THE UNIVERSITY OF MANITOBA

A SENSITIVITY STUDY BASED ON A CONTINUOUS VARIABLE m IN POLYNOMIAL DECOMPOSITION

BY

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Abstract

This thesis develops a new approach to the optimum decomposition of the second order polynomial in active RC network synthesis.

The main feature of this approach is that the decomposition is based on a new parameter $m = \tan\phi$, where ϕ is the angle of pole sensitivity with respect to the real axis. The angle ϕ renders qualitative estimation of the center frequency sensitivity and quality factor sensitivity.

The decomposition of three different types and various design criteria are expressed explicitly as a function of m, so that the optimum decomposition, with respect to a certain specified performance criterion, is easily obtained by a simple substitution. It is shown that in the decompositions with real phantom zeros the value of m varies continuously within a certain constrained interval; however, in the decomposition with complex phantom zeros, no such constraint exists.

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Contents

1 Introduction

Since J. G. Linvill [18] demonstrated the possibility of constructing inductorless filters in his pioneering paper in 1954, numerous methods of active RC network synthesis have been developed. The prominent aspect of the existing techniques [1] - [4], however, may be characterized by the fact that a certain performance criterion being specified, the optimum decomposition of the polynomial $D(s)$, with respect to the criterion, is obtained in the form of

$$D(s) = a\, D_1(s) \pm x\, b\, D_2(s)\,, \tag{1.1}$$

where x is the parameter of interest, s is the complex variable, a and b are nonnegative real numbers, and $D_1(s)$ and $D_2(s)$ are monic polynomials with certain constraints. Sensitivity minimization by means of polynomial decomposition, therefore, has been the main subject of interest in system design, particularly in active RC network synthesis. In addition to sensitivity, post-design adjustment considerations have made it common practice to realize higher order active RC filters by cascading second order sections [1], [3], [19]. Accordingly, in this study, the decomposition of the second order polynomials is considered exclusively.

The novel feature of the present method is the introduction of a versatile variable $m = \tan\phi$, where ϕ is the angle of pole sensitivity, in the decomposition before specifying performance criteria in such a way that m will be varied in a constrained range of real numbers. It has been shown that the continuous variation of m not only covers the existing optimum decompositions [4], [12] but also offers a variety of decompositions that may be chosen by a designer to be optimum in different senses. The proposed method, therefore, allows us to view overall performance taking the diverse criteria into account simultaneously.

Development of this work is based on the recognition that the magnitude of pole sensitivity depends on the angle ϕ, and also it is the angle ϕ that renders various measures of practical significance such as center frequency sensitivity, quality factor sensitivity and stability margin. After straightforward calculations, it is shown that the polynomial $D(s)$ can be expressed in the form of

$$D(s) \doteq a'(m)\, D_1'(s,m) \pm \chi\, b'(m)\, D_2'(s,m) \,. \quad (1.2)$$

Subsequently, the design criteria such as the magnitude of pole sensitivity, center frequency sensitivity, quality factor sensitivity, stability margin (S.M.) are written in terms of m, respectively, as

$$|S_x^P| = f_1(m) ,$$

$$S_x^{\omega_p} = f_2(m) , \tag{1.3}$$

$$S_x^Q = f_3(m) ,$$

$$S.M._p = f_4(m) ,$$

where $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$, $f_4(\cdot)$ are relatively simple functions.

The well-known Horowitz decomposition [4] turned out to be the consequence of minimizing $f_1(m)$ and $f_2(m)$ leaving the possibility of optimizing other criteria, and in Calahan case [12] which is optimum in terms of $f_1(m)$, but not unique, it is possible to optimize any combination of these criteria simultaneously within the permissible range of m.

Usefulness of the proposed method stems from the fact that the variable m can be varied continuously within a considerably wide range. Examples are provided to illustrate the versatility of the proposed method in chapter 3, where the main part of the work is presented. Curves of $|S_x^P|$, $S_x^{\omega_p}$, S_x^Q vs. m are plotted to demonstrate the significant effects of m as a design parameter.

The fundamental concepts of various sensitivities and related topics are described in chapter 2. They are prerequisites to the main development of this work.

2 Sensitivity in Active RC Circuits

In an active filter, the realization of complex natural frequencies is achieved by decomposing the characteristic polynomial, that is, the denominator polynomial of a specified network function, into the difference or sum of two suitably constrained polynomials.

The polynomials are substracted in filters using negative-impedance converter (NIC) [1] - [3], while they are added in filters employing gyrator or controlled sources with negative feedback [1] - [5].

In both cases, the filter becomes increasingly sensitive to active parameter changes as the complex pole pair moves closer to the imaginary axis. The question of sensitivity to parameter changes is, therefore, a matter of great importance in active RC filter design.

There are various sensitivity measures by means of which the performance of various types of active RC circuits can be evaluated and compared.

In this chapter, the most commonly used sensitivity measures, viz., the sensitivity of network function, pole, quality factor and center frequency, are introduced.

A. NETWORK FUNCTION SENSITIVITY

It has been shown that in a linear time-invariant circuit, the network functions, in general, have a bilinear dependence on a single parameter of the given network [3], [6]. There are some exceptions to this relationship [1], [21]. Such exceptions, however, are of relatively little significance to this study. Therefore, the transfer function of a linear active filter can be written as

$$T(s) = \frac{N(s)}{D(s)} = \frac{N_1(s) + x N_2(s)}{D_1(s) + x D_2(s)}, \qquad (2.1)$$

where x is a variable parameter of interest and polynomials $N_1(s)$, $N_2(s)$, $D_1(s)$ and $D_2(s)$ are functions of all other fixed parameters and complex frequency s.

In the operation of active filters, it is very important to investigate the effects of variations of active parameters on the transfer function performance. The useful definition of network function sensitivity with respect to the variation of a single parameter is generally taken as the reciprocal of Bode's sensitivity function [7].

Definition 2-1. The sensitivity function $S_x^{T(s)}$ of the network function $T(s,x)$ due to variations of the network parameter x is defined as

$$S_{x}^{T(s)} \triangleq \frac{d[\ln T(s)]}{d[\ln x]} = \frac{dT(s) / T(s,x)}{dx / x} . \qquad (2.2)$$

In other words, the sensitivity of T(s), with respect to x is the percentage change in T(s) divided by the percentage change in x , with all changes considered differentialy small. It should be noted that $S_{x}^{T(s)}$ is a function of complex variable s.

From Equation (2.1) and Definition 2-1, it immediately follows that

$$S_{x}^{T(s)} = \frac{D_1(s)}{D(s)} - \frac{N_1(s)}{N(s)} , \qquad (2.3)$$

or

$$S_{x}^{T(s)} = x \left[\frac{N_2(s)}{N(s)} - \frac{D_2(s)}{D(s)} \right] . \qquad (2.4)$$

B. POLE SENSITIVITY

In some situations, for example, in a band pass filter having a high selectivity, or in a notch filter with a sharp depth, the variation of a resonant frequency or the change of a notch frequency could be critical. As an estimate of the change in pole or zero locations of network function, due to an incremental variation in a network parameter, pole or zero sensitivity can be defined.

Definition 2-2. Let $s = p$ be a pole or zero of the network function $T(s,x)$ when x takes its nominal value. The pole or zero sensitivity of $T(s,x)$ is then defined as [8],

$$S_x^p \triangleq \left.\frac{dp}{dx/x}\right|_{s=p} . \tag{2.5}$$

The concepts of pole and zero sensitivity, being based on the roots of polynomial, are obviously identical, and often they are simply referred to as " root sensitivity". Root sensitivity is a number in contrast to $S_x^{T(s)}$ which is a function of the complex frequency, s. If the root is real, the corresponding root sensitivity is real, otherwise it is a complex number.

C. QUALITY FACTOR AND CENTER FREQUENCY SENSITIVITY

Since active filters are most practical when realized as a cascade of second order sections [1], [3], only a second order transfer function is considered.

The two system parameters commonly used to characterize a second order filter section are the quality factor (usually denoted by Q .) and the center, or peak frequency (usually denoted by ω_p).

Let

$$D(s) = s^2 + 2\sigma_c s + \omega_p^2 \tag{2.6}$$

be the denominator of a second order transfer function where $\sigma_c^2 - \omega_p^2 < 0$ for complex roots. In filter theory, $2\sigma_c$ is termed as the "3-db bandwidth" of a bandpass filter realized by a pair of complex poles and ω_p is called "center or peak frequency" of the filter. The quality factor of a network is usually defined as the magnitude of the distance from either pole to the origin devided by twice the magnitude of the real part of the pole location, i.e.,

$$Q \triangleq \frac{\omega_p}{2\sigma_c} . \tag{2.7}$$

Thus, the Q factor can be used to get a rough estimate of the position of a pair of complex poles. The larger the Q is, the closer the poles are to the $j\omega$ axis for a given ω_p. In case of frequency selective filters, the quality factor of a pole pair also determines the shape of resonance. Therefore, Q sensitivity and ω_p sensitivity which give a measure of the effect of parameter changes on the resonance shape and possibly on the stability margin for high Q cases, are very important.

Definition 2-3. The Q and ω_p sensitivities are defined [3] as

$$S_x^Q \triangleq \frac{d[\ln Q]}{d[\ln x]} = \frac{dQ/Q}{dx/x} , \tag{2.8}$$

$$S_x^{\omega_p} \triangleq \frac{d[\ln \omega_p]}{d[\ln x]} = \frac{d\omega_p/\omega_p}{dx/x} . \tag{2.9}$$

where x is a variable parameter.

These sensitivities can be shown to be closely related to the corresponding pole sensitivity [9].

From Equation (2.6), the pole p and its conjugate p* can be found as

$$p\,,\,p^* \;=\; -\sigma_c \;\pm\; j\,\omega_c\;, \tag{2.10}$$

where
$$\omega_c^2 \;=\; \omega_p^2 - \sigma_c^2\,. \tag{2.11}$$

The pole variation can be written as

$$\frac{dp}{p} \;=\; Re\left(\frac{dp}{p}\right) + j\,Im\left(\frac{dp}{p}\right)\,. \tag{2.12}$$

It is observed that when dp/p is real, dp/p moves along a radial direction; when dp/p is imaginary, it is tangential to the circle through p, which is centered at the origin as shown in Figure 2.1. Clearly, this defines an

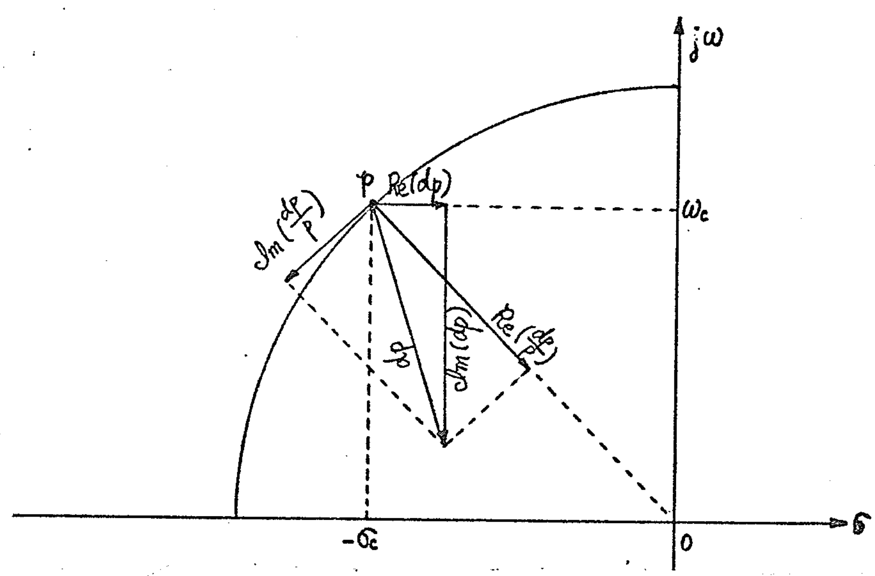

Fig. 2.1 Pole Variation

alternative orthogonal coordinate system $dp = Re(dp) + j\,\mathfrak{Im}(dp)$.

Equation (2.12) can be written as

$$\frac{dp}{p} = \frac{-d\sigma_c + j\,d\omega_c}{-\sigma_c + j\,\omega_c} . \tag{2.13}$$

Thus, from Equations (2.6) and (2.13), we have

$$Re\left(\frac{dp}{p}\right) = \frac{\sigma_c\,d\sigma_c + \omega_c\,d\omega_c}{\omega_p^2} = \frac{d\omega_p}{\omega_p} , \tag{2.14}$$

$$\mathfrak{Im}\left(\frac{dp}{p}\right) = \frac{\omega_c\,d\sigma_c - \sigma_c\,d\omega_c}{\omega_p^2} = \frac{\sigma_c}{\omega_c} \cdot \frac{dQ}{Q} \tag{2.15}$$

$$= - \frac{dQ/Q}{\sqrt{4Q^2-1}} . \tag{2.15}$$

Therefore,

$$\frac{dp}{p} = \frac{d\omega_p}{\omega_p} - j\,\frac{dQ/Q}{\sqrt{4Q^2-1}} . \tag{2.16}$$

From Equation (2.5), it follows that

$$S_x^p = \frac{-d\sigma_c + j\,d\omega_c}{dx} \cdot x ,$$

therefore,

$$\frac{S_x^p}{p} = \frac{x}{dx}\left\{ \frac{\sigma\,d\sigma + \omega_c\,d\omega_c}{\omega_p^2} + j\,\frac{\omega_c\,d\sigma - \sigma\,d\omega_c}{\omega_p^2} \right\} . \tag{2.17}$$

Substituting Equations (2.14) and (2.15) into Equation (2.17), and from Definition 2-3, Equation (2.17) can be written as,

$$\frac{S_x^p}{p} = S_x^{\omega_p} - j\,\frac{1}{\sqrt{4Q^2-1}}\cdot S_x^Q \quad . \tag{2.18}$$

Let

$$X = S_x^{\omega_p} \ , \tag{2.19}$$

and

$$Y = \frac{1}{\sqrt{4Q^2-1}}\cdot S_x^Q \quad . \tag{2.20}$$

Then, at $p = -\sigma_c + j\,\omega_c$, S_x^p becomes

$$S_x^p = M\,e^{j\phi} \ , \tag{2.21}$$

where

$$M = [(\sigma_c X - \omega_c Y)^2 + (\omega_c X + \sigma_c Y)^2]^{\frac{1}{2}} \ , \tag{2.22}$$

$$\phi = \tan^{-1}\frac{\omega_c X + \sigma_c Y}{\omega_c Y - \sigma_c X} \quad . \tag{2.23}$$

Define

$$m \triangleq \tan\phi \tag{2.24}$$

Then,

$$\frac{Y}{X} = \frac{\omega_c/\sigma_c + m}{m\,\omega_c/\sigma_c - 1} \quad . \tag{2.25}$$

Now it is clear that network synthesis, with prescribed angle of pole sensitivity, can be achieved by decomposing a polynomial in such a way that Equation (2.25) is satisfied for a given σ_c and ω_c.

In the following chapter, polynomial decomposition techniques are developed for the three cases such that the decompositions are represented in terms of the angle parameter m.

3 Decomposition Methods

Many active RC network synthesis techniques start with decomposition of the denominator polynomial of a specified network function $T(s)$ in the form of

$$D(s) = a D_1(s) \pm x b D_2(s) \qquad (3.1)$$

where a and b are positive real numbers, x is the parameter of interest, and $D_1(s)$ and $D_2(s)$ are monic polynomials in s.

In this chapter, the effects of continuous variation of pole sensitivity angle will be investigated to obtain useful information on performance criteria such as center frequency sensitivity and pole-Q sensitivity. The optimal angle in terms of the specified criterion and corresponding decomposition will be shown explicitly.

Three types of decomposition, having practical significance, are chosen in this study. For obvious reasons, we will assume that the given polynomial is of second order,

$$\begin{aligned} D(s) &= s^2 + 2\sigma_c s + (\sigma_c^2 + \omega_c^2) \\ &= s^2 + 2\sigma_c s + \omega_p^2 , \end{aligned} \qquad (3.2)$$

where σ_i and ω_i are the real and imaginary parts of the pole respectively, and ω_p is the center frequency.

For convenience, the sum decomposition with each component polynomial having negative real roots only is discussed first and the difference decomposition with each component polynomial having negative real roots and the sum decomposition with one component polynomial having complex roots in the left-half s-plane are presented later.

A. SUM DECOMPOSITION : REAL ROOTS

Consider the decomposition

$$D(s) = a D_1(s) + x b\, D_2(s) , \tag{3.3}$$

where x is assumed to be positive, and the roots of the monic polynomials, $D_1(s)$ and $D_2(s)$, are restricted to the negative real axis including the origin. Let the degree of $D_1(s)$ be the same as that of $D(s)$, and let the degree of $D_2(s)$ be equal to or less than that of $D(s)$. Then, there are there possible cases;

(i) $D_1(s)^\circ = 2 \qquad D_2(s)^\circ = 2$,

(ii) $D_1(s)^\circ = 2 \qquad D_2(s)^\circ = 1$,

(iii) $D_1(s)^\circ = 2 \qquad D_2(s)^\circ = 0$, where $D_1(s)^\circ$ and $D_2(s)^\circ$ denote the degree of $D_1(s)$ and $D_2(s)$, respectively. If we assume that $D(s)$ has no real roots, then the roots of $D_1(s)$ and $D_2(s)$ must have the pattern as shown in Figure 3.1 [10].

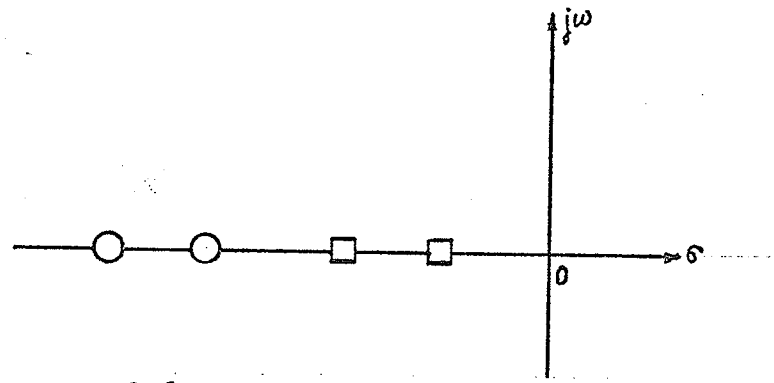

Fig. 3.1 A plot of root pattern.
□ : root of $D_1(s)$, ○ : root of $D_2(s)$

The root closest to the origin (it may lie at the origin) is a root of $D_1(s)$. The two roots closest to infinity (either or both of them may lie at infinity) must belong to $D_2(s)$. It should be noted that since a summation of polynomials is considered in this case, it is possible to interchange the designations of $D_1(s)$ and $D_2(s)$. If the network parameter x deviates from its nominal value, the roots of Equation (3.3) are determined by the root locus of the equation

$$\frac{K\,D_2(s)}{D_1(s)} = -1 \quad , \qquad (3.4)$$

where $K = \frac{b}{a} \cdot x$.

For the second order polynominal, all the root loci are either straight lines or circles and are symmetric with respect to the real axis [11]. The root loci for the three possible cases are shown in Figures 3.2, 3.3 and 3.4. Let us examine the relationship between pole sensitivity angle and root locus in the neighbourhood of nominal value of K.

Let $s = p = -\sigma_c + j\omega_c$ be a root of the polynomial $D(s)$ for the nominal value of K. For an incremental change dK, the root at $s = p$ will shift by an incremental amount $dp = -d\sigma_c + jd\omega_c$. The pole sensifivity S_K^p is

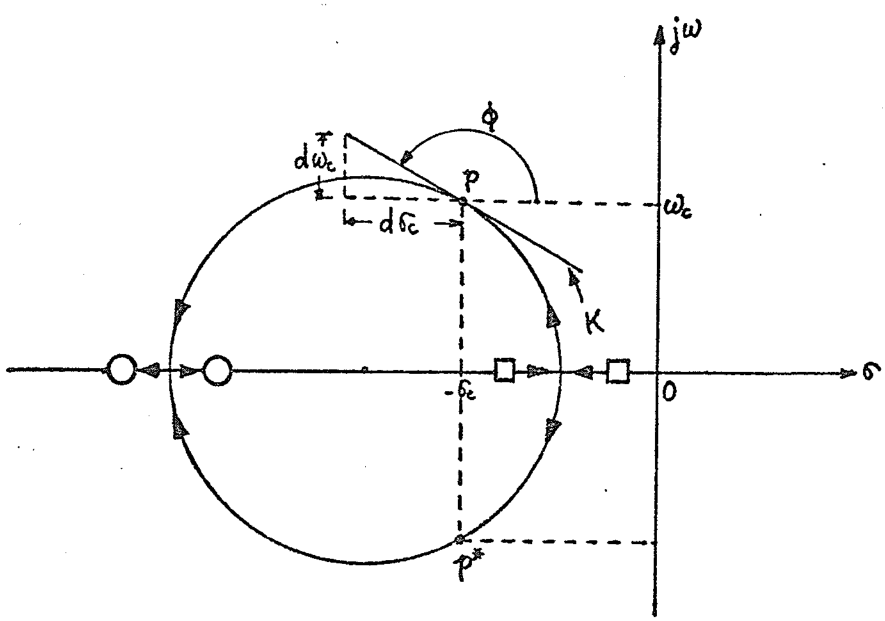

Fig. 3.2 Root locus for $D_1(s)^\circ = 2$ & $D_2(s)^\circ = 2$

□ : root of D (s) O: root of D (s)

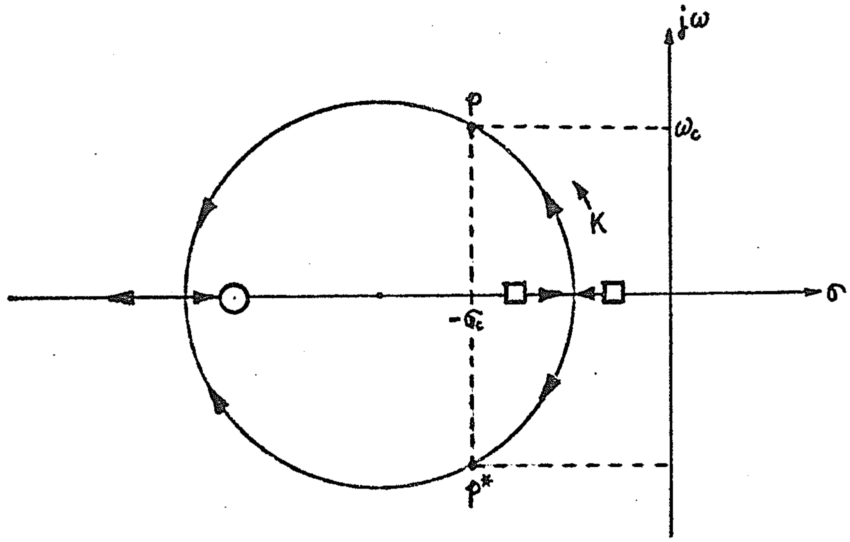

Fig. 3.3 Root locus for $D_1(s)^{\circ} = 2$ & $D_2(s)^{\circ} = 1$.

□: root of $D_1(s)$ O: root of $D_2(s)$

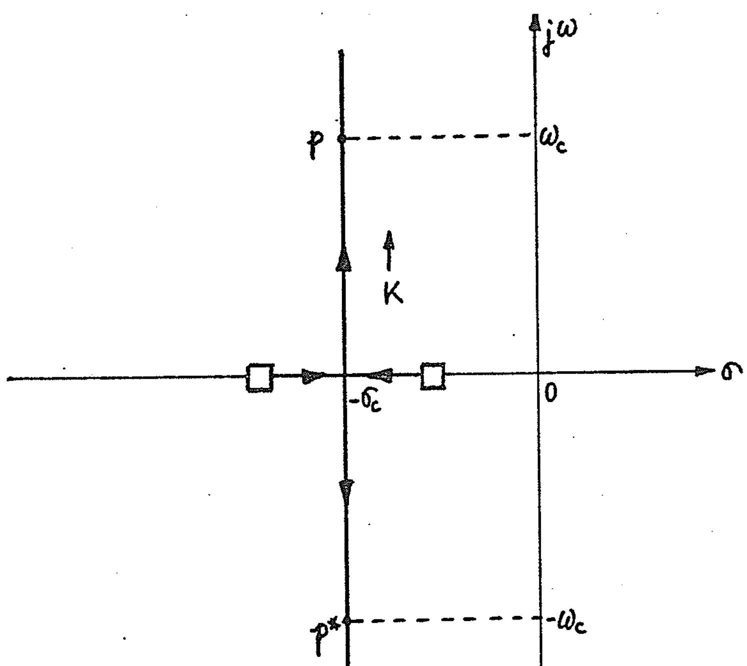

Fig.3.4 Root locus for $D_1(s)^{\circ} = 2$ & $D_2(s) = 0$.

□: root of D (s)

then given by

$$S_K^p = \frac{dp}{dK/K} = \frac{-d\sigma_c + j\,d\omega_c}{dK/K} \quad . \tag{3.5}$$

Since the parameter K is real, the angle ϕ of pole sensitivity is

$$\phi = \tan^{-1} \frac{d\,\omega_c}{-d\,\sigma_c} \quad . \tag{3.6}$$

From the root locus shown in Figure 3.2, it is obvious that the angle of the tangential line to the root locus at $s = p$ is also $\tan^{-1}\frac{d\omega_c}{-d\sigma_c}$.

On the other hand, from Equation (3.3) the pole sensitivity with respect to parameter x at $p = -\sigma_c + j\omega_c$ can be derived as

$$S_x^p = \frac{a\, D_1(p)}{j\, 2\, \omega_c} \quad . \tag{3.7}$$

Thus, the magnitude of pole sensitivity is

$$|S_x^p| = \frac{a\, |D_1(p)|}{2\, \omega_c} \quad . \tag{3.8}$$

Therefore, the magnitude and angle of the pole sensitivity, which are of interest, depend on how the component polynomials $D_1(s)$ and $D_2(s)$ are chosen in the decomposition. It is shown in Appendix A that $|D_1(p)|$

has minimum value when $D_1(s)$ has a double root. For the general sum type of decomposition, Calahan [12] has shown that the minimum magnitude of pole sensitivity occurs when the component polynomials have double roots. Thus, the decomposition that gives minimum magnitude of the pole sensitivity should be written as

$$D(s) = a(s+a_0)^2 + x\,b(s+b_0)^2 , \qquad (3.9)$$

where a, a_0 , b and b_0 are nonnegative real numbers.

The root loci corresponding to this type of decomposition are shown for two different angles in Figure 3.5. It should be noted that the interchange of roots of $D_1(s)$ and $D_2(s)$ would create an identical locus with opposite sense and 1/K replacing K.

Such a decomposition is optimum in the sense that it gives a minimum magnitude of pole sensitivity. It is, however, not unigue because a_0 and b_0 can assume various values, and as can be seen from Figure 3.5, the values of a_0 and b_0 depend on the angle of pole sensitivity.

Now, for a prescribed value m , or for a prescribed angle of pole sensitivity, the optimum decomposition can be found uniquely as follows :

(i) Find a straight line which passes through the given pole p and whose slope is perpendicular to the given m.

The equation of this straight line is

$$\omega = -\frac{\sigma}{m} + \omega_c - \frac{\sigma_c}{m} \qquad (3.10)$$

(ii) Determine the center, $-\sigma_o$, of the root locus which passes through p [See Figure 3.5].

From Equation (3.10), it follows that

$$-\sigma_o = -\sigma_c\left(1 - m\sqrt{4Q^2-1}\right) . \qquad (3.11)$$

(iii) Find the radius r of the root locus.

This can be found as

$$r = \sigma_c\sqrt{(4Q^2-1)(1+m^2)} . \qquad (3.12)$$

(iv) From (3.11) and (3.12), the desired decomposition is obtained by

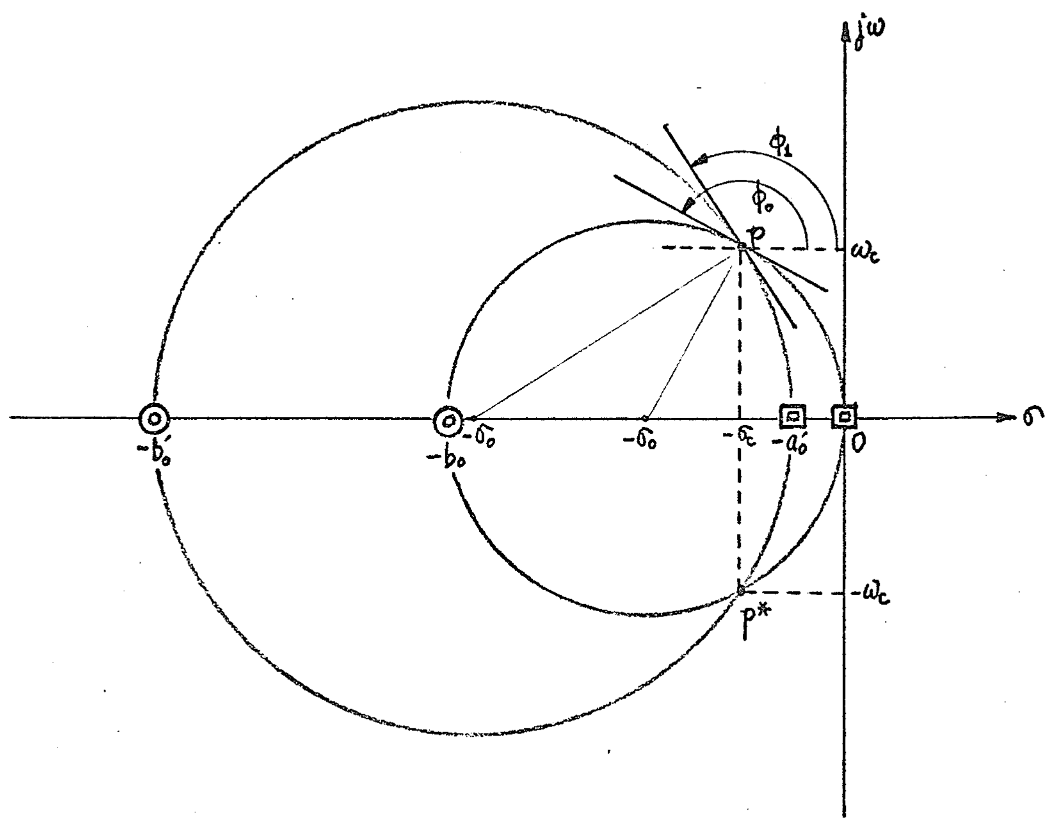

Fig. 3.5 Root locus for optimum decomposition.

$$D(s) = a\left[s + \sigma_c\left\{1 - \sqrt{4Q^2-1}\left(\sqrt{1+m^2} + m\right)\right\}\right]^2$$

$$+ x \cdot b\left[s + \sigma_c\left\{1 + \sqrt{4Q^2-1}\left(\sqrt{1+m^2} - m\right)\right\}\right]^2$$

Therefore, we have

$$a_0 = \sigma_c\left\{1 - \sqrt{4Q^2-1}\ \left(m + \sqrt{1+m^2}\right)\right\}$$

$$b_0 = \sigma_c\left\{1 + \sqrt{4Q^2-1}\ \left(\sqrt{1+m^2} - m\right)\right\} \quad .$$

(v) Calculate a and b.

With (3.2) and (3.13), we can identify

$$a = \frac{1}{2}\left(1 - \frac{m}{\sqrt{1+m^2}}\right) ,$$

$$b = \frac{1}{2x}\left(1 + \frac{m}{\sqrt{1+m^2}}\right) .$$

Note that a, b and b_0 are nonnegative for any real m. For a_0 to be nonnegative we require

$$\sqrt{4Q^2-1}\ \left(m + \sqrt{1+m^2}\right) \leqslant 1 \quad ,$$

from which we derive the constraint

$$m \leq \frac{1 - 2Q^2}{\sqrt{4Q^2 - 1}} \quad . \tag{3.13}$$

The sum decomposition is now written in terms of m with constraint (3.13) as follows.

$$D(s) = \frac{1}{2}\left(1 - \frac{m}{\sqrt{1+m^2}}\right)\left[s + \sigma_c\left\{1 - \sqrt{4Q^2-1}\left(\sqrt{1+m^2} + m\right)\right\}\right]^2$$
$$+ \ x \cdot \frac{1}{2x}\left(1 + \frac{m}{\sqrt{1+m^2}}\right)\left[s + \sigma_c\left\{1 + \sqrt{4Q^2-1}\left(\sqrt{1+m^2} - m\right)\right\}\right]^2 \tag{3.14}$$

With this decomposition, from Equation (3.7) we have

$$S_x^p = -\frac{\sigma_c\sqrt{4Q^2-1}}{2} \; \frac{1 + j\,m}{\sqrt{1+m^2}} \quad , \tag{3.15}$$

$$\therefore \quad \left|S_x^p\right| = \frac{\sigma_c\sqrt{4Q^2-1}}{2} \quad . \tag{3.16}$$

Thus, the decomposition of (3.14) gives the magnitude of pole sensitivity $\sigma_c\sqrt{4Q^2-1}/2$ which is independent of m. This, as expected, is the same result as shown by Calahan [12]. Note that in all types of sum decomposition, regardless of multiple roots in component polynomials, the angle constraint (3.13) always holds.

Therefore, it can be concluded that in the sum decomposition, it is always possible to minimize the magnitude of pole sensitivity while the angle of pole sensitivity is varied to satisfy another design criterion.

In practical design, the characteristics of $S_x^{\omega_p}$ and S_x^Q versus m render very useful information. From Equations (2.18) and (3.15), it follows that

$$S_x^{\omega_p} = \frac{4Q^2-1\;(1-m\;\;4Q^2-1)}{8Q^2\;\;1+m^2}, \tag{3.17}$$

$$S_x^Q = -\frac{(4Q^2-1)\,(m+\;4Q^2-1)}{8Q^2\;\;1+m^2}. \tag{3.18}$$

and for large Q,(3.17) and (3.18) are reduced to

$$S_x^{\omega_p} \doteq \frac{1-2mQ}{4Q\sqrt{1+m^2}}, \tag{3.19}$$

$$S_x^Q \doteq \frac{m+2Q}{2\sqrt{1+m^2}}, \tag{3.20}$$

respectively.

The decomposition which gives zero Q sensitivity was first suggested by Kim and Phan [13] and later Phung, et al. [14] added the technique of simultaneously minimizing the magnitude of pole sensitivity. In this work, the same result can be easily obtained by finding m which minimizes Q sensitivity in Equation (3.18).

Obviously, $S_x^Q = 0$ when $m = -\sqrt{4Q^2-1}$. Substituting this value into Equation (3.14), it is always possible to obtain the optimum decompesition with zero Q sensitivity as

$$D(s) = \frac{1}{2}\left(1+\frac{\sqrt{4Q^2-1}}{2Q}\right)\left[s + 2\sigma_c Q\left(2Q - \sqrt{4Q^2-1}\right)\right]^2$$

$$+ x \cdot \frac{1}{2x}\left(1-\frac{\sqrt{4Q^2-1}}{2Q}\right)\left[s + 2\sigma_c Q\left(2Q + \sqrt{4Q^2-1}\right)\right]^2 , \quad (3.21)$$

and therefore, from Equations (3.17) and (3.18) we have

$$S_x^{\omega_p} = \frac{\sqrt{4Q^2-1}}{4Q} ,$$

$$S_x^Q = 0 . \quad (3.22)$$

This result is the same as shown in [14]. It should be noted that it is impossible to find a decomposition with zero ω_p sensitivity since this would require a value of m outside its permissible range. However, minimum $S_x^{\omega_p}$ will be found with either $m = -\frac{1-2Q^2}{\sqrt{4Q^2-1}}$ or $m = -\infty$, and maximum $S_x^{\omega_p}$ occurs for $m = -\sqrt{4Q^2-1}$. Note that maximum $S_x^{\omega_p}$ occurs when S_x^Q is minimum.

Due to stability considerations, it is sometimes desirable to have $\phi = \frac{\pi}{2}$ or equivalently $m = -\infty$. By substituting this value into Equations(3.14), (3.17) and (3.18), we have

$$D(s) = (s+\sigma_c)^2 + \sigma_c^2(4Q^2-1) ,$$

$$S_x^{\omega_p} = \frac{4Q^2-1}{8Q^2} , \qquad (3.23)$$

$$S_x^Q = \frac{4Q^2-1}{8Q^2} .$$

Examples: Consider the quadratic polynomial

$$D(s) = s^2 + 6s + 25 ,$$

where we have

$$\sigma_c = 3 ,$$
$$\omega_c = 4 ,$$
$$\omega_p = 5 ,$$
$$Q = \frac{5}{6} .$$

(i) For zero Q sensitivity, from Equation (3.21), we have

$$D(s) = \frac{9}{10}\left[s+\frac{5}{3}\right]^2 + x \cdot \frac{1}{10x}\left[s+15\right]^2 ,$$

and therefore from Equations (3.16) and (3.22), it follows that

$$\left|S_x^{\rho}\right| = 2$$

$$S_x^{\omega_p} = 0.4$$

$$S_x^{Q} = 0$$

(ii) For minimum $S_x^{\omega_p}$, by substituting $m = \frac{1-2Q^2}{\sqrt{4Q^2-1}} = -\frac{7}{24}$ into Equation (3.14), we have

$$D(s) = \frac{16}{25} s^2 + x \cdot \frac{9}{25x}\left(s + \frac{25}{3}\right)^2 ,$$

and consequently we obtain

$$\left|S_x^{P}\right| = 2 ,$$

$$S_x^{\omega_p} = 0.32 ,$$

$$S_x^{Q} = -0.32 ,$$

and with $m = -\infty$, from Equation (3.23) it follows that

$$D(s) = (s+3)^2 + x \cdot \frac{16}{x} ,$$

$$\left|S_x^{P}\right| = 2 ,$$

$$S_x^{\omega_p} = 0.32 ,$$

$$S_x^{Q} = 0.32 .$$

Continuous curves, $S_x^{\omega_p}$ and S_x^{Q} vs. m, are plotted for this example and for $Q = 10$ of the frequency normalized case in Figs. 3.6 and 3.7, respectively.

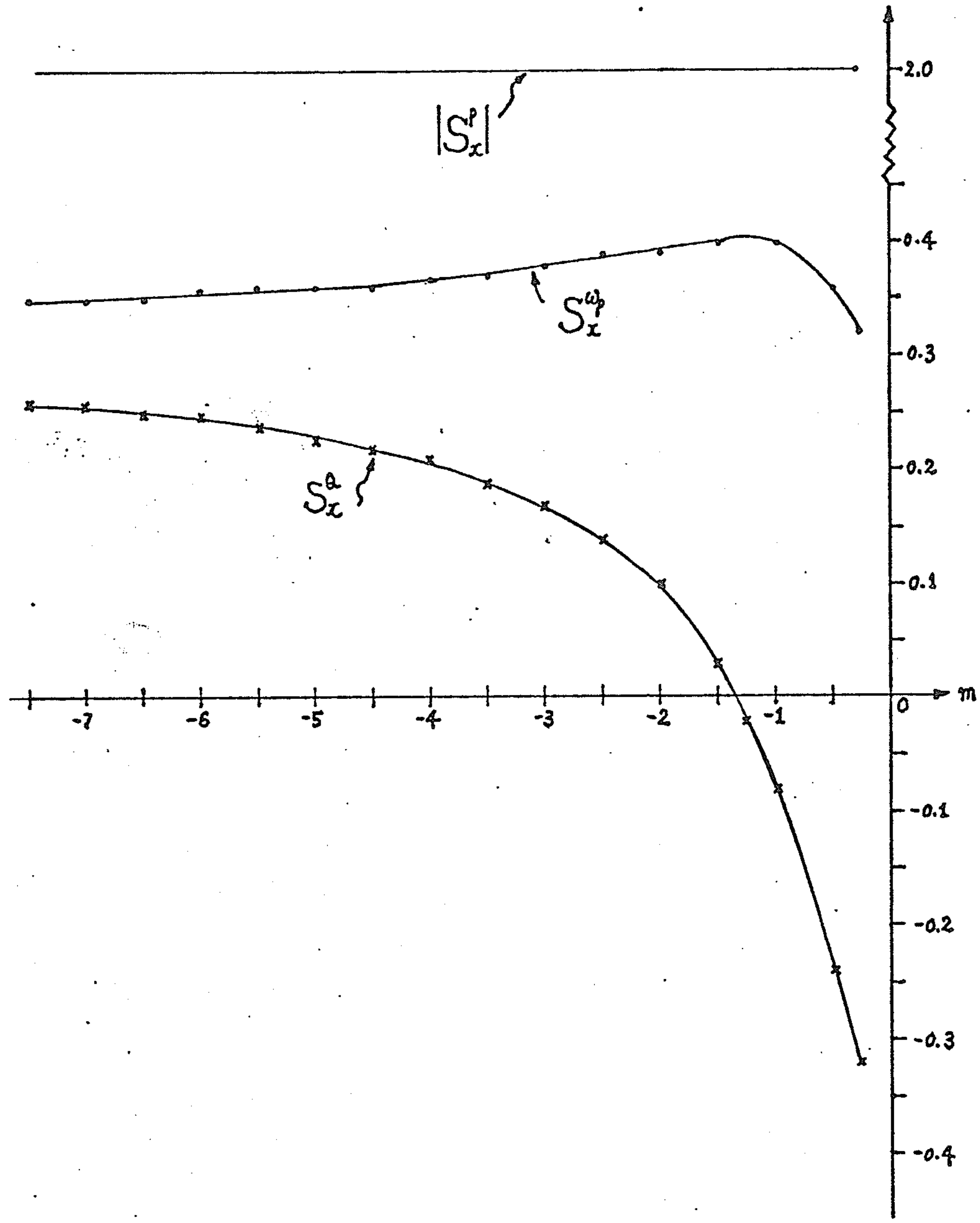

Fig. 3.6 $\left|S_x^\rho\right|$, $S_x^{\omega_\rho}$ & S_x^Q vs. m

(Q=5/6).

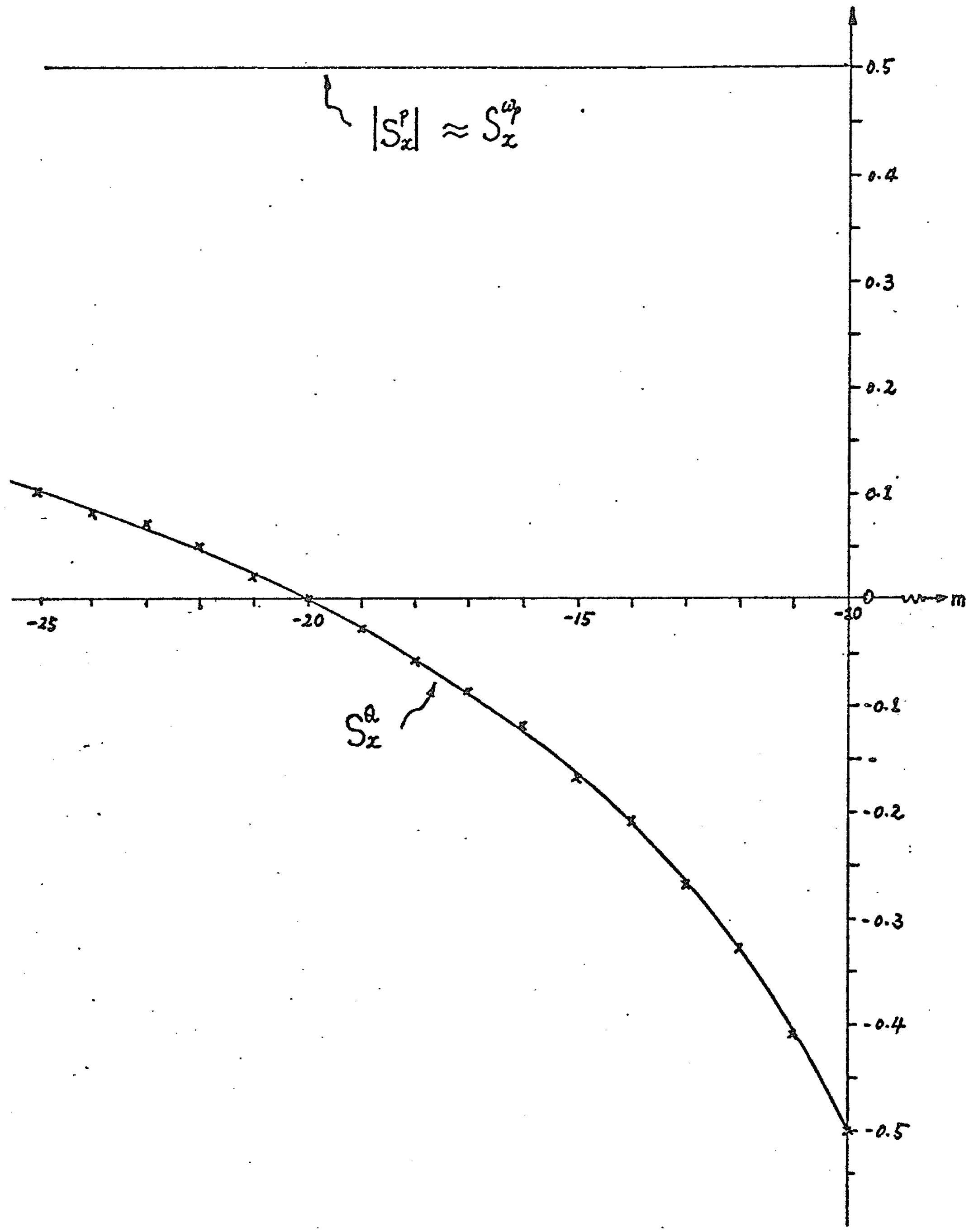

Fig. 3.7 $|S_x^{\rho}|$, $S_x^{\omega_p}$ & S_x^{Q} vs. m

(Q=10, frequency normalized case).

B. DIFFERENCE DECOMPOSITION : REAL ROOTS

The difference decomposition is written as

$$D(s) = a\,D_1(s) - x\,b\,D_2(s)\,, \tag{3.24}$$

where a and b are positive real numbers and x is the variable network parameter of interest. x is assumed to be a positive real number.

Since the difference of polynomials is considered in this decomposition, the roots of $D(s)$ are determined by a root locus satisfying the equation

$$\frac{K\,D_2(s)}{D_1(s)} = 1\,, \tag{3.25}$$

where $K = \frac{b}{a}\cdot x$.

Note that the Equation (3.25) defines a 0° root locus. For such a locus, the roots of $D_1(s)$ and $D_2(s)$ must be located on the negative real axis in a pattern shown in Figure 3.8 [10]. The root closest to the origin (it may

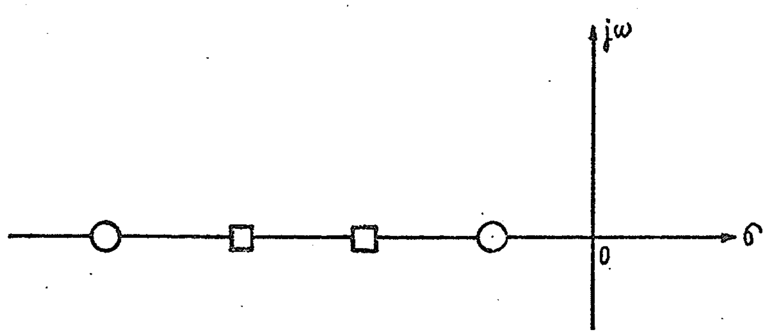

Fig. 3.8 A plot of the roots of $D_1(s)$ and $D_2(s)$

□ : root of $D_1(s)$ ○ : root of $D_2(s)$.

lie at the origin) is a root of $D_2(s)$ and the root closest to infinity (it may lie at infinity) must also belong to $D_2(s)$.

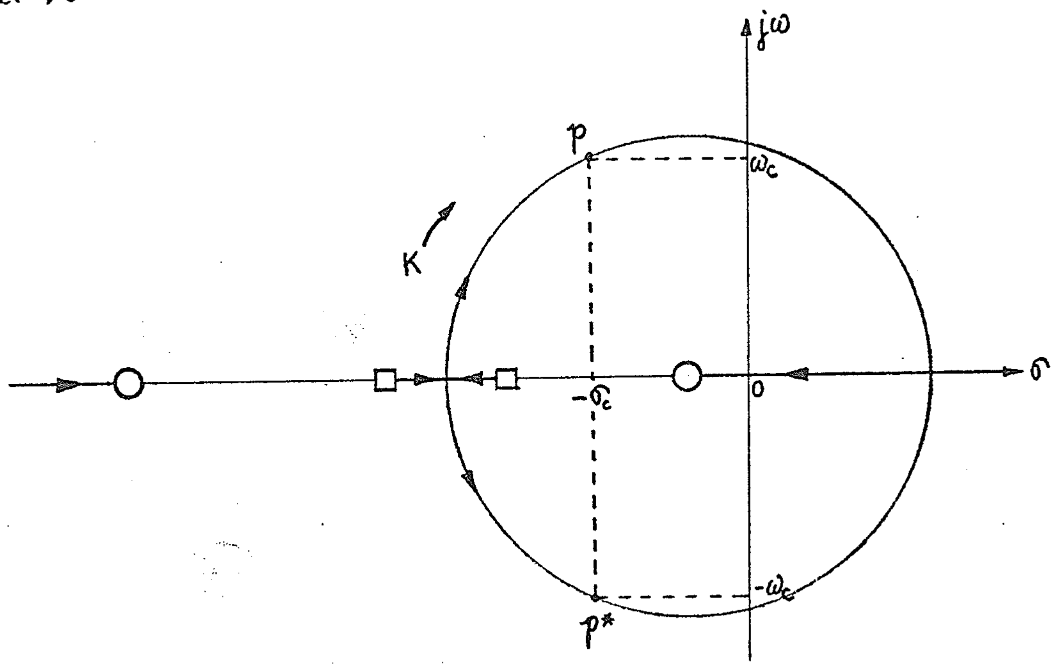

Fig. 3.9 Root locus. □: root of $D_1(s)$ O: root of $D_2(s)$.

The root locus with this root pattern is shown in Figure 3.9.

Here again, the minimization of the magnitude of pole sensitivity is our main interest. From Equation (3.8), it is obvious that the minimization of $|S_x^p|$ is achieved by minimizing $a|D_1(p)|$. In Equation (3.24), we can observe that a has minimum value of unity when $D_2(s)$ has a zero at infinity, i.e., the degree of $D_2(s)$ is lower than that of $D_1(s)$ by one. Furthermore, it is always desirable to have $D_1(s)$ in the form of $(s + a_0)^2$ rather than $(s + \alpha)(s + \beta)$ in order to minimize $|D_1(p)|$. [See Appendix

A]. Therefore, the decomposition which gives the minimum magnitude of pole sensitivity should be of the form

$$D(s) = (s + a_0)^2 - x b (s + \alpha), \tag{3.26}$$

where a_0 and α are nonnegative real numbers.

Such a decomposition shall be referred to as the optimum decomposition in the sense that it gives a minimum magnitude of pole sensitivity. The root locus of Equation (3.26) is shown in Figure 3.10, and we can easily see the direct relationship between the angle ϕ and the single root of $D_2(s)$.

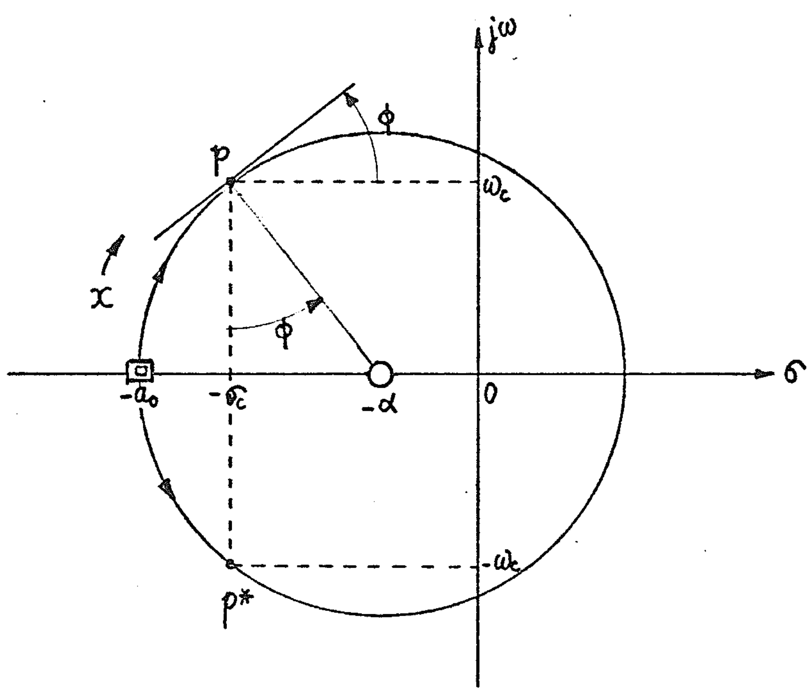

Fig. 3.10 Root locus.

⊡: double root of $D_1(s)$

O: root of $D_2(s)$.

Using Equations (3.11) and (3.12), the values of α and a_o are fuond as

$$\alpha = \sigma_c\,(1 - m\;4Q^2-1)\,, \tag{3.27}$$

$$a_o = \sigma_c\left\{1 - (m - \sqrt{1+m^2})\sqrt{4Q^2-1}\right\}. \tag{3.28}$$

Substituting Equations (3.27) and (3.28) into (3.26), and equating Equation (3.26) with Equation (3.2), the value of b can be obtained. Now, the optimum decomposition can be written in terms of m as follows.

$$D(s) = \left[s + \sigma_c\left\{1 + \sqrt{4Q^2-1}\,(\sqrt{1+m^2} - m)\right\}\right]^2$$
$$- x\cdot\frac{2\sigma_c}{x}\sqrt{4Q^2-1}\,(\sqrt{1+m^2} - m)\left[s + \sigma_c(1 - m\sqrt{4Q^2-1})\right]. \tag{3.29}$$

a_o and b are positive for any real m. For α to be nonnegative, it is required that

$$m \leqslant \frac{1}{\sqrt{4Q^2 - 1}}\,. \tag{3.30}$$

With this decomposition, from Equations (3.7) and (3.8) we have

$$S_x^p = \sigma_c\sqrt{4Q^2-1}\,(\sqrt{1+m^2})(1 + jm)\,,$$
$$|S_x^p| = \sigma_c\sqrt{4Q^2-1}\;\sqrt{1+m^2}\,(\sqrt{1+m^2} - m)\,. \tag{3.31}$$

It is shown in Appendix B that $|S_x^p|$ is a monotonically decreasing function of m. Hence, the least value of $|S_x^\rho|$ occurs when m has the upper limit value, i.e.,

$$m = \frac{1}{\sqrt{4Q^2 - 1}} \,. \tag{3.32}$$

By substituting this value into Equations (3.29) and (3.31), we have

$$D(s) = (s + 2\sigma_c Q)^2 - x \cdot \frac{2\sigma_c}{x}(2Q - 1)s \,,$$

$$S_x^\rho = \sigma_c(2Q - 1)\left(1 + j\frac{1}{\sqrt{4Q^2 - 1}}\right) \,, \tag{3.33}$$

$$|S_x^\rho| = \frac{2\sigma_c Q(2Q - 1)}{\sqrt{4Q^2 - 1}} \,,$$

respectively.

Note that this coincides with Horowitz decomposition [15].

As in the case of the sum decomposition $S_x^{\omega_p}$ and S_x^Q can be expressed as functions of m. From Equations (2.18) and (3.31), it follows that

$$S_x^{\omega_p} = \frac{\sqrt{4Q^2-1}}{4Q^2}\left(\sqrt{1+m^2}-m\right)\left(m\sqrt{4Q^2-1}-1\right), \tag{3.34}$$

$$S_x^{Q} = \frac{(4Q^2-1)}{4Q^2}\left(\sqrt{1+m^2}-m\right)\left(m+\sqrt{4Q^2-1}\right). \tag{3.35}$$

For large Q, Equatons (3.34) and (3.35) are reduced to

$$S_x^{\omega_p} \doteq \frac{(\ 1+m^2-m)(2mQ-1)}{2Q}, \tag{3.36}$$

$$S_x^{Q} \doteq \left(\sqrt{1+m^2}-m\right)(m+2Q), \tag{3.37}$$

respectively.

From Equations (3.34) and (3.35), it is observed that

$$S_x^{\omega_p} = 0 \quad \text{for} \quad m = \frac{1}{\sqrt{4Q^2-1}}, \tag{3.38}$$

and $$S_x^{Q} = 0 \quad \text{for} \quad m = -\sqrt{4Q^2-1}. \tag{3.39}$$

Obviously, Equation (3.38) coincides with the Horowitz decomposition.

Substituting Equation (3.39) into Equation (3.29), we obtain the decomposition of zero Q sensitivity as,

$$D(s) = \left[s + 2\sigma_c Q\left(2Q + \sqrt{4Q^2-1}\right)\right]^2$$
$$-x \cdot \frac{2\sigma_c}{x}\sqrt{4Q^2-1}\left(2Q + \sqrt{4Q^2-1}\right)\left(s + 4\sigma_c Q^2\right). \qquad (3.40)$$

Thus, from Equation (3.31), we have

$$\left|S_x^\rho\right| = 2\sigma_c Q\sqrt{4Q^2-1}\left(2Q + \sqrt{4Q^2-1}\right), \qquad (3.41)$$

and from Equation (3.34) and (3.35), we find

$$S_x^{\omega_\rho} = -\sqrt{4Q^2-1}\left(2Q + \sqrt{4Q^2-1}\right),$$
$$S_x^Q = 0, \qquad (3.42)$$

as expected.

Note that this magnitude is smaller than that of the decomposition suggested in [13].

Examples : Consider again the quadratic polynomial

$$D(s) = s^2 + 6s + 25.$$

(i) For min $\left|S_x^\rho\right|$ and $S_x^{\omega_\rho} = 0$, we choose

$$m = \frac{1}{\sqrt{4Q^2-1}} = 0.75,$$

then from Equation (3.33), we have

$$D(s) = (s+5)^2 - x \cdot \frac{4}{x} s,$$

$$\left|S_x^\rho\right| = 2.5.$$

Thus, from Equation (3.34) and (3.35), it follows that

$$S_{\chi}^{\omega_p} = 0,$$

$$S_{\chi}^{Q} = 0.667.$$

(ii) For $S_{\chi}^{Q} = 0$, we take $m = -\sqrt{4Q^2 - 1} = -4/3$, and then from Equation (3.40), it follows that

$$D(s) = (s + 15)^2 - \chi \cdot \frac{24}{\chi}(s + 25/3),$$

and subsequently we obtain

$$|S_{\chi}^{p}| = 20,$$

$$S_{\chi}^{\omega_p} = -4,$$

$$S_{\chi}^{Q} = 0.$$

(iii) For $m = 0$, from Equation (3.29) we have

$$D(s) = (s + 7)^2 - \chi \cdot \frac{8}{\chi}(s + 3),$$

and the pertinent values are found as

$$|S_{\chi}^{p}| = 4,$$

$$S_{\chi}^{\omega_p} = -12/25,$$

$$S_{\chi}^{Q} = 64/75.$$

$|S_{\chi}^{p}|$, $S_{\chi}^{\omega_p}$ and S_{χ}^{Q} vs. m are plotted in Fig. 3.11 and 3.12 for this example and Q = 10 of the frequency normalized case, respectively.

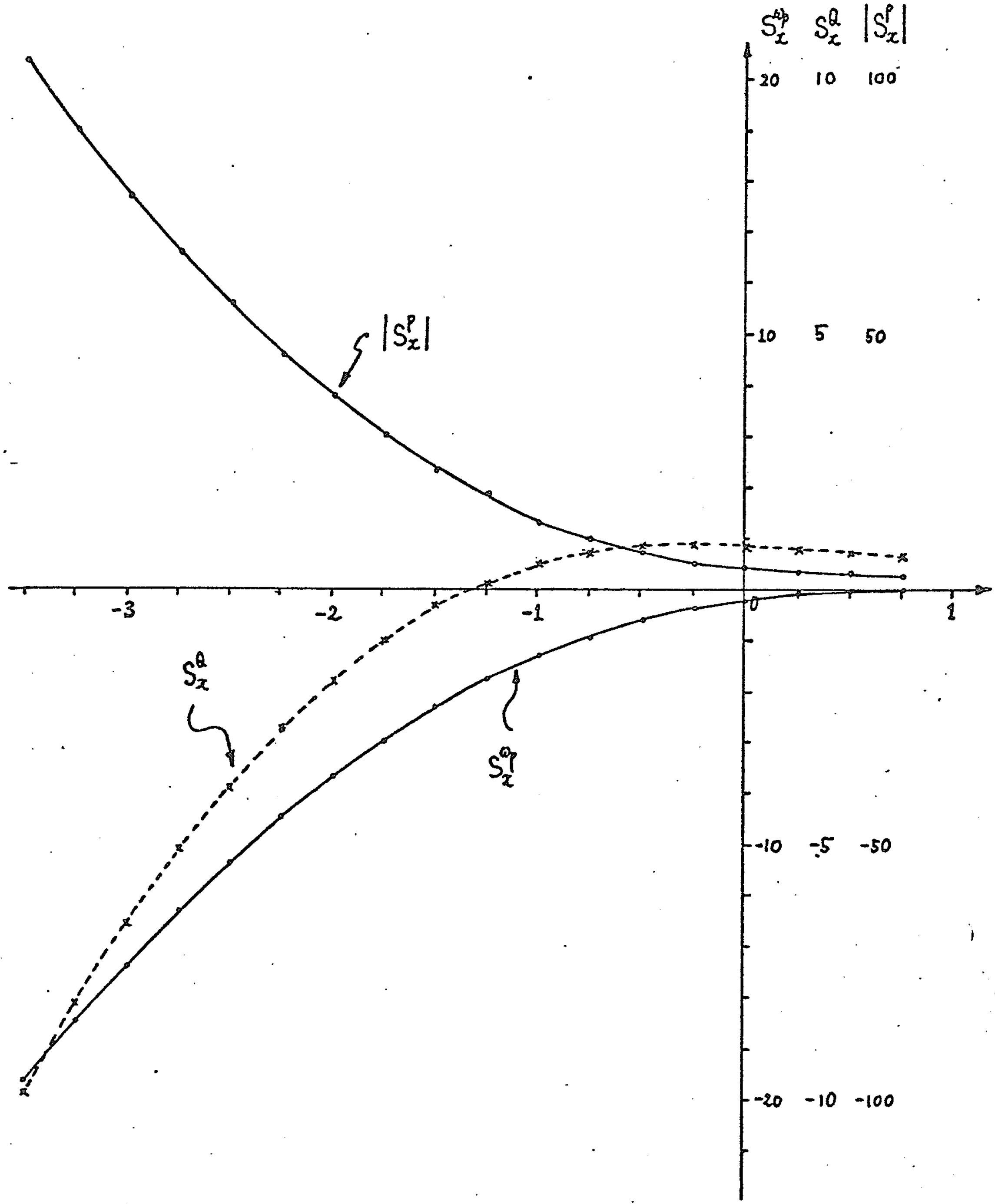

Fig. 3.11 $|S_x^\rho|$, $S_x^{\omega_p}$ & S_x^Q vs. m

(Q = 5/6).

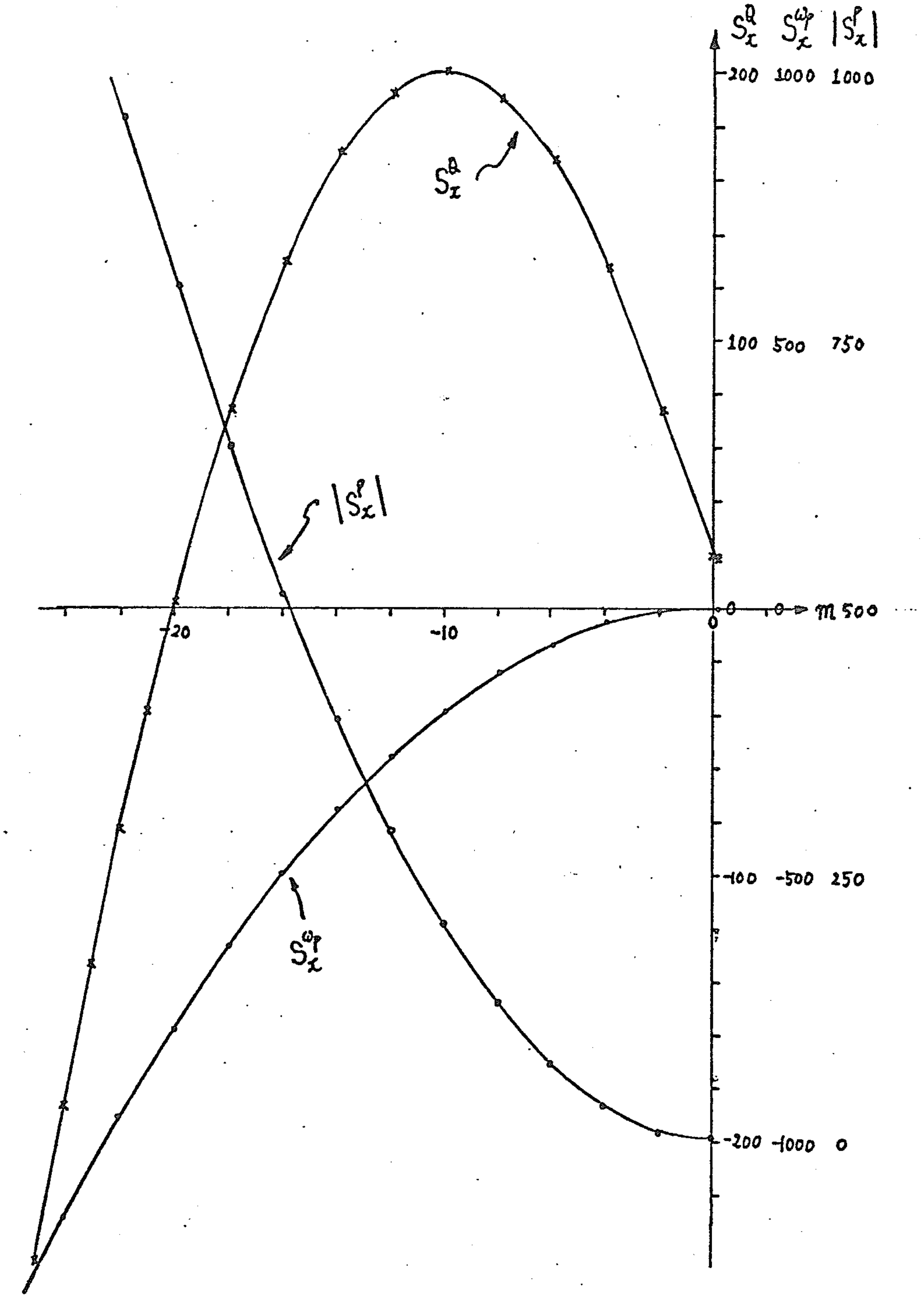

Fig. 3.12 $|S_x^\rho|$, $S_x^{\omega_\rho}$ & S_x^Q vs. m

(Q = 10, frequency normalized case).

C. DECOMPOSITION WITH COMPLEX ROOTS

In the preceeding two sections, sensitivity studies have been conducted under the constraint that the component polynomials $D_1(s)$ and $D_2(s)$ have only negative real roots. In this section, the constraint will be relaxed so that $D_2(s)$ may have a pair of complex roots.

Herbst [16] demonstrated that smaller pole sensitivities can be achieved if the phantom zeros are allowed to be complex. In order to minimize a pole sensitivity, he considers a polynomial decomposition with phantom zeros on the imaginary axis only.

Haykin [2] advanced new network configurations which can be realized by a polynomial decomposition with complex phantom zeros in the left-half s-plane. With this decomposition it is possible to prescribe the magnitudes of pole sensitivities as small as desired with any Q factor. However, he restricts the angle of pole sensitivity to two cases only. Sometimes, this restriction causes a large spread of passive element values and/or very large values for the active parameters in the case of high Q factors.

Burkhardt, et al. [17] presented a polynomial decomposition which allows the specification of magnitude as well as the angle of pole sensitivity in advance. He shows that if only the magnitude is prescribed, the active gain parameter can be minimized by proper choice of the angle of pole sensitivity.

In the following, a method of polynomial decomposition is presented for the prescribed magnitude and angle of pole sensitivity.

Consider the decomposition

$$D(s) = a\,D_1(s) + x\,b\,D_2(s)\,, \tag{3.43}$$

where the monic polynomial $D_2(s)$ may have a pair of complex roots in the left-half s-plane. Note that since a summation of polynomials is considered, the designations of $D_1(s)$ and $D_2(s)$ are interchangable.

As in the sum decomposition case before, the roots of Equation (3.43) are determined by the root locus of

$$\frac{K\,D_2(s)}{D_1(s)} = -1\,, \tag{3.44}$$

where $K = \frac{b}{a}x$.

From the sensitivity point of view [See Appendix A], it is always advantageous for $D_1(s)$ to have a double root, therefore, this particular decomposition will be discussed. The root loci of such decompositions are shown in Figure 3.13 for two different angles of the pole sensi-

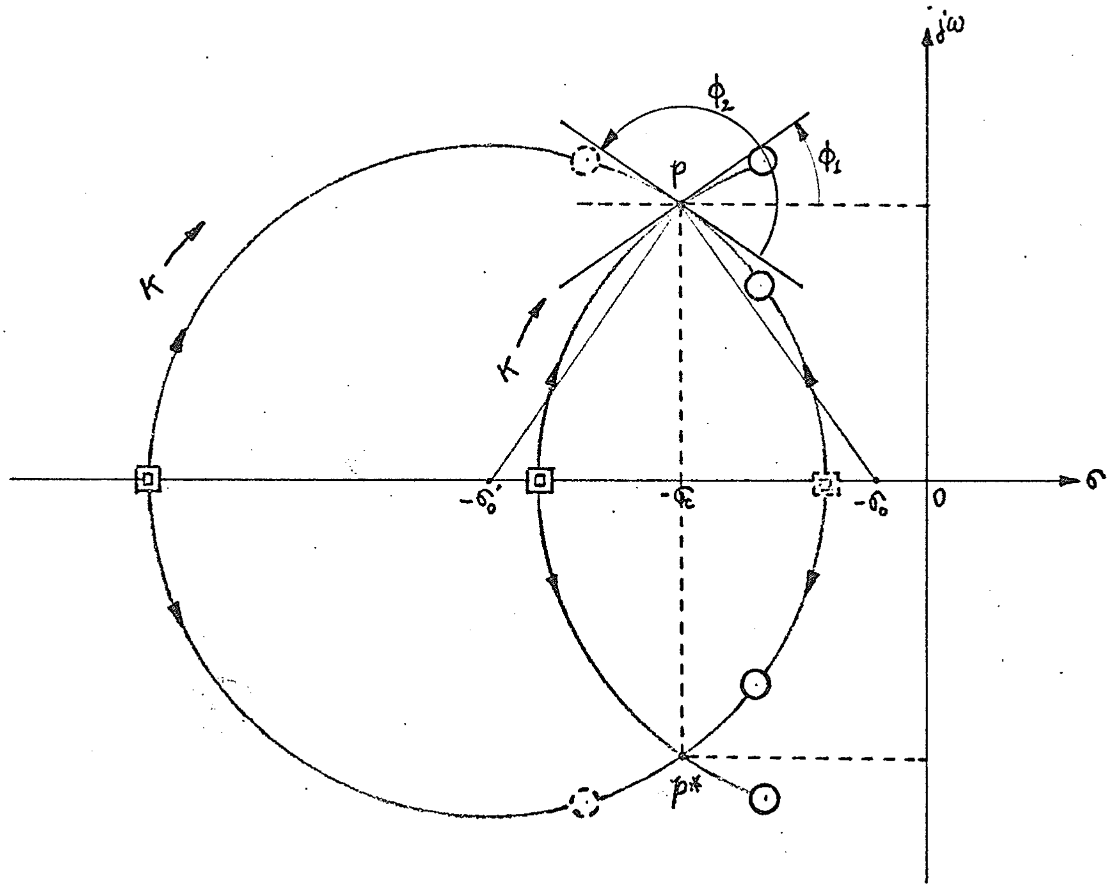

Fig. 3.13 Root locus for given angles ϕ_1 and ϕ_2.
⊡: double root of $D_1(s)$ ○: root of $D_2(s)$.

tivity. As can be seen, if an angle of pole sensitivity is given, the circle portion of the root locus is uniquely determined, and hence $D_1(s)$ is determined if it is to have a double root on the negative real axis. Now, the magnitude of pole sensitivity with prescribed pole sensitivity angle depends only on the locations of phantom zeros.

Therefore, a decomposition with prescribed angle and magnitud can be obtained as follows;

(i) For a given m, find the center, $-\sigma_0$, of the circle portion of the root locus.

As in the previous cases, it is

$$-\sigma_0 = -\sigma_c(1 - m\sqrt{4Q^2-1}) . \tag{3.45}$$

(ii) Write the equation of the circle.

From Equations (3.12) and (3.45), it can be represented as

$$\{\sigma + \sigma_c(1 - m\sqrt{4Q^2-1})\}^2 + \omega^2 = \sigma_c^2(4Q^2-1)(1+m^2). \tag{3.46}$$

From Figure 3.13, it can be observed that there are two different cases;

(a) Both intersections of the circle and the real axis occur in the left-half s-plane. This happens for values of m such that

$$m \leqslant \frac{1 - 2Q^2}{\sqrt{4Q^2-1}} \tag{3.47}$$

(b) Only one of the intersections is on the negative real axis. This happens when m is constrained by

$$m \geqslant \frac{1 - 2Q^2}{\sqrt{4Q^2-1}} \tag{3.48}$$

(iii) Write the form of decomposition.

Designating phantom zeros as $-\alpha_0 + j\beta_0$, there are two possible decompositions in case(a);

$$D(s) = a\left[s + \sigma_c\left\{1 - \sqrt{4Q^2-1}\left(\sqrt{1+m^2} + m\right)\right\}\right]^2 + xb\left\{(s+\alpha_0)^2 + \beta_0^2\right\}, \qquad (3.49.a)$$

for $\alpha_0 > \sigma_c$,

and

$$D(s) = a\left[s + \sigma_c\left\{1 - \sqrt{4Q^2-1}\left(m - \sqrt{1+m^2}\right)\right\}\right]^2 + xb\left\{(s+\alpha_0)^2 + \beta_0^2\right\} \qquad (3.49.b)$$

for $\alpha_0 < \sigma_c$.

Note that the root loci of (3.49.a) and (3.49.b) are on the same circle with the direction opposite to each other.

In case(b), the only possible decomposition is of the form

$$D(s) = a\left[s + \sigma_c\left\{1 - \sqrt{4Q^2-1}\left(m - \sqrt{1+m^2}\right)\right\}\right]^2 + xb\left\{(s+\alpha_0)^2 + \beta_0^2\right\}, \qquad (3.50)$$

for $\alpha_0 < \sigma_c$.

Hereafter, only the decomposition with phantom zeros such that $\alpha_0 < \sigma_c$ will be discussed. The treatment of the other case is identical.

(iv) Represent a and b in terms of m and the magnitude of pole sensitivity.

From Equations (3.7), (3.49.a) and (3.50), the pole sensitivity can be written as

$$S_x^p = a\,\sigma_c \sqrt{4Q^2-1}\,(\sqrt{1+m^2}-m)(1+jm)\,. \tag{3.51}$$

Thus,

$$\left|S_x^p\right| = a\,\sigma_c \sqrt{4Q^2-1}\,(\sqrt{1+m^2}-m)\sqrt{1+m^2}\,. \tag{3.52}$$

If the magnitude of pole sensitivity with prescribed m is given as M, from Equations (3.50) and (3.43), a and b can be obtained as,

$$a = \frac{M\,(m+\sqrt{1+m^2})}{\sigma_c\sqrt{4Q^2-1}\,\sqrt{1+m^2}}\,,$$
$$b = \frac{1}{x}\left\{1 - \frac{M\,(m+\sqrt{1+m^2})}{\sigma_c\sqrt{4Q^2-1}\,\sqrt{1+m^2}}\right\}. \tag{3.53}$$

Now Equation (3.50) can be written as

$$D(s) = \frac{M(m+\sqrt{1+m^2})}{\sigma_c\sqrt{4Q^2-1}\sqrt{1+m^2}}\left[\,s + \sigma_c\left\{1-\sqrt{4Q^2-1}\,(m-\sqrt{1+m^2})\right\}\right]^2$$
$$+\, x\,\frac{1}{x}\left\{1-\frac{M\,(m+\sqrt{1+m^2})}{\sigma_c\sqrt{4Q^2-1}\,\sqrt{1+m^2}}\right\}\left\{(s+\alpha_0)^2+\beta_0^2\right\}\,. \tag{3.54}$$

(v) Find α_o and β_o in terms of m and M.

By equating Equation (3.2) with (3.54), α_o can be obtained as

$$\alpha_o = \frac{\sigma_c\left\{\sqrt{4Q^2-1}\,(\sigma_c\sqrt{1+m^2}-M) - M(\sqrt{1+m^2}+m)\right\}}{(\sigma_c\sqrt{4Q^2-1}-M)\sqrt{1+m^2} - mM}, \quad (3.55)$$

Thus, from Equation (3.46), β_o is

$$\beta_o = \left[\sigma_c^2(4Q^2-1)(1+m^2) - \left\{\sigma_c(1-m\;4Q^2-1)-\alpha_o\right\}^2\right]^{\frac{1}{2}}. \quad (3.56)$$

It is interesting to note that on plotting appropriate root loci, the nature of alternative decompositions can be easily seen without carrying out the detailed mathematics. With this decomposition, from Equation (2.18) we have

$$S_x^{\omega_p} = \frac{M\sigma_c(1+m\sqrt{4Q^2-1})}{\sqrt{1+m^2}}, \quad (3.57)$$

$$S_x^{Q} = \frac{M\sigma_c\sqrt{4Q^2-1}\,(m-\sqrt{4Q^2-1})}{\sqrt{1+m^2}}. \quad (3.58)$$

Example: Consider the quadratic polynomial

$$D(s) = s^2 + 6s + 25.$$

Let it be desired for a decomposition to have zero sensitivity and magnitude of pole sensitivity, M = 1.

From Equation (3.57), it follows that

$$S_{o}^{\omega_p} = 0 \qquad \text{for} \quad m = -\frac{1}{\sqrt{4Q^2-1}} = -3/4 .$$

Substituting these values of m and M into (3.54), (3.55) and (3.56) we obtained the desired decomposition as,

$$D(s) = \frac{1}{10}(s+11)^2$$
$$+ x\,\frac{9}{10x}\left\{\left(s+\frac{19}{9}\right)^2 + \left(\frac{20\sqrt{2}}{9}\right)^2\right\} .$$

4 Conclusions

The technique of obtaining various decompositions of the second order polynomial is presented. The main feature of this thesis is that the decomposition is based on a new parameter m which varies continuously within a certain interval. It is shown that in the sum and difference type decompositions with real phantom zeros, the interval is constrained by Equations (3.13) and (3.30) respectively, and no such constrain exists if the phantom zeros are allowed to be complex numbers. Once the performance criterion is specified, the parameter m may be properly chosen to yield the decomposition which corresponds to the optimum performance in terms of sensitivities. The well-known decompositions, such as Horowitz's and Calahan's, are only the special cases of the general decomposition. Also, it is shown that in the sum case with real roots, the optimum decomposition with same minimum magnitude of pole sensitivity is always possible for any angles within the constraint. In the difference case, however, the Horowitz decomposition gives the least minimum magnitude of pole sensitivity, therefore, a decomposition with desired angle can be obtained only at the expense of the magnitude. The pertinent decompositions are presented in the form of explicit functions of prescribed angle so as to be of practical use in Equations (3.14) and (3.29), respectively. Furthermore, for the sum case with complex phantom zeros, the decomposition is expressed in the form of explicit function of pole sensitivity angle, m , and the magnitude, M , as well in Equation (3.54). The desired decomposition can be easily obtained by simple substitution of the given values. Also, the useful design criteria, such as $S_x^{\omega_p}$ and S_x^Q , are expressed as a function of the angle of pole sensitivity, m , in Equations (3.17), (3.18), (3.34), (3.35), (3.57) and (3.58).

These useful expressions enable a circuit designer to obtain a decomposition that may satisfy other design criteria, e.g., stability margin, spread of element values, zero Q or zero ω_p sensitivity, etc.. As shown in [17], it could be used to minimize a gain factor with prescribed magnitude of pole sensitivity.

It is worth observing that this method could apply to the network configurations which can be realized by the coefficient matching technique [5], [20].

APPENDIX A

Consider the second order polynomial

$$D_1(s) = (s+\alpha)(s+\beta), \qquad (A-1)$$

where α and β are positive real numbers.

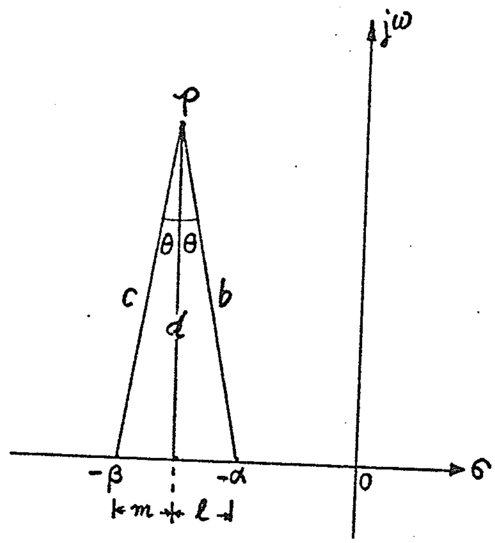

Fig.1 Case for $b = c$

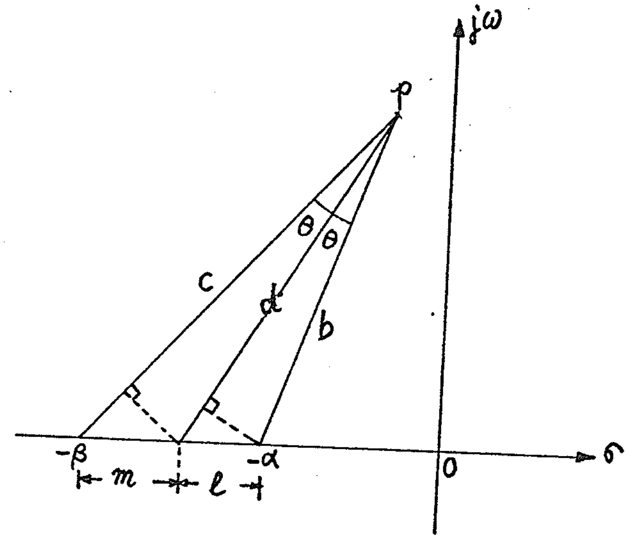

Fig.2 Case for $b \neq c$

The magnitude of $D_1(p)$ is

$$|D_1(p)| = |p+\alpha|\,|p+\beta| \qquad (A-2)$$

Letting $|p+\alpha| = b$ and $|p+\beta| = c$, obviously there are two possibilities, i.e., either $b = c$ or $b \neq c$ as shown in Figures 1 and 2, respectively.

Draw a straight line d which bisects the angle $\angle(-\beta)p(-\alpha)$ and denote the bisected angle as θ.

Then, in Figure 1, it is clear that

$$d^2 < bc \qquad (A-3)$$

From Figure 2, assuming $c > b$, we have

$$l^2 = b^2 + d^2 - 2bd\cos\theta,$$

$$m^2 = c^2 + d^2 - 2cd\cos\theta,$$

$$\therefore\ cl^2 - bm^2 = (c-b)(d^2 - bc),$$

$$\therefore\ d^2 - bc = \frac{cl^2 - bm^2}{c-b}. \qquad (A-4)$$

Also,

$$\frac{m}{l} = \frac{cd\sin\theta}{db\sin\theta} = \frac{c}{b}. \qquad (A-5)$$

Clearly the sign of Equation (A-4) depends on the sign of $(cl^2 - bm^2)$. From Equations (A-4) and (A-5), it follows that

$$cl^2 - bm^2 = cl^2\left(\frac{b-c}{b}\right) < 0. \qquad (A-6)$$

Hence,

$$d^2 < bc. \qquad (A-7)$$

Therefore, as long as $\alpha \neq \beta$, we can always choose d such that $d^2 < bc$. Any possible advantage is lost when $\alpha = \beta$.

APPENDIX B

$$M = \left| S_x^p \right| = \omega_c \sqrt{1+m^2}\left[\sqrt{1+m^2} - m\right] . \quad (B - 1)$$

Differentiating M with respect to m, it follows that

$$\frac{dM}{dm} = \frac{\omega_c}{\sqrt{1+m^2}}\left[2m\sqrt{1+m^2} - 2m^2 - 1\right] . \quad (B - 2)$$

For $m \leqslant 0$, it is always true that

$$2m\sqrt{1+m^2} < 2m^2 + 1 . \quad (B - 3)$$

For $m > 0$, let us assume that Expression (B-3) is true. Since both sides are positive, the inequality must hold when they are squared. Squaring both sides, we have

$$4m^2 + 4m^4 < 4m^2 + 4m^4 + 1 ,$$

which means that the assumption is true.

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