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THE RATIO OF NON-CENTRAL F DISTRIBUTIONS AND ITS APPLICATIONS

by

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ABSTRACT

The distribution of the ratio of two independent non-central F variables arises as the test statistic in comparing experiments in terms of sensitivity, in testing for interaction in contingency tables and in comparing multiple correlation coefficients.

Bradley and Schumann (1957a, 1957b, 1959) have studied the distribution when the degrees of freedom are equal. In this thesis an extension is made to the cases where the degrees of freedom are unequal. The exact distribution and an approximation to it are found. The upper percentage points for the approximate distribution are tabulated for equal degrees of freedom ($\alpha = .01$) and for unequal degrees of freedom ($\alpha = .05$).

Examples are given in the three areas of application and the particular values of the test statistics required are calculated.

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I Introduction and Summary

1.1 Problem and Summary of the Results

The distribution of the ratio of two independent non-central F variables arises as the distribution of the test statistic in several practical situations; for example, the comparison of experiments in terms of sensitivity, tests for interaction in a 2x2 contingency table and comparison of multiple correlation coefficients.

Schumann and Bradley (1957a, 1957b, 1959) have studied this distribution in detail when the degrees of freedom are equal. Their work includes an approximation to this distribution and a tabulation of the percentage points of the approximate distribution for selected α.

If the restriction of equal degrees of freedom were removed, this distribution could be applied in more general situations. In this thesis, the exact distribution of the ratio of two non-central F distributions is found. Also, an approximation is considered and the percentage points for unequal degrees of freedom are tabulated. Consideration is then given to the applications with equal and unequal degrees of freedom. The three applications considered are a test of sensitivity, a test of equality of regression coefficients and lastly, a test for interaction in the 2x2 contingency table.

1.2 Review of Literature

The ratio of two non-central F distributions has been studied by various people, namely Cochran (1943), Schumann and Bradley (1957a, 1957b, 1959), Dar (1962, 1964), Gart (1966) and Schoemann and Schumann (1959).

Cochran (1943) first discussed the comparison of experiments in terms of their sensitivities. In this context one experiment is said to be more sensitive than another if it better demonstrates treatment effects (Model I) or the existence of a between treatment component of variance Model II. He noted that a comparison of sensitivities should depend on the magnitude of treatment effects and the errors associated with them. Cochran, using a result of Pitman, showed how sensitivities could be compared and stated that, in general, the comparison should depend on a significance test of a hypothesis on two non-central variance-ratio distributions.

The most extensive work on this problem has been done by Bradley and Schumann. In their paper on the theory of the comparison of sensitivities (1957a), they first obtain the distribution of the ratio of two F distributions.

$$g(w;a,b,\lambda_1,\lambda_2) = e^{-\lambda_1-\lambda_2} \sum_{rs}\sum \frac{\lambda_1^r}{r!}\frac{\lambda_2^s}{s!}\,[B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot w^{a+r-1} B(2a+r+s,b)\,{}_2F_1(a+b+r,\ 2a+r+s,\ 2a+2b+r+s,\ 1-w)$$

$$0 \le w \le 1$$

$$= e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r}{r!} \frac{\lambda_2^s}{s!} [B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot w^{a+r-1} w^{-(2a+r+s)} B(2a+r+s,\ 2b)\,{}_2F_1(a+b+s,\ 2a+r+s,\ 2a+2b+r+s,\ \tfrac{w-1}{w})$$

$$1 \leq w \leq \infty \qquad (1.1)$$

where 2a, 2b are the degrees of freedom and λ_1, λ_2 are non-centrality parameters.

In order to find the distribution function for the ratio of two non-central F distributions, an expansion in a binomial series is used and then the integration for $G(w_0)$ is done.

$$G(w_0) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r}{r!} \frac{\lambda_2^s}{s!} [B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot \int_0^\infty y^{a+s-1}(1+y)^{-(a+b+s)}(w_0 y)^{a+r}(1+w_0 y)^{-(a+r)}$$

$$\cdot \left[\frac{1}{a+r} - \frac{(b-1)w_0 y}{(a+r+1)(1+w_0 y)} + \frac{(b-1)(b-2)(w_0 y)^2}{2!(a+r+2)(1+w_0 y)^2} - \cdots\cdots\right] dy.$$

Also considered in some detail is the ratio of central F distributions.

$$g(w;a,b) = \frac{w^{a-1}}{[B(a,b)]^2} B[2a,\ 2b]\,{}_2F_1(a+b,\ 2a,\ 2a+2b,\ 1-w)$$

$$0 \leq w \leq 1$$

$$= w^{-2a} B[2a,\ 2b]\,{}_2F_1(a+b,\ 2a,\ 2a+2b,\ \tfrac{w-1}{w})$$

$$1 \leq w \leq \infty \qquad (1.2)$$

where 2a, 2b are the degrees of freedom.

For the ratio of central F distributuions, the distribution function is:

$$G(w_0) = [B(a,b)]^2 \int_0^\infty y^{a-1}(1+y)^{-(a+b)}(w_0y)^a(1+w_0y)^{-a}$$

$$\cdot[\frac{1}{a} - \frac{(b-1)w_0y}{(a+1)(1+w_0y)} + \frac{(b-1)(b-2)(w_0y)^2}{2!(a+2)(1+w_0y)^2} - \ldots\ldots]dy.$$

Also developed by Bradley and Schumann were tests of hypotheses on sensitivity. Since Model I of the analysis of variance is considered ($\sum_{i=1}^{t} \tau_{ij} = 0$, where τ_{ij} is the effect of the j^{th} of t treatments in experiment i), if σ is the population experimental error, then

$$\lambda_i = k \sum_{i=1}^{t} \tau_{ij}^2/2\sigma_i^2$$

where k is the number of observations in each treatment mean. λ, therefore, incorporates the magnitudes of treatment effects in the scale and the experimental error associated with the scale. When the experiments are similar (i.e., have the same degrees of freedom), the null hypothesis of equal sensitivities is equivalent to the hypothesis that $\frac{\lambda_1}{k_1} = \frac{\lambda_2}{k_2}$.

Also considered were tests of hypotheses on multiple correlation coefficients. Since $F_i = \frac{bR_i^2}{a(1-R_i^2)}$ (where R_i is the multiple correlation coefficient), a ratio of two such distributions has the form (1.1), with

$$\lambda_i = \frac{(a+b)\rho_i^2}{(1-\rho_i)^2}$$

$$a = p/2$$

$$b = (N-p-1)/2$$

where p is the number of independent variables. Also,

$$w = \frac{R_1^2(1-R_2^2)}{R_2^2(1-R_1^2)} \; .$$

From the two distributions derived ((1.1) and (1.2)), tables of w_0 are derived for various values of the degrees of freedom and the non-centrality parameter.

In their second paper on the comparison of sensitivities, they discuss areas of application for sensitivity comparison: taste testing, chemistry, life testing and agronomy. It is noted that if experimental techniques interact with the treatments under comparison, difficulties in interpretation may arise.

In their third paper, Bradley and Schumann consider ways of comparing sensitivities when Model II of the analysis of variance applies and give extended tables. With expected variability among groups, the comparison should be done through a comparison of the relative group variabilities. $\sigma_0^2 \neq 0$ is assumed and $\gamma_j = k_j\sigma_{0j}^2/\sigma^2$, j = 1, 2, is defined as the parameter of relative group variability in similar experiments. The null hypothesis then is H_0: $\gamma_1/k_1 = \gamma_2/k_2$ where k_j is the number of observations in each group mean in experiment j.

Dar (1962) has considered large sample tests based on normal approximation and also (1964) discussed certain dependent experiments of similar structure. He gives a test statistic for the case in which there are more than two dependent experiments.

Gart (1966) considers a 2x2 contingency table with no fixed marginals. He states that the null hypothesis of no interaction may be stated $\psi = \dfrac{p_{11}p_{22}}{p_{12}p_{21}}$. Using results derived in his paper, he states that $\psi \sim (u_{11}u_{22})/(u_{12}u_{21})$ where the u_{ij} are mutually independent chi-square variates with degrees of freedom $2n_{ij}+1$. Therefore, this may be written:

$$\frac{(2n_{12}+1)(2n_{21}+1)}{(2n_{11}+1)(2n_{22}+1)}\,\psi \sim \frac{F(2n_{11}+1,\ 2n_{12}+1)}{F(2n_{21}+1,\ 2n_{22}+1)}$$

where the F's are independent variates. It is suggested that the null hypothesis of no interaction: $\psi = 1$, should be tested by using

$$\{(2n_{12}+1)(2n_{21}+1)\}/\{(2n_{11}+1)(2n_{22}+1)\}$$

as a test statistic, distributed as the ratio of two independent F distributions.

Schoemann and Schumann (1969) considered the comparison of two methods of measurement in the case where the methods are applied to the same experimental units. The exact distribution, as well as approximations, are found for the test statistic ($w = \dfrac{w_1}{w_2}$ where

$w_s = (\frac{A_{ss}}{m_s} \frac{m_s}{B_{ss}})$ and $\frac{A_{ss}}{m_1}$ and $\frac{B_{ss}}{m_1}$ are mean sums of squares for property Y_s). As well, tests of hypotheses and critical values are discussed. Further, it is shown that these results may be used in the case where the methods are applied to separate subsamples of the same experimental units and to the case where the different methods are applied to independent experimental units.

II. Exact Distribution

2.1 Introduction

In this chapter the exact distribution of the ratio of two non-central F distributions with unequal degrees of freedom is derived.

Two independent non-central F distributions are considered and therefore the joint distribution is obtained by taking the product of the two probability density functions. Through suitable transformations, the joint distribution of the ratio (w) of the two non-central F distributions and a variable, y, is obtained. In order to write the joint distribution in terms of confluent hypergeometric series, one transformation is made over the area $0 < w < c$ and a second for $c < w < \infty$. Then by integrating over y in each area, the marginal distribution of w is found.

Finally, the distribution function is obtained. Again this is found for the two areas $0 < w < c$ and $c < w < \infty$. In the area $0 < w < c$ the case is also considered in which the entire area is integrated over.

2.2 Derivation of the Exact Distribution

A "doubly" non-central F distribution is derived by considering the ratio of two independently distributed non-central χ^2 variates. Consider two such variates z_1, z_2 with degrees of freedom ν_1, ν_2 and non-centrality parameters $\frac{\tau_1^2}{2}$, $\frac{\tau_2^2}{2}$ respectively. Let

$$u = z_1/z_2 \qquad \tau^2 = \tau_1^2 + \tau_2^2 \qquad \nu = \nu_1 + \nu_2$$

$$h(u,v) = \frac{e^{-\frac{1}{2}(uv+\tau_1^2)}(uv)^{\frac{1}{2}\nu_1-1}}{2^{\frac{\nu_1}{2}}\Gamma\{\frac{1}{2}(\nu_2-1)\}\Gamma(\frac{1}{2})} \sum_{r=0}^{\infty} \frac{(\tau_1^2)^r}{(2r)!}(uv)^r B\{\tfrac{1}{2}(\nu_1-1), \tfrac{1}{2}+r\}$$

$$\cdot \frac{e^{-\frac{1}{2}(v+\tau_2^2)} v^{\frac{1}{2}\nu_2-1}}{2^{\frac{\nu_2}{2}}\Gamma\{\frac{1}{2}(\nu_2-1)\}\Gamma(\frac{1}{2})} \sum_{s=0}^{\infty} \frac{(\tau_2^2)^s v^s}{(2s)!} B\{\tfrac{1}{2}(\nu_2-1), \tfrac{1}{2}+s\}v$$

$$h(u) = \frac{e^{-\frac{1}{2}\tau^2}}{2^{\frac{\nu}{2}}} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\tau_1^r}{(2r)!}\,\frac{\tau_2^s}{(2s)!}\,\frac{\Gamma(\frac{1}{2}+r)\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+r)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+s)}$$

$$\cdot\left\{\int_0^\infty e^{-\frac{1}{2}v(1+u)} v^{\frac{1}{2}\nu+r+s-1}\,dv\right\} u^{\frac{1}{2}\nu_1+r-1}\,du. \tag{2.1}$$

Since the integral in (2.1) is equal to

$$\frac{\Gamma(\frac{1}{2}\nu+r+s)}{(\frac{1+u}{2})^{\frac{1}{2}\nu+r+s}}$$

and

$$\frac{\Gamma(\frac{1}{2}+r)\Gamma(\frac{1}{2}+s)2^{r+s}}{(2r)!(2s)!\{\Gamma(\frac{1}{2})\}^2} = \frac{1}{2^{r+s}r!s!}$$

$$h(u) = e^{-\frac{1}{2}\tau^2} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!}\,\frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!}\, u^{\frac{1}{2}\nu_1+r-1}\left(\frac{1}{1+u}\right)^{\frac{1}{2}\nu+r+s}$$

$$\cdot \frac{1}{B(\frac{1}{2}\nu_1+r, \frac{1}{2}\nu_2+s)}\,.$$

If $\tau_2^2 = 0$, then we have a (singly) non-central $F(\nu_1, \nu_2; \tau_1^2)$.

$$h(u) = e^{-\frac{\tau_1^2}{2}} \sum_{r=0}^{\infty} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} u^{\frac{1}{2}\nu_1+r-1} \left(\frac{1}{1+u}\right)^{\frac{1}{2}\nu+r} \frac{du}{B(\frac{1}{2}\nu_1+r, \frac{1}{2}\nu_2)} .$$

This can be re-written in the form

$$h(u) = \frac{e^{-\frac{\tau_1^2}{2}} u^{\frac{\nu_1}{2}-1} (1+u)^{-(\frac{\nu_1+\nu_2}{2})}}{B[\frac{\nu_1}{2}, \frac{\nu_2}{2}]} {}_1F_1\left[\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \frac{\frac{\tau_1^2}{2} u}{1+u}\right] \tag{2.2}$$

where ${}_1F_1$ is the hypergeometric series of the form

$${}_1F_1(\alpha, \beta, X) = 1 + \frac{\alpha}{\beta} X + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{X^2}{2!} + \ldots . \qquad \text{(Erdelyi, 1953, p.202).}$$

Suppose we consider now the ratio of two non-central F distributions. Define $u_1 = \frac{\nu_1}{\nu_2} F$ and $u_2 = \frac{\gamma_1}{\gamma_2} F$ where (ν_1, ν_2) and (γ_1, γ_2) are the sets of degrees of freedom for the distribution and $\tau_1^2/2$ and $\tau_2^2/2$ are the corresponding non-centrality parameters.

u_1 has the density (from (2.2))

$$h(u_1) = \frac{e^{-\frac{\tau_1^2}{2}} u_1^{\frac{\nu_1}{2}-1} (1+u_1)^{-(\frac{\nu_1+\nu_2}{2})}}{B[\frac{\nu_1}{2}, \frac{\nu_2}{2}]} {}_1F_1\left[\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \frac{\frac{\tau_1^2}{2} u_1}{1+u_1}\right]$$

and u_2 has the density

$$h(u_2) = \frac{e^{-\frac{\tau_2^2}{2}} u_2^{\frac{\gamma_1}{2}-1} (1+u_2)^{-(\frac{\gamma_1+\gamma_2}{2})}}{B[\frac{\gamma_1}{2}, \frac{\gamma_2}{2}]} {}_1F_1\left[\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_2}{2}, \frac{\frac{\tau_2^2}{2} u_2}{1+u_2}\right].$$

Since u_1 and u_2 are independent, then the joint distribution of u_1

and u_2 is the product of $h(u_1)$ and $h(u_2)$:

$$g(u_1,\ u_2) = h(u_1)\cdot h(u_2)$$

$$= e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \left\{ \frac{1}{B[\frac{\nu_1}{2},\ \frac{\nu_2}{2}]} u_1^{\frac{\nu_1}{2}-1} (1+u_1)^{-(\frac{\nu_1+\nu_2}{2})} \right.$$

$$\left. \cdot {}_1F_1\left[\frac{\nu_1+\nu_2}{2},\ \frac{\nu_2}{2},\ \frac{\frac{\tau_1^2}{2} u_1}{1+u_1}\right] \frac{1}{B[\frac{\gamma_1}{2},\ \frac{\gamma_2}{2}]} u_2^{\frac{\gamma_1}{2}-1} (1+u_2)^{-(\frac{\gamma_1+\gamma_2}{2})} \right\}$$

$$\cdot {}_1F_1\left[\frac{\gamma_1+\gamma_2}{2},\ \frac{\gamma_2}{2},\ \frac{\frac{\tau_2^2}{2} u_2}{1+u_2}\right] .$$

Since

$${}_1F_1\left[\frac{\nu_1+\nu_2}{2},\ \frac{\nu_1}{2},\ \frac{\frac{\tau_1^2}{2} u_1}{1+u_1}\right] = \sum_{r=0}^{\infty} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{u_1^r}{(1+u_1)^r} \frac{(\frac{\nu_1+\nu_2}{2}+r-1)!(\frac{\nu_1}{2}-1)!}{(\frac{\nu_1+\nu_2}{2}-1)!(\frac{\nu_1}{2}+r-1)!}$$

and similarly,

$${}_1F_1\left[\frac{\gamma_1+\gamma_2}{2},\ \frac{\gamma_1}{2},\ \frac{\frac{\tau_2^2}{2} u_2}{1+u_2}\right] = \sum_{s=0}^{\infty} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{u_2^s}{(1+u_2)^s} \frac{(\frac{\gamma_1+\gamma_2}{2}+s-1)!(\frac{\gamma_1}{2}-1)!}{(\frac{\gamma_1+\gamma_2}{2}-1)!(\frac{\gamma_1}{2}+s-1)!} .$$

$$g(u_1,u_2) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{rs}\sum \left\{ \frac{1}{B(\frac{\nu_1}{2},\ \frac{\nu_2}{2})} u_1^{\frac{\nu_1}{2}-1} (1+u_1)^{-(\frac{\nu_1+\nu_2}{2})} \right.$$

$$\left. \cdot \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{u_1^r}{(1+u_1)^r} \frac{(\frac{\nu_1+\nu_2}{2}+r-1)!(\frac{\nu_1}{2}-1)!}{(\frac{\nu_1+\nu_2}{2}-1)!(\frac{\nu_1}{2}+r-1)!} \right\} \left\{ \frac{1}{B(\frac{\gamma_1}{2},\ \frac{\gamma_2}{2})} u_2^{\frac{\gamma_1}{2}-1} \right.$$

$$\cdot (1+u_2)^{-(\frac{\gamma_1+\gamma_2}{2})} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{u_2^s}{(1+u_2)^s} \frac{(\frac{\gamma_1+\gamma_2}{2}+s-1)!(\frac{\gamma_1}{2}-1)!}{(\frac{\gamma_1+\gamma_2}{2}-1)!(\frac{\gamma_1}{2}+s-1)!}$$

$$= e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} u_1^{\frac{\nu_1}{2}+r-1} u_2^{\frac{\gamma_1}{2}+s-1} (1+u_1)^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)}$$

$$\cdot (1+u_2)^{-(\frac{\gamma_1+\gamma_2}{2}+s)} \frac{1}{B[\frac{\nu_1}{2}+r, \frac{\nu_2}{2}]} \frac{1}{B[\frac{\gamma_1}{2}+s, \frac{\gamma_2}{2}]} .$$

We now wish to find the distribution of the ratio $w = c \frac{u_1}{u_2}$, where $c = \frac{\gamma_1\nu_2}{\nu_1\gamma_2}$. We shall consider the range of w in two regions (i) and (ii). For region (i) with $0 < w < c$, let

$$y = \frac{x-1}{w-x} \frac{\gamma_1}{\gamma_2}$$

$$\frac{\partial y}{\partial x} = \frac{\gamma_1}{\gamma_2} \frac{(w-1)}{(w-x)^2}$$

$$g(w,y) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \left\{\frac{wy}{c}\right\}^{\frac{\nu_1}{2}+r-1} y^{\frac{\gamma_1}{2}+s-1}$$

$$\cdot \left\{1+\frac{wy}{c}\right\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)} \{1+y\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)} \frac{1}{B[\frac{\nu_1}{2}+r, \frac{\nu_2}{2}]} \frac{1}{B[\frac{\gamma_1}{2}+s, \frac{\gamma_2}{2}]} \frac{y}{c}$$

$$= e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} w^{\frac{\nu_1}{2}+r-1} y^{\frac{\nu_1+\gamma_1}{2}+r+s-1} c^{-(\frac{\nu_1}{2}+r)}$$

$$\cdot\{1+\frac{wy}{c}\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)}\{1+y\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)}\frac{1}{B[\frac{\nu_1}{2}+r,\frac{\nu_2}{2}]}\frac{1}{B[\frac{\gamma_1}{2}+s,\frac{\gamma_2}{2}]}\;.$$

Now the marginal distribution of w can be obtained by integrating over y.

$$g(w) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)}\sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!}\frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!}\frac{w^{\frac{\nu_1}{2}+r-1}c^{-(\frac{\nu_1}{2}+r)}}{B[\frac{\nu_1}{2}+r,\frac{\nu_2}{2}]B[\frac{\gamma_1}{2}+s,\frac{\gamma_2}{2}]}$$

$$\cdot\int_0^\infty y^{\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s-1}(1+\frac{wy}{c})^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)}(1+y)^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)}dy. \quad (2.3)$$

From Erdelyi et al (1953, p.60), we have

$$\int_0^\infty y^{\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s-1}\{1+y\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)}\{1+\frac{wy}{c}\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)}dy$$

$$=\frac{\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s)}\,{}_2F_1(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r,\ \frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s;$$

$$\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s;\ 1-\frac{w}{c}).$$

Since

$${}_2F_1(a_1,a_2,b_1,x) = \sum_{n=0}^{\infty}\frac{\frac{\Gamma(a_1+n)}{\Gamma(a_1)}\ \frac{\Gamma(a_2+n)}{\Gamma(a_2)}}{\frac{\Gamma(b_1+n)}{\Gamma(b_1)}}\frac{x^n}{n!}\,, \qquad \begin{array}{l}\text{Re } a > \text{Re } b > 0\\ \arg z < \pi\end{array}$$

$${}_2F_1(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r,\ \frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s;\ \frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s;\ 1-\frac{w}{c})$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r+n)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s+n)\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s)}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s)\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s+n)}$$

$$\cdot \frac{(1- \frac{w}{c})^n}{n!}$$

in equation (2.3). Thus we obtain

$$g(w) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{r}\sum_{s} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{w^{\frac{\nu_1}{2}+r-1} c^{-(\frac{\nu_1}{2}+r)}}{B[\frac{\nu_1}{2}+r, \frac{\nu_2}{2}]B[\frac{\gamma_1}{2}+s, \frac{\gamma_2}{2}]}$$

$$\cdot \frac{\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s)\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_2}{2} + r+s)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r+n)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s+n)}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s+n)}$$

$$\cdot \frac{(1- \frac{w}{c})^n}{n!}, \qquad 0 < w < c.$$

For region (ii) with $c < w < \infty$, let

$$u_1 = y = \frac{w(x-1)}{(w-x)} \frac{\nu_1}{\nu_2}$$

$$\frac{\partial y}{\partial x} = \frac{w(w-1)}{(w-x)^2} \frac{\nu_1}{\nu_2}$$

$$g(w:y) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{r}\sum_{s} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{1}{B[\frac{\nu_1}{2} + r, \frac{\nu_2}{2}]B[\frac{\gamma_1}{2} +s, \frac{\gamma_2}{2}]}$$

$$\cdot y^{\frac{\nu_1}{2}+r-1} \{\frac{cy}{w}\}^{\frac{\gamma_1}{2}+s-1} \{1+y\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)} \{1+ \frac{cy}{2}\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)} \frac{cy}{w^2}$$

$$= e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{r}\sum_{s} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{c^{\frac{\gamma_1}{2}+s}\, w^{-(\frac{\gamma_1}{2}+s+1)}}{B[\frac{\nu_1}{2}+r,\ \frac{\nu_2}{2}]B[\frac{\gamma_1}{2}+s,\ \frac{\gamma_2}{2}]}$$

$$\cdot y^{\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s-1} \{1+y\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)} \{1+\frac{cy}{w}\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)} .$$

Again the marginal distribution of w can be obtained by integrating over y.

$$g(w) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{r}\sum_{s} \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{r!} \frac{c^{\frac{\gamma_1}{2}+s}\, w^{-(\frac{\gamma_1}{2}+s+1)}}{B[\frac{\nu_1}{2}+r,\ \frac{\nu_2}{2}]B[\frac{\gamma_1}{2}+s,\ \frac{\gamma_2}{2}]}$$

$$\cdot \int_0^\infty y^{\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s-1} \{1+y\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)} \{1+\frac{cy}{w}\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)}\, dy.$$

Now

$$\int_0^\infty y^{\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s-1} \{1+y\}^{-(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)} \{1+\frac{cy}{w}\}^{-(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)}\, dy$$

$$= \frac{\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s)}\, {}_2F_1(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s,\ \frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s;$$

$$\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s;\ 1-\frac{c}{w})$$

by the same reasoning as previously, and

$${}_2F_1(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s,\ \frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s;\ \frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s;\ 1-\frac{c}{w})$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s+n)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s+n)}{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s)}$$

$$\cdot \frac{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s)(1-\frac{c}{w})^n}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s+n)} .$$

So

$$g(w) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{rs}\sum \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{w^{-(\frac{\gamma_1}{2}+s+1)} c^{\frac{\gamma_1}{2}+s}}{B[\frac{\nu_1}{2}+r, \frac{\nu_2}{2}]B[\frac{\gamma_1}{2}+s, \frac{\gamma_2}{2}]}$$

$$\cdot \frac{\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s)\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s+n)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s+n)}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s+n)} \frac{(1-\frac{c}{w})^n}{n!}, \qquad c < w < \infty$$

We now need to find the distribution functions for $g(w)$ in each of the regions:

(i) $0 < w < c$

$$G(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_{rs}\sum \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s)\ (\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2}+s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot \sum_n \frac{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r+n)\ (\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r+s+n)}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r+s+n)} \int_0^{w_0} \frac{1}{c}\cdot(\frac{w}{c})^{\frac{\nu_1}{2}+r-1} \frac{(1-\frac{w}{c})^n}{n!}\, dw.$$

To facilitate computations we now want to cast this in the form of an incomplete Beta distribution. To do this, the following transformation is made:

$$t = \frac{w}{c} \Rightarrow w = ct$$

$$\frac{dw}{dt} = cdt$$

$$G(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r \sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{\Gamma(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2}+s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot \sum_n \frac{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r+n)\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s+n)}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s+n)\Gamma(n+1)} \int_0^{\frac{w_0}{c}} t^{\frac{\nu_1}{2}+r-1}(1-t)^n dt \tag{2.4}$$

(ii) $c < w < \infty$

$$G(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r \sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2}+s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot \sum_n \frac{\Gamma(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s+n)\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s+n)}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s+n)} \int_c^{w_0} \frac{1}{c}\cdot\left(\frac{c}{w}\right)^{\frac{\gamma_1}{2}+s+1} \frac{(1-\frac{c}{w})^n}{n!} dw. \tag{2.5}$$

Again equation (2.5) can be cast into the form of an incomplete Beta distribution by making two transformations.

$$v = \frac{c}{w} \Rightarrow w = \frac{c}{v}$$

$$\frac{dw}{dv} = \frac{-c}{v^2} dv$$

$$G(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2}+s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot\sum_n \frac{\Gamma(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s+n)\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s+n)}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s+n)} \int_0^{\frac{c}{w_0}} -v^{\frac{\gamma_1}{2}+s-1} \frac{(1-v)^n}{n!}\, dv.$$

Now let

$$q = 1 - v \Rightarrow v = 1 - q$$

$$\frac{dv}{dq} = -1\, dq$$

$$G(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r\sum_s \frac{\left(\frac{\tau_1^2}{2}\right)^r}{r!} \frac{\left(\frac{\tau_2^2}{2}\right)^s}{s!} \frac{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+r)\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{\nu_1}{2}+r)\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2}+s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot\sum_n \frac{\Gamma(\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+s+n)\Gamma(\frac{\nu_1}{2}+\frac{\gamma_1}{2}+r+s+n)}{\Gamma(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{\gamma_1}{2}+\frac{\gamma_2}{2}+r+s+n)n!} \int_0^{1-\frac{c}{w_0}} (1-q)^{\frac{\gamma_1}{2}+s-1} q^n dq.$$

In order to do significance tests, it is necessary to construct tables of w_0 such that $G(w_0) = 1 - \alpha$. w_0 may fall in either of the two regions $0 < w < c$ or $c < w < \infty$. If w_0 falls in the second region, the integral over the entire first region must be obtained; i.e., the integral is from 0 to c. In this case, the Beta function in (2.4) is complete and may be written

$$\frac{\Gamma(\frac{\nu_1}{2}+r)\Gamma(n+1)}{\Gamma(\frac{\nu_1}{2}+r+n+1)}.$$

This part of $G(w_0)$ will be denoted $G_p(w_0)$

$$G_p(w_0) = e^{-\left(\frac{\tau_1^2+\tau_2^2}{2}\right)} \sum_r \sum_s \left(\frac{\tau_1^2}{2}\right)^r \left(\frac{\tau_2^2}{2}\right)^s \frac{\Gamma(\frac{\gamma_1}{2} + \frac{\gamma_2}{2} + s)\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{\nu_2}{2})\Gamma(\frac{\gamma_1}{2} + s)\Gamma(\frac{\gamma_2}{2})}$$

$$\cdot \frac{\sum_n \Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + r + n)\Gamma(\frac{\nu_1}{2} + \frac{\gamma_1}{2} + r + s + n)}{\Gamma(\frac{\nu_1}{2} + \frac{\nu_2}{2} + \frac{\gamma_1}{2} + \frac{\gamma_2}{2} + r + s + n)\Gamma(\frac{\nu_1}{2} + r + n + 1)} .$$

In most cases, w_0 falls in the second region. Notable exceptions occur in the last row of Table 4.3.2. With such large degrees of freedom, most of the contribution to $G(w_0)$ comes from the first region.

III Approximate Distribution

3.1 Introduction

The calculation of w_0 for various degrees of freedom requires a large amount of computer time when the exact distribution is considered. In order to get a table of values of w_0, an approximation to the exact distribution is needed.

Patnaik (1949) found that a non-central F distribution can be approximated by a central F distribution by the following relation:

$$f\left(u;\ \nu_1,\ \nu_2,\ \frac{\tau_1^2}{2}\right) = \frac{1}{k}\, f\left(\frac{u}{k};\ a_1,\ \nu_2\right)$$

where

$$a_1 = \frac{\left(\nu_1 + \frac{\tau_1^2}{2}\right)^2}{\left(\nu_1 + 2\,\frac{\tau_1^2}{2}\right)} \quad \text{and} \quad k = \frac{\left(\nu_1 + \frac{\tau_1^2}{2}\right)}{\nu_1}.$$

The ratio of two non-central F distributions can then be found by considering the ratio of two such approximations. As for the exact distribution, the non-central F variates are independent and the joint distribution is just the product of the two probability density functions. And again transformations are made over the areas $0 < w < c$ and $c < w < \infty$, to obtain the joint distribution of w and y. Integration is then performed over y and the marginal distribution of w is found. Using methods similar to those discussed in Section 2.2, the distribution function is obtained for the approximate distribution.

3.2 Derivation of the Approximate Distribution

A non-central F may be approximated by a function of a central F. [See Patnaik (1949) for a more detailed discussion of the approximation.]

If $u_1 \sim F^*(\nu_1, \nu_2)$ where * denotes non-centrality, then

$$u_1 \sim \frac{1}{k_1} F\left(\underbrace{\frac{\left(\nu_1 + \frac{\tau_1^2}{2}\right)^2}{\nu_1 + 2\frac{\tau_2^2}{2}}}_{a_1}, \nu_2\right)$$

and

$$g(u_1) = \frac{1}{k_1} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})} \frac{1}{u_1} \frac{(\frac{a_1}{\nu_2} u_1)^{\frac{a_1}{2}}}{(1 + \frac{a_1}{\nu_2} u_1)^{\frac{a_1+\nu_2}{2}}}$$

where

$$k_1 = \frac{\nu_1 + \frac{\tau_1^2}{2}}{\nu_1} .$$

Similarly, if $u_2 \sim F^*(\gamma_1, \gamma_2)$, then

$$u_2 \sim \frac{1}{k_2} F\left(\underbrace{\frac{\left(\gamma_1 + \frac{\tau_2^2}{2}\right)^2}{\gamma_1 + 2\frac{\tau_2^2}{2}}}_{a_2}, \gamma_2\right)$$

and

$$g(u_2) = \frac{1}{k_2} \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \frac{\frac{1}{u_2}(\frac{a_2}{\gamma_2} u_2)^{\frac{a_2}{2}}}{(1+ \frac{a_2}{\gamma_2} u_2)^{\frac{a_2+\gamma_2}{2}}}.$$

Since u_1 and u_2 are independent,

$$h(u_1, u_2) = g(u_1)\cdot g(u_2)$$

$$= \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})\frac{1}{u_1 u_2}(\frac{a_1}{\nu_2} u_1)^{\frac{a_1}{2}}(\frac{a_2}{\gamma_2} u_2)^{\frac{a_2}{2}}}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})(1+ \frac{a_1}{\nu_2} u_1)^{\frac{a_1+\nu_2}{2}}(1+ \frac{a_2}{\gamma_2} u_2)^{\frac{a_2+\gamma_2}{2}}}$$

Let

$$w = \frac{u_1}{u_2}$$

$$u_1 = \frac{w(x-1)}{(w-x)} \qquad u_2 = \frac{(x-1)}{(w-x)}$$

$$|J| = \frac{(x-1)(w-1)}{(w-x)^3}$$

$$h(w,x) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})(\frac{a_1}{\nu_2})^{\frac{a_1}{2}}(\frac{a_2}{\gamma_2})^{\frac{a_2}{2}}}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})[1+ \frac{a_1}{\nu_2} \frac{w(x-1)}{(w-x)}]^{\frac{a_2+\nu_2}{2}}}$$

$$\cdot \frac{\left[\frac{w(x-1)}{w-x}\right]^{\frac{a_1}{2}-1}\left[\frac{x-1}{w-x}\right]^{\frac{a_2}{2}-1}(x-1)(w-1)}{\left[1+\frac{a_1}{\gamma_2}\frac{x-1}{w-x}\right]^{\frac{a_2}{2}+\frac{\gamma_2}{2}}(w-x)^3} .$$

In order that this may be written in terms of confluent hypergeometric series, transformations are made over two areas: $0 < w < c$ and $c < w < \infty$. For $0 < w < c$, let

$$y = \frac{(x-1)}{(w-x)}\frac{a_2}{\gamma_2} \qquad \frac{\partial y}{\partial x} = \frac{a_2}{\gamma_2}\frac{(w-1)}{(w-x)^3}$$

and

$$c = \frac{a_2\nu_2}{a_1\gamma_2} .$$

$$h(w,y) = \frac{1}{k_1k_2}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})\Gamma(\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})}\frac{\left[\frac{wy}{c}\right]^{\frac{a_1}{2}-1}y^{\frac{a_2}{2}-1}y}{\left[1+\frac{wy}{c}\right]^{\frac{a_1}{2}+\frac{\nu_2}{2}}\left[1+y\right]^{\frac{a_2}{2}+\frac{\gamma_2}{2}}c}$$

$$= \frac{1}{k_1k_2}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})\Gamma(\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})}\frac{w^{\frac{a_1}{2}-1}c^{-\frac{a_1}{2}}y^{\frac{a_1}{2}+\frac{a_2}{2}-1}}{\left[1+\frac{wy}{c}\right]^{\frac{a_1}{2}+\frac{\nu_2}{2}}\left[1+y\right]^{\frac{a_2}{2}+\frac{\gamma_2}{2}}} .$$

The marginal distribution of w can be obtained by integrating over y.

$$h(w) = \frac{1}{k_1k_2}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})\Gamma(\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} w^{\frac{a_1}{2}-1}c^{-\frac{a_1}{2}}\int_0^\infty y^{\frac{a_1}{2}+\frac{a_2}{2}-1}$$

$$\cdot[1+y]^{-(\frac{a_2}{2}+\frac{\gamma_2}{2})}[1+\frac{wy}{c}]^{-(\frac{a_1}{2}+\frac{\nu_2}{2})}dy.$$

Using the formula given by Erdelyi et al (1953, p.60)

$$\int_0^\infty y^{\frac{a_1}{2}+\frac{a_2}{2}-1}[1+y]^{-(\frac{a_2}{2}+\frac{\gamma_2}{2})}[1+\frac{wy}{c}]^{-(\frac{a_1}{2}+\frac{\nu_2}{2}]}dy$$

$$= {}_2F_1(\frac{a_1}{2}+\frac{\nu_2}{2}, \frac{a_1}{2}+\frac{a_2}{2}; \frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2}; 1-\frac{w}{c})\frac{\Gamma(\frac{a_1}{2}+\frac{a_2}{2})\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2})}$$

$$= \frac{\Gamma(\frac{a_1}{2}+\frac{a_2}{2})\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+n)}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})}\frac{\Gamma(\frac{a_1}{2}+\frac{a_2}{2}+n)}{\Gamma(\frac{a_1}{2}+\frac{a_2}{2})}$$

$$\cdot\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2}+n)}\frac{(1-\frac{w}{c})^n}{n!}.$$

Therefore, for $0 < w < c$, the marginal distribution is

$$h(w) = \frac{1}{k_1k_2}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})\Gamma(\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})}w^{\frac{a_1}{2}-1}c^{-\frac{a_1}{2}}\frac{\Gamma(\frac{a_1}{2}+\frac{a_2}{2})\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2})}$$

$$\cdot\sum_n\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+n)\Gamma(\frac{a_1}{2}+\frac{a_2}{2}+n)}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2})\Gamma(\frac{a_1}{2}+\frac{a_2}{2})}\frac{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2}+\frac{\nu_2}{2}+\frac{a_2}{2}+\frac{\gamma_2}{2}+n)}\frac{(1-\frac{w}{c})^n}{n!}$$

$$= \frac{1}{k_1k_2}\frac{\Gamma(\frac{a_2}{2}+\frac{\gamma_2}{2})\Gamma(\frac{\nu_2}{2}+\frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})}w^{\frac{a_1}{2}-1}c^{-\frac{a_1}{2}}$$

$$\cdot \sum_n \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)(1 - \frac{w}{c})^n}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!} .$$

For $c < w < \infty$, let

$$y = \frac{w(x-1)}{(w-x)} \frac{a_1}{\nu_2}$$

$$\frac{\partial y}{\partial x} = \frac{a_1}{\nu_2} \frac{w(w-1)}{(w-x)^2}$$

$$h(w,y) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \frac{y^{\frac{a_1}{2} - 1} [\frac{cy}{w}]^{\frac{a_2}{2} - 1}}{[1+y]^{\frac{a_1}{2} + \frac{\nu_2}{2}} [1 + \frac{cy}{w}]^{\frac{a_2}{2} + \frac{\gamma_2}{2}}} w^2$$

$$= \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \frac{y^{\frac{a_1}{2} + \frac{a_2}{2} - 1} c^{\frac{a_2}{2}} w^{-(\frac{a_2}{2} + 1)}}{[1+y]^{\frac{a_1}{2} + \frac{\nu_2}{2}} [1 + \frac{cy}{w}]^{\frac{a_2}{2} + \frac{\gamma_2}{2}}} .$$

Again the marginal distribution of w can be obtained by integrating over y.

$$h(w) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} c^{\frac{a_2}{2}} w^{-(\frac{a_2}{2} + 1)} \int_0^\infty y^{\frac{a_1}{2} + \frac{a_2}{2} - 1}$$

$$\cdot [1+y]^{-(\frac{a_1}{2} + \frac{\nu_2}{2})} [1 + \frac{cy}{w}]^{-(\frac{a_2}{2} + \frac{\gamma_2}{2})} dy.$$

Now again this can be put in series form,

$$\int_0^\infty y^{\frac{a_1}{2} + \frac{a_2}{2} - 1} [1+y]^{-(\frac{a_1}{2} + \frac{\nu_2}{2})} [1 + \frac{cy}{w}]^{-(\frac{a_2}{2} + \frac{\gamma_2}{2})} dy$$

$$= {}_2F_1\left(\frac{a_2}{2} + \frac{\gamma_2}{2}, \frac{a_1}{2} + \frac{a_2}{2}; \frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2}; 1 - \frac{c}{w}\right) \frac{\Gamma(\frac{a_1}{2} + \frac{a_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2})}$$

$$= \frac{\Gamma(\frac{a_1}{2} + \frac{a_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{a_1}{2} + \frac{\gamma_2}{2} + n)}{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})}$$

$$\cdot \frac{\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2})(1 - \frac{c}{w})^n}{\Gamma(\frac{a_1}{2} + \frac{a_2}{2})\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!} .$$

Therefore for $c < w < \infty$, the marginal distribution of w is

$$h(w) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} c^{\frac{a_2}{2}} w^{-(\frac{a_2}{2} + 1)}$$

$$\cdot \sum_n \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)(1 - \frac{c}{w})^n}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!}$$

It is now necessary to find the distribution function for $h(w)$ in each of the regions:

(i) $0 < w < c$

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)}$$

$$\cdot \int_0^{w_0} \left(\frac{w}{c}\right)^{\frac{a_1}{2} - 1} \frac{1}{c}\left(1 - \frac{w}{c}\right)^n dw.$$

The following transformation is made in order to cast this in the form of an incomplete Beta distribution. Let

$$t = \frac{w}{c} \qquad w = ct \qquad dw = c \cdot dt$$

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!}$$

$$\cdot \int_0^{\frac{w_0}{c}} t^{\frac{a_1}{2} - 1} (1-t)^n dt.$$

If $w_0 = c$, the integral reduces to $B(\frac{a_1}{2}, n+1)$. In that case

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2})\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)\Gamma(\frac{a_1}{2} + n + 1)} .$$

(ii) $c < w < \infty$

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{\nu_2}{2} + \frac{\gamma_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!}$$

$$\cdot \int_0^{w_0} (\frac{c}{w})^{\frac{a_2}{2} + 1} \frac{1}{c} (1 - \frac{w}{c})^n dw.$$

Again this form is cast into an incomplete Beta by making suitable transformations. Let

$$v = \frac{c}{w} \qquad w = \frac{c}{v} \qquad \frac{dw}{dv} = \frac{-c}{v^2}$$

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!}$$

$$\cdot \int_1^{\frac{c}{w_0}} - v^{\frac{a_2}{2} - 1} (1-v)^n dv.$$

Let

$$q = 1 - v \qquad\qquad v = 1 - q \qquad\qquad dv = -1 \cdot dq$$

$$H(w_0) = \frac{1}{k_1 k_2} \frac{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2})\Gamma(\frac{\gamma_2}{2} + \frac{\nu_2}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{\nu_2}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{\gamma_2}{2})} \sum_n \frac{\Gamma(\frac{a_2}{2} + \frac{\gamma_2}{2} + n)\Gamma(\frac{a_1}{2} + \frac{a_2}{2} + n)}{\Gamma(\frac{a_1}{2} + \frac{\nu_2}{2} + \frac{a_2}{2} + \frac{\gamma_2}{2} + n)n!}$$

$$\cdot \int_0^{1-\frac{c}{w_0}} q^n (1-q)^{\frac{a_2}{2} - 1} dq.$$

IV Computer Methods

4.1 Introduction

In this chapter flowcharts and programs are given for the determination of w_0 in both the exact distribution and the approximate distribution. There are two programs for each case, one which calculates the contribution to $G(w_0)$ from the region $0 < w < c$ and a second which calculates the contribution from the region $c < w < \infty$. Usually w_0 lies in the second region and so the iterative procedure to find w_0 is most often required in this region.

The recursive relationships used in the programs in order to reduce computer time and space are discussed in Section 4.2.

4.2 Recursive Relationships

In order to calculate values of w_0 for the ratio of the non-central F distribution, integrals must be found for values of s and $\ddot{n}$. To cut down on computer time and storage space, recursive relationships are used to minimize the number of calls to the subroutine DLGAM and to eliminate entirely the need for a subroutine for integration.

The integrals needed may be thought of as a matrix of integrals with indices ($\ddot{n}$, s). The first entry in the "matrix" is calculated by integration. Subsequent entries in the column are obtained through the use of a recursive relationship found in the Handbook of Mathematical Functions, p.944 (equation 26.5.16)

$$I_x(a+1,\ b) = I_x(a,b) - \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a(1-x)^b \tag{4.2.1}$$

where

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)}$$

and $B_x(a,b)$ is an incomplete Beta distribution and $B(a,b)$ is a complete Beta distribution. Therefore,

$$\frac{B_x(a+1,b)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b)} = \frac{B_x(a,b)\Gamma(a+b)}{\Gamma(a)\Gamma(b)} - \frac{\Gamma(a+b)x^a(1-x)^b}{\Gamma(a+1)\Gamma(b)} .$$

A more convenient form is

$$B_x(a+1,\ b) = \frac{a}{a+b} B_x(a,b) - \frac{x^a(1-x)^b}{(a+b)} .$$

In each column, the first two elements are saved until that column is complete. Other values are used as soon as they are created and then replaced by the next value. When the column is complete, the first two elements are used in a second recursive relationship to generate the first element in the next column. This value and the next one created are then the two values kept to obtain the initial value in the next column. The relationship is again taken from the Handbook of Mathematical Functions, p.944 (equation 26.5.13)

$$(a+b)I_x(a,b) = aI_x(a+1,\ b) + bI_x(a,\ b+1)$$

$$(a+b)\ \frac{B_x(a,b)}{B(a,b)} = a\ \frac{B_x(a+1,\ b)}{B(a+1,\ b)} + \frac{bB_x(a,\ b+1)}{B(a,\ b+1)}$$

$$\frac{(a+b)\Gamma(a+b)B_x(a,b)}{\Gamma(a)\Gamma(b)} - \frac{a(a+b)\Gamma(a+b)B_x(a+1, b)}{a\Gamma(a)\Gamma(b)} = \frac{b(a+b)\Gamma(a+b)B_x(a, b+1)}{\Gamma(a)b\Gamma(b)}$$

OR

$$B_x(a, b) - B_x(a+1, b) = B_x(a, b+1). \tag{4.2.2}$$

In this way, a large "matrix" of integrals can be found and only three variable spaces are needed to store it.

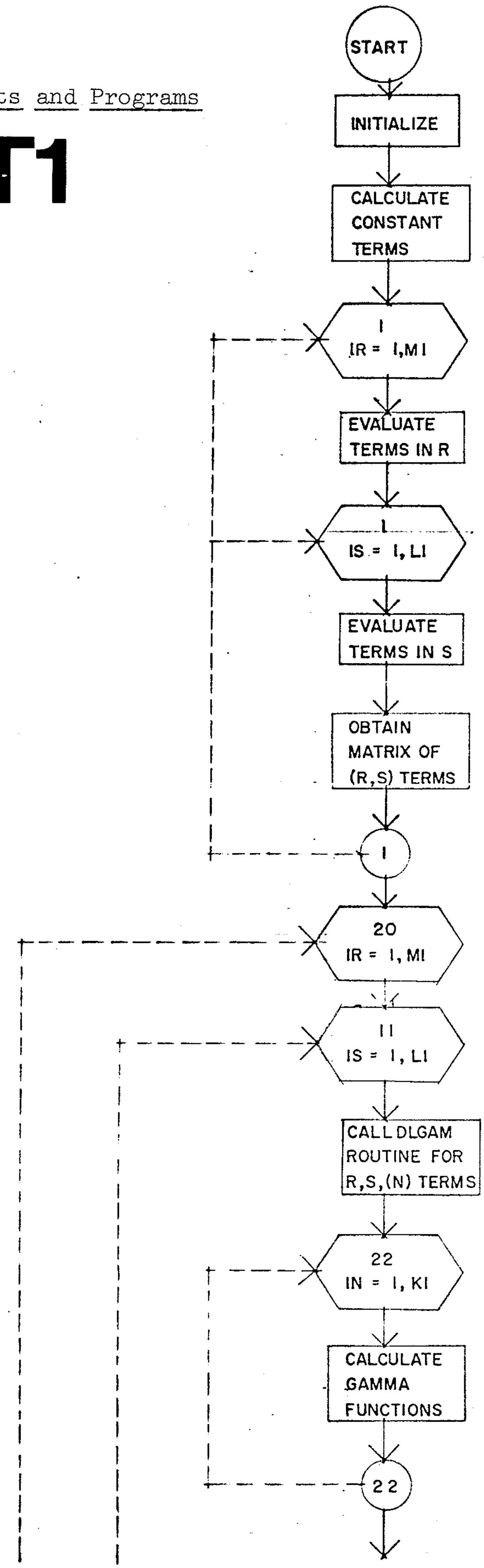
A further saving in time is obtained by using recursive relationships in the calculation of gamma functions. The DLGAM subroutine which evaluates these functions, returns the answer in log form. When these functions appear in DO loops, the number of calls to the subroutine for each function can be reduced to one by making the call outside the DO loop, using a parameter one less than that desired for calcuation and a statement of the form

```
DL = DL + DLØG(PARAMETER + s + n - 1.)
```

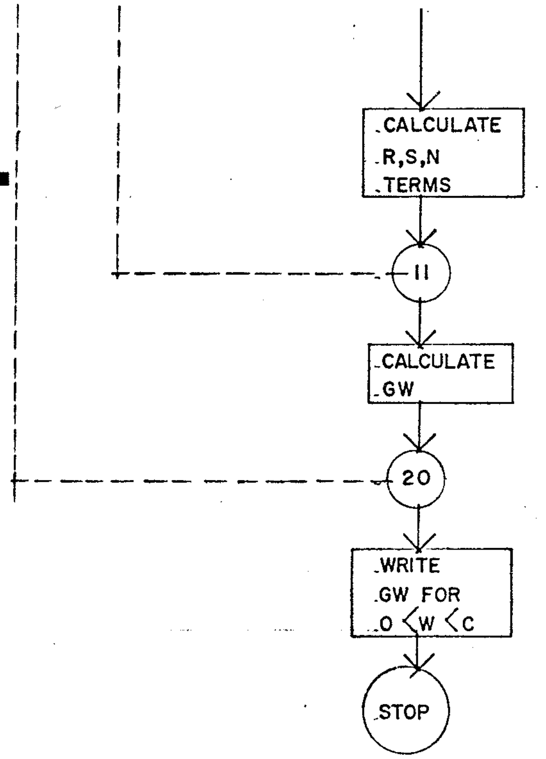
(where DL is the log value originally returned by DLGAM) inside the loop. When the loop has been satisfied, the log gamma function is complete and it is antilogged outside the loop.

4.3 Flowcharts and Programs

PART1



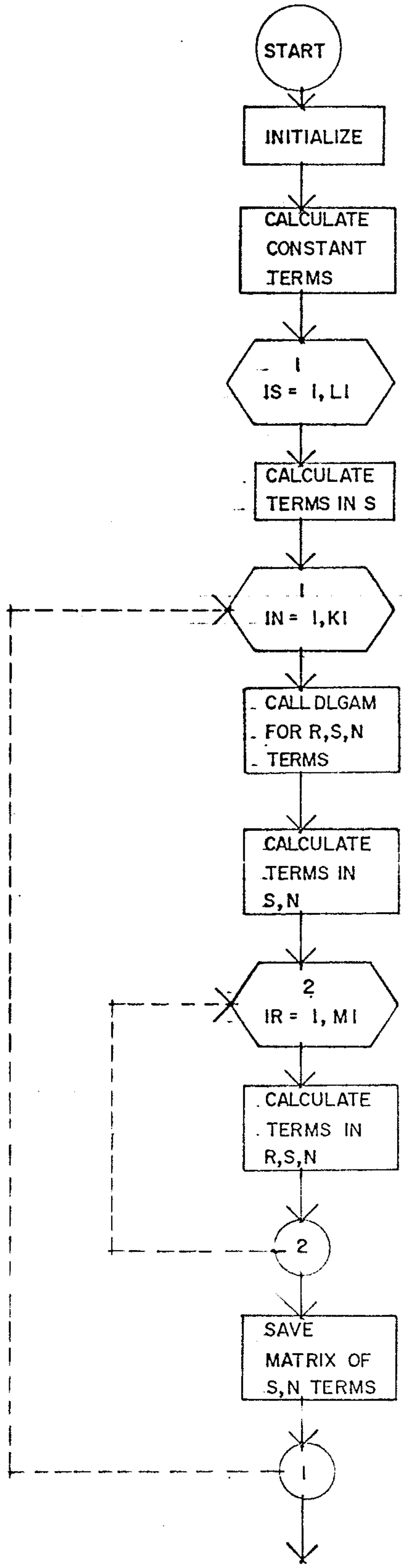
PART 1 CONT.



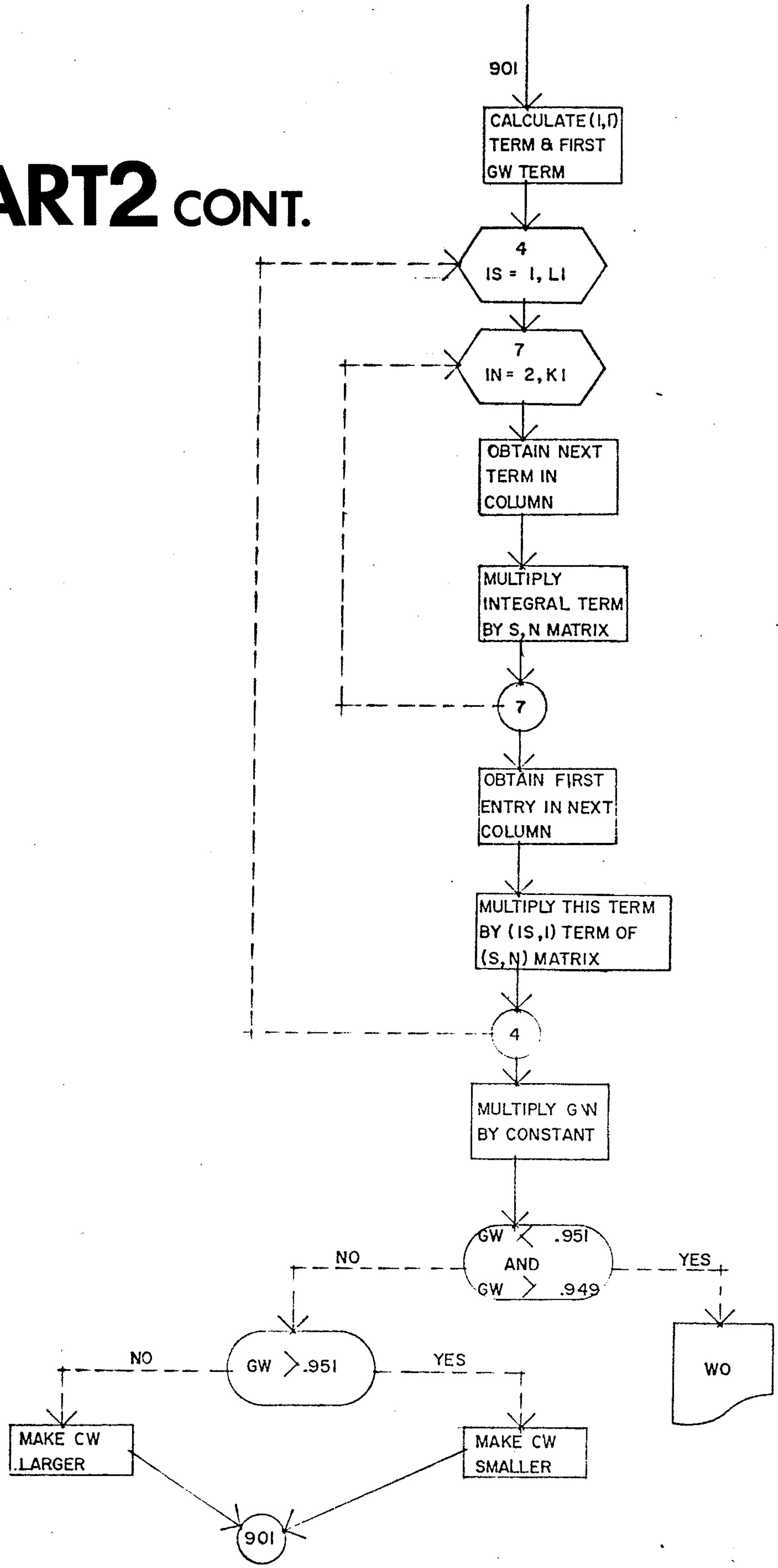
```
      C
      C
      C       PROGRAM CALCULATES THE PART OF G(W) IN REGION ONE, USING THE EXACT
      C       DISTRIBUTION OF THE RATIO OF NON-CENTRAL F DISTRIBUTIONS
      C
      C
              DOUBLE PRECISION U1,U2,TAU1,TAU2,GAMMA1,GAMMA2,CONST
              DOUBLE PRECISION DEXP,DC1,DC2,DC3,DC4,DC5,DC6,DC7,DC8,DC9
              DOUBLE PRECISION DR,DS,R,S,DG1,DG2,DG3,DG4,DG
              DOUBLE PRECISION DLOG,RSN,RSS
              DOUBLE PRECISION DLC,GG,GW,WG
              DOUBLE PRECISION TAUO,TAUT,TAU
              DOUBLE PRECISION RS(36,36)
      C
      C       READ IN NUMBER OF TERMS IN INFINITE SUMS,DEGREES OF FREEDOM,
      C       AND NON-CENTRALITY PARAMETERS
      C
              READ(5,100)M,L,K,U1,U2,GAMMA1,GAMMA2,TAUO,TAUT
        100   FORMAT(3I3,6D4.0)
              WRITE(6,200)U1,U2,GAMMA1,GAMMA2,TAUO,TAUT
        200   FORMAT('1','PARAMETERS ARE',2X,3(D16.8,',',D16.8,5X))
              WRITE(6,205)M,L,K
        205   FORMAT('0','M=',2X,I2,2X,'L=',2X,I2,2X,'K=',2X,I3)
              TAU=TAUO+TAUT
              U=U1/2.+U2/2.
              GAMMA=GAMMA1/2.+GAMMA2/2.
              C=(GAMMA1*U2)/(U1*GAMMA2)
      C
      C
      C       EVALUATE CONSTANT TERMS
      C
      C
              CALL DLGAM(GAMMA2/2.+U2/2.,DC1,IER)
              CALL DLGAM(U2/2.,DC2,IER)
              CALL DLGAM(GAMMA2/2.,DC3,IER)
              CONST=DC1-DC2-DC3
      C
      C
      C       CALCULATE THE R,S TERMS
      C
      C
              M1=M+1
              L1=L+1
              K1=K+1
              RSN=0.
              DO 1 IR=1,M1
              R=IR-1
              DC6=R*DLOG(TAUO)
              CALL DLGAM(R+1.,DR,IER)
              DO 1 IS=1,L1
              S=IS-1
              DC8=S*DLOG(TAUT)
              CALL DLGAM(S+1.,DS,IER)
              CALL DLGAM(GAMMA1/2.+GAMMA2/2.+S,DC5,IER)
              CALL DLGAM(GAMMA1/2.+S,DC7,IER)
              RSS=CONST+DC6-DR+DC8-DS+DC5-DC7
          1   RS(IR,IS)=DEXP(RSS)
      C
```

```
C
C     EVALUATE THE TERMS IN R,S,N
C
C
      GW=0.D0
      DO 20 IR=1,M1
      R=IR-1
      WG=0.D0
      DO 11 IS=1,L1
      GG=0.D0
      S=IS-1
      DLC=0.D0
      DG=0.
      CALL DLGAM(U1/2.+L2/2.+R-1.,DG1,IER)
      CALL DLGAM(U1/2.+GAMMA1/2.+R+S-1.,DG2,IER)
      CALL DLGAM(U1/2.+U2/2.+GAMMA1/2.+GAMMA2/2.+R+S-1.,DG3,IER)
      CALL DLGAM(U1/2.+R,DG4,IER)
      DO 22 IN=1,K1
      N=IN-2
      DG1=DG1+DLOG(U1/2.+L2/2.+R+N)
      DG2=DG2+DLOG(U1/2.+GAMMA1/2.+R+S+N)
      DG3=DG3+DLOG(U1/2.+U2/2.+GAMMA1/2.+GAMMA2/2.+R+S+N)
      DG4=DG4+DLOG(U1/2.+R+N+1.)
      DG=DG1+DG2-DG3-DG4
   22 DLC=DLC+DEXP(DG)
      GG=DEXP(-TAU)*RS(IR,IS)*DLC
   11 WG=WG+GG
C
C     SUM TERMS TO GET FINAL VALUE OF G(W) IN THE FIRST REGION
C
      GW=WG+GW
   20 CONTINUE
C
C     CHECK IF G(W) < .95 OR >.95
C
  934 CONTINUE
      IF(GW.LE.0.951.AND.GW.GE.0.949) GO TO 930
      IF(GW.LT.0.949) GOTO 932
      WO=(.99)*C
      GW=WO/C*GW
      GO TO 934
C
C     WRITE THE VALUE OF G(W) FOR THE FIRST REGION
C
  930 WRITE(6,931)WO
  931 FORMAT('0','VALUE OF WO FOR G(W) = .95 IS',F8.4)
      GO TO 950
  932 WRITE(6,933) GW
  933 FORMAT('0','CONTRIBUTION TO G(W) FROM FIRST REGION IS',5X,D16.8)
  950 CONTINUE
      STOP
      END
```

PART2


START
INITIALIZE
CALCULATE CONSTANT TERMS
1
IS = 1, L1
CALCULATE TERMS IN S
1
IN = 1,K1
CALL DLGAM FOR R,S,N TERMS
CALCULATE TERMS IN S,N
2
IR = 1, M1
CALCULATE TERMS IN R,S,N
2
SAVE MATRIX OF S,N TERMS
1

PART2 CONT.


901
CALCULATE (I,I) TERM & FIRST GW TERM
4
IS = I, LI
7
IN = 2, KI
OBTAIN NEXT TERM IN COLUMN
MULTIPLY INTEGRAL TERM BY S, N MATRIX
7
OBTAIN FIRST ENTRY IN NEXT COLUMN
MULTIPLY THIS TERM BY (IS, I) TERM OF (S, N) MATRIX
4
MULTIPLY GW BY CONSTANT
GW < .951 AND GW > .949
NO
YES
WO
GW > .951
NO
YES
MAKE CW LARGER
MAKE CW SMALLER
901

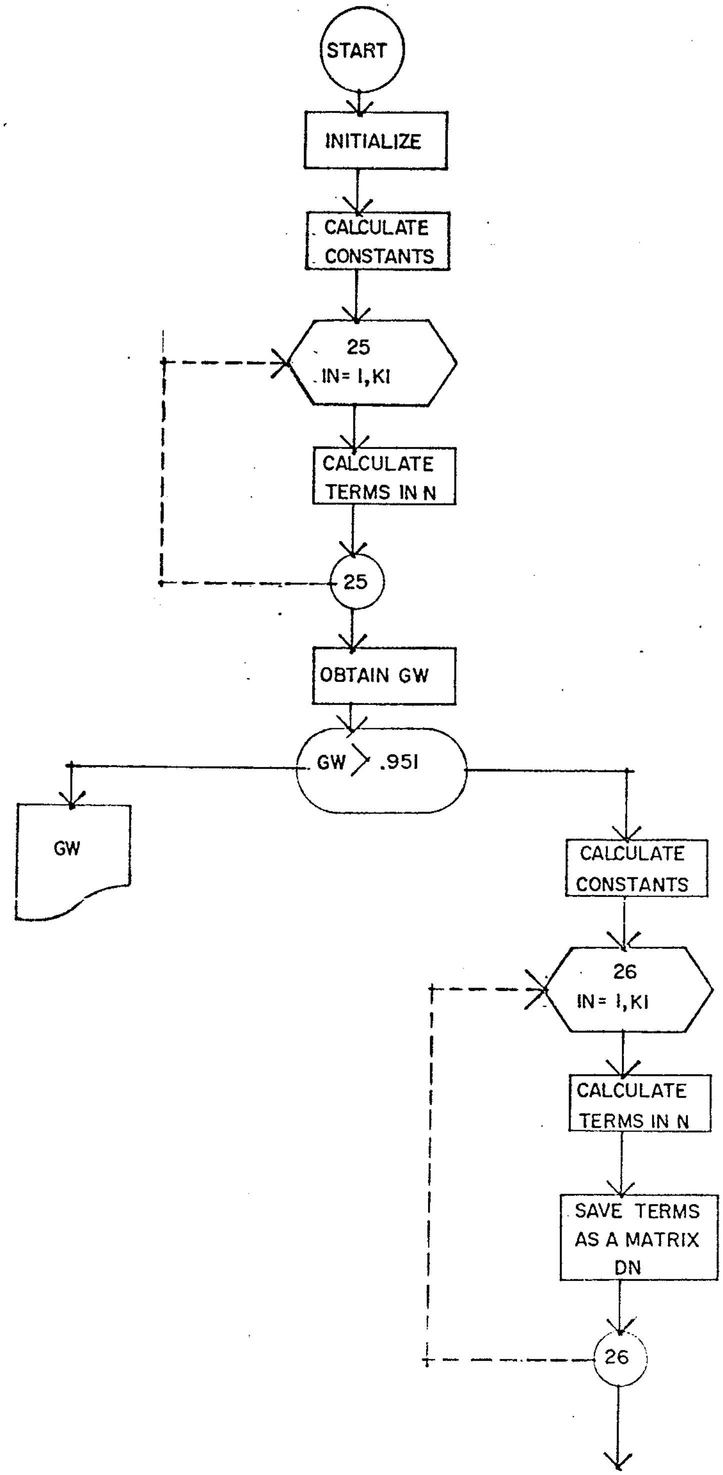
PART 2


```
      C
      C
      C        PROGRAM CALCULATES THE PART OF G(W) IN REGION TWO, USING THE
      C        EXACT DISTRIBUTION OF THE RATIO OF NON-CENTRAL F DISTRIBUTIONS
      C
      C
               DOUBLE PRECISION DLM(26,501),MREC(2)
               DOUBLE PRECISION DLL,TINT
               DOUBLE PRECISION AA,BB,CW
               DOUBLE PRECISION U1,U2,TAU1,TAU2,GAMMA1,GAMMA2,CONST
               DOUBLE PRECISION DEXP,DC1,DC2,DC3,DC4,DC5,DC6,DC7,DC8,DC9
               DOUBLE PRECISION DR,DS,S,R,N
               DOUBLE PRECISION DQ
               DOUBLE PRECISION DLOG,CN,RSQ,DLC
               DOUBLE PRECISION GW,DL,DL1,DL2,DL3,DL4
               DOUBLE PRECISION  TAU0,TAUT,TAU,U,GAMMA,WG,GG,REC,AB
      C
      C        READ IN NUMBER OF TERMS IN INFINITE SUMS, DEGREES OF FREEDOM,
      C        AND NON-CENTRALITY PARAMETERS
      C
               READ(5,100)M,L,K,U1,U2,GAMMA1,GAMMA2,TAU0,TAUT
        100    FORMAT(3I3,6D4.0)
               WRITE(6,200)U1,U2,GAMMA1,GAMMA2,TAU0,TAUT
        200    FORMAT('1','PARAMETERS ARE',2X,3(D16.8,',',D16.8,5X))
               WRITE(6,201)M,L,K
        201    FORMAT('0','M=',I2,2X,'L=',I2,2X,'K=',I3)
               TAU=TAU0+TAUT
               GAMMA=(GAMMA1+GAMMA2)/2.
               U=(U1+U2)/2.
               C=(GAMMA1*U2)/(U1*GAMMA2)
               W0=5.5
               P1=.5000000D0
      C
      C
      C        CALCULATE TERMS UP TO THE INTEGRAL
      C        EVALUATE CONSTANT TERMS
      C
      C
               CALL DLGAM(GAMMA2/2.+L2/2.,DC1,IER)
               CALL DLGAM(U2/2.,DC2,IER)
               CALL DLGAM(GAMMA2/2.,DC3,IER)
               CONST=DC1-DC2-DC3
      C
      C
      C        OBTAIN MATRIX DLM(IS,IN)
      C
      C
               M1=M+1
               L1=L+1
               K1=K+1
               CALL DLGAM(GAMMA1/2.-1.,DC9,IER)
               DO 1 IS=1,L1
               S=IS-1
               DC8=S*DLOG(TAUT)
               DC9=DC9+DLOG(GAMMA1/2.+S-1.)
               CALL DLGAM(S+1.,DS,IER)
               CALL DLGAM(GAMMA+S-1.,DL1,IER)
               DO 1 IN=1,K1
```

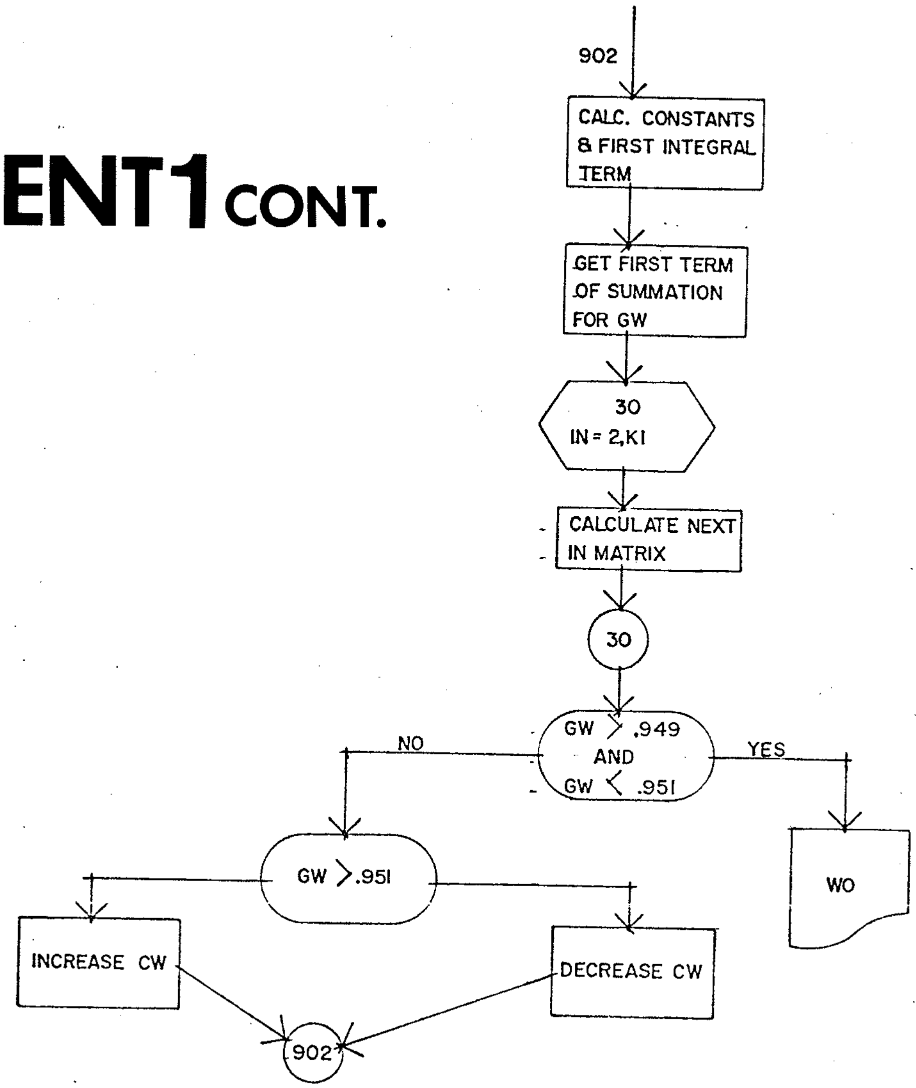
```
              N=IN-1
              CN=IN-1
              CALL DLGAM(U-1.,DC5,IER)
              CALL DLGAM(U1/2.-1.,DC4,IER)
              DL1=DL1+DLOG(GAMMA+S+N-1.)
              CALL DLGAM(U1/2.+GAMMA1/2.+S+N-1.,DL2,IER)
              CALL DLGAM(U+GAMMA+S+N-1.,DL3,IER)
              CALL DLGAM(CN+1.,DQ,IER)
              DL=0.D0
              DO 2 IR=1,M1
              R=IR-1
              DC5=DC5+DLOG(U+R-1.)
              DC4=DC4+DLOG(U1/2.+R-1.)
              DL2=DL2+DLOG(U1/2.+GAMMA1/2.+S+N+R-1.)
              DL3=DL3+DLOG(U+GAMMA+S+N+R-1.)
              DC6=R*DLOG(TAU0)
              CALL DLGAM(R+1.,DR,IER)
     2        DL=DL+DEXP(DL2+DC6+DC5-DL3-DR-DC4)
              DLC=DL1+CONST-(DC9+DS+DQ)
              DLM(IS,IN)=DEXP(DLC)*DL*DEXP(DC8)
      1       CONTINUE
    C
    C
    C         PORTION OF DISTRIBUTION THAT WILL CHANGE WHEN WO IS CHANGED
    C
    C
              CW=1.0-C/WO
    C
    C
    C         FIND INTEGRAL VALUES AND MULTIPLY BY GAMMA TERMS
    C
    C
    901       GW=0.D0
              BB=GAMMA1/2.
              K1=K+1
              L1=L+1
    C
    C         FIND THE FIRST INTEGRAL DIRECTLY
    C
              TINT=(1.0-(1.0-CW)**BB)/BB
              MREC(1)=TINT
              GW=GW+DLM(1,1)*TINT
              DO 4 IS=1,L1
              S=IS-1
              BB=GAMMA1/2.+S
              DO 7 IN=2,K1
    C
    C         FIND INTEGRAL FOR NEXT VALUE IN COLUMN
    C
              N=IN-1
              AA=N
              AB=AA+BB
              REC=(AA*DLOG(CW)+BB*DLOG(1.0-CW))-DLOG(AB)
              REC=DEXP(REC)
              TINT=AA*TINT/AB-REC
              IF(IN.EQ.2)MREC(2)=TINT
    C
    C         MULTIPLY INTEGRAL BY GAMMA TERMS IN S,N
```

```
          C
                GW=GW+DLM(IS,IN)*TINT
          7     CONTINUE
                IF(IS+1.GT.L1) GC TO 4
          C
          C     MOVE TC NEXT COLUMN
          C
                MREC(1)=MREC(1)-MREC(2)
                TINT=MREC(1)
                GW=GW+DLM(IS,1)*TINT
          4     CCNTINUE
                GW=GW*DEXP(-TAU)
          C
          C     ADC PART CF G(W) FRCM FIRST REGICN TC THAT FRCM THE SECCND
          C
                GW=GW+P1
                WRITE(6,904)GW,WO
          904   FORMAT('0','VALUE CF G(W) IS',D16.8,2X,'FOR WO=',F8.4)
                IF(CW.GE.0.9999) GC TO 555
          C
          C     CHECK TO SEE IF VALUE OF GW IS AS REQUIRED
          C     IF IT IS, WRITE VALUE CF WO AND GC ON TO NEW PARAMETERS
          C
                IF(GW.LE.0.951.ANC.GW.GE.0.949) GO TO 902
                IF(GW.GT.0.951) GC TO 9C0
          C
          C     IF GW IS TOO SMALL, INCREASE CW
          C
                CW=CW+((.95-GW)/.95*CW)
                IF(CW.GE.1.0) CW=C.9999
                WO=C/(1.0-CW)
                GC TO 901
          C
          C     IF GW IS TOO LARGE, DECREASE CW
          C
          900   CW=CW-((GW-.95)/.95*CW)
                WO=C/(1.0-CW)
                GC TC 901
          902   WRITE(6,903)WO
          903   FCRMAT('0','VALUE CF WO FOR G(W)=.95IS',F8.4)
          555   STOP
                END
```

CENT1


START
INITIALIZE
CALCULATE CONSTANTS
25 IN= 1,K1
CALCULATE TERMS IN N
25
OBTAIN GW
GW > .951
GW
CALCULATE CONSTANTS
26 IN= 1,K1
CALCULATE TERMS IN N
SAVE TERMS AS A MATRIX DN
26

CENT1 CONT.


902
CALC. CONSTANTS & FIRST INTEGRAL TERM
GET FIRST TERM OF SUMMATION FOR GW
30
IN = 2,KI
CALCULATE NEXT IN MATRIX
30
GW > .949 AND GW < .951
NO
YES
WO
GW >.951
INCREASE CW
DECREASE CW
902

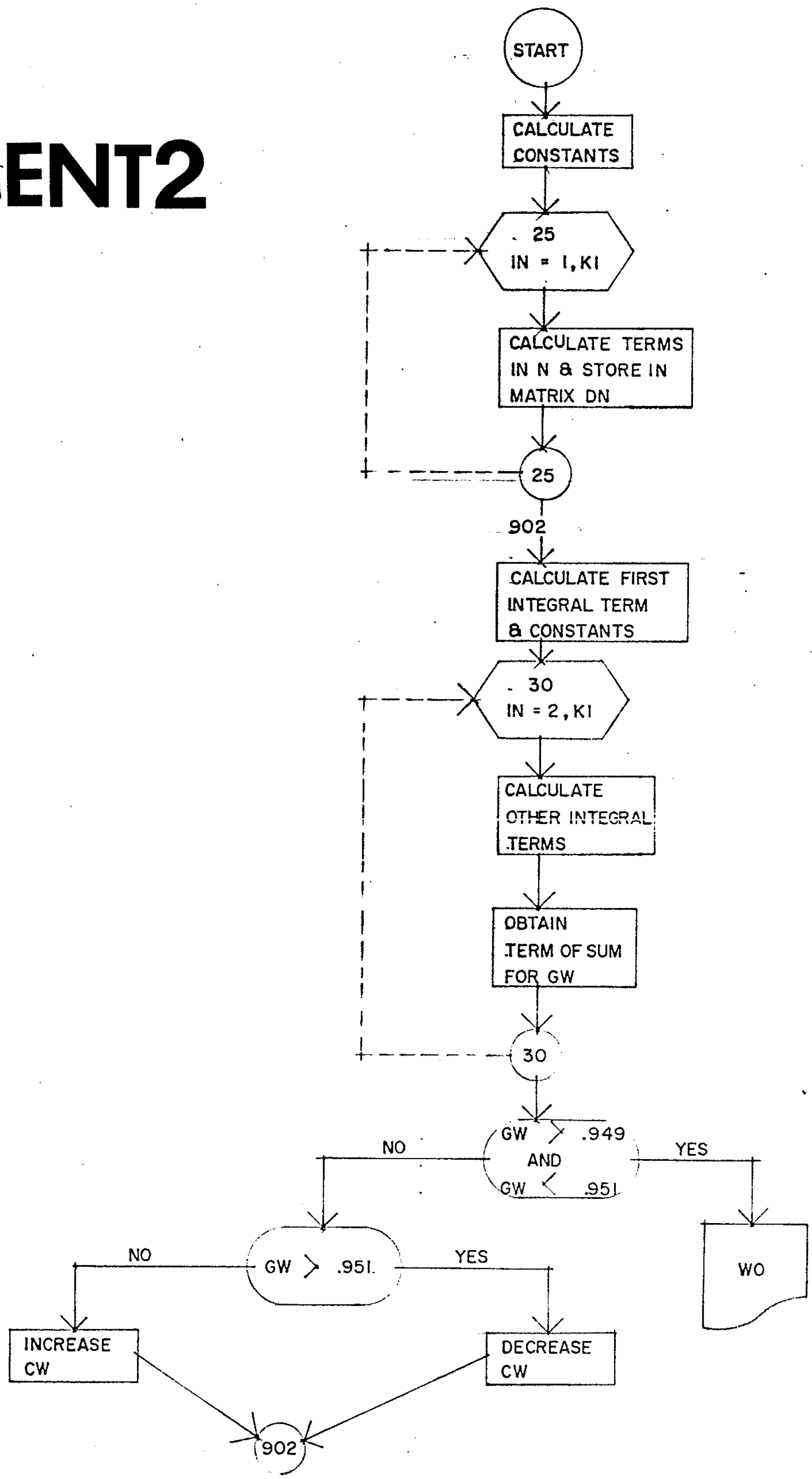
CENT1


```
      C
      C
      C      PROGRAM CALCULATES THE PART OF G(W) IN REGION ONE OR FINDS WO
      C      IF G(W) IS GREATER THAN 1-ALPHA, USING AN APPROXIMATION FOR THE
      C      RATIO OF NON-CENTRAL F DISTRIBUTIONS
      C
      C
      C
             DOUBLE PRECISION U2,GAMMA2,CONST,A1,A2
             DOUBLE PRECISION DLOG,DC1,DC2,DC3,DC4,DC5,DC6
             DOUBLE PRECISION DL1,DL2,DL3,DL4,DEXP,DL,GW
             DOUBLE PRECISION DN(501),DW,AA,BB,AB,TINT,REC,QN
      C
      C      READ IN THE NUMBER OF TERMS IN INFINITE SUM,DEGREES OF FREEDOM,
      C      AND INITIAL VALUE FOR WO
      C
             READ(5,101)K
       101   FORMAT(I3)
             WRITE(6,205)K
       205   FORMAT('1','K=',2X,I3)
       901   READ(5,100,END=900) A1,U2,A2,GAMMA2,WO
       100   FORMAT(4D5.0,D5.2)
             WRITE(6,200)A1,U2,A2,GAMMA2
       200   FORMAT('0','A1=',F5.1,2X,'U2=',F5.1,2X,'A2=',F5.1,2X,'GAMMA2=',
            XF5.1)
      C
      C
      C      CALCULATE CONSTANTS
      C
      C
             K1=K+1
             C=(A2*U2)/(A1*GAMMA2)
             CALL DLGAM(A2/2.+GAMMA2/2.,DC1,IER)
             CALL DLGAM(U2/2.+GAMMA2/2.,DC2,IER)
             CALL DLGAM(U2/2.,DC3,IER)
             CALL DLGAM(A2/2.,DC4,IER)
             CALL DLGAM(GAMMA2/2.,DC5,IER)
             CONST=DC2-DC3-DC4-DC5
      C
      C
      C      CALCULATE TERMS INVOLVING N
      C
      C
             CALL DLGAM(A1/2.+U2/2.-1.,DL1,IER)
             CALL DLGAM(A1/2.+A2/2.-1.,DL2,IER)
             CALL DLGAM(A1/2.+U2/2.+A2/2.+GAMMA2/2.-1.,DL3,IER)
             CALL DLGAM(A1/2.,DL4,IER)
             DL=0.D0
             DO 25 IN=1,K1
             N=IN-1
             DL1=DL1+DLOG(A1/2.+U2/2.+N-1.)
             DL2=DL2+DLOG(A1/2.+A2/2.+N-1.)
             DL3=DL3+DLOG(A1/2.+U2/2.+A2/2.+GAMMA2/2.+N-1.)
             DL4=DL4+DLOG(A1/2.+N)
        25   DL=DL+DEXP(DL1+DL2-DL3-DL4)
      C
      C      OBTAIN FINAL VALUE FOR GW
      C
             GW=DEXP(CONST)*DEXP(DC1)*DL
```

```
              C
              C       IF VALUE OF GW FOR FIRST REGION IS GREATER THAN 1-ALPHA, FIND
              C       WO SUCH THAT GW IS APPRCX. 1-ALPHA
              C
 0034                 IF(GW.GT.C.951) GC TC 888
              C
              C       PRINT CCNTRIBUTICN TO GW FRCM THE FIRST REGICN
              C
 0035                 WRITE(6,300)GW
 0036         300     FORMAT('0','VALUE OF GW FOR 0<W<C IS',D16.8)
 0037                 GO TC 901
              C
              C
              C       IF WO IS LESS THAN C, THIS PART OF THE PROGRAM IS USED
              C
              C
 0038         888     CCNTINUE
 0039                 CALL CLGAM(A1/2.,CC6,IER)
 0040                 CCNST=CCNST-DC6
              C
              C
              C       CALCULATE TERMS INVOLVING N (INCLUDING INTEGRAL)
              C
              C
 0041                 CALL CLGAM(A1/2.+U2/2.-1.,DL1,IER)
 0042                 CALL DLGAM(A1/2.+A2/2.-1.,CL2,IER)
 0043                 CALL CLGAM(A1/2.+U2/2.+A2/2.+GAMMA2/2.-1.,DL3,IER)
 0044                 DL=C.DO
 0045                 DO 26 IN=1,K1
 0046                 N=IN-1
 0047                 GN=IN-1
 0048                 DL1=CL1+DLOG(A1/2.+U2/2.+N-1.)
 0049                 DL2=DL2+DLOG(A1/2.+A2/2.+N-1.)
 0050                 DL3=CL3+DLOG(A1/2.+A2/2.+U2/2.+GAMMA2/2.+N-1.)
 0051                 CALL CLGAM(GN+1.,DC,IER)
 0052                 DL=DL2+CL2-DL3-DQ
 0053         26      DN(IN)=DEXP(DL)
              C
              C
              C       PCRTICN CF DISTRIBUTICN THAT WILL CHANGE WHEN WO IS CHANGED
              C
              C
 0054                 CW=WO/C
 0055         902     AA=A1/2.
 0056                 GW=C.CO
 0057                 TINT=CW**AA/AA
 0058                 GW=GW+DN(1)*TINT
 0059                 DC 30 IN=2,K1
 0060                 BB=IN-1
 0061                 AB=AA+BB
 0062                 REC=(AA*DLOG(CW)+BB*DLOG(1.0-CW))-DLOG(AB)
 0063                 REC=DEXP(REC)
 0064                 TINT=AA*TINT/AB-REC
 0065         30      GW=GW+TINT*DN(IN)
 0066                 GW=GW*CCNST
              C
              C       CHECK TO SEE IF VALUE CF GW IS AS REQUIRED
              C       IF IT IS, WRITE VALUE CF WO AND GO ON TO NEW PARAMETERS
```

```
      C
            IF(GW.GE.0.949.AND.GW.LE.0.951) GO TO 301
            IF(GW.GT.0.951) GO TO 302
      C
      C     IF GW IS TOO SMALL, INCREASE CW
      C
            CW=CW+((.95-GW)/.95*CW)
            IF(CW.GE.1.0) CW=0.9999
            WO=C*CW
            GO TO 902
      C
      C     IF GW IS TOO LARGE, DECREASE CW
      C
      302   CW=CW-((GW-.95)/.95*CW)
            WO=C*CW
            GO TO 902
      301   WRITE(6,997) A1,U2,A2,GAMMA2,WO
      997   FORMAT('0','VALUE OF WO FOR A1=',F5.1,2X,'U2=',F5.1,2X,'A2=',
           XF5.1,2X,'GAMMA2=',F5.1,2X,'IS',D16.8)
            GO TO 901
      900   STOP
            END
```

CENT2


START
CALCULATE CONSTANTS
25
IN = 1, KI
CALCULATE TERMS IN N & STORE IN MATRIX DN
25
902
CALCULATE FIRST INTEGRAL TERM & CONSTANTS
30
IN = 2, KI
CALCULATE OTHER INTEGRAL TERMS
OBTAIN TERM OF SUM FOR GW
30
GW > .949 AND GW < .951
NO
YES
WO
GW > .951
NO
YES
INCREASE CW
DECREASE CW
902

```
      C
      C
      C       PROGRAM FINDS THE VALUE OF WO SUCH THAT G(W) EQUALS 1-ALPHA, USING
      C       AN APPROXIMATION FOR THE RATIO OF NON-CENTRAL DISTRIBUTIONS
      C
      C
              DOUBLE PRECISION U2,GAMMA2,CONST,A1
              DOUBLE PRECISION A2,DLOG,DEXP,DC1,DC2,DC3,DC4,DC5,DC6,DN(476),P1
              DOUBLE PRECISION DL1,DL2,DL3,DQ,DLC,QN,AA,BB,AB,REC,TINT,GW,CW
      C
      C       READ IN NUMBER OF TERMS IN INFINITE SUM, DEGREES OF FREEDOM
      C       INITIAL VALUE FOR WO AND CONTRIBUTION TO G(W) FROM THE FIRST
      C       REGION
      C
              READ(5,999)K
      999     FORMAT(I3)
      901     READ(5,100,END=900)A1,U2,A2,GAMMA2,WO,P1
      100     FORMAT(4D5.0,D5.2,D11.8)
      C
      C
      C       CALCULATE CONSTANTS
      C
      C
              K1=K+1
              C=(A2*U2)/(A1*GAMMA2)
              CALL DLGAM(A1/2.+U2/2.,DC1,IER)
              CALL DLGAM(GAMMA2/2.+U2/2.,DC2,IER)
              CALL DLGAM(A1/2.,DC3,IER)
              CALL DLGAM(U2/2.,DC4,IER)
              CALL DLGAM(A2/2.,DC5,IER)
              CALL DLGAM(GAMMA2/2.,DC6,IER)
              CONST=DC2-DC3-DC4-DC5-DC6
              CONST=DEXP(CONST)*DEXP(DC1)
      C
      C
      C       CALCULATE TERMS INVOLVING N AND STORE IN MATRIX DN
      C
      C
              CALL DLGAM(A2/2.+GAMMA2/2.-1.,DL1,IER)
              CALL DLGAM(A1/2.+A2/2.-1.,DL2,IER)
              CALL DLGAM(A1/2.+A2/2.+U2/2.+GAMMA2/2.-1.,DL3,IER)
              DLC=0.D0
              DO 25 IN=1,K1
              N=IN-1
              QN=IN-1
              DL1=DL1+DLOG(A2/2.+GAMMA2/2.+N-1.)
              DL2=DL2+DLOG(A1/2.+A2/2.+N-1.)
              DL3=DL3+DLOG(A1/2.+A2/2.+U2/2.+GAMMA2/2.+N-1.)
              CALL DLGAM(QN+1.,DQ,IER)
              DLC=DL1+DL2-DL3-DQ
      25      DN(IN)=DEXP(DLC)
      C
      C
      C       PORTION OF DISTRIBUTION THAT WILL CHANGE WHEN WO IS CHANGED
      C
      C
              CW=1.0-C/WO
      902     BB=A2/2.
```

```
                GW=0.D0
                TINT=(1.0-(1.0-CW)**BB)/BB
                GW=GW+DN(1)*TINT
                DO 30 IN=2,K1
                AA=IN-1
                AB=AA+BB
                REC=(AA*DLOG(CW)+BB*DLOG(1.0-CW))-DLOG(AB)
                REC=DEXP(REC)
                TINT=AA*TINT/AB-REC
C
C               SUM THE TERMS OF GW
C
  30            GW=GW+TINT*DN(IN)
                GW=GW*CONST
C
C               ADD THE CONTRIBUTION TO GW FROM THE FIRST REGION
C
                GW=GW+P1
C
C               CHECK TO SEE IF GW IS AS REQUIRED
C               IF IT IS, WRITE VALUE OF WO AND GO ON TO NEW PARAMETERS
C
                IF(GW.GE.0.949.AND.GW.LE.0.951) GO TO 301
                IF(GW.GT.0.951) GO TO 302
C
C               IF GW IS LARGE,DECREASE CW
C
                CW=CW+((.95-GW)/.95*CW)
                IF(CW.GE.1.0) CW=0.9999
                WO=C/(1.0-CW)
                GO TO 902
C
C               IF GW IS TOO SMALL,INCREASE CW
C
 302            CW=CW-((GW-.95)/.95*CW)
                WO=C/(1.0-CW)
                GO TO 902
 301            WRITE(6,997)A1,U2,A2,GAMMA2,WO
 997            FORMAT('0','VALUE OF WO FOR A1=',F5.1,2X,'U2=',F5.1,2X,'A2=',F5.1
               X,2X,'GAMMA2=',F5.1,2X,'IS',D16.8)
                GO TO 901
 900            STOP
                END
```

```
0001          SUBROUTINE DLGAM ( XX , DLNG , IER )
        C     IER = 0 MEANS NO ERROR
        C     IER = -1 MEANS XX IS TOO NEAR ZERO
        C     IER = +1 MEANS XX IS TOO LARGE
0002          DOUBLE PRECISION XX , ZZ , TERM , RZ2 , DLNG , DLOG
0003          IER = 0
0004          ZZ = XX
0005          IF( XX - 1.0D10 ) 2 , 2 , 1
0006        1 IF( 1.0D70 - XX ) 9 , 9 , 8
        C     SEE IF XX IS NEAR ZERO OR NEGATIVE
0007        2 IF( XX - 1.0D-9 ) 3 , 3 , 4
0008        3 IER = -1
0009          DLNG = - 1.0D75
0010          GO TO 10
        C     XX IS .GT. ZERO AND .LE. 1.0D+10
0011        4 TERM = 1.0D0
0012        5 IF( ZZ - 18.D0 ) 6 , 6 , 7
0013        6 TERM = TERM * ZZ
0014          ZZ = ZZ + 1.0D0
0015          GO TO 5
0016        7 RZ2 = 1.0D0 / ZZ**2
0017          DLNG = ( ZZ - 0.5D0 ) * DLOG( ZZ ) - ZZ + 0.9189385332046727 -
             1    DLOG( TERM ) + ( 1.0D0 / ZZ ) * ( .8333333333333333D-1 - ( RZ2
             2     *( .2777777777777777D-2 + ( RZ2 *( .7936507936507936D-3 - (
             3 RZ2 * ( .5952380952380952D-3 )))))))
0018          GO TO 10
        C     XX .GT. 1.0D+10 AND .LT. 1.0D+70
0019        8 DLNG = ZZ * ( DLOG( ZZ ) - 1.0D0 )
0020          GO TO 10
        C     XX .GE. 1.0D+20
0021        9 IER = +1
0022          DLNG = 1.0D75
0023       10 RETURN
0024          END
```

V. Examples

5.1 Comparing Sensitivity of Experiments

Consider the example found in Bradley and Schumann (1957b) in which they compare the sensitivity of pasture yields to irrigation.

Mean Yields of Orchard Grass and Ladino Clover in lbs./acre under Fertilizer Treatments

Treatment	A	B	C	D	E	F	G	H
No Irrigation	499	627	642	756	884	494	597	377
Irrigation	831	836	931	1137	1346	730	903	583

The analysis of the comparison of sensitivities is as follows:

Analysis of Variance for the Two Fertilizer Experiments

Source	d.f.	Mean Square	F
Experiment 1 - No Irrigation			
Replications	3	17, 899	
Treatments	7	102, 104	8.86
Error	21	11, 517	
Experiment 2 - Irrigation			
Replications	3	93, 155	
Treatments	7	224, 462	7.10
Error	21	31, 599	

1. $F_1 = 8.86$ $\qquad F_2 = 7.10$

   $\nu_1 = \gamma_1 = 3.5$ $\qquad \nu_2 = \gamma_2 = 10.5$

2. $\hat{\lambda}_1 = 3.5(8.86 - 1) = 27.5$

   $\hat{\lambda}_2 = 3.5(7.10 - 1) = 21.4$

   $\hat{\lambda} = \frac{1}{2}(27.5 + 21.4) = 24.4$

3. $H_0: \lambda_1 = \lambda_2 = \lambda$

   $H_A: \lambda_1 \neq \lambda_2$

4. $\alpha = .05$

5. $w = \dfrac{8.86}{7.10} = 1.25$

6. $\dfrac{a_1}{2} = \dfrac{a_2}{2} = (3.5 + 24.4)^2/(3.5 + 48.8) = 14.9$

7. Using the program developed in 4.3, the upper 5% point for the approximate distribution was determined: $w_0(.05) = 2.61$.

8. Therefore, w is not significant when compared with $w_0(.05)$ nor is $\frac{1}{w}$. That is, it does not appear that these experiments differ much in senstivity to fertilizer treatment effects.

5.2 Comparing Regression Coefficients

Consider an example constructed from data on water usage in rural and urban communities with variables

X_1 = average monthly temperature (°F)

X_2 = amount of production (in pounds)

X_3 = number of plant operating days in the month

X_4 = number of persons on the monthly plant payroll.

For urban usage variables X_1, X_2, X_3 and X_4 are considered with 8 data points. For rural usage variables X_1, X_2 and X_3 are considered with 9 data points. The analysis of the comparison of regression coefficients is as follows:

Analysis of Variance for Water Usage

Source	d.f.	Mean Square	F
		Urban	
Regression	4	405654.2500	7.821
Deviation	3	51869.0000	
Total	7		
		Rural	
Regression	3	136140.5000	1.568
Deviation	5	86813.5625	
Total	8		

1. $$\frac{\frac{\gamma_1}{2} R_1^2 / \frac{a_1}{2} (1 - R_1^2)}{\frac{\gamma_2}{2} R_2^2 / \frac{a_2}{2} (1 - R_2^2)} = \frac{(\frac{1.5}{2.0})(.95525^2)/(1.0 - .95525^2)}{(\frac{2.5}{2.0})(.69626^2)/(1.0 - .69626^2)} = \frac{1.8217}{1.5682} = 4.9878$$

$$\frac{\nu_1}{2} = 2.0 \qquad \frac{\nu_2}{2} = 1.5$$

$$\frac{\gamma_1}{2} = 1.5 \qquad \frac{\gamma_2}{2} = 2.5$$

2. $$\hat{\rho}_1^2 = 1 - (1 - R_1^2)(2.0 + 1.5)/1.5 = .79584$$

$$\hat{\rho}_2^2 = 1 - (1 - R_2^2)(1.5 + 2.5)/2.5 = .17564$$

$$\hat{\lambda}_1 = (2.0 + 1.5)\hat{\rho}_1^2/(1 - \hat{\rho}_1^2) = 13.643$$

$$\hat{\lambda}_2 = (1.5 + 2.5)\hat{\rho}_2^2/(1 - \hat{\rho}_2^2) = 0.85342$$

$$\hat{\lambda} = \frac{1}{2}(\hat{\lambda}_1 + \hat{\lambda}_2) = 14.496$$

3. H_0: $\rho_1^2 = \rho_2^2$

   H_A: $\rho_1^2 \neq \rho_2^2$

4. $\alpha = .05$

5. $w = 4.9878$

6. $$\frac{a_1}{2} = \frac{(\frac{\nu_1}{2} + \hat{\lambda}_1)^2}{(\frac{\nu_1}{2} + 2\hat{\lambda}_1)} = 8.3556$$

   $$\frac{a_2}{2} = \frac{(\frac{\gamma_1}{2} + \hat{\lambda}_2)^2}{(\frac{\gamma_1}{2} + 2\hat{\lambda}_2)} = 1.7271$$

7. Using the program developed in 4.3, the upper 5% point for the approximate distribution was determined: $w_0(.05) > 4.9878$.

8. Therefore, w is not significant when compared with $w_0(.05)$ nor is $\frac{1}{w}$. That is, it does not appear that these regression coefficients differ.

5.3 Testing Interaction in 2x2 Contingency Tables

Gart (1966) states that the null hypothesis of no interaction may be stated in terms of the cross-product ratio $\psi = \frac{p_{11}p_{22}}{p_{21}p_{22}}$ (which is the non-centrality parameter of the exact test). It is found that ψ is approximately distributed as $\frac{u_{11}u_{22}}{u_{21}u_{22}}$ where the u_{ij}

$(i, j = 1, 2)$ are mutually independent chi-squared variates with corresponding degrees of freedom $2n_{ij} + 1$. Therefore

$$\frac{(2n_{12} + 1)(2n_{21} + 1)}{(2n_{11} + 1)(2n_{22} + 1)} \psi \sim \frac{F(2n_{11} + 1,\ 2n_{12} + 1)}{F(2n_{21} + 1,\ 2n_{22} + 1)} . \qquad (5.1)$$

The hypothesis $\psi = 1$ (no interaction) can then be tested by setting $\psi = 1$ and treating $\{(2n_{12} + 1)(2n_{21} + 1)\}/\{(2n_{11} + 1)(2n_{22} + 1)\}$ as a test statistic distributed as the ratio of two independent central F's. The programs developed in 4.3 then give an exact value for $w_0(.05)$.

Consider a 2x2 contingency table with sites and types of tumour.

Type of Tumour

		A	B	
Sites	I	11	7	18
	II	16	3	19
		27	10	37

The null hypothesis is H_0: $\psi = 1$ (no interaction) and the alternate hypothesis is $\psi \neq 1$. The test statistic is

$$\frac{(2n_{12} + 1)(2n_{21} + 1)}{(2n_{11} + 1)(2n_{22} + 1)} = \frac{(15)(33)}{(23)(7)} = 3.0745.$$

Using the program in 4.3, the upper 5% point was found to be $w_0(.05) = 3.25$. Therefore, the conclusion is: Do not reject H_0: $\psi = 1$.

For large degrees of freedom in the denominator of both F distributions, the distribution is asymptotically an F distribution. Because n_{12} and n_{22} are large, equation (5.1) becomes

$$\frac{(2n_{12} + 1)(2n_{21} + 1)}{(2n_{11} + 1)(2n_{22} + 1)} \psi \sim F(2n_{11} + 1,\ 2n_{21} + 1).$$

For example, consider a 2x2 contingency table showing results of a clinical trial.

		Outcome		
		Death	Survival	Total
Treatment	A	41	216	257
	B	64	180	244
		105	396	501

The test statistic

$$\frac{(2n_{12} + 1)(2n_{21} + 1)}{(2n_{11} + 1)(2n_{22} + 1)} = \frac{(433)(129)}{(83)(361)} \doteq 1.864$$

$F(83,\ 129) \doteq 1.42.$

Therefore, the conclusion is to reject H_0: $\psi = 1$.

TABLE I

Values of w_0 in the Approximate Distributions Such That $1 - G(w_0) = .01$

a_2, γ_2 \ a_1, ν_2	(4, 4)	(6, 6)	(8, 8)	(10, 10)	(12, 12)	(14, 14)	(16, 16)	(18, 18)	(20, 20)	(30, 30)	(∞, ∞)
(4, 4)	43.91	29.95	24.95	22.14	20.97	19.87	19.09	18.87	18.43	17.16	15.98
(5, 5)	34.57	22.53	18.35	16.39	15.05	14.39	13.77	13.31	13.12	12.11	10.97
(6, 6)		18.86	15.10	13.32	12.29	11.53	11.10	10.70	10.39	9.60	8.47
(7, 7)		16.70	13.19	11.52	10.56	9.86	9.45	9.09	8.80	8.07	7.00
(8, 8)			11.94	10.35	9.36	8.78	8.32	8.04	7.77	7.08	6.03
(9, 9)			10.97	9.46	8.58	8.02	7.58	7.26	7.05	6.38	5.35
(10, 10)				8.87	8.01	7.42	7.04	6.73	6.49	5.87	4.85
(11, 11)				8.36	7.55	7.00	6.59	6.32	6.09	5.45	4.46
(12, 12)					7.15	6.63	6.27	5.97	5.74	5.15	4.16
(13, 13)					6.89	6.33	5.98	5.69	5.49	4.88	3.91
(14, 14)						6.12	5.74	5.48	5.26	4.68	3.70
(15, 15)						5.91	5.56	5.28	5.08	4.49	3.51
(16, 16)							5.39	5.11	4.92	4.33	3.37
(17, 17)							5.40	5.10	4.95	4.35	3.24
(18, 18)								5.05	4.88	4.27	3.13
(19, 19)								4.98	4.78	4.16	3.03
(20, 20)									4.67	4.06	2.94
(21, 21)									4.58	3.98	2.86
(30, 30)										3.47	2.38

TABLE II

Contribution to $G(w_0)$ Found in the Region $0 < w < c$ for the Approximate Distribution (With Unequal Degrees of Freedom)

a_2, γ_2 / a_1, ν_2	(4, 5)	(4, 10)	(6, 10)	(8, 20)	(8, 30)	(10, 30)	(10, 60)
(4, 5)	.499989	.281734	.405573	.273333	.168607	.219927	.085264
(4, 10)	.718220	.500000	.657343	.520382	.366460	.452852	.214658
(6, 10)	.594381	.342657	.500000	.335913	.193676	.263454	.082049
(8, 20)	.726453	.479618	.664087	.500000	.309158	.413163	.133190
(8, 30)	.830851	.633540	.806323	.690842	.500000	.621265	.272042
(10, 30)	.779533	.547148	.736545	.586837	.387350	.500000	.165300
(10, 60)	.912107	.785337	.917937	.866810	.727958	.834700	.500000
(10, 120)	.960400	.920638	.981883	.975692	.934326	.973759	.844382

TABLE III

Values of w_0 in the Approximate Distribution
(With Unequal Degrees of Freedom) Such That $1 - G(w_0) = .05$

a_2, ν_2 / a_1, ν_2	(4, 5)	(4, 10)	(6, 10)	(8, 20)	(8, 30)	(10, 30)	(10, 60)
(4, 5)	12.000000	11.754856	8.852629	7.690272	7.500000	7.058336	7.000000
(4, 10)	8.979434	8.706291	6.480137	5.469527	5.345645	4.895445	4.900000
(6, 10)	8.591558	8.386263	6.232943	5.191547	5.135302	4.690725	4.653612
(8, 20)	7.580584	7.293737	5.227657	4.282369	4.212484	3.826460	3.722127
(8, 30)	7.307749	6.979904	4.985229	4.000000	4.000000	3.612117	3.500000
(10, 30)	7.237404	6.907783	4.810286	3.915244	3.909849	3.513048	3.421517
(10, 60)	7.000000	6.613768	4.674112	3.766099	3.697359	3.321956	3.246171

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