THE USE OF LEGENDRE POLYNOMIALS
TO APPROXIMATE THE IDEAL
LOW PASS FILTER

A THESIS

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ABSTRACT

The expansion of the ideal normalized amplitude response of a low-pass filter in a series of Legendre polynomials was investigated. This series was then converted to a rational function by the use of a Padé approximant. The range of integration, K, was varied from 0 to 1. The effect of this was to shift the cut off frequency of the filter. For the second order case the Butterworth filter, Chebychev filter and the Bessel filter could all be obtained by a suitable choice for the range of integration.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

THE PROBLEM AND METHODS OF SOLUTION

I. The Problem

<u>The ideal low-pass filter</u>. Any coupling network has a transfer function which can be written in the following form.

$$T_{12}(j\omega) = |T_{12}(j\omega)| e^{j\beta(\omega)} \qquad (1)$$

If the magnitude and phase have the characteristics shown in equation 2 and 3, respectively, the coupling network is said to be an ideal low-pass filter.

$$|T_{12}(j\omega)|^2 = \begin{cases} 1 & 0 < \omega < \omega_c \\ 0 & \omega > \omega_c \end{cases} \qquad (2)$$

$$\beta(\omega) = -t_0 \omega \qquad 0 < \omega < \omega_c \qquad (3)$$

These characteristics are shown graphically in figures 1 and 2.

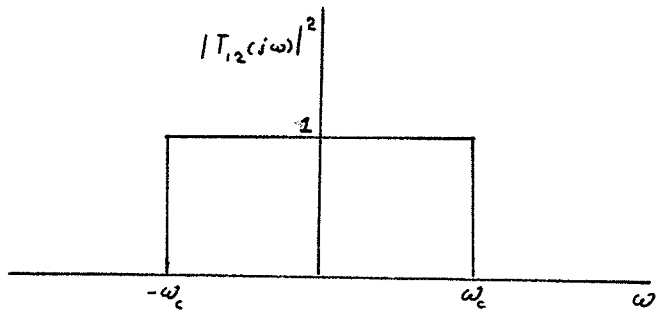

Figure 1

Magnitude Characteristic

of an Ideal Low-Pass Filter

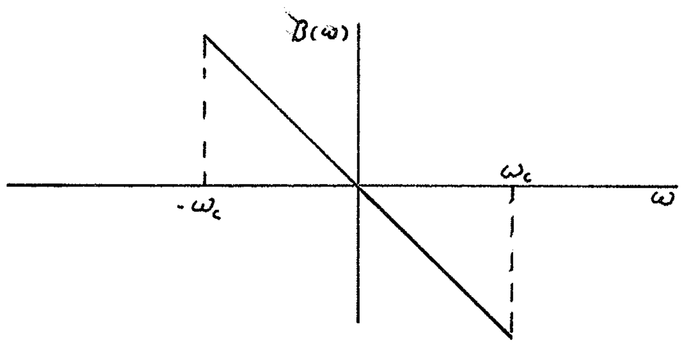

Figure 2

Phase Characteristic of an

Ideal Low-Pass Filter

The "time delay", $T_d(w)$, is defined by equation 4.

$$T_d(\omega) = -\frac{dB(\omega)}{d\omega} \tag{4}$$

The time delay for the ideal low-pass filter is obtained by differentiating equation 3.

$$T_d(\omega) = t_0 \qquad 0 < \omega < \omega_c \tag{5}$$

J. L. Stewart[1] developed an error criterion that includes both the phase and magnitude characteristics of a transfer function. This method could be used to approximate equations 2 and 5 simultaneously. The usual approach, however, is to approximate only one of these equations, depending on the given specifications. The approach taken in this work was to use the magnitude characteristic and then to examine the phase of the resulting transfer function.

<u>Realizability</u>. The transfer function of a network consisting of passive elements is said to be realizable. Therefore the transfer function resulting from some method of approximation must be realizable to be useful in practice. A transfer function is realizable if the denominator has no zeros in the right half s plane[2].

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1 J. L. Stewart, "Generalized Padé Approximation," <u>Proceedings I. R. E.</u>, Volume 48 (December, 1960) pp. 2003-2005.

2 N. Balabanian, <u>Network Synthesis</u> (Englewood Cliffs, N.J. Prentice-Hall, Inc. 1958) p. 143.

Polynomials whose zeros all lie in the left half s plane or are simple on the $j\omega$ axis are Hurwitz polynomials. The Hurwitz test[1] on the denominator of a transfer function determines whether or not the function is realizable.

II Methods of Solving The Problem

The common approach. The most common method of approximating the magnitude characteristic of an ideal low-pass filter is to assume a solution of the form.

$$|T_{12}(\omega)|^2 = \frac{1}{1 + \epsilon^2 F_n^2(\omega)} \qquad (6)$$

Where:

$$F_n^2(\omega) = \begin{cases} \ll 1 & \omega < 1 \\ \gg 1 & \omega > 1 \end{cases} \qquad (7)$$

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1 S. Karni, Network Theory: Analysis and Synthesis (Boston: Allyn and Bacon, Inc. 1966), pp. 109 - 114.

Many polynomials have been used to approximate $F_n(w)$. The most common in use today are, Butterworth[1], Chebychev[2], Legendre[3] and Hermite[4]. The use of Ultraspherical polynomials was examined by Johnson and Johnson[5] and was shown to include as special cases the Chebychev filter, the Butterworth filter, and also the associated Legendre filters.

The Bessel filter[6] is a good approximation of the phase characteristics of an ideal low-pass filter.

<u>The use of orthogonal polynomials</u>. Another method of approximating the magnitude characteristic of an ideal low-pass filter was shown by Karni[7]. The magnitude function, $|T_{12}(\omega)|^2$, was expanded directly in terms of orthogonal polynomials. The resulting approximating polynomial must be

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1 *Ibid*, pp. 344 - 347

2 *Ibid*, pp. 349 - 358

3 Sheila Prasad and others "Filter Synthesis Using Legendre Polynomials" *Proceedings I.E.E.* (Vol., 114 No. 8 August 1967), pp. 1 - 12.

4 Y. H. Ku and M. Drubin, "Network Synthesis Using Legendre and Hermite Polynomials". *J. Franklin Inst.* (February 1962) pp. 138 - 57.

5 D. E. Johnson and J. R. Johnson, "Low-Pass Filters Using Ultraspherical Polynomials", *IEEE Transactions on Circuit Theory*, Vol. CT - 13, No. 4,(December 1966), pp. 364 - 69.

6 S. Karni, *op. cit.*, pp. 370 - 372

7 *Ibid*, pp. 365 - 368

converted into a rational function. Karni suggests the use of a Padé approximation for this conversion.

The approach taken in this work was the use of Legendre polynomials to approximate the ideal magnitude characteristic and then using a Padé approximant to convert it into a rational function.

CHAPTER II

APPROXIMATION PROBLEM

The first step in network synthesis is the approximation of given ideal specifications. It may occur that the transfer function obtained in this manner is not realizable. If this is the case another approximation must be made until a realizable transfer function is obtained. This procedure is shown pictorially in figure 3.

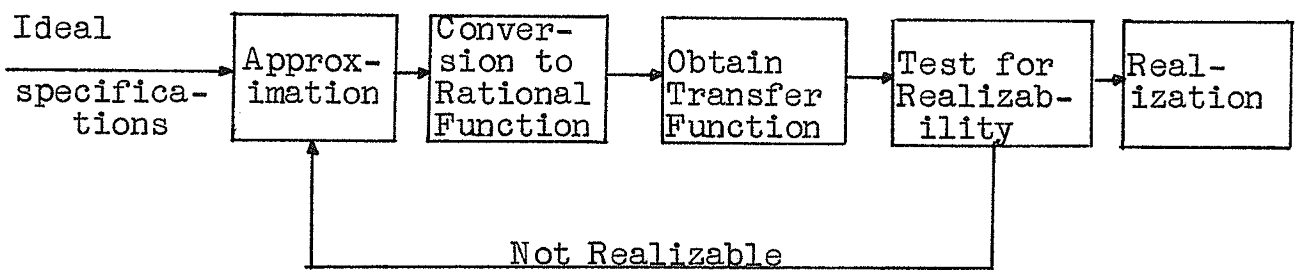

Figure 3

Steps In Network Synthesis

The "ideal specifications" were previously defined and now the method of approximation and conversion to a rational function will be discussed.

I. Function to be Approximated

The method of approximation is to expand the function into a series of orthogonal polynomials. The function to be approximated is the magnitude squared. Therefore it is necessary that no terms of the form $w^{(2p+1)}$ appear in the approximation. This is achieved by making the ideal specifications an even function as shown in figure 4. The orthogonal polynomials to be used are the Legendre polynomials. The range of integration is varied from 0 to 1. The effect of this is to vary the cut off frequency of the approximation. This cut-off frequency is scaled back after the rational function conversion is performed. The ideal specification to be approximated is shown in figure 4.

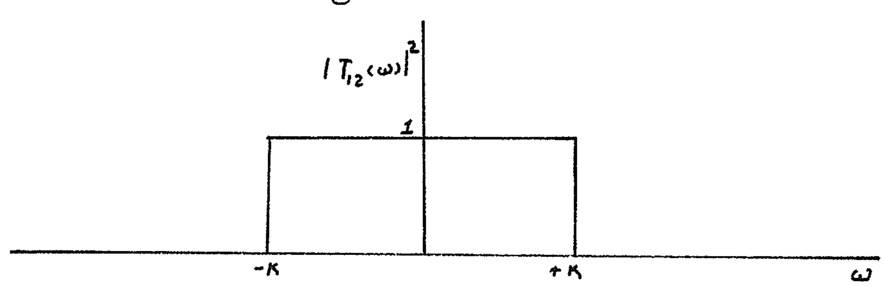

Figure 4

Magnitude Characteristic of
an Ideal Low-Pass Filter With
a Variable Cut-Off Frequency

II Legendre Approximation

The expansion for any function in terms of orthogonal polynomials is given by.[1]

$$|T_{12}(\tilde{\omega})|^2 = \sum_{L=0}^{\tilde{N}} C_L P_L(\omega) \qquad (8)$$

where:

$$C_L = \frac{\int_A^B |T_{12}(\omega)|^2 P_L(\omega)\,d\omega}{\int_A^B P_L^2(\omega)\,d\omega} \qquad (9)$$

The interval of orthogonality is from A to B, or in the case of Legendre polynomials, from -1 to +1.

Equation 9 reduces to:

$$C_L = \frac{\int_{-K}^{K} P_L(\omega)\,d\omega}{\int_{-1}^{+1} P_L^2(\omega)\,d\omega} \qquad (10)$$

For the Legendre polynomials:[2]

$$\int_{-1}^{+1} P_L^2(\omega)\,d\omega = \frac{2}{2L+1} \qquad (11)$$

The odd terms are zero because the ideal function is an even function. Therefore the complete solution becomes:

$$C_L = (2L+1)\int_0^K P_L(\omega)\,d\omega \qquad (12)$$

$$L = 0, 2, 4 \cdots\cdots n$$

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1 The bar on top of $|T_{12}(\tilde{\omega})|^2$ is used to show that it is an approximation of the ideal function $|T_{12}(\omega)|^2$.

2 Erwin Kreyszig, Advanced Engineering Mathematics (New York: John Wiley and Sons, Inc. 1962) p. 515.

This can be written as:

$$|T_{12}(\omega)|^2 = \sum_{m=0}^{n} C_{2m} P_{2m}(\omega) \qquad (13)$$

where:

$$C_{2m} = (4m+1) \int_0^K P_{2m}(\omega)\, d\omega \qquad (14)$$

The first ten even Legendre polynomials and their first integral between 0 and K are given in Appendix I.

III Conversion of the Approximation into a Rational Function.

A Padé approximant[1] is used to convert the Legendre approximation into a rational function. This conversion is very accurate near the origin but the accuracy decreases as w approaches unity. To improve the approximation in the range of frequency which is greater than unity it was decided to have the largest possible difference between the orders of the numerator and denominator of the Padé approximant. That is the (0, 2n) Pade approximant.

$$|T_{12}(\tilde{\omega})|^2 \overset{(0+2n)}{=\!=} \frac{1}{B_{2n}(\omega)} \qquad (15)$$

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1 S. Karni, Network Theory: Analysis and Synthesis (Boston: Allyn and Bacon, Inc. 1966) pp. 367 - 368.

These functions can be written down more explicitly as

$$|T_{12}(\tilde{\omega})|^2 = a_0 + a_2\omega^2 - - - - - - - - - - a_{2n}\omega^{2n} \qquad (16)$$

The coefficients of equation 16 are determined from equations 13 and 14.

$$B_{2n}(\omega) = b_0 + b_2\omega^2 - - - - - - - - - - b_{2n}\omega^{2n} \qquad (17)$$

In appendix II it was shown that the coefficients of equation 17 were given by

$$b_0 = \frac{1}{a_0} \qquad (18)$$

$$b_{2n} = -\frac{1}{a_0}\left[\sum_{k=1}^{n} a_{2k}\, b_{2(n-k)}\right] \qquad (19)$$

The problem is solved by choosing values for n and K and determining the approximation of the magnitude squared using the previous equations. The effect of K is very difficult to determine explicitly so the method in this work is to chose specific values for K and examine the resulting magnitude approximation.

CHAPTER III

THE SECOND ORDER FILTER

The second order case is relatively simple but it gave a good idea of the effect of varying K and the problems involved with this method of filter design.

I. Solution of Magnitude Characteristics

Solving equations 13 and 14 using n = 2 yields:

$$|T_{12}(\bar{\omega})|^2 = a_0 + a_2\omega^2 + a_4\omega^4$$

where:

$$a_0 = 2.95\,K[K^4 - 1.85K^2 + 1.19] \qquad (20)$$

$$a_2 = 29.53\,K(1-K^2)(K^2 - .556) \qquad (21)$$

$$a_4 = 34.45\,K(K^2-1)(K^2 - .428) \qquad (22)$$

Substituting these into equations 18 and 19 it is found that:

$$b_0 = \frac{1}{a_0} \qquad (23)$$

$$b_2 = \frac{a_2}{a_0^2} \qquad (24)$$

$$b_4 = \frac{770.3[K^2-1][K^2-1.09][K^4-.99K^2+.257]}{a_0^3} \qquad (25)$$

The second order transfer function is given by:

$$T_{12}(s) = \frac{1}{d_0 + d_1 s + d_2 s^2} \qquad (26)$$

Where:

$$d_0 = \sqrt{b_0} \qquad (27)$$

$$d_1 = \sqrt{b_2 + 2\sqrt{b_0 b_4}} \qquad (28)$$

$$d_2 = \sqrt{b_4} \qquad (29)$$

II. Realizability of the Transfer Function

It was previously stated that all practical transfer functions must be realizable. A second order transfer function is realizable if the following conditions are met.

$$d_0 > 0 \qquad (30)$$

$$d_1 > 0 \qquad (31)$$

$$d_2 > 0 \qquad (32)$$

In appendix III it was shown that the second order function is realizable for all values of K in the region.

$0 < K < 1$

The coefficients b_o and b_4 are plotted, in figure 4A, as a function of K. From this graph the effect of the variation of K for the second order case can be seen. For a frequency greater than one the dominent term is b_4. Therefore the largest attenuation in the passband occurs where b_4 is largest, or at a K of about .9 . The region where a is less than one b_2 is negative and there is a ripple in the passband. This occurs for the region . The ripple is a maximum for a K of .9 . It is interesting to note that this is also where the attenuation is a maximum in the stop band. For a K less than .7 b_4 is relatively small and therefore the attenuation in the stop band is relatively poor. Therefore the form of the solution can be determined, in terms of K, from figure 4A.

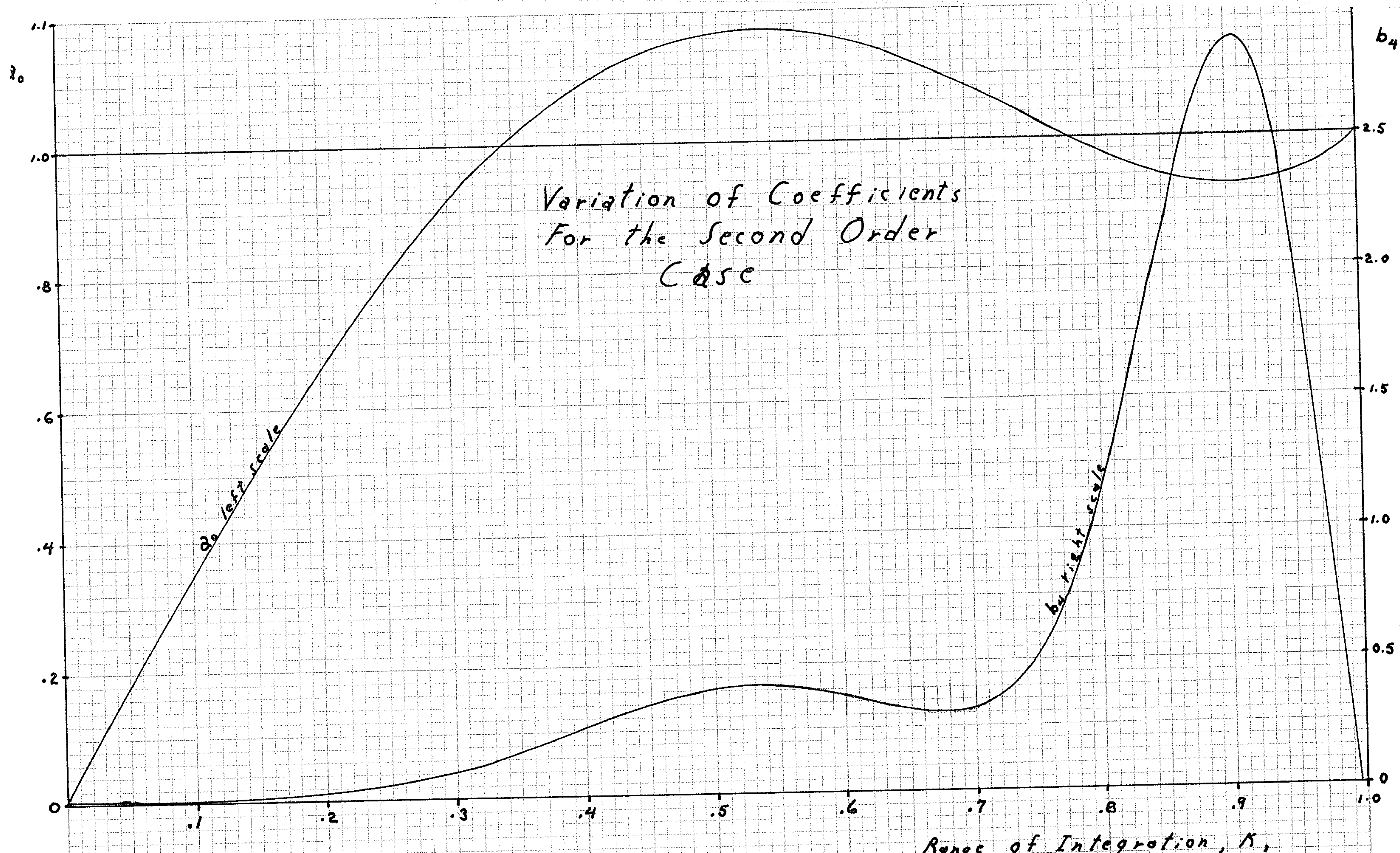


FIGURE 4A

P.T.
24/4/68

III. Specific Second Order Filters

Specific values of K are used in equations 20 to 25 yielding the following results:

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.09 + 4.5\,\omega^2 + 14.6\,\omega^4} \qquad K = .3$$

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{.925 + .593\,\omega^2 + 1.13\,\omega^4} \qquad K = .7$$

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{.96 + .765\,\omega^4} \qquad K = .745$$

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.02 - .742\,\omega^2 + 2.74\,\omega^4} \qquad K = .8$$

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.08 - 1.50\,\omega^2 + 4.74\,\omega^4} \qquad K = .9$$

The problem now is to determine a suitable frequency shift for these filters. The method used is to convert the filter for $K = .9$ into a Chebychev filter. The other filters are given the same maximum error in the pass band. This is done so that the filters could be compared. The magnitude of the filters is also scaled down a little so that it never exceeded one. The results are as follows:

I Using a K of .9 the following Chebychev filter with ϵ of .354 was obtained.

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.125 - .5\,\omega^2 + .5\,\omega^4}$$

II. Using a K of .8 the following filter is obtained.

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.04 - .267\omega^2 + .356\omega^4}$$

III. Using a K of .745 the following Butterworth filter is obtained.

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1 + .125\omega^4}$$

IV. Using a K of .7 the following filter, similar to a Bessel filter, is obtained.

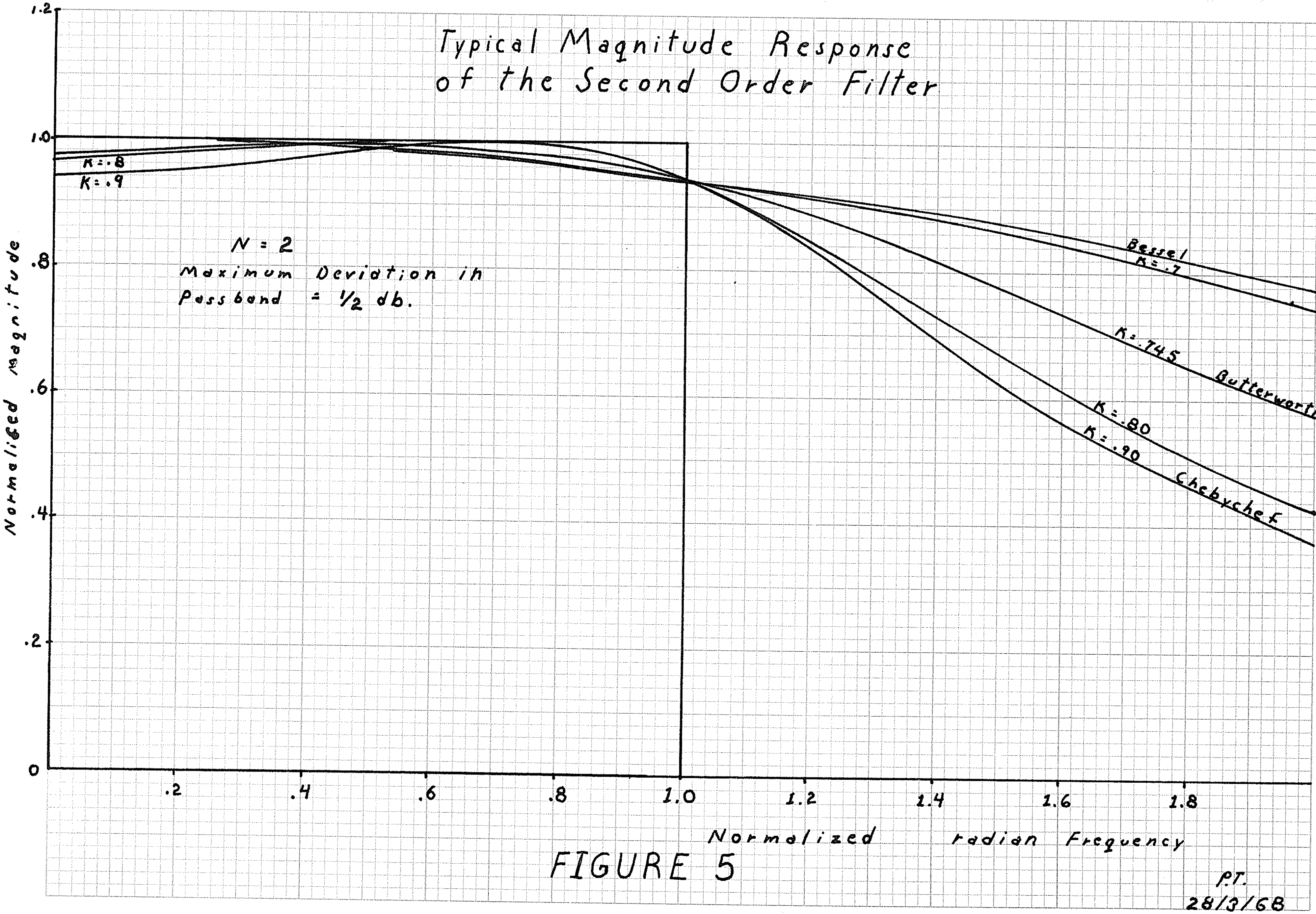
$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1 + .09\omega^2 + .026\omega^4}$$

The magnitude of these filters is plotted in figure 5.

The maximum rate of attenuation in the stop band is obtained for a K of .9. The best approximation near the origin is obtained for a K of approximatly .7. For K less than .7 the approximation became exceedingly poorer in the stop band, but this is not plotted.

IV. The Phase Characteristics of the Second Order Transfer Function

For the previous second order filters the actual transfer functions are obtained from equations 26 to 29.



Typical Magnitude Response
of the Second Order Filter
N = 2
Maximum Deviation in
Passband = 1/2 db.
K = .8
K = .9
Bessel
K = .7
K = .745
Butterworth
K = .80
K = .90
Chebychef
Normalized Magnitude
Normalized radian Frequency
1.2
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
P.T.
28/3/68

FIGURE 5

The results are as follows:

I. Using a K of .9

$$T_{12}(s) = \frac{1}{1.06 + s + .707 s^2}$$

II. Using a K of .8

$$T_{12}(s) = \frac{1}{1.02 + .973 s + .596 s^2}$$

III. Using a K of 745

$$T_{12}(s) = \frac{1}{1 + .84 s + .354 s^2}$$

IV. Using a K of .7

$$T_{12}(s) = \frac{1}{1 + .642 s + .1612 s^2}$$

For the second order transfer function the phase is determined from equations 1 and 4.

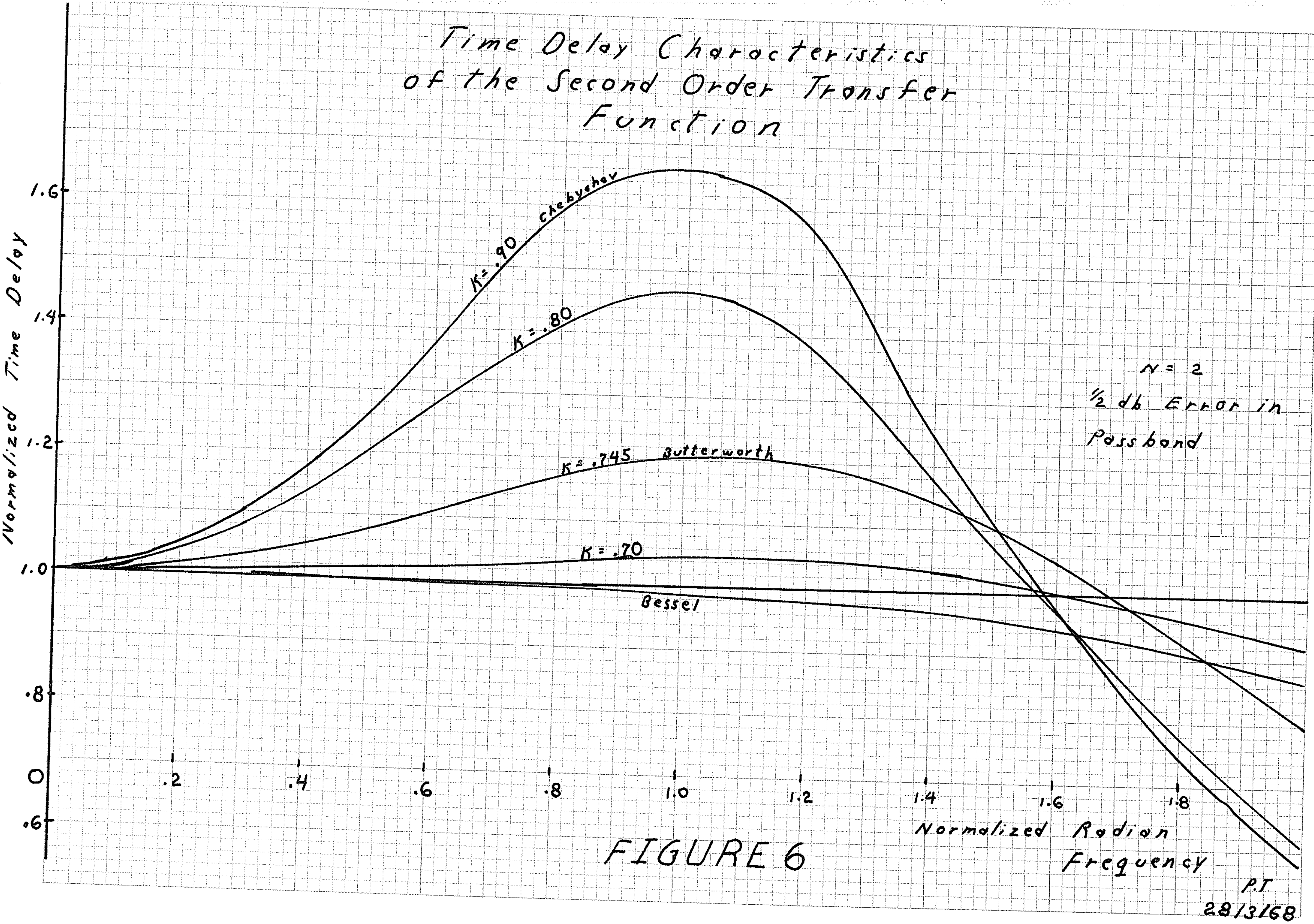
$$\beta(\omega) = -\tan^{-1}\left[\frac{\omega d_1}{d_0 - d_2 \omega^2}\right] \qquad (33)$$

which yields the following time delay

$$T_d(\omega) = \frac{d_1 [d_0 + d_2 \omega^2]}{[d_0 - d_2 \omega^2]^2 + \omega^2 d_1^2} \qquad (34)$$

The time delay for the second order filters is plotted in figure 6.

It can be seen from figure 6 that by a suitable choice of K the time delay can be varied from that of a Chebychev filter to that of a Bessel filter.



Time Delay Characteristics
of the Second Order Transfer
Function
Normalized Time Delay
1.6
1.4
1.2
1.0
.8
.6
O
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
chebychev
K=.90
K=.80
K=.745
Butterworth
K=.70
Bessel
N= 2
½ db Error in
Passband
Normalized Radian
Frequency
P.T.
28/3/68

FIGURE 6

In the synthesis of the second order filters realizability conditions are satisfied for $0 < K \leq 1$. The problem of choosing the value of K for a specified filter may be solved from curves of the most dominant prameters vs. K, as in figure 4A.

CHAPTER IV

HIGHER ORDER FILTERS

The method used for the second order filter is to solve the equations for a continuous range of K. This becomes very difficult for higher order filters. Therefore the method used is to choose specific values of K and examine the resulting filters.

I. Solution of the Third Order Filter

It is found for the third order case that the transfer functions are realizable for only a small range of K. The conditions for realizability for the third order case are shown in the book by Shlomo Karni[1].

The following third order cases are taken as examples. Equations 13 to 19 are used for the solution.

Using a K of .867

$$|T_{12}(\hat{\omega})|^2 = \frac{1}{1.015 - 1.36\,\omega^4 + 2.84\,\omega^6}$$

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[1] S. Karni, <u>Network Theory: Analysis and Synthesis</u> (Boston: Allyn and Bacon, Inc. 1966) pp. 109 - 114.

Using a K of .5

$$|T_{12}(\omega)|^2 = \frac{1}{.885 + 1.965\,\omega^2 + 4.62\,\omega^4 + 9.3\,\omega^6}$$

A frequency shift is again applied such that the maximum deviation in the pass band is $\frac{1}{2}$ db. The results are as follows:

For a K of .867

$$|T_{12}(\omega)|^2 = \frac{1}{1.04 - .465\,\omega^4 + .562\,\omega^6}$$

For a K of .5

$$|T_{12}(\omega)|^2 = \frac{1}{1 + .111\,\omega^2 + .013\,\omega^4 + .00132\,\omega^6}$$

The magnitude of these filters is plotted in figure 7 with the Butterworth, Chebychev and Bessel Filters. All five filters are given, by frequency shifting, the same maximum error in the pass band.

The maximum attenuation in the stop band is for the Chebychev filter, but in the pass band the total mean squared error is less for the filter obtained for a K of .867. The filter obtained for a K of .5 had a magnitude response very similar to that of the Bessel filter.

Typical Magnitude Response of the Third Order Filter

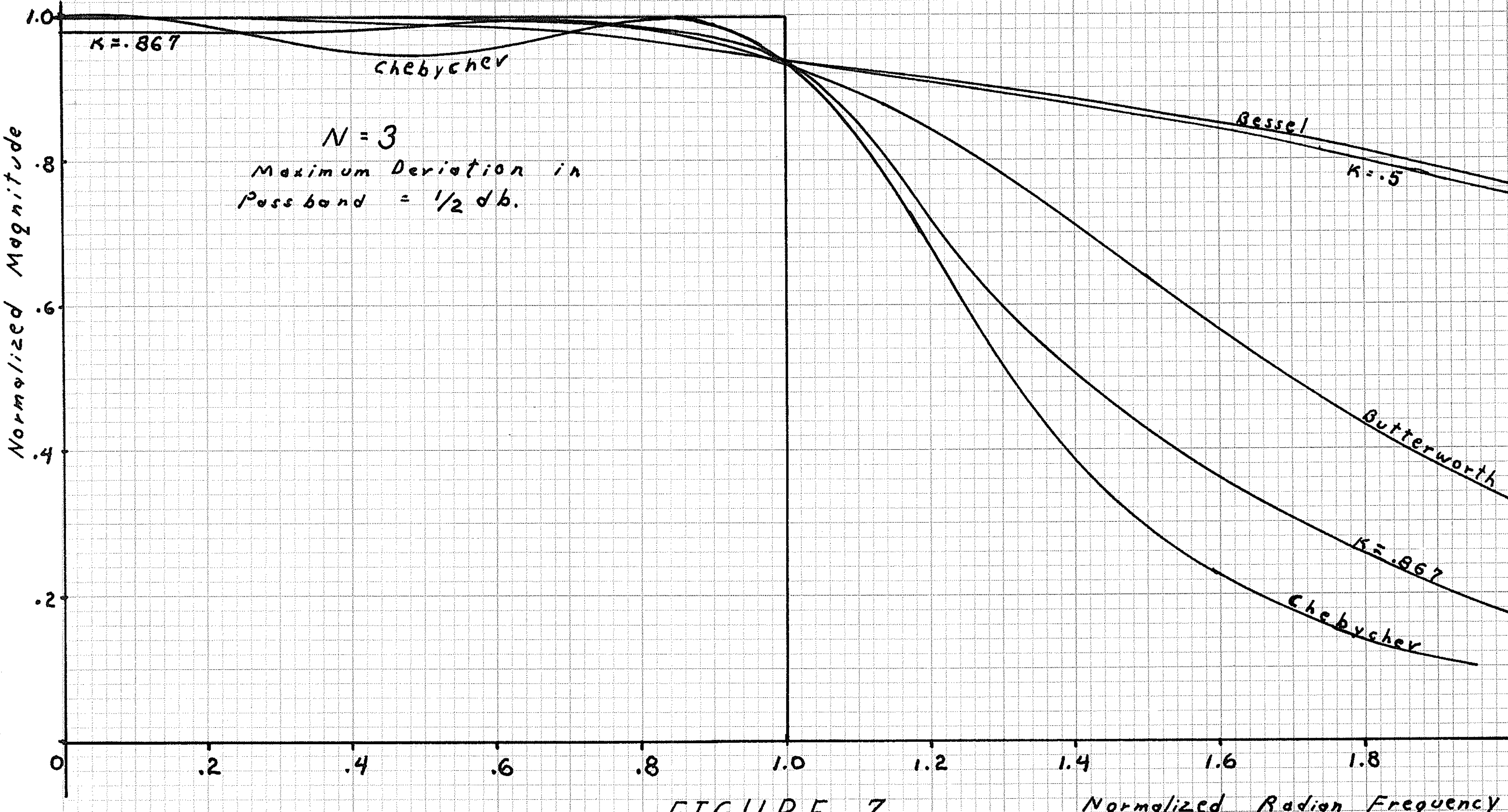

FIGURE 7

P.T.
28/3/68

II. The Phase Characteristics of the Third Order Function.

The actual transfer function is determined for these specific third order filters.

For a K of .867

$$T_{12}(s) = \frac{1}{1.02 + 1.72s + 1.45s^2 + .75s^3}$$

For a K of .5

$$T_{12}(s) = \frac{1}{1.0 + .71s + .196s^2 + .0363s^3}$$

From equations 1, 4 and 26 the time delay is determined for the third order case.[1]

$$T_d(\omega) = \frac{[d_0 - d_2\omega^2][d_1 - 3d_3\omega^2] + 2d_2\omega^2[d_1 - d_3\omega^2]}{[d_0 - d_2\omega^2]^2 + [d_1 - d_3\omega^2]\,\omega^2}$$

The normalized time delay for these filters is plotted in figure 8.

The filter obtained using a K of .5 is very similar to the Bessel filter. The filter obtained for a K of .867 has a time delay similar to the Chebychev filters.

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1 Y. H. Ku and M. Drubin "Network Synthesis Using Legendre and Hermite Polynomials," J. Franklin Inst., (February 1962) p. 147.

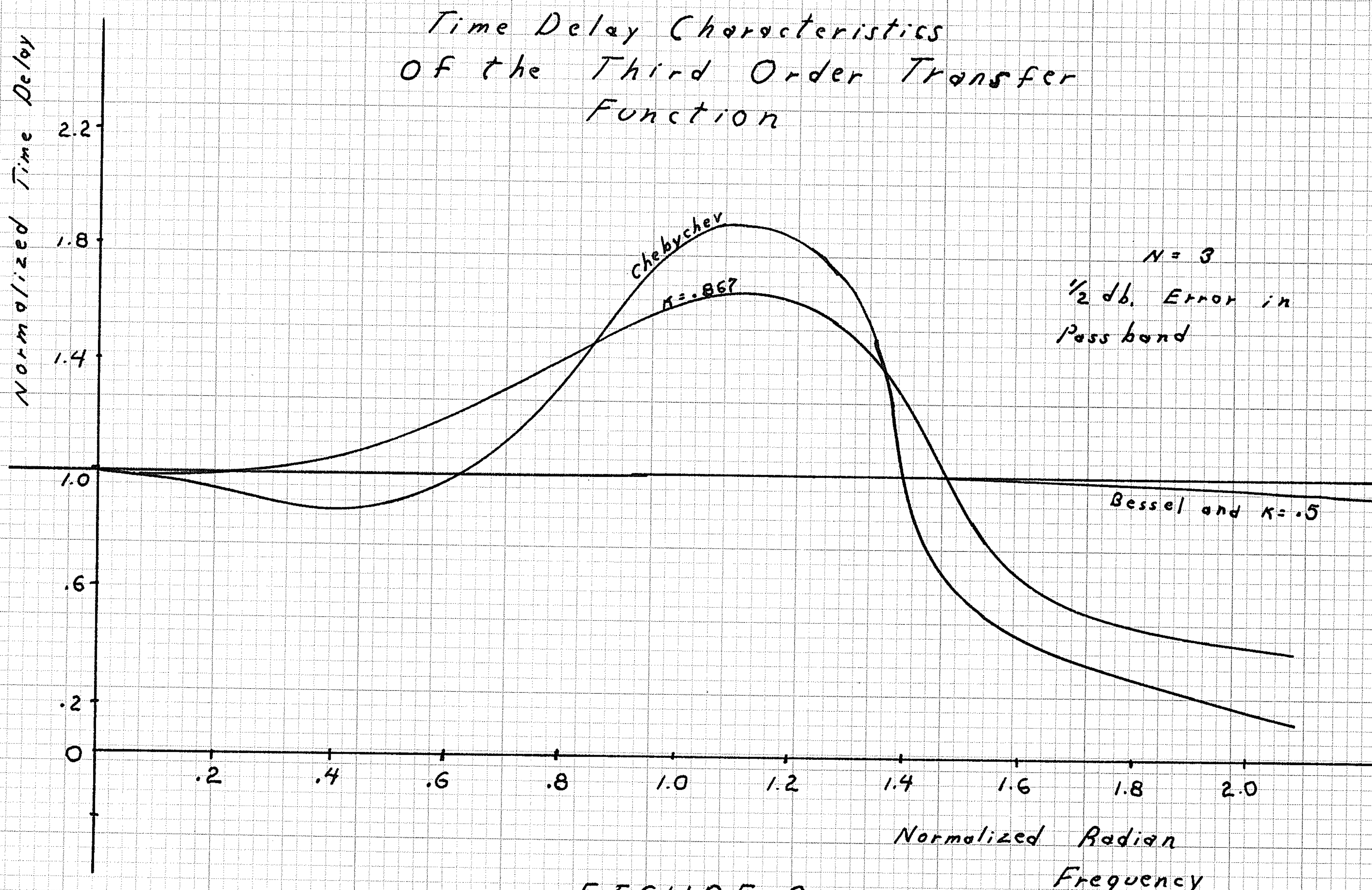

FIGURE 8

P.T.
28/3/68

III. Solution of the Fourth Order Filter

The fourth order filter is found to be realizable over a larger range of K than the third order filter. The transfer functions are tested for realizability in the same manner as the third order filter.

The following examples are obtained from equations 13 to 19. A frequency shift is used to limit the maximum error in the pass band to 1 db.

For a K of .95

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.056 - .305\omega^2 + .542\omega^4 - .503\omega^6 + .459\omega^8}$$

For a K of .8

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1 + .654\omega^2 - .918\omega^4 - .688\omega^6 + 1.168\omega^8}$$

For a K of .6

$$|T_{12}(\tilde{\omega})|^2 = \frac{1}{1.17 - .625\omega^2 + .738\omega^4 - .67\omega^6 + .65\omega^8}$$

The magnitude characteristic of these filters is plotted in figure 9. Again a Butterworth and Chebychev filter are plotted with a 1 db. maximum error in the pass band.

Typical Magnitude Response
Of the Forth Order Filter

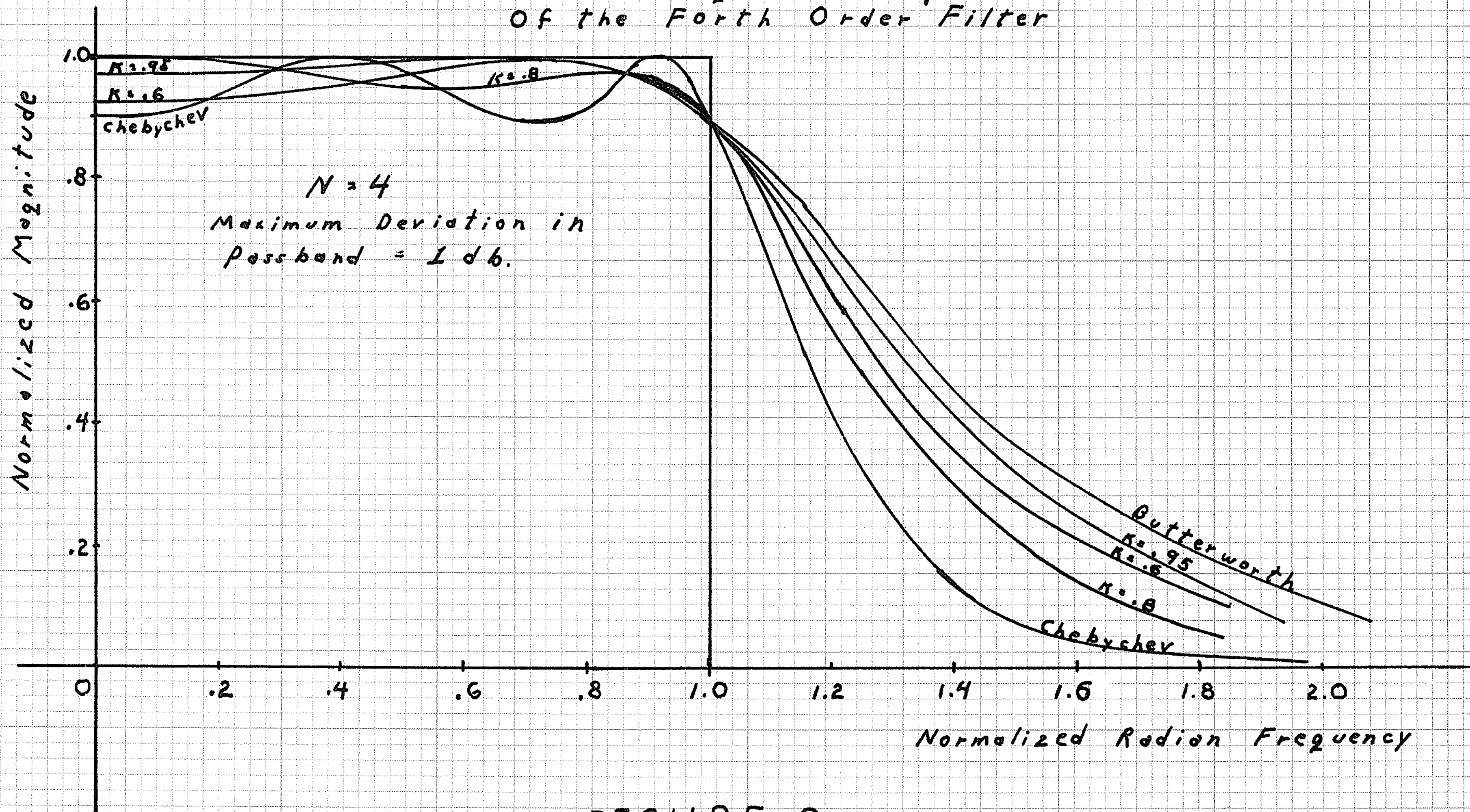

FIGURE 9

P.T.
28/3/68

It is well known that the Chebychev filter has the maximum attenuation in the stop band for a prescribed maximum error in the pass band. It is interesting to note that all the chosen filters have a rate of attenuation which falls between that of a Butterworth filter and Chebychev filter.

The actual transfer functions are determined for these filters.

For a K of .95

$$T_{12}(s) = \frac{1}{1.026 + 2.29s + 2.72s^2 + 1.78s^3 + .666s^4}$$

For a K of .8

$$T_{12}(s) = \frac{1}{1 + 2.66s + 3.2s^2 + 2.49s^3 + 1.08s^4}$$

For a K of .6

$$T_{12}(s) = \frac{1}{1.08 + 2.41s + 2.98s^2 + 2.04s^3 + .807s^4}$$

These filters were tested by the Hurwitz test and were all found to be realizable.

CHAPTER V

CONCLUSIONS

The object of this study was to approximate the ideal normalized amplitude response of a low-pass filter using the Legendre polynomials and convert this to a rational function by the use of a Padé approximant. The most difficult problem was the choice of the range of integration to meet given filter specifications. This problem was solved for the second order case by plotting the denominator coefficients in terms of K. Therefore the only solution was to determine filters for specific values of K.

The filter determined had two variables, the order of the filter, and the range of integration, K, in the process of approximating the low-pass filter by Legendre polynomials. By a suitable choice of K it was possible to obtain a phase response similar to that of the Bessel filter of the same order, or an amplitude response similar to that of the Chebychev filter of the same order. This method of approximation minimizes the mean squared error in the pass band. Therefore the pass band ripple is usually smaller than that of a Chebychev filter of the same order. The attenuation in the stop band, however, is less than that for the Chebychev filter, of the same maximum deviation in the pass band.

Only for the second order filter was realizability determined in terms of K. For higher order cases this problem may be solved by the use of a computer. This, however, would still be difficult on a computer because equations of the same order as the filter, must have the roots determined.

In filter design certain characteristics must be met, or approximated, such as passband ripple, rate of attenuation in the stop band, or phase characteristics. The use of this method of filter design gave no method of checking to see if the specified condition were met before the actual filter was determined except for the second order case.

For cases with n greater than 2 the roots of the filter's magnitude characteristic were hard to determine and therefore the transfer function was difficult to obtain. The problem of a proper frequency shift was made difficult because of the use of a Padé approximant. The Padé approximant becomes very poor as the radian frequency approaches one. Because of this fact the rational function's amplitude often cuts off before the expected value, K. This makes it hard to determine the proper frequency shift necessary to give a desirable passband ripple. The highest order coefficients in the denominator of the filter determine the amplitude response in the vicinity of cut-off. Therefore the use of a Padé approximant in some cases yields unsuitable values for these coefficients. Because of this fact the third order filter is not realizable for a large range of K.

If some other method was used to obtain a rational function the problems of frequency shifting and higher order cases may be avoided. This method might be one which approximates most accuratly in the centre of the passband. The Padé approximant may be altered to involve numerator terms or a different amount of denominator terms. A possible solution would be to determine a lot of filters using different values of K and n, using a computor. The results could be graphed for different values of n in terms of bandpass ripple, stop band attenuation and phase characteristics. Then if certain specifications were given a suitable solution could be determined from the graphs. There would still be the problem of determining a suitable frequency shift. This could be solved if a relationship could be found between bandpass ripple, stopband attenuation and the frequency shift.

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APPENDIX I

The first ten even terms of the Legendre polynomial are as follows.

$$P_0(\omega) = 1$$
$$P_2(\omega) = \frac{3}{2}\omega^2 - \frac{1}{2}$$
$$P_4(\omega) = \frac{1}{8}\left[35\omega^4 - 30\omega^2 + 3\right]$$
$$P_6(\omega) = \frac{1}{16}\left[231\omega^6 - 315\omega^4 + 105\omega^2 - 5\right]$$
$$P_8(\omega) = \frac{1}{128}\left[6435\omega^8 - 12012\omega^6 + 6930\omega^4 - 1260\omega^2 + 35\right]$$
$$P_{10}(\omega) = \frac{1}{256}\left[46189\omega^{10} - 109395\omega^8 + 90090\omega^6 - 30030\omega^4 + 3465\omega^2 - 63\right]$$

The value of $\int_0^{\kappa} P_{2n}(\omega)\,d\omega$ for these polynomials are as follows.

$$\int_0^{\kappa} P_0(\omega)\,d\omega = \kappa$$
$$\int_0^{\kappa} P_2(\omega)\,d\omega = \frac{\kappa}{2}\left[\kappa^2 - 1\right]$$
$$\int_0^{\kappa} P_4(\omega)\,d\omega = \frac{\kappa}{8}\left[7\kappa^4 - 10\kappa^2 + 3\right]$$
$$\int_0^{\kappa} P_6(\omega)\,d\omega = \frac{\kappa}{16}\left[33\kappa^6 - 63\kappa^4 + 35\kappa^2 - 5\right]$$
$$\int_0^{\kappa} P_8(\omega)\,d\omega = \frac{\kappa}{128}\left[715\kappa^8 - 1716\kappa^6 + 1386\kappa^4 - 420\kappa^2 + 35\right]$$
$$\int_0^{\kappa} P_{10}(\omega)\,d\omega = \frac{\kappa}{256}\left[4199\kappa^{10} - 12155\kappa^8 + 12870\kappa^6 - 6006\kappa^4 + 1155\kappa^2 - 63\right]$$

APPENDIX II

The determination of the coefficients of the Pade approximant given by equations (15), (16) and (17).

$$a_0 + a_2\omega^2 - - - - - a_{2n}\omega^{2n} \overset{0+2n}{=} \frac{1}{b_0 + b_2\omega^2 - - - - b_{2n}\omega^{2n}}$$

This can be rewritten as:

$$b_0 + b_2\omega^2 - - - - - b_{2n}\omega^{2n} \overset{0+2n}{=} \frac{1}{a_0 + a_2\omega^2 - - - - - a_{2n}\omega^{2n}}$$

Then the denominator is divided into the numerator on the Right hand side and equated to the coefficients on the left hand side.

$$b_0 + b_2\omega^2 + b_4\omega^4 - - - - - - b_{2n}\omega^{2n}$$

$$\frac{1}{a_0} - \frac{b_0 a_2}{a_0}\omega^2 - \frac{1}{a_0}\left[b_0 a_4 + a_2 b_2\right]\omega^4 - - - -$$

$$a_0 + a_2\omega^2 - - - - - - a_{2n}\omega^{2n} \,\big)\, 1$$

$$1 + b_0 a_2\omega^2 + b_0 a_4\omega^4 - - - - - b_0 a_{2n}\omega^{2n}$$

$$-b_0 a_2\omega^2 - b_0 a_4\omega^4 - - - - - - b_0 a_{2n}\omega^{2n}$$

$$-b_0 a_2\omega^2 + a_2 b_2\omega^4 - - - - - - b_2 a_{2n}\omega^{2n}$$

$$-\left[b_0 a_4 + a_2 b_2\right]\omega^4 - - - - \left[b_0 a_{2n} + b_2 a_{2n}\right]\omega^{2n}$$

An observation of the form of the solution yields

$$b_0 = \frac{1}{a_0} \qquad (18)$$

$$b_{2n} = -\frac{1}{a_0}\left[\sum_{k=1}^{n} a_{2k}\, b_{2(n-k)}\right] \qquad (19)$$

APPENDIX III

Test For the Realizability of the Second Order Transfer Function

Equations 30, 31 and 32 give the conditions for the realizability of a second order transfer function.

It can be seen from equation 20 that for K between zero and one that a_0 is always positive. Therefore from equation 23 and 25 it can also be seen that b_0 and b_4 are positive for K in this region. Using equations 27 - 32 the conditions for realizability become.

$$b_0 > 0 \qquad \text{III - 1}$$

$$b_2 + 2\sqrt{b_0 b_4} > 0 \qquad \text{III - 2}$$

$$b_4 > 0 \qquad \text{III - 3}$$

It has already been shown that conditions III - 1 and 3 are met. Condition III - 2 is met when b_2 is positive. This occurs in the range $0 < K < .556$ as can be seen from equations 21 and 24. It is necessary to prove that

that when $.556 < K < 1$ condition III - 2 is still met.

From equations, 23, 24, 25, 20, 21 and 22 condition III - 2 becomes.

$$4(770.3)[K^2-1][K^2-1.09][K^4-.99K^2+.257] > [29.53K(1-K^2)(K^2-.556)]^2$$

which becomes:

$$(K^2-1.137)(K^4-.946K^2+.241) < 0$$

This equation is met for $0 < K < \sqrt{1.137}$ and is therefore met for the range $0 < K < 1$ in particular. Therefore equations 30, 31 and 32 are always met for $0 < K < 1$ and the second order transfer function is always realizable.