A SOLUTION TO A CLASS OF PULSE-FREQUENCY MODULATED CONTROL SYSTEMS

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ABSTRACT

Pulse-frequency modulation (PFM) involves representing a signal or function by a series of standard pulses with variable spacing and polarity. PFM finds applications in different areas. Among these are communication systems, physiological systems and control systems.

The object of the present study is to calculate the optimum control function u(t) which consists of a series of standard pulses. This problem is reduced, by the application of the Modified Maximum Principle, to extremizing a certain function of the control function denoted by I*(u).

As a first step to finding the optimum control u(t), a search for common properties among different I*(u) is made, for different systems having different performance indices. Finally, a method for extremizing I*(u) is developed, programmed and illustrated by a number of examples.

TABLE OF CONTENTS

Table of Contents (Continued)

LIST OF FIGURES

List of Figures (Continued) Page

LIST OF TABLES

LIST OF SYMBOLS

a_{ij}	Elements of the coefficient matrix of the system equations.
a_i	Initial condition for $x_i(t)$.
AI	I* for a single pulse.
C	Pulse width plus minimum dead time.
C_i	Constants used in the definition of the Pontryagin function S when there are no final value constraints.
c_i	Variables used in the definition of the Pontryagin function S when final value constraints are present.
D	Minimum allowable dead time between the pulses constituting the control function.
$f_i(x\ ,\ldots x_n,u,t)$	Time derivative of $x_i(t)$.
$\bar{f}(\bar{x},u,t)$	Time derivative of $\bar{x}(t)$.
$F_k(\bar{x}(T))$	A function used in expressing the final value constraints.
$F(a,u)$	A function used in the definition of the performance index.
FJ	Ending point of an interval.
H	Inner product of $\bar{f}$ and $\bar{P}$, the auxiliary vector.
I'	Time integral of H.
I*	Time integral of $\sum_i P_i \phi_i$.
I	I* for a single section of P(t).
J	The performance index.

n	System dimension.
N	Number of pulses in a certain interval.
$P_i(t)$	Auxiliary variables.
$P(t)$	Function defined in Equation 2.27.
S	Pontryagin function.
t	Denotes time.
T	Final time.
u	Control function.
V	Interval of time under consideration.
W	Pulse width.
$x_i(t)$	State variable.
$\bar{x}(t)$	State vector.
X	Magnitude of the standard pulse.
$\bar{Z}$	Starting point of an interval.
Z_i	Zeros of $P(t)$.
α	Cost applied on pulses.
β	Function defined in Equation 4.4.
ϕ	A function of the control variable used in the description of the system.
λ_k	Lagrange multipliers.
δ	Magnitude of the shift from $\bar{Z}$.
δ'	Variation in δ.
δ^*	δ corresponding to minimum (maximum) I.

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CHAPTER 1

INTRODUCTION

There are four basic different ways of representing a signal in terms of a series of pulses. This signal may be sampled by varying the height of the pulses in an equally spaced train of pulses. This manner of sampling is called pulse-amplitude modulation. If the width of pulses, initiated at equal intervals is varied with the signal, pulse width modulation is obtained. On the other hand, if the distance of the pulses varies from their reference positions in the equally spaced pulse train, we get the pulse-position modulation.

Pulse-frequency modulation (PFM) is obtained by representing the modulating signal by a series of identical pulses of variable spacing, and possibly, different polarities.

1.1 Pulse-frequency Modulation

Pulse-frequency modulation is a method of modulation where the modulating signal, which in the case of control systems often is the error signal, is represented by a series of identical pulses. The information is contained in the time between the initiation of these pulses, and possibly, their polarities. In most practical cases, the shape of the polses is of minor significance. Rectangular pulses of predetermined magnitude and width are used exclusively in this study. However, it would require minor modification to consider other pulse shapes.

The frequency is limited to some maximum value since it is not desirable to consider the case where the pulses overlap. This is because, usually, a "pulse-generator" can only produce one pulse at a time. In addition, a reset time, called the dead time, may be required for the generation of the next pulse. However, in the case of two consecutive pulses of different polarity the overlapping of pulses may be considered. This would be the case where different "pulse-generators" are used for the different polarities; for example, the "attitude control jets on a space vehicle".

In pulse-amplitude modulation the sampling frequency is signal-independent and the analysis of the system can be carried out using linear techniques. The pulse representation of the signal in pulse-width and pulse-position modulation is a nonlinear process. The sampling frequency in the pulse-width modulation process is signal independent and thus the analysis in this case is somewhat easier than in the case of pulse-position modulation, where the sampling frequency is signal dependent even though it is referred to a fixed reference frequency. Of the four basic schemes, PFM is probably the most difficult since the sampling frequency is completely signal dependent and the modulating process is nonlinear which introduces complexity in the analysis.

1.2 Application of PFM

The interest in PFM scheme originates from different areas. Among these are communication systems, physiological systems and control systems.

In communication systems, PFM is used primarily in the transmission of messages or signals [12]. In the biological field the interest in PFM began with the realization that parts of the nervous system use the PFM scheme to transmit information [13]. In the development of a quantitative understanding of physiological reflex arcs, it becomes apparent that the concept of PFM is bound to play an important role [3]. The concept of PFM in physiological systems serves as a tool for analysis. It provides a means for the construction of mathematical models for the nervous systems of animals.

There exists a variety of applications of PFM in the area of control systems. For example, in remote control systems pulses may be generated by the system itself and only the triggering is remote. This is advantageous when the application of the pulses requires more energy than can be transmitted economically from the controlling source.

Another application is in systems where the fuel consumption of the controlling gear is critical. In this case, PFM is particularly applicable in lightly damped systems because less energy is required to correct the errors in these systems so that the corrections can be made in pulses. An example of this is in space vehicle attitude control, where the correcting thrust comes from gas jets which are designed to shut off at a given time after starting. This type of controlling system uses much less fuel than a continuous system and at the same time it does not require the sending of a continuous controlling signal to the space vehicle, thus, a saving in telemetry.

Increasing interest is being shown in the area of adaptive control systems. Murphy and West [7] employed PFM in an adaptive outer loop around a basic system in which only time invariant compensation was used to obtain an adaptive autopilot for high performance contemporary military aircraft.

1.3 Background

Since the appearance of the first work on pulse-frequency modulators in 1949 [12] many theoretical and practical aspects of this scheme of modulation have been investigated. Ross [12] was mainly concerned with the reconstruction of sinusoidal signals from a PFM approximation.

The next few articles on PFM originated from studies of physiological systems. Jones, Li, Meyer and Pinter [3] have studied these systems and found the existence of pulse trains along nerve fibers that possesses both logarithmic and direct relationships between frequency and stimulus intensity. Pavlidis [9] was concerned with the applications of PFM in the modelling of neural nets.

The problem of PFM feedback systems has been investigated by Li [4], Li and Jones [5] and Meyer [6]. They have defined integral PFM and presented analytical methods for feedback systems incorporating IPFM. The application of IPFM to space vehicle attitude control has been made by Farrenkoff, Sabroff and Wheeler [2].

In addition to IPFM, Meyer [6] has defined the relaxation pulse-frequency modulation (RPFM) which represents a first order relaxation oscillator. Pavlidis and Jury [10] considered a generalization of IPFM and referred to as Sigma PFM which has some advantages over other schemes, such as IPFM. Most significant advantages are improved stability and simpler physical implementation of the modulator. Two general areas were investigated [10]: the study of PFM in control systems and the design of models for physiological systems. The study in control system included the dynamic response analysis of a pulse frequency modulator and the stability of feedback PFM systems by means of a modification of Lyapunov's second method. Also, the existence of sustained oscillations was studied by developing a quasi-describing function for the modulation.

A study by Clark [1] in 1965 included the analysis of a PFM scheme. The interval between pulses was determined by the instantaneous value of the error at the beginning of the pulse just prior to the interval in question.

Stability was investigated by Clark [1] and Tchekhovoi [14-16]. A recent work of Tchekhovoi [16] is a study of the application of Lyapunov's direct method for the synthesis of pulse-frequency systems for the automatic stabilization of space craft position.

Limited work has been done in the area of optimum PFM control systems. Pavlidis [11] presented a solution of the minimum time and minimum fuel problems for PFM systems. The derivation of the optimal control was done by a heuristic argument.

The problem of finding the optimum control function for the PFM system was considered by Onyshko and Noges [8]. The control function discussed consisted of a series of standard pulses. The optimization procedure consisted of determining the polarity and position of the pulses which make up the control function. The performance index was assumed to be a linear combination of the final values of the state variables but not necessarily of the control function. In the pulse frequency modulated system considered, the control function was fixed for a period of time following the initiation of each pulse. This fact precluded the direct application of the existing standard optimizing techniques.

Onyshko and Noges [8] solved the optimization problem using two different techniques. One is the Modified Maximum Principle (M M P) based on Pontryagin's Maximum principle. It is applicable to open-loop systems with linear plants with fixed operating time, and valid for systems with and without final value constraints on the state variables.

The dynamic programming technique was also presented. It may be used with nonlinear systems. However, to make this technique applicable a restriction had to be placed on the control function u. This restriction, which is required to make the performance index Markovian, allows the pulses to be initiated only at predetermined instant of time. This technique can be applied to open-loop and closed-loop systems and gives solutions to a number of different initial conditions and time intervals.

1.4 Outline of Analysis

Onyshko and Noges [8] showed that for the system

$$\dot{x}_i = f_i = \sum_{j=1}^{n} a_{ij}(t)x_j + \phi_i(u) \qquad i=1,\ldots,n$$

the control function u(t), which extremizes a certain performance index J, of the form

$$J = \int_0^T F(x_1, \ldots, x_n, u)\, dt$$

may be found by extremizing the function

$$I' = \int_0^T H\, dt$$

where T is the final time and H is a scalar function defined as

$$H = H(x_1, \ldots, x_n; P_1, \ldots, P_n, u, t) = \sum_{i=1}^{n} P_i f_i$$

where P_i's are auxiliary functions of time satisfying the following set of differential equations

$$\dot{P}_i(t) = - \sum_{s=1}^{n} P_s \frac{\partial f_s}{\partial x_i}, \quad i=1, \ldots, n$$

They also proved that the particular control u(t) which extremizes the function

$$I^*(u) = \int_0^T \sum_{i=1}^{n} P_i\, \phi_i(u)\, dt$$

also extremizes the function

$$I'(u) = \int_0^T H\, dt$$

The variables P_i are functions of time only, and consequently, the function I* is dependent only on the control function

$u(t)$ over the given time interval. Finding the control $u(t)$ which extremizes the function $I^*(u)$ was done in [8] by inspection.

The object of the present analysis is to find the optimum control $u(t)$ by extremizing $I^*(u)$. This is achieved in two principal parts:

(a) a search for common properties among different $I^*(u)$ for different systems having different performance indices;

(b) a description of a method for extremizing $I^*(u)$ and the development of a computer program for finding the optimum control function $u(t)$.

In Chapter 2, a brief description of the Modified Maximum Principle is presented, followed by a remark on the formulation of $I^*(u)$. A search of the different forms of $I^*(u)$ for different systems and performance indices is presented in Chapter 3. In Chapter 4, a description of the method of extremizing $I^*(u)$ is presented along with a number of illustrating examples.

CHAPTER 2

MODIFIED MAXIMUM PRINCIPLE

2.1 Brief Description

The problem under consideration can be summarized: given a physical system which can be described by a system of differential equations it is necessary to find the control function u which extremizes a given performance criterion. The control function u(t) is restricted to a series of identical pulses. In this study the pulses are assumed to be rectangular, of predetermined magnitude and time duration, but may be of either polarity.

In a more mathematical form the problem may be depicted as: given the system of differential equations

$$\dot{\bar{x}} = \bar{f}(\bar{x},u,t) \;, \quad \bar{x}(0) = \bar{a} \qquad 0 \le t \le T \tag{2.1}$$

it is necessary to find the control function u(t) which extremizes some given scalar function J called the performance index, where

$$J = \int_0^T F(x_1,\ldots,x_n,u)\, dt \tag{2.2}$$

The Modified Maximum Principle (MMP) requires that the function F be separable and linear with respect to the state variables $x_i (i=1,\ldots,n)$. Also, when applying this principle, it is necessary that the system equations can be expressed in the following form:

$$\begin{aligned} \dot{x}_i &= f_i(x_1,\ldots,x_n,u,t) \\ &\triangleq \sum_{j=1}^{n} a_{ij}x_j + \phi_i(u) \quad i=1,\ldots,n \end{aligned} \tag{2.3}$$

Using this method, it is convenient first to incorporate the performance index into the system equations by defining a new state variable, and then optimize a linear combination of the final values of the state variables. That is, it is required to extremize the scalar function,

$$S = \sum_{i=1}^{n+1} C_i x_i(T) \tag{2.4}$$

where T is the final time and the constants C_i depend on the variables to be extremized.

To obtain a solution by this method one must form the following set of auxiliary differential equations, which are similar to the adjoint set of the system

$$\dot{P}_i = -\sum_{s=1}^{n} P_s \frac{\partial f_s}{\partial x_i} \tag{2.5}$$

The boundary conditions for the above equations are

$$P_i(T) = - C_i \ , \quad i=1,\ldots,n \tag{2.6}$$

provided there are no final value constraints on the state variables.

For the case when final value constraints on the state variables are present and are expressed by a set of linear equations of the form

$$F[x_1(T),\ldots,x_n(T)] = 0 \tag{2.7}$$

the boundary conditions are given by

$$P_i(T) = - \left[C_i + \lambda_k \sum_{k=1}^{m} \frac{\partial F_k}{\partial x_i(T)} \right] , \quad m<n, i=1,\ldots,n \tag{2.8}$$

The λ_k represent Lagrange multipliers. The extra equations needed to determine the λ_k are the constraint equations given above.

From the auxiliary functions, a scalar function H is defined as

$$H = H(x_1,\ldots,x_n;P_1,\ldots,P_n,u,t) = \sum_{i=1}^{n} P_i f_i \tag{2.9}$$

where the functions f_i are defined by Equation 2.1.

It is shown in [8] that the control function u(t) may be found by extremizing the function

$$I' = \int_0^T H \, dt \tag{2.10}$$

with respect to u(t). It is also shown that if the function S is to be minimized (maximized) then the function I' must be maximized (minimized). Since H appears as a function of the variables x_i and P_i, and since the x_i's cannot be found without knowing u(t), the extremization of I' appears to be a formidable task. However, extremizing the function I' is equivalent to extremizing the function

$$I^* = \int_0^T \sum_{i=1}^{n} P_i \phi_i \, dt \tag{2.11}$$

provided the system equations are of the form given by Equation 2.3. In this case, the variables P_i are functions of time, and consequently the function I* is only dependent on the control function u(t) over the given time interval.

The detailed development of the MMP described above is given in [8].

2.2 Remark on the Formulation of I*(U)

Applying the MMP to any open-loop system of order n of the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 \dot{x} + a_0 x = f(u) \tag{2.12}$$

with the performance index to be maximized or minimized of the form

$$J = b_0 x(T) + b_1 \dot{x}(T) + \ldots + b_{n-2} x^{(n-2)}(T) \mp \alpha_0 \int_0^T F(u)\, dt \tag{2.13}$$

where the minus (positive) sign preceeding the last term in the right hand side is used whenever J is to be maximized (minimized). It can be shown that I*(u) will take the form

$$I^*(u) = \int_0^T [\frac{1}{a_n} P_n(t)\ f(u) \pm \alpha\ F(u)]\ dt \tag{2.14}$$

To prove the relationship (2.14), let $x_1 = x$ and define the other state variables as

$$\begin{aligned}
x_2 &= \dot{x}_1 = \dot{x} \\
x_3 &= \dot{x}_2 = \ddot{x} \\
&\cdots \\
x_n &= \dot{x}_{n-1} = x^{(n-1)} \\
\dot{x}_n &= x^{(n)} \\
x_{n+1} &= J = b_0 x(T) + b_1 \dot{x}(T) + \ldots + b_{n-2} x^{(n-2)}(T) \\
&\quad \mp \alpha \int_0^T F(u)\ dt
\end{aligned} \tag{2.15}$$

Then, the system equation becomes

$$\begin{aligned}
\dot{x}_1 &= f_1 = x_2 \\
\dot{x}_2 &= f_2 = x_3 \\
&\cdots \\
\dot{x}_{n-1} &= f_{n-1} = x_n \\
\dot{x}_n &= f_n = [-a_{n-1}x_n - \ldots - a_1 x_2 - a_0 x_1 + f(u)]\ \frac{1}{a_n} \\
\dot{x}_{n+1} &= f_{n+1} = b_0 x_2 + b_1 x_3 + \ldots + b_{n-2} x_n \mp \alpha\ F(u)
\end{aligned} \tag{2.16}$$

when Equation 2.16 is compared with the general form of Equation 2.3, one obtains

$$
\begin{aligned}
\phi_1 &= 0 \\
&\cdots \\
\phi_{n-1} &= 0 \\
\phi_n &= \frac{f(u)}{a_n} \\
\phi_{n+1} &= \mp \alpha F(u)
\end{aligned}
\tag{2.17}
$$

In order that the Pontryagin function $S = \Sigma C_i x_i(T)$ be equal to the performance index J, the constants C_i must take the values

$$
\begin{aligned}
C_1 &= 0 \\
&\cdots \\
C_n &= 0 \\
C_{n+1} &= 1
\end{aligned}
\tag{2.18}
$$

From their definition the auxiliary variables P_i must satisfy the equations

$$
\begin{aligned}
\dot{P}_1 &= -\sum_{s=1}^{n+1} P_s \frac{\partial f_s}{\partial x_1} = \frac{a_o}{a_n} P_n(t) \\
&\cdots \\
\dot{P}_n &= -\sum_{s=1}^{n+1} P_s \frac{\partial f_s}{\partial x_n} = -P_{n-1} + \frac{a_{n-1}}{a_n} P_n - b_{n-2} P_{n+1} \\
\dot{P}_{n+1} &= -\sum_{s=1}^{n+1} P_s \frac{\partial f_s}{\partial x_{n+1}} = 0
\end{aligned}
\tag{2.19}
$$

with the boundary conditions

$$\begin{aligned} P_1(T) &= 0 \\ &\cdots \\ P_n(T) &= 0 \\ P_{n+1}(T) &= -1 \end{aligned} \tag{2.20}$$

Substituting Equation 2.17 into Equation 2.11 the function $I^*(u)$ becomes

$$I^*(u) = \int_0^T [P_n(t)\ \phi_n(u) + P_{n+1}(t)\ \phi_{n+1}(u)]\ dt$$

$$I^*(u) = \int_0^T [\frac{1}{a_n} P_n(t)\ f(u) \mp \alpha\ P_{n+1}(t)\ F(u)]\ dt$$

Solving for the auxiliary variable $P_{n+1}(t)$ using Equations 2.19 and 2.20, it is found that

$$P_{n+1}(t) = -1 \tag{2.21}$$

and hence, the final form of $I^*(u)$ is

$$I^*(u) = \int_0^T [\frac{1}{a_n} P_n(t)\ f(u) \pm \alpha\ F(u)]\ dt \tag{2.22}$$

It should be noted that Equation 2.22 is applicable to any performance index consisting of a linear combination of any number of final values of state variables ranging from x_1 up to and including x_{n-1}.

If the function J includes the term $x^{(n-1)}$, i.e., the state variable x_n, it can be proved by following the same argument that

$$I^*(u) = \int_0^T [\frac{1}{a_n}(P_n(t) - b_{n-1})\ f(u) \pm \alpha\ F(u)]\ dt \qquad (2.23)$$

Equation 2.13 is expressed as

$$J = b_0 x(T) + b_1 \dot{x}(T) + \ldots + b_{n-1} x^{(n-1)}(T) \mp \alpha \int_0^T F(u)\ dt \quad (2.24)$$

The system Equation 2.16 becomes

$$\begin{aligned}
&\dot{x}_1 = f_1 = x_2 \\
&\ldots \\
&\dot{x}_{n-1} = f_{n-1} = x_n \\
&\dot{x}_n = f_n = \frac{1}{a_n}[-a_{n-1}x_n - a_{n-2}x_{n-1} \ldots - a_1 x_2 - a_0 x_1 + f(u)] \\
&x_{n+1} = f_{n+1} = -b_{n-1}\frac{a_o}{a_n} x_1 + (b_0 - \frac{b_{n-1}a_1}{a_n})x_2 + \ldots \\
&\qquad + (b_{n-2} - \frac{b_{n-1}a_{n-1}}{a_n})x_n + \frac{b_{n-1}}{a_n} f(u) \mp \alpha\ F(u)
\end{aligned}$$

and Equation 2.17 becomes

$$\begin{aligned}
&\phi_1 = 0 \\
&\ldots \\
&\phi_{n-1} = 0 \\
&\phi_n = \frac{f(u)}{a_n} \\
&\phi_{n+1} = \frac{b_{n-1}}{a_n} f(u) \mp \alpha\ F(u)
\end{aligned} \qquad (2.25)$$

Substituting into Equation 2.11 from Equation 2.26 and then from Equation 2.22 we get

$$I^*(u) = \int_0^T [P_n(t) \frac{f(u)}{a_n} \mp \alpha P_{n+1}(t) F(u) + \frac{b_{n-1}}{a_n} P_{n+1}(t) f(u)] dt$$

$$I^*(u) = \int_0^T [\frac{1}{a_n}\{P_n(t) - b_{n-1}\} f(u) \pm \alpha F(u)] dt \qquad (2.23)$$

To obtain a general form for $I^*(u)$, let

$$P(t) = \frac{1}{a_n}(P_n(t) - b_{n-1}) \quad \text{if J includes the } x^{(n-1)} \text{ term}$$

$$= \frac{1}{a_n} P_n(t) \quad \text{if J does not include the } x^{(n-1)} \text{ term} \qquad (2.27)$$

then, in general

$$I^*(u) = \int_0^T [P(t) f(u) \pm \alpha F(u)] dt \qquad (2.28)$$

It is noticed that the only auxiliary variable contained in $I^*(u)$ is $P_n(t)$. The advantage of this property will be taken into consideration later.

CHAPTER 3

FORMULATION OF I*(U) FOR DIFFERENT SYSTEMS

It has been shown in the previous chapter that, for an n^{th} order system of the form assumed, I* depends on a function of the control function u(t), which is given by the system equation and the performance index. Also, the only explicit function of time contained in I* is the n^{th} auxiliary function $P_n(t)$.

Therefore, in order to find a way of extremizing I*(u), it is necessary first to examine various shapes of $P_n(t)$ and to find common properties among them. This can be done by solving, for I*(u), some systems of the first, second and third order using different performance indices.

3.1 Formulation of I*(U)

3.1.1 First order system

Given the open loop system described by the differential equation

$$\dot{x} + ax = f(u) \quad , \qquad 0 \leq t \leq T$$

it is required to maximize the performance index

$$J = x(T) - \alpha \int_0^T F(u) \, dt$$

That is, it is required to minimize $I^*(u)$ in Equation 2.28

$$I^*(u) = \int_0^T [f(u)\ P(t) + \alpha\ F(u)]\ dt \tag{2.28}$$

where $P(t) = P_1(t) - 1$, since J includes the $x(T)$ term.
Letting

$$x_1 = x$$
$$x_2 = x_1 - \int_0^t \alpha\ F(u)\ dt$$

one obtains

$$\dot{x}_1 = f_1 = -\ ax_1 + f(u)$$

$$\dot{x}_2 = f_2 = -\ ax_1 + f(u) - \alpha\ F(u)$$

The constants C_i in the Pontryagin function $S = \Sigma\ C_i x_i(T)$ are,

$$C_1 = 0$$

$$C_2 = 1$$

and consequently, the boundary conditions of the auxiliary variables P_i are

$$P_1(T) = 0$$

$$P_2(T) = -1$$

The auxiliary variables P_i must satisfy the equations

$$\dot{P}_1 = aP_1 + aP_2$$

$$\dot{P}_2 = 0$$

The resulting $P_1(t)$ is

$$P_1(t) = 1 - e^{a(t-T)}$$

Substituting in Equation 2.28

$$I^*(u) = \int_0^T [\alpha\, F(u) - f(u)\, e^{a(t-T)}]\, dt$$

Following the same development $I^*(u)$ can be found for different performance indices. The results are shown in Table 3.1.1.

3.1.2 Second order system

Given the open loop system described by the differential equation

$$\ddot{x} + a\dot{x} + bx = f(u)\;, \quad 0 \leq t \leq T$$

it is required to maximize

$$J = x(T) - \alpha \int_0^T F(u)\, dt$$

As J does not include the $x^{(n-1)}$ term, Equation 2.28 is used, where $P(t) = P_n(t) = P_2(t)$; that is, it is required

Table 3.1.1: I*(u) for first order systems.

System Equation	Performance Index	I*(u)
$\dot{x} = f(u)$	$x(T) \mp \alpha \int_0^T F(u)\, dt$	$\int_0^T [-f(u) \pm \alpha F(u)]\, dt$
	$\alpha \int_0^T F(u)\, dt$	$\int_0^T [f(u) - \alpha F(u)]\, dt$
$\dot{x} + ax = f(u)$	$x(T) \mp \alpha \int_0^T F(u)\, dt$	$\int_0^T [-f(u)\, e^{a(t-T)} \pm \alpha F(u)]\, dt$
	$\alpha \int_0^T F(u)\, dt$	$\int_0^T [f(u)\, e^{a(t-T)} - \alpha F(u)]\, dt$

to minimize I*(u) where

$$I^*(u) = \int_0^T [f(u)\ P_2(t) + \alpha\ F(u)]\ dt$$

Letting

$$x_1 = x$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2 = -\ ax_2 - bx_1 + u$$

the system equation (Equation 2.16) becomes

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\ ax_2 - bx_1 + u$$

$$\dot{x}_3 = x_2 - \alpha\ F(u)$$

The constants C_i , in Pontryagin function $S = \Sigma\ C_i x_i$ are

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = 1$$

and the boundary conditions of the auxiliary variables P_i are

$$P_1(T) = 0$$

$$P_2(T) = 0$$

$$P_3(T) = -1$$

The auxiliary variables P_i must satisfy the equations

$$\dot{P}_1 = bP_2$$

$$\dot{P}_2 = - P_1 + aP_2 - P_3$$

$$\dot{P}_3 = 0$$

which means that the resulting $P_2(t)$ depends on the relation between a and b. Three cases are considered:

(a) Under damped systems

This corresponds to the case when $a<2\sqrt{b}$. $P_2(t)$ is found to be

$$P_2(t) = 2(4b-a^2)^{-\frac{1}{2}} e^{\frac{a}{2}(t-T)} \sin \tfrac{1}{2}(4b-a^2)^{\frac{1}{2}} (t-T)$$

(b) Critically damped systems

This is the case when $a=2\sqrt{b}$. The resulting $P_2(t)$ is

$$P_2(t) = (t-T) e^{\frac{a}{2}(t-T)}$$

(c) Over damped systems

This corresponds to the case when $a>2\sqrt{b}$ and $P_2(t)$ is given by

$$P_2(t) = (a^2-4b)^{-\frac{1}{2}} [e^{\frac{1}{2}[a+(a^2-4b)^{\frac{1}{2}}](t-T)} - e^{\frac{1}{2}[a-(a^2-4b)^{\frac{1}{2}}](t-T)}]$$

Table 3.1.2 presents $I^*(u)$ for different performance

Table 3.1.2: $I^*(u)$ for second order systems.

System Equation	Performance Index	$I^*(u)$
$\ddot{x} = f(u)$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [f(u)(t-T) \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u) \pm \alpha F(u)]dt$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [f(u)(t-T-1) \pm \alpha F(u)]dt$
$\ddot{x} + bx = f(u)$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [f(u)\, b^{-\frac{1}{2}} \sin b^{\frac{1}{2}}(t-T) \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u) \cos b^{\frac{1}{2}}(t-T) \pm \alpha F(u)]dt$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u)\, (1+\frac{1}{b})^{\frac{1}{2}} \cos(b^{\frac{1}{2}}(t-T)-\theta) \pm \alpha F(u)]dt$ where $\theta = \tan^{-1}(b^{-\frac{1}{2}})$
$\ddot{x} + a\dot{x} = f(u)$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [\frac{f(u)}{a} (e^{a(t-T)}-1) \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u)\, e^{a(t-T)} \pm \alpha F(u)]dt$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-\frac{f(u)}{a} \{1-(1-a)e^{a(t-T)}\} \pm \alpha F(u)]dt$

Table 3.1.2 (continued)

System Equation	Performance Index	I*(u)
$\ddot{x} + a\dot{x} + bx = f(u)$ $a < 2\sqrt{b}$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [2f(u)\ (4b-a^2)^{-\frac{1}{2}}\ e^{\frac{a}{2}(t-T)} \sin \frac{1}{2}(4b-a^2)^{\frac{1}{2}}(t-T) \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-2f(u)\ b^{\frac{1}{2}}(4b-a^2)^{-\frac{1}{2}} e^{\frac{a}{2}(t-T)} \cos[\frac{1}{2}(4b-a^2)^{\frac{1}{2}}(t-T)-\theta] \pm \alpha F(u)]dt$, $\theta = \tan^{-1} a(4b-a^2)^{-\frac{1}{2}}$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [2f(u)(b-a+1)^{\frac{1}{2}}(4b-a^2)^{-\frac{1}{2}}\ e^{\frac{a}{2}(t-T)} \sin[\frac{1}{2}(4b-a^2)^{\frac{1}{2}}(t-T)-\theta] \pm \alpha F(u)]dt$, $\theta = \tan^{-1}(4b-a^2)^{\frac{1}{2}}(2-a)^{-1}$
$\ddot{x} + a\dot{x} + bx = f(u)$ $a = 2\sqrt{b}$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [f(u)(t-T)\ e^{\frac{a}{2}(t-T)} \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u)\{1+\frac{a}{2}(t-T)\}\ e^{\frac{a}{2}(t-T)} \pm \alpha F(u)]dt$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u)\{1+(\frac{a}{2}-1)(t-T)\}\ e^{\frac{a}{2}(t-T)} \pm \alpha F(u)]dt$

Table 3.1.2 (continued)

System Equation	Performance Index	I*(u)
$\ddot{x} + a\dot{x} + bx = f(u)$ $a > 2\sqrt{b}$	$x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [f(u)(a^2-4b)^{-\frac{1}{2}}\{e^{\frac{1}{2}[a+(a^2-4b)^{\frac{1}{2}}](t-T)} - e^{\frac{1}{2}[a-(a^2-4b)^{\frac{1}{2}}](t-T)}\} \pm \alpha F(u)]dt$
	$\dot{x}(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-f(u)\ (\frac{1}{2}+a(a^2-4b)^{-\frac{1}{2}})\ e^{(\frac{a}{2}+\frac{1}{2}(a^2-4b)^{\frac{1}{2}})(t-T)} + (\frac{1}{2}-a(a^2-4b)^{-\frac{1}{2}})\ e^{\frac{1}{2}[a-(a^2-4b)^{\frac{1}{2}}](t-T)} \pm \alpha F(u)]dt$
	$\dot{x}(T) + x(T) \mp \alpha \int_0^T F(u)dt$	$\int_0^T [-\frac{1}{2}f(u)\{(1-(2-a)(a^2-4b)^{-\frac{1}{2}})\ e^{\frac{1}{2}[a+(a^2-4b)^{\frac{1}{2}}](t-T)} + [1+(2-a)(a^2-4b)^{-\frac{1}{2}}]\ e^{\frac{1}{2}[a-(a^2-4b)^{\frac{1}{2}}](t-T)}\} \pm \alpha F(u)]dt$

indices.

3.1.3 Third order system

For the open loop system described by the 3^{rd} order differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = f(u)$$

it is required to maximize

$$J = x(T) - \alpha \int_0^T F(u)\, dt$$

that is, it is required to minimize

$$I^*(u) = \int_0^T [f(u)\, P(t) + \alpha\, F(u)]\, dt$$

where $P(t) = P_3(t)$ as J does not include the $x^{(n-1)}$ term. Let

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$x_3 = \dot{x}_2$$

$$x_4 = J$$

one obtains

$$\dot{x}_1 = f_1 = x_2$$

$$\dot{x}_2 = f_2 = x_3$$

$$\dot{x}_3 = f_3 = -ax_3 - bx_2 - cx_1 + f(u)$$

$$\dot{x}_4 = f_4 = x_2 - \alpha F(u)$$

The constants C_i in the Pontryagin function are

$$C_1 = 0$$

$$C_2 = 0$$

$$C_3 = 0$$

$$C_4 = -1$$

and the auxiliary variables P_i must satisfy the equations

$$\begin{aligned} \dot{P}_1 &= cP_3 \\ \dot{P}_2 &= -P_1 + bP_3 - P_4 \\ \dot{P}_3 &= -P_2 + aP_3 \\ \dot{P}_4 &= 0 \end{aligned} \tag{3.1}$$

The boundary conditions of the auxiliary variables are

$$\begin{aligned} P_1(T) &= P_2(T) = P_3(T) = 0 \\ P_4(T) &= -1 \end{aligned} \tag{3.2}$$

and it is required to calculate $P_3(t)$.

Two cases are considered according to the roots of the auxiliary equation of the system. The equation has three roots which can be all real or one real and two complex. Specific numbers are used to solve for the auxiliary variable $P_3(t)$ to facilitate the calculations.

(a) Case of real roots

Letting $a = 4$, $b = 2$, $c = -1$ the auxiliary variables P_i must satisfy the equations

$$\dot{P}_1 = -P_3$$

$$\dot{P}_2 = -P_1 + 2P_3 - P_4$$

$$\dot{P}_3 = -P_2 + 4P_3$$

$$\dot{P}_4 = 0$$

with the boundary conditions given by Equation 3.2.

Solving these equations leads to complex calculations. This is a result of the knowledge of the final conditions rather than the initial conditions.† In this case, the following form of auxiliary variables could be obtained:

$$\begin{aligned} P_1(t) = {} & (-0.703 + 0.491\ A_1 + 0.213\ A_2 + 0.176\ A_3)e^{-0.3t} \\ & + (+0.037 - 0.157\ A_1 + 0.12\ A_2 + 0.157\ A_3)e^{3.3t} \\ & + 0.333(-1 + 2A_1 - A_2 - A_3)e^{t} + 1 \end{aligned}$$

$$\begin{aligned} P_2(t) = {} & (-0.213 + 0.213\ A_1 + 0.277\ A_2 + 0.084\ A_3)e^{-0.3t} \\ & + (-0.12 + 0.12\ A_1 - 0.277\ A_2 + 0.916\ A_3)e^{3.3t} \\ & + 0.333(1 - A_1 + 3A_2 - 3A_3)e^{t} \end{aligned}$$

$$\begin{aligned} P_3(t) = {} & (-0.213 + 0.21\ A_1 + 0.065\ A_2 + 0.019\ A_3)e^{-0.3t} \\ & + (-0.12 + 0.12\ A_1 - 0.398\ A_2 + 1.314\ A_3)e^{3.3t} \\ & + 0.333(1 - A_1 + A_2 - A_3)e^{t} \end{aligned}$$

---

† These equations can also be solved using reverse time method.

where $A_n = P_n(0)$, $n = 1,2,3$. Substituting for the final conditions from Equation 3.2, A_1, A_2 and A_3 can be found as a function of the final time T; however, in a complicated form. The above equations can be easily solved for any specified final time T. Thus, $P_3(t)$ will be of the general following form

$$P_3(t) = Ae^{-0.3t} + Be^{3.3t} + Ce^{t}$$

where A, B and C are constants depending on the final time. Substituting into Equation 2.27

$$I^*(u) = \int_0^T [f(u)\ (Ae^{-0.3t} + Be^{3.3t} + Ce^{t}) + \alpha\ F(u)]\ dt$$

(b) Case of one real and two complex roots

Letting a = 2, b = 2, c = -1, the same difficulties encountered in case (a) will arise in the solution of the differential equations (Equation 3.1). $P_3(t)$ in this case is found to be of the form

$$P_3(t) = Ae^{-0.35t} + Be^{1.175t}\sin 1.2t + Ce^{1.175t}\cos 1.2t$$

where A, B and C are constants depending on the value of the final time and the boundary conditions.

3.2 Remarks on the Form of I*(U)

Examining the I*(u)'s computed for the systems considered and the graphs plotted for the P(t)'s, for example Fig. 3.2.1, Fig. 3.2.2, and Fig. 3.2.3, two important remarks can be concluded:

(a) The function P(t), the only explicit function of time contained in I*(u), can be

(i) Extremum-one†, between any two consecutive zeros.

(ii) Extremum-one or monotonic, between the initial time and the first zero, or, between the last zero and the final time, or, between the two end points.

(b) Another remark is about the integrand in the expression of I*(u). It is noticed that the control function u(t) can always be factored out. For example, Equation 2.28 can be written as

$$I^*(u) = \int_0^T u\{\frac{f(u)}{u} P(t) \pm \alpha \frac{F(u)}{u}\} dt$$

where usually

$$F(u) = \alpha u^2 \quad \text{or} \quad \alpha|u|$$

$$f(u) = u$$

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† An extremum-one function is defined here as a function which is non-monotonic and has only one maximum (minimum) in the interval considered.

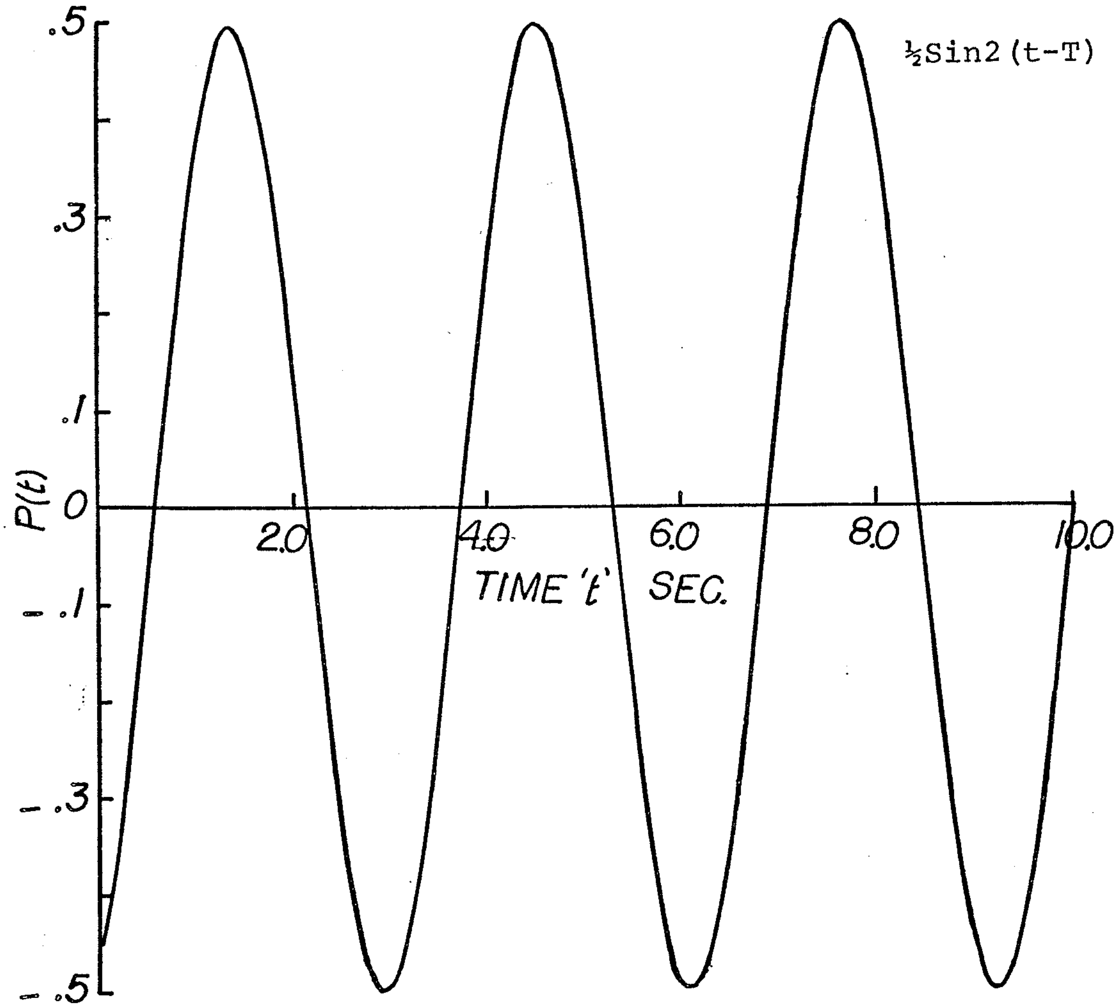

Fig. 3.2.1 The auxiliary function P(t) for the system $\ddot{x} + 4x = f(u)$ and the performance index $x(T) - \int_0^T \alpha F(u)dt, T = 10.0$ sec.

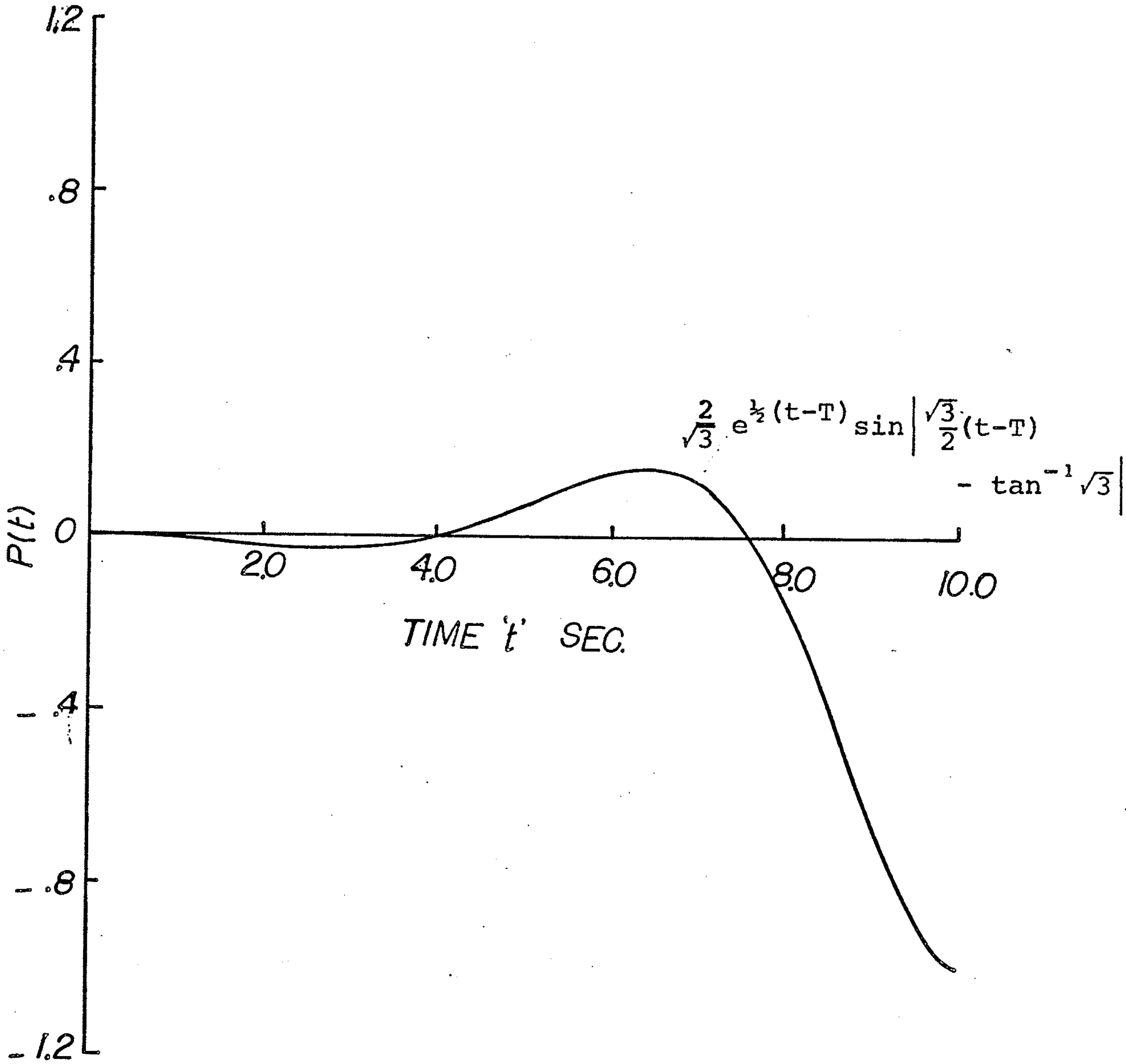

Fig. 3.2.2. The auxiliary function P(t) for the system $\ddot{x} + \dot{x} + x = f(u)$ and the performance index $\dot{x}(T) + x(T) - \alpha_0\int_0^T F(u)\,dt, T = 10.0$ sec.

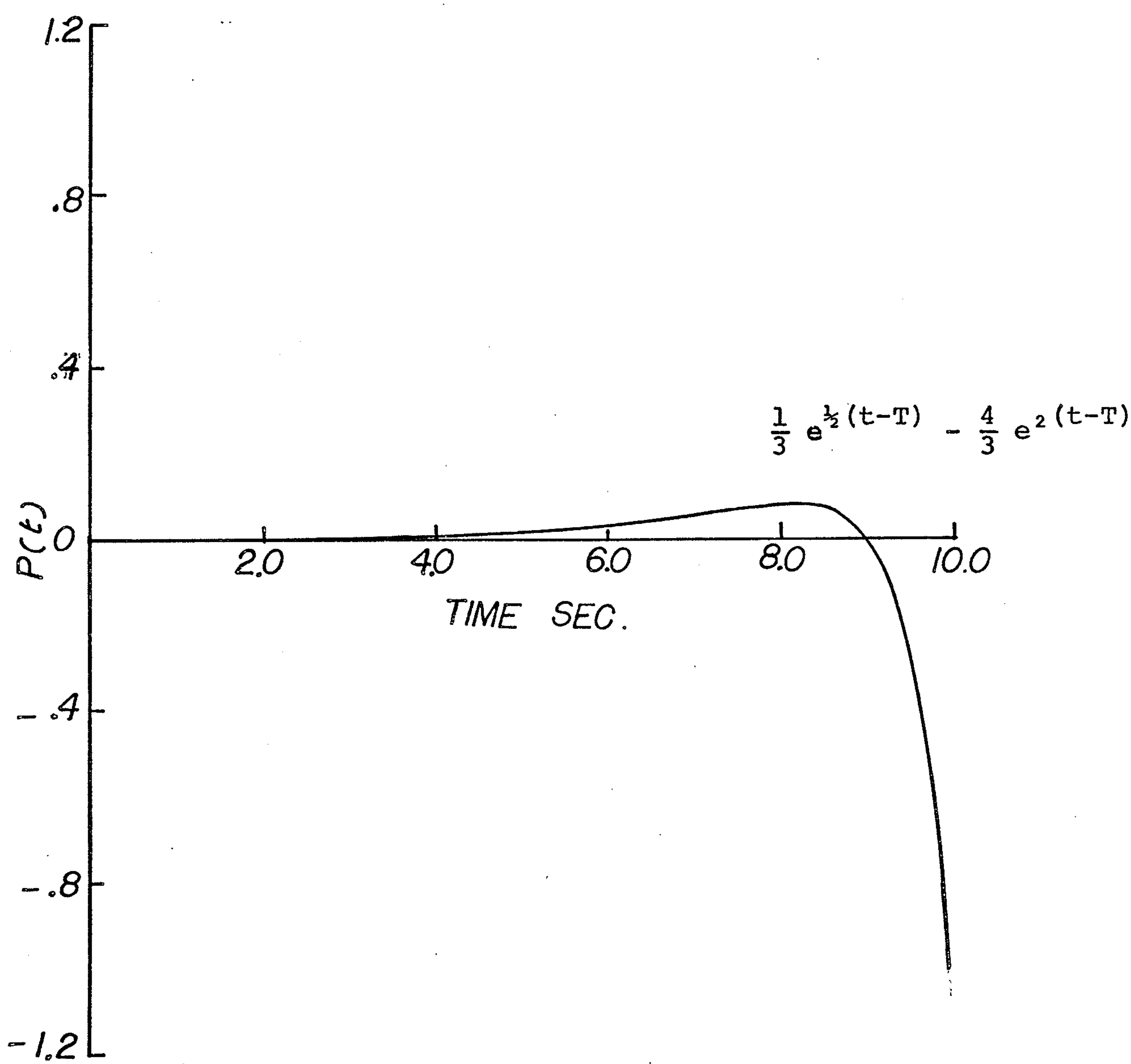

Fig. 3.2.3 The auxiliary function P(t) for the system $\ddot{x} + \frac{5}{2}\dot{x} + x = f(u)$ and the performance index $\dot{x}(T) - \alpha\int_0^T F(u)dt$, T = 10.0 sec.

The above mentioned properties will be used in defining the method of optimization of $I^*(u)$ presented in Chapter 4.

CHAPTER 4

OPTIMIZATION PROCEDURE

Standard methods could not be applied directly in the present study because the process of extremizing the function $I^*(u)$, which is used to find the optimum control $u(t)$, is a "non-analytic process". By this, it is meant that differentiation with respect to u is not possible since the function $u(t)$ is a series of pulses of fixed duration and amplitude. To extremize the function $I^*(u)$ it is required to find the optimum control function $u(t)$, i.e., to find the optimum position and polarity of each pulse in the train of pulses constituting $u(t)$. In other words, with

$$I^*(u) = \int_0^T [P(t)\ f(u) \pm \alpha\ F(u)]\ dt$$

the aim of the present study is to find the $u(t)$ which maximizes the absolute value of the area representing $I^*(u)$. For example, if the minimum (maximum) of $I^*(u)$ is sought, it is desirable to have as many pulses as possible which might add negative (positive) contributions to $I^*(u)$.

4.1 Brief Description of the Optimization Method

In the previous chapter some conclusions have been drawn from the various I*(u) originating from different order systems and with different performance indices. One of them is that "u" can always be factored out from the integrand. The importance of this fact is that whenever one of the pulses constituting the control u(t) has a magnitude other than zero, the integration of $\Sigma P_i \phi_i$ over the pulse width has a non-zero value. When the control u(t) is equal to zero, i.e., during the dead time between pulses, a zero contribution is added to I*(u). Hence, the integration over the interval [0,T] can be transformed to a summation of integrals over the pulse width only. That is, if I*(u) is of the form

$$I^*(u) = \int_0^T (FCT)\, dt \qquad (4.1)$$

where FCT is the integrand represented by

$$FCT = P(t)\ f(u) \pm \alpha\ F(u) \qquad (4.2)$$

then I*(u) can have the following form

$$I^*(u) = \sum_{n=1}^{N} \int_{t_n}^{t_n+W} (FCT)\, dt \qquad (4.3)$$

where t_n = initiation time of the n^{th} pulse

N = number of pulses available.

The discussion in this section will be concerned with the parts of the function P(t) included between (i) two consecutive zeros of P(t); (ii) the end points if no zero exists; and (iii) one of the end points and the nearest zero. That is, the function P(t) has the same polarity during the time interval considered. Since minimizing (maximizing) the function I*(u) is equivalent to minimizing (maximizing) every subinterval of the function P(t), then the optimum solution u(t) for the interval [0,T] is the summation of the optimum solutions of the different subintervals. Special consideration is given to the boundary points of the adjacent subintervals of P(t) to allow interaction between them.

4.1.1 Monotonic Function P(t)

To extremize I*(u) over any part of the function P(t), it is clear that the concentration of pulses is required in the proximity of the extremum points of this function.

In the case of a monotonic function P(t), the procedure starts by searching for the extremum point of the function to locate the first pulse. Other pulses are then placed as close as possible to each other, that is, having the minimum allowable dead time between any two consecutive pulses. Figure 4.1.1 illustrates the procedure.

W is the pulse width and D is the minimum dead time that must be present between pulses.

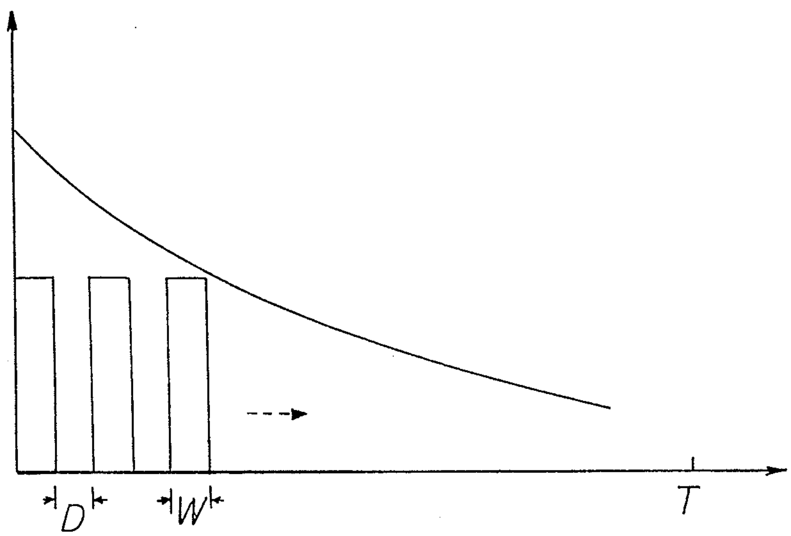

Fig. 4.1.1 Distribution of pulses for a monotonic function

4.1.2 Extremum-one function P(t)

For this case, the search for the extremum point is also done first. If the function is symmetric, there must be either a pulse or a dead time located around the extremum. But if P(t) is not symmetric, for the interval considered, the only known information is that the distribution of pulses should be concentrated around the extremum point. An iterative method is needed to find this distribution. The detailed description for the search of such distribution will be presented in Section 4-2.

4.1.3 Case of cost applied on pulses

When a cost α is imposed on the pulses, it is found that it is undesirable to have any pulses applied during certain intervals of the function P(t). The reason is that during these intervals, the contribution of the pulses to I*(u) would be of opposite sign to the optimum sought.

For example, if the minimum of I*(u) is wanted, a zero contribution to the value of I*(u) is better than some positive contribution. For the remaining parts of the function duration the pulses are distributed as required. That is, if the minimum of

$$I^*(u) = \int_0^T u(-e^{2(t-T)} + \alpha u)\, dt$$

is sought, it is found that negative pulses will contribute positively to the value of I*(u). Positive pulses, for $e^{2(t-T)}$ less than αu, would also contribute positively. But for $e^{2(t-T)}$ greater than αu the contribution will be negative. If $e^{2(t-T)}$ is equal to αu, the limit where there are no contributions at all, positive or negative, to the value of I*(u) is obtained.

The limits can be determined by locating the zeros of the integrand of I*(u). Therefore

$$P(t)\ f(u) \pm \alpha F(u) = 0$$

$$\text{i.e.} \qquad P(t) = \mp \frac{\alpha F(u)}{f(u)}$$

where the upper (lower) sign is used when the minimum (maximum) of $I^*(u)$ is required. Thus, letting

$$\beta = \mp \frac{\alpha\ F(u)}{f(u)} \tag{4.4}$$

the intersection between $P(t)$ and β will specify the regions where pulses should and should not be applied.

4.2 Detailed Description of the Optimization Method

In the previous section, a brief description of the extremizing method of monotonic and extremum-one functions is given. It has been mentioned that an iterative method is needed. The optimization procedure has been programmed, and the computation done on the IBM 360 digital computer, in the University of Manitoba computer centre. The complete flow diagrams for the solution are given in Appendix A, and the detailed Fortran program is presented in Appendix B. The principal flow charts are for (i) the MONO subroutine which gives the algorithm of solution for monotonic functions; (ii) the SHIFT subroutine which includes the algorithm of solution for extremum-one functions; (iii) the main program which treats the function P(t) as a whole; it takes care of the shift between the subintervals of P(t) and the determination of the beginning point "$\bar{Z}$" and the ending point "FJ" of the interval in question. The interval depends on the cost applied on the pulses and on the allowable dead time around the zeros of the function P(t), where P(t) is changing sign. In the present study two cases are discussed: (i) pulses of different polarity are allowed to overlap partially at the end points; and (ii) the minimum dead time is imposed between all pulses, including those of different polarity.

4.2.1 <u>Solution to a monotonic function</u>

The following procedure is introduced to deal with any monotonic subinterval of P(t). A summary is given in Fig. 4.2.1, where AI(u) is defined as

$$AI(u) = \int_{t_{2n}}^{t_{2n+1}} (FCT)\, dt \qquad (4.5)$$

The flow diagram for the method is presented in Appendix A as the MONO subroutine.

In more detail, let

V ≡ interval of time under consideration

W ≡ pulse width

D ≡ minimum allowable dead time

then, the maximum number of allowed pulses can be calculated as follows:

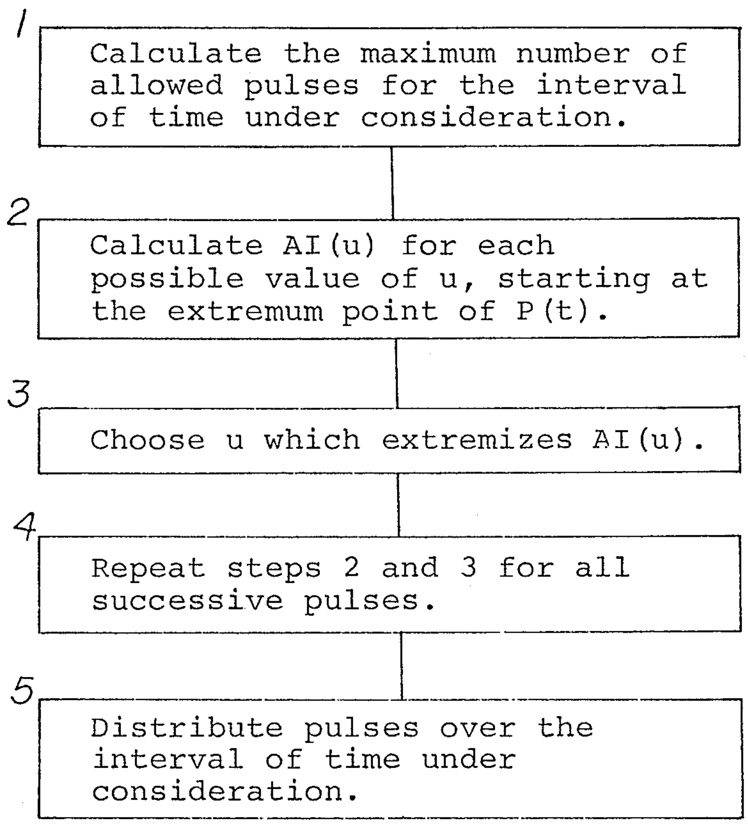

Fig. 4.2.1 Major operations to be carried out when dealing with a monotonic function.

$$N = \left[\frac{V - W}{D + W} + 1\right] \quad (4.6)$$

The pulse width is subtracted from the interval V and a one is then added since the last pulse does not need to be followed by a dead time. The brackets [ ] means that the number N is approximated to the nearest lower integer, since no portion of pulse is to be used. Once the pulse is initiated, it cannot be turned off for an interval of time equal to the pulse width.

In the present study any pulse constituting the function u(t) is allowed to have the following values: +X, -X, or zero. Knowing the extremum point of P(t), the calculation of AI(u) over the first pulse, starting or ending at the extremum point, is performed for each value of u given above. The value of u which gives minimum (maximum) AI(u) is then chosen. The operation is then repeated for the consecutive pulses which are located as close as possible to each other.

4.2.2 Solution to an extremum-one function

The summary of the solution procedure for any extremum-one subinterval of P(t) is given in Fig. 4.2.2. As in the case of a monotonic function, the first step is the calculation of the maximum number of allowed pulses for the interval in question, Equation 4-6. The distribution of pulses, over the interval, is then done locating the leading edge of the first pulse at the beginning point $\bar{Z}$

1. Calculate the maximum number of pulses allowed for the interval of time under consideration
2. Distribute the pulses over the interval in question such that the leading edge of the first pulse coincides with the starting point.
3. Calculate AI(u) for each possible value of u starting by the first pulse.
4. Choose u which extremizes AI(u). Let MAI(u) = minimum (maximum) AI(u).
5. Repeat steps 2 to 4 for all pulses.

Fig. 4.2.2 Major operations to be carried out when dealing with an extremum-one function.

6 Calculate $I(u) = \Sigma MAI(u)$

7 Shift all pulses by δ'

$\delta = \delta + \delta'$

8 If $\delta \leq W + D$

Repeat steps 3 to 7

9 If $\delta > W + D$

Choose δ which gives minimum (maximum) $I(u)$

10 Distribute the pulses over the interval of time under consideration.

Fig. 4.2.2 (continued)
Major operations to be carried out when dealing with an extremum-one function.

of the interval. The following step is the calculation of the area AI for each of the three values that a pulse can take, and then choose the value which gives minimum (maximum) AI for each pulse in the interval considered. I(u) is then calculated by summing the chosen AI's.

All the pulses are then shifted by an increment of time δ', the magnitude of which determines the accuracy of the optimization calculation. I(u) is then calculated for each shift δ' of the pulses.

Before calculating each I(u), a check must be done on the coincidence of the trailing edge of the last pulse and the ending point FJ of the interval. If the first is less than or equal to the second, I(u) will be the summation over the N chosen AI's. If the first is greater than the second, N is set to N-1, since once a pulse is initiated it cannot be turned off before a given period of time W. Hence, the last pulse is disregarded, and I(u) will be the summation of the N-1 first AI's. All the following calculated AI's will be for N-1 pulses.

The shift δ' is done a number of time equals to the ratio of the "pulse width pulse dead time" to δ'. In other words, the limit on δ, which is the total shift from $\bar{Z}$, is that it cannot be greater than the sum of the "pulse width plus dead time". If δ is greater than this sum, the calculations will be a repetition of the previous ones with the pulse number reduced by one.

After each calculation of I(u), a check must be done on δ. If δ is greater than the sum of the "pulse width pulse dead time", the iteration is terminated. The optimum solution is then obtained by choosing δ^* corresponding to the minimum (maximum) I(u). Then the optimum control function can be found by adding δ^* to the original distribution of u(t).

The flow diagram for this method is presented in Appendix A as the SHIFT subroutine.

4.2.3 Solution procedure

A summary of the solution procedure is given in Fig. 4.2.3 where

$\bar{Z}$ is the starting point of a subinterval

FJ is the ending point of the subinterval

The calculation of the zeros and local extremum points of P(t) must be done first. Once the zeros are determined, computations can be performed over the subintervals where the function P(t) has the same polarity. The next step is the search for the shape of the function P(t) in the first subinterval. It can be either monotonic or extremum-one. The main program flow chart in Appendix A gives the detailed description of the search method.

(a) *Monotonic Function*

If the function P(t) is monotonic and has the same polarity over the interval [0,T], two cases are considered. (i) If there is no cost on pulses, the optimum solution will

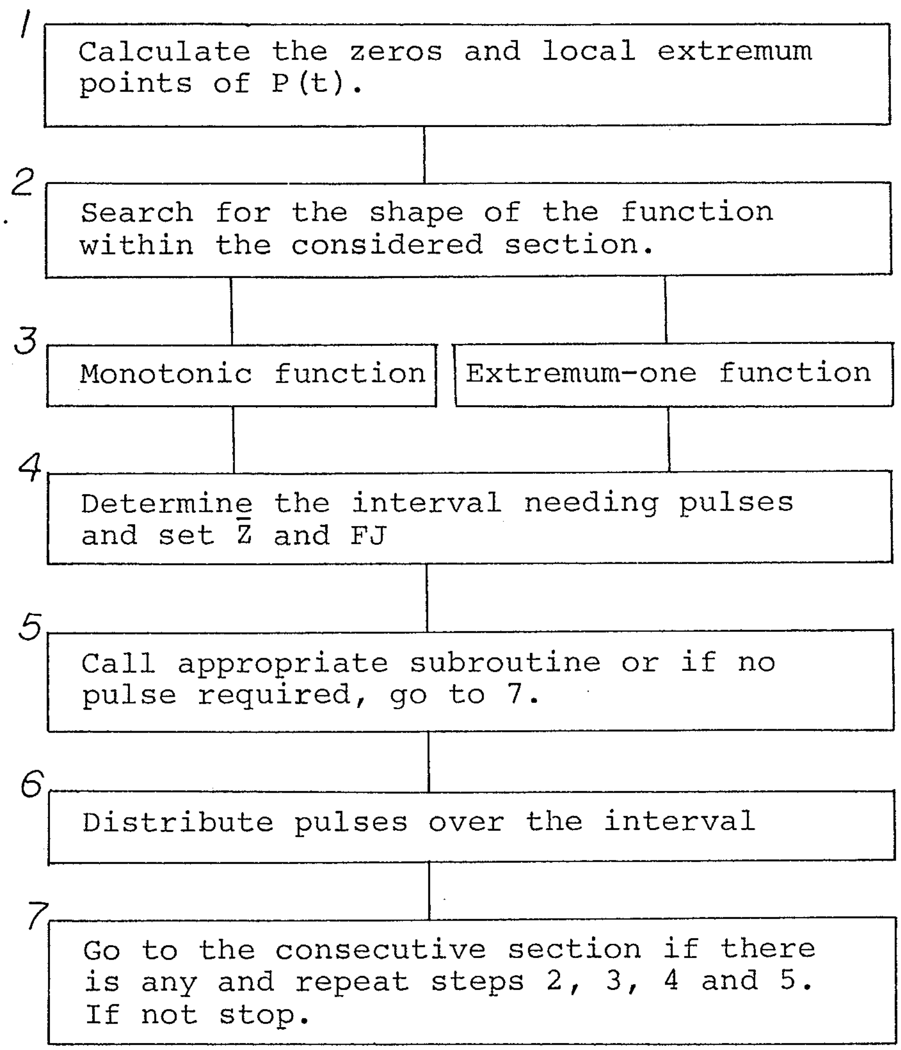

Fig. 4.2.3 Major operations to be carried out when dealing with a function consisting of more than one portion.

be obtained using the method suggested in Section 4.2.1 taking the interval V equal to the time T.

(ii) If a cost is applied on pulses, the interval V will be a part of the total interval T. To find V the points of intersection of β and P(t) should be calculated. Figure 4.2.4 presents the different cases that could be obtained. Figure 4.2.4, case (a,i) shows the case where the cost applied on pulses is too high and the solution with no pulses gives the optimum solution. Case (c) presents the case of a low cost, where no point of intersection can be found, and no reversed contributions are added to I(u). This case is, therefore, similar to the one without cost. For the general case, cases (a,ii) and (b,ii), a pulse around the point of intersection has almost zero contribution to I(u). The interval V is taken as

$$V = A + \frac{W}{2} \qquad \text{For case (a,ii)}$$

and

$$V = T - A + \frac{W}{2} \qquad \text{For case (b,ii)}$$

where A is the time at which the intersection of β and P(t) occurs.

The addition of $\frac{W}{2}$ to the interval allows all pulses, which contribute to the extremization of I(u), to be contained in the calculations. The limit for such pulses is the pulse centered at A.

Similar reasoning is applied to any monotonic section

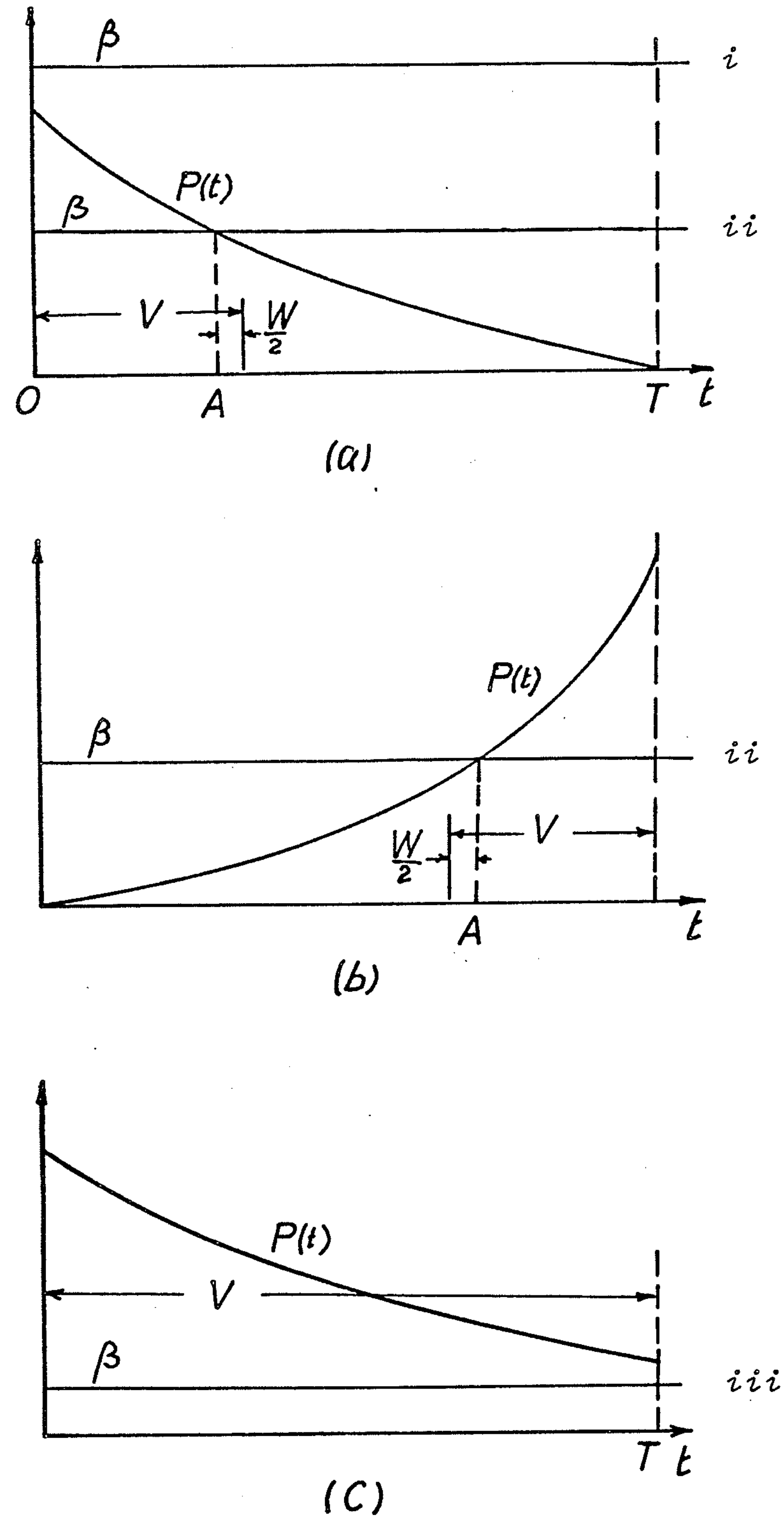

Fig. 4.2.4 The interval V for a monotonic function when cost is applied on pulses.

considered of the function, when there are one or more zeros within the interval (0,T). The main difference is that the dead time, around the zeros of the function, must be taken into consideration. Different cases are studied: (i) when no cost is applied on pulses and overlapping of pulses around the zeros of P(t) is allowed, the interval and the starting point are taken as

$$V = Z_1 + \frac{W}{2}\ , \quad \bar{Z} = 0.0 \qquad \text{For Fig. 4.2.5}$$

and

$$V = T - Z_n + \frac{W}{2}\ , \quad \bar{Z} = Z_n - \frac{W}{2} \qquad \text{For Fig. 4.2.6}$$

For the consecutive portion of P(t), in case of Fig. 4.2.5, the starting point $\bar{Z}$ will be

$$\bar{Z} = Z_1 - \frac{W}{2}$$

(ii) if no cost is applied on pulses but a minimum dead time D is essential between any two consecutive pulses, for the first zero Z_1, the dead time is assumed to be equally divided around Z_1, and hence

$$V = Z_1 - \frac{D}{2}\ , \quad \bar{Z} = 0.0 \qquad \text{For Fig. 4.2.5}$$

But, for the case of Fig. 4.2.6, the interval is equal to

$$V = T - \bar{Z}$$

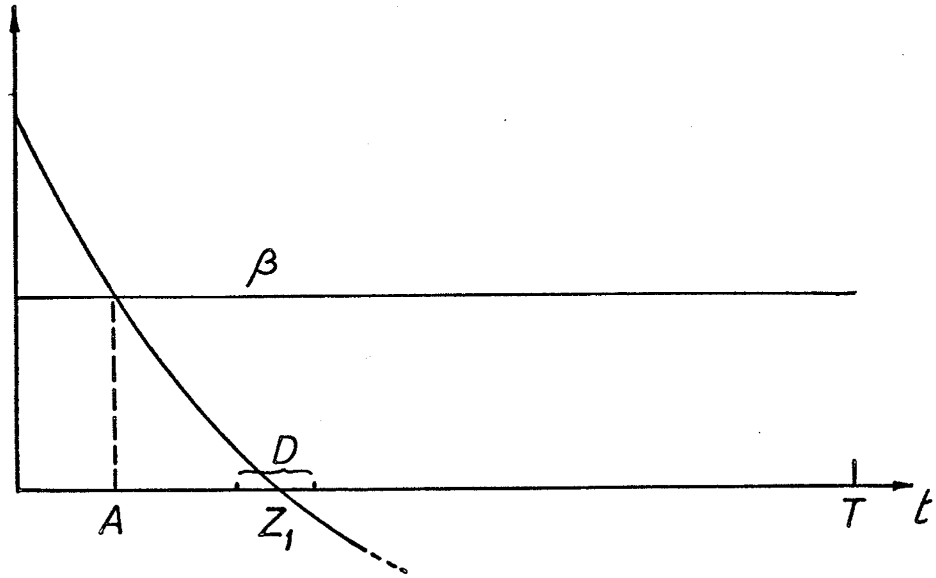

Fig. 4.2.5 Monotonic function as a first subinterval.

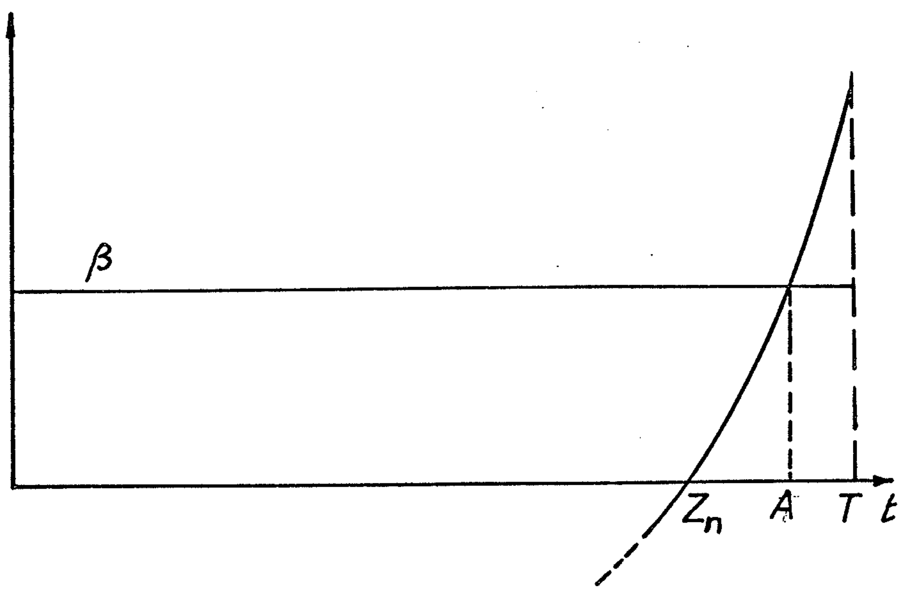

Fig. 4.2.6 Monotonic function as a last subinterval.

where $\bar{Z}$ is the time of the trailing edge of the last pulse in the preceding interval + dead time.

For the consecutive section of P(t), in case of Fig. 4.2.5, the starting point $\bar{Z}$ is taken as

$$\bar{Z} = NC$$

where · N = number of pulses contained in the first section

C = pulse width plus dead time.

(iii) if a cost is applied on the pulses and overlapping is allowed, two cases are considered:

- no pulses are required if no point of intersection exists between β and P(t) within the considered interval.
- if a point of intersection exists at the time A then the interval and the starting point are taken as

$$V = A + \frac{W}{2}\ , \quad \bar{Z} = 0.0 \quad \text{For Fig. 4.2.5}$$

and

$$V = T + \frac{W}{2} - A\ , \quad \bar{Z} = A - \frac{W}{2} \quad \text{For Fig. 4.2.6}$$

For the consecutive subinterval, Fig. 4.2.5, $\bar{Z}$ will depend on the points of intersection of β and P(t) within this subinterval. This case will be considered in a later section on the extremum-one function.

(iv) if a cost is applied on the pulses, and a minimum dead time is essential, two cases arise:

- no pulses are required if no point of intersection exists.
- if a point of intersection exists, the interval V will vary according to the time A. For Fig. 4.2.5, if

$$Z_1 - A - \frac{W}{2} < \frac{D}{2}$$

then

$$V = Z_1 - \frac{D}{2}$$

but if

$$Z_1 - A - \frac{W}{2} \geq \frac{D}{2}$$

then

$$V = A + \frac{W}{2}$$

For Fig. 4.2.6, if

$$A - \frac{W}{2} < \bar{Z}$$

then

$$V = T - \bar{Z}$$

but if

$$A - \frac{W}{2} \geq \bar{Z}$$

then

$$V = T - A + \frac{W}{2}$$

For the case of Fig. 4.2.5, $\bar{Z}$ of the consecutive subinterval will depend on the points of intersection of β and $P(t)$, within this subinterval. The flowchart representing the above described procedure is given in Appendix A-2, pages 14, 15, 16 and 19.

(b) *Extremum-one Function*

When no cost is applied on pulses, the method described in Sec. 4.2.2 is used to find the optimum solution. However, when a cost is applied, different cases (Fig. 4.2.7) arise. These cases are treated in the same way as in the case of the monotonic function.

For case (i), Fig. 4.2.7, no pulses are required. The interval calculated for case (ii) depends on the values of $(A_1 - \frac{W}{2})$ and $(A_2 + \frac{W}{2})$ with respect to the zero time and the final time, respectively. For case (iii), V depends on the value of $(A + \frac{W}{2})$ with respect to T, while for case (iv), the interval is equal to T.

The same method is applied to any section of the function included between one of the end points and the nearest zero, or, between two consecutive zeros. As the solution shifts to the next part of the function P(t), new values must be assigned to $\bar{Z}$. For example, for the case where there are no costs on pulses, and a minimum dead time is essential between any two consecutive pulses, $\bar{Z}$ is taken as

$$\bar{Z}_2 = \bar{Z}_1 + \delta^* + NC$$

This is illustrated in Fig. 4.2.8

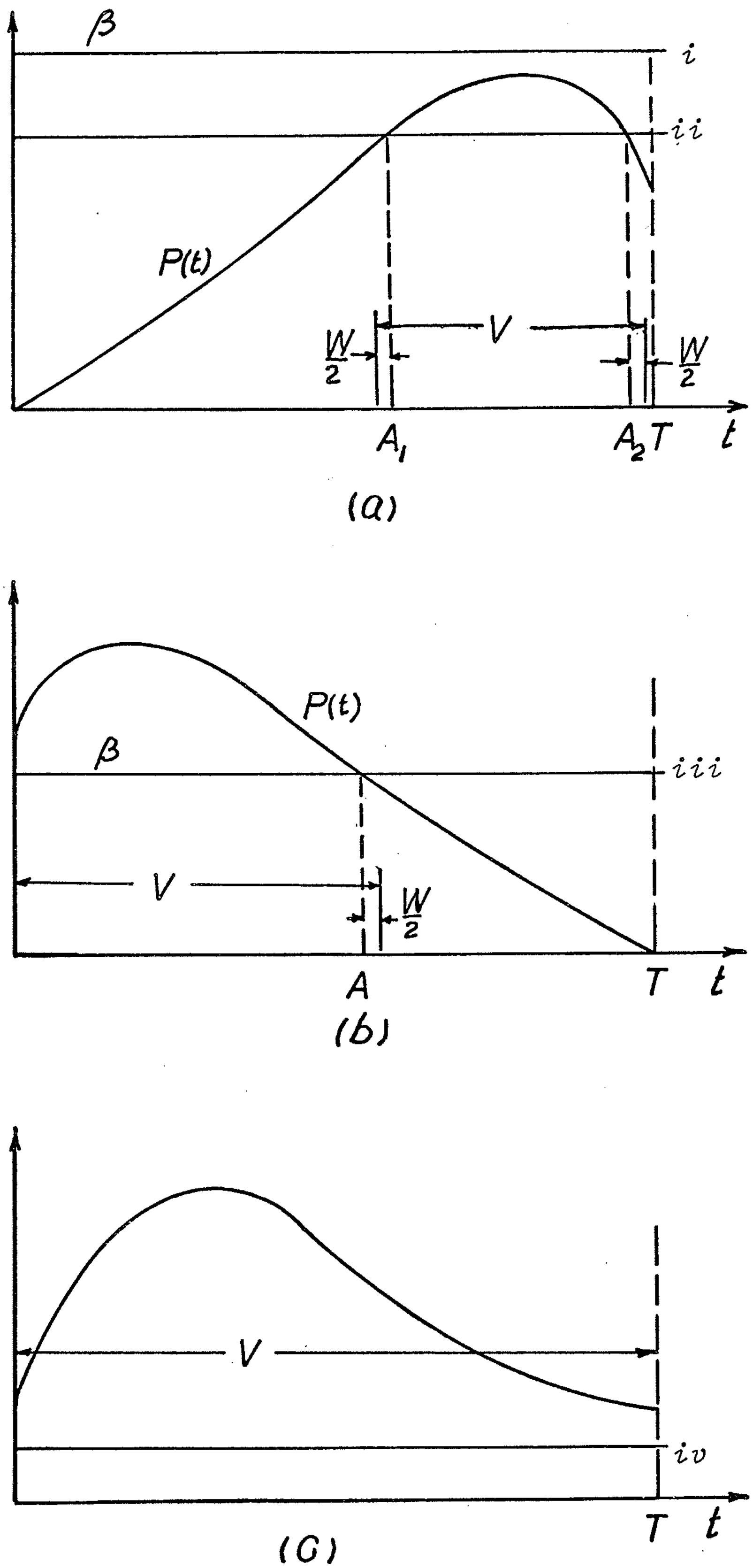

Fig. 4.2.7 The interval V for an extremum-one function when a cost is applied on pulses.

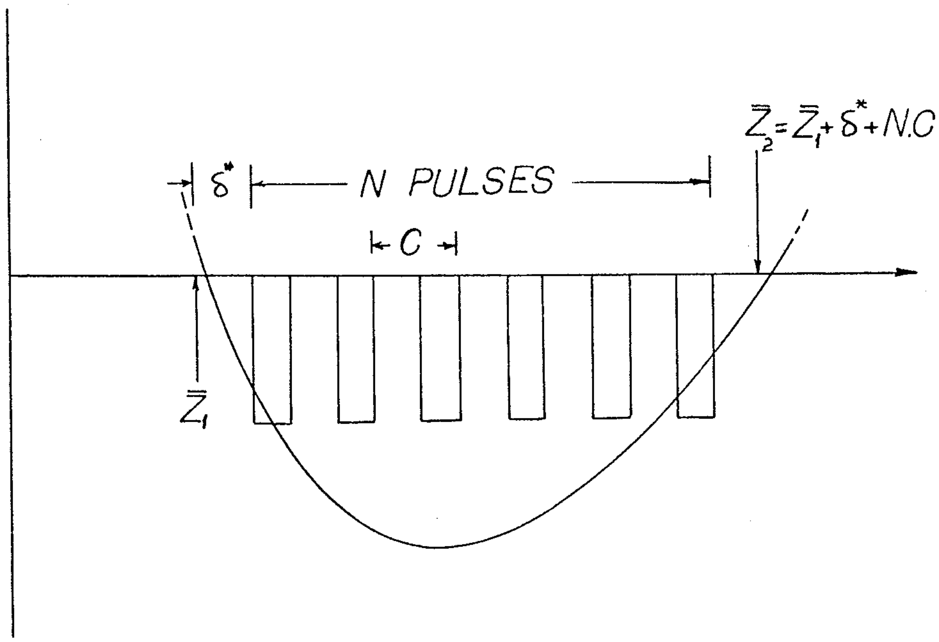

Fig. 4.2.8 Starting point of an interval preceeded by an extremum-one function.

For the detailed expressions of the interval V, the starting point $\bar{Z}$, and the ending point FJ, the reader is referred to Appendix A.2.

For both cases, monotonic or extremum-one functions, the ending point FJ, in case of no cost applied to pulses, is taken as

$$FJ = Z(\cdot) + \frac{W}{2}$$

in the case of allowing overlap of pulses in the neighbourhood of $Z(\cdot)$, and

$$FJ = Z(\cdot) - \frac{D}{2}$$

in the case of essential dead time between pulses of any polarity. This is true for all the sections of P(t), except the last where FJ is taken equal to T.

If a cost is applied, FJ will be dependent again on the value of $(A_2 + \frac{W}{2})$. This is fully described in the flow-chart, Appendix A.2.

Once all the information of a section is known, the appropriate subroutine is applied and the pulses distributed within the interval to give I(u). The optimum control function u(t) is then obtained by summing the optimum solution of each subinterval.

4.3 Remarks

(a) When the constraint of essential dead time around the zeros of P(t) is imposed on the problem, it is observed that the pulse distribution may differ slightly if the search for the solution is started at the final time and worked backwards in time, instead of starting at the initial time. In this case, it is proposed to solve for the control function starting at each end. The distribution which gives better results is then chosen.

For example, for the function shown in Fig. 4.3.1, starting the search for the optimum control u(t) from point a in the direction A, the interval of the first portion is taken equal to $(Z - \frac{D}{2})$. After distributing the pulses on this subinterval, the dead time D is added to the trailing edge M of the last pulse, thus obtaining

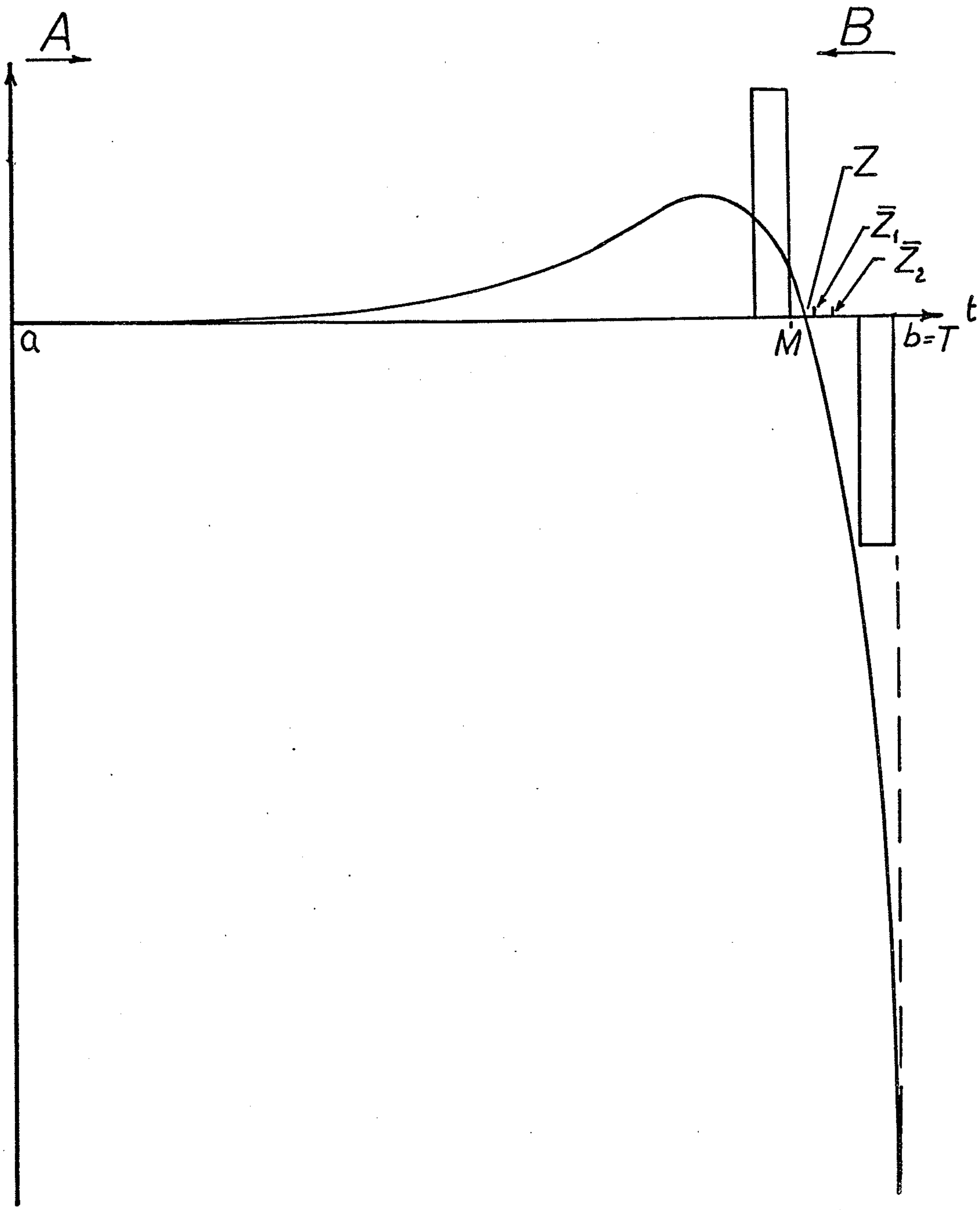

Fig. 4.3.1 The auxiliary function P(t) for the system $\ddot{x} + 2\dot{x} + x = u$ and the performance index $\dot{x} \pm \alpha \int_0^T (u)dt$

the starting point $\bar{Z}_1$ of the second interval. Solving for the maximum number N of pulses allowed for this interval it is found that N is equal to unity and located such that its trailing edge coincides with point b.

Repeating the same procedure but beginning from b in the direction B, the first interval is equal to $(b - Z \; - \frac{D}{2})$ and, as before, one pulse is obtained in the same position. $\bar{Z}_2$ is the starting point of the second interval. For this interval a second negative pulse, D seconds apart from the first, is obtained, and then positive ones.

For the two cases, $I^*(u)$ is calculated and the maximum chosen. This example is illustrated numerically as Example 3 in Sec. 4.4.

For the examples solved in Sec. 4.4, it is found that there is less than 9% difference in the calculation of minimum (maximum) $I^*(u)$ between the two cases.

(b) For high order system, third and up, the function $P(t)$ can have more than one extremum point within a subinterval. An example of a function having three local extremum points is shown in Fig. 4.3.2.

In this case, the interval in question is not taken as $[0, Z_1 + F]$, where

$F = \frac{W}{2}$ if overlapping of pulses of different polarity is allowed, or

$F = -\frac{D}{2}$ if a minimum dead time D is essential between any two consecutive pulses

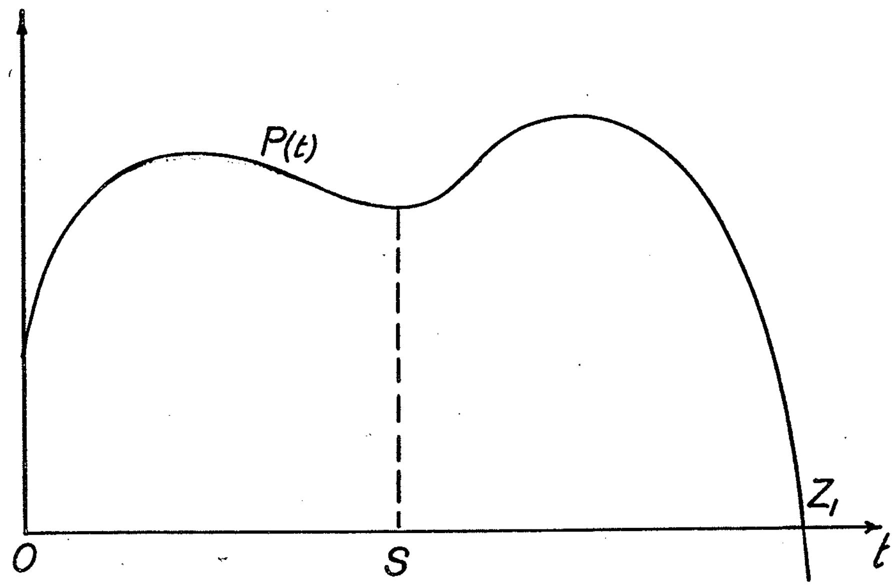

Fig. 4.3.2 Example of a function having three local extremum points.

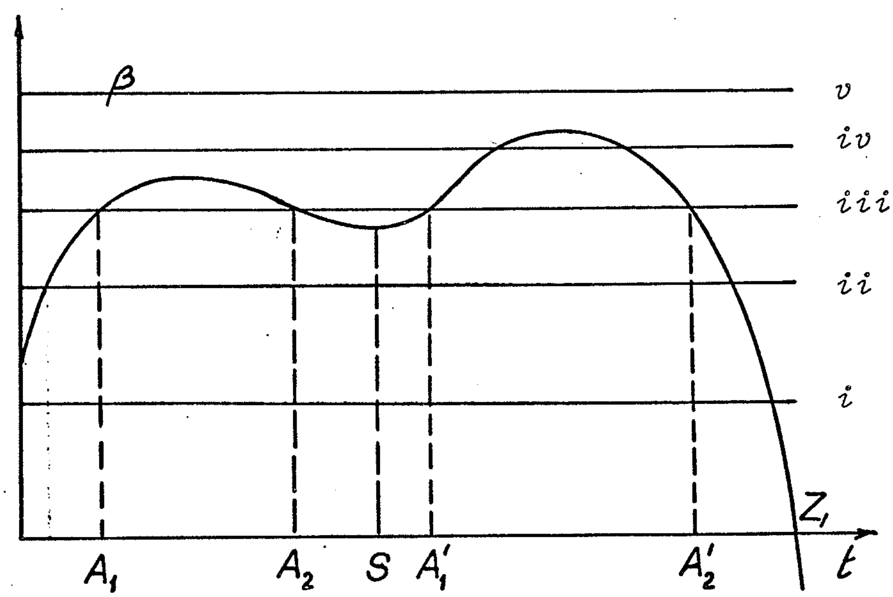

Fig. 4.3.3 Intervals for a function having three local extremum points when cost is applied on pulses.

but is divided into two parts. The first subinterval is $[0, S-\frac{D}{2}]$ and the second is $[\bar{Z}, Z_1+F]$, where S is the time of the even extremum point. The SHIFT subroutine is used for each subinterval separately.

When a cost is applied on pulses, the cases shown in Fig. 4.3.3 may arise.

Case (i): The cost is too high and no pulses are required to obtain the optimum solution.

Cases (ii),(iv): The method described in Sec. 4.2.3 for the solution of an extremum-one function is directly applied.

Case (iii): The same method is applied on two intervals. These intervals are dependent on the position of $(A_1-\frac{W}{2})$, $(A_2+\frac{W}{2})$ and $(A_1'-\frac{W}{2}), (A_2'+\frac{W}{2})$ with respect to the zero, $(S-\frac{D}{2})$ and $\bar{Z}, (Z_1+F)$ times respectively.

(c) For the problem with final value constraints and referring to [8], the differential equations, defining the auxiliary variables, are the same as for the case with no final value constraints. However, the boundary conditions are not the same.

$$P_i(T) = -c_i = -\left[C_i + \sum_{k=1}^{m} \lambda_k \frac{\partial F_k}{\partial x_i(T)}\right]; \; i=1, \ldots n$$

with the constraints

$$F_k[x_1(T), \ldots x_n(T)] = \varepsilon_k , \quad k = 1, \ldots m; \quad m<n$$

where the λ_k are Lagrange multipliers.

The process used to calculate the Lagrange multipliers is also described in [8]. It is an iterative process, using linear extrapolation or interpolation, which is continued until the constraints are satisfied as closely as possible.

Once λ_k are known, the optimum control $u(t)$ can be found using the techniques described in the previous sections.

4.4 Examples

Example 1

Given an open loop system, which is described by the differential equation,

$$\dot{x} + ax = u(t), \quad x(0)=0, \quad 0 \le t \le T$$

it is required to maximize the performance index

$$J = x(T) - \int_0^T \alpha\, u^2\, dt$$

Solving for $I^*(u)$, as described in Chapter 3, it is found that it is required to minimize

$$I^*(u) = \int_0^T [- u\, e^{a(t-T)} + \alpha\, u^2]\, dt$$

Using numerical values for the parameters involved, let

the final time T = 5 sec.

the pulse magnitude X = 0.1 volts

the pulse width W = 0.2 sec.

the dead time D = 0.3 sec

the cost on pulses $\alpha = 1$

then

$$\beta = -\alpha \frac{F(u)}{f(u)} = -\alpha u = -\alpha X$$

$$\beta = -0.1$$

The optimum control u(t) is found to be as shown in Fig. 4.4.1, for a = 1. The polarity and time of application of each of these pulses are given in Table 4.4.1.

Example 2

Given the second order system described by the differential equation

$$\ddot{x} + \dot{x} + x = u, \; \dot{x}(0) = x(0) = 0, \; 0 \le t \le T$$

it is required to maximize

$$J = x(T) - \int_0^T \alpha u^2 \, dt$$

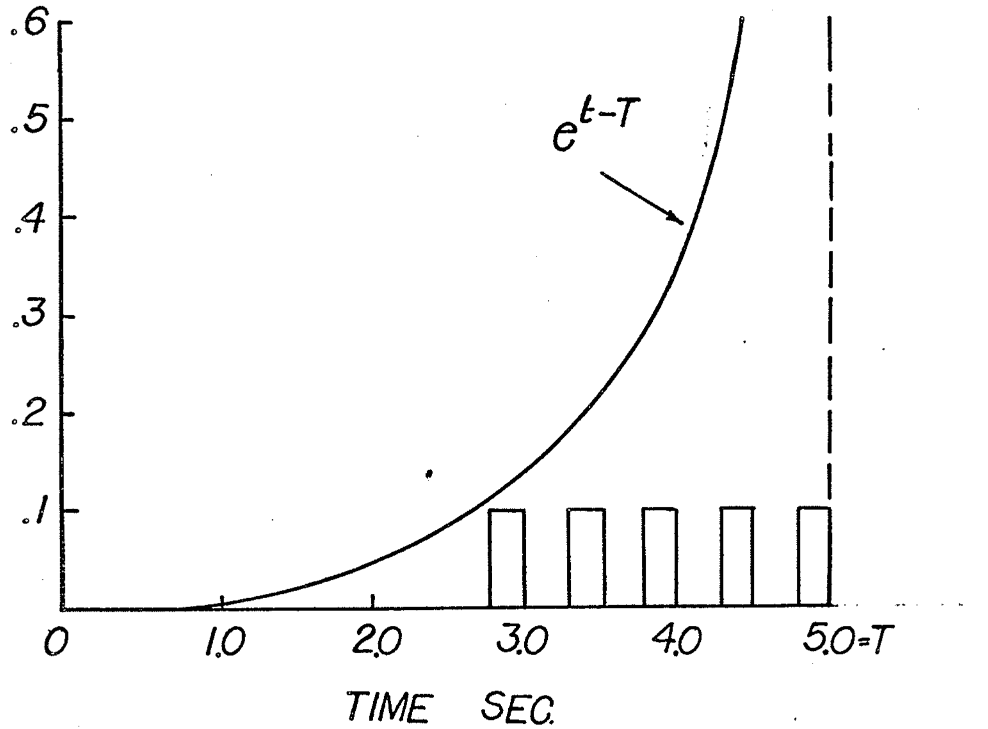

Fig. 4.4.1 Optimum control u(t) for Example 1

Table 4.4.1

Pulses constituting the optimum control u(t); Example 1

Pulse Number	Pulse Polarity	Time of Application
1	+	2.8
2	+	3.3
3	+	3.8
4	+	4.3
5	+	4.8

Table 4.4.2

Pulses constituting the optimum control u(t); Example 2.a

Pulse Number	Pulse Polarity	Time of Application
1	-	4.340
2	-	4.940
3	-	5.540
4	+	6.608
5	+	7.208
6	+	7.808
7	+	8.408
8	+	9.008
9	+	9.608

That is, it is required to minimize

$$I^*(u) = \int_0^T \left(\frac{2}{\sqrt{3}}\, u\, e^{\frac{1}{2}(t-T)} \sin \frac{\sqrt{3}}{2}(t-T) + \alpha u^2\right) dt$$

Using the following numerical values

$T = 10$ sec.

$X = 0.1$ volts

$W = 0.2$ sec.

$D = 0.4$ sec.

different cases will be considered according to the value of the cost and to the dead time around the zeros of $P(t)$. Letting IDT = 0 means overlapping of pulses of different polarity around the zeros of $P(t)$ is allowed, and, IDT $\neq$ 0 means the minimum dead time D must be present between any two consecutive pulses.

(a) Let $\alpha = 0.5$

then $\beta = -0.05$ for $u = x$

$= 0.05$ for $u = -x$

Applying the method of solution developed in the present study, it is found that an identical solution is obtained for the two cases IDT = 0 and IDT $\neq$ 0.

The optimal control $u(t)$ is as shown in Fig. 4.4.2 and the polarity and time of application of pulses are given in Table 4.4.2.

(b) For this case, let β and IDT equal to zero. Polarity and time of application of pulses are given in Table 4.4.3.

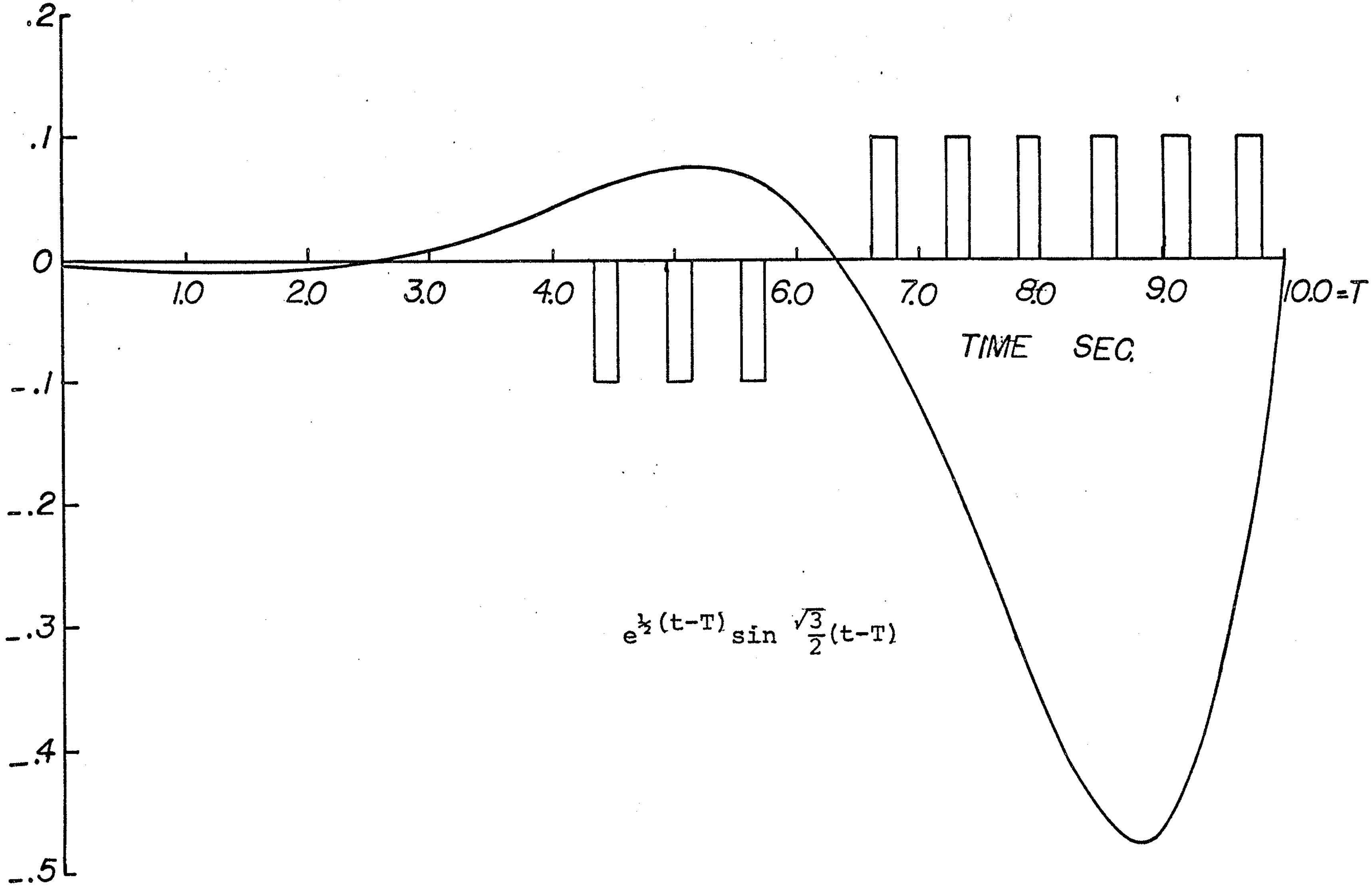

Fig. 4.4.2 Optimum control u(t) for Example 2.a.

Table 4.4.3

Pulses constituting the optimum control u(t); Example 2.b

Pulse Number	Pulse Polarity	Time of Application
1	+	0.000
2	+	0.600
3	+	1.200
4	+	1.800
5	+	2.400
6	-	2.985
7	-	3.585
8	-	4.185
9	-	4.785
10	-	5.385
11	-	5.985
12	+	6.613
13	+	7.213
14	+	7.813
15	+	8.413
16	+	9.013
17	+	9.613

From this table, it can be remarked that the dead time between the pulses 5 and 6 is less than the allowable minimum dead time.

(c) The polarity and time of application of the pulses constituting the optimal control u(t) for the case of β equal to zero and, IDT other than zero, are given in Table 4.4.4. It is remarked that the dead time between any two consecutive pulses is equal to or greater than the minimum required dead time. Repeating this case, but starting from the final time and increasing backward, a little change in pulse position is realized. This is given in Table 4.4.4.

Table 4.4.5 gives the values of $I^*(u)$ corresponding to the different cases solved above. It is clear that the minimum of $I^*(u)$ occurs when IDT = 0. For the two cases solved in (c) it can be seen that the difference in the value of $I^*(u)$ is negligible.

Example 3

Given an open loop system, which is described by the differential equation

$$\ddot{x} + 2\dot{x} + x = u, \ \dot{x}(0) = x(0) = 0, \ 0 \leq t \leq T$$

it is required to minimize

$$J = \dot{x}(T) + \alpha \int_0^T |u| \, dt$$

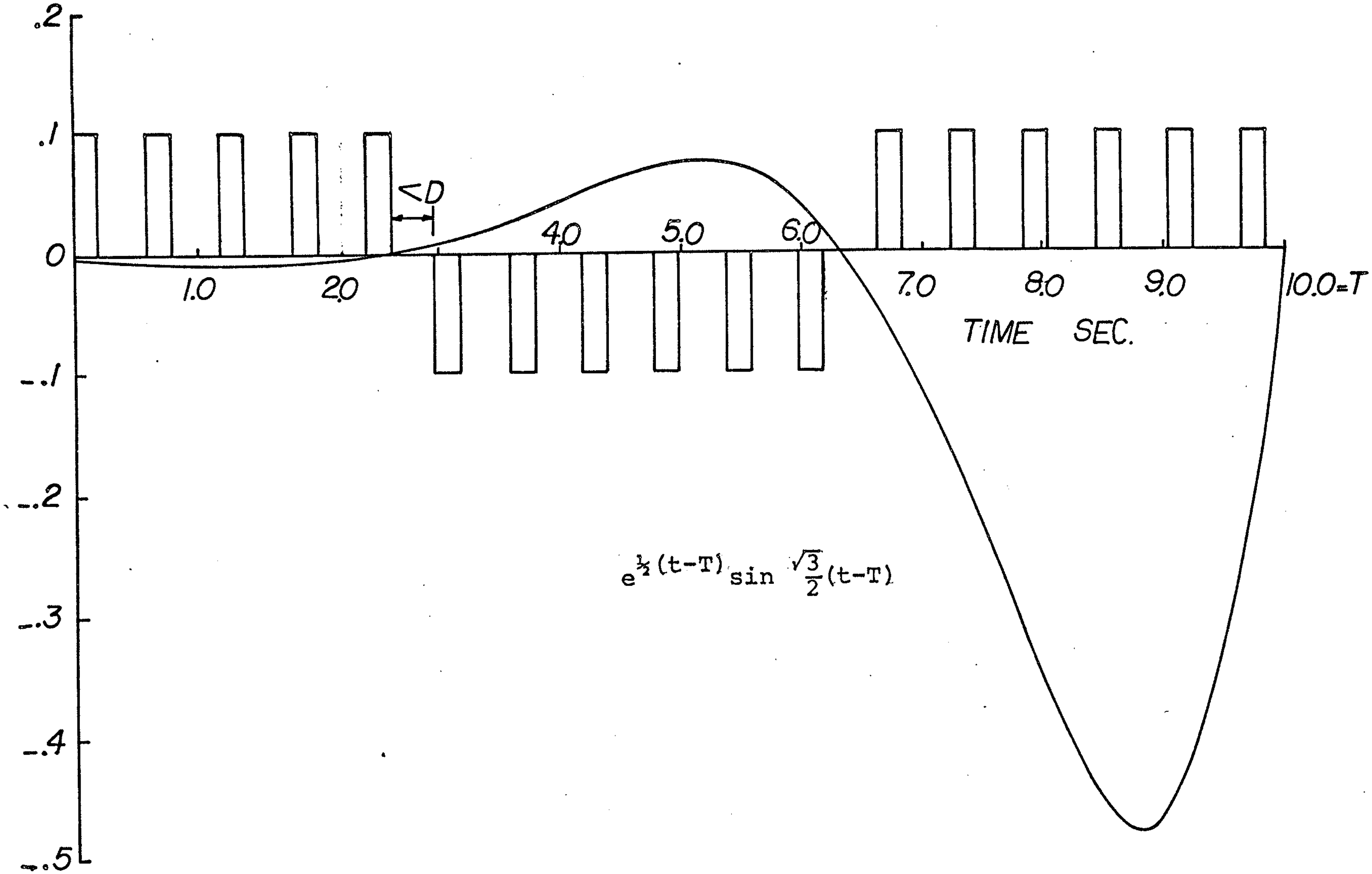

Fig. 4.4.3 Optimum control u(t) for Example 2.b.

Table 4.4.4

Pulses constituting the optimum control u(t); Example 2.c

Pulse Number	Pulse Polarity	Time of Application	
		Starting at t=0	Starting at t=T
1	+	0.30	0.30
2	+	0.90	0.90
3	+	1.50	1.50
4	+	2.10	2.10
5	−	2.97	2.98
6	−	3.57	3.58
7	−	4.17	4.18
8	−	4.77	4.78
9	−	5.37	5.38
10	−	5.97	5.98
11	+	6.61	6.61
12	+	7.21	7.21
13	+	7.81	7.81
14	+	8.41	8.41
15	+	9.01	9.01
16	+	9.61	9.61

Table 4.4.5

Value of I*(u) for the different cases treated in Example 2

β	0.05	0.0	0.0	0.0
IDT	−	0	$\neq 0$	$\neq 0$
Starting Point	0	0	0	T
I*(u)	−0.0430922	−0.0466664	−0.0465898	−0.0465902

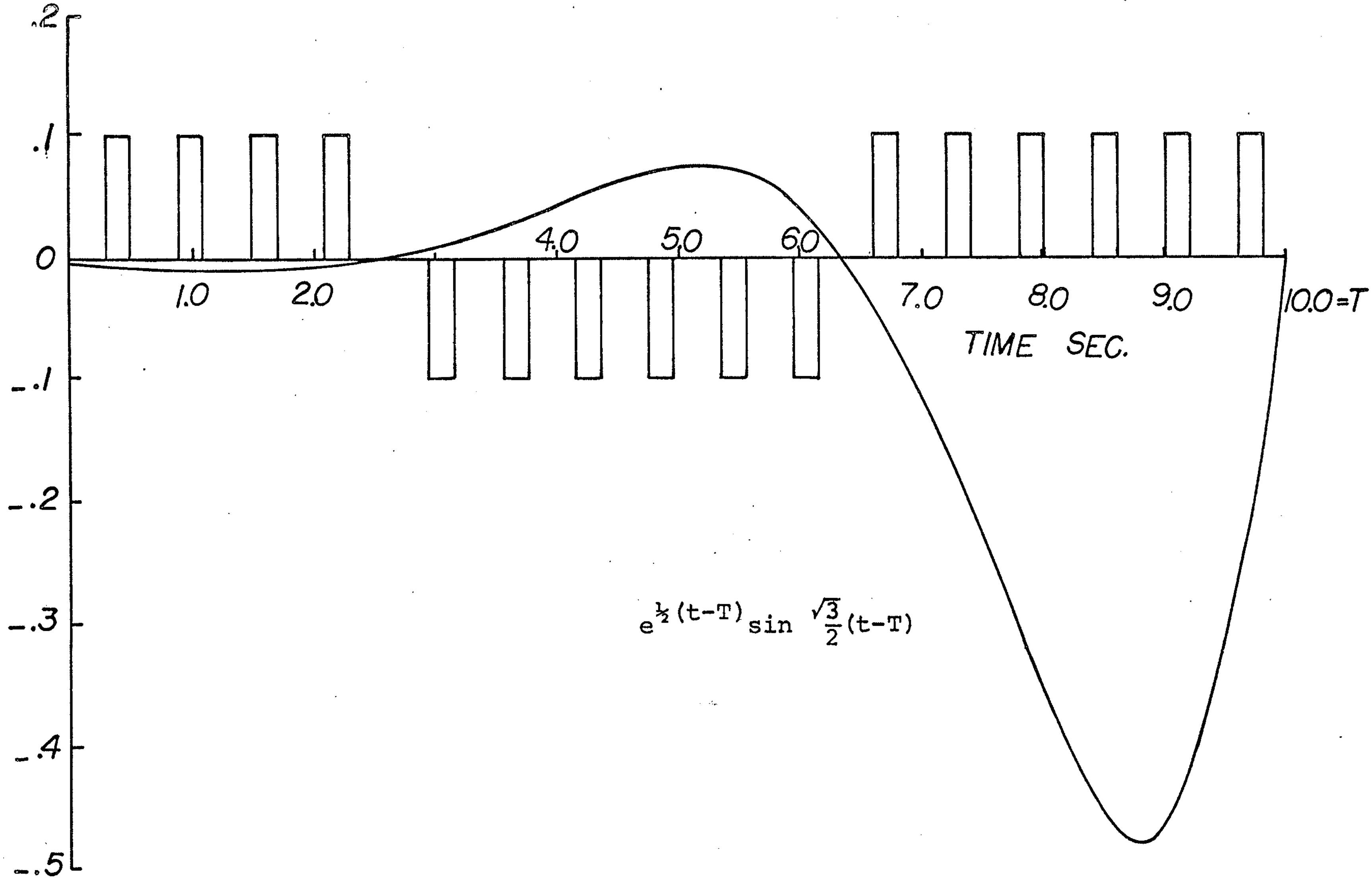

Fig. 4.4.4 Optimum control u(t) for Example 2.c.
(starting at t=T)

That is, it is required to maximize

$$I^*(u) = \int_0^T [u(T-t-1)\ e^{(t-T)} - \alpha|u|]\ dt$$

As in Example 2, different cases will be considered using the following numerical values

T = 10 sec.

X = 0.2 volts

W = 0.3 sec.

D = 0.3 sec.

(a) For β equal to ± 0.1, an identical solution is obtained for any value of IDT. The optimum control distribution is shown in Fig. 4.4.5 and the polarity and time of application of these pulses is given in Table 4.4.6.

(b) Let β and IDT be equal to zero, the optimum control function polarity and time of application are obtained and given in Table 4.4.7.

It is observed that the dead time between the pulses 15 and 16 is less than the minimum dead time D.

(c) For the case where $\beta = 0$ and IDT $\neq 0$, the result given in Table 4.4.8 is obtained.

The different values of I*(u) calculated in this example are given in Table 4.4.9. For the two cases solved in part (c) it can be seen that the difference in the value of I*(u) is of about 8.1%, and the second distribution is chosen.

Table 4.4.6

Pulses constituting the optimum control u(t); Example 3.a

Pulse Number	Pulse Polarity	Time of Application
1	+	7.019
2	+	7.619
3	+	8.219
4	-	9.100
5	-	9.700

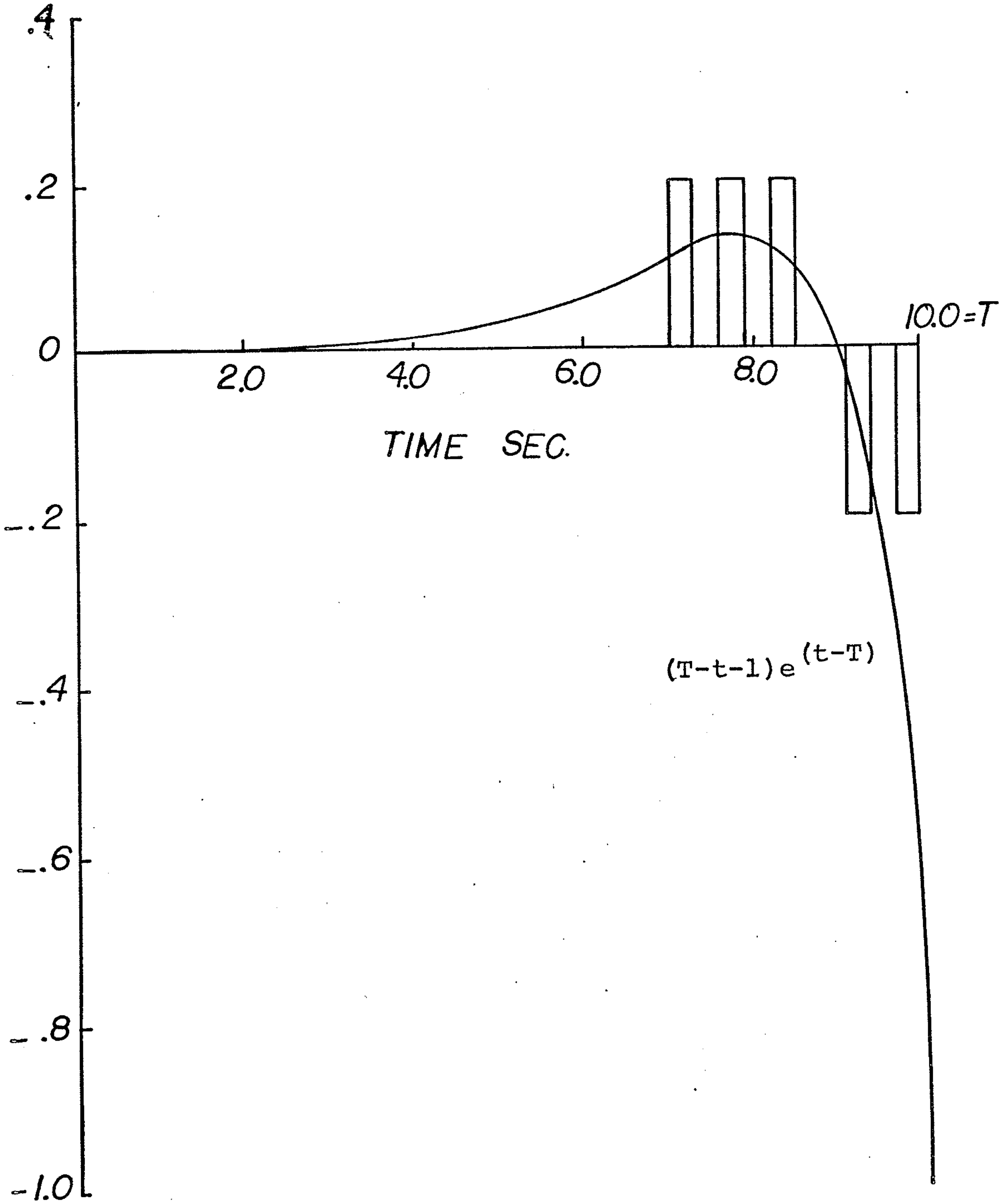

Fig. 4.4.5 Optimum control u(t) for Example 3.a.

Table 4.4.7

Pulses constituting the optimum control u(t); Example 3.b

Pulse Number	Pulse Polarity	Time of Application
1	+	0.17
2	+	0.77
3	+	1.37
4	+	1.97
5	+	2.57
6	+	3.17
7	+	3.77
8	+	4.37
9	+	4.97
10	+	5.57
11	+	6.17
12	+	6.77
13	+	7.37
14	+	7.97
15	+	8.57
16	-	9.10
17	-	9.70

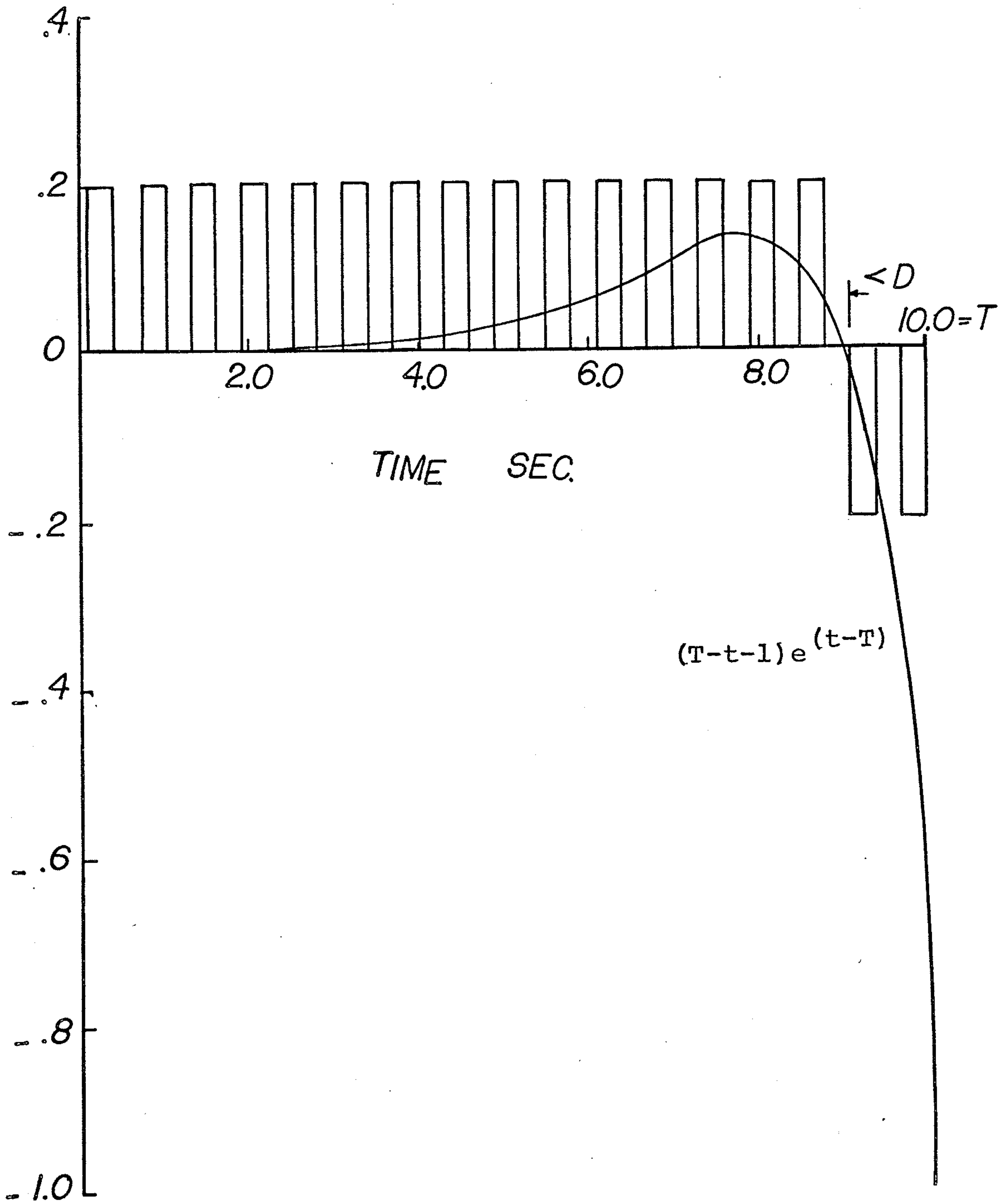

Fig. 4.4.6 Optimum control u(t) for Example 3.b.

Table 4.4.8

Pulses constituting the optimum control u(t); Example 3.c

Pulse Number		Pulse Polarity		Time of Application	
Starting at t=0	Starting at t=T	Starting at t=0	Starting at t=T	Starting at t=0	Starting at t=T
1	1	+	+	0.15	0.1
2	2	+	+	0.75	0.7
3	3	+	+	1.35	1.3
4	4	+	+	1.95	1.9
5	5	+	+	2.55	2.5
6	6	+	+	3.15	3.1
7	7	+	+	3.75	3.7
8	8	+	+	4.35	4.3
9	9	+	+	4.95	4.9
10	10	+	+	5.55	5.5
11	11	+	+	6.15	6.1
12	12	+	+	6.75	6.7
13	13	+	+	7.35	7.3
14	14	+	+	7.95	7.9
15	15	+	+	8.55	8.5
16	16	-	-	9.70	9.1
	17		-		9.7

Table 4.4.9

Value of I*(u) for the different cases treated in Example 3

β	0.1	0	0	0
IDT	-	0	$\neq 0$	$\neq 0$
Starting Point	0	0	0	T
I*(u)	0.0435136	0.089270	0.0815939	0.0888377

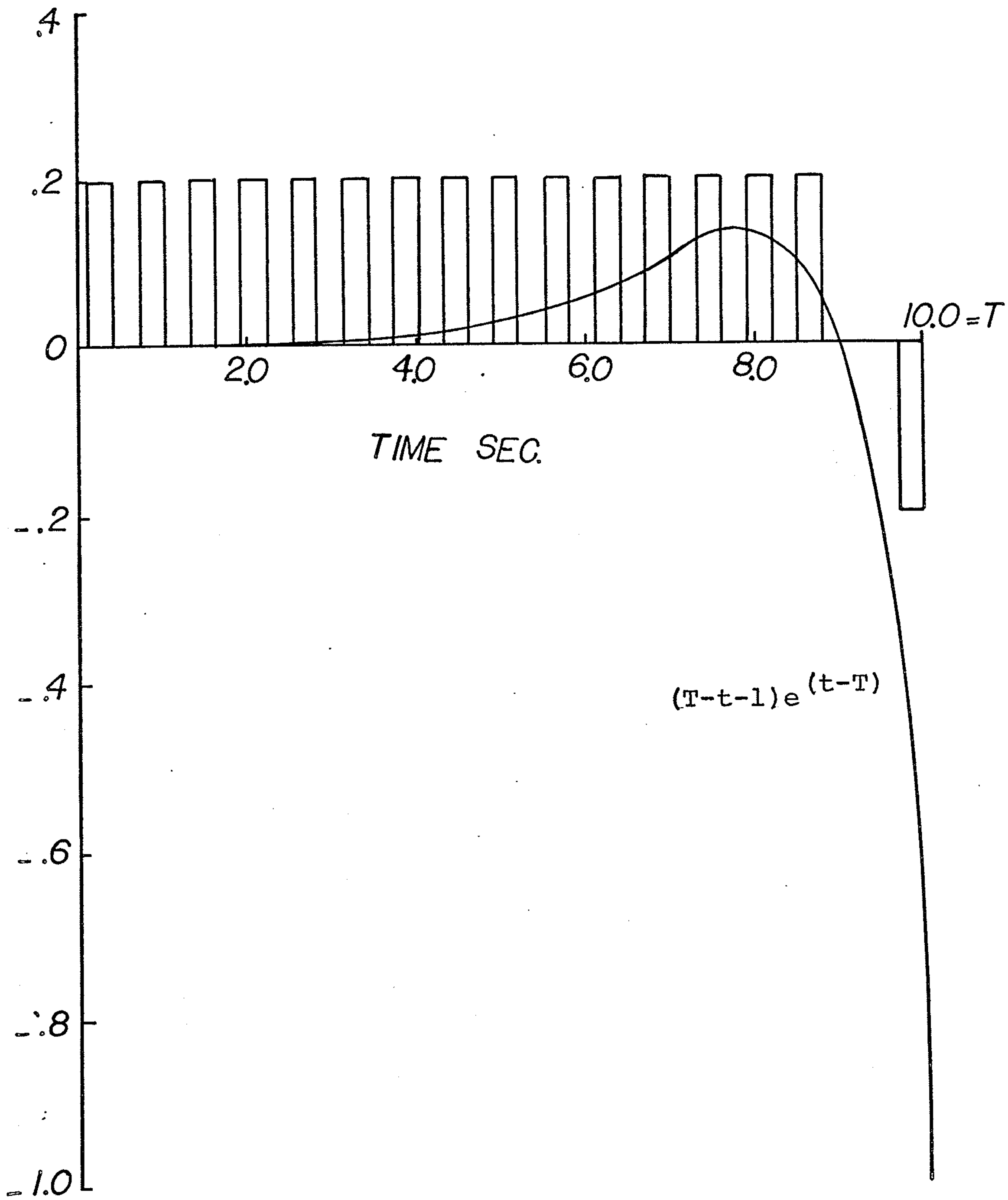

Fig. 4.4.7 Optimum control u(t) for Example 3.c. (starting at t=0)

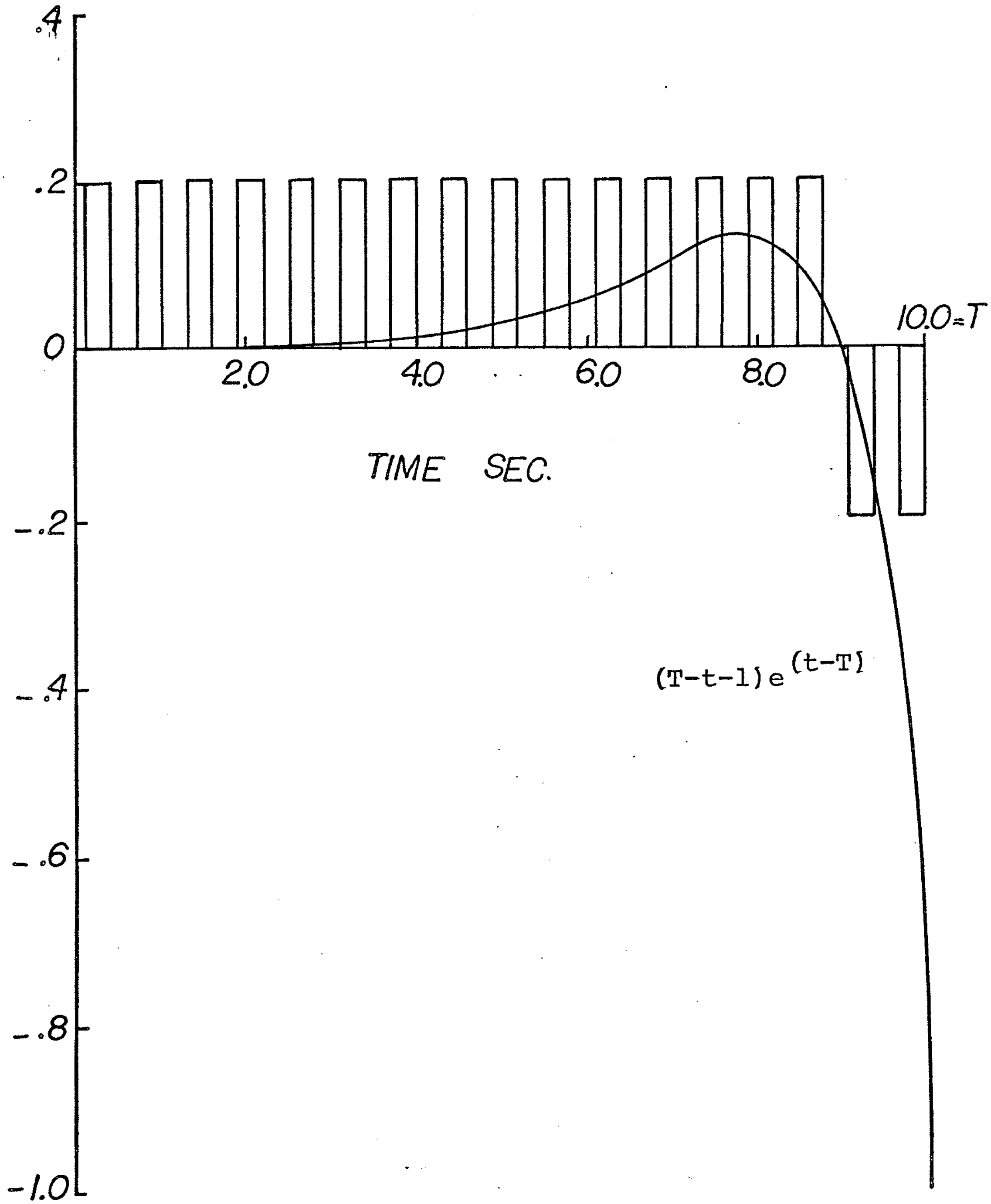

Fig. 4.4.8 Optimum control u(t) for Example 3.c (starting at t=T)

Example 4

Given the third order system

$$\dddot{x} + 2\ddot{x} + 2\dot{x} - x = u, \quad \ddot{x}(0) = \dot{x}(0) = x(0) = 0, \quad 0 \le t \le T$$

it is required to maximize

$$J = x(T) - \int_0^T \alpha |u| \, dt$$

Solving as described in Chapter 3, it is found that it is required to minimize

$$I^*(u) = \int_0^T \{u[- 0.36564e^{-0.35t} - 0.007786e^{1.175t} \sin 1.2t + 0.0014009e^{1.175t} \cos 1.2t] + \alpha |u|\} \, dt$$

Corresponding to the parameter values

T = 8.0 sec

X = 5 volts

W = 0.2 sec

D = 0.3 sec

the optimum control u(t) is found for the following cases:

(a) When β equal to ± 0.5, little difference in the solutions is obtained for IDT = 0 and IDT $\neq$ 0. The polarity and time application of the pulses is given in Table 4.4.10.

(b) Let β and IDT be equal to zero, the optimum control function polarity and time of application are obtained and given in Table 4.4.11. It is observed that pulses 7 and 8

Table 4.4.10

Pulses constituting the optimum control u(t); Example 4.a

Pulse Number	Pulse Polarity	Time of Application	
		IDT=0	IDT≠0
1	-	3.930	3.930
2	-	4.430	4.430
3	-	4.930	4.930
4	+	5.669	5.670
5	+	6.169	6.170
6	+	6.669	6.670
7	+	7.169	7.170
8	+	7.669	7.670

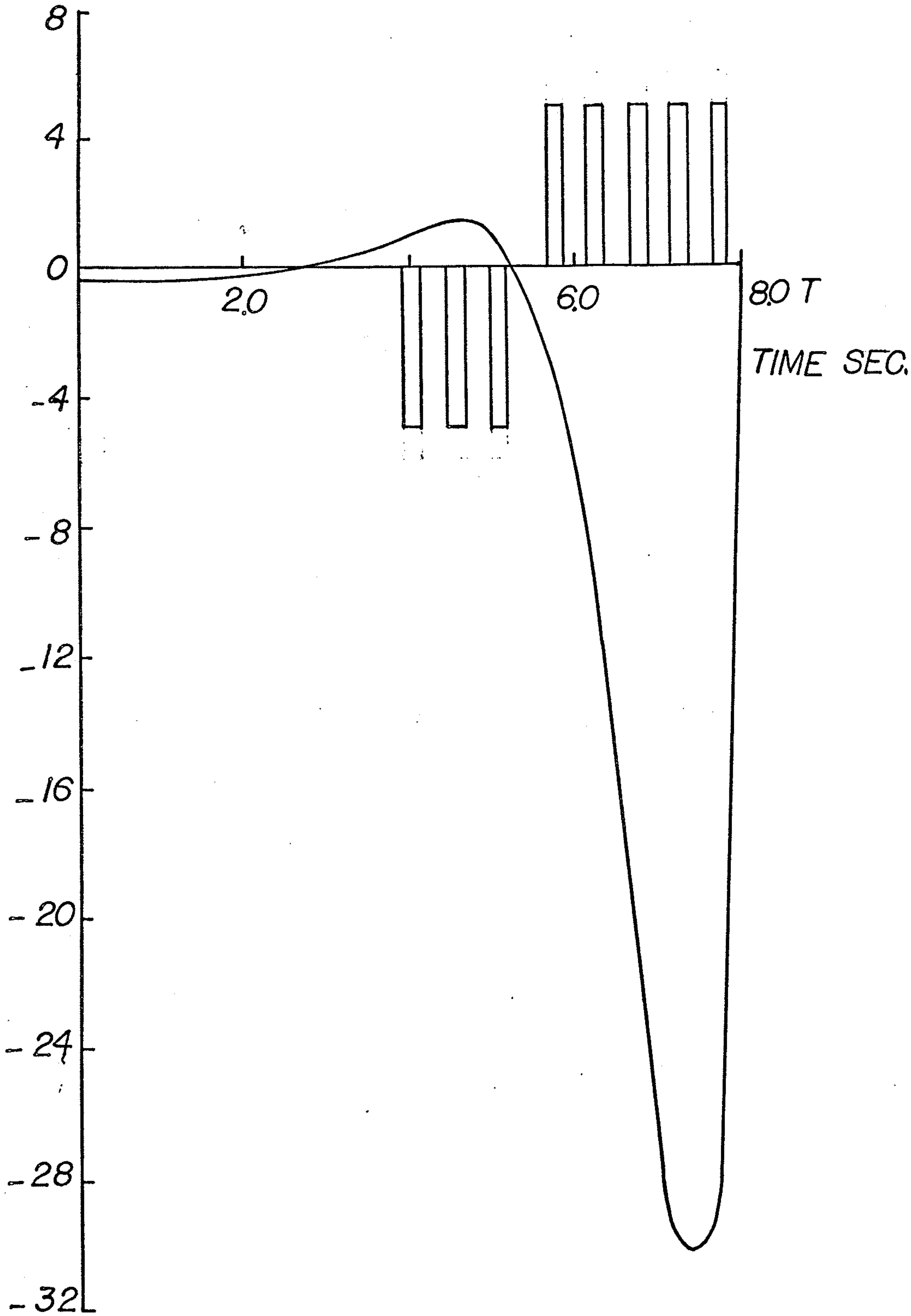

Fig. 4.4.9 Optimum control u(t) for Example 4.a.

Table 4.4.11

Pulses constituting the optimum control u(t); Example 4.b

Pulse Number	Pulse Polarity	Time of Application
1	+	0.0000
2	+	0.5000
3	+	1.0000
4	+	1.5000
5	+	2.0000
6	+	2.5000
7	+	3.0000
8	-	3.0600
9	-	3.5600
10	-	4.0600
11	-	4.5600
12	-	5.0600
13	+	5.6637
14	+	6.1637
15	+	6.6637
16	+	7.1637
17	+	7.6637

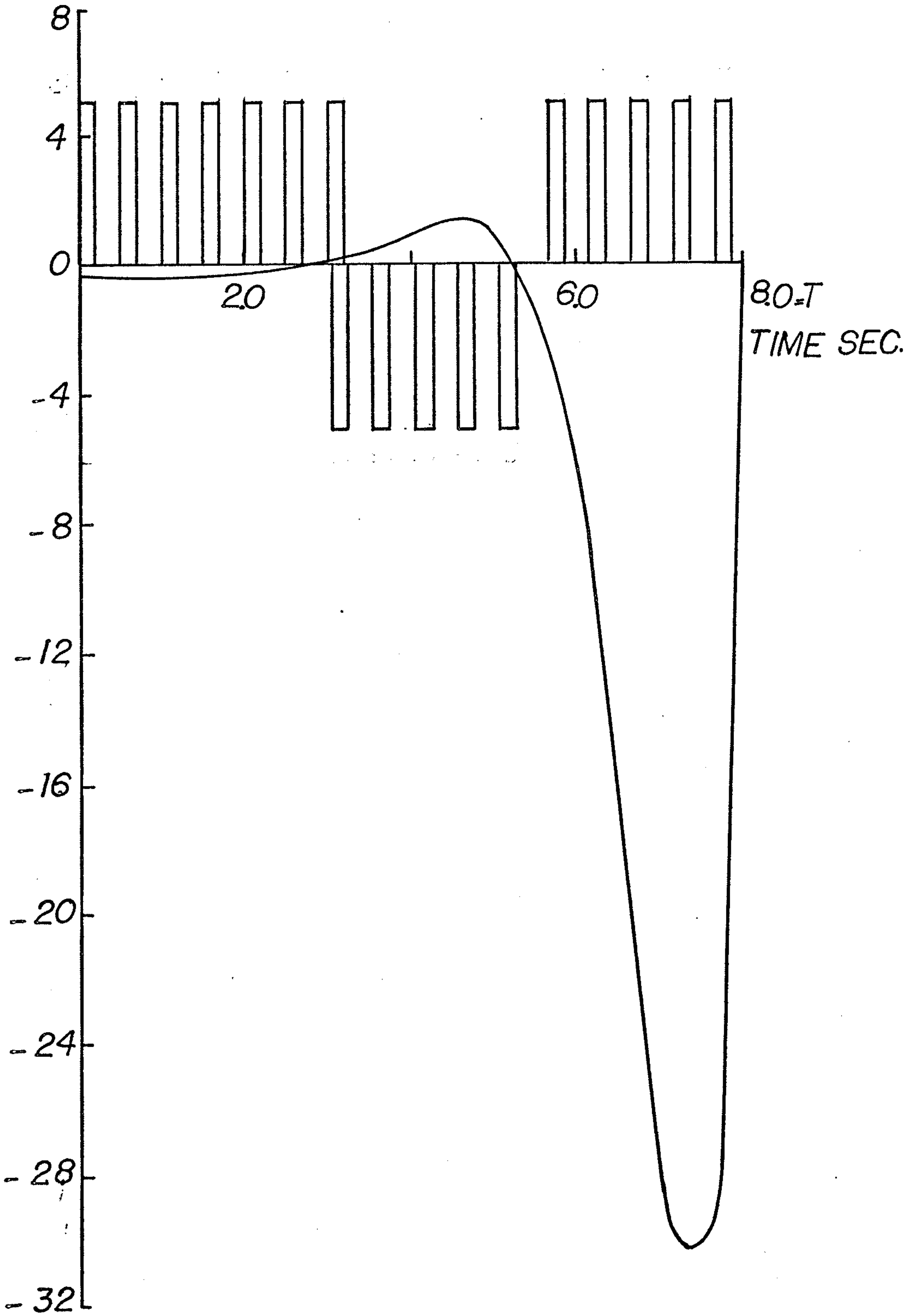

Fig. 4.4.10 Optimum control u(t) for Example 4.b.

Table 4.4.12

Pulses constituting the optimum control u(t); Example 4.c

Pulse Number	Pulse Polarity		Time of Application	
	Starting at t=0	Starting at t=T	Starting at t=0	Starting at t=T
1	+	+	0.00	0.00
2	+	+	0.50	0.50
3	+	+	1.00	1.00
4	+	+	1.50	1.50
5	+	+	2.00	2.00
6	+	+	2.50	2.50
7	-	+	3.02	3.00
8	-	-	3.52	3.51
9	-	-	4.02	4.01
10	-	-	4.52	4.51
11	-	-	5.02	5.01
12	+	+	5.67	5.67
13	+	+	6.17	6.17
14	+	+	6.67	6.67
15	+	+	7.17	7.17
16	+	+	7.67	7.67

Table 4.4.13

Value of I*(u) for the different cases treated in Example 4

β	0.5	0.5	0.0	0.0	0.0
IDT	0	0	0	0	0
Starting Point	0	0	0	0	T
I*(u)	-86.81897	-86.81831	-92.78376	-92.76884	-92.76761

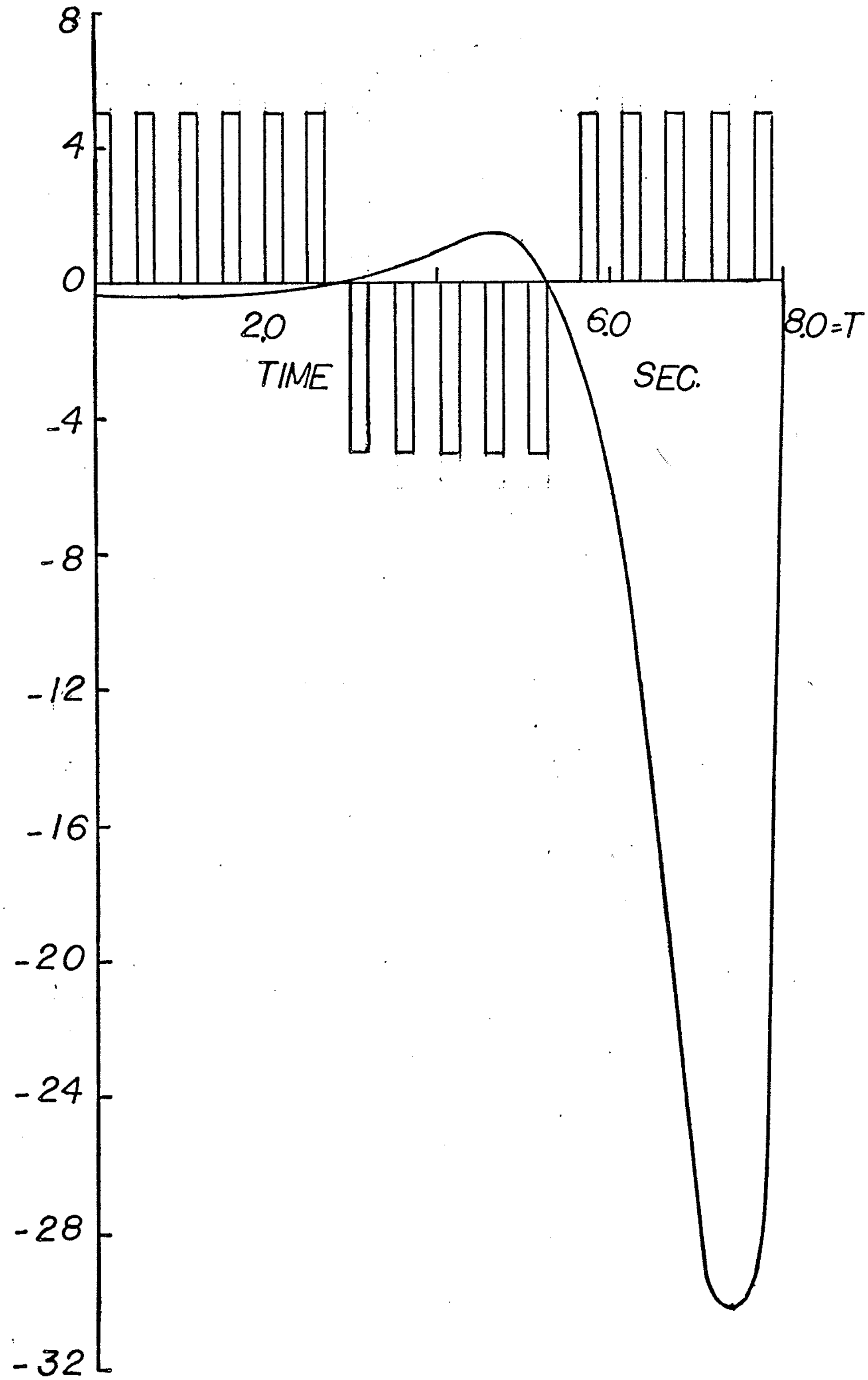

Fig. 4.4.11 Optimum control u(t) for Example 4.c.
(starting at t=0)

are overlapping for minimum I*(u).

(c) For the case where β is equal to zero but IDT not, the result given in Table 4.4.12 is obtained.

Table 4.4.13 is presenting the different calues of I*(u) computed for Example 4. For the two cases solved in part (c), it can be seen that the difference in the values of I*(u) is of about 0.0013%. The first distribution is chosen.

CHAPTER 5

CONCLUSION

In this present study, a method of extremizing the function I*(u) with respect to the control function u(t) has been developed. First, a general form of I*(u) has been formulated. It has been proven that I*(u) contains, as explicit function of time, the n^{th} auxiliary variable $p_n(t)$ only, where n is the order of the system. Different open loop systems, up to and including the third order, with linear system equations and with fixed operating time, have then been investigated for the form of I*(u), using different performance indices. Some useful common properties of the shape of P(t) and the integrand of I*(u) have then been deduced. Finally, in the previous chapter, a method of extremizing I*(u) has been illustrated. It is found that the pulses, constituting the optimal control u(t), must be located as close as possible to each other in the vicinity of the extremum points of P(t), that is, the minimum dead time is present between any two consecutive pulses.

The method developed in Chapter 4 is applicable when a cost is applied on the pulses as well as when the cost is equal to zero. Also, it is applicable if the dead time between two consecutive pulses of different polarity is allowed to be equal to zero, or when overlapping between these two pulses is permitted.

A comparison of the values of $I^*(u)$ and the variation in the distribution of pulses, for the cases above, are illustrated by some examples in Section 4.4. The complete flow diagrams for the solution are given in Appendix A, and the detailed Fortran program is presented in Appendix B.

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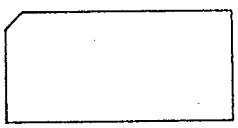
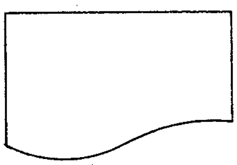
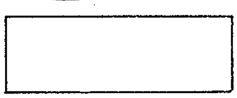
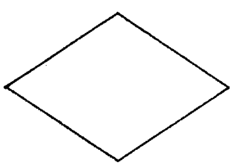
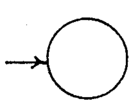
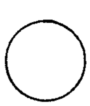
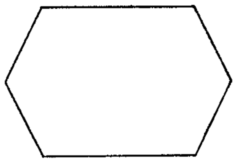
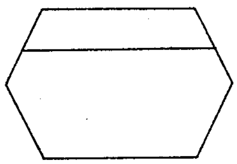
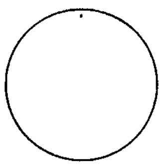
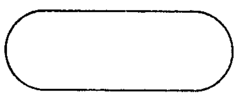
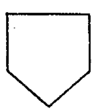
APPENDIX A

The optimization method presented in this study required lengthy calculations; thus, the use of the digital computer was essential. This appendix presents the detailed flow diagrams for this method.

A.1 Flow Chart Conventions Used

Symbol Shape	Meaning	Information Inside Symbol
	Input statement	List of items inputted
	Output statement	List of items outputted, or message contained in a Hollerith field of FORMAT
	Assignment statement	One or more statements of the form $V \leftarrow \xi$
	Conditional or IF statement	Condition which is true or false, or Algebraic value which is less than, equal to, or greater than zero.
	Unconditional transfer or GØTØ statement	Numerical statement label
	Statement label used as a junction point	Numerical value, normally appears ahead of the statement which it represents. The dummy statement CØNTINUE is represented only by its label.

Symbol Shape	Meaning	Information Inside Symbol
	Iteration of DØ statement	Parameters of the repetition
	Calling a subroutine subprogram	Name of function and the calling arguments in parentheses (if any)
	Identification for the point of entry to a subprogram	Name of function followed by dummy arguments in parentheses (if any)
	Exist from a subprogram	The word RETURN
	Special OFFPAGE CONNECTOR for entry to or exit from a page	Page number

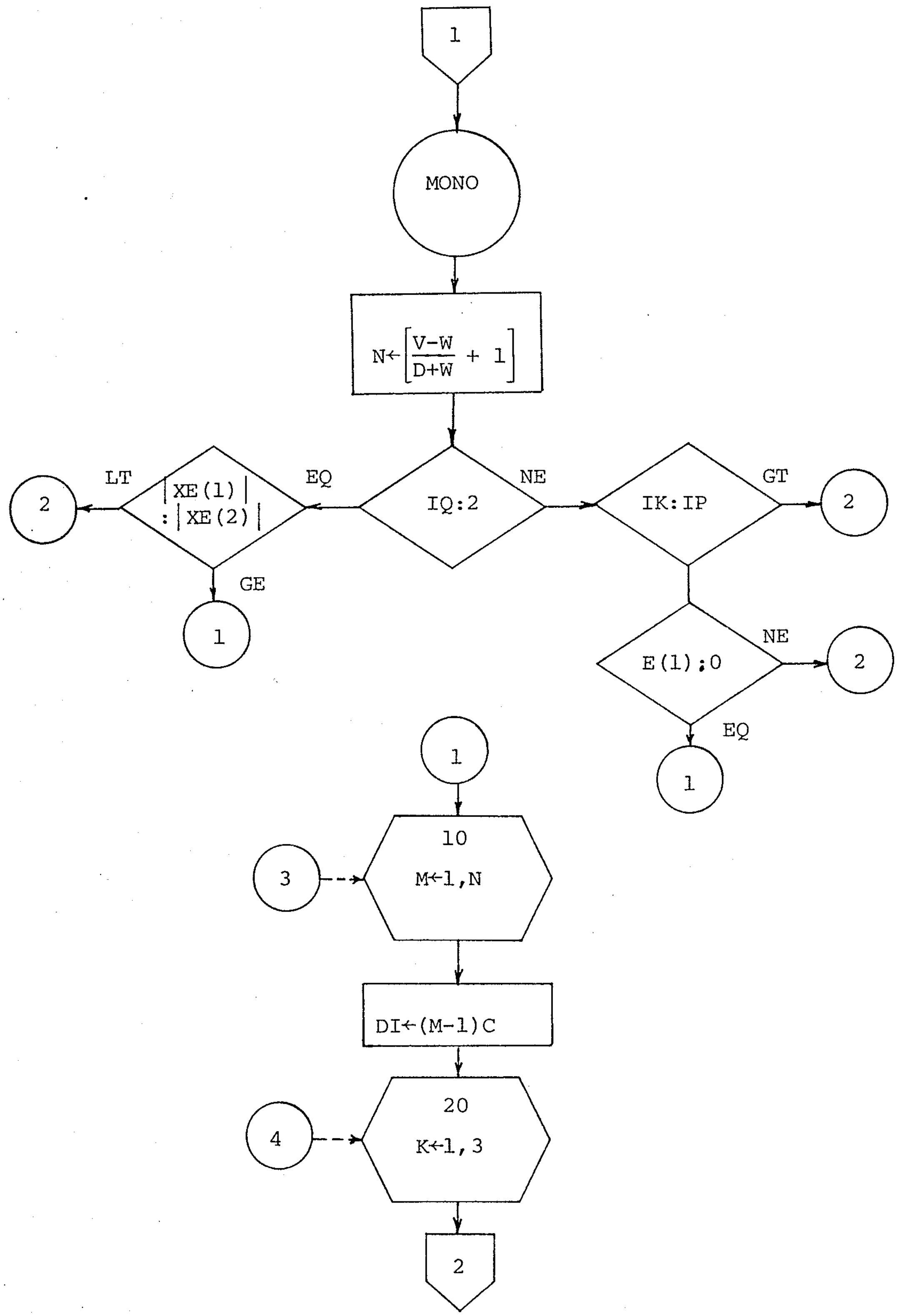
A.2 Flow Diagrams

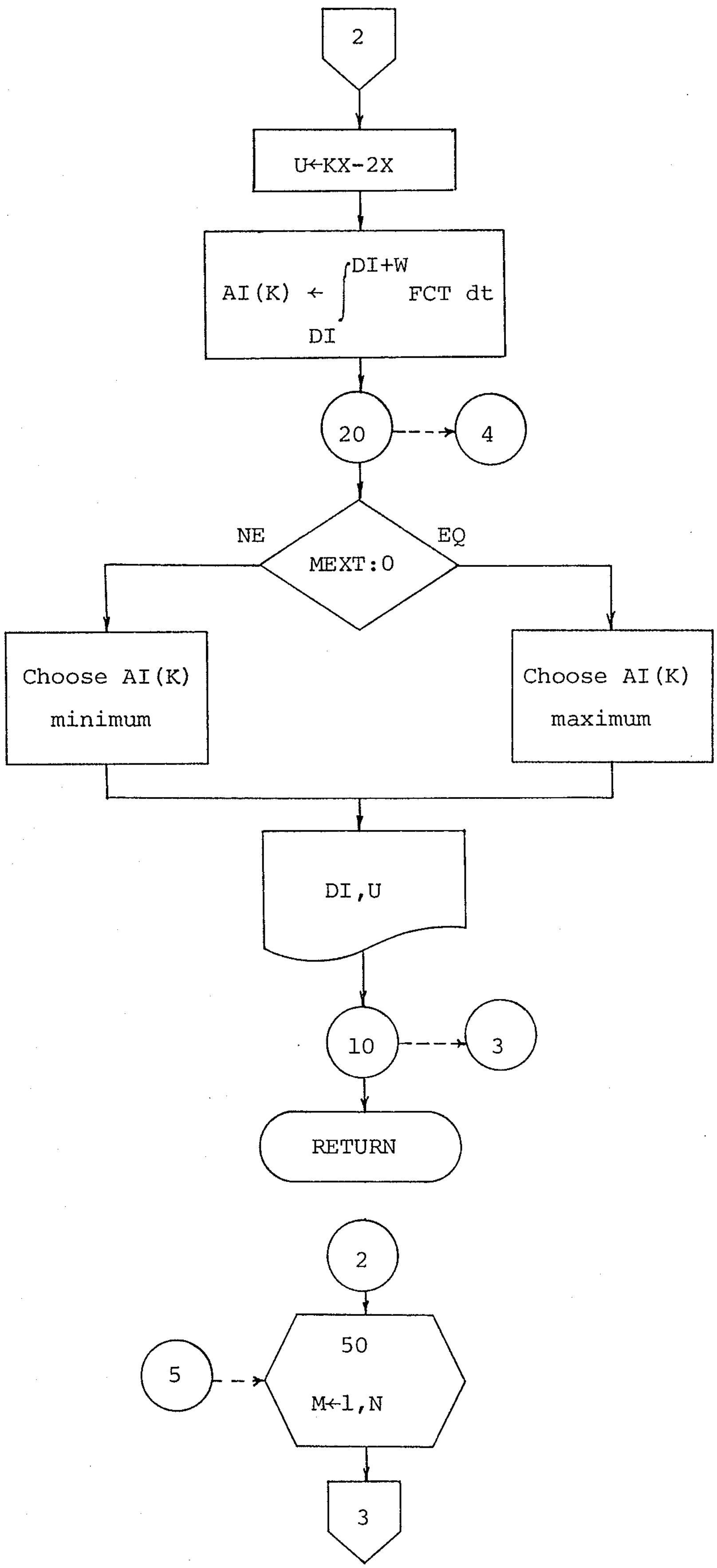
The various symbols used in the flow diagrams, other than those included in the list of symbols, have meanings described as follows:

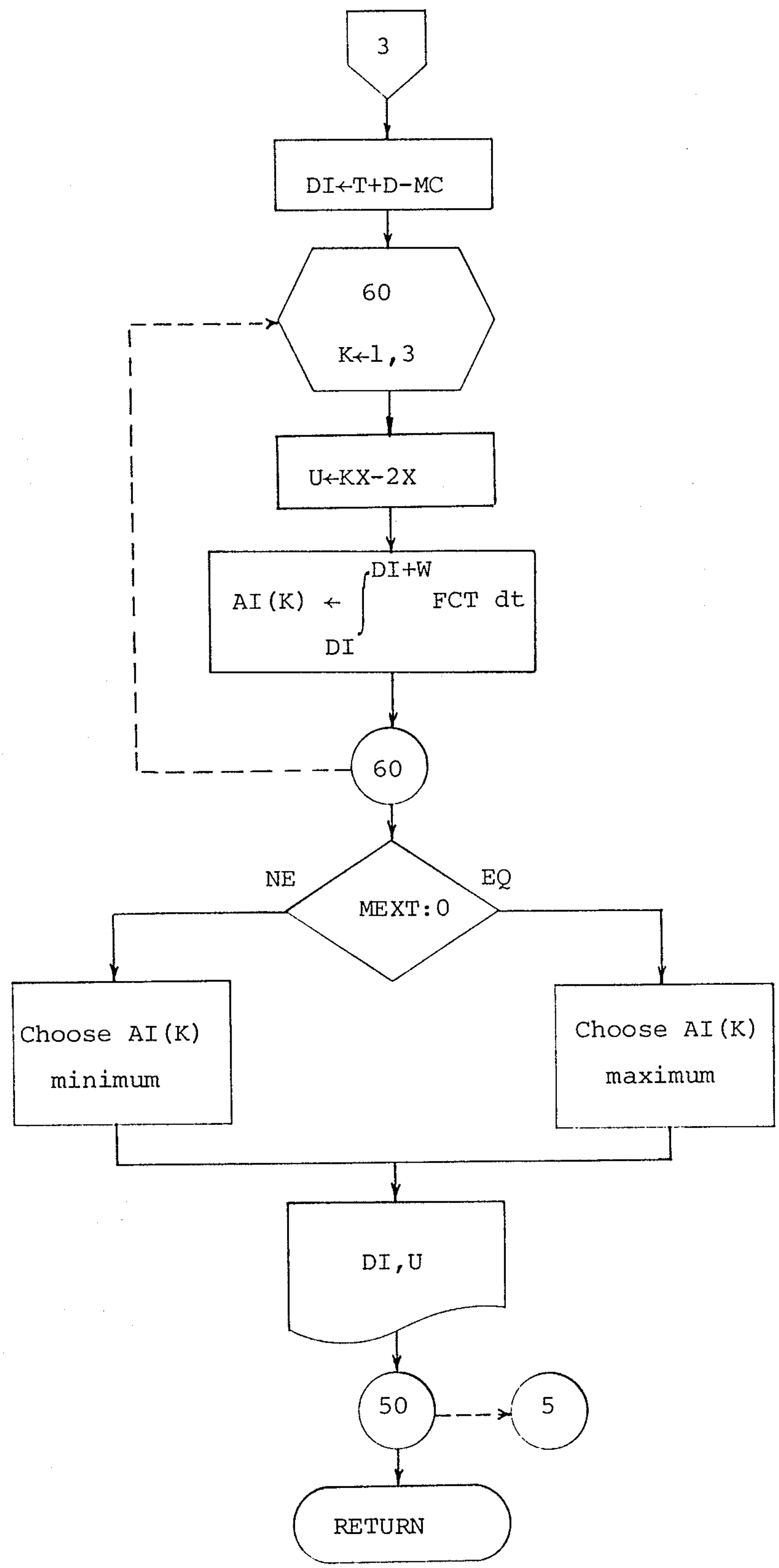
A	Time at which an intersection between β and $P(t)$ occurs.
DI	Place of the leading edge of a pulse.
E	Time at which an extremum occurs.
FCT	Integrand of $I^*(u)$
IA,IB	Arguments of the subroutine INTERS representing the starting and ending points of an interval respectively.
IDT	A location indicating, when containing a zero, that consecutive pulses of different polarity can overlap. The minimum dead time must be present otherwise.
IK	Index
IP	Number of zeros of $P(t)$
IQ	Number of extremum points of $P(t)$
K	Number of points of intersection of β and $P(t)$ within an interval
KC	Number of times a shift δ' is applied
MEXT	A location indicating maximization when containing a zero, minimization otherwise.
PNT	The function $P(t)$

POSIT	Pulse distribution within an interval
XE	Magnitude of an extremum

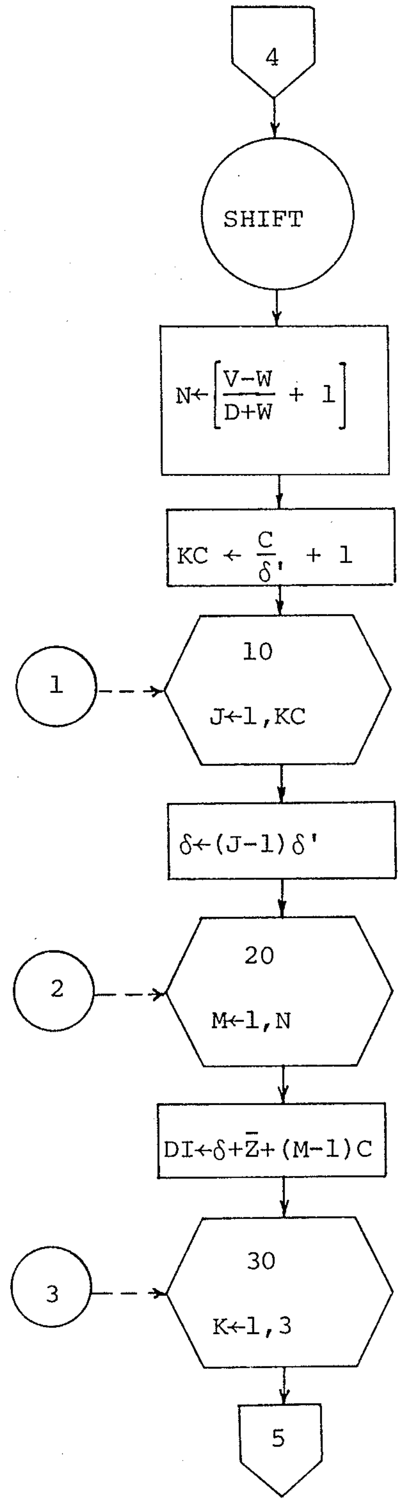
MONO Subroutine Flow Diagram

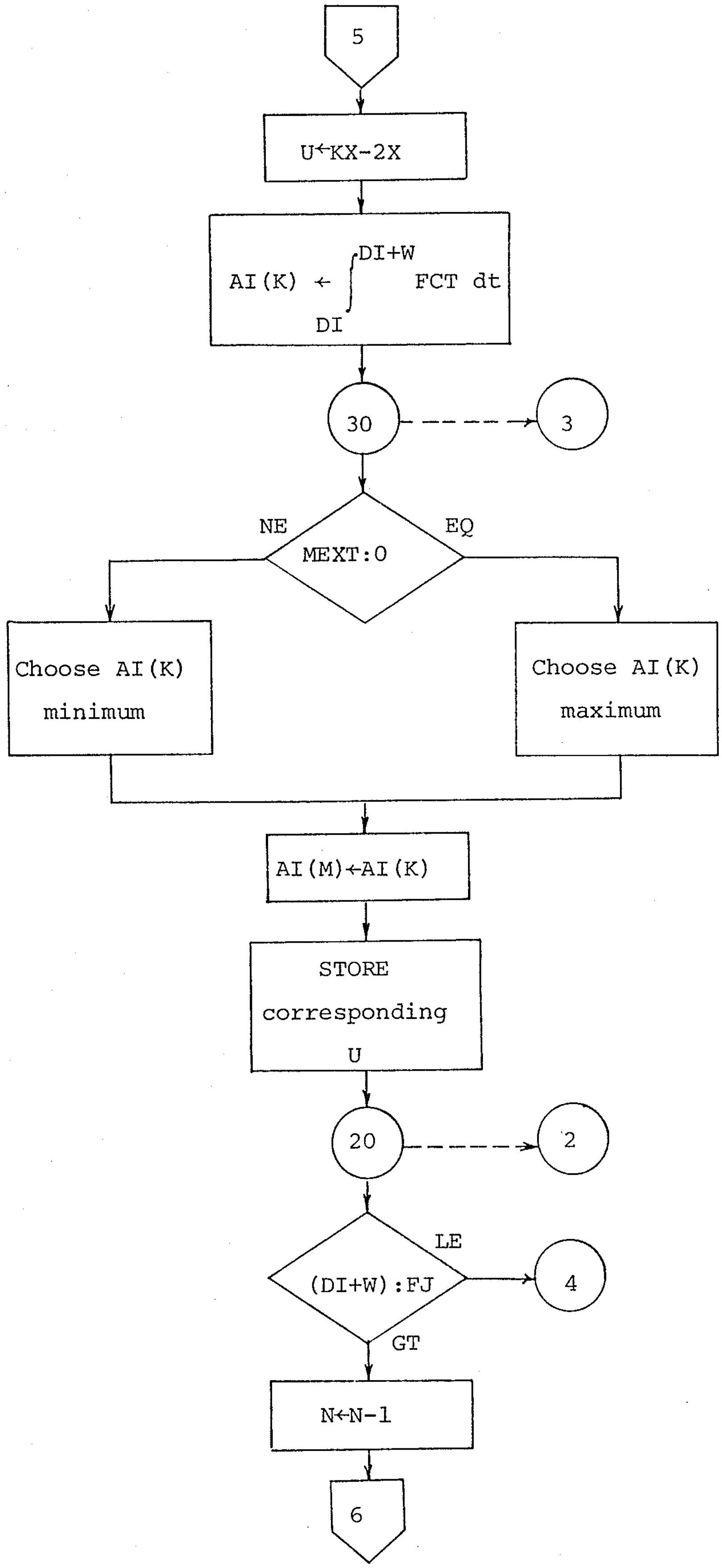


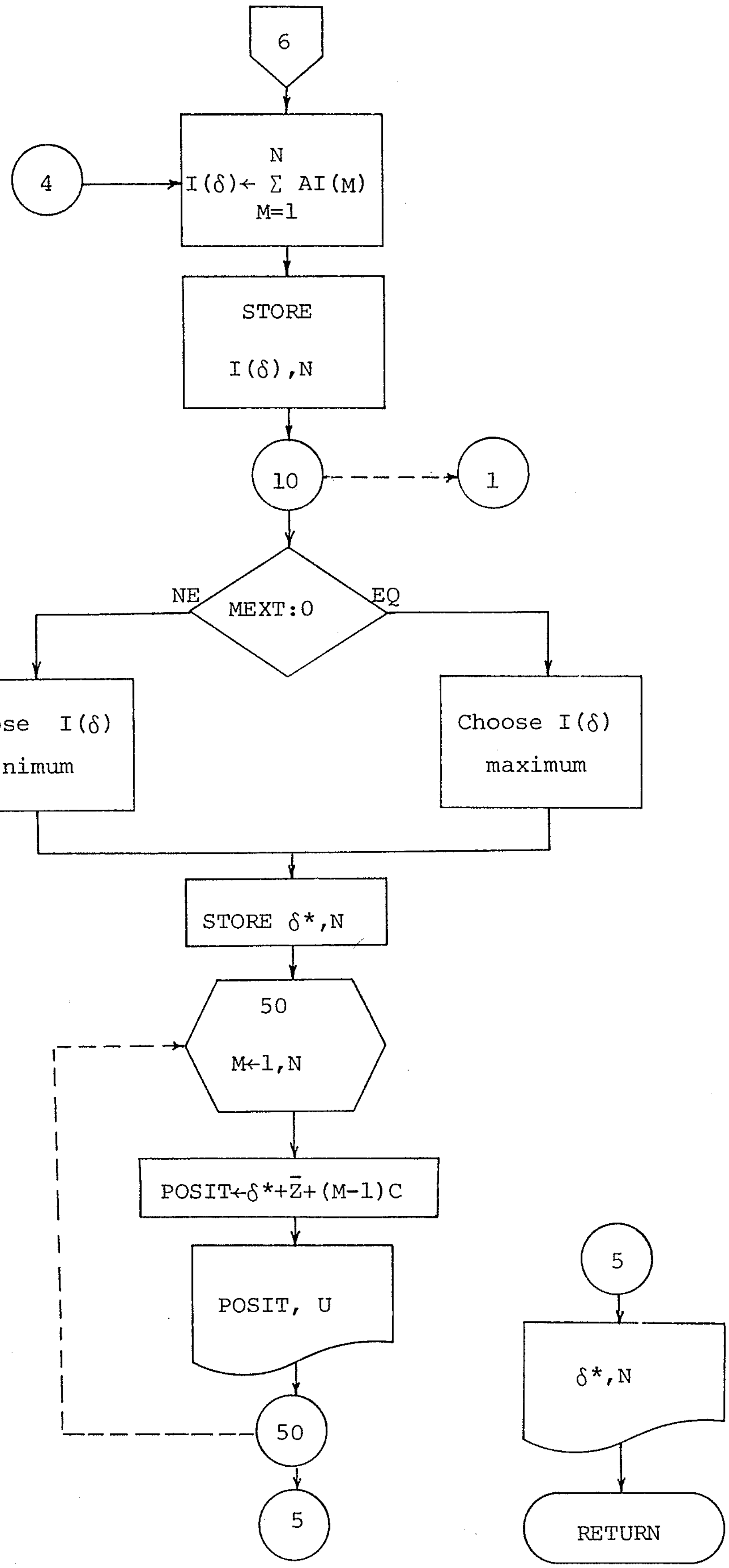

2
U←KX-2X
AI(K) ← ∫ FCT dt
DI+W
DI
20
4
NE
MEXT:0
EQ
Choose AI(K) minimum
Choose AI(K) maximum
DI,U
10
3
RETURN
2
5
50
M←1,N
3


3
DI←T+D-MC
60
K←1,3
U←KX-2X
AI(K) ← ∫ DI+W FCT dt
DI
60
NE
EQ
MEXT:0
Choose AI(K) minimum
Choose AI(K) maximum
DI,U
50
5
RETURN

SHIFT Subroutine Flow Diagram

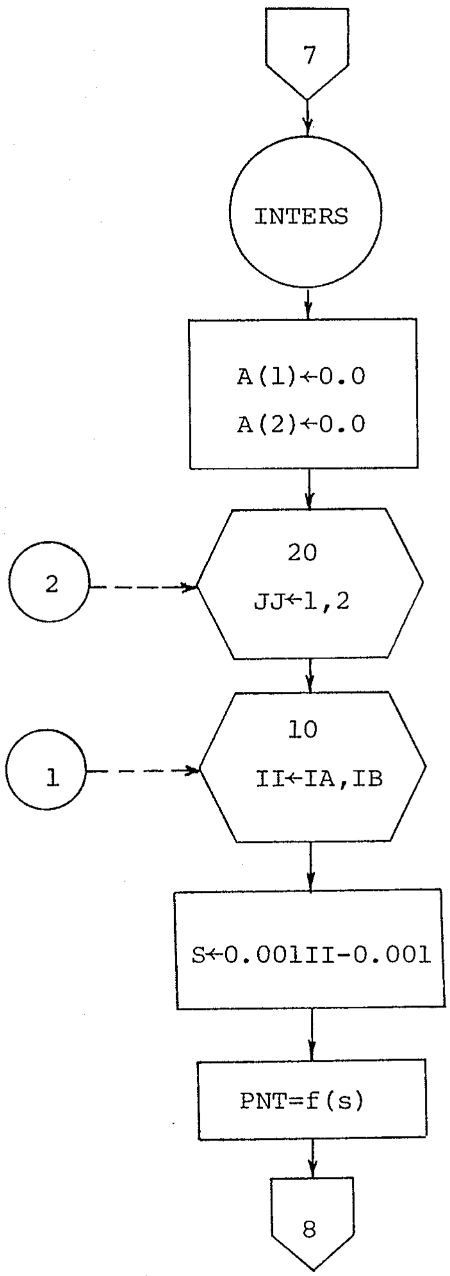


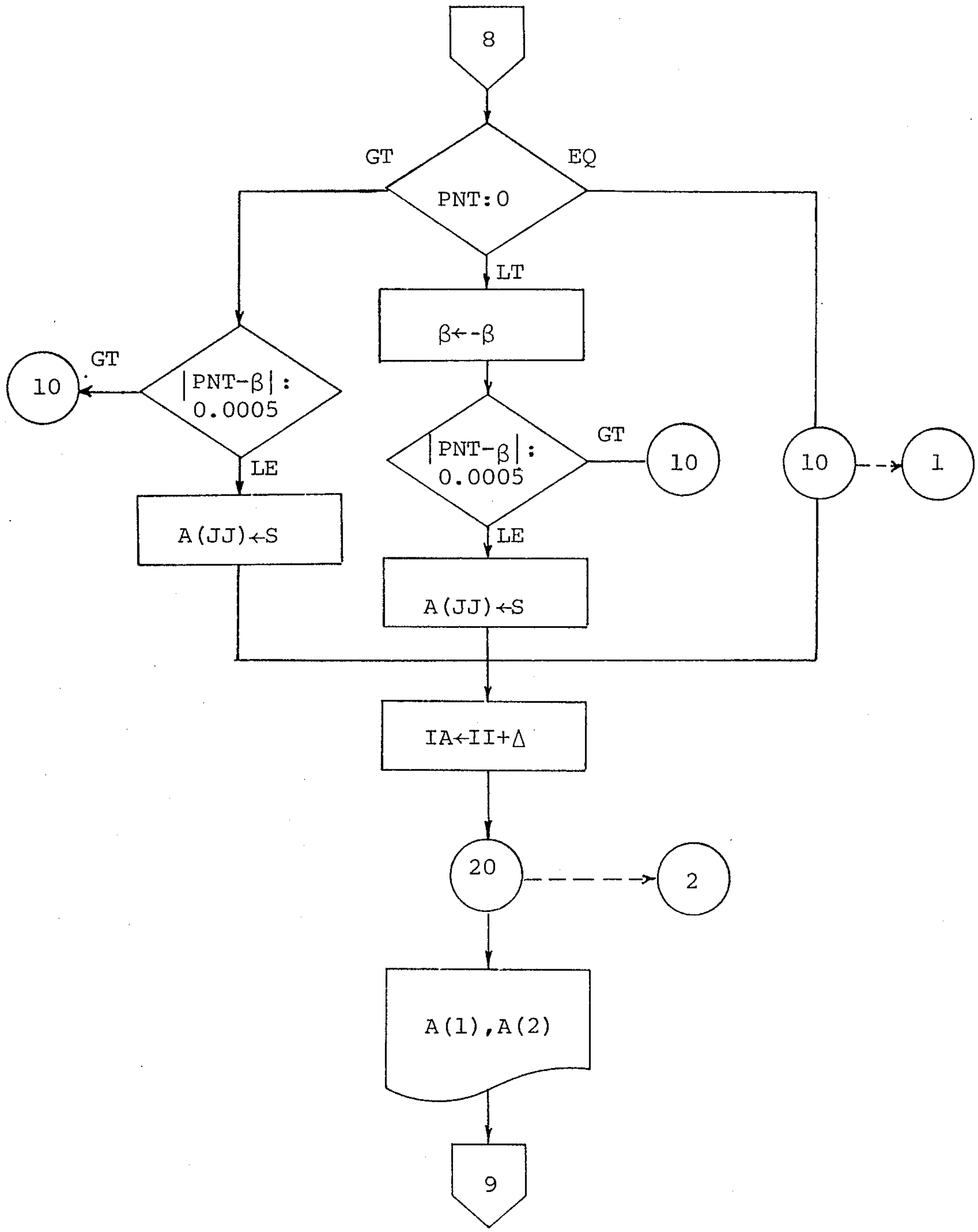

5
U←KX-2X
AI(K) ← ∫ DI to DI+W FCT dt
30
3
NE
EQ
MEXT:0
Choose AI(K) minimum
Choose AI(K) maximum
AI(M)←AI(K)
STORE corresponding U
20
2
LE
(DI+W):FJ
4
GT
N←N-1
6

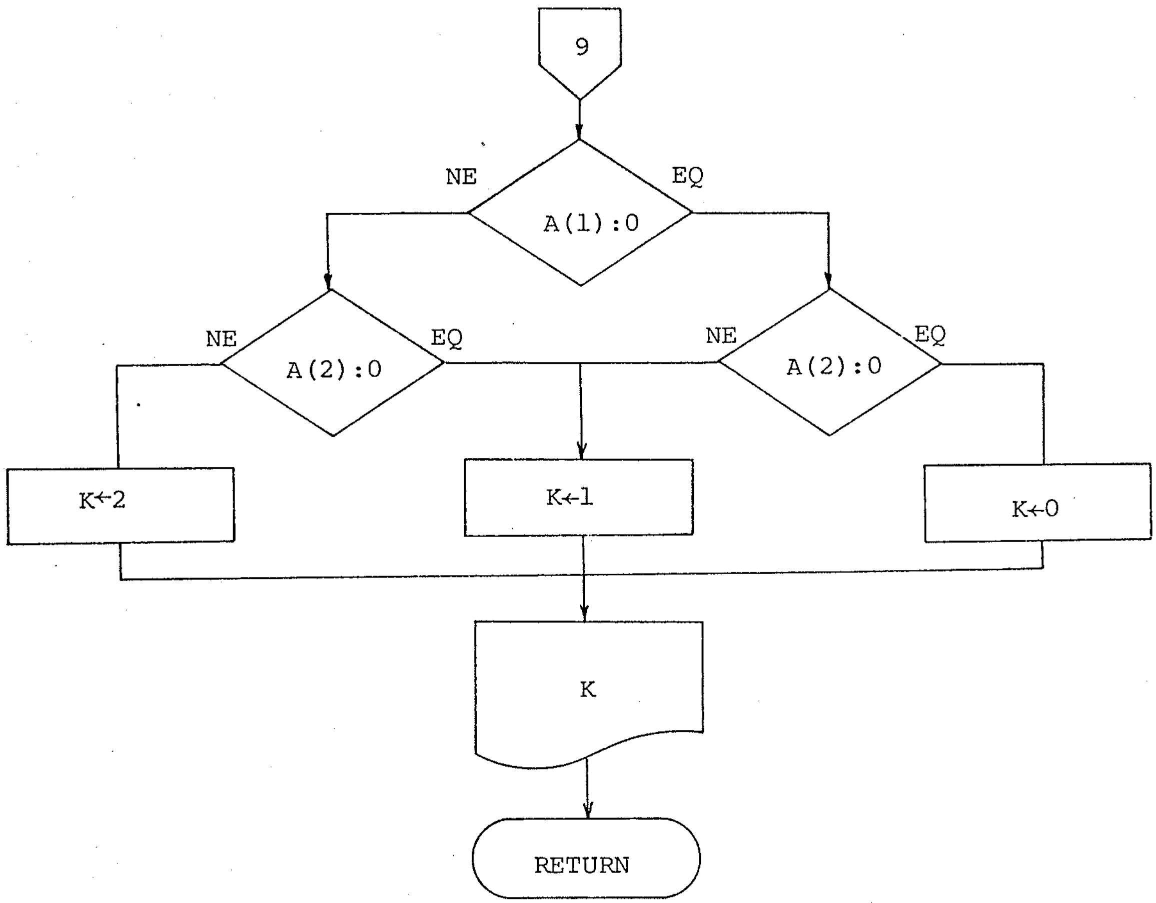

6
N
I(δ)← Σ AI(M)
M=1
4
STORE
I(δ),N
10
1
NE
MEXT:0
EQ
Choose I(δ)
minimum
Choose I(δ)
maximum
STORE δ*,N
50
M←1,N
POSIT←δ*+Z̄+(M-1)C
POSIT, U
50
5
5
δ*,N
RETURN

INTERS Subroutine Flow Diagram

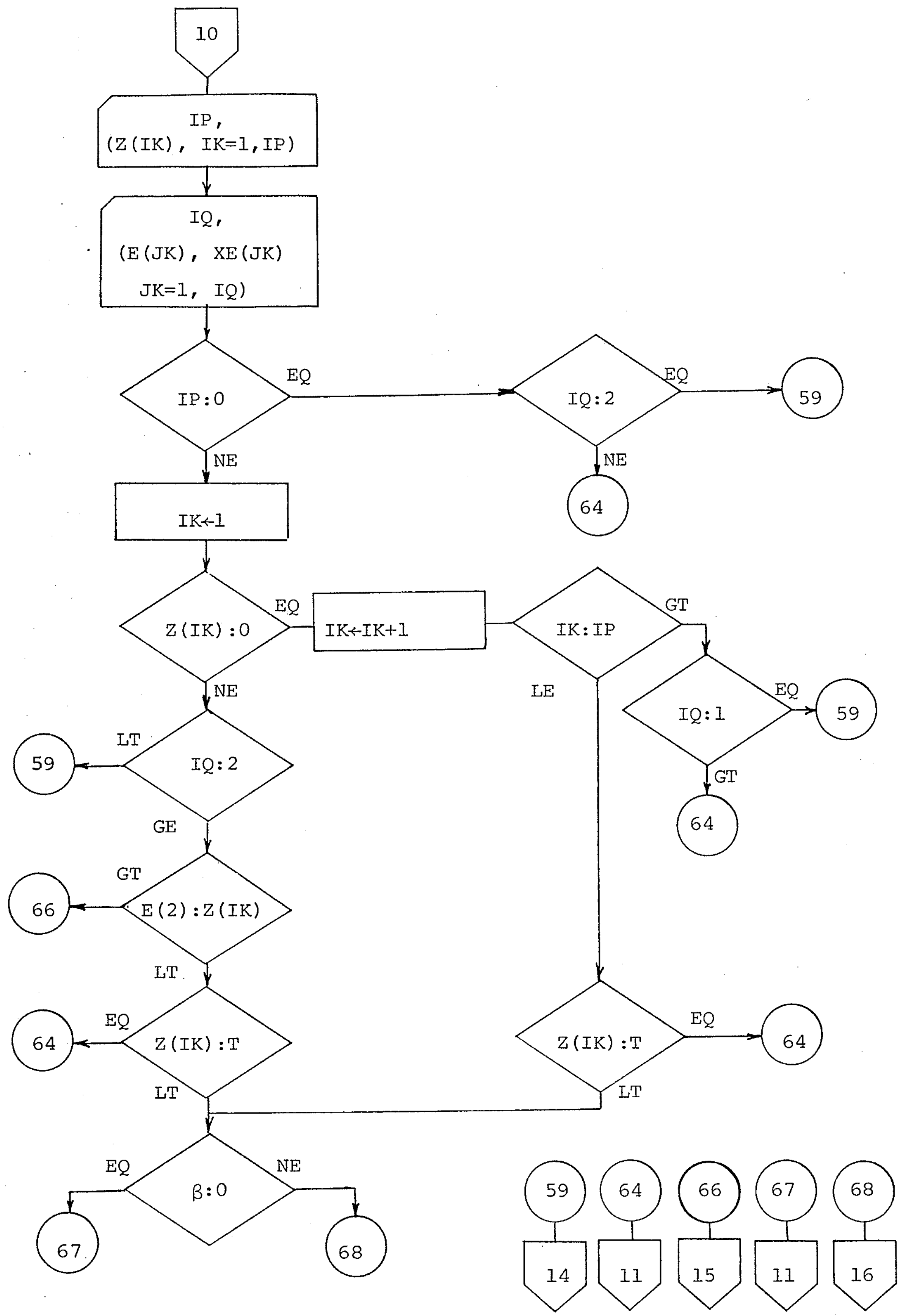
This subroutine is used to find the time at which β and P(t) intersect, within a certain interval.

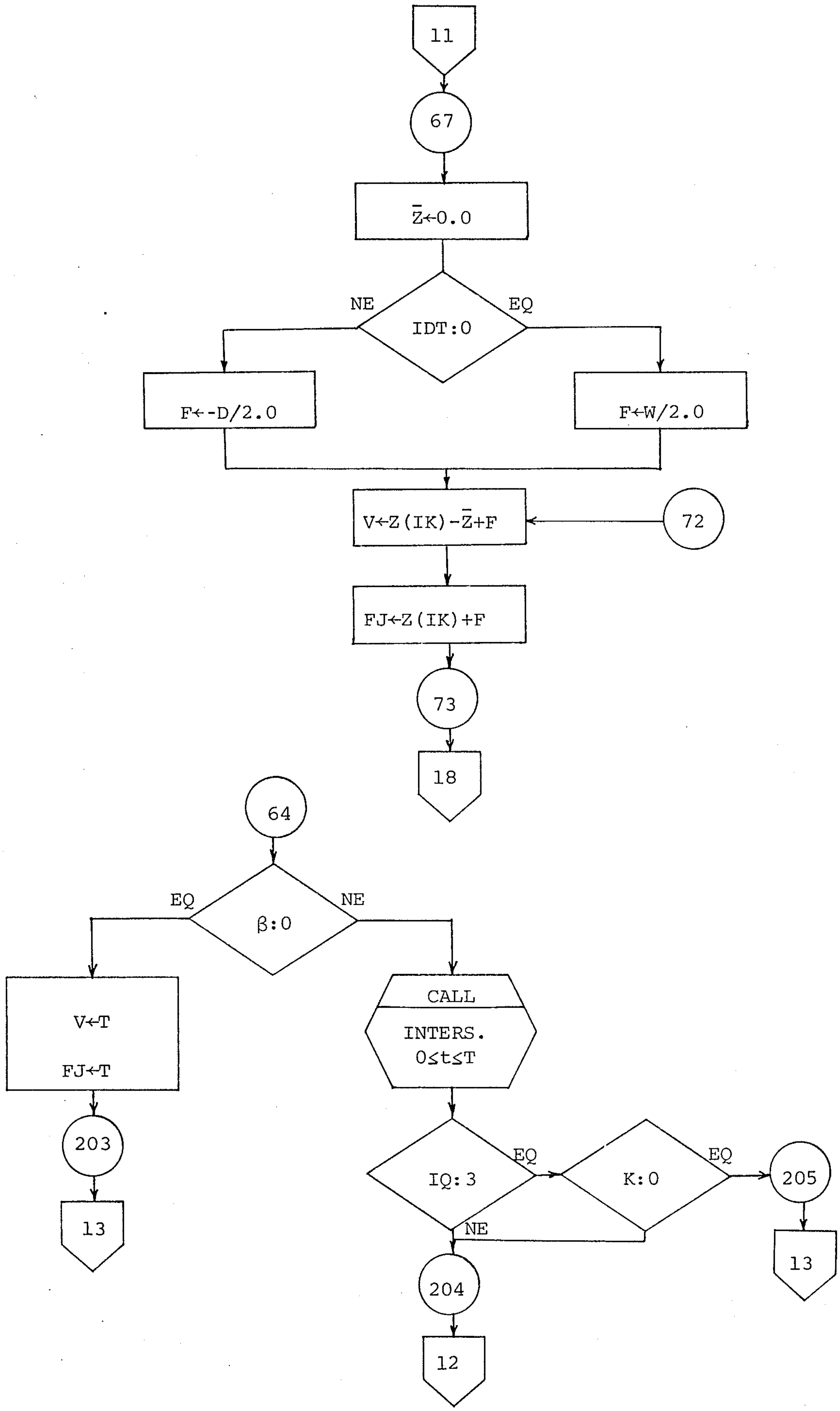


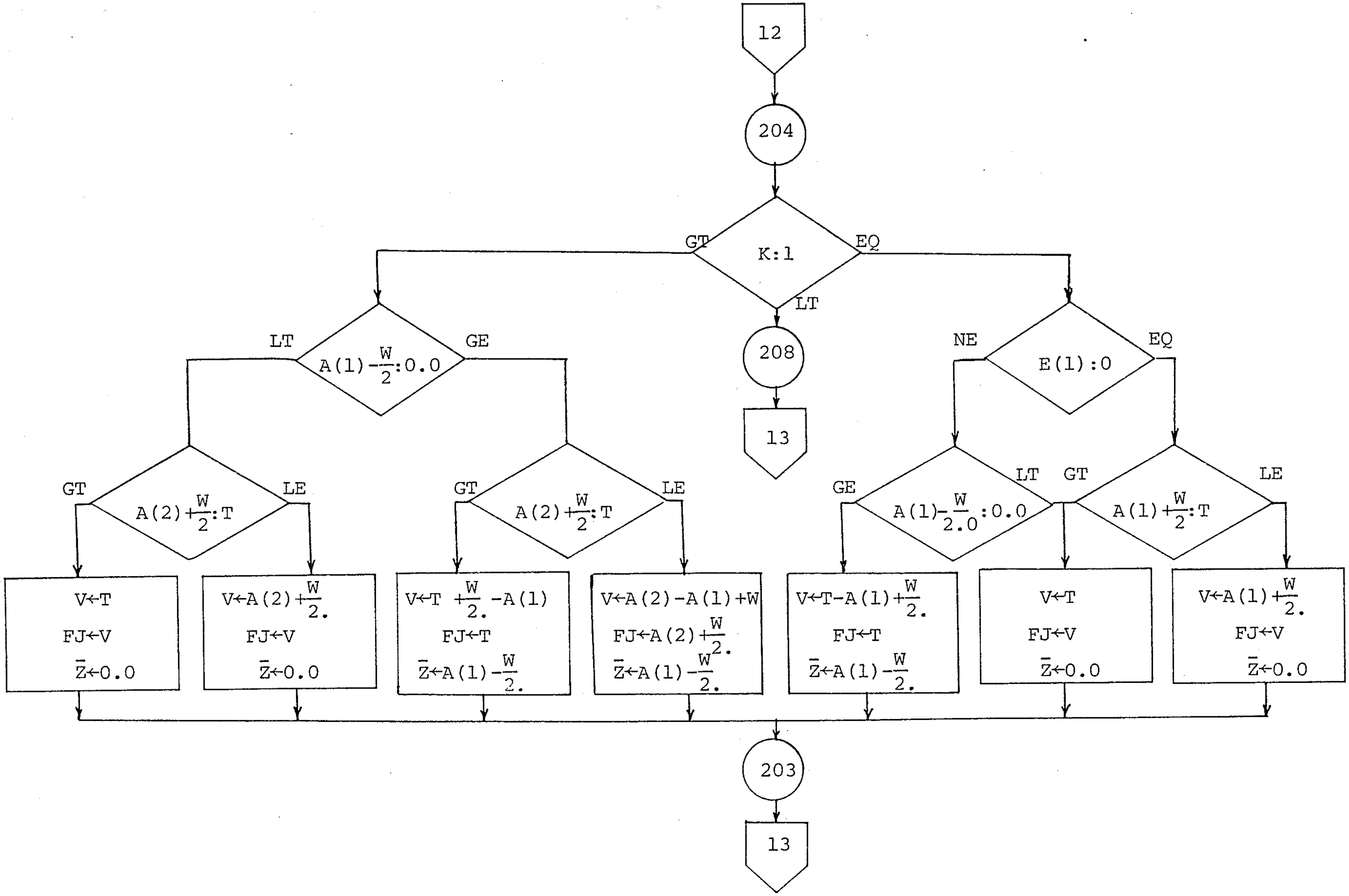

8
PNT:0
GT
EQ
LT
β←-β
|PNT-β|: 0.0005
GT
10
LE
A(JJ)←S
|PNT-β|: 0.0005
GT
10
LE
A(JJ)←S
10
1
IA←II+Δ
20
2
A(1),A(2)
9

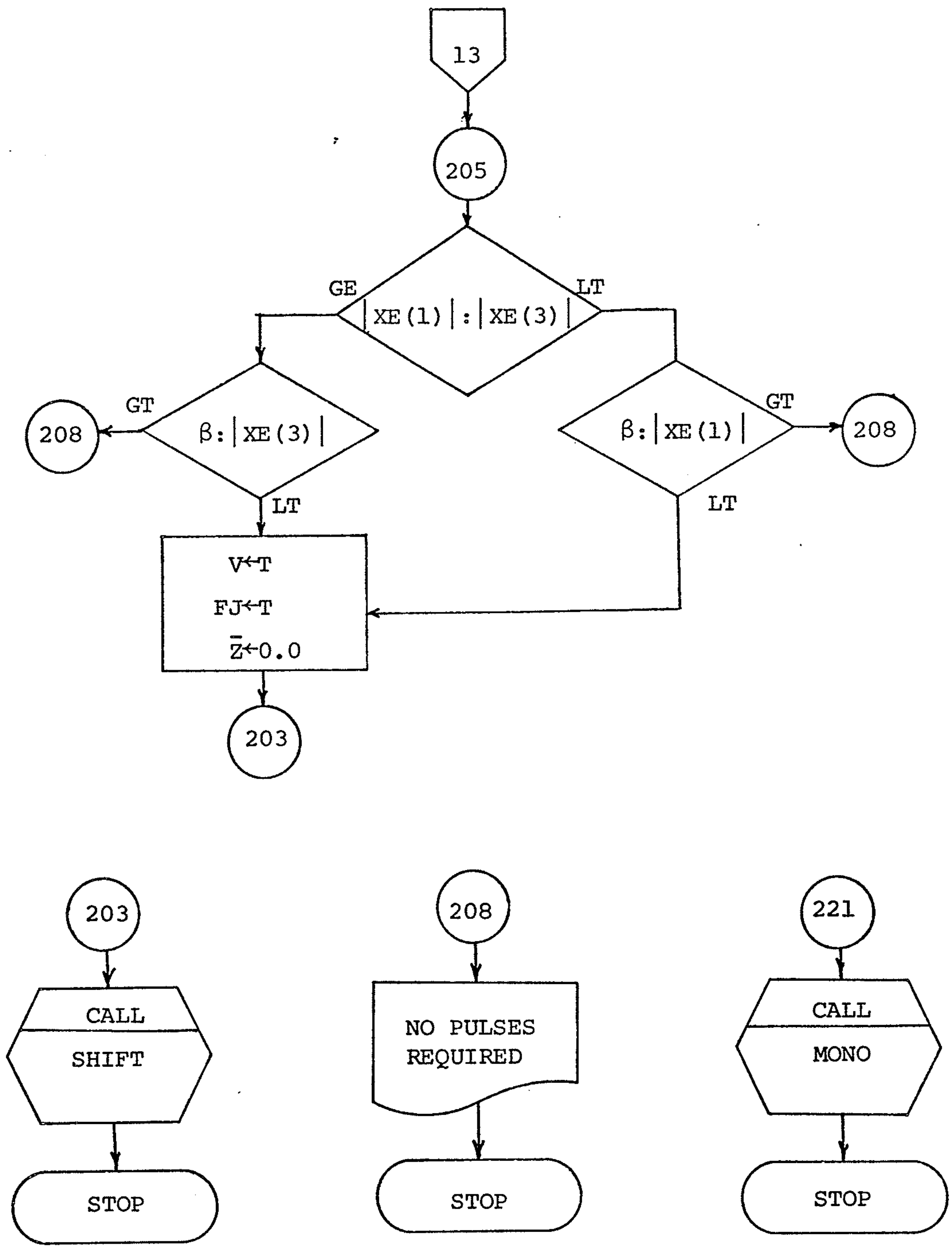

9
NE
EQ
A(1):0
NE
EQ
A(2):0
NE
EQ
A(2):0
K←2
K←1
K←0
K
RETURN

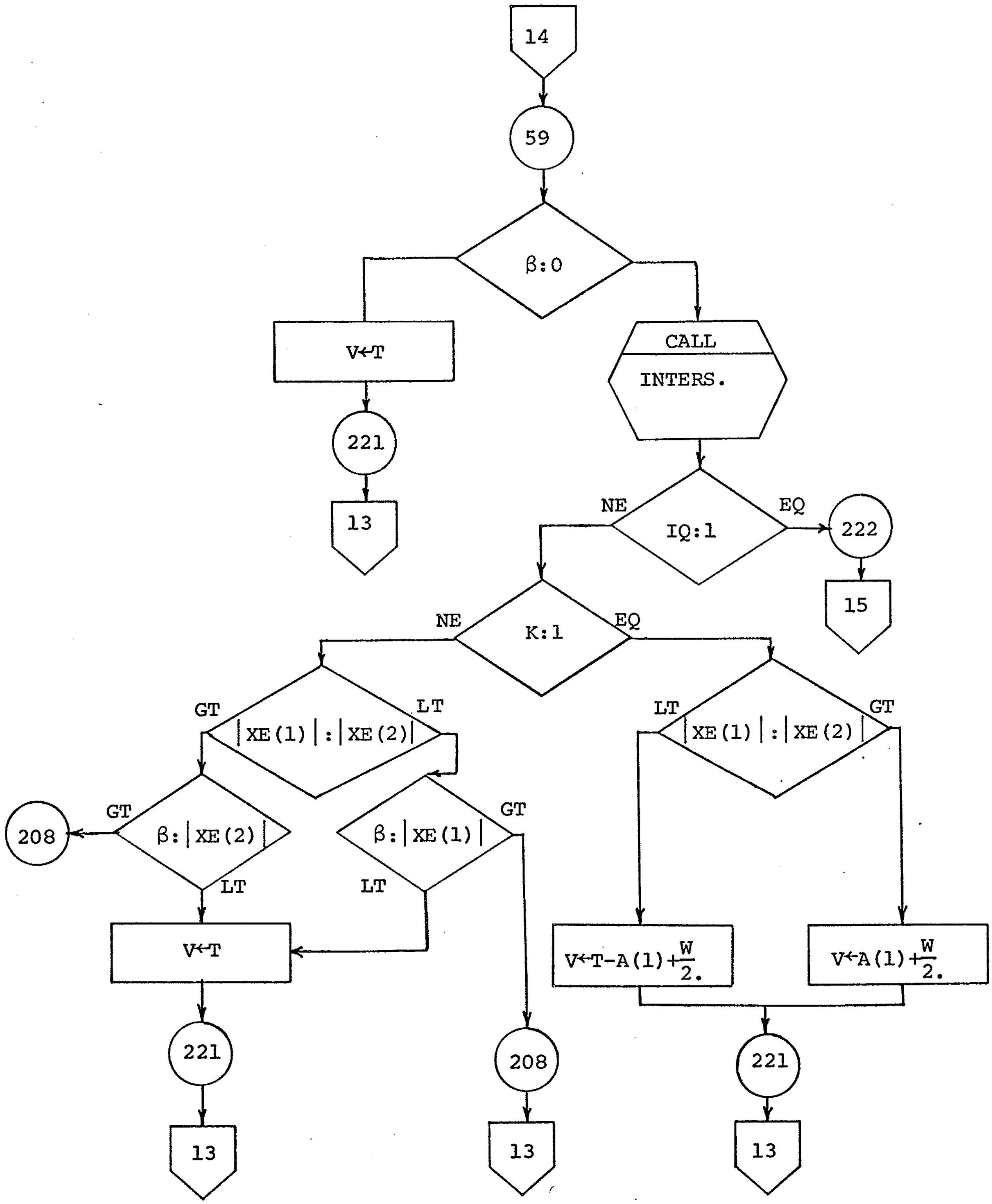
Main Program

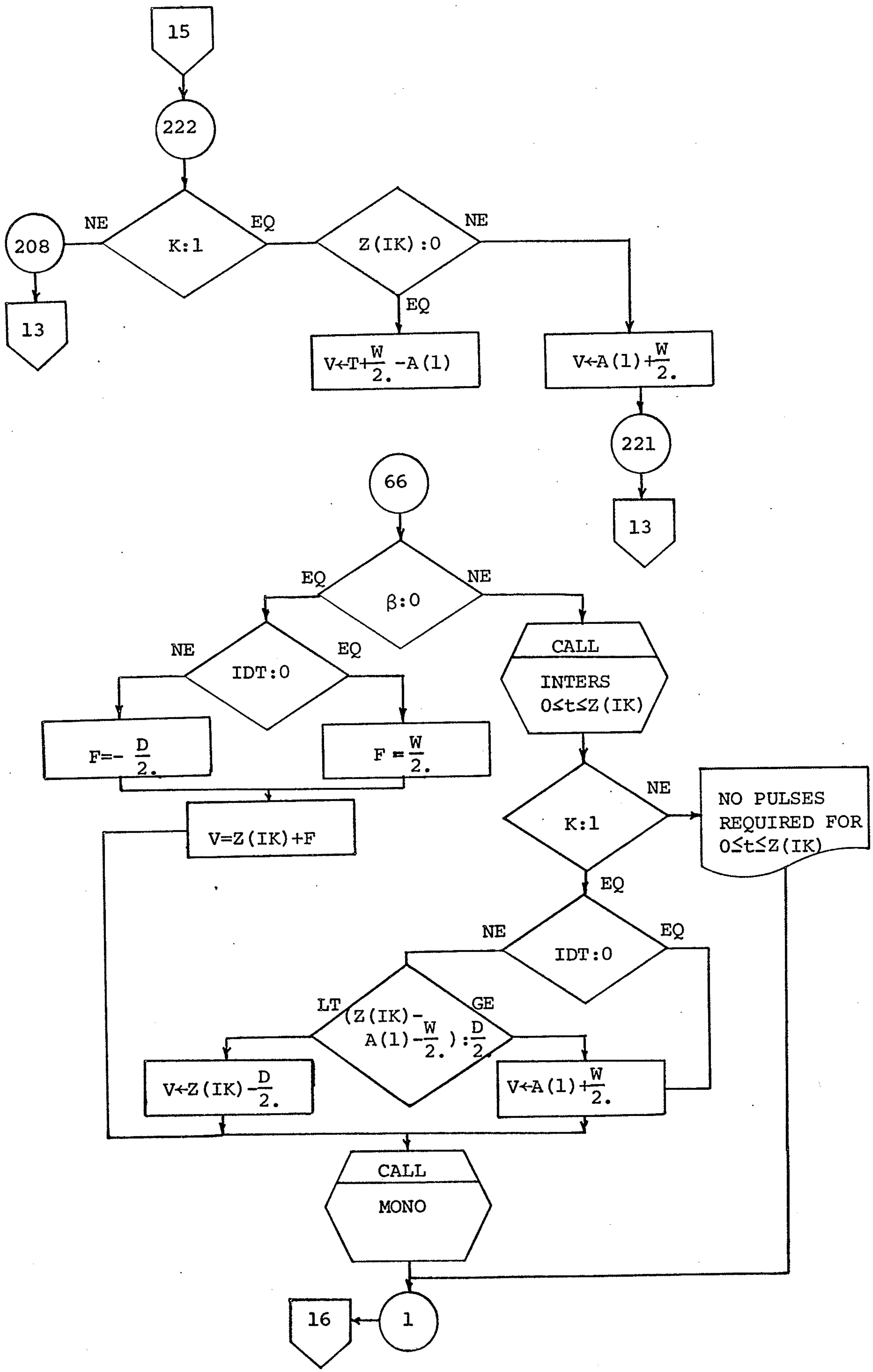


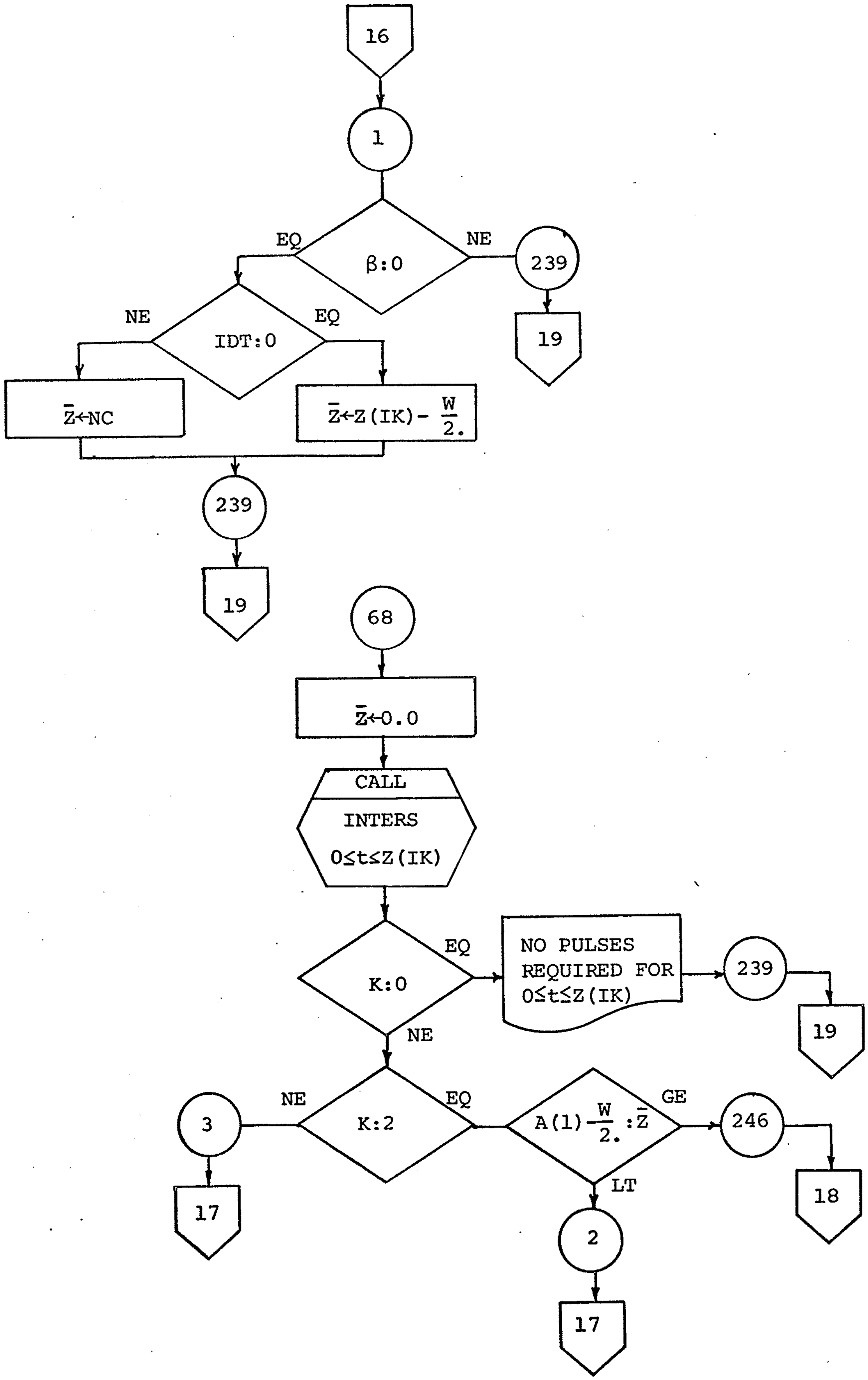

11
67
Z̄←0.0
IDT:0
NE
EQ
F←-D/2.0
F←W/2.0
V←Z(IK)-Z̄+F
72
FJ←Z(IK)+F
73
18
64
β:0
EQ
NE
V←T
FJ←T
CALL
INTERS.
0≤t≤T
203
13
IQ:3
EQ
K:0
EQ
205
NE
13
204
12


12
204
K:1
GT
EQ
LT
208
13
A(1)-W/2:0.0
LT
GE
E(1):0
NE
EQ
A(2)+W/2:T
GT
LE
A(2)+W/2:T
GT
LE
A(1)-W/2.0:0.0
GE
LT
A(1)+W/2:T
GT
LE
V←T
FJ←V
Z̄←0.0
V←A(2)+W/2.
FJ←V
Z̄←0.0
V←T +W/2. -A(1)
FJ←T
Z̄←A(1)-W/2.
V←A(2)-A(1)+W
FJ←A(2)+W/2.
Z̄←A(1)-W/2.
V←T-A(1)+W/2.
FJ←T
Z̄←A(1)-W/2.
V←T
FJ←V
Z̄←0.0
V←A(1)+W/2.
FJ←V
Z̄←0.0
203
13

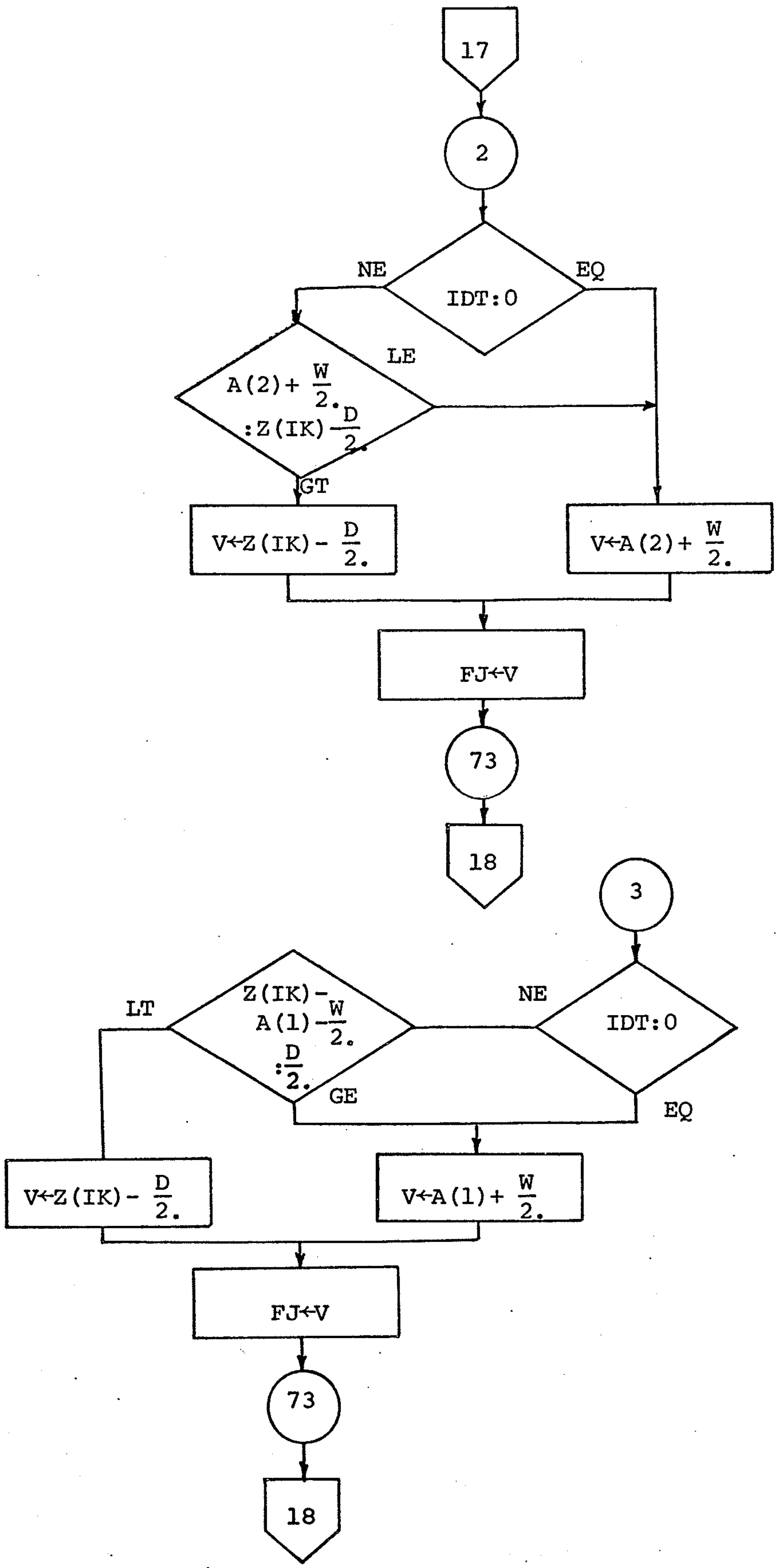

13
205
GE
LT
|XE(1)|:|XE(3)|
GT
208
β:|XE(3)|
LT
GT
β:|XE(1)|
208
LT
V←T
FJ←T
Z̄←0.0
203
203
CALL
SHIFT
STOP
208
NO PULSES
REQUIRED
STOP
221
CALL
MONO
STOP

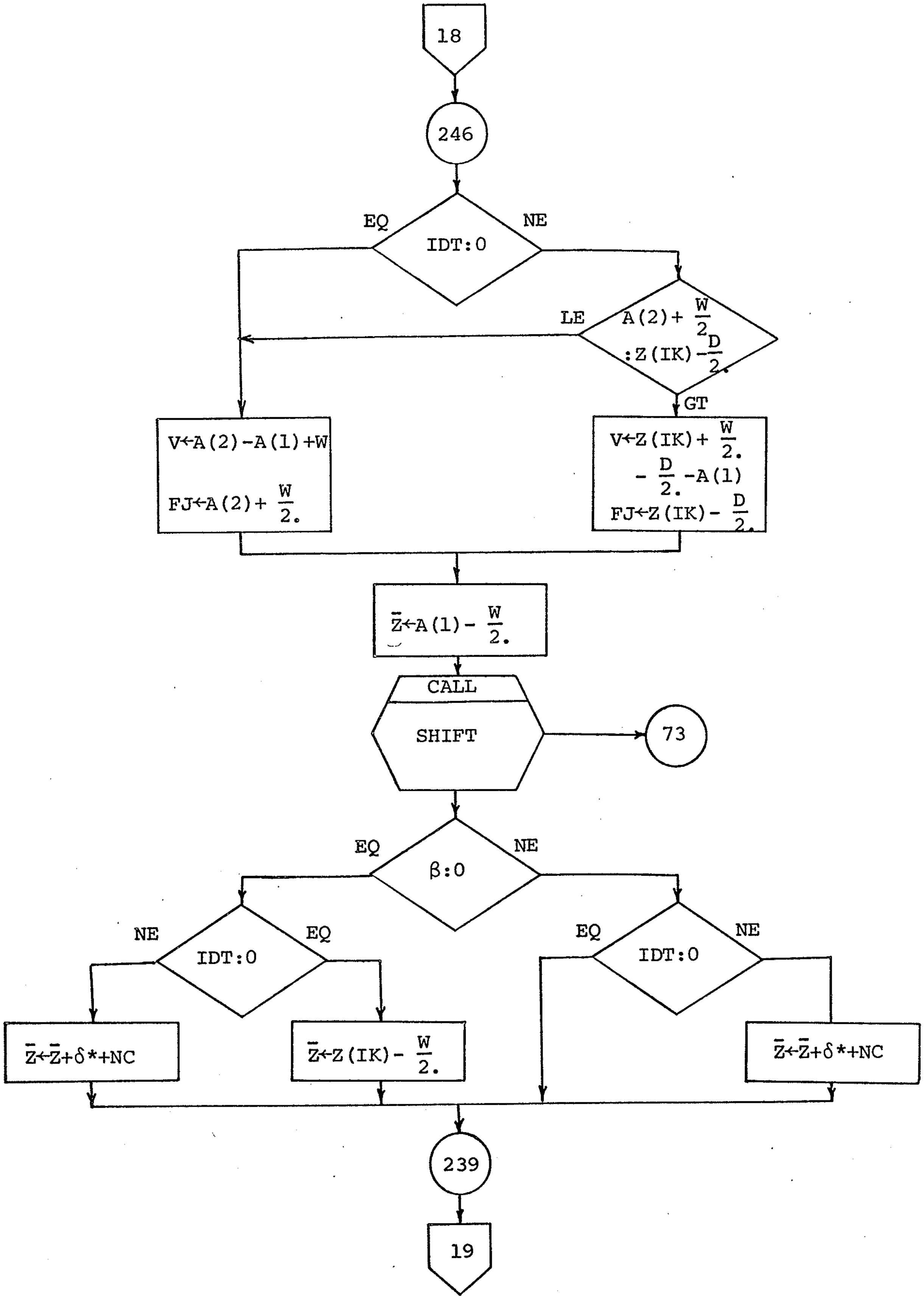

14
59
β:0
V←T
221
13
CALL
INTERS.
NE
IQ:1
EQ
222
15
NE
K:1
EQ
GT
|XE(1)|:|XE(2)|
LT
LT
|XE(1)|:|XE(2)|
GT
GT
β:|XE(2)|
208
LT
β:|XE(1)|
GT
LT
V←T
221
13
208
13
V←T-A(1)+W/2.
V←A(1)+W/2.
221
13

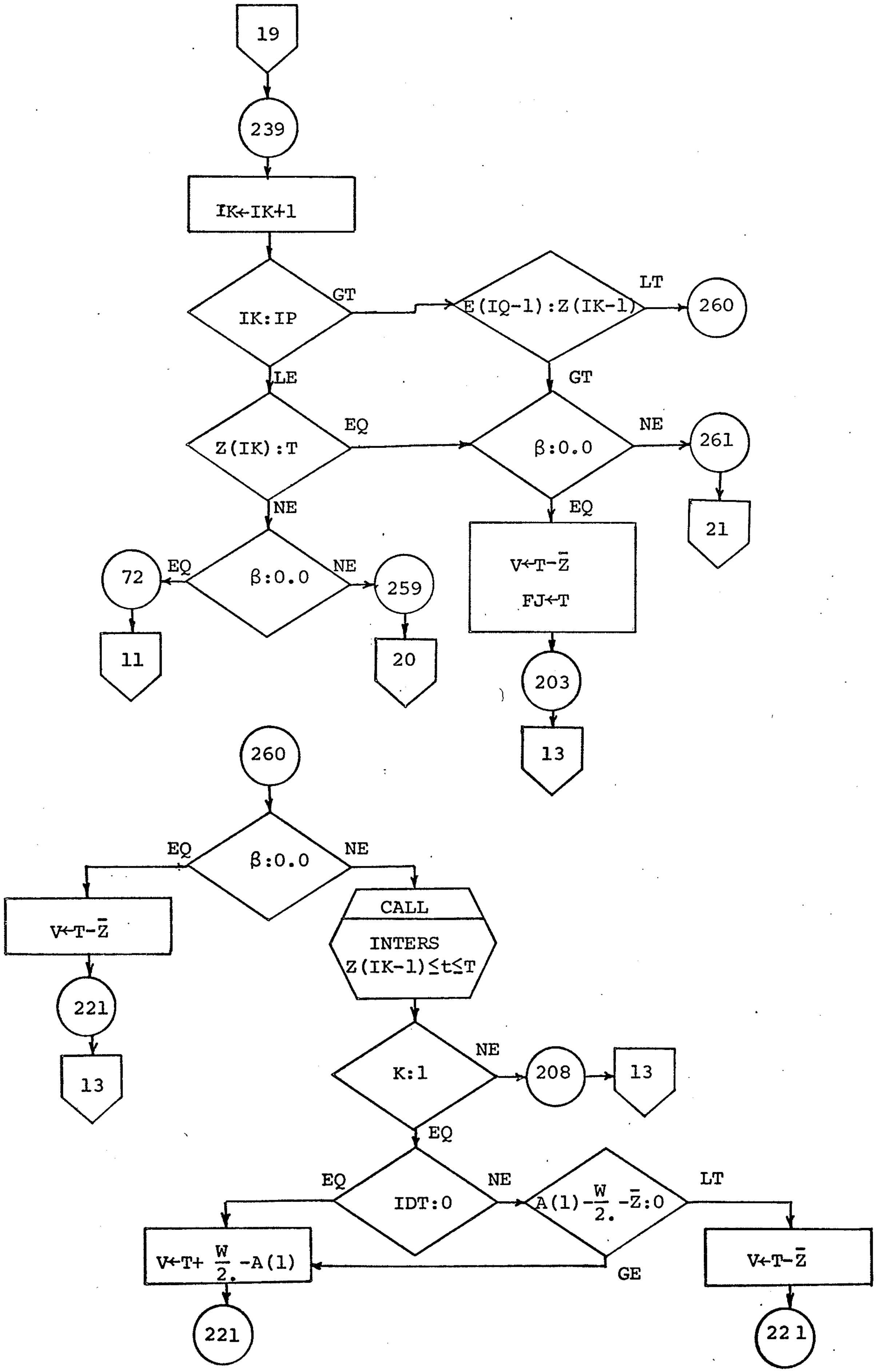

15
222
K:1
NE
EQ
208
13
Z(IK):0
NE
EQ
V←T+W/2. -A(1)
V←A(1)+W/2.
221
13
66
β:0
EQ
NE
IDT:0
NE
EQ
F=- D/2.
F = W/2.
V=Z(IK)+F
CALL
INTERS
0≤t≤Z(IK)
K:1
NE
NO PULSES
REQUIRED FOR
0≤t≤Z(IK)
EQ
IDT:0
NE
EQ
(Z(IK)-A(1)-W/2.):D/2.
LT
GE
V←Z(IK)-D/2.
V←A(1)+W/2.
CALL
MONO
1
16

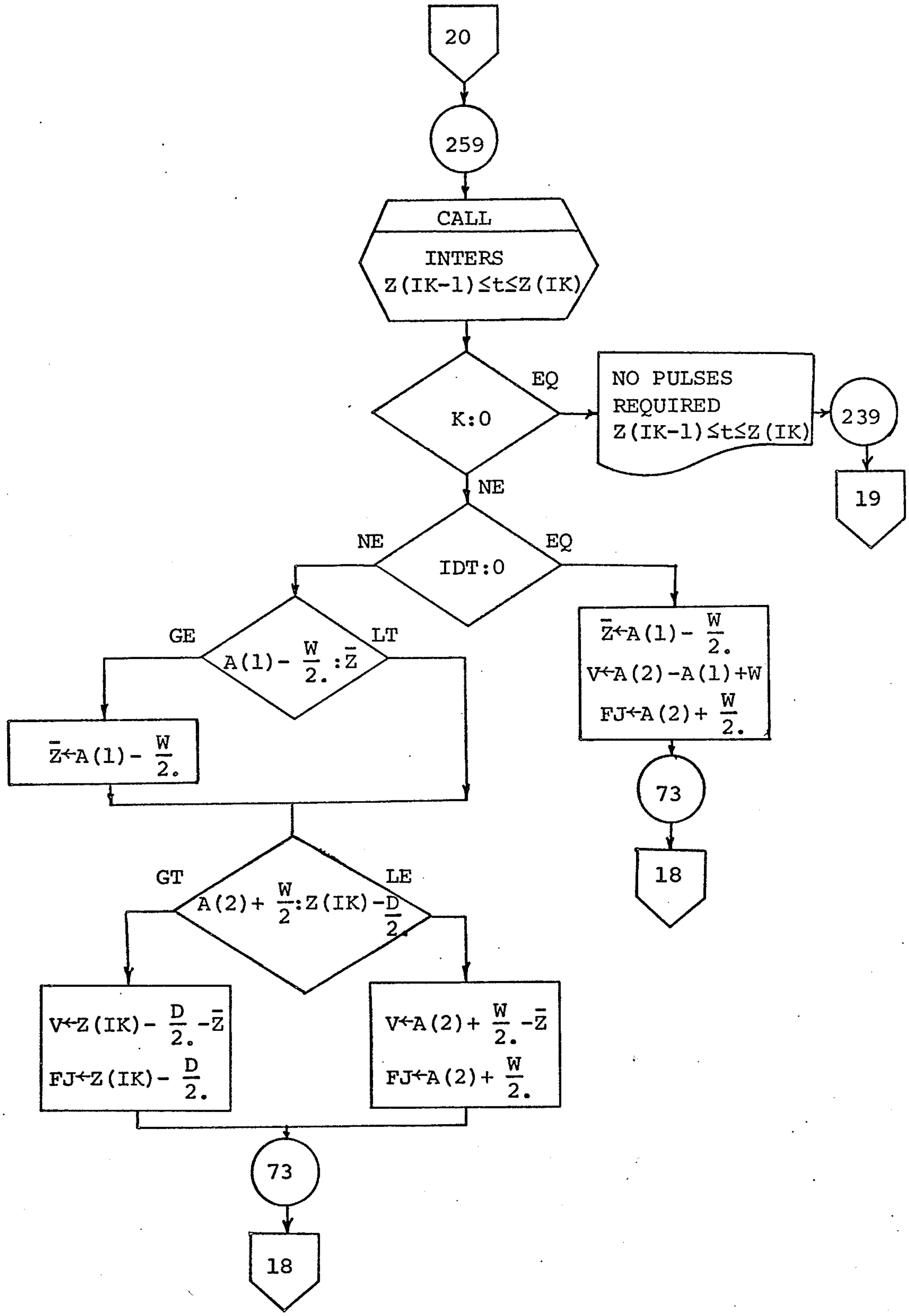


16
1
β:0
EQ
NE
239
19
IDT:0
NE
EQ
Z̄←NC
Z̄←Z(IK)- W/2.
239
19
68
Z̄←0.0
CALL
INTERS
0≤t≤Z(IK)
K:0
EQ
NO PULSES REQUIRED FOR 0≤t≤Z(IK)
239
19
NE
K:2
NE
EQ
3
17
A(1)-W/2. :Z̄
GE
LT
246
18
2
17

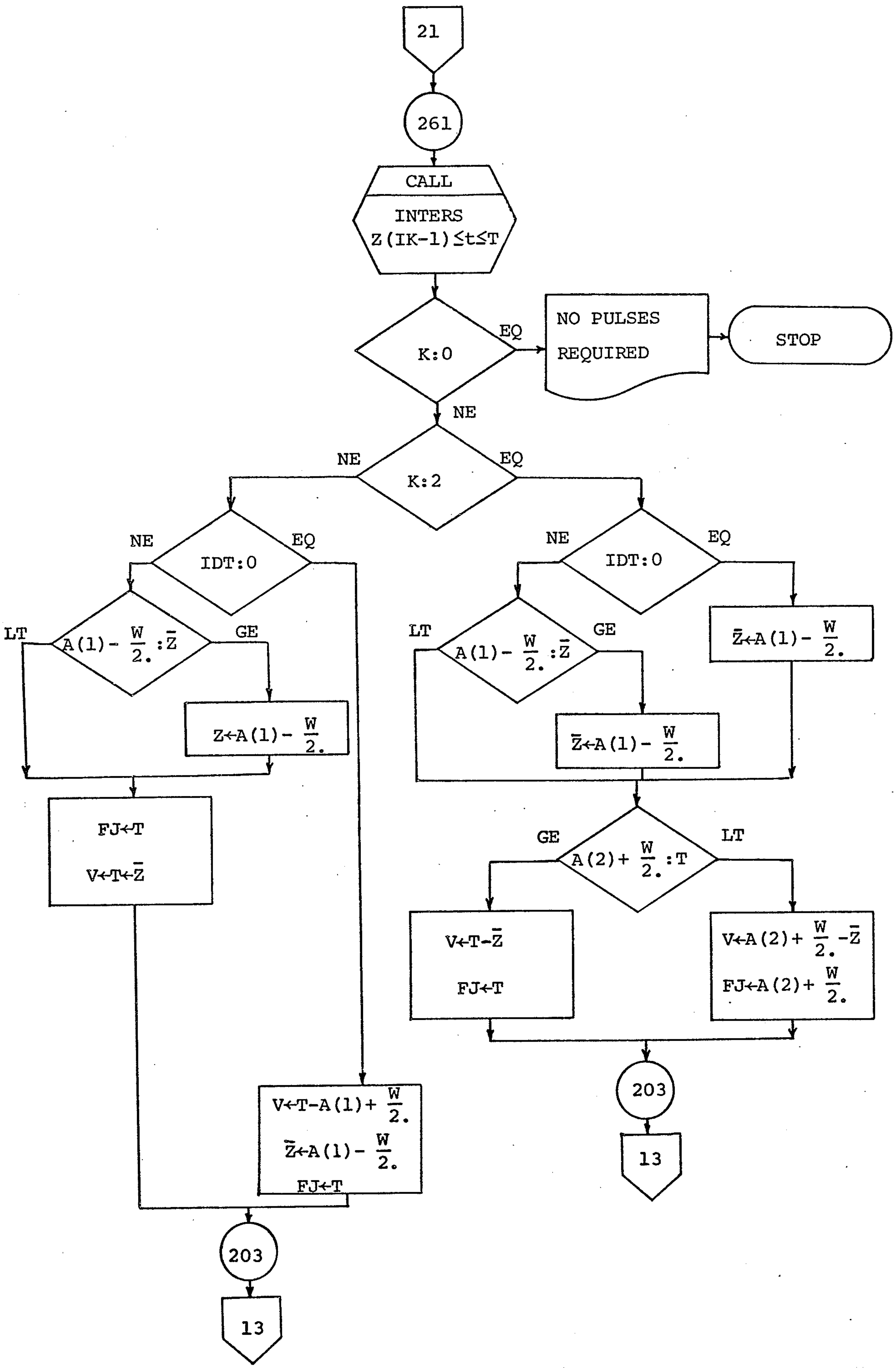

17
2
IDT:0
NE
EQ
A(2)+ W/2. :Z(IK) − D/2.
LE
GT
V←Z(IK) − D/2.
V←A(2)+ W/2.
FJ←V
73
18
3
IDT:0
NE
EQ
Z(IK) − A(1) − W/2. : D/2.
LT
GE
V←Z(IK) − D/2.
V←A(1)+ W/2.
FJ←V
73
18


18
246
EQ
IDT:0
NE
LE
A(2)+ W/2.
:Z(IK) - D/2.
GT
V←A(2)-A(1)+W
FJ←A(2)+ W/2.
V←Z(IK)+ W/2. - D/2. -A(1)
FJ←Z(IK)- D/2.
Z̄←A(1)- W/2.
CALL
SHIFT
73
EQ
β:0
NE
NE
IDT:0
EQ
EQ
IDT:0
NE
Z̄←Z̄+δ*+NC
Z̄←Z(IK)- W/2.
Z̄←Z̄+δ*+NC
239
19


19
239
IK←IK+1
IK:IP
GT
LE
E(IQ-1):Z(IK-1)
LT
260
GT
Z(IK):T
EQ
β:0.0
NE
261
21
EQ
NE
β:0.0
EQ
NE
72
259
11
20
V←T-Z̄
FJ←T
203
13
260
β:0.0
EQ
NE
V←T-Z̄
CALL
INTERS
Z(IK-1)≤t≤T
221
13
K:1
NE
208
13
EQ
IDT:0
EQ
NE
A(1)-W/2.-Z̄:0
LT
GE
V←T+W/2.-A(1)
V←T-Z̄
221
221



20
259
CALL
INTERS
Z(IK-1)≤t≤Z(IK)
K:0
EQ
NO PULSES
REQUIRED
Z(IK-1)≤t≤Z(IK)
239
19
NE
IDT:0
NE
EQ
Z̄←A(1)- W/2.
V←A(2)-A(1)+W
FJ←A(2)+ W/2.
73
18
A(1)- W/2. :Z̄
GE
LT
Z̄←A(1)- W/2.
A(2)+ W/2:Z(IK)-D/2.
GT
LE
V←Z(IK)- D/2. -Z̄
FJ←Z(IK)- D/2.
V←A(2)+ W/2. -Z̄
FJ←A(2)+ W/2.
73
18



21
261
CALL
INTERS
Z(IK-1)≤t≤T
K:0
EQ
NO PULSES REQUIRED
STOP
NE
K:2
NE
EQ
IDT:0
NE
EQ
A(1)- W/2. :Z̄
LT
GE
Z←A(1)- W/2.
FJ←T
V←T←Z̄
V←T-A(1)+ W/2.
Z̄←A(1)- W/2.
FJ←T
203
13
IDT:0
NE
EQ
A(1)- W/2. :Z̄
LT
GE
Z̄←A(1)- W/2.
Z̄←A(1)- W/2.
A(2)+ W/2. :T
GE
LT
V←T-Z̄
FJ←T
V←A(2)+ W/2. -Z̄
FJ←A(2)+ W/2.
203
13

APPENDIX B

COMPUTER PROGRAMS

This appendix gives a sample of the computer programs which finds the optimum control function u(t) for any system of the form assumed in Chapter 2. Two cards defining the functions FCT and PNT need to be changed depending on specified problem.

The symbols used in this program are the same symbols mentioned in Appendix A.2, except for the following changes:

ALPHA	Positive value of β
BETA	Negative value of β
DSTAR	Equivalent to δ^*
FT	Final time T
LD	Inverse of δ'
NISTAR	Number of pulses N
QG10	Library routine for integration using Gaussian quadrature method
T	Time T
USTAR	Optimum control function u(t)
ZBAR	Starting point of an interval $\bar{Z}$

```
C
C
C     COMPUTER PROGRAM
C
C
C
      DIMENSION Z(10),XE(10),E(10)
      COMMON FT,U,MEXT,W,D,X,V,C,ALPHA
      COMMON /IET/ZBAR,FJ,LD,CSTAR,NISTAR
      COMMON /TON/IQ,XE,E,N,IK,IP
      COMMON /SEC/A(2),K
      READ(5,54) MEXT,IDT,W,D,FT,X,ALPHA,LD
      READ(5,53) IP,(Z(IK),IK=1,IP)
      READ(5,55) IQ,(E(JK),JK=1,IQ)
      READ(5,56)     (XE(IK),IK=1,IQ)
      IF(IP) 58,57,58
   57 IF(IQ.EQ.2) GO TO 59
   64 IF(ALPHA) 202,201,202
  201 V=FT
      FJ=FT
      GO TO 203
  202 IB=1000*FT+1
      IA=1
      CALL INTERS(IA,IB)
      IF(IQ.NE.3) GO TO 204
      IF(K) 204,205,204
  205 IF(ABS(XE(1)).LT.ABS(XE(3))) GO TO 206
      IF(ALPHA.LT.ABS(XE(3))) GO TO 207
  208 WRITE(6,290)
      GO TO 234
  207 V=FT
      FJ=FT
      ZBAR=0.0
      GO TO 203
  206 IF(ALPHA.LT.ABS(XE(1))) GO TO 207
      GO TO 208
  204 IF(K-1)208,209,210
  209 IF(E(1)) 211,212,211
  211 IF(A(1)-W/2.0) 214,213,213
  213 V=FT-A(1)+W/2.
      FJ=FT
      ZBAR=A(1)-W/2.
      GO TO 203
  214 V=FT
      FJ=V
      ZBAR=0.0
      GO TO 203
  212 IF((A(1)+W/2.).GT.FT)GO TO 214
      V=A(1)+W/2.
      FJ=V
      ZBAR=0.
      GO TO 203
  210 IF(A(1)-W/2.)215,216,216
  215 IF((A(2)+W/2.).GT.FT)GO TO 214
      V=A(2)+W/2.
      FJ=V
      ZBAR=0.0
      GO TO 203
```

```
216 IF((A(2)+W/2.).GT.FT)GO TO 218
    V=A(2)-A(1)+W
    FJ=A(2)+W/2.
    ZBAR=A(1)-W/2.
    GO TO 203
218 V=FT-A(1)+W/2.
    FJ=FT
    ZBAR=A(1)-W/2.
203 CALL SHIFT
    GO TO 234
 59 IF(ALPHA)220,219,220
219 V=FT
    GO TO 221
220 IB=1000*FT+1
    IA=1
    CALL INTERS(IA,IB)
    IF(IQ.EQ.1)GO TO 222
    IF(K.EQ.1)GO TO 223
    IF(ABS(XE(1)).LT.ABS(XE(2)))GO TO 224
    IF(ALPHA.GT.ABS(XE(2)))GO TO 208
225 V=FT
    GO TO 221
224 IF(ALPHA.GT.ABS(XE1))GO TO 208
    GO TO 225
223 IF(ABS(XE(1)).GT.ABS(XE(2)))GO TO 226
227 V=FT-A(1)+W/2.
    GO TO 221
226 V=A(1)+W/2.
    GO TO 221
222 IF(K.NE.1)GO TO 203
    IF(Z(1))226,227,226
221 CALL ICNO
    GO TO 234
 58 IK=1
    IF(Z(IK))62,61,62
 61 IK=IK+1
    IF(IK.LE.IP)GO TO 63
    IF(IQ.EQ.1)GO TO 59
    GO TO 64
 63 IF(Z(IK).EQ.FT)GO TO 64
    GO TO 65
 62 IF(IQ.LT.2)GO TO 59
    IF(E(2).GT.Z(IK))GO TO 66
    GO TO 63
 66 IF(ALPHA)229,228,229
228 IF(IDT)230,231,230
230 F=-D/2.
    GO TO 232
231 F=W/2.
232 V=Z(IK)+F
    GO TO 233
229 IB=1000*Z(IK)+1
    IA=1
    CALL INTERS(IA,IB)
    IF(K.EQ.1)GO TO 234
    WRITE(6,291) Z(IK)
    GO TO 235
234 IF(IDT)236,237,236
```

```
236 IF((Z(IK)-A(1)-W/2.).GE.(D/2.))GO TO 237
    V=Z(IK)-D/2.
    GO TO 233
237 V=A(1)+W/2.
233 CALL MONO
235 IF(ALPHA)239,238,239
238 IF(IDT)241,240,241
240 ZBAR=Z(IK)-W/2.
    GO TO 239
241 ZBAR =N*C
    GO TO 239
 65 IF(ALPHA)68,67,68
 67 ZBAR=0.0
    IF(IDT)71,69,71
 69 F=W/2.
    GO TO 72
 71 F=-D/2.
 72 V=Z(IK)-ZBAR+F
    FJ=Z(IK)+F
    GO TO 73
 68 ZBAR=0.
    IA=1
    IB=1000*Z(IK)+1
    CALL INTERS(IA,IB)
    IF(K)286,285,286
285 WRITE(6,291)Z(IK)
    GO TO 239
286 IF(K.EQ.2)GO TO 242
    IF(IDT)244,243,244
243 V=A(1)+W/2.
245 FJ=V
    GO TO 73
244 IF((Z(IK)-A(1)-W/2.).GE.(D/2.))GO TO 243
    V=Z(IK)-D/2.
    GO TO 245
242 IF((A(1)-W/2.).GE.ZBAR)GO TO 246
    IF(IDT)247,248,247
247 IF((A(2)+W/2.).LE.(Z(IK)-D/2.))GO TO 248
    V=Z(IK)-D/2.
    GO TO 245
248 V=A(2)+W/2.
    GO TO 245
246 IF(IDT)249,250,249
249 IF((A(2)+W/2.).LE.(Z(IK)-D/2.))GO TO 250
    V=Z(IK)-D/2.-A(1)+W/2.
    FJ=Z(IK)-D/2.
251 ZBAR=A(1)-W/2.
    GO TO 73
250 V=A(2)-A(1)+W
    FJ=A(2)+W/2.
    GO TO 251
 73 CALL SHIFT
    IF(ALPHA) 252,253,252
252 IF(IDT) 254,235,254
254 ZBAR=ZBAR+DSTAR+NISTAR*C
    GO TO 239
253 IF(IDT) 255,256,255
256 ZBAR=Z(IK)-W/2.
```

```
            GO TO 239
        255 ZBAR=ZBAR+DSTAR+NISTAR*C
        239 IK=IK+1
            IF(IK.GT.IP) GO TO 257
            IF(Z(IK).EQ.FT) GO TO 258
            IF(ALPHA) 259,72,259
        257 IF(E(IO-1).LT.Z(IK-1)) GO TO 260
        258 IF(ALPHA) 261,262,261
        262 V=FT-ZBAR
            FJ=FT
            GO TO 203
        260 IF(ALPHA) 264,263,264
        263 V=FT-ZBAR
            GO TO 221
        264 IA=1000*Z(IK-1)+2
            IB=1000*FT+1
            CALL INTERS(IA,IB)
            IF(K) 265,209,265
        265 IF(IDT) 266,267,266
        266 IF(A(1)-W/2.-ZBAR) 263,267,267
        267 V=FT+W/2.-A(1)
            GO TO 221
        259 IA=1000*Z(IK-1)+2
            IB=1000*Z(IK)+1
            CALL INTERS(IA,IB)
            IF(K)268,269,268
        269 WRITE(6,292)Z(IK-1),Z(IK)
            GO TO 239
        268 IF(IDT)271,270,271
        270 ZBAR=A(1)-W/2.
            V=A(2)-A(1)+W
            FJ=A(2)+W/2.
            GO TO 73
        271 IF((A(1)-W/2.).LT.ZBAR)GO TO 272
            ZBAR=A(1)-W/2.
        272 IF((A(2)+W/2.).GT.(Z(IK)-D/2.))GO TO 273
            V=A(2)-ZBAR+W/2.
            FJ=A(2)+W/2.
            GO TO 73
        273 V=Z(IK)-D/2.-ZBAR
            FJ=Z(IK)-D/2.
            GO TO 73
        261 IA=1000*Z(IK-1)+2
            WRITE(6,600) Z(IK-1)
        600 FORMAT (1X,'Z(IK-1)=',F10.4)
            IB=1000*FT+1
            WRITE(6,502) IA,IB
        502 FORMAT(1X,'IA=',I12,5X,'IB=',I12)
            CALL INTERS(IA,IB)
            IF(K)275,274,275
        274 WRITE(6,292)Z(IK-1),FT
            GO TO 284
        275 IF(K.EQ.2)GO TO 276
            IF(IDT)278,277,278
        277 V=FT-A(1)+W/2.
            ZBAR=A(1)-W/2.
            FJ=FT
            GO TO 203
```

```
0227      278 IF((A(1)-W/2.).LT.ZBAR) GO TO 279
0228          ZBAR=A(1)-W/2.
0229      279 FJ=FT
0230          V=FT-ZBAR
0231          GO TO 203
0232      276 IF(IDT) 281,280,281
0233      281 IF((A(1)-W/2.).LT.ZBAR) GO TO 282
0234          ZBAR=A(1)-W/2.
0235          GO TO 282
0236      280 ZBAR=A(1)-W/2.
0237      282 IF((A(2)+W/2.).LT.FT) GO TO 283
0238          V=FT-ZBAR
0239          FJ=FT
0240          GO TO 203
0241      283 V=A(2)+W/2.-ZBAR
0242          FJ=A(2)+W/2.
0243          GO TO 203
0244       54 FORMAT (2I2,4F5.2,F4.2,I4)
0245       53 FORMAT (I2,3F11.7)
0246       55 FORMAT (I2,4F11.7)
0247       56 FORMAT (4F11.7)
0248      290 FORMAT (/5X,18HNO PULSES REQUIRED)
0249      291 FORMAT (/5X,'NO PULSES REQUIRED FOR THE',
             C'INTERVAL 0<T<',F8.4)
0250      292 FORMAT (/5X,'NO PULSES REQUIRED FOR THE',
             C'INTERVAL',F8.4,'<T<',F8.4)
0251      284 STOP
0252          END
```

```
C
C       MONOTONIC FUNCTION SUBROUTINE
        SUBROUTINE MONO
        DIMENSION AI(3),XE(10),E(10)
        COMMON FT,U,MEXT,W,D,X,V,C,ALPHA
        COMMON /TON/IO,XE,E,N,IK,IP
        EXTERNAL FCT
        B=V-W
        C=W+D
        N=B/C+1
        WRITE(6,30)
        IF(IO.NE.2)GO TO 3
        IF(ABS(XE(1)).LT.ABS(XE(2)))GO TO 2
        GO TO 1
      3 IF(IK.GT.IP)GO TO 2
        IF(E(1))2,1,2
      1 DO 10 M=1,N
        DI=(M-1)*C
        DO 20 K=1,3
        U=K*X-2*X
        XL=DI
        XU=DI+W
        CALL QG10(XL,XU,FCT,Y)
        AI(K)=Y
     20 CONTINUE
        IF(MEXT)4,5,4
      4 STORE=AI(1)
        U=-X
        IF(STORE.LT.AI(3))GO TO 8
        STORE=AI(3)
        U=X
      8 IF(STORE.LT.AI(2))GO TO 6
        STORE=AI(2)
        U=0
        GO TO 6
      5 STORE =AI(1)
        U=-X
        IF (STORE.GT.AI(3))GOTO 7
        STORE =AI(3)
        U=X
      7 IF(STORE.GT.AI(2))GO TO 6
        STORE=AI(2)
        U=0
      6 WRITE(6,40)DI,U,STORE
     10 CONTINUE
        GO TO 9
      2 DO  50 M=1,N
        DI=FT+D-M*C
        DO 60 K=1,3
        U=K*X-2*X
        XL=DI
        XU=DI+W
        CALL QG10 (XL,XU,FCT,Y)
        AI(K)=Y
     60 CONTINUE
        IF(MEXT)11,12,11
     11 STORE=AI(1)
        U=-X
```

```
0057          IF(STORE.LT.AI(3))GO TO 13
0058          STORE=AI(3)
0059          U=X
0060       13 IF(STORE.LT.AI(2))GO TO 14
0061          STORE=AI(2)
0062          U=0
0063          GO TO 14
0064       12 STORE=AI(1)
0065          U=-X
0066          IF(STORE.GT.AI(3))GO TO 15
0067          STORE=AI(3)
0068          U=X
0069       15 IF(STORE.GT.AI(2))GOTO 14
0070          STORE=AI(2)
0071          U=0
0072       14 WRITE(6,40)DI,U,STORE
0073       50 CONTINUE
0074       30 FORMAT(/20X,25H TIME OF APPLICATION(SEC),
             C20X,15H PULSE MAGNITUDE,10X,6HAI*(U))
0075       40 FORMAT(30X,F10.5,30X,F5.2,10X,F11.7)
0076        9 RETURN
0077          END
```

```
C
C       SHIFT SUBROUTINE
        SUBROUTINE SHIFT
        DIMENSION DELTA(80),AIDELT(80),AISTAR(20),
       CUM(80,20),NI(80)
        COMMON FT,U,MEXT,W,D,X,V,C,ALPHA
        COMMON /IFT/ZBAR,FJ,LD,DSTAR,NISTAR
        EXTERNAL FCT
        B=V-W
        C=W+D
        N=B/C+1
        KC=LD*C+1
        WRITE(6,60)
        AIDSTA=0.0
        DO 10 J=1,KC
        AJ=J-1
        ALD=LD
        DELTA(J)=AJ/ALD
        AIDELT(J)=0.0
        DO 20 M=1,N
        AISTAR(M)=0.0
        DI=DELTA(J)+ZBAR+(M-1)*C
        UM(J,M)=0.0
        DO 30 K=1,3
        U=K*X-2*X
        XL=DI
        XU=DI+W
        CALL QG10 (XL,XU,FCT,Y)
        AI=Y
        IF(MEXT) 1,2,1
      1 IF(AISTAR(M).LT.AI) GO TO 30
        GO TO 31
      2 IF(AISTAR(M).GT.AI) GO TO 30
     31 AISTAR(M)=AI
        UM(J,M)=U
     30 CONTINUE
     20 CONTINUE
        IF(XU.LE.FJ)GO TO 3
        N=N-1
      3 NI(J)=N
        DO 40 M=1,N
        AIDELT(J)=AIDELT(J)+AISTAR(M)
     40 CONTINUE
        IF(MEXT)4,5,4
      4 IF(AIDSTA.LT.AIDELT(J))GO TO 10
        GO TO 11
      5 IF(AIDSTA.GT.AIDELT(J))GO TO 10
     11 AIDSTA=AIDELT(J)
        NISTAR=NI(J)
        DSTAR=DELTA(J)
        JSTAR=J
     10 CONTINUE
        DO 50 M=1,NISTAR
        POSIT=DSTAR+ZBAR+(M-1)*C
        USTAR=UM(JSTAR,M)
        WRITE(6,70) POSIT,USTAR
     50 CONTINUE
        WRITE(6,80) DSTAR,NISTAR,AIDSTA
```

```
0056              60 FORMAT(/20X,25H TIME OF APPLICATION(SEC),
                    C20X,16H PULSE MAGNITUDE)
0057              70 FORMAT(30X,F10.5,30X,F5.2)
0058              80 FORMAT (5X,'DELTA*=',F8.4,15X,'NUMBER OF ',
                    C'PULSES=',I2,15X,'I*(U)=',F11.7)
0059                 RETURN
0060                 END
```

```
C
C     SUBROUTINE  INTERSECTION
      SUBROUTINE  INTERS(IA,IB)
      COMMON FT,U,MEXT,W,D,X,V,C,ALPHA
      COMMON /SEC/A(2),K
      A(1)=0.
      A(2)=0.
      DO 20 JJ=1,2
      DO 10 II=IA,IB,1
      S=.001*II-.001
      PNT=EXP(.5*(S-FT))*SIN((1.7321/2.)*(S-FT))
      IF (PNT)1,10,2
    1 BETA=-ALPHA
      IF(ABS(PNT-BETA).GT.0.0200)GO TO 10
      A(JJ)=S
      GO TO 21
    2 IF(ABS(PNT-ALPHA).GT.0.0200)GO TO 10
      A(JJ)=S
      GO TO 21
   10 CONTINUE
   21 IA=II+40
   20 CONTINUE
      WRITE(6,30) A(1),A(2)
      IF(A(1))4,3,4
    3 IF(A(2))8,7,8
    4 IF(A(2))9,8,9
    7 K=0
      GO TO 11
    8 K=1
      GO TO 11
    9 K=2
   11 WRITE(6,31)K
   31 FORMAT(/10X,2HK=I1)
   30 FORMAT(/5X,5HA(1)=F8.4,10X,5HA(2)=F8.4)
      RETURN
      END
```

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```
            C
            C        SUBROUTINE FCT
0001                 FUNCTION FCT(T)
0002                 COMMON FT,U,MEXT,W,D,X,V,C,ALPHA
0003                 FCT=1.154*U*EXP(.5*(T-FT))*SIN((1.7321/2.)
                    C*(T-FT))+ALPHA*U*U
0004                 RETURN
0005                 END
```