, z, p, z_1, q\}$, where F is a fence in (p, q) that connects b and w.) Finally, if $w > b$, let P be the subposet of L that is isomorphic to the dual of Poset (f) for some $n \geq 0$ with a, u, and v replaced by b, q, and p, respectively. In this case, $\underset{\sim}{I}_{n+1} \cong (P - \{p, q\}) \cup \{0, c, u, z_1, v\}$. Thus, we conclude that D can have no dangles.

We have now shown how to obtain a planar embedding $e_2(L)$ of L for which b is on the boundary of $[r_2, s_2]$, the maximum element of S (which was previously denoted by $[r, s]$). Applying the dual of this procedure to $e_2(L)$ for the down-dangles on $[a, b]$, we arrive at a planar embedding $e_3(L)$ of L for which the following conditions are satisfied:

(a) $r_1 \leq s_1 \leq r_2 \leq s_2$, where $r_1 = z_1 \wedge a$, $s_1 = z_1 \vee a$, $r_2 = z_2 \wedge b$, and $s_2 = z_2 \vee b$ for some down-dangle z_1 (up-dangle z_2) on $[a, b]$;

(b) all down-dangles on $[a, b]$ are in $[r_1, s_1]$ and all up-dangles on $[a, b]$ are in $[r_2, s_2]$;

(c) a is on the boundary of $[r_1, s_1]$ and b is on the boundary of $[r_2, s_2]$.

Clearly, both $[r_1, s_1]$ and $[r_2, s_2]$ are indecomposable intervals. We now choose intervals $[u_1, v_1]$, $[u_2, v_2]$ containing $[r_1, s_1]$, $[r_2, s_2]$ subject to the following conditions:

(1) $u_1 \leq r_1 \leq s_1 \leq v_1 \leq u_2 \leq r_2 \leq s_2 \leq v_2$;

(2) $[u_1, v_1]$ and $[u_2, v_2]$ are indecomposable;

(3) u_1 and u_2 (v_1 and v_2) are minimal (maximal) with respect to (1) and (2).

In fact, the following four properties are also satisfied:

(4) all down-dangles (up-dangles) on $[a, b]$ are in $[u_1, v_1]$ ($[u_2, v_2]$);

(5) a is on the boundary of $[u_1, v_1]$ and b is on the boundary of $[u_2, v_2]$;

(6) there are no down-dangles on $[u_1, v_1]$ and no up-dangles on $[u_2, v_2]$;

(7) if x is a down-dangle on $[u_2, v_2]$ (up-dangle on $[u_1, v_1]$) , then $x \parallel v_1$ $(x \parallel u_2)$.

Indeed, (4) follows trivially from (b) . If a is not on the left boundary of $[u_1, v_1]$, then there is $z \in [u_1, v_1]$ with $z \lambda a$. Since z is a left dangle on $[a, b]$, $z \in [r_1, s_1]$ which, in turn, implies that a could not be on the left boundary of $[r_1, s_1]$; thus, (5) now follows by (c) . If there were a down-dangle y on $[u_1, v_1]$, then $[y \wedge u_1, v_1]$ would also be an indecomposable interval, contradicting (3) . Also, if x were a down-dangle on $[u_2, v_2]$ with x comparable with v_1 , then $x \geq v_1$ which, since $[u_2 \wedge x , v_2]$ is indecomposable, contradicts (3) .

If a and b were both on the left (right) boundary of $[u_1, v_1]$ and $[u_2, v_2]$, respectively, then $[a, b]$ would have no left (right) dangles; hence, b would be visible from a . Therefore, without loss of generality, a is on the left boundary of $[u_1, v_1]$ and b is on the right boundary of $[u_2, v_2]$. Furthermore, neither $[u_1, v_1]$ nor $[u_2, v_2]$ can be reflected, since a and b would then be on the same boundary of $[u_1, v_1]$ and $[u_2, v_2]$, respectively. Thus, there must be a dangle on both $[u_1, v_1]$ and $[u_2, v_2]$. In view of (6) , there must be an up-dangle on $[u_1, v_1]$ and a down-dangle on $[u_2, v_2]$. In order to complete the proof of the first assertion of Theorem 1, we show that, whichever way these dangles occur, M contains a lattice in $\mathcal{L}$ as a subposet.

We first suppose that there is a left down-dangle z on $[u_2, v_2]$ with $w_2 = z \vee u_2$ and $w_1 = z \wedge v_1$.

(i) $w_2 \leq b$: In this case $w_1 \geq a$, since otherwise z would be a down-dangle on $[a, b]$. Let P be a subposet of L , guaranteed by Lemma 4.3, which is isomorphic to Poset (a) for $n = k$ with u and v replaced by u_1 and v_1 , respectively; and let Q be a subposet of L which is isomorphic to Poset (d) for $n = m$ with v deleted, and a and u replaced by b and v_2 , respectively. Note that $P \cap Q = \{z\}$. $P \cup Q \cup \{c\}$ is a subposet of M which is isomorphic to $\underset{\sim}{H}_{k+m+1}$.

(ii) $w_2 \parallel b$ and $z > a$: If $w_1 > a$, then z would be an up-dangle on $[a, b]$; hence, $w_1 = a$. Let P and Q be the subposets of L provided by Lemma 4.3 which correspond to Poset (a) for $n = 0$ and the dual of Poset (e), respectively, with the replacements and deletion as in (i) . Then, $\underset{\sim}{D} \cong P \cup Q \cup \{c\}$.

(iii) $w_2 \parallel b$ and $u_1 < w_1 < a$: Let P_1 (P_2) and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (e) , respectively, with the replacements and deletion as in (i) . Then, $D \cong (P - \{a\}) \cup Q \cup \{c\}$ $(\underset{\sim}{H}_{k+1} \cong (P - \{v_1\}) \cup (Q - \{b\}) \cup \{c\})$.

(iv) $w_2 > b$ and $z > a$: As in (ii) , $w_1 = a$. If P and Q are the subposets of L that correspond to Poset (a) for $n = 0$ and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (i) , then $\underset{\sim}{H}_{m+1} \cong P \cup Q \cup \{c\}$.

(v) $w_2 > b$ and $u_1 < w_1 < a$: Let P_1 (P_2) and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (i) . In this case, $\underset{\sim}{H}_{m+1} \cong (P_1 - \{a\}) \cup Q \cup \{c\}$ $(\underset{\sim}{H}_{k+m+1} \cong P \cup Q \cup \{c\})$.

We have shown that no left down-dangle on $[u_2, v_2]$ satisfies any of the conditions (i) to (v), and, by duality, that no right up-dangle on $[u_1, v_1]$ satisfies any of the corresponding dual conditions. Let z_2 be a left down-dangle on $[u_2, v_2]$ with attachment point $w_2 = z_2 \vee u_2$; by (i), $w_2 \nleq b$. There is also an up-dangle z_1 on $[u_1, v_1]$ with attachment point $w_1 = z_1 \wedge v_1$.

We now show that z_1 must be a right up-dangle on $[u_1, v_1]$. It suffices to show that, if z_1 were a left up-dangle on $[u_1, v_1]$, then $z_1 \leq w_2$ because z_1 would then be a left down-dangle on $[u_2, v_2]$ satisfying one of (i) to (v) , contrary to assumption. Let C be a maximal chain from 0 to w_2 through z_2 , let D_1 be a maximal chain passing through the left boundaries of $[u_1, v_1]$ and $[u_2, v_2]$, and let $D = D_1 \cap [0, w_2]$. Suppose $z_1 \nleq w_2$; then, by Lemma 1.3, z is not in the region defined by C and D . Since z_1 is on the left of D_1 , z_1 is on the left of $C_1 = C \cup (D_1 \cap [w_2, 1])$. Let $x \in C_1$ be such that $z_1 \geq x \geq w_1$. Since $z_1 \nleq u_2$ and $z_2 \parallel u_2$, $x \parallel u_2$; x is thus a left down-dangle on $[u_2, v_2]$ that satisfies one of (i) to (v) .

Since, by duality, we can assume that $w_2 > b$ implies $w_1 < a$, there are the following three cases to consider.

(vi) <u>$w_2 \parallel b$ and $w_1 \parallel a$:</u> Let P be Poset (e) with u, v, and z replaced by 0, v_1, and z_1 , respectively, and let Q be the dual of Poset (e) with v deleted, and u and z replaced by 1 and z_2 , respectively, which, by Lemma 4.3, occur as subposets of L . $P \cup Q \cup \{c\}$ is a subposet of M which is isomorphic to $\underset{\sim}{F}_2$.

(vii) $\underline{w_2 \parallel b \text{ and } w_1 < a}$: Let P and Q be the subposets of L that correspond to Poset (f) for $n = k$ and the dual of Poset (e), respectively, with the replacements and deletion as in (vi). Then, $\underset{\sim}{I}_{k+1} \cong P \cup Q \cup \{c\}$.

(viii) $\underline{w_2 > b \text{ and } w_1 < a}$: Let P and Q be the subposets of L that correspond to Poset (f) for $n = k$ and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (vi). Then, $\underset{\sim}{I}_{k+m} \cong P \cup Q \cup \{c\}$.

We can now assume there are no right dangles on $[u_1, v_1]$ and no left dangles on $[u_2, v_2]$. Let z_1 be a left up-dangle on $[u_1, v_1]$ with attachment point $w_1 = z_1 \wedge v_1$ and let z_2 be a right down-dangle on $[u_2, v_2]$ with attachment point $w_2 = z_2 \vee u_2$. For all $x \in [u_2, v_2)$, $z_1 \nleq x$, since we have already observed that $z_1 \parallel u_2$, and otherwise z_1 would be a left dangle on $[u_2, v_2]$; similarly, $z_2 \ngeq x$ for all $x \in (u_1, v_1]$. It follows that $w_1 \leq a$ and $w_2 \geq b$ since otherwise z_1 or z_2 would be a dangle on $[a, b]$. By duality, there are only three cases left to consider.

(ix) $\underline{w_2 = b \text{ and } w_1 = a}$: Let P and Q be the subposets of L that correspond to Poset (a) for $n = 0$ and the dual of Poset (a) for $n = 0$, respectively, with the replacements and deletion as in (vi). Then, $\underset{\sim}{I}_1 \cong P \cup Q \cup \{c\}$.

(x) $\underline{w_2 = b \text{ and } w_1 < a}$: Let P_1 (P_2) and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of

Poset (a) for $n = 0$, respectively, with the replacements and deletion as in (vi). Then, $\underset{\sim}{I}_1 \cong (P_1 - \{a\}) \cup Q \cup \{c\}$ ($\underset{\sim}{I}_{k+1} \cong P_2 \cup Q \cup \{c\}$).

(xi) $w_2 > b$ and $w_1 < a$: Let P_1 (P_2) and Q_1 (Q_2) be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (b) (the dual of Poset (c) for $n = m$), respectively, with the replacements and deletion as in (vi). By duality, M has one of the following subposets: $\underset{\sim}{I}_1 \cong (P_1 - \{a\}) \cup (Q_1 - \{b\}) \cup \{c\}$; $\underset{\sim}{I}_{k+1} \cong (P_1 - \{a\}) \cup Q_2 \cup \{c\}$; or $\underset{\sim}{I}_{k+m+1} \cong P_2 \cup Q_2 \cup \{c\}$.

This completes the proof of the fact that a finite lattice is planar if and only if it does not have a subposet isomorphic to a lattice in $\mathcal{L}$. Furthermore, it is easy to verify that no lattice is repeated in the list given for $\mathcal{L}$.

Let $\mathfrak{F}$ be a set of finite lattices such that a lattice is planar if and only if it does not have a subposet isomorphic to a lattice in $\mathfrak{F}$. We will show that $\mathcal{L} \subseteq \mathfrak{F}$. Obviously, every lattice in $\mathfrak{F}$ is nonplanar. If $L \in \mathcal{L}$, there is a subposet K of L that is isomorphic to a lattice in $\mathfrak{F}$. If we show that no proper subposet of L is a nonplanar lattice, it will follow that $K = L$, and therefore, $\mathcal{L} \subseteq \mathfrak{F}$. An element of a finite lattice will be called <u>irreducible</u> if it is a join-irreducible element of the lattice distinct from 0, or a meet-irreducible element distinct from 1. We will apply the following general observation.

PROPOSITION 5.3. Let L be a finite lattice and K be a subposet of L which is a lattice. If $K \subset L$, then there is $x \in L - K$ that is irreducible in L. Moreover, $L - \{x\}$ is a lattice.

<u>Proof</u>[1]. Let $y \in L - K$ and suppose that K contains every irreducible element of L. Then, y can be expressed as $y = \bigvee_L(a_i \mid 1 \leq i \leq m) = \bigwedge_L(b_j \mid 1 \leq j \leq n)$, where a_i $(1 \leq i \leq m)$ are join-irreducibles of L distinct from 0, and b_j $(1 \leq j \leq n)$ are meet-irreducibles of L distinct from 1. By assumption, all the a_i's and b_j's are in K. Hence, $\bigvee_K(a_i \mid 1 \leq i \leq m) \geq y \geq \bigwedge_K(b_j \mid 1 \leq j \leq n)$. If both m and n are nonzero, then $a_i \leq b_j$ for all i and j which implies that $\bigvee_K(a_i \mid 1 \leq i \leq m) \leq \bigwedge_K(b_j \mid 1 \leq j \leq n)$. Therefore, $y = \bigvee_K(a_i \mid 1 \leq i \leq m) \in K$, contrary to assumption. If $m = 0$, then $y = 0 \in L$ and $n \geq 1$. In this case, $\bigwedge_K(b_j \mid 1 \leq j \leq n) \leq 0$; since equality must hold, we again obtain a contradiction. The second statement is trivial. □

For every lattice L in Figure 1, it is easy to check that $L - \{x\}$ is planar for any irreducible element of L. If the lattice K is a proper subposet of $L \in \mathcal{L}$, then, by the above proposition, K is a subposet of $L - \{x\}$ for an irreducible x. Since $L - \{x\}$ is planar, so is K. This completes the proof of Theorem 1.

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1) Simplified by B. Wolk.

6. Some results related to Theorem 1.

For finite lattices, Proposition 5.2 showed that planarity and dimension ≤ 2 are equivalent properties. Using the compactness property of finite dimension, Theorem 1 can be extended to all lattices of dimension ≤ 2.

THEOREM 6.1. A lattice has dimension ≤ 2 if and only if it does not contain any lattice in $\mathcal{L}$ as a subposet. Moreover, $\mathcal{L}$ is the minimum such list of lattices.

Proof. Since the dimension of each lattice in $\mathcal{L}$ exceeds 2 (in fact, equals 3), one direction is immediate. If K is a lattice whose dimension exceeds 2, then there is a finite subset S of K whose dimension exceeds 2. The join-semilattice L of K generated by $S \cup \{\bigwedge S\}$ is a finite lattice of dimension > 2; hence, by Proposition 5.2, L is nonplanar. Since, by Theorem 1, L contains a lattice in $\mathcal{L}$ as a subposet, so does K. The second statement of the theorem follows immediately from the corresponding statement of Theorem 1. $\square$

K.A. Baker has shown that the dimension of the lattice obtained by completion by cuts of a poset of dimension n also has dimension n (cf. [1, Theorem 4.1]).

COROLLARY 6.2. A poset has dimension ≤ 2 if and only if its completion by cuts does not contain any lattice in $\mathcal{L}$ as a subposet.

The following result is proved in R. Wille [9].

THEOREM 6.3. A modular lattice has dimension ≤ 2 if and only if it does not contain $\underset{\sim}{A}_3$, $\underset{\sim}{B}$, $\underset{\sim}{B}^d$, $\underset{\sim}{C}$ or $\underset{\sim}{C}^d$ as a subposet.

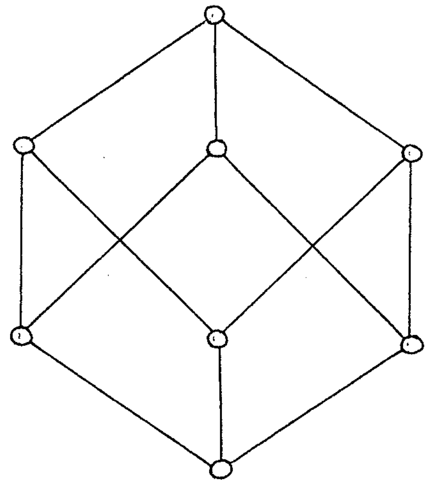
We will prove a more general theorem in which only four <u>modular</u> lattices are mentioned. In particular, this will show that the list in Theorem 6.3 is redundant since one of $\underset{\sim}{B}$ or $\underset{\sim}{B}^d$ can be omitted. To this end let $\underset{\sim}{J}$ and $\underset{\sim}{K}$ be the modular lattices illustrated in Figure 3. Since $\underset{\sim}{J}$ contains $\underset{\sim}{B}$ (and $\underset{\sim}{B}^d$) and $\underset{\sim}{K}$ contains $\underset{\sim}{C}$, both $\underset{\sim}{J}$ and $\underset{\sim}{K}$ are nonplanar.

A sublattice S of a finite lattice L <u>can be obtained by dismantling</u> L if there is a sequence $L = L_0 \supset L_1 \supset \cdots \supset L_n = S$ of sublattices of L satisfying $|L_i| = |L_{i+1}| + 1$ for $0 \leq i \leq n-1$.

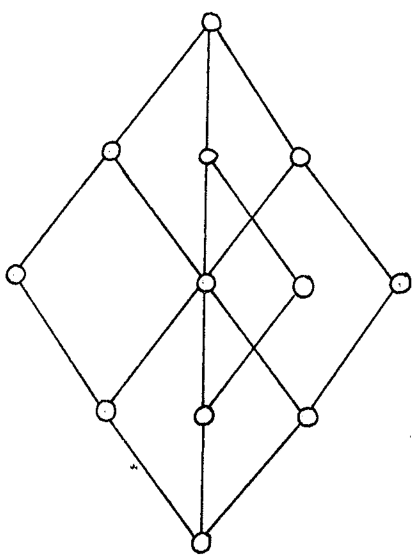
LEMMA 6.4. Let M be a finite modular lattice and let M_1 be a sublattice of M that can be obtained by dismantling M. If P is a cover-preserving sublattice of M_1 such that x and y are not both splitting elements of P whenever $x \prec y$ in P, then P is a cover-preserving sublattice of M.

<u>Proof.</u> By induction, we can assume that $M = M_1 \cup \{c\}$, where c is doubly irreducible in M and $a \prec c \prec b$ in M. Let P be a cover-preserving sublattice of M_1 that satisfies the condition of the lemma. The only cover of P that c could destroy is $a \prec b$. Suppose that $a \prec b$ in P. Without loss of generality, there is a cover d of a in P with $d \neq b$; hence, $\{a, c, b, d, b \vee d\}$ would be a nonmodular sublattice of M, a contradiction. $\square$

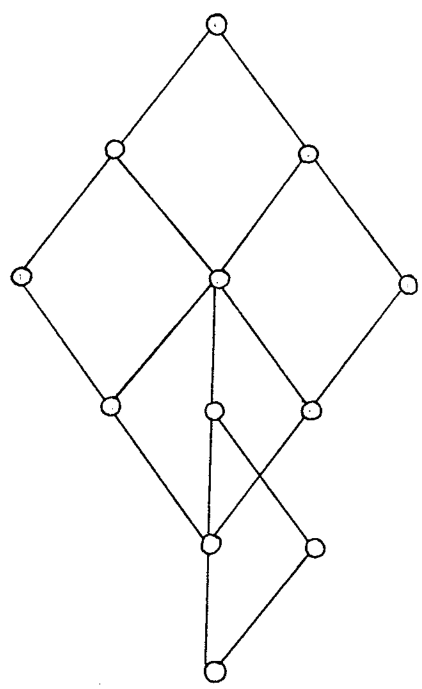
Figure 3.



$\underset{\sim}{A}_3$



$\underset{\sim}{J}$



$\underset{\sim}{K}$

THEOREM 6.5 A finite modular lattice is planar if and only if it does not contain $\mathbf{A}_3$, $\mathbf{J}$, $\mathbf{K}$ or $\mathbf{K}^d$ as a cover-preserving sublattice. Moreover, if $\mathfrak{F}$ is a set of finite modular lattices such that a modular lattice is planar if and only if it does not contain any lattice in $\mathfrak{F}$ as a subposet, then $\mathfrak{F}$ contains $\mathbf{A}_3$, $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{K}^d$.

Proof. As in the proof of Theorem 1, one direction is immediate since $\mathbf{A}_3$, $\mathbf{J}$ and $\mathbf{K}$ are nonplanar.

Let M be a nonplanar finite modular lattice. If M does not contain $\mathbf{A}_3$ as a cover-preserving sublattice, then it follows from the proof of Theorem 3.5 of [6] that M is dismantlable. M can be dismantled down to a nonplanar sublattice M_1 so that $M_1 = L \cup \{c\}$ for a planar lattice L. By Lemma 6.4, we can assume that $M = M_1$. If $a \prec c \prec b$ in M, then, by virtue of Lemma 5.1, b is not visible from a in any planar embedding of L. Since M is modular, there is d in L such that $a \prec d \prec b$. Let $r_1 \lambda r_2 \lambda \cdots \lambda r_m$ ($s_1 \lambda s_2 \lambda \cdots \lambda s_n$) be all the lower (upper) covers of d in L with respect to a planar embedding $e(L)$ of L. The join of any two distinct s_j's is the same since otherwise $\mathbf{A}_3$ would be a cover-preserving sublattice of M; the dual statement holds for the r_i's. If b is not s_1 or s_n, and there is both a left dangle z_1 and a right dangle z_2 on $[d, s_1 \vee s_n]$, then a cover-preserving sublattice of M is isomorphic to $\mathbf{J}$, $\mathbf{K}$ or $\mathbf{K}^d$. For example, if z_1 and z_2 are both down-dangles, they can be chosen so that $z_1 \prec s_1$ and $z_2 \prec s_n$. If $a = r_1$, then $\{a, z_1, c, b, s_1 \vee s_n\}$ would be a nonmodular sublattice of M which is impossible; therefore, $\mathbf{J} \cong \{r_1 \wedge r_m, r_1, a, r_m, d, z_1, c, z_2, s_1, b, s_n, s_1 \vee s_n\}$.

We now show that the above situation or its dual actually must occur. Otherwise, $a = r_1 \neq r_m$ and $b = s_n \neq s_1$ in some planar embedding of L. For example, if there is no right dangle on $[d, s_1 \vee s_n]$ then b and s_n can be interchanged, giving a planar embedding of L in which b is visible from a. If d were a splitting element of L, then b would be visible from a in some planar embedding of L, a contradiction. We can assume that there is y in L with $y \lambda d$. By Lemma 1.11, there is a face containing d and some x in L with $x \lambda d$. In order that this face be modular, $x \wedge d \prec x, d \prec x \vee d$; then, since $a = x \wedge d$, $\{a, x, c, b, s_1 \vee b\}$ is a nonmodular sublattice of M, a contradiction.

None of the lattices $\underset{\sim}{A}_3$, $\underset{\sim}{J}$, $\underset{\sim}{K}$ or $\underset{\sim}{K}^d$ contain one of the others as a subposet since each of the latter three consist of 12 elements and are dismantlable. The second statement now follows immediately. □

In order to extend Theorem 6.5 to infinite lattices, we need the following lemma.

LEMMA 6.6. If M is a modular lattice that does not contain $\underset{\sim}{A}_3$, $\underset{\sim}{J}$, $\underset{\sim}{K}$ or $\underset{\sim}{K}^d$ as a subposet, then any finitely generated sublattice of M is finite.

This lemma and its proof are based on an idea of R. Wille [9, Theorem 5]. The only real novelty here is Lemma 6.8.

LEMMA 6.7. (cf. [9, Lemma 4]) Let M be a modular lattice with no subposet isomorphic to $\underset{\sim}{J}$. If $\{a, c_1, c_2, c_3, e\}$ is a nondistributive sublattice of M with $a < c_i < e$, then $a \prec c_i \prec e$ $(i = 1, 2, 3)$.

Proof. If the conclusion were false, then, by modularity, there is b_1 in M with $a < b_1 < c_1$. Let $b_2 = (b_1 \vee c_3) \wedge c_2$, $b_3 = (b_1 \vee c_2) \wedge c_3$, $c_4 = (b_1 \vee c_2) \wedge (b_1 \vee c_3)$, and $d_i = c_i \vee c_4$ for $i = 1, 2, 3$. As noted in Lemma 4 of [9], $a < b_i < c_i$ and $c_i \wedge c_4 = b_i$ for $i = 1, 2, 3$. Since for $i = 2$ or 3 , $d_i = b_1 \vee c_i$, it follows similarly that $c_i < d_i < e$. Also, $c_1 < d_1 < e$ since otherwise $c_1 \vee c_4 = e$, implying that both c_4 and d_2 are comparable relative complements of c_1 in $[b_1, e]$. Therefore, $\{a, b_1, b_2, b_3, c_1, c_2, c_3, c_4, d_1, d_2, d_3, e\}$ is isomorphic to $\underset{\sim}{J}$. □

LEMMA 6.8. Let M be a modular lattice with no subposet isomorphic to $\underset{\sim}{A}_3$. If a, b_1, b_2, c, z_1 and z_2 are elements of M such that $b_1 \parallel b_2$, $b_1 \wedge b_2 = a$, $b_1 \vee b_2 = c$, $z_1 \wedge c = b_1$ and $z_2 \wedge c = b_2$, then $z_1 \vee c \parallel z_2 \vee c$.

Proof. Note that $z_1 \parallel b_2$ and $z_2 \parallel b_1$. If $z_1 \wedge z_2 \parallel c$, then $\{a, z_1, b_1, c, b_2, z_2, z_1 \wedge z_2, z_1 \vee z_2 \vee c\}$ would be isomorphic to $\underset{\sim}{A}_3$. Therefore, $z_1 \wedge z_2 \leq c$; hence, $z_1 \wedge z_2 = (z_1 \wedge c) \wedge (z_2 \wedge c) = b_1 \wedge b_2 = a$. Suppose $z_1 \vee c \leq z_2 \vee c$. Then, $z_2 \vee z_1 = z_2 \vee b_2 \vee z_1 \vee b_1 = z_2 \vee c$, and $z_2 \vee b_1 = z_2 \vee b_2 \vee b_1 = z_2 \vee c$. Hence, both b_1 and z_1 are comparable relative complements of z_2 in $[a, z_2 \vee c]$, a contradiction. □

Proof of Lemma 6.6. If $M_5 = \{a, b_1, b_2, b_3, c\}$ is a sublattice of M with $a \prec b_i \prec c$ $(i = 1, 2, 3)$, then some b_i is doubly irreducible in M . For example, if all the b_i's are join-reducible, then there are z_1, z_2, z_3 in M such that $z_i \wedge c = b_i$ $(i = 1, 2, 3)$. By Lemma 6.7 , $z_1 \vee c$, $z_2 \vee c$, and $z_3 \vee c$ are distinct pairwise incomparable elements; therefore, $J \cong \{a, b_1, b_2, b_3, c, z_1, z_2, z_3, z_1 \vee c, z_2 \vee c, z_3 \vee c, z_1 \vee z_2 \vee z_3 \vee c\}$. The other cases are similar.

Let S be a finite subset of M. Let T be the set of all elements x of S that are doubly irreducible in M and appear as some b_1 in some sublattice of M of the form M_5 as above; set $x_0 = a$ and $x_1 = c$. Let $T_k = \{x_k \mid x \in T\}$ for $k = 0, 1$. By Lemmas 6.7 and 6.8, $(S - T) \cup T_0 \cup T_1$ generates a distributive sublattice N of M. Then, the finite sublattice $N \cup T$ of M includes S. □

THEOREM 6.9. A modular lattice has dimension ≤ 2 if and only if it does not contain one of the modular lattices $\underset{\sim}{A}_3$, $\underset{\sim}{J}$, $\underset{\sim}{K}$ or $\underset{\sim}{K}^d$ as a subposet. Moreover, this is the minimum such list of modular lattices.

Proof. The proof of Theorem 6.1 is used with Theorem 6.5 replacing Theorem 1. The only other difference is that L is the sublattice generated by S; L is finite by Lemma 6.6. □

In [6], we proved that every finite dismantlable lattice contains two incomparable doubly irreducible elements. We now show that this can be sharpened for nonplanar finite dismantlable lattices. To this end, we need the following lemma.

LEMMA 6.10. Let S be a sublattice of a finite lattice L that can be obtained by dismantling L. If S contains n pairwise incomparable doubly irreducible elements, then so does L.

Proof. Let $a_1, a_2, \cdots, a_n$ be n pairwise incomparable doubly irreducible elements in S. By induction, we can assume that $L = S \cup \{c\}$, where c is doubly irreducible in L. All the elements $a_1, a_2, \cdots, a_n$ would be doubly irreducible in L unless c covers or is covered by one of them. Therefore, we can assume that $a_1 \prec c$; a_1 is thus the unique lower cover of c in L. It follows from $a_1 \| a_i$ that $c \| a_i$ for every $i = 2, 3, \cdots, n$. Thus $\{a_2, a_3, \cdots, a_n, c\}$ is an n-element set of pairwise incomparable doubly irreducible elements in $S \cup \{c\}$. $\square$

THEOREM 6.11. Any nonplanar finite dismantlable lattice contains at least three pairwise incomparable doubly irreducible elements.

<u>Proof.</u> Let M be a nonplanar finite dismantlable lattice. By virtue of Lemma 6.10, we can assume that $M = L \cup \{c\}$, where c is doubly irreducible in M, $a \prec c \prec b$ in M, b is not visible from a in L, and L is planar with a planar embedding $e(L)$. We denote the left (right) boundary of a region R in L by $\ell(R)$ $(r(R))$.

Let $C = \ell([0, a] \cup [a, b] \cup [b, 1])$, $D = r([0, a] \cup [a, b] \cup [b, 1])$, S be the left side of C, and T be the right side of D. It suffices to show that $S \cup T - ([0, a] \cup [b, 1])$ contains two incomparable doubly irreducible elements x and y in L, since $\{x, y, c\}$ would then consist of three pairwise incomparable doubly irreducible elements in M. We now carry out the proof in a sequence of simple steps, each of which we elaborate upon at most briefly.

(i) <u>$\ell(S) = \ell(L)$</u>.

(ii) <u>There is $d \in \{a, b\}$ such that $d \notin \ell(L)$</u>. Since b is not visible from a, there is a left dangle on $[a, b]$.

(iii) <u>There is $z \in \ell(S) - C$ which is doubly irreducible in L</u>. By Proposition 2.6, there is $z \in \ell(L)$ such that z is doubly irreducible in L and $z \parallel d$.

Let u be a minimal such z and choose v analogously in $r(T) - D$.

(iv) <u>Without loss of generality, $u > v$</u>. If $u \parallel v$, we are done.

Let $E = r([v, u])$, $h \in C \cap E$, and choose k maximal in $D \cap E$. Let $H = (C \cap [0, h]) \cup (E \cap [h, u]) \cup (\ell(L) \cap [u, 1])$ and $K = (r(L) \cap [0, v]) \cup (E \cap [v, k]) \cup (D \cap [k, 1])$.

(v) <u>$b > h \geq k > a$</u>.

(vi) <u>$\ell(L) = H$</u>. If there were $x \in H$, $x < u$, and $x \notin \ell(L)$, then there would be a $y \in \ell(S) - C$, doubly irreducible in L, $y \parallel x$, and $y < u$, contradicting the minimality of u.

(vii) <u>Without loss of generality, $r(L) = K$</u>. If $r(L) \neq K$, there is $x \in K$ such that $x \notin r(L)$. Then $x > v$ and there is $y \in r(T) - D$, doubly irreducible in L, $y \parallel x$, and $y > v$. If $y \parallel u$, we are done. If $y < u$, then $y \in [v, u]$ which is impossible since $y \notin K$; if $y > u$, then a maximal chain from u to y crosses C at $z \leq b$ so that $u \in [a, b]$, which is impossible.

(viii) <u>Without loss of generality, $[k, b]$ is a chain</u>. Otherwise, there is $y \in (k, b)$ such that $y \notin r([k, b])$. By the proof of Proposition 2.6, there is a doubly irreducible element x of L such that $x \parallel y$ and $x \in r(L)$; thus, $x \in r([k, d]) \subseteq D$. Since $x > k$, $x \parallel u$ by the choice of k and we would be done.

(ix) <u>h is a splitting element of L ; that is, $L = [0, h] \cup [h, 1]$</u> .

Finally, by reflecting $[h, 1]$ in $e(L)$ we obtain a planar embedding of L in which b is visible from a , a contradiction which completes the proof. $\square$

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