ENUMERATION OF ANTICHAINS

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Enumeration of Antichains

1. Introduction.

In the course of this discussion, certain terms in set theory, graph theory, and the theory of partially-ordered sets will be used. Some of these - for example, SET, SUBSET, ORDERED n-TUPLE, UNION - are standard terms, and there can be no confusion as to their meaning. However, without further clarification, other terms would be meaningless, or at best ambiguous. This section will define and explain these particular terms.

'n-VECTOR' is the name used throughout for the concept of ordered n-tuple.

Let $\langle x_1, x_2, \ldots, x_n \rangle$ be an n-vector. Then, for each $1 \le i \le n$, i an integer, we define the i-TH CONSTITUENT of $\langle x_1, x_2, \ldots, x_n \rangle$ as x_i .

Let $S = \langle s_1, s_2, \ldots, s_n \rangle$ be an n-vector of variables, and let $\Sigma = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_n \rangle$ be an n-vector of constants. By SUBSTITUTING Σ FOR S , we will mean substituting Σ_i for each s_i in the n-vector S .

A SIMPLE DIGRAPH is defined as an ordered pair of sets $\langle V,E \rangle$. V is a set whose elements are called VERTICES. E is a set whose elements are ordered pairs of vertices called EDGES.

Let one of the edges be $\langle i,j \rangle$, where $i,j \in V$. Then i is the TAIL VERTEX of the edge, and j is the HEAD VERTEX. Let H be a simple digraph. Then V(H) represents the set of vertices of H , and E(H) represents the set of edges of H .

If X and Y are sets of vertices of a simple digraph G , $X \overset{G}{\nabla} Y$ is the set of edges $\langle i,j \rangle$ such that $i \in X$, $j \in Y$, and $\langle i,j \rangle \in E(G)$. The set $X \overset{G}{\nabla} Y$ is called the COBOUNDARY OF X TOWARD Y . The set $X \overset{G}{\nabla} [V(G) - X]$ is called the COBOUNDARY OF X .

A simple digraph H is called a SUBGRAPH of another simple digraph G when the vertex set and edge set of H are respectively subsets of the vertex set and edge set of G . If U is a set of vertices of G , then G - U is the simple digraph such that V(G-U) is the set of vertices of G not in U , and E(G-U) is the set of edges of G none of whose constituents is an element of U . If K is a set of edges of G , then G-K is the simple digraph such that E(G-K) is the set of edges of G not in K , and V(G-K) is the set of vertices v such that there is at least one edge having v as a constituent.

If G and H are simple digraphs, the UNION of G and H written $G \cup H$, has vertex set $V(G \cup H) = V(G) \cup V(H)$, and has edge set $E(G \cup H) = E(G) \cup E(H)$.

A DITRACK $\langle v_1, v_2, \ldots, v_k \rangle$ of G is a k-vector, for some $k \geq 2$, such that for all integral i, with $1 \leq i \leq k - 1$, $\langle v_i, v_{i+1} \rangle$

is an edge of G .

A TRACK $v_1, v_2, \ldots, v_k\rangle$, $k \geq 2$, of G is a k-vector such that, for all integral i with $1 \leq i \leq k-1$, either $\langle v_i, v_{i+1}\rangle$ or $\langle v_{i+1}, v_i\rangle$ is an edge of G . If $v_1 = v_k$ in a ditrack, we call that ditrack a DICIRCUIT.

An ACYCLIC DIGRAPH is a simple digraph with no dicircuits. A SOURCE of an acyclic digraph is a vertex which is not the head vertex of any edge of the acyclic digraph.

Let $\langle v_1, v_2, \ldots, v_k\rangle$ be a ditrack of a simple digraph G such that $k \geq 3$. Then the ordered pair $\langle v_1, v_k\rangle$, if v_1 and v_k are distinct, is called a TRANSVERSAL of G .

A PARTIAL ORDER is an acyclic digraph whose edge set contains all its transversals.

A simple digraph is CONNECTED if, for any two of its vertices i, j, there is a track from i to j . It is STRONGLY CONNECTED if there is always a ditrack from any vertex to any other.

A simple graph M is MAXIMAL WITH PROPERTY P if no simple digraph other than M which contains M as a subgraph has property P .

A STRONG COMPONENT of G is a maximal strongly connected subgraph of G .

A COMPONENT of G is a maximal connected subgraph of G .

A TAIL COMPONENT T of G is a strong component of G such that no edge of $G-T$ has a head vertex in the vectors set of T .

Define a q-DIVISION of a set Q as an ordered q-vector of disjoint subsets of Q where the union of all the subsets is Q.

Define a PARTITION of Q as a collection of non-null disjoint subsets of Q where the union of all subsets in the collection is Q.

Define the n-SPREAD of a partition of a set Q with n elements as the n-vector $\langle i_1, i_2, \ldots, i_n \rangle$, where, for all k, i_k is the number of sets in the partition with k elements. We must have $i_1 + 2i_2 + \ldots + ki_k + \ldots + ni_n = n$.

Define the STRONG COMPONENT PARTITION of a simple digraph G as the collection of vertex sets of all the strong components of G.

Define the COMPONENT PARTITION of a simple digraph G as the collection of vertex sets of the components of G.

In this paper $(1 + x)^{\{m\}}$ will be equal to $(1 + x)^m$ if $m > 0$ or identically 1 if $m = 0$.

A relation R on a set S is a PARTIAL ORDER RELATION if it satisfies the three properties:

(i) $a R a$ for all $a \in S$ (REFLEXIVITY)

(ii) $a R b$ and $b R a$ imply a and b are the same element of S (ANTISYMMETRY)

(iii) $a R b$ and $b R c$ imply $a R c$ for all $a,b,c \in S$. (TRANSITIVITY)

S is called the PARTIALLY-ORDERED SET.

An ANTICHAIN U is a subset of S such that for all $a,b \in U$, $a R b$ if and only if a and b are the same element of U. If U has k elements, it is called a k-ANTICHAIN.

Define the POWER SET of a set A as a collection of all the subsets of A, including A and $\emptyset$ (the null set). It can be seen that the power set of any set under the subset relation is a partially-ordered set. A k-antichain of the power set of a set with n elements under the subset relation is called an n-BOOLEAN k-ANTICHAIN.

Let f_n be the number of simple digraphs with n edges of a certain type. Then we define the EDGE SERIES for the type of simple digraph being counted by f_n as the series

$$\sum_{n=0}^{\infty} f_n x^n \quad .$$

Let g_m be the number of simple digraphs with m vertices of a certain type. Then the VERTEX SERIES for the type of simple digraph being counted by g_m is the series

$$\sum_{m=0}^{\infty} g_m \frac{g^m}{m!} \quad .$$

Let h_{mn} be the number of simple digraphs of a certain type with the restriction that each simple digraph have m vertices and n edges. Then we define the COMPLETE SERIES for that type of simple digraph as

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h_{m,n} x^n \frac{y^m}{m!} \quad .$$

Let a simple digraph with property C possess either property A or property B , but not both.

Let the edge series for simple digraphs with properties A, B, and C be $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$, and $\sum_{n=0}^{\infty} c_n x^n$ respectively.

For all n, $c_n = a_n + b_n$. Therefore

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \quad .$$

There are analogous results for vertex series and complete series. This result will be referred to as the ADDITION THEOREM.

Let the edge set of a simple digraph of type A be the union of the edge sets of two simple digraphs of types B and C respectively, where the union of any two simple digraphs of types B and C respectively will be a simple digraph of type A . Further, suppose that any simple digraph of type A is respresentable in one and only one way as a union of simple digraphs of types B and C . Let the edge series for simple digraphs of types A, B, and C be

$$\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n , \sum_{n=0}^{\infty} c_n x^n \quad \text{respectively .}$$

A simple digraph of type A with n edges must be the union of a simple digraph of type B with c edges and a simple digraph of type C with n-i edges in one and only one way. Let such a simple digraph be of type A_i . These are $b_i c_{n-i}$ such simple digraphs. Because all the types $A_0, A_1, \ldots, A_n$ are mutually exclusive, by repeated application of the addition theorem we obtain $a_n = b_0 c_n + b_1 c_{n-1} + \ldots + b_n c_0$ for all n. Thus

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) = \left(\sum_{n=0}^{\infty} b_n x^n \right) \left(\sum_{n=0}^{\infty} c_n x^n \right).$$

This result will be called the MULTIPLICATION THEOREM.

The STIRLING NUMBER $s(n,m)$ OF THE FIRST KIND is the coefficient of x^m in the expansion of $x(x+1) \ldots (x+n-1)$. The binomial theorem gives the result

$$(1 - y)^{-x} = 1 + xy + \frac{x(x+1)}{2} y^2 + \ldots + \frac{x(x+1) \ldots (x+n-1)}{n!} y^n + \ldots$$

But

$$(1 - y)^{-x} = \exp \left[x \log \frac{1}{1-y} \right],$$

that is,

$$(1 - y)^{-x} = 1 + x \log \frac{1}{1-y} + \frac{x^2}{2} \log^2 \frac{1}{1-y} + \ldots + \frac{x^m}{m!} \log^m \frac{1}{1-y} + \ldots$$

$s(n, m)$ is the coefficient of $x^m \frac{y^n}{n!}$ in $(1-y)^{-x}$. It is also, by the last equality, the coefficient of $\frac{y^n}{n!}$ in $\frac{1}{m!} \log^m \frac{1}{1-y}$. This will be called the STIRLING FORMULA.

In the following discussion, we investigate the problem of enumerating n-Boolean k-antichains. Some preliminary work is done on the more general problem of enumerating k-antichains on any given partially-ordered set. As a by-product of these investigations, we discover a method fo formulating a complete series for strongly connected simple digraphs, as well as a complete series for acyclic digraphs.

2. History of the Problem

A one-to-one correspondence between n-Boolean k-antichains and elements of a free distributive lattice on n generators can be established as follows. Let $\{A_1, A_2, \ldots, A_k\}$ be the set of sets which form the antichain. Then

$$\bigcup_{a=1}^{k} \bigcap_{\alpha \in A_\alpha} \alpha$$

is an element of the free distributive lattice on n generators (the partial order relation being formal set inclusion), and conversely.

The problem of determining the number of elements of the free distributive lattice on n generators was essentially posed by Dedekind in 1897 [3], and solved by him for $n = 1,2,3,4$. R. Church [12] obtained the solution for $n = 5,6$. His solution for $n = 7$ is disputed by F. Lennon [7]. In [1], Church presents an analysis of the free distributive lattices on 5 or fewer generators by rank and by conjugacy class.

In 1954, E. N. Gilbert [4] established the lower bound $2^{\binom{n}{[\frac{n}{2}]}}$ on the number of elements in the free distributive lattice, where $[\frac{n}{2}]$ is the greatest integer less than $\frac{n}{2}$. He also obtained the upper bound $n^{\binom{n}{[n/2]+2}}$.

B. Korobkov [8] improved the upper bound to $2^{4.23\binom{n}{[\frac{n}{2}]}}$.

In 1966, G. Hansel [5] reduced the upper bound to

$$3^{\binom{n}{[\frac{n}{2}]}} .$$

In 1968, D. Kleitman [6] showed that the logarithm of the size of the free distributive lattice on n generators is asymptotic to $\binom{n}{[\frac{n}{2}]}$.

If we designate the total number by N_n , these results thus show that:

(a) $2^{\binom{n}{[\frac{n}{2}]}} < N_n < 3^{\binom{n}{[\frac{n}{2}]}}$,

(b) $N_n \sim e^{\binom{n}{[\frac{n}{2}]}}$.

3. The Complete Series for the Strongly Connected Simple Digraph.

Let the complete series for the strongly connected simple digraph be represented by

$$D(x,y) = \sum_{n=1}^{\infty} D_n(x) \frac{y^n}{n!} .$$

Let a simple digraph with n vertices and k components have the same component partition as it has strong component partition, and let the spread of the common partitions be $\langle i_1, i_2, \ldots, i_n \rangle$. The edge series for digraphs of this type is the product of the edge series for each component, since the components satisfy the conditions of the multiplication theorem. This product is

$$D_1^{i_1}(x)\, D_2^{i_2}(x) \ldots D_n^{i_n}(x) .$$

The number of component partitions with spread $\langle i_1, i_2, \ldots, i_n \rangle$ is

$$\frac{n!}{1!^{i_1} 2!^{i_2} \ldots n!^{i_n}\, i_1!\, i_2! \ldots i_n!} .$$

Since a simple digraph can have only one component partition, we can use the addition theorem to obtain the edge series for simple digraphs with identical component and strong component partitions and k components as

$$\Delta_n^{[k]}(x) = \sum \frac{n! \quad D_1^{i_1}(x) D_2^{i_2}(x) \ldots D_n^{i_n}(x)}{1!^{i_1} 2!^{i_2} \ldots n!^{i_n}\, i_1!\, i_2! \ldots i_n!} ,$$

all n-spreads $\langle i_1, i_2, \ldots, i_n \rangle$ such that $i_1 + i_2 + \ldots + i_n = k$.

With the same range for $\sum$, we can write

$$\sum_{n=1}^{\infty} \Delta_n^{[k]}(x) \frac{y^n}{n!}$$

$$= \sum_{n=1}^{\infty} \sum \prod_{j=1}^{n} \frac{[D_j(x)y^j/j!]^{i_j}}{i_j!}$$

$$= \frac{1}{k!} \sum_{i_1+i_2+\ldots+i_n=k} \binom{k}{i_1, i_2, \ldots, i_n, \ldots} \prod_{j=1}^{\infty} \left[\frac{D_j(x)y^j}{j!} \right]^{i_j}$$

$$= \frac{1}{k!} D^k(x,y) ,$$

where we use $\binom{k}{i_1, i_2, \ldots, i_n, \ldots}$ to denote the multinomial coefficient.

Any simple digraph G can be constructed by taking the union of a simple digraph on t vertices for some t with identical component and strong component partitions, where the simple digraph is a subgraph of tail components of G , and any subgraph of the graph whose edge set has no head vertex among any of the t vertices. There are (n-1)(n-t) such edges if G has n vertices.

Consider the edge series

$$(1+x)^{\{(n-1)n\}} + \sum_{k=1}^{n} (-1)^k \sum_{t=1}^{n} \binom{n}{t} \Delta_t^{[k]}(x) \cdot (1+x)^{\{(n-1)(n-t)\}} .$$

In this expression, each simple digraph with exactly r tail components on exactly t vertices is counted $(-1)^k\binom{r}{k}$ times by each edge series

$$\sum_{t=1}^{n} \binom{n}{t} \Delta_t^{[k]}(x)(1+x)^{\{(n-1)(n-t)\}} ,$$

and so is counted 0 times altogether, r being always at least one. Since $\Delta_t^{[k]}(x)$ is the coefficient of $\frac{y^t}{t!}$ in $\frac{1}{k!} D^k(x,y)$, $\sum_{k=1}^{n} (-1)^k \Delta_t^{[k]}(x)$ is the coefficient of $\frac{y^t}{t!}$ in $\exp[-D(x,y)] -1$; we shall write this coefficient as $\epsilon_t(x)$.

We then obtain

$$(1+x)^{\{(n-1)n\}} + \sum_{t=1}^{n} \binom{n}{t} \epsilon_t(x)\cdot(1+x)^{\{(n-1)(n-t)\}} = 0 \quad .$$

This result can also be written as

$$\frac{(1+x)^{(n-1)n}}{n!(1+x)^{n^2/2}} + \sum_{t=1}^{n} \frac{{}_t(x)}{t!(1+x)^{t^2/2}} \circ \frac{(1+x)^{(n-t-1)(n-t)}}{(n-t)!(1+x)^{(n-t)^2/2}} = 0 ,$$

when $x \neq -1$.

Therefore we obtain

<u>Theorem 1</u>. Let $\epsilon_t(x)$ be the coefficient of $\frac{y^t}{t!}$ in $\exp[-D(x,y)] - 1$. Let $x \neq -1$ and

$$D(x,y) = 1 + \sum_{m=1}^{\infty} \frac{(1+x)^{(n-1)(n)}}{(1+x)^{n^2/2}} \circ \frac{y^n}{n!} \quad .$$

Then we have

Corollary 1.1. $E(x,y) = 1/\delta(x,y)$.

Also, take the identity

$$(1+x)^{\{(n-1)n\}} + \sum_{t=1}^{n} \binom{n}{t} \epsilon_t(x)\cdot(1+x)^{\{(n-1)(n-t)\}} = 0 .$$

Substitute $n = 1$ and $x = -1$. The result is

$$1 + \epsilon_1(-1) = 0 .$$

If $n > 1$ and $x = -1$, we obtain

$$0 + \epsilon_n(-1) = 0 .$$

Therefore

$$\exp[-D(-1,y)] - 1 = -y$$

or $D(-1,y) = \log \frac{1}{1-y}$

Thus, we have

Lemma 1. $D(-1,y) = \log \frac{1}{1-y} = -\log(1-y)$.

Further, observe that any acyclic digraph A on n vertices can be constructed by taking the union of any acyclic digraph B on $n-k$ vertices and any simple digraph whose edge set is a subset of $(V(A) - V(B)) \overset{A}{\nabla} V(B)$ which has $k(n-k)$ edges. We note that $V(A) - V(B)$ is a subset of the set of sources of A .

Consider the edge series

$$A_n(x) - n\, A_{n-1}(x)(1+x)^{\{n(-1)\}} + \dots$$

$$+ (-1)^k \binom{n}{k} A_{n-k}(x)(1+x)^{\{k(n-k)\}} + \ldots$$

$$+ (-1)^n A_0(x) \quad .$$

Each acyclic digraph with exactly r sources is counted $(-1)^k \binom{r}{k}$ times by each edge series $\binom{n}{k} A_{n-k}(x)(1+x)^{\{k(n-k)\}}$ and, because r is at least one, is not counted at all in the edge series. Thus the series is identically zero, or if

$$A(x,y) = 1 + \sum_{m=1}^{\infty} \frac{A_m(x)}{(1+x)^{m^2/2}} \cdot \frac{y^m}{m!}$$

and

$$I(x,y) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(1+x)^{k^2/2}} \cdot \frac{y^k}{k!} \quad ,$$

then $A(x,y) = 1/I(x,y)$.

4. The Inclusion-Exclusion Formula Applied to k-Antichains of a Partially-Ordered Set.

Let L be a partially-ordered set, and let L be the corresponding partial order relation (a L b is to be read as "a is L-related to b").

Let G be a simple digraph such that V(G) is the set of integers {1,2,...,k} . Let $S = \langle s_1, s_2, \ldots, s_k \rangle$ be a k-vector of variable constituents. Define the PROPERTY SET of G on S as the set of all expressions of the form $s_i \; L \; s_j$, $\langle i,j \rangle \in E(G)$. Define the G-COLLECTION of L as the set of all k-vectors $\sum$ of elements of L whose constiticents satisfy the conditions imposed by the members of the property set of G on S , upon substitution of $\sum$ for S .

For every k-vector of the G-collection of L , if there is a ditrack from i to j in G , then the i-th constituent of the k-vector is L-related to the jth constituent. This follows naturally from the transititivity property of the partial order relation L . Further, if i and j are vertices in the same strong component of G , then the i-th and j-th constituents of the k-vectors in the G-collection of L must be the same element of L , by the antisymmetry property of the partial order relation L .

Now, because the vertices are designated by integers, we can speak of the smallest vertex in a given simple digraph. Let $\{v_1, v_2, \ldots, v_m\}$ be the set of vertices, one from the vertex set of each strong component of a simple digraph G, which are the smallest vertices in their respective strong components. Let v_1 be the smallest of these, v_2 the next smallest, and so on. Let $T = \langle t_1, t_2, \ldots, t_m \rangle$ be a vector of variable constituents, and let r be the set of all expressions of the form $t_p \; L \; t_q$, where i is in the same strong component as v_p, j is in the same strong component as v_q, $s_i \; L \; s_j$, and $p \neq q$. Then any m-vector θ of elements of L satisfying all the properties of r upon substituting θ for T corresponds to an elements of the G-collection of L by the substitution of t_p for all s_i, where i is in the same strong component as v_p, and conversely.

Now r is the property set for some acyclic digraph K, which we call the IMPLOSION of G. We see that the C-collection and the K-collection of L have the same number of elements. K is the implosion of G if and only if: G has strong component partition $\{V_1, V_2, \ldots, V_m\}$ if $m = |V(k)|$, where v_i is the smallest vertex in V_i; and, if $\langle i,j \rangle \in E(K)$, then $V_i \overset{G}{\nabla} V_j$ does not have a null edge set.

The edge series for simple digraphs G with the above conditions satisfied for a fixed acyclic digraph K is the product of the edge series for strong components on each vertex set in the partition multiplied by the product of the edge series for a non-null coboundary $V_i \overset{G}{\nabla} V_j$, $<i,j> \in E(K)$, since all these edge sets are disjoint and independent, as required by the multiplication theorem. This product is dependent only on the number of vertices in each V_i .

Since the number of m-divisions such that $|V_i| = m_i$ is $\frac{n!}{\prod_{i=1}^{m} m_i!}$, we see that the edge-series for simple digraphs on n vertices whose implosion is K is

$$\sum_{m_1+m_2+\ldots+m_m = n} \frac{n!}{\prod_{i=1}^{m} m_i!} \prod_{i=1}^{m} D_{m_i}(x) \cdot \prod_{<i,j> \in E(K)} [(1+x)^{m_i - m_j} - 1] \ .$$

Upon substituting $x = -1$, this expression becomes the co-efficient of $\frac{y^n}{n!}$ in

$$\frac{(-1)^{|E(K)|}}{m!} D^m(-1,y) = \frac{(-1)^{|E(K)|}}{m!} \log^m \frac{1}{1-y}$$

$$= (-1)^{|E(K)|} s(n,m) \ ,$$

by employing the Stirling Formula. Hence, we have

<u>Lemma 2</u>. If $x = -1$ is substituted in the formula for the edge series for simple digraphs on n vertices whose implosion is an acyclic digraph on m vertices with q edges, the result is $(-1)^q s(n,m)$, where $s(n,m)$ is a Stirling number of the first kind.

By direct application of the inclusion-exclusion method, we have

<u>Lemma 3.</u> To enumerate k-antichains of L , we must divide k! into the sum of $(-1)^{|E(G)|}$ times the G-size of L , the sum being over all simple digraphs G on the vertex set $\{1, 2, \ldots, k\}$.

Suppose we consider all the terms involving any G such that the implosion of G is K in the summation of $x^{|E(G)|}$ times the G-size of L over all G with k vertices. These terms represent precisely the edge series for all simple digraphs on k vertices with implosion K , multiplied by the K-size of L , where the K-size of L is equal to the G-size of L over all G with implosion K . These considerations, along with Lemmas 2 and 3, lead us to

<u>Lemma 4.</u> The number of k-antichains of L is 1/k! times the sum over all acyclic digraphs K on $1 \le p \le k$ vertices and q edges of $(-1)^q s(k,p)$ times the K-size of L .

There is a unique partial order resulting from the addition of all transversals to a given acyclic digraph. Also, by the transitivity of the partial order relation L , addition of transversals to an acyclic digraph U does not affect the U-size of L . The edge series for acyclic digraphs U such that the partial order formed by adding all t transversals of U to its edge set is a given partial order P is $(1+x)^{\{t\}}$. Upon substituting $x = -1$, as required to eliminate acyclic digraphs from consideration in Lemma 4 unless they are partial orders, we get a coefficient of $s(k,p)(-1)^q$, where P has p vertices and q edges ; of 0, if P has any transversals; and of 1 if P has no transversals, that is, if every edge of P has as tail vertex a source of P. Let such a simple digraph be called a BASIC DIGRAPH. Then we have

Theorem 2. The number of k-antichains of L is $1/k!$ times the sum over all basic digraphs B on $1 \le p \le k$ vertices and q edges of $(-1)^q s(k,p)$ times the B-size of L.

These results apply regardless of the partially-ordered set and partial order relation considered in enumerating antichains.

Further, let E_p be the sum for a fixed p of $(-1)^q$ times the B-size of L over all basic digraphs on p vertices and q edges. Then we can write the formula of Theorem 2 as

$$N_k = \sum_{m=1}^{k} \frac{1}{k!} s(k,m)E_m \quad .$$

Let the maximum number of elements of any antichain of L be w. Then the total number of all antichains of L is

$$\sum_{k=1}^{w} N_k = \sum_{m=1}^{w} \sum_{k=m}^{w} \frac{1}{k!} s(k,m)E_m$$

$$= \sum_{m=1}^{w} E_m \left[\sum_{k=1}^{w} \frac{1}{k!} s(k,m) \right] \quad .$$

As indicated in the introduction,

$$(1-y)^{-x} = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{n!} s(n,m)\, x^m y^n \quad .$$

Consequently,

$$(1 + y + y^2 + \ldots)(1-y)^{-x} = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \left[\sum_{k=1}^{\infty} \frac{1}{(n-k)!} \; s(k,m) \right] x^m y^n \; .$$

Hence, $$(1-y)^{-x-1} = 1 + \sum_{w=1}^{\infty} \sum_{m=1}^{n} \left[\sum_{k=1}^{w} k! \; s(k,m) \right] x^m y^n \quad ,$$

that is $$\sum_{k=1}^{w} N_k = \sum_{m=1}^{w} E_m \frac{1}{(w+1)!} \; s(w+1,m+1) \quad .$$

5. Methods for Simplifying the Enumeration of n-Boolean k-Antichains.

Define the (A,P) - SIZE and the (A,P) - COLLECTION as the P-size and P-collection respectively of any power set of a set A under the subset relation.

Let $S = \langle s_1, s_2, \ldots, s_{v(p)} \rangle$ and $T = \langle t_1, t_2, \ldots, t_{1v(p)} \rangle$ be two $|V(P)|$-vectors of variable constituents. Define the PRIME SET of S and T as the collection of all expressions of the form

$$\left[\bigcap_{x \in v_1} s_x \right] \cap \left[\bigcap_{y \in v_2} t_y \right],$$

where $\langle v_1, v_2 \rangle$ is a 2-division of V(P) . Such an expression will be called a PRIME.

Let $\Sigma = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_{|v(p)|} \rangle$ and $\Phi = \langle A-\Sigma_1, \ldots, A-\Sigma_{|v(p)|} \rangle$, where Σ is in the (A,P)-collection. The result of the substitutions $S = \Sigma$ and $T = \Phi$ in a prime will be called its Σ-VALUE .

Define a REGIONAL of P as a prime whose X-value is not the null set for some X in the (A,P)-collection. Let the number of regionals of P be H(P) .

Let each of the regionals of P be set equal to a different set from any H(P)-division of A . Then the system of equalities can be solved in one and only one way, and that solution must be in the (A,P)-collection, by definition of the term regional. Hence, we obtain

Lemma 5. The (A,P)-size is the n-th power of the numer of regionals of P , where A has n-elements.

Suppose the property set of P on S contains the property $s_i \subseteq s_j$. Then

$$s_i \cap [A-s_j] = s_i \cap t_j = \emptyset \quad ,$$

and all primes of any member of the (A,P)-collection which contain the symbols s_i and t_j can not be regionals since they are $\emptyset$ for every member Σ of the (A,P) collection, where Σ_i is always a subset of Σ_j . Conversely, if for every $s_i \subseteq s_j$ in the property set of P on S a prime ρ does not simultaneously contain both s_i and t_j , then we may construct an element Σ of the (A,P)-collection which assigns a non-null value (in fact, the value A) to ρ as follows. If ρ contains s_i , then set $\Sigma_i = A$; otherwise, Σ_i is the null set (for, if we assume that $\Sigma_i \subseteq \Sigma_j$ is not satisfied in Σ , then $\Sigma_i = A$ and $\Sigma_j = \emptyset$, and thus contains s_i and does not contain s_j , that is, contains t_j , contrary to the conditions imposed on ρ) . Thus, we have

<u>Lemma 6</u>. A prime ρ on S and T is a regional of P if and only if, for all $s_i \subseteq s_j$ in the property set of P on S , ρ does not simultaneously contain both s_i and t_j , where $S = \langle s_1, s_2, \ldots, s_{|v(p)|} \rangle$, $T = \langle t_1, \ldots, t_{|v(p)|} \rangle$.

Let K_i be the set of all vertices j of P such that $s_i \subseteq s_j$ is in the property set of P . Then, if a regional of P contains s_i , it must also contain all s_j , $j \in K_i$. Therefore, we can state

Lemma 7. $H(P) = H(P - \{i\}) + H(P - K_i - \{i\})$, where each i is a source of P ; also, if P is null, $H(P) = 1$.

By induction on the number of vertices in the smallest component of P , i a vertex in that component, and by employment of Lemma 7, we otbain

Lemma 8. The number of regionals of P is the product of the number of regionals of each component of P .

6. Numerical Results

Herein, $S_i(n)$ will represent the sum for a fixed i of $(-1)^q$ times the B-size of a power set of a set with n elements over all basic digraphs on i vertices and q edges. Let $N_k(n)$ represent the number of k-antichains of such a power set. Then, from Section 4 , we write

$$N_k(n) = \sum_{m=1}^{k} \frac{1}{k!} s(k,m) S_m(n) \quad .$$

Table 1 gives the computed values of $S_m(n)$ for m from 1 to 7 as a function of n , the size of the set whose subsets form the power set on which we are counting antichains.

Table 2 gives the values of $N_k(n)$ for k from 1 to 10 and n from 1 to 5 , and the total number of antichains for each n from 1 to 5 , thus verifying previous results on the size of the small free distributive lattices.

Table 1

$$S_1 = 2^n$$

$$S_2 = 4^n - 2.3^n$$

$$S_3 = 8^n - 6.6^n + 6.5^n$$

$$S_4 = 16^n - 12.12^n + 24.10^n + 4.9^n - 24.8^n + 6.7^n$$

$$S_5 = 32^n - 20.24^n + 60.20^n + 20.18^n + 10.17^n - 120.16^n$$
$$-120.15^n + 150.14^n + 120.13^n - 120.12^n + 20.11^n$$

$$S_6 = 64^n - 30.48^n + 120.40^n + 60.36^n + 60.34^n - 12.33^n$$
$$-360.32^n - 720.30^n + 810.28^n + 120.27^n + 480.26^n$$
$$+ 360.25^n - 180.24^n - 720.23^n - 240.22^n - 720.21^n + 600.20^n$$
$$+ 750.19^n - 360.18^n - 360.17^n + 180.16^n - 20.15^n$$

$$S_7 = 128^n - 42.96^n + 210.80^n - 490.72^n + 210.68^n$$
$$-84.66^n - 840.64^n - 2520.60^n + 2730.56^n + 840.54^n$$
$$+ 840.52^n - 420.51^n + 2960.50^n + 1260.48^n - 5040.46^n$$
$$+ 840.45^n - 1260.44^n + 1680.43^n - 10920.42^n$$
$$+ 1260.41^n - 760.40^n - 7560.39^n + 8610.38^n$$
$$+ 5880.37^n + 7140.36^n + 1302.35^n - 2520.34^n$$
$$+ 3360.33^n + 7560.32^n - 3360.31^n + 6580.30^n$$
$$+ 13860.29^n + 7560.28^n - 7560.27^n + 420.26^n$$
$$+ 2520.25^n - 840.24^n + 70.23^n$$

Table 2

$N_k(n)$

k\n	1	2	3	4	5	6
1	2	4	8	16	32	64
2	0	1	9	55	285	1351
3	0	0	2	64	1090	14000
4	0	0	0	25	2020	82115
5	0	0	0	6	2146	304752
6	0	0	0	1	1380	
7	0	0	0	0	490	
8	0	0	0	0	115	
9	0	0	0	0	20	
10	0	0	0	0	2	
TOTAL NOT COUNTING THE NULL SET AS A ZERO ANTICHAIN	2	5	19	167	7580	

References

[1] R. CHURCH, Numerical Analysis of Certain Free Distributive Structures, Duke Math. J. 6(1940), 732-734.

[2] R. CHURCH, Enumeration by Rank of the Elements of the Free Distributive Lattice with Seven Generators, Notices of the Amer. Math. Soc. 12(1965), 724.

[3] R. DEDEKIND, Ueber Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler, Fest. Hoch. Braun. U. GES. Werke (1897), II, 103-148.

[4] E. N. GILBERT, Lattice-Theoretic Properties of Frontal Switching Functions, J. Math. Phys. 33(1954), 57-67.

[5] G. HANSEL, Sur le nombre des fonctions booteenes monotones de n variables, C. R. Acad. Sci. Paris 262(1966), 1088-1090.

[6] D. KLEITMAN, On Dedekind's Problem: The Number of Monotone Boolean Functions, Proc. Amer. Math. Soc. 21(1969), 677-682.

[7] B. H. KOROBKOV, Problemg Hibernet 13(1965), 5-28.

[8] F. LUNNON, The IU Function: The Size of a Free Distributive Lattice, in Combinatorial Mathematics and its Applications (1971), ed. J. A. Welsh, Academic Press, 173-181.

[9] J. REILLY and R. G. STANTON, The Intersection-Union Problem, Congressus Numerantium $\overline{x}$, Proceedings of the Manitoba Conference on Numerical Mathematics (1971), Utilitas Math. Publ. Inc., 551-555.

[10] K. YAMAMOTO, Logarithmic Order of Free Distributive Lattices, J. Math. Soc. Japan 6(1964), 343-353.