ON SINGULAR TIME- AND FUEL-OPTIMAL SOLUTIONS

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ABSTRACT

The time-optimal control processes are considered for a class of linear time-invariant systems which are singular. It is shown that singularity occurs when there is a cancellation in the system transfer function. For single-input systems, a unique solution exists which is bang-bang if the controllable states are constrained. A generalized non-unique bang-bang control law is shown to exist for multi-input systems.

For fuel-optimal control processes, it is proved that for linear time-invariant systems, singular controls cannot be optimal. In the case of non-linear systems, optimal singular controls may occur quite often, in particular, when the problems considered are subjected to non-linear friction forces.

Conditions characterizing singular problems are derived and several examples are presented.

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CHAPTER 1

GENERAL INTRODUCTION

The structure and synthesis of optimal control processes is the subject of much current research. The Pontryagin's Maximum (Minimum) Principle (PMP) [28] has been one of the main tools in these studies. This principle makes it possible to find optimal control steering functions for a wide class of optimal processes in terms of a linear differential equation called the adjoint system of the original system equation.

A difficulty arises when the control variable enters linearly in the system equation or the index of performance. In such problems, situations may occur when PMP fails to provide effective optimality conditions. The corresponding Hamiltonian function ceases to be an explicit function of the control variables and thus yields no information about the desired optimal control. This class of problems is referred to as "singular" problems or "non-normal" problems , in the sense of LaSalle.

In this thesis, the necessary and sufficient (if possible) conditions are established for the existence of singular problems. The optimization under study is time-optimal, fuel-optimal and time-weighted fuel-optimal problems.

REVIEW OF LITERATURE

In the earlier literature of the calculus of variations, curves with singular subarcs received little analytical attention except to be explicitly excluded from the hypothesis of most of the established theorems. Evidently the concept "singular control" is of a very recent origin.

LaSalle[20] (1960) observes the computational difficulties in constructing optimal solutions to a class of time-optimal problems, for which there exist many control laws, each equally optimal. He establishes a set of theorems which characterize a normal and a proper system.

At this stage, a carefully expressed definition of the term "singular" extremals does not appear to exist in the literature. Haynes and Hermes[13] (1963) put forward a general definition of the singular problem using the associated Pfaffian system approach. They are concerned only with problems which are mildly nonlinear.

Following their works, a number of contributions (Kelly [16 , 17], Snow [30], Kopp and Moyer [18], Johnson et al. [14,15], Athans et al. [4, 5], Thau [31], and Goh [8, 9]) have appeared which throw considerable light on the difficulties of these problems.

Kelly [17] produces a set of necessary conditions for singular arcs via the second variation of the function to be minimized. This set of necessary conditions is the generalized Clebsch (Legendre) necessary condition:

$$\frac{\partial}{\partial u}\left[\frac{d^k}{dt^k}\left(\frac{\partial H}{\partial u} \right) \right] \geqq 0 \qquad (1.1)$$

$$k=2,4,\ldots\ldots\ldots\ldots$$

Kopp and Moyer [18] extended this result by considering a specially chosen class of perturbations on the singular control u_s. They have shown that, for a special class of piecewise continuous perturbations the second variation of the index of performance (IP) along singular arcs is strictly positive if and only if the inequality in (1.1) holds along the singular arc.

Johnson et al work out a class of singular problems to illustrate the significance of singular solutions. Goh extends the test of singular extremals for conventional Bolza problems. In a series of papers he indicates the procedures for the derivation of singular extremals.

Most of the works mentioned above consider problems in which the IP is quadratic .

The works of Thau, Snow and Athans et al. are closest to the material presented in this thesis . Some of their theorems are discussed, amplified, and extended to cover the problems considered here.

CHAPTER 2

SINGULAR EXTREMALS IN THE PROBLEMS OF OPTIMAL CONTROL

The main objective of this chapter is to review and discuss the fundamental relationships existing between the extremal of PMP, the solution of the adjoint system equation induced by PMP and the max or min Hamiltonian scalar function. During the course of discussion a scheme is suggested for classifying extremals of PMP.

It is now necessary to formulate the control process to establish a quantitative basis for discussion of the material to follow.

The dynamical system which is controlled is assumed to satify the following vector differential equation

(2.1) $$\dot{\underline{x}}(t) = \underline{f}\,[\underline{x}(t);\underline{u}(t)]$$

where $\underline{x}(t)$ is a vector with n components representing the state of the system at time t, and $\underline{u}(t)$ is a vector with r components representing the control input to the system at time t and $\underline{f}[\underline{x}(t);\underline{u}(t)]$ is a function of $\underline{x}(t)$ and $\underline{u}(t)$. For the remainder of this thesis, $\underline{x}$ and $\underline{u}$ or other symbols are functions of t explicitly unless otherwise stated.

For reasons of mathematical expediency, it is assumed that

(a) the function $\underline{f}$ is continuous,

(b) for each fixed $(t,\underline{u})$ the function $\underline{f}$ is differentiable and its derivatives are continuous.

The precise statement of optimization is as follows:

Given the system (2.1), the boundary conditions $\underline{x}(t_o)$, the constraint set Ω(where Ω is a set of r-dimensional space of the control variables $\underline{u}$, usually closed, bounded and convex), the target set S and the IP

(2.2) $$J(\underline{u}) = \int_{t_o}^{T} L(\underline{x},\underline{u})\, dt$$

then find the control $\underline{u}$ that

(a) satisfies the constraint $\underline{u} \varepsilon \Omega$

(b) transfers the state of the system from $\underline{x}(t_o)$ of (2.1) to $\underline{x}(T)$ so that $\underline{x}(T) \varepsilon S$ and in so doing

(c) minimizes the IP (2.2)

The search for the optimal control u* is facilitated via the PMP which will be stated.

PONTRYAGIN'S MAXIMUM (MINIMUM) PRINCIPLE

If $\underline{u}^*$ is the optimal control and $\underline{x}^*$ is the generated optimal trajectory, then corresponding to $\underline{u}^*$ and $\underline{x}^*$ there exists a co-state (adjoint) vector p* such that the following relationships hold:

(a) Canonical equations

(2.3) $$\dot{x}_i = \frac{\partial H}{\partial p_i}\bigg|_* \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}\bigg|_*$$

where H is the Hamiltonian function given by

$$H(\underline{x},\underline{p},\underline{u}) = L(\underline{x},\underline{u}) + \langle \underline{p}, \underline{f}(\underline{x},\underline{u}) \rangle$$

and $|_*$ means the partial derivatives must be evaluated at the optimal values.

(b) Boundary conditions

$$\underline{x}^*(t_o) = \underline{\xi}$$

$$\underline{x}^*(T) \in S$$

$$\underline{p}^*(T) \perp S$$

(c) Minimization of the Hamiltonian

(2.4) $$H^\circ = \min_{u \in \Omega} H(\underline{x}^*,\underline{p}^*,\underline{u}) = H(\underline{x}^*,\underline{p}^*,\underline{u}^*) \leq H(\underline{x}^*,\underline{p}^*,\underline{u})$$

for every t, $t_o \leq t \leq T$ and $\forall$ $\underline{u}$.

The proof of the PMP can be found in [28].

The conditions provided by PMP are local in nature and in general not sufficient.

Athans and Falb [4] have shown for $u \leq 1$ and $\underline{x}^*(t_o) = \underline{\xi}$ and $\underline{x}^*(T) = \underline{0}$ the optimal control law is given by

(2.5) $$\underline{u}^* = -\text{SGN}^\dagger \{B^T \underline{p}^*\}$$
$$\equiv -\text{SGN} \{\underline{q}^*\} \qquad \text{for time-optimal}$$

(2.6) $$\underline{u}^* = -\text{DEZ}^\dagger \{B^T \underline{p}^*\}$$
$$\equiv -\text{DEZ} \{\underline{q}^*\} \qquad \text{for fuel-optimal}$$

These equations provide a unique solution to time- or fuel-optimal problems if $q_j^* \neq 0$ in time-optimal and $|q_j^*| \neq 1$ in fuel-optimal problems over any finite interval of positive length. Such problems are called normal. If, however, $\underline{q}^* \equiv 0$ in

† The signum and deadzone functions are defined in pp. 380 and 438 of Athans and Falb [4].

time-optimal or $|q_j^*| \equiv 1$ in fuel-optimal problems over some finite interval of positive length, the signum and the deadzone functions are not defined, and accordingly an arbitrary $|u| \leq 1$ will satisfy PMP. The control problems are called non-normal in the sense of LaSalle, or totally singular.

With these preliminary facts in mind, a clear-cut definition of singularity for a class of control processes is advanced.

Before attempting this, the following definition and theorem of Lebesgue measure are reviewed.

DEFINITION 1

Suppose $M \in E_1$. By a Lebesgue covering of a set M, it is meant a countable sequence $I = \{I_1, I_2, \ldots\ldots\}$ of open intervals which covers M. If $L(I_k)$ is the length of I_k, covering L(I) is defined to be the number

$$L(I) = \sum_{k=1}^{\infty} L(I_k)$$

whenever the series on the right converges. The number

$$m(M) = \inf\{\, L(I) \mid I \text{ is a Lebesgue covering of } M\}$$

is called the Lebesgue measure of M.

When M is bounded, say $M \subset [a, b]$, then

$$0 \leq m(M) \leq b-a$$

If $m(M) = 0$, M is said to be a set of measure zero.

THEOREM 1.

If F is a countable collection of sets in E_1 say

$$F = \{F_1, F_2, \ldots\ldots\ldots F_n\}$$

such that $m(F_k) = 0$ for each k, let

$$M = \bigcup_{k=1}^{\infty} F_k$$

then $m(M) = 0$.

DEFINITION 2.

For each value j, j=1,2,.......r, designate

$$q_j = \sum_{i=1}^{n} p_i \, b_{ij}$$

to define a switching function and denote the set

$$\Gamma(k) = \{\underline{p}, \underline{x} \mid q_k = 0 \text{ (time-optimal) or } |q_k| - 1 = 0 \text{ (fuel-optimal)}\}$$

and denote by Γ the set

$$\Gamma = \bigcup_{k=1}^{r} \Gamma(k)$$

then an extremal $(\underline{p}, \underline{x})$ given on an interval I is called "totally singular" if the set

$$\beta = \{ t \mid t \in I \text{ and } \underline{p}, \underline{x} \in \Gamma \}$$

have positive Lebesgue measure in I, where I denotes the time interval $[t_o, T]$

In other words, a totally singular arc or subarc occurs whenever the inequality in (2.4) becomes an equality over a finite interval of time and H thus becomes independent of $\underline{u}$. If for example H is linear in u_k, $\frac{\partial H}{\partial \bar{u}_k}$ vanishes over a finite interval; the PMP fails to select the optimal control in this case. It is clear that if a $\underline{u}^*$ exists ,it must satisfy the requirement that the system remains on a path such that $\frac{\partial H}{\partial u_k} = 0$.

In general, the determination of whether or not a totally singular control satisfies the necessary conditions of PMP is not a difficult task; the difficulty is to prove that this extremal is optimal.

In this thesis, a distinction is made between a singular and a totally singular problem." By total singularity" it is meant that the PMP fails to provide any information about all the control variables. If one or more control variables are defined, then the problem is said to be singular.

CHAPTER 3

TIME-OPTIMAL SINGULAR PROBLEMS

§1. Introduction

It is well known that if a control system is to be operated from a limited source of power, and if the transition time from some initial state to some final state is to be minimum, then the control strategy is to utilize all available power. This hypothesis [21] is called "The Bang-Bang Principle". For a normal time-optimal problem the control law is uniquely defined by this hypothesis. However, if the problem is singular, the validity of this hypothesis is open to conjecture. Moreover, the time-optimal solution(s) is non-unique even if the solution(s) exists. It is found that under certain conditions and restrictions, a generalized "Bang-Bang Principle" may be established with regard to the singular problem.

§2. The Singular Problem

The problem posed is to determine the control law which steers the system from some initial state to some final state $\underline{x}_F$ (in particular, the origin $\underline{0}$) in minimum time satisfying the dynamical system of the form

$$(3.1) \qquad \dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x}$$

where A is an nxn and B is an nxr constant matrix , with $\underline{u}$ constrained in magnitude by the relation

(3.2) $|u_j| \leq 1$

Treating the problem via the PMP method of optimization, the Hamiltonian is

(3.3) $H(\underline{x},\underline{p},\underline{u}) = 1 + \langle \underline{p}, A\underline{x} \rangle + \langle \underline{p}, B\underline{u} \rangle$

where $\underline{p}$ is the co-state vector satisfying the canonical equation

(3.4) $$\underline{p} = -\frac{\partial H}{\partial \underline{x}} = -A^T\underline{p}$$

and the control law is given by

(3.5) $$\underline{u} = -\mathrm{SGN}\{B^T\underline{p}\} = -\mathrm{SGN}\{\underline{q}\}$$

which can be written in component form as

(3.6) $$u_j = -\mathrm{sgn}\{\sum_{i=1}^{n} b_{ij} p_i\} = -\mathrm{sgn}\{ q_j \} \qquad j = 1,2,\ldots\ldots\ldots r$$

The condition for totally singular control occurs when $B^T\underline{p} \equiv 0 \ \forall \ t \in [t_1,t_2] \in [0,T]$; then $\underline{u}$ is arbitrary and may be optimal if $|\underline{u}| \leq 1$. If for some j, j=1,2,.....r, $q_j \equiv 0 \ \forall t \in [t_1,t_2] \in [0,T]$ then the problem is said to be singular. With these preliminaries the following definitions are made.

<u>DEFINITION 3.</u>Normal time-optimal problem

If for all j, j=1,2,.....r, $q_j \neq 0$ except at $t = t_{\gamma j}$ where $t_{\gamma j}$ is a countable set of times t_{1j}, t_{2j}, t_{3j} ....

$t_{\gamma j} \in [0,T]$ $\qquad \gamma=1,2,\ldots\ldots m$

$\qquad j=1,2,\ldots\ldots r$

then the problem is said to be normal time-optimal.

DEFINITION 4. Singular time-optimal problem

Suppose in the interval $I=[0,T]$ there is one or more proper subinterval $[t_1,t_2]_j$, $[t_1,t_2]_j \varepsilon [0,T]$ such that $q_j \equiv 0$ for all $t \varepsilon [t_1,t_2]$ and for some j, then the problem is said to be singular time-optimal. If the condition holds for all j then the problem is totally singular.

§ 3. Controllability and Existence of Optimal Control

It has been shown [4] that a necessary and sufficient condition that a linear system defined by (3.1) be completely controllable at time $t > t_o$ is that the matrix

(3.7) $\quad G= [\ B,\ AB,\ A^2B,\ldots\ldots\ldots\ldots A^{n-1}B]$ has a rank n for all $t > t_o$.

This condition can be shown to be equivalent to the statement that the matrix [28]

(3.8) $\quad W(t_o,t_1) = \int_{t_o}^{t_1} \Phi(t_o,t)BB^T\Phi(t_o,\ t)^T\ dt$

be positive definite for some time $t_1 \geq t_o$, where the superscript T denotes the transpose of the matrix. The matrix $\Phi(t,\ t_o)$ is the fundamental transition matrix of (3.1) with the following properties:

(3.9) $\dot{\Phi}(t,t_o) = A\Phi(t,t_o)$

$\Phi(t_o,t_o) = I$ the identity matrix

$\Phi(t,t_o)\Phi(t_o,t) = I$

and consequently $\Phi(t_o,t)$ satisfies the equation

(3.10) $\dot{\Phi}(t_o,t) = -\Phi(t_o,t)A$

From linear algebra, in order that the matrix $W(t_o,t_1)$ be positive definite at time t_1, the quadratic form

(3.11) $\underline{z}^T W(t_o,t_1)\underline{z}$ (where $\underline{z}$ is a constant vector) must be greater that zero for all $\underline{z} \neq 0$.

Substitute (3.11) into (3.8) this condition can be written as

(3.12) $$\int_{t_o}^{t_1} \underline{z}^T\Phi(t_o,t)BB^T\Phi(t_o,t)^T\underline{z} \;dt > 0 \quad \forall\ \underline{z}\neq 0$$

Now partition the B matrix into column vectors, i.e.

$$B = [\underline{b}_1\ \underline{b}_2\ \ldots\ldots\ldots\ldots\underline{b}_r]$$

or

$$B^T = \begin{bmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \underline{b}_r^T \end{bmatrix}$$

and

$$BB^T = \sum_{i=1}^{r} \underline{b}_i\underline{b}_i^T$$

and (3.12) reduces to

(3.13) $$\sum_{i=1}^{r} \left[\int_{t_o}^{t_1} \underline{z}^T\Phi(t\ ,t)\underline{b}_i\right]^2 dt > 0 \qquad \forall\ \underline{z} \neq 0$$

Note this condition implies that

(3.14) $\underline{z}^T\Phi(t_o,t)\underline{b}_i$ $\qquad$ i=1,2, ...r

are not identically zero over the interval $[t_o,t_1]$. For linear time invariant system there is no loss of generality if $t_o=0$ and $t_1 = T$.

LaSalle [20] in his treatment of time-optimal control systems discusses the concept of normal system and proper system. In the notation of the present formulation the following definitions are made.

DEFINITION 5.

A system is called proper if $\underline{z}^T\Phi(0,T)B = 0$ on an interval of positive length implies $\underline{z} = 0$, otherwise it is improper.

DEFINITION 6.

A system is called normal if for each i, $\underline{z}^T\Phi(0,T)\underline{b}_i = 0$ on an interval of non-zero length implies $\underline{z} = 0$ on that interval, otherwise it is non-normal or singular.

From these definitions it is obvious that every normal system is proper, but not every proper system is normal. Thus a proper system is equivalent to complete controllability of the system [20].

To relate the definition of normality of a linear system to the definition of a normal time-optimal control problem, take the transpose of (3.14) to obtain

(3.15) $[\underline{z}^T\Phi(0,T)\underline{b}_i] = 0$

Since z is any arbitrary vector, $\underline{z}$ may be viewed as the initial co-state vector $\underline{\pi}$ which is also arbitrary as will be shown later in this chapter.

Thus the system being normal is equivalent to the normality of time-optimal problem.

A criterion for testing a linear system described by (3.1) to be singular or totally singular is given by the following theorems.

THEOREM 1.

The time-optimal problem is totally singular if and only if the system is not completely controllable.

Proof: Assume the system is not completely controllable; then the matrix defined by (3.7) has a rank less than n. It follows that there exists a non-zero vector $\underline{z}$ such that

(3.16) $\underline{z}^T G = 0^T$

$\underline{z}^T [B, AB, \ldots\ldots\ldots\ldots A^{n-1}B] = 0^T$

or $\underline{z}^T B = \underline{z}^T AB = \ldots\ldots\ldots\ldots = \underline{z}^T A^{n-1} B = 0^T$

By the Cayley-Hamilton Theorem matrix A satisfies its own characteristic equation, i.e.

$$A^n = \sum_{i=0}^{n-1} c_i A^i$$

for certain real numbers c_i, i = 0,1,.........n-1.

Thus

$$\underline{z}^T A^n B = \underline{z}^T \sum_{i=0}^{n-1} c_i A^i B = 0^T$$

By induction

$$\underline{z}^T A^{n+k} B = 0 \qquad \forall\ k = 0, 1, 2, \ldots\ldots\ldots\ldots$$

Therefore

(3.17) $$\underline{z}^T e^{-At} B = \underline{z}^T [I - At + \frac{A^2 t^2}{2!} - \ldots\ldots\ldots\ldots] B = 0^T$$

for all real t.

Since $\underline{z}$ is non-zero (3.17) implies that

$$B^T e^{-At} \underline{z} = 0 \quad \text{or}$$

$$B^T \underline{p} = 0.$$

Hence the time-optimal problem is totally singular.

Assume the time-optimal problem to be totally singular.

By definition

(3.18) $$\underline{q} = B^T \underline{p} = 0 \qquad \forall\ t\ \varepsilon\ [t_1, t_2]$$

It follows that $\underline{q}$ and its first n-1 time derivatives are zero, i.e.

(3.19) $$\begin{aligned} B^T \underline{p} &= 0 \\ B^T A^T \underline{p} &= 0 \\ B^T A^{T^2} \underline{p} &= 0 \qquad \forall\ t\ \varepsilon\ [t_1, t_2] \\ \ldots\ldots &\ \ldots \\ B^T A^{T^{n-2}} \underline{p} &= 0 \\ B^T A^{T^{n-1}} \underline{p} &= 0 \end{aligned}$$

Denoting G as the matrix defined (3.7), (3.19) can be written as

(3.18) $$G^T \underline{p} = 0$$

Since $\underline{p} \neq 0$, a singular G^T => the system is uncontrollable.

A conjecture that one might be tempted to make is that if the system is completely controllable, then it admits no totally singular extremals. This is not true as shown in Example 2. However, it may be asserted that totally singular extremals cannot be optimal. This is because G has a rank= n, in which case $\underline{z}$ has to be zero. But by PMP, the extremals are not optimal.

In order to prove the next theorem, it is assumed that the matrix is of simple structure [33]. In other words, the eigenvectors associated with each eigenvalue of A are linearly independent. If A has distinct eigenvalues or if A has indistinct eigenvalues which can be linked in a Jordan chain, then their eigenvectors are linearly independent, and they constitute a basis for the state space.

THEOREM 2.

Assume A is of simple structure . The time-optimal problem is singular if and only if for some j, j=1,2,.....r the matrix given by

(3.21) $$G_j = [\underline{b}_j \; A\underline{b}_j \; \ldots\ldots\ldots\ldots \; A^{n-1}\underline{b}_j]$$

is singular.

Proof: Necessity-Mimic the proof of Theorem 1 using (3.18) and re-writing it as

(3.22) $$\underline{q}_j = \langle \underline{b}_j, \underline{p} \rangle = 0$$

to show G_j has a rank less than n.

Sufficiency: Since A is of simple structure, G_j can be written as [26]

(3.23) $$[Ib_{kj} \quad (A-\lambda_k I)b_{kj} \ldots\ldots\ldots\ldots (A-\lambda_k I)^{n-1} b_{kj}]$$

where λ_k is the kth eigenvalue and $k = 1,2,\ldots\ldots\ldots n$ and b_{kj} is the row component of the column vector b_j .

Since G_j has a rank less than n, it implies that there exist certain real numbers c_i, $i=0,1,\ldots\ldots\ldots,n-1$ such that

(3.24) $$c_o I b_{kj} + c_1(A-\lambda_k I)b_{kj} + \ldots\ldots\ldots + c_{n-1}(A-\lambda_k I)^{n-1} b_{kj} = 0$$

i.e. $$[c_o I + c_1(A-\lambda_1 I) + \ldots\ldots\ldots + c_{n-1}(A-\lambda_1 I)^{n-1}] b_{1j} = 0$$

$$[c_o I + c_1(A-\lambda_2 I) + \ldots\ldots\ldots + c_{n-1}(A-\lambda_2 I)^{n-1}] b_{2j} = 0$$

$$\cdot\cdot \qquad \cdot\cdot \qquad \cdots\cdots\cdots \qquad \cdot\cdot \qquad \cdot\cdot \qquad \cdot\cdot$$

$$[c_o I + c_1(A-\lambda_n I) + \ldots\ldots\ldots + c_{n-1}(A-\lambda_n I)^{n-1}] b_{nj} = 0$$

By assumption , the matrices $(A-\lambda_i I)^i$, $i=0,1, \ldots\ldots n-1$ are linearly independent, the coefficients are uniquely determined: unique in the sense that n-1 of them can be solved in terms of the remaining one.

The n equations [27] which determine the coefficients c_i can be written in matrix form as

(3.25) $$\begin{bmatrix} F(\lambda_1) \\ F(\lambda_2) \\ \cdot \\ F(\lambda_n) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots\ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots\ldots & \lambda_2^{n-1} \\ \cdot & & & \ldots\ldots & \\ 1 & \lambda_n & \lambda_n^2 & \ldots\ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_o \\ c_1 \\ \cdot \\ c_{n-1} \end{bmatrix} = 0$$

or $$\underline{F}(\lambda\ ) = \underline{\Lambda}\ \underline{c}$$

Therefore

(3.26) $\underline{c} = \Lambda^{-1} \underline{F}(\lambda)$

Since $e^{-\Lambda} = \sum_{i=0}^{n-1} \alpha_i \Lambda^i$, the coefficients α_i are precisely those c_i when one computes $e^{-\Lambda}$ by Sylvester's Theorem [26].

Thus $e^{-\Lambda t}\underline{b}_j = 0$ and for any non-zero $\underline{z}$, $\underline{z}^T e^{-\Lambda t}\underline{b}_j = 0 \forall t \in [t_1, t_2]$.

Therefore the time-optimal problem is singular.

Note that the above proof uses the fact that the eigenvaules are distinct. The same result can be obtained if the eigenvalues are indistinct, provided that they can be linked in a Jordan chain.

For properties of matrix A other than that A is of simple structure, (3.21) is, in general, insufficient to guarantee that the time-optimal problem is singular. In such problems(e.g. A is singular), there exist some initial states for which the time-optimal is normal, while for other initial states, it is singular.

The fact that G or G_j is independent of $\underline{x}$, $\underline{p}$, $\underline{u}$, and t implies that a time-optimal problem is singular or totally singular $\forall$ t, t$\in [t_1,t_2] \in [0,T]$, then it is singular or totally singular for all t, t >0. The verification of this statement may be made by invoking the Principle of Optimality [4], which states that any portion of the optimal trajectory is also optimal.

Regarding the existence of singular optimal control, a few facts and the terminology of linear algebra are used to provide a framework to discuss the results which are to follow.