PARTIAL LATTICES

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ABSTRACT

If we consider a subset of a lattice and the lattice operations restricted to that subset, the resulting algebraic structure is not necessarily a lattice, but instead a partial algebra which is called a partial lattice. Partial lattices are of interest because their study solves certain problems in lattice theory.

This thesis began as a review of three papers on partial lattices. The first was an attempt by Yu. I. Sorkin, Dokl. Akad. Nauk SSSR 95 (1954), 931 to establish a system of identities that would characterize partial lattices. Using the paper of N. Funayama (3) we realize that Sorkin's result is in error. Chapter one is an extension of Funayama's results which gives a minimal system of identities to characterize partial lattices.

The second paper reviewed is "On the problem of isomorphism of lattices" by M. M. Gluhov (5) in which Gluhov relied heavily on the incorrect identities given in Sorkin's paper. Chapter two characterizes the free extension of a partial lattice using the identities of Chapter One. Unfortunately, the final result that a partial lattice has a unique basis could not be proven although it is believed to be true.

The third paper reviewed, "On a lattice-theoretical theorem of a kind similar to Grusko's theorem" by M. M. Gluhov (4) studies the free product of lattices. The main result

is that if a lattice is a free product of k lattices, each with a finite number of generators, then the minimum number of generators of the lattice is equal to the sum of the corresponding number of generators of the k free factors.

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CHAPTER ONE

Let $\mathcal{L}$ be a lattice. We may consider subsets of L and the lattice operations restricted to this subset. If we have chosen a sublattice, then the resulting algebraic structure is a lattice. But, in general, when we consider a subset of the lattice, the restricted operations are merely partial operations and this algebraic structure is called a <u>partial lattice</u>.

Funayama (3) characterized partial lattices but did not present a minimal system of defining equations for partial lattices. The object of Chapter One is to extend Funayama's results.

<u>Definition</u>. Let $\mathcal{L} = \langle L; \underset{L}{\vee}, \underset{L}{\wedge} \rangle$ be a lattice. We consider P, a subset of L, and define two partial binary operations, $\vee$ and $\wedge$ on P as follows:

if $a, b \in P$

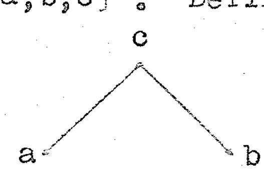
1) $a \vee b$ exists if and only if $a \underset{L}{\vee} b \in P$ and then

$a \vee b = a \underset{L}{\vee} b$

2) $a \wedge b$ exists if and only if $a \underset{L}{\wedge} b \in P$ and then

$a \wedge b = a \underset{L}{\wedge} b$ then $\mathcal{P} = \langle P; \vee, \wedge \rangle$ is called a <u>partial lattice</u>.

<u>Example</u>. Let P be the three element set $\{a, b, c\}$. Define $a \vee b = c$. Do not define $a \wedge b$.



We shall now define two partial orderings on P.

Define a $\leq_J$ b if and only if a $\vee$ b exists and a $\vee$ b = b.

a $\leq_M$ b if and only if a $\wedge$ b exists and a $\wedge$ b = a.

We shall now prove that these two partial orderings are equivalent.

<u>Lemma</u>. a $\leq_J$ b if and only if a $\leq_M$ b.

<u>Proof</u>. Identify $\leq_J$ and $\leq_M$ with the existing partial order in the lattice L.

Define lub $\{a,b\}$ = c if and only if a $\leq$ c, b $\leq$ c, and if there exists d such that a $\leq$ d, b $\leq$ d, then c $\leq$ d.

In a similar manner we can define glb $\{a,b\}$ = c if and only if c $\leq$ a, c $\leq$ b, and if there exists d such that d $\leq$ a, d $\leq$ b, then d $\leq$ c.

<u>Imbedding Theorem for Partial Lattices</u>

It is known that any partially ordered set can be embedded in a complete lattice preserving the inclusion relation and all glbs and lubs.

Consider an algebraic structure with two binary partial operations $\vee$ and $\wedge$, (P; $\vee$,$\wedge$). If (P; $\vee$,$\wedge$) satisfies the following eight identities then it is called a <u>weak partial lattice</u>:

1) $a \vee a = a$

2) $a \wedge a = a$

3) $a \vee (b \vee c) = (a \vee b) \vee c$

4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

5) $a \vee b = b \vee a$

6) $a \wedge b = b \wedge a$

7) $a \vee (a \wedge b) = a$

8) $a \wedge (a \vee b) = a$

The above identities are read as follows: if the left hand side exists and if the inner segment of the right hand side exists then the entire right hand side exists and equals the left hand side.

Example. If $a \vee (b \vee c)$ and $a \vee b$ exist, then $(a \vee b) \vee c$ exists and $a \vee (b \vee c) = (a \vee b) \vee c$.

Note that in the terminology of Funayama such an algebraic structure is called a partial lattice. What we call a partial lattice, Funayama called a strong partial lattice.

It is clear that any partial lattice is a weak partial lattice. Recall that a partial order on an algebraic structure $(P; \vee, \wedge)$ was defined as follows:

for all $a, b \in P$

$a \leq_J b$ if and only if $a \vee b$ exists and $a \vee b = b$.

$a \leq_M b$ if and only if $a \wedge b$ exists and $a \wedge b = a$.

Lemma. $a \leq_J b$ if and only if $a \leq_M b$ in a weak partial lattice.

Proof. Assume $a \leq_J b$. Then $a \vee b$ exists and $a \vee b = b$. Thus $a \vee b$ exists and so $a = a \wedge (a \vee b) = a \wedge b$. Therefore $a \leq_M b$.

Assume $a \leq_M b$. Then $a \wedge b$ exists and $a \wedge b = a$.

Thus $b = b \vee (a \wedge b) = b \vee a$.

Therefore $a \leq_J b$.

Define $a \leq b$ if and only if $a \leq_J b$ or if $a \leq_M b$.

To clarify the idea of imbedding theorems of algebraic structures we state the following:

<u>Definition</u>. If φ is a one-to-one mapping from an algebraic structure $(P; \vee, \wedge)$ into a lattice L, then φ is said to be a <u>weak embedding</u> of P into L if

$x \vee y = z$ in P implies that $\varphi(x) \vee \varphi(y) = \varphi(z)$ in L.

$x \wedge y = z$ in P implies that $\varphi(x) \wedge \varphi(y) = \varphi(z)$ in L.

is said to be a <u>strong embedding</u> of P into L if in addition to the above

$\varphi(x) \vee \varphi(y) = \varphi(z)$ in L implies that $x \vee y = z$ in P.

$\varphi(x) \wedge \varphi(y) = \varphi(z)$ in L implies that $x \wedge y = z$ in P.

<u>Example</u>. Let $P = \{a, b, c\}$ be an algebraic structure $(P; \vee, \wedge)$ defined by $a \vee b = a$ and $a \vee c = a$. Let $L = \{\varphi(a), \varphi(b), \varphi(c), 0\}$ be a four element lattice defined by $\varphi(b) \vee \varphi(c) = \varphi(a)$. Then φ embeds P into L in the weak sense but not in the strong sense.

<u>Definition</u>. I is an <u>ideal</u> of a partial lattice P if I is a subset of P satisfying the two conditions:

(i) $x \in I$, $y \leq x$, implies that $y \in I$;

(ii) if $x, y \in I$ and $x \vee y$ is defined then $x \vee y \in I$.

I is a <u>prime ideal</u> if in addition

(iii) $x \wedge y$ exists and is in I implies that $x \in I$ or $y \in I$.

<u>Definition.</u> D is a <u>dual ideal</u> of a partial lattice P if D is a subset of P satisfying the two conditions:

(i) $x \in D$, $y \geq x$, implies that $y \in D$;

(ii) if $x, y \in D$ and $x \wedge y$ exists then $x \wedge y \in D$.

D is a <u>prime dual ideal</u> if in addition

(iii) $x \vee y$ exists and is in D then $x \in D$ or $y \in D$.

We shall establish a partial order on ideals and dual ideals of a partial lattice.

If I_1, I_2 are ideals of a partial lattice then $I_1 \leq I_2$ if and only if $I_1 \subseteq I_2$.

If D_1, D_2 are dual ideals then $D_1 \leq D_2$ if and only if $D_1 \supseteq D_2$.

We shall now establish a minimal system of identities on an algebraic structure $(P; \vee, \wedge)$ such that it can be embedded strongly in a lattice.

Iv) $a \vee a = a$	I∧) $a \wedge a = a$
Av) $a \vee (b \vee c) = (a \vee b) \vee c$	A∧) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
Cv) $a \vee b = b \vee a$	C∧) $a \wedge b = b \wedge a$
D^1v) $a \vee (a \wedge b) = a$	D^1∧) $a \wedge (a \vee b) = a$
D^2v) $(a \wedge b) \vee a = a$	D^2∧) $(a \vee b) \wedge a = a$
D^3v) $a \vee (b \wedge a) = a$	D^3∧) $a \wedge (b \vee a) = a$
D^4v) $(b \wedge a) \vee a = a$	D^4∧) $(b \vee a) \wedge a = a$
Pv) $(a] \vee (b] = (c]$ implies that $a \vee b$ exists and $a \vee b = c$	P∧) $[a) \wedge [b) = [c)$ implies that $a \wedge b$ exists and $a \wedge b = c$

The above identities are to be read as before.

Which of the above identities form a minimal system for the embedding of an algebraic structure into a lattice?

<u>Theorem.</u> $\sum_{v}^{1} = \{$Iv), Av), A∧), Cv), C∧), D^1v), D^1∧), Pv), P∧)$\}$

$\sum_{\wedge}^{1} = \{$I∧), Av), A∧), Cv), C∧), D^1v), D^1∧), Pv), P∧)$\}$

Each of $\sum_{v}^{1}$ and $\sum_{\wedge}^{1}$ is a minimal system of identities which ensures that an algebraic structure $(P; \vee, \wedge)$ can be strongly embedded in a lattice L.

<u>Proof.</u> First, to show that $\sum$ is minimal.

Without loss of generality we will show that the system $\sum_{v}^{1}$ is minimal. To establish the minimality of $\sum_{v}^{1}$ we must exhibit nine algebraic structures each of which fails to satisfy one of the identities of $\sum_{v}^{1}$ but which satisfies the remaining eight identities of $\sum_{v}^{1}$.